Nested set complexes of Dowling lattices and complexes of Dowling trees

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Trivia

For a finite set P with a partial order $<$ (i.e., a *poset*):

$\Delta(P)$ is the simplicial complex with vertex set P and simplices given by the totally ordered subsets (*chains*) of P.

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Writing $\alpha = A_0|A_1|\cdots|A_k$ for the partition $\biguplus_{i=1}^{k} A_i = [n]$,

$$\alpha \leq \alpha' \iff \text{for all } i \text{ there is } j \text{ s.t. } A_i \subseteq A'_j.$$

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Definition. *A partition $\sigma = \sigma_0 \mid \sigma_1 \mid \cdots \mid \sigma_k$ of ${}^nG$, where $0 \in \sigma_0$, is called G-symmetric (n-)partition if $\circlearrowright$ induces an action of G on the blocks such that the orbit of every σ_i with $i > 0$ has length $|G|$.*

$$\sigma := \left\{\begin{matrix} 0, & (3,id) \\ & (3,g) \\ & (3,g^2) \end{matrix}\right\} \begin{matrix} \{(1,id),(6,g)\} & \{(2,id)\} & \{(4,id),(5,g^2)\} \\ \{(1,g),(6,g^2)\} & \{(2,g)\} & \{(4,g),(5,id)\} \\ \{(1,g^2),(6,id)\} & \{(2,g^2)\} & \{(4,g^2),(5,g)\} \end{matrix}$$

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$$\underline{\sigma} = \{0 \;,\; 3\} \quad \{1 \;,\; 6\} \quad \{2\} \quad \{4 \;,\; 5\}$$

Is the associated partition of $[n] \cup \{0\}$

THE DOWLING LATTICE

[Dowling '71]

Definition. *The Dowling lattice* $\mathcal{Q}_n(G)$ *is the set of* G*-symmetric partitions, partially ordered by refinement:*

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Theorem. [Dowling '71] *$\mathcal{Q}_n(G)$ is a geometric, supersolvable lattice. $\widetilde{\Delta}(\mathcal{Q}_n(G))$ is homotopy equivalent to a wedge of*

$$(|G|+1)(2|G|+1)\cdots((n-1)|G|+1)$$

spheres of dimension $(n-2)$.

The Dowling lattice

Example. Two comparable elements in $\mathcal{Q}_3(\mathbb{Z}_3)$.

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$$\bigvee$$

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Lemma. [Dowling '71] *For every* $\sigma = \sigma_0|\sigma_1|\cdots|\sigma_k \in \mathcal{Q}_n(G)$ *with associated partition* $\underline{\sigma} = S_0|S_1|\cdots|S_\ell$, *we have*

$$\mathcal{Q}_n(G)_{\geq\sigma} \simeq \mathcal{Q}_k(G)$$

and

$$\mathcal{Q}_n(G)_{\leq\sigma} \simeq \mathcal{Q}_{|S_0|-1}(G) \times \Pi_{|S_1|} \times \cdots \times \Pi_{|S_\ell|}.$$

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In particular, the subposet given by all elements having $\sigma_0 = \{0\}$ is a lower ideal of $\mathcal{Q}_n(G)$.

The poset $\mathcal{Q}_n^0(G)$

[Hultman '06]

Definition. *Let* $\mathcal{Q}_n^0(G)$ *be the subposet of* $\mathcal{Q}_n(G)$ *given by the partitions with trivial zero-block* σ_0. *Thus, for* $\sigma \in \mathcal{Q}_n^0(G)$ *we will let* $\underline{\sigma} \in \sigma_n$.

Remark. In Hultman's definition $\hat{0} \notin \mathcal{Q}_n^0(G)$. Our choice makes $\mathcal{Q}_n^0(G)$ a *meet-semilattice*.

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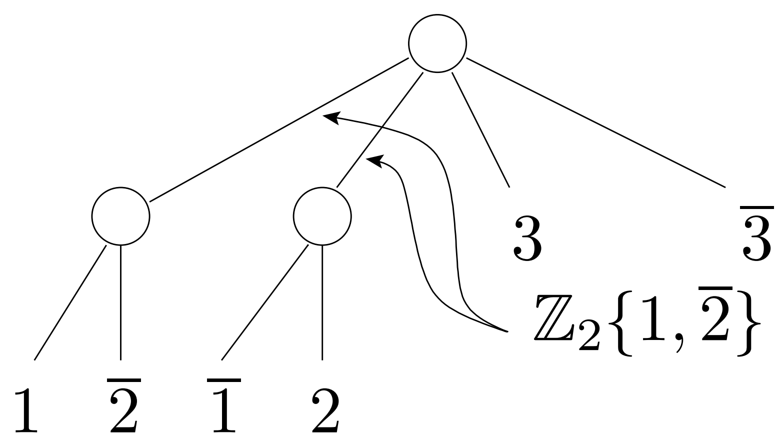
Definition. *A G-symmetric phylogenetic tree is a G-tree satisfying:*
(1) Every internal vertex (except the root) has degree ≥ 3.
(2) The tree is invariant under $\circlearrowright$.
(3) For all $g, h \in G$, $g \neq h$, and $i \in [n]$, the shortest path connecting (i, g) to (i, h) contains the root.

A $\mathbb{Z}_2$-symmetric phylogenetic tree with $n = 3$:



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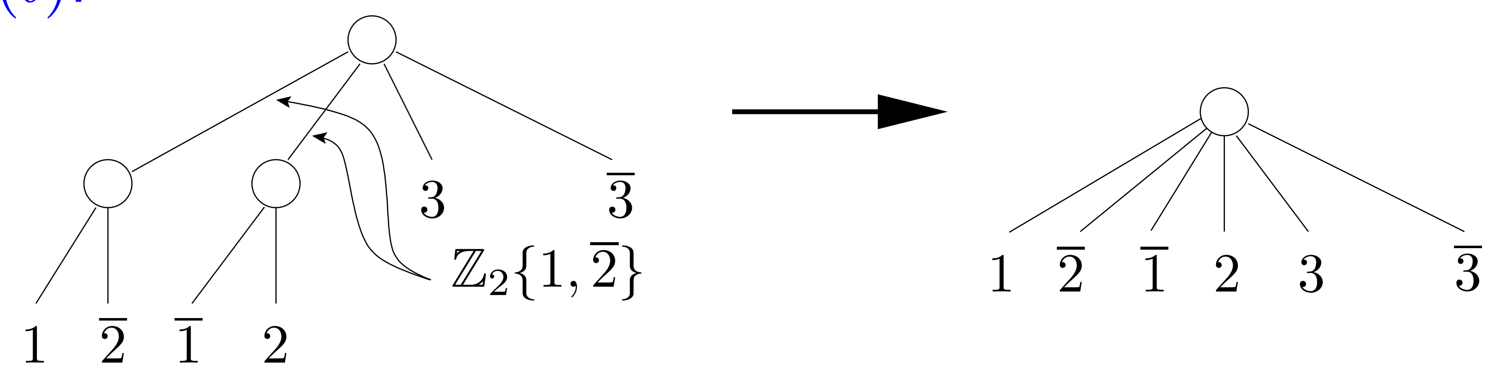
Definition. *Let $\mathcal{T}_n^G$ denote the set of G-symmetric phylogenetic trees.*

If t is an inner edge of a $T \in \mathcal{T}_n^G$, we define the contraction of its orbit $o(t)$:



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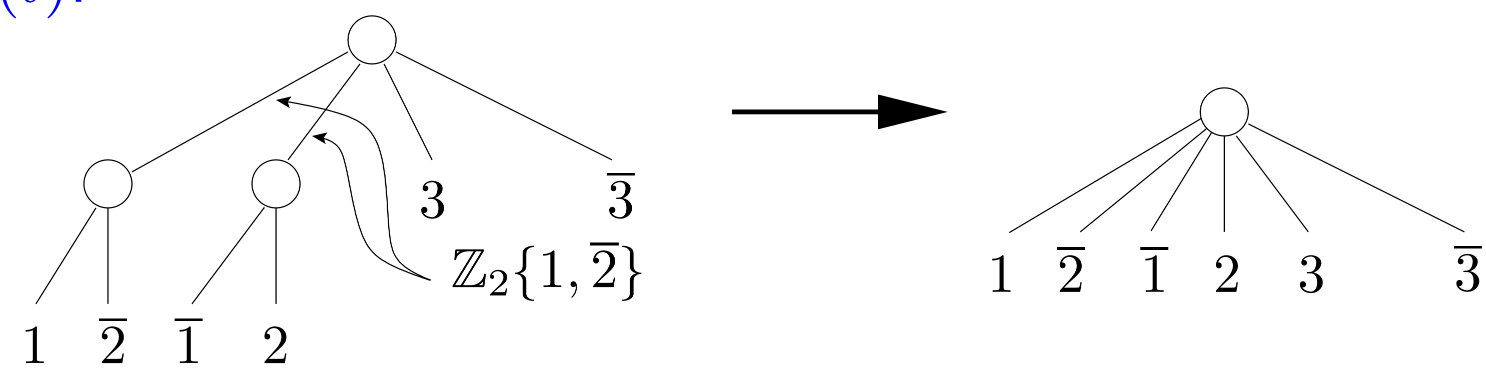

With the contraction of inner orbits as boundary operation, $\mathcal{T}_n^G$ is a pure simplicial complex of dimension $n-2$.

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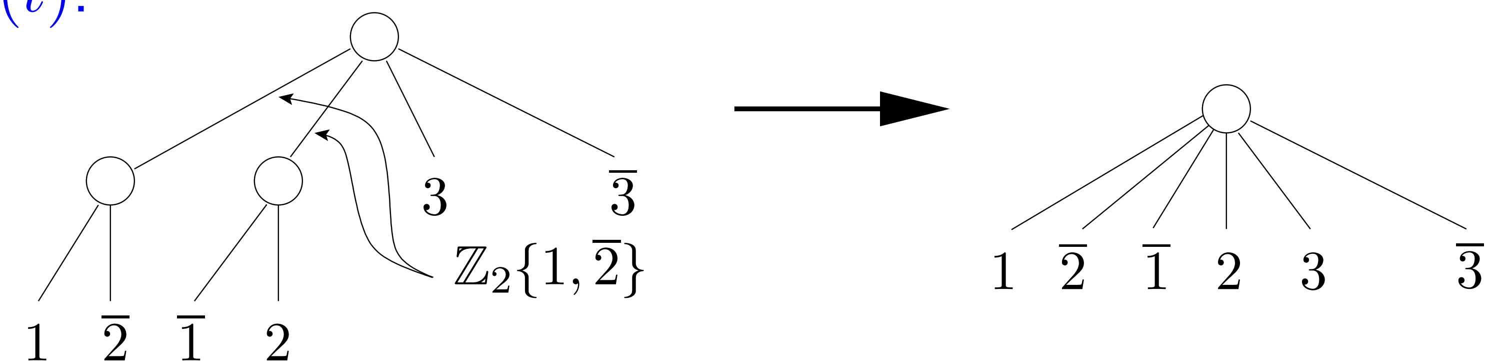

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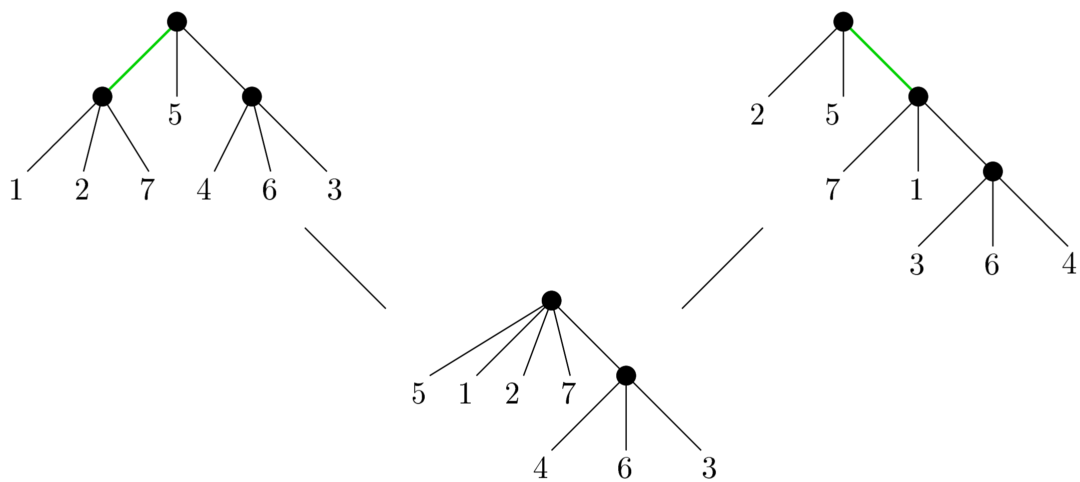
Theorem. [Hultman '06] *There is a homotopy equivalence*

$$\widetilde{\Delta}(\mathcal{Q}_n^0(G)) \simeq \mathcal{T}_n^G.$$

Complexes of trees

Definition. *The complex of (philogenetic) trees $\mathcal{T}_N$ is the abstract simplicial complex of rooted trees on N leaves with vertex degrees at least 3, except possibly for the root vertex.*
T_1 is a face of T_2 if T_1 is obtained from T_2 by a contraction of some set of internal edges.



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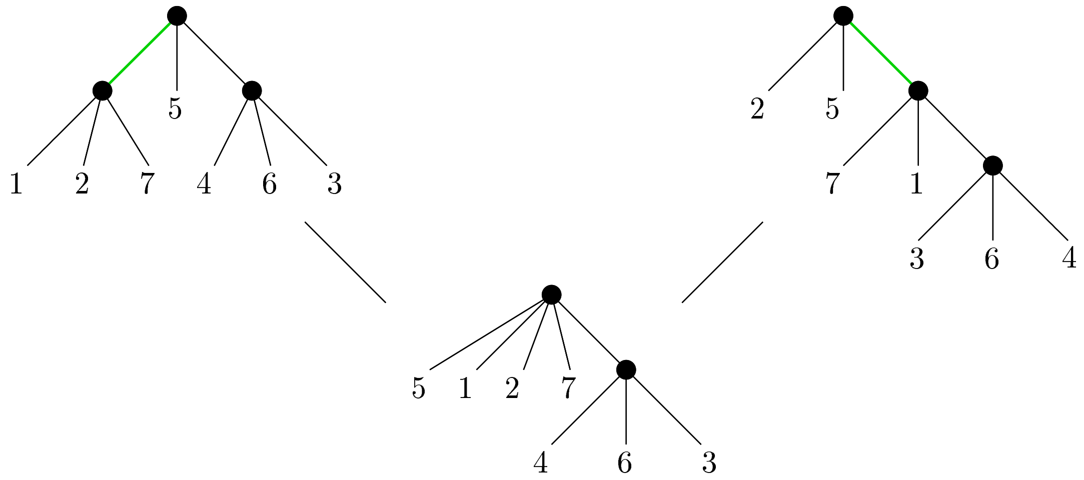

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Theorem. [Feichtner '04] $\Delta(\Pi_N)$ *can be obtained from $\mathcal{T}_N$ by a sequence of stellar subdivisions.*

BUILDING SETS

[De Concini, Procesi '95],
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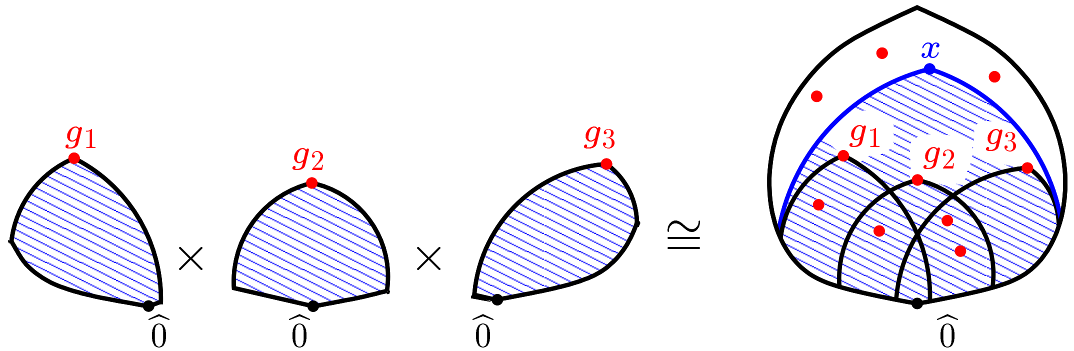
$\mathcal{L}$: finite meet-semilattice.

Definition. $\mathcal{G} \subseteq \mathcal{L}_{>\hat{0}}$ *is a building set if for any* $x \in \mathcal{L}_{>\hat{0}}$ *and* $\max \mathcal{G}_{\leq x} = \{g_1, \ldots, g_k\}$, *there exists an isomorphism*

$$\varphi_x : \prod_{i=1}^{k} [\hat{0}, g_i] \to [\hat{0}, x];$$



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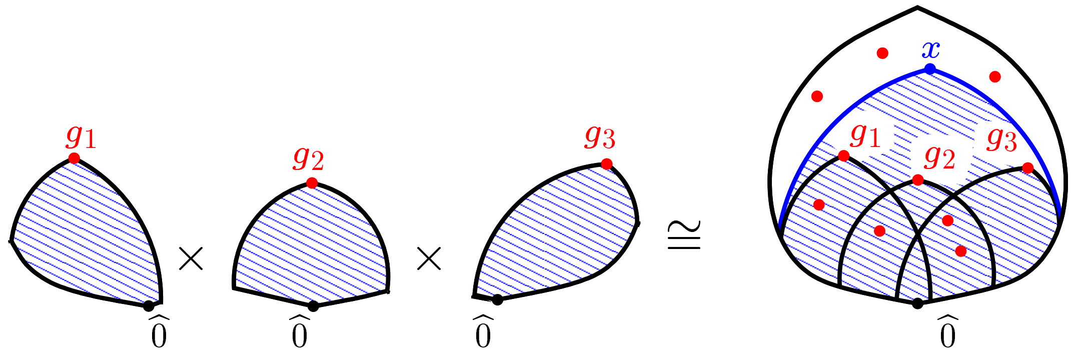

$\mathcal{G} := \mathcal{L}_{>\hat{0}}$ is the *maximal* building set
$\mathcal{G} := \mathcal{I} = \{x \in \mathcal{L}_{>\hat{0}} \mid x \text{ is irreducible}\}$ is the *minimal* building set.

Example. For Π_n, $\mathcal{I}$ is the set of one-nonsingleton-block partitions.

Nested set complexes

[De Concini, Procesi '95],
[Feichtner, Kozlov '01]

Definition. *Let* $\mathcal{G} \subseteq \mathcal{L}_{>\hat{0}}$ *be a building set.* $U \subset \mathcal{G}$ *is a nested set if for any incomparable elements* $x_1, \ldots, x_t \in U$, $t \geq 2$,

$$x_1 \vee \cdots \vee x_t \text{ exists, and is not an element of } \mathcal{G}.$$

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Remark. For $\mathcal{G} = \mathcal{L}_{>\hat{0}}$, $\mathcal{N}(\mathcal{L}, \mathcal{G}) = \Delta(\mathcal{L}_{>\hat{0}})$ and $\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{G}) = \widetilde{\Delta}(\mathcal{L})$.

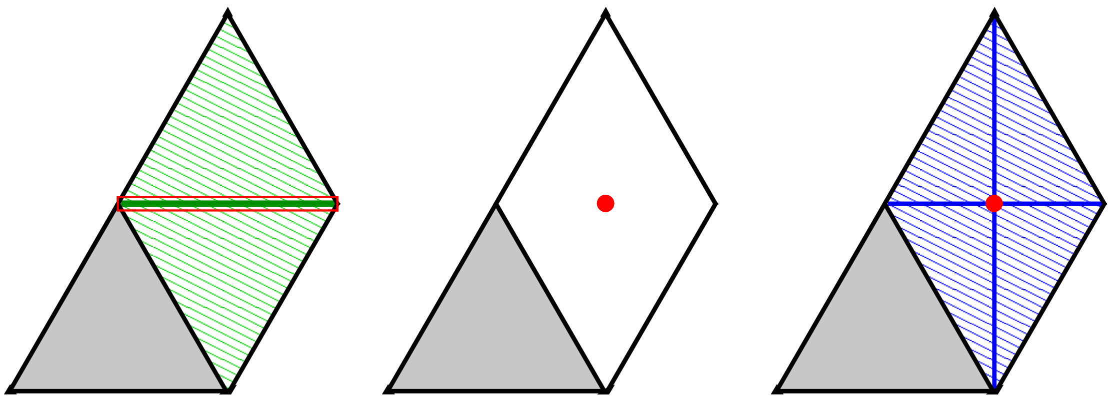
Example. We have $U \in \mathcal{N}(\Pi_n, \mathcal{I})$ if, and only if,

for any $\alpha, \alpha' \in U$, either $A \cap A' = \emptyset$ or $A \subset A'$ or $A' \subset A$.

where A (A') is the only nonsingleton block of α (α')

STELLAR SUBDIVISIONS

Theorem. [Feichtner, Müller '03; Čukić, D. '05] *Let $\mathcal{L}$ be a meet-semilattice, $\mathcal{H} \subseteq \mathcal{G}$ building sets of $\mathcal{L}$. The simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is obtained from $\mathcal{N}(\mathcal{L}, \mathcal{H})$ by a sequence of stellar subdivisions .*



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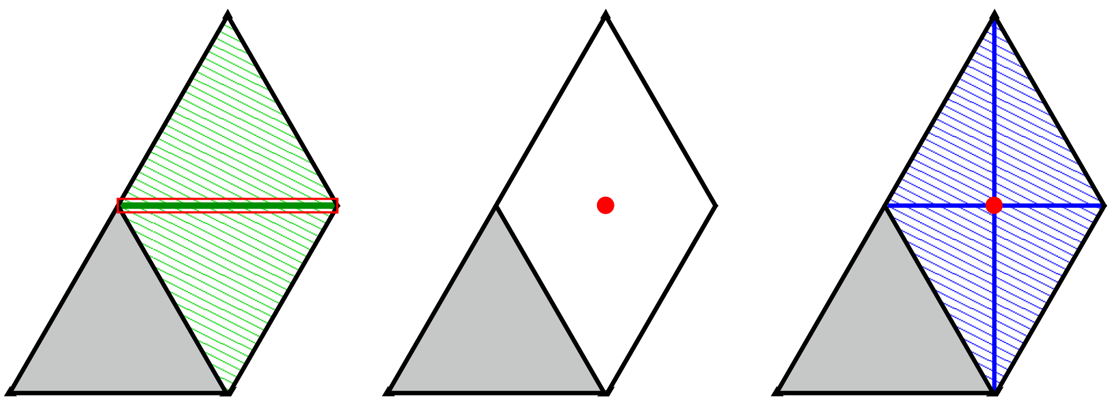

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Note: The abstract simplicial complexes $\mathcal{N}(\mathcal{L}, \mathcal{G})$, $\mathcal{N}(\mathcal{L}, \mathcal{H})$ are homeomorphic : there are geometric realizations $|K_{\mathcal{G}}|$ of $\mathcal{N}(\mathcal{L}, \mathcal{G})$ and $|K_{\mathcal{H}}|$ of $\mathcal{N}(\mathcal{L}, \mathcal{H})$ with a PL-homeomorphism $\phi : |K_{\mathcal{G}}| \rightarrow |K_{\mathcal{H}}|$.

SUBDIVISION OF $\mathcal{T}_n^G$

Recall the partition lattice Π_n with its minimal building set $\mathcal{I}$.

Definition. *Let*

$$\mathcal{I}^G := \{\sigma \in \mathcal{Q}_n^0(G) \mid \underline{\sigma} \in \mathcal{I}\}.$$

Since lower intervals in $\mathcal{Q}_n^0(G)$ have one factor for every block,

$\mathcal{I}^G$ is the minimal building set of $\mathcal{Q}_n^0(G)$.

Theorem. [D. '06] *The simplicial complexes $\mathcal{N}(\mathcal{Q}_n^0(G), \mathcal{I}^G)$ and $\mathcal{T}_n^G$ are isomorphic.*

Corollary. $\boxed{\mathcal{T}_n^G \overset{sss}{\rightsquigarrow} \widetilde{\Delta}(\mathcal{Q}_n^0(G)).}$

What about $\mathcal{Q}_n(G)$?

Recall that, for $\sigma = \sigma_0|\sigma_1|\cdots|\sigma_k \in \mathcal{Q}_n(G)$, we have an isomorphism

$$\mathcal{Q}_n(G)_{\leq\sigma} \xrightarrow{\sim} \mathcal{Q}_{\frac{|\sigma_0|-1}{|G|}}(G) \times \Pi_{|\underline{\sigma}_1|} \times \cdots \times \Pi_{|\underline{\sigma}_\ell|}.$$

Thus, the minimal building set of $\mathcal{Q}_n(G)$ is

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Remark. We have $\mathcal{J}^G \cap \mathcal{Q}_n^0(G) = \mathcal{I}^G$. Moreover:

- If $X \in \mathcal{N}(\mathcal{Q}_n(G), \mathcal{J}^G)$, then $X \cap \mathcal{I}^G \in \mathcal{N}(\mathcal{Q}_n^0(G), \mathcal{I}^G)$, and
- $X \setminus \mathcal{I}^G$ *is a chain* .
- $\mathcal{N}(\mathcal{Q}_n^0(G), \mathcal{I}^G) \subset \mathcal{N}(\mathcal{Q}_n(G), \mathcal{J}^G)$.

Dowling trees?

Natural question: "is" $\mathcal{N}(\mathcal{Q}_n(G), \mathcal{J}^G)$ some 'bigger' complex of trees containing $\mathcal{N}(\mathcal{Q}_n^0(G), \mathcal{I}^G) = \mathcal{T}_n^G$?

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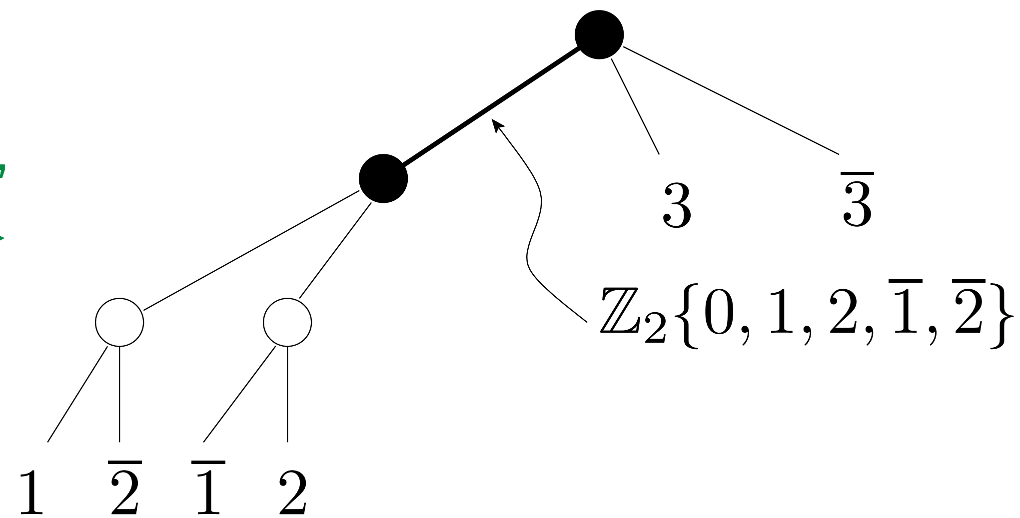
For example, consider the nested set

$$U := \big\{\{0,1,\overline{1},2,\overline{2}\}|\{3\}|\{\overline{3}\}, \{0\}|\{1,\overline{2}\}|\{\overline{1},2\}|\{3\}|\{\overline{3}\}\big\}$$

in $\mathcal{N}(\mathcal{Q}_n(G), \mathcal{J}^G)$.



The formal definition is now at hand.

Dowling trees!

Recall G-trees from before.

Definition. *A* *Dowling tree* *is a* G*-tree with some distinguished vertices, called* 0*-vertices, satisfying:*

(0) The subgraph of the 0*-vertices is a path starting at the root.*

(1) Every internal vertex (except the root) has degree ≥ 3*.*

(2) The tree is invariant under $\circlearrowright$*, which fixes the* 0*-vertices.*

(3) For all $g, h \in G$*,* $g \neq h$*, and* $i \in [n]$*, the shortest path connecting* (i, g) *to* (i, h) *contains exactly one* 0*-vertex.*

The set of Dowling trees is denoted $\mathcal{T}_n(G)$*.*

THE COMPLEX $\mathcal{T}_n(G)$

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Remark. The operation of contracting an orbit of inner edges is still well defined, and makes $\mathcal{T}_n(G)$ into a simplicial complex.

Theorem. [D.'06] *The simplicial complexes $\widetilde{N}(\mathcal{Q}_n(G), \mathcal{J}^G)$ and $\mathcal{T}_n(G)$ are isomorphic.*

...in particular, $\boxed{\mathcal{T}_n(G) \overset{sss}{\rightsquigarrow} \widetilde{\Delta}(\mathcal{Q}_n(G)).}$

$\mathcal{T}_n^G$ INSIDE $\mathcal{T}_n(G)$

For $X \in \mathcal{N}(\mathcal{Q}_n(G), \mathcal{J}^G)$, let $X_0 := X \setminus \mathcal{I}^G$, $X_1 := X \cap \mathcal{I}^G$.

Recall that X_0 'is' a chain $x_1 \subset x_2 \subset \cdots \subset x_m$ of 'zero blocks'.

$$\mathcal{K}_m := \{X \in \mathcal{N}(\mathcal{Q}_n(G), \mathcal{J}^G) \mid X_0 \leq m\}.$$

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Fix X_0. Then X_1 can contain any nested set of partitions whose nonsingleton blocks are contained into one of the $x_i \setminus x_{i+1}$'s.

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Lemma. [D.'06] *The link of X_0 in $\mathcal{K}_m$ is*

$$lk_{\mathcal{K}_m}(X_0) = \widetilde{\Delta}(B_m) * \widetilde{\Delta}(\mathcal{Q}^0_{p_1}(G)) * \cdots * \widetilde{\Delta}(\mathcal{Q}^0_{p_m}(G)).$$

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For $Y = x_0 \subset x_1 \subset \cdots \subset x_{m+1}$ as before, let

$$q_i^Y := \prod_{j=1}^{p_i-1}(j|G|-1), \qquad Q(Y) := q_1^Y q_2^Y \cdots q_m^Y$$

Theorem. [D.'06] *$lk_{\mathcal{K}_m}(Y)$ is a wedge of $Q(Y)$ spheres of dimension $(m-3)$, and all these spheres bound in $\mathcal{K}_m$.*

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Thus, every chain $Y \in \widetilde{\Delta}(B_n)$ indexes a simplex that contributes $Q(Y)$ times to the difference between the number of spheres in the homotopy types of $\mathcal{Q}^0_n(G)$ and $\mathcal{Q}_n(G)$.

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Corollary. *For every $k \geq 1$ and $n \geq 2$,*

$$\prod_{j=1}^{n}(jk+1) - \prod_{j=1}^{n}(jk-1) = \sum_{\omega \in \widetilde{\Delta}(B_n)} \prod_{j=1}^{n}(jk-1)^{h(\sigma,j)},$$

where $h(\omega, j)$ is the height of the j-th row in the Young tableau of the partition determined by ω.

Related topics I

Theorem. [Feichtner, Sturmfels '05] *Let* $\mathcal{L}$ *be a geometric lattice,* $\mathcal{M}$ *the associated matroid.*
Suppose that, for every irreducible $x \in \mathcal{L}$ *and every* $y < x$*, the interval* $[y, x]$ *is irreducible. Then, the Bergman fan of* $\mathcal{M}$ *equals the reduced nested set complex* $\widetilde{\mathcal{N}}(\mathcal{L}, \mathcal{I})$*.*

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If $\mathcal{L} = \mathcal{Q}_n(G)$, the irreducibles are the $\sigma \in \mathcal{J}^G$. So let $\sigma \in \mathcal{J}^G$ and consider any $\sigma' \leq \sigma$. Say $\underline{\sigma}'$ has k blocks. Then

either $[\sigma', \sigma] \simeq \Pi_k$ (if $0 \notin S$) or else $[\sigma', \sigma] \simeq \mathcal{Q}_k$.

Corollary. [D'06] *The Bergman complex of the Dowling geometry equals the corresponding complex of Dowling trees.*

Related topics II

Gottlieb & Wachs ['00] defined a complex of rooted forests $\mathcal{F}(n, G)$ with leaves labelled by elements of ${}^nG$. These forests are used to encode generators of the multilinear component of the enveloping algebra of the fixed point subalgebra of the free Lie superalgebra on $[n] \times G$ on the one side, and of the cohomology of $\mathcal{Q}_n(G)$ on the other side.

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Remark. There is a natural isomorphism of simplicial complexes

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Lemma. [Ardila, Wachs] $\mathcal{F}(n, G)$ *equals the Bergman complex of the Dowling lattice.*

Moreover,

Zaslavsky ['89, '91, ...] developed a theory of biased graphs as a matroidal generalization of Dowling lattices. In that context, the poset $\mathcal{Q}_n^0(G)$ has a natural counterpart: the so-called *semilattice of balanced flats*.

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