Catalan lattices
and
realizers of triangulations

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Joint work with Nicolas Bonichon

FPSAC, Tianjin, July 2007

Question [Chapoton] :

Why does the number of intervals in the Tamari lattice of size n equal the number of triangulations of size n?

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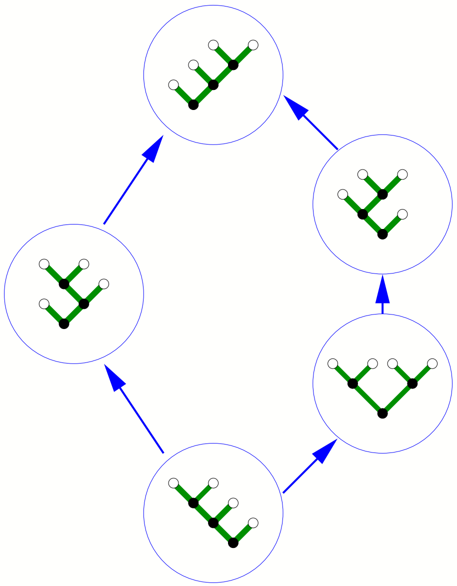

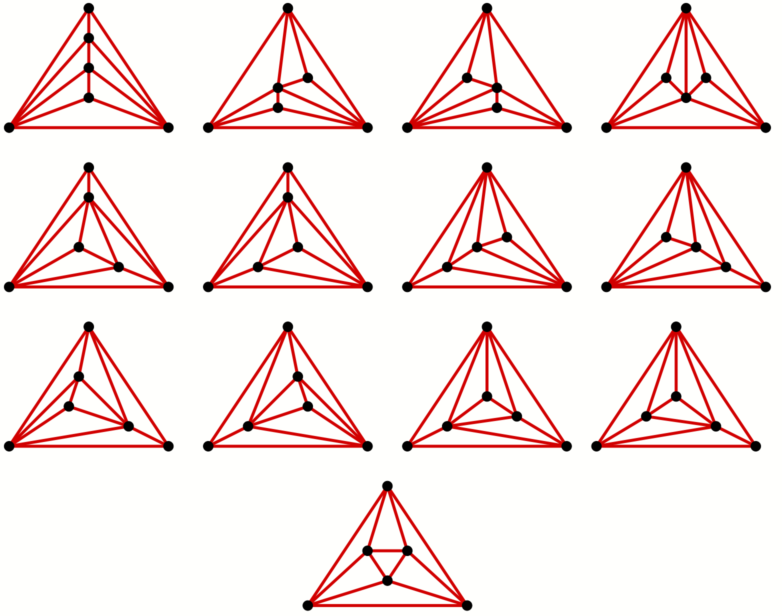

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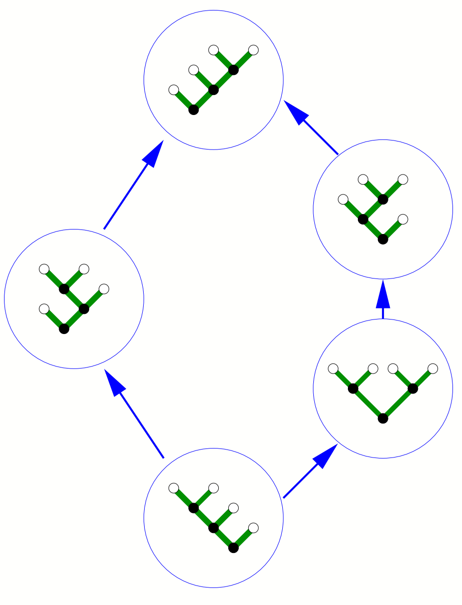
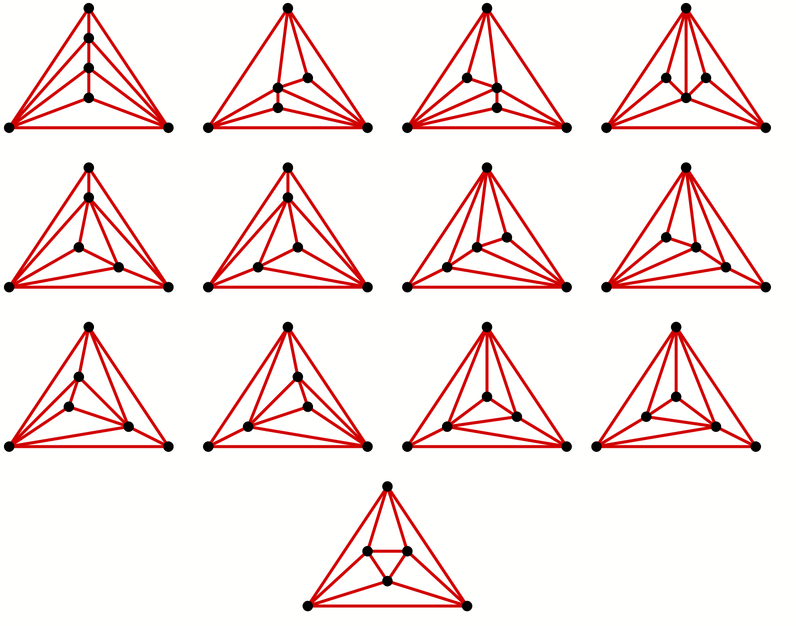




$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

[Chapoton 06] **[Tutte 62, Poulalhon & Schaeffer 03]**

Broader picture :

Stanley intervals $\Longleftrightarrow$ Realizers of triangulations.

$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

Tamari intervals $\Longleftrightarrow$ Triangulations.

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

Kreweras intervals $\Longleftrightarrow$ Stack triangulations.

$$\frac{1}{2n+1}\binom{3n}{n}$$

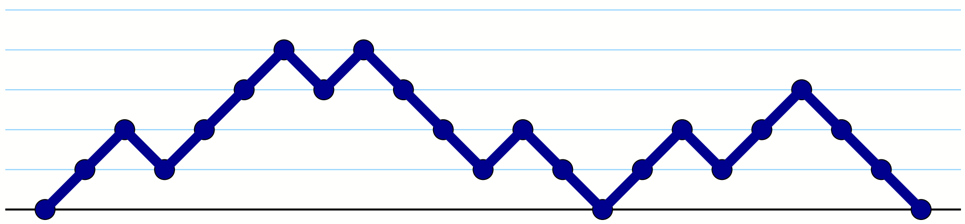
Broader picture :

Stanley intervals	$\Longleftrightarrow$ $\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$	Realizers of triangulations. **[Bonichon '02]**
Tamari intervals	$\Longleftrightarrow$ $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$	Triangulations.
Kreweras intervals	$\Longleftrightarrow$ $\frac{1}{2n+1}\binom{3n}{n}$	Stack triangulations. **[Edelman '82]**

Catalan lattices

Dyck paths

A Dyck path is made of +1, -1 steps, starts from 0, remains non-negative and ends at 0.



There are $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ Dyck paths of size n (length $2n$).

Catalan objects

Dyck paths :

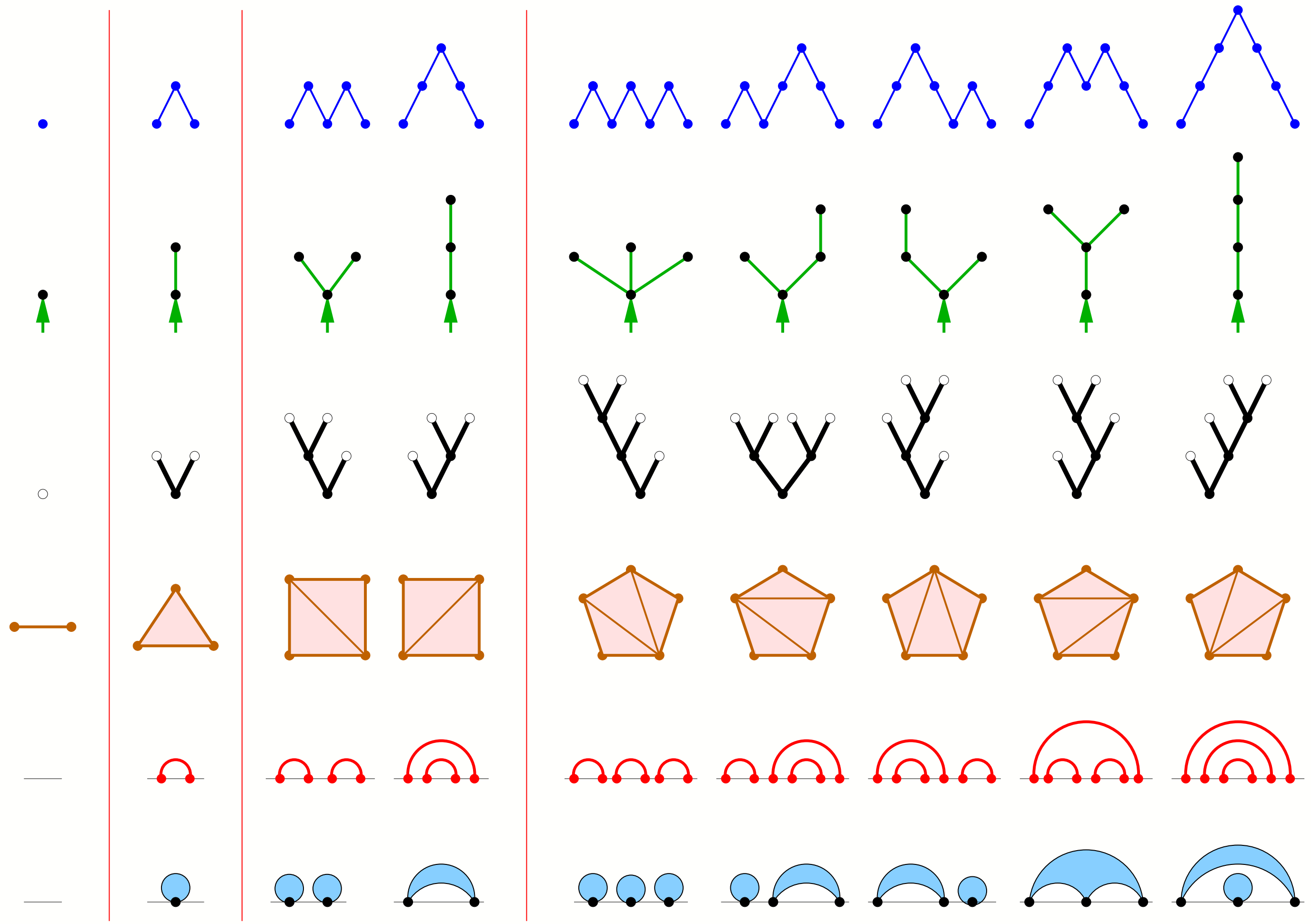
Plane trees :

Binary trees :

Decomposition of polygons :

Parenthesis systems :

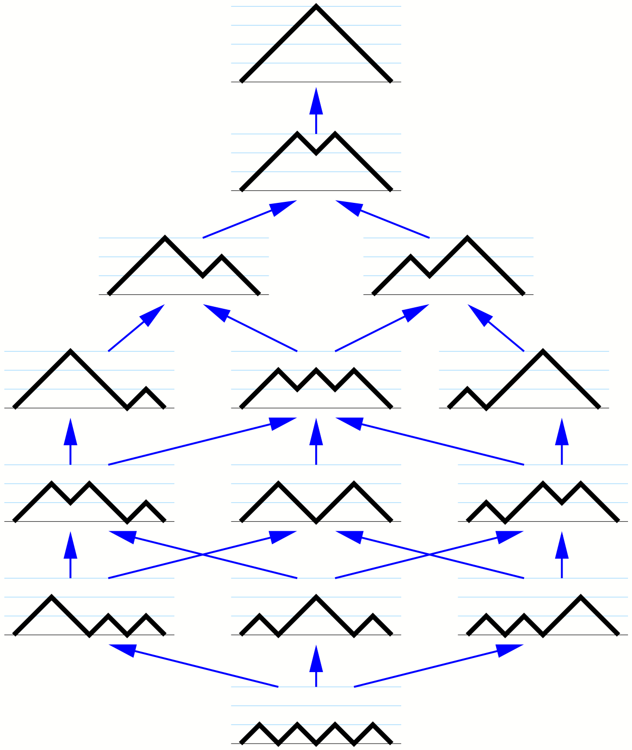
Non-crossing partitions :



Stanley lattice

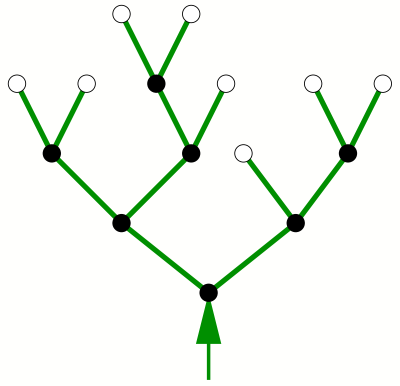
The relation of being above defines the Stanley lattice on the set of Dyck paths of size n.

Hasse Diagram $n = 4$:



Tamari lattice

The Tamari lattice is defined on the set of binary trees of size n (n nodes).



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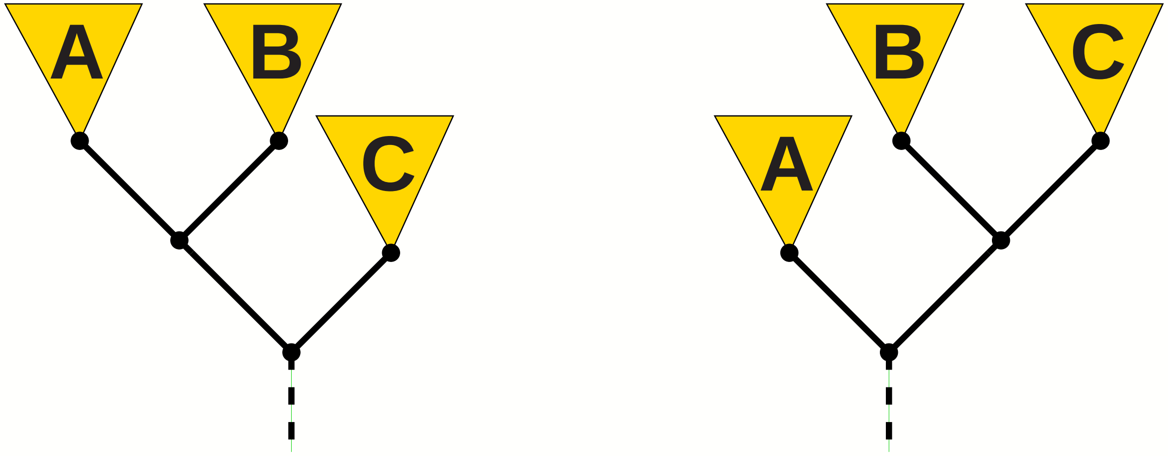
The covering relation corresponds to right-rotation.



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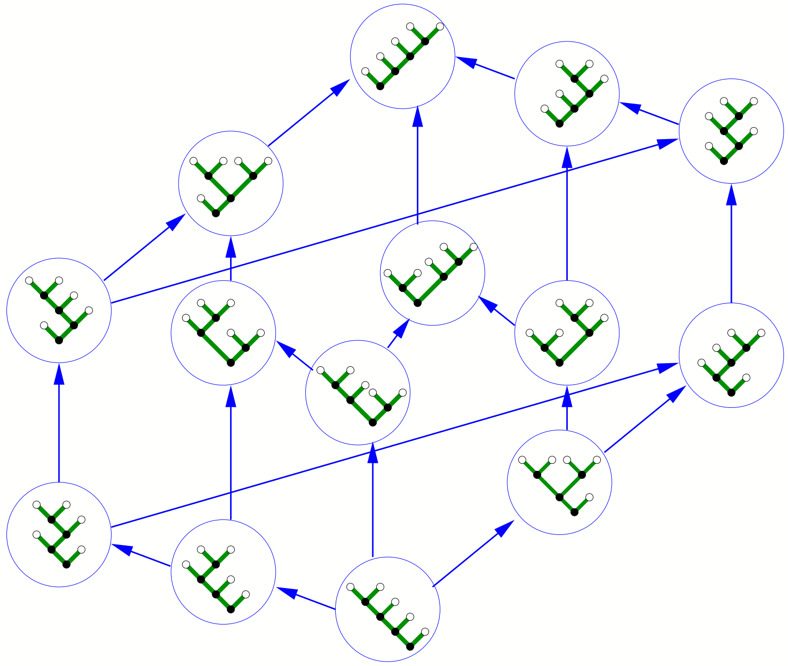
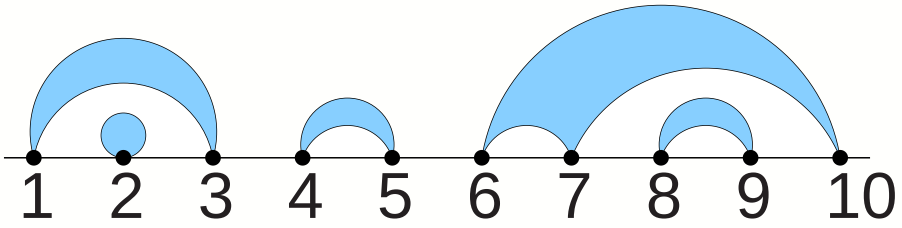


Kreweras lattice

The Kreweras lattice is defined on the set of non-crossing partitions of $\{1, \ldots, n\}$.



Kreweras lattice

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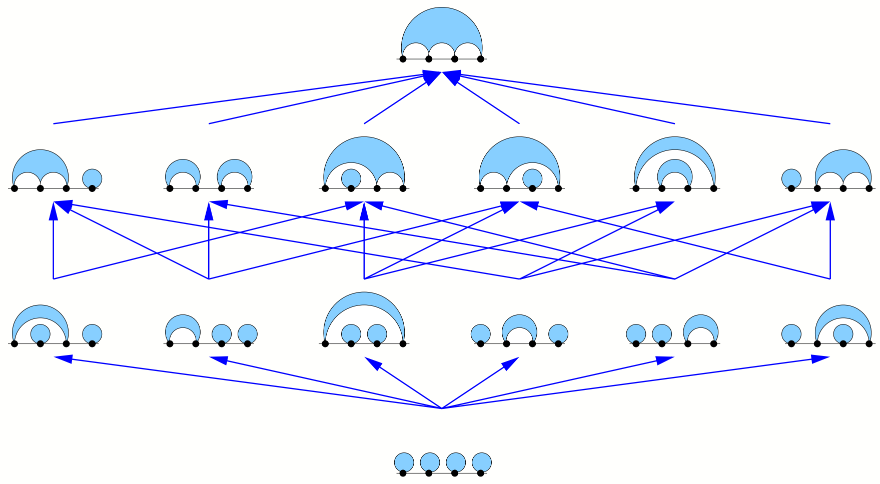
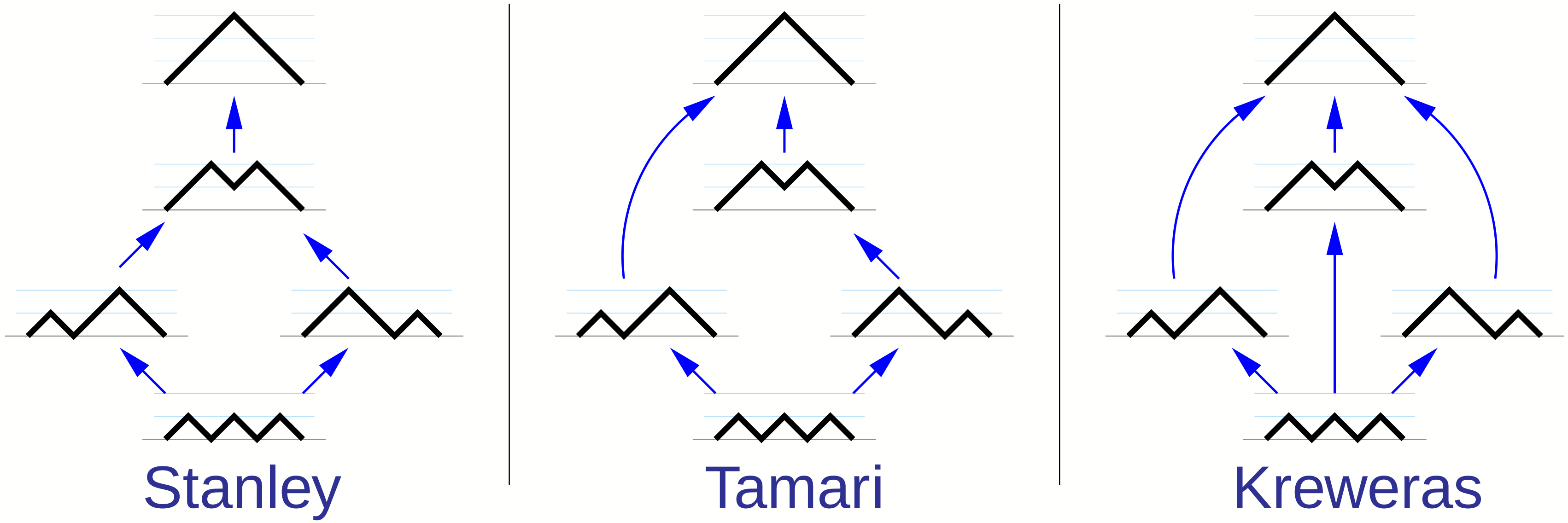
Kreweras relation corresponds to refinement:

Hasse Diagram $n = 4$**:**



Stanley, Tamari and Kreweras

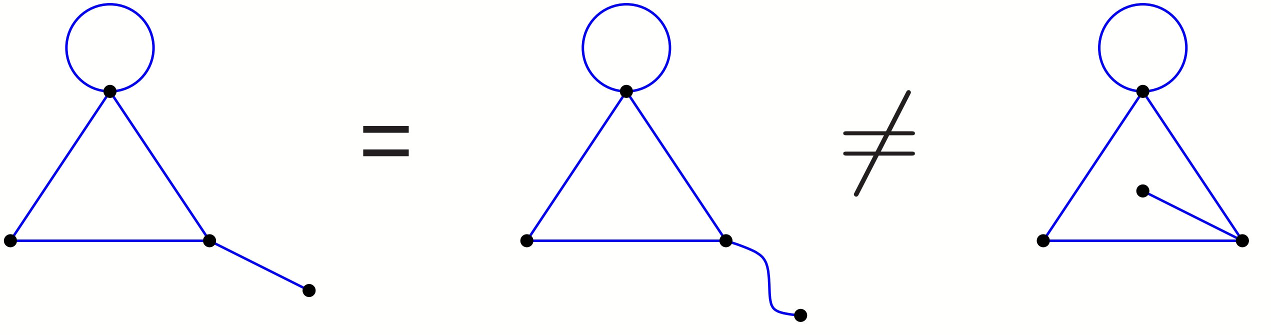


[Knuth 06] The Stanley lattice is an extension of the Tamari lattice which is an extension of the Kreweras lattice.

Triangulations and realizers

Maps

A map is a connected planar graph embedded in the sphere and considered up to homeomorphism.



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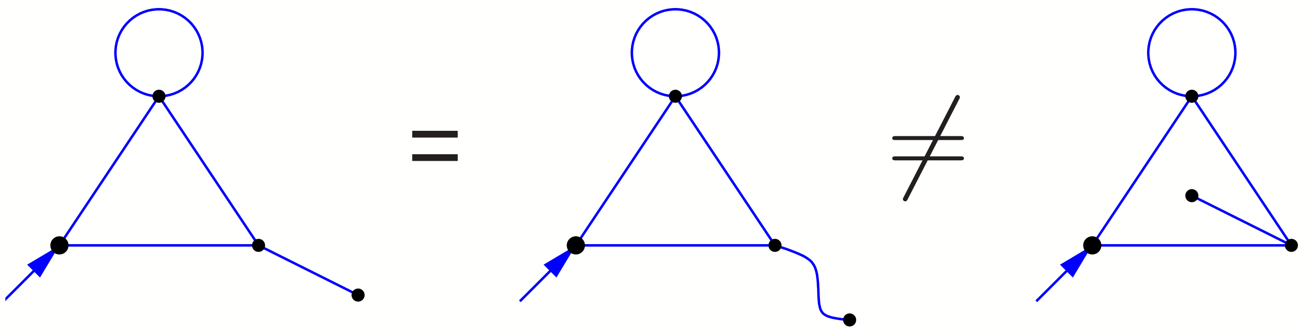
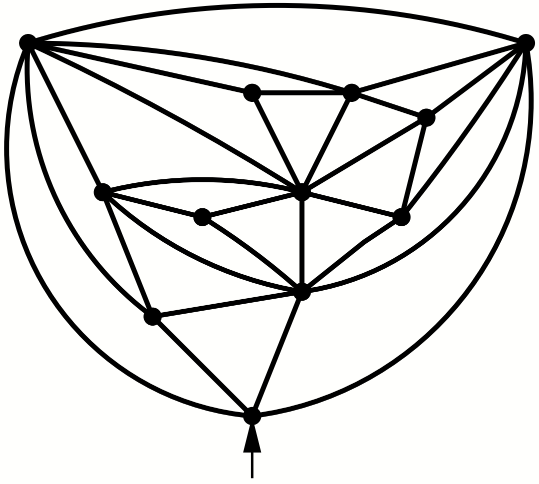


A map is rooted by distinguishing a corner.

Triangulations

A map is a triangulation if every face has degree 3.



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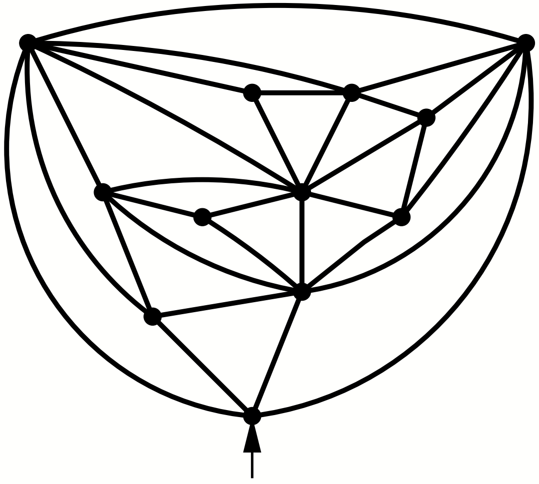
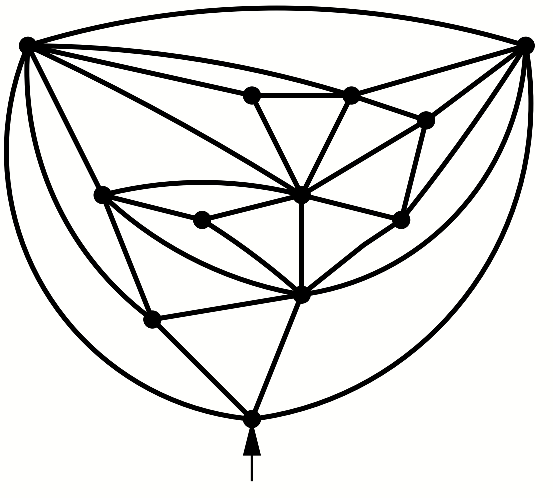


A triangulation of size n has n internal vertices, $3n$ internal edges, $2n+1$ internal triangles.

Realizers [Schnyder 89,90]

Example:



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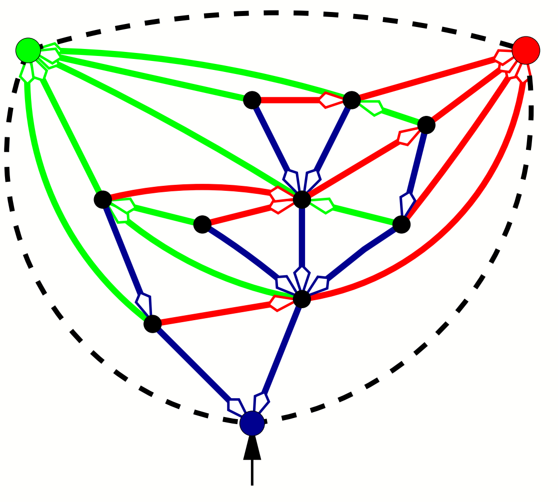

Realizers [Schnyder 89,90]

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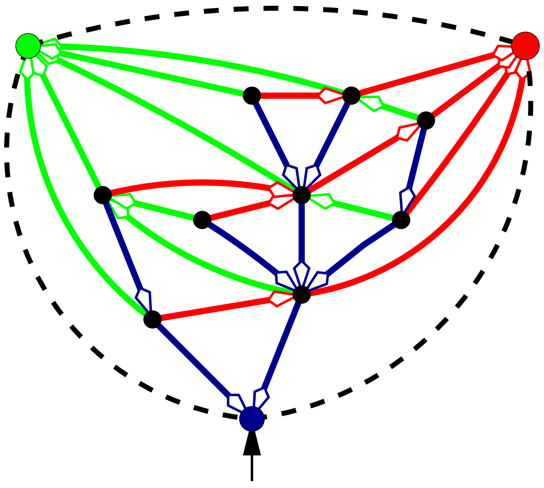
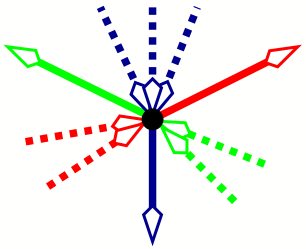


A realizer is a partition of the internal edges in 3 trees satisfying the Schnyder condition:



Lattice of realizers

Prop [Schnyder 89], [Propp 93, Ossona de Mendez 94]:
For any triangulation, the set of realizers is non-empty and can be endowed with a lattice structure.

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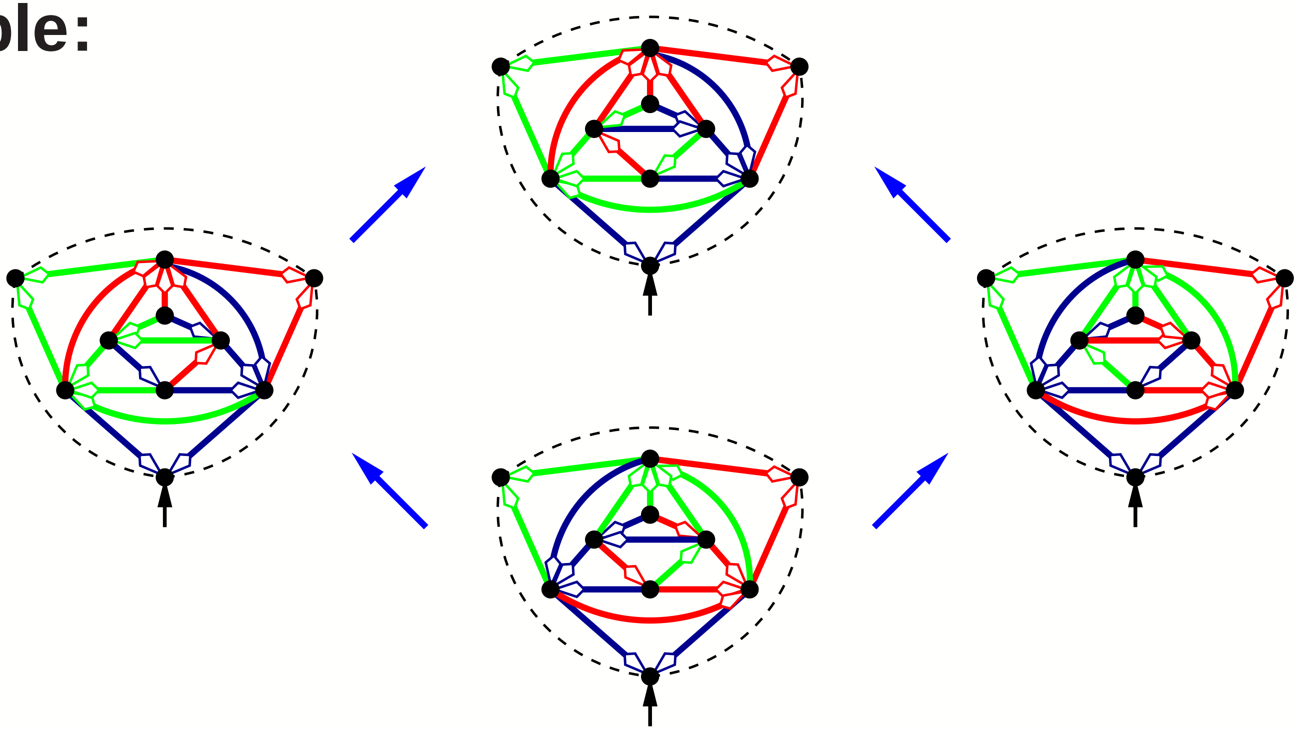
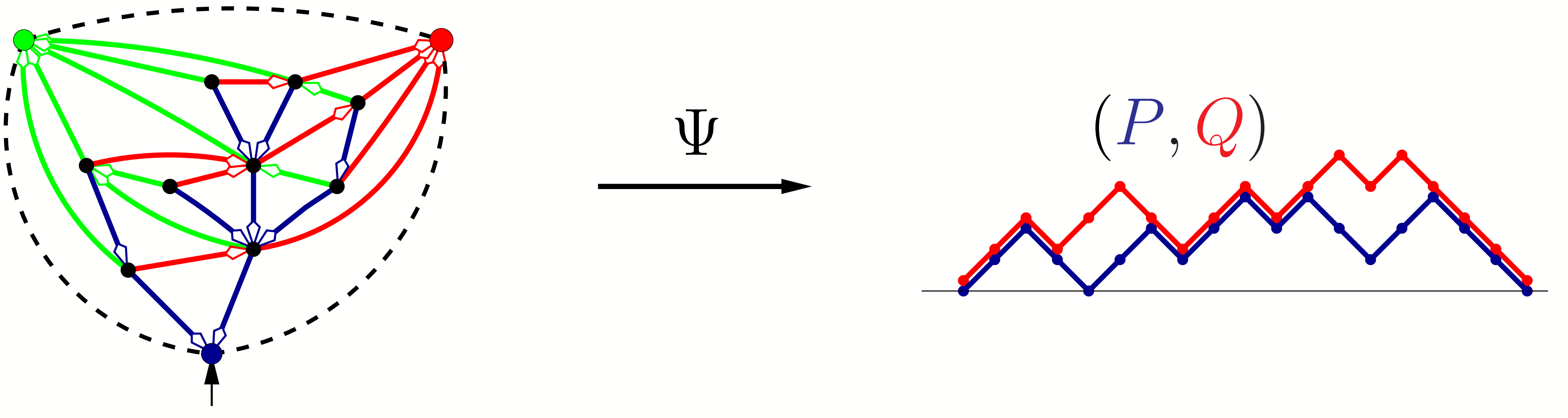


Bijections

Stanley intervals	$\Longleftrightarrow$	Realizers.
Tamari intervals	$\Longleftrightarrow$	Minimal realizers.
Kreweras intervals	$\Longleftrightarrow$	Minimal and maximal realizers.

From realizers to pairs of Dyck paths



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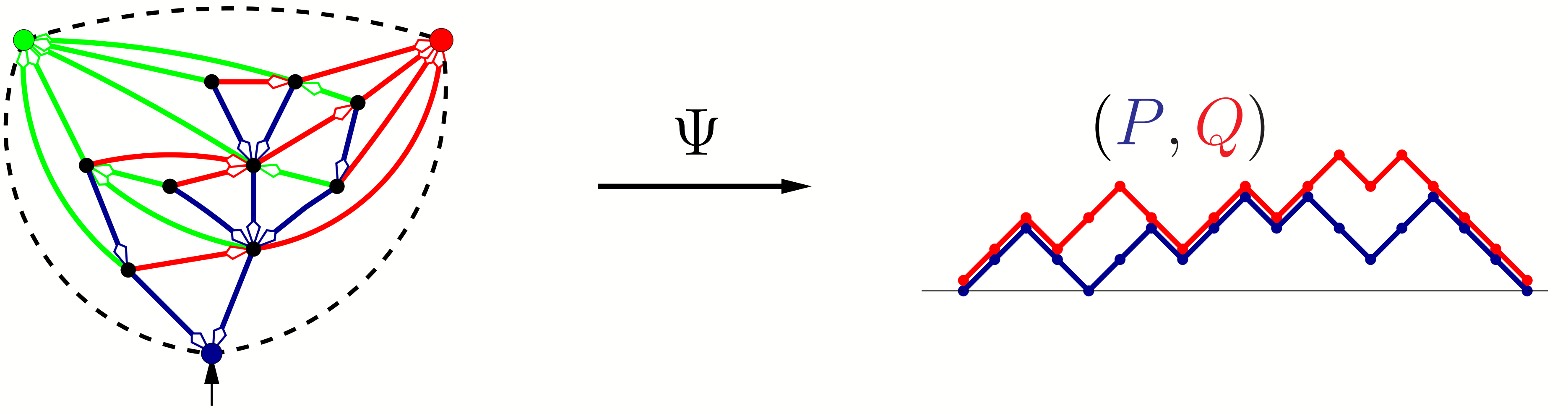
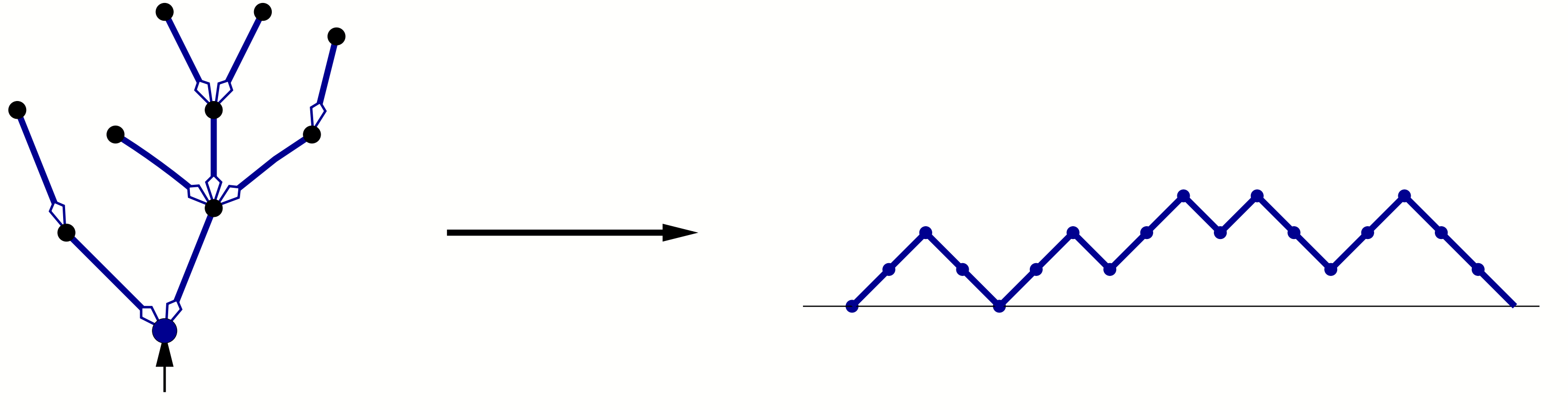


- P is the Dyck path associated to the blue tree.



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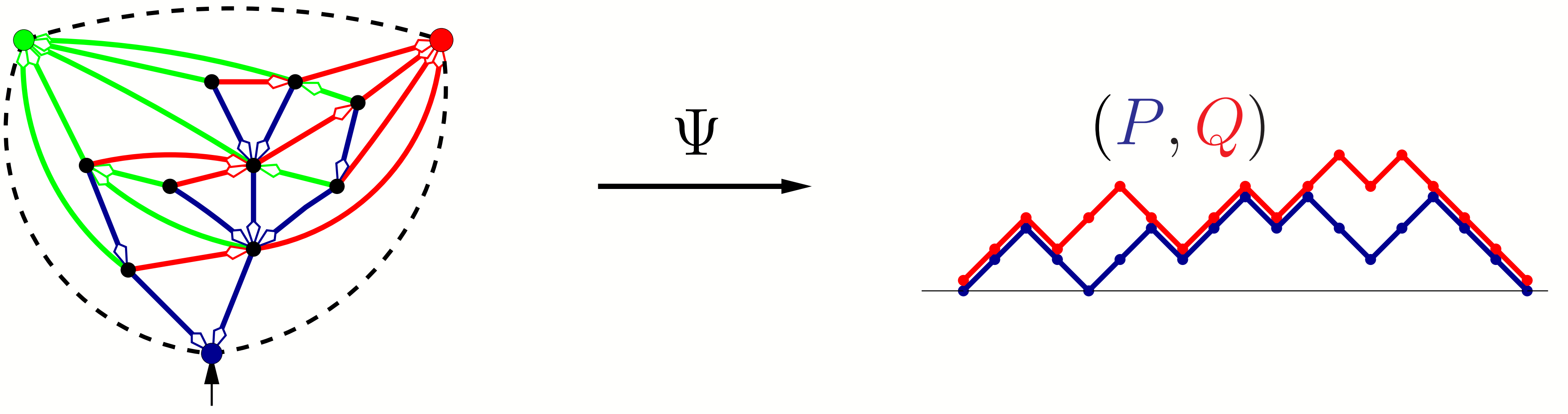
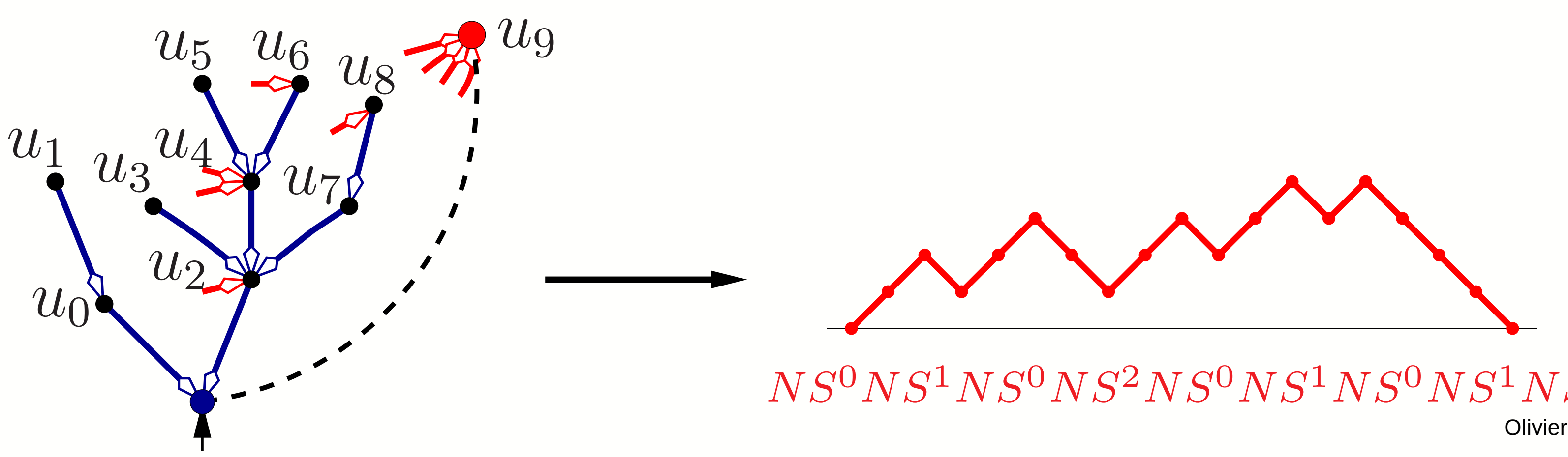
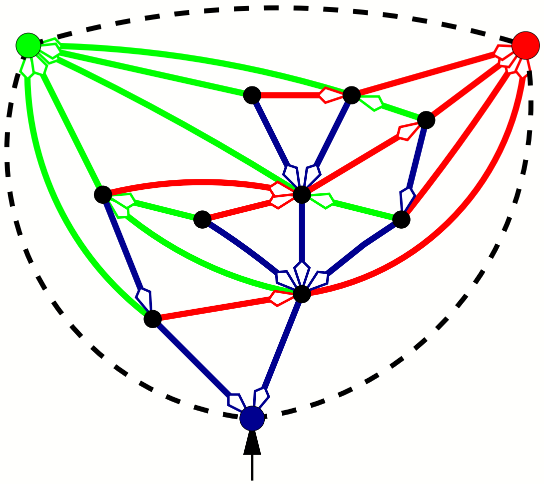


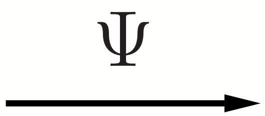
• Q is the Dyck path $NS^{\beta_1} \dots NS^{\beta_n}$, where β_i is the number of red ingoing edges incident to the vertex u_i, $i = 1 \dots n$.

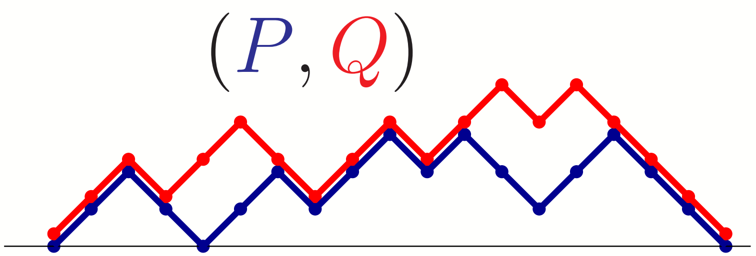


Bijection

Theorem: The mapping Ψ is a bijection between realizers of size n and pairs of non-crossing Dyck paths of size n.

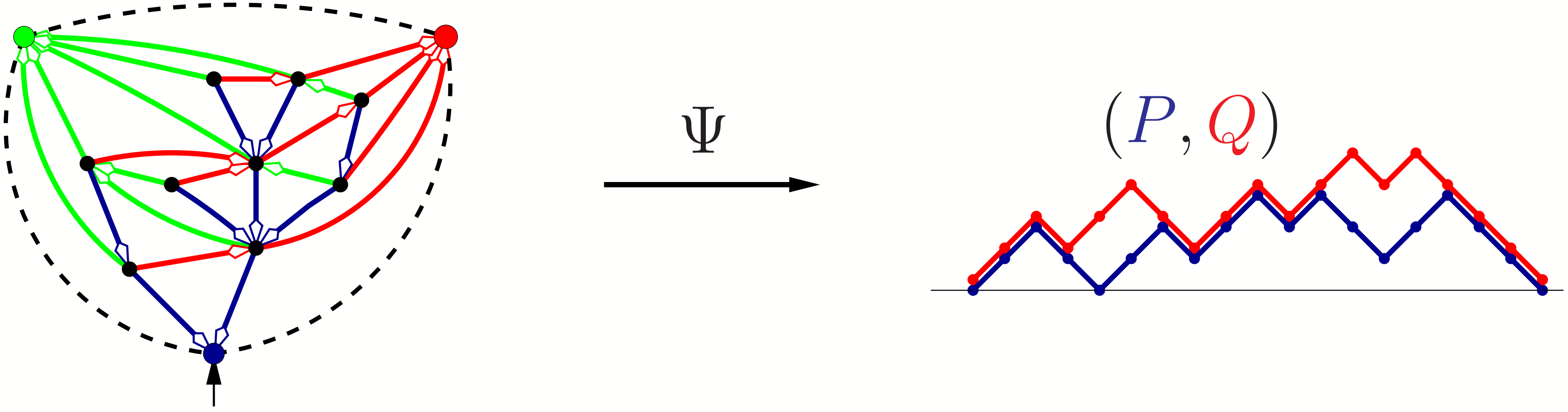






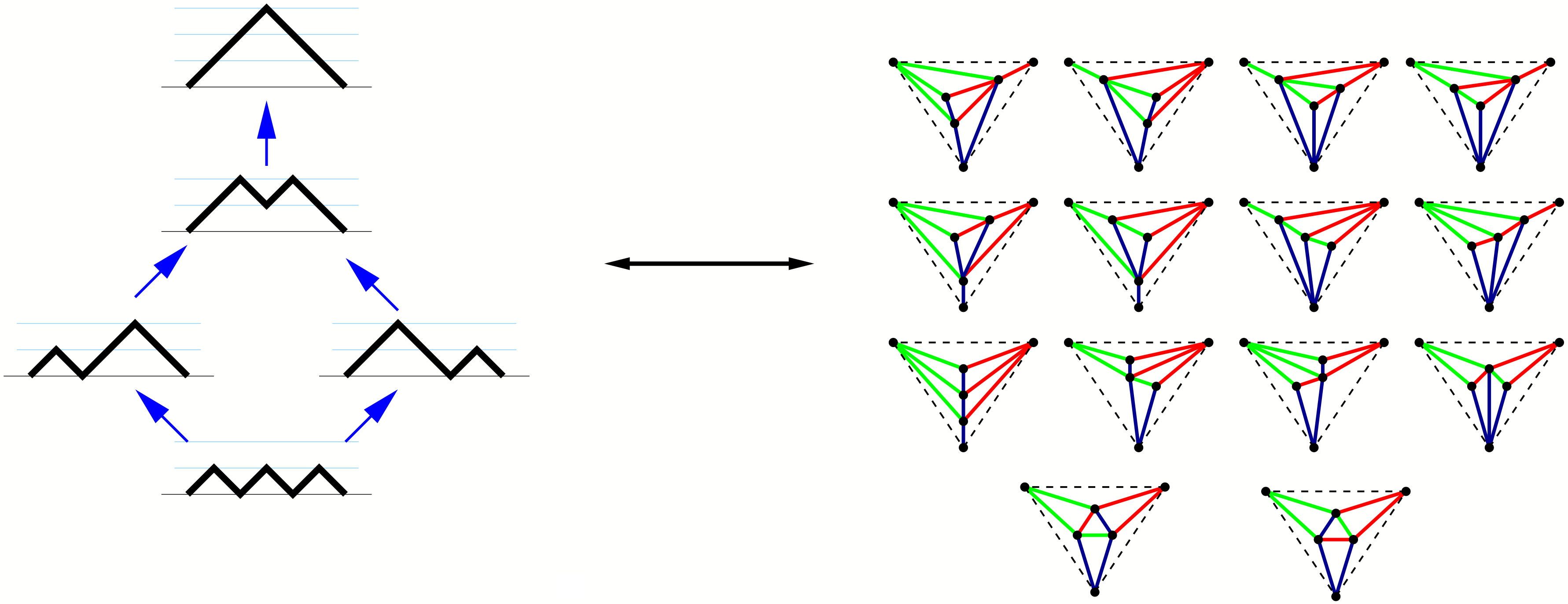
Bijection

Theorem: The mapping Ψ is a bijection between realizers of size n and intervals in the n^{th} Stanley lattice.



Bijection

Stanley intervals $\Longleftrightarrow$ Realizers



Refinement Tamari

Theorem: The mapping Ψ induces a bijection between minimal realizers of size n and intervals in the n^{th} Tamari lattice.

Refinement Tamari

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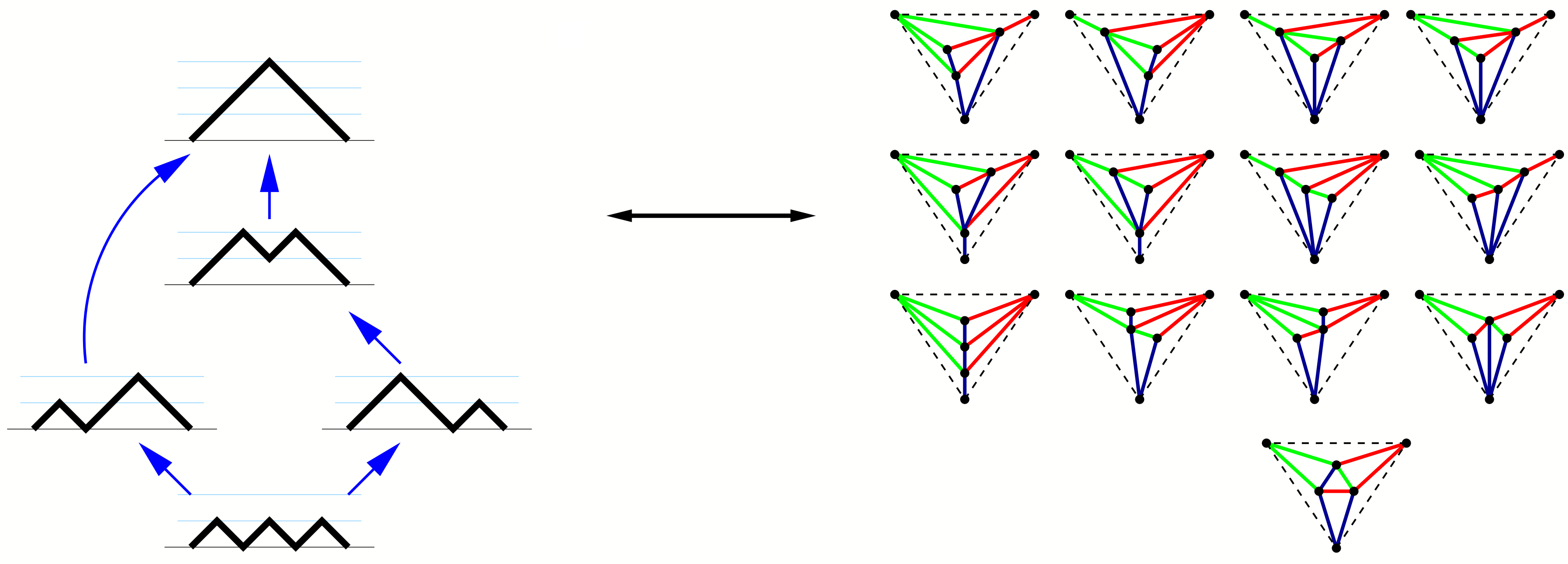
Corollary: We obtain a bijection between triangulations of size n and intervals in the n^{th} Tamari lattice.

Refinement Tamari

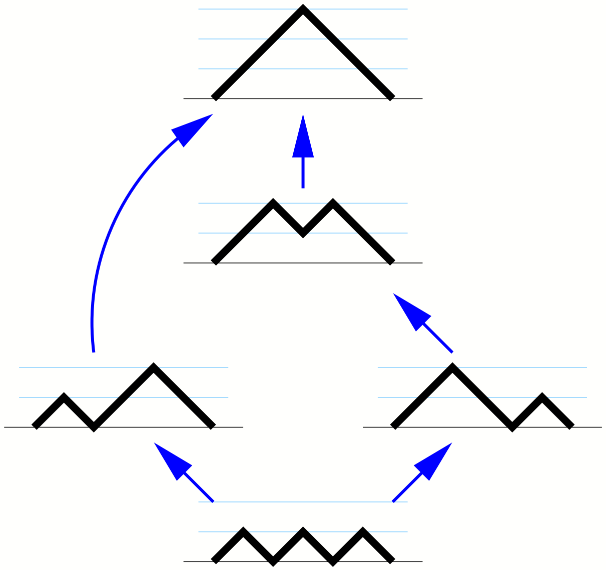
Tamari intervals $\Longleftrightarrow$ Minimal realizers

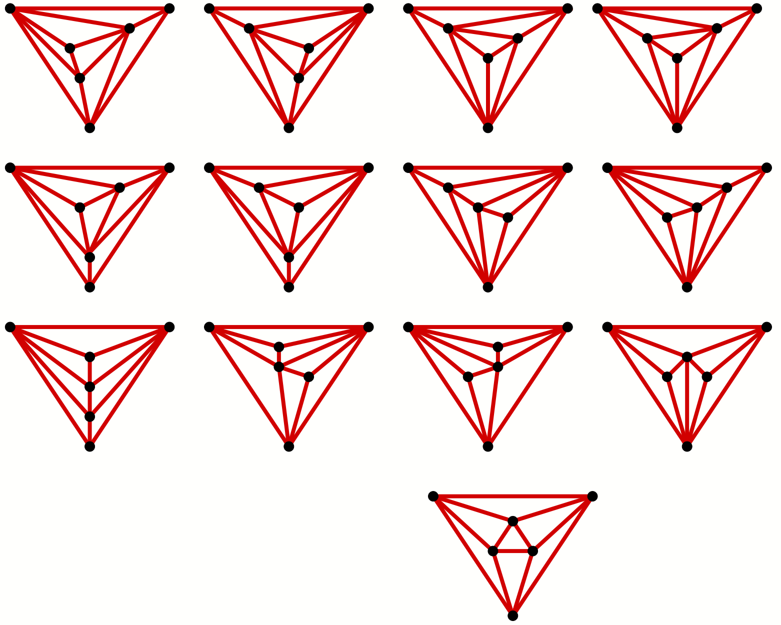


Refinement Tamari

Tamari intervals $\iff$ Minimal realizers

$\iff$ Triangulations





Refinement Kreweras

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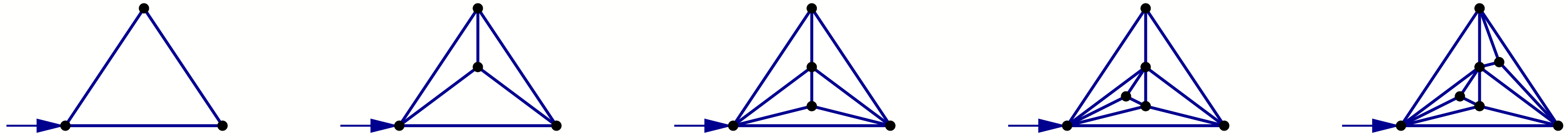
Proposition: A triangulation has a unique realizer if and only if it is stack.



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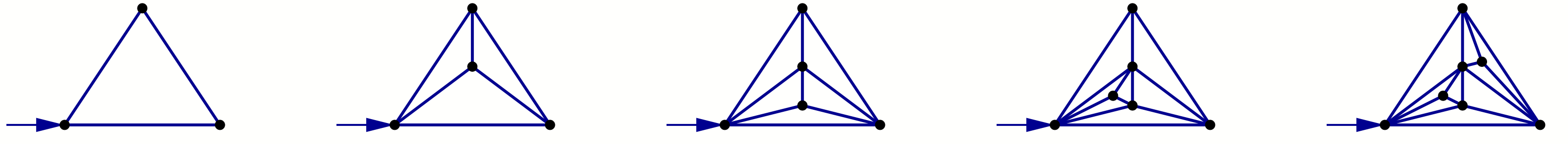
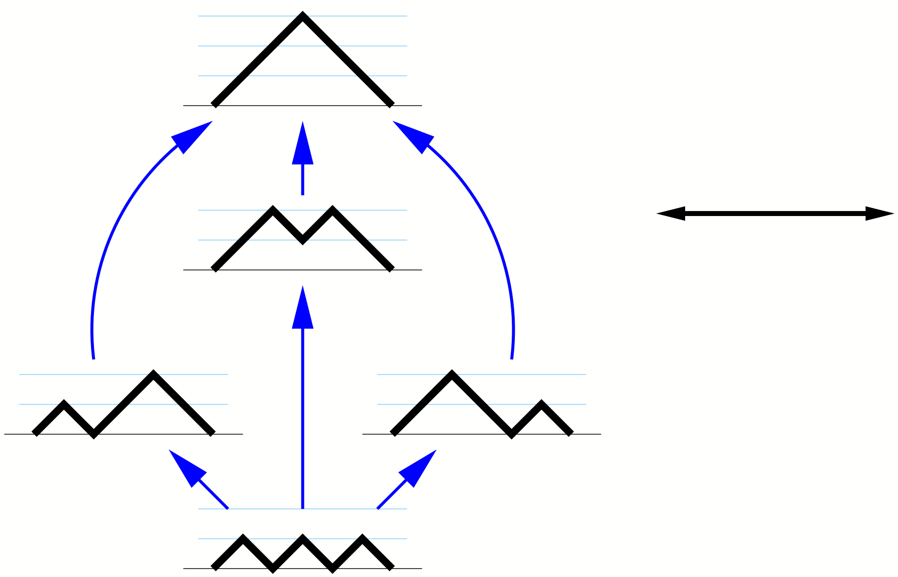
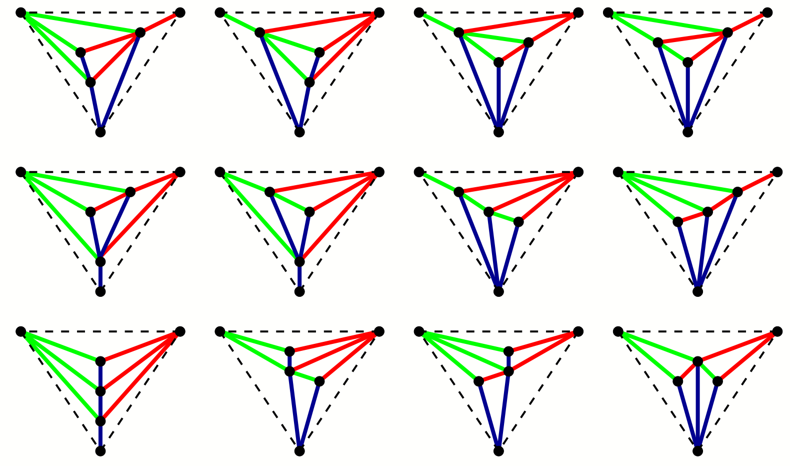


Corollary: We obtain a bijection between stack triangulations ($\Leftrightarrow$ ternary trees) of size n and intervals in the n^{th} Kreweras lattice.

Refinement Kreweras

Kreweras intervals $\Longleftrightarrow$ Minimal and maximal realizers

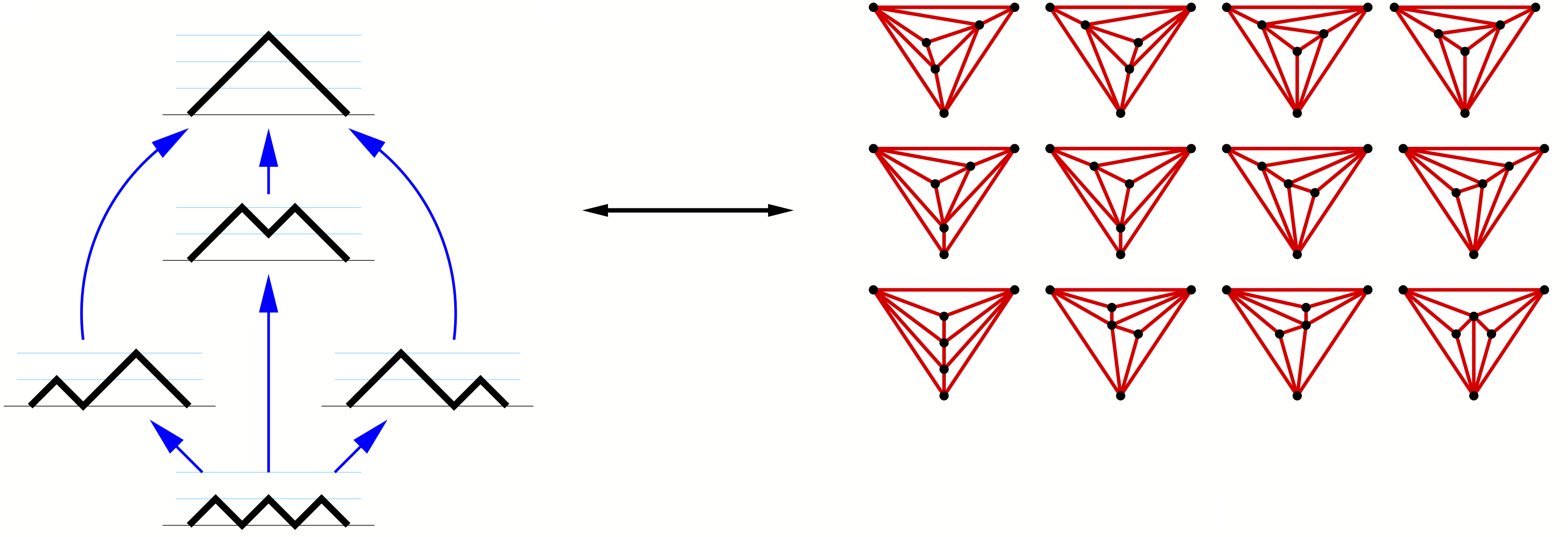




Refinement Kreweras

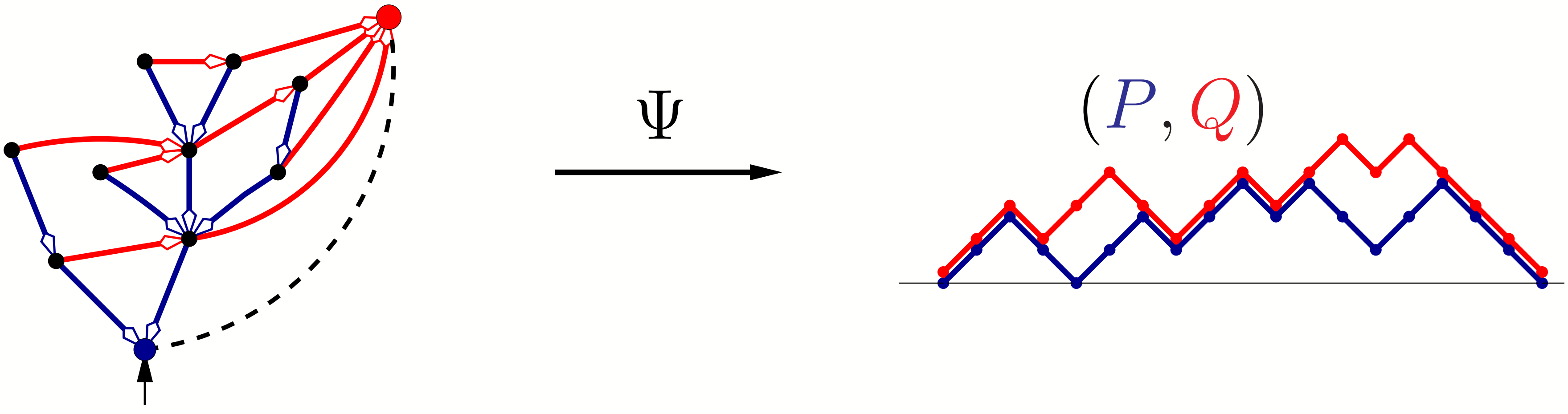
Kreweras intervals $\Longleftrightarrow$ Minimal and maximal realizers

$\Longleftrightarrow$ Stack triangulations ($\Leftrightarrow$ Ternary trees)



Elements of proofs

Claim : The image of any realizer is a pair of non-crossing Dyck paths.



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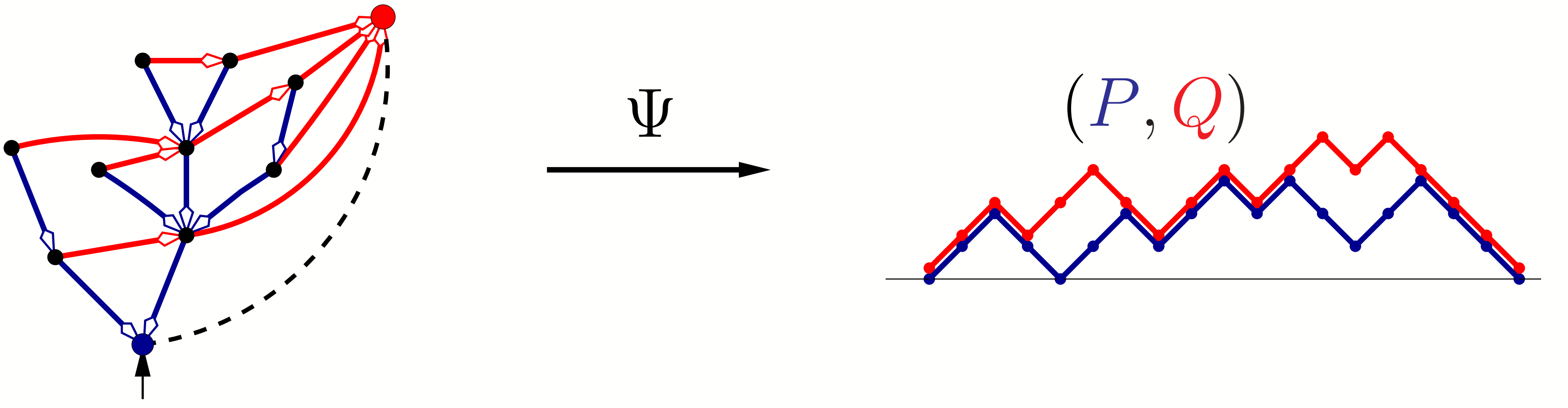


Key property :

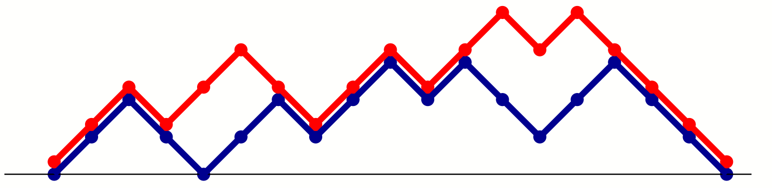
The path Q is above the path P.

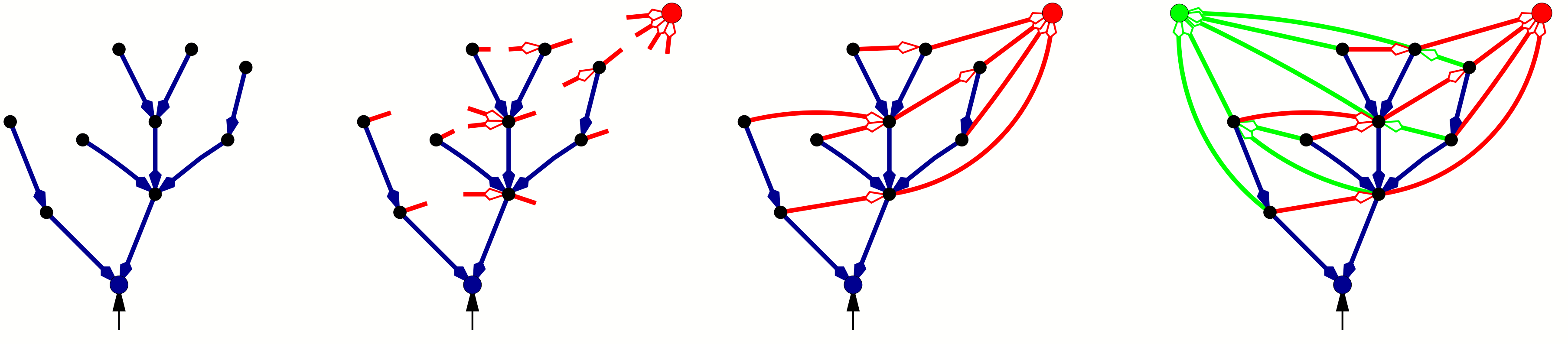
$\Longleftrightarrow$

Red outgoing half-edges appear before red ingoing helf-edges around the blue tree.

Inverse mapping

(P, Q)





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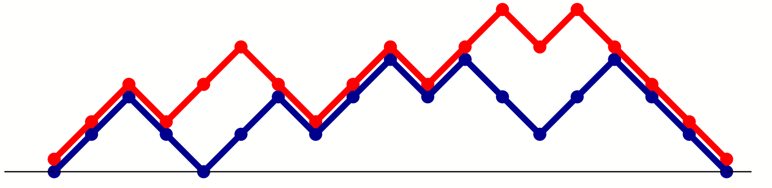
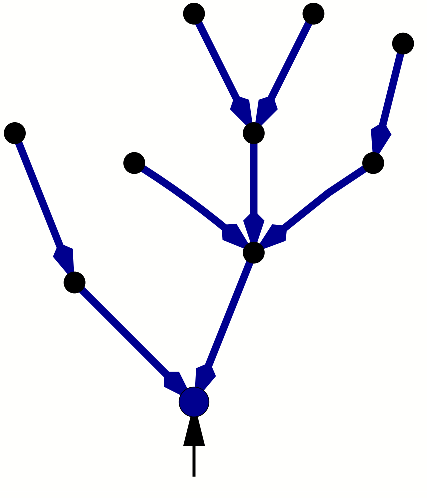


Step 1: Construct the blue tree (using P).



Inverse mapping

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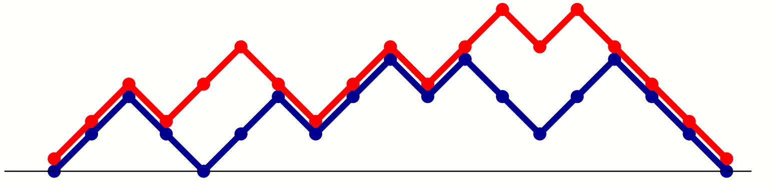
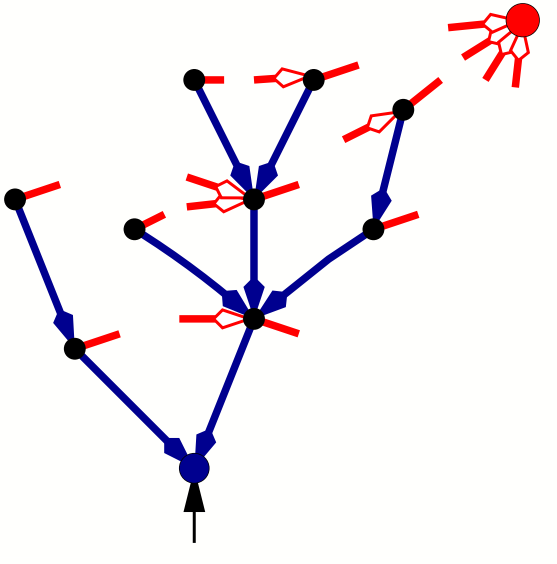


Step 2: Add red ingoing and outgoing half-edges (using Q).



Inverse mapping

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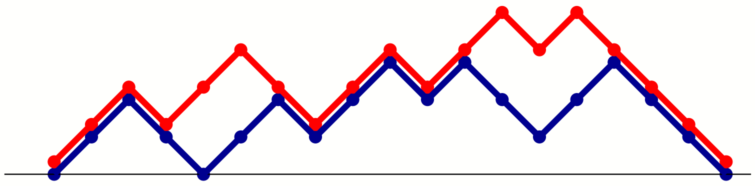
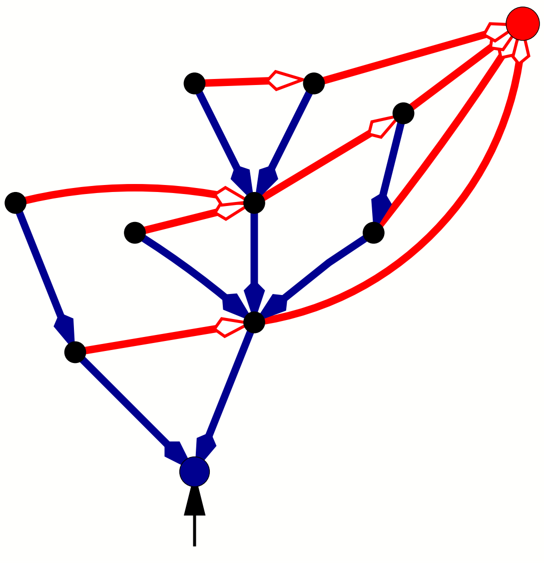


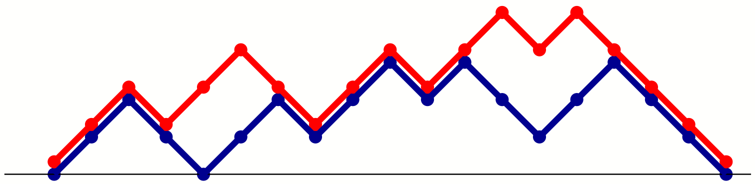
Step 3: Match ingoing and outgoing half-edges.



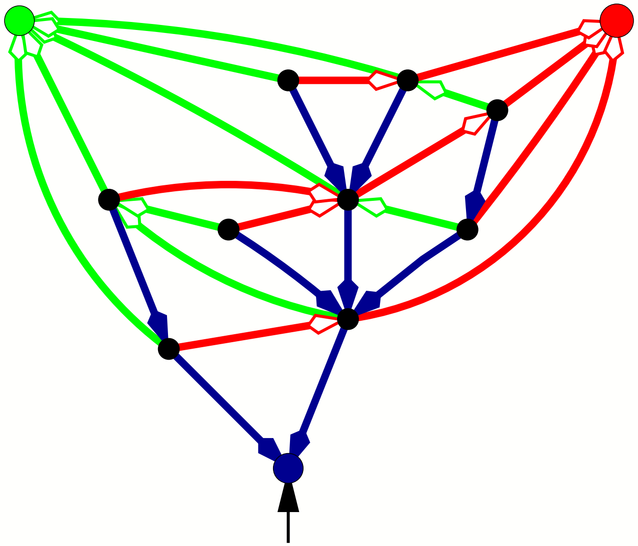
Claim : There is a unique way of matching ingoing and outgoing half-edges. This creates a tree.

Inverse mapping

(P, Q)



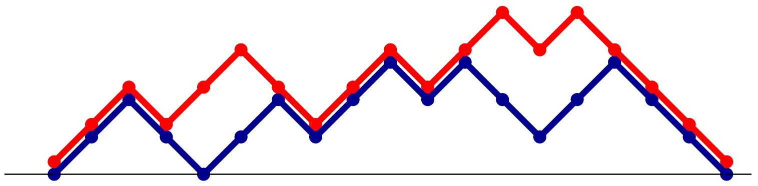
Step 4: Construct the green tree.



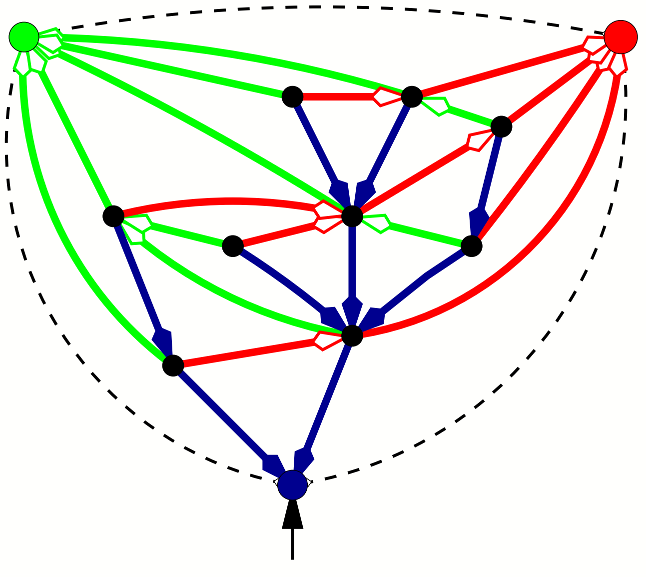
Claim : There is a unique green tree completing the realizer.

Inverse mapping

(P, Q)

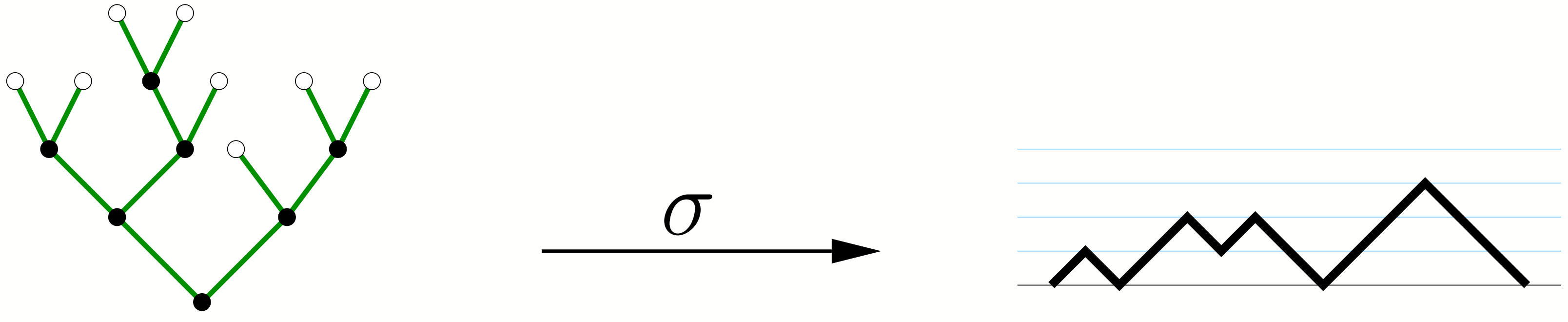


Step 5: Close the map.



Refinement Tamari

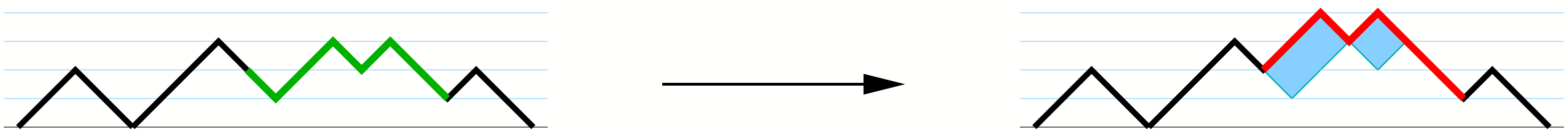
- Chose a good bijection binary-trees $\mapsto$ Dyck paths.



Refinement Tamari

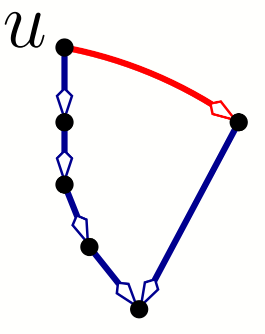
• Chose a good bijection binary-trees $\mapsto$ Dyck paths.

• Characterize the covering relation of the Tamari lattice in terms of Dyck paths.



Refinement Tamari

- Chose a good bijection binary-trees $\mapsto$ Dyck paths.
- Characterize the covering relation of the Tamari lattice in terms of Dyck paths.
- Characterize the minimal realizers **[Bonichon et al.]**.



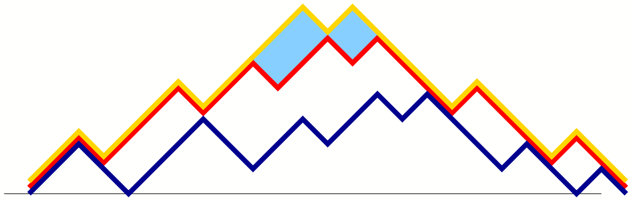
Refinement Tamari

• Chose a good bijection binary-trees $\mapsto$ Dyck paths.

• Characterize the covering relation of the Tamari lattice in terms of Dyck paths.

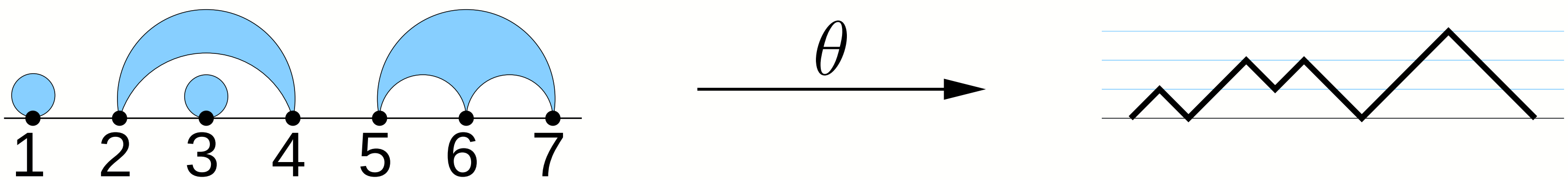
• Characterize the minimal realizers **[Bonichon et al.]**.

• Make an induction on $\Delta(P, Q)$ to prove that P and Q are comparable in the Tamari lattice if and only if the realizer $\Phi(P, Q)$ is minimal.



Refinement Kreweras

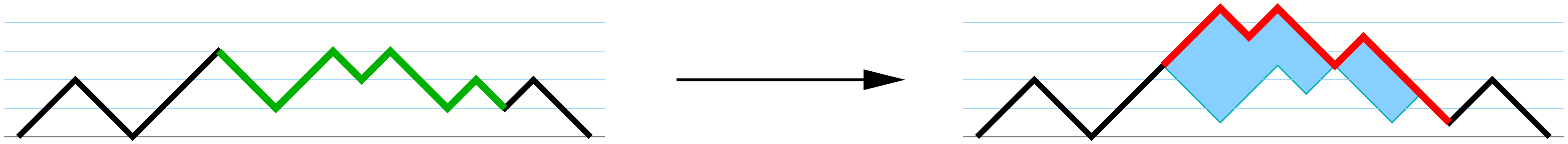
• Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.



Refinement Kreweras

• Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.

• Characterize the covering relation of the Kreweras lattice in terms of Dyck paths.

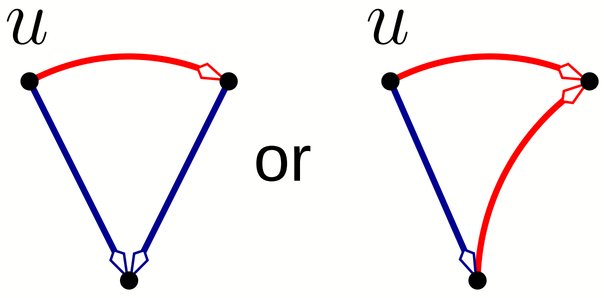


Refinement Kreweras

• Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.

• Characterize the covering relation of the Kreweras lattice in terms of Dyck paths.

• Characterize the minimal and maximal realizers **[Bonichon et al.]**.



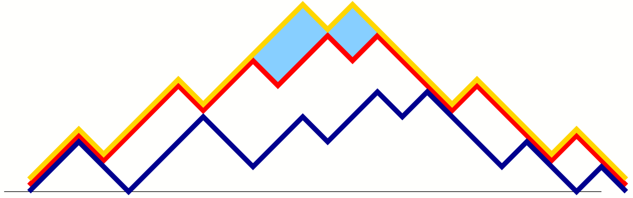
Refinement Kreweras

• Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.

• Characterize the covering relation of the Kreweras lattice in terms of Dyck paths.

• Characterize the minimal and maximal realizers **[Bonichon et al.]**.

• Make an induction on $\Delta(P, Q)$ to prove that P and Q are comparable in the Kreweras lattice if and only if the realizer $\Phi(P, Q)$ is minimal and maximal.



Refinement Kreweras

- Chose a good bijection non-crossing partitions $\mapsto$ Dyck paths.

- Characterize the covering relation of the Kreweras lattice in terms of Dyck paths.

- Characterize the minimal and maximal realizers **[Bonichon et al.]**.

- Make an induction on $\Delta(P,Q)$ to prove that P and Q are comparable in the Kreweras lattice if and only if the realizer $\Phi(P,Q)$ is minimal and maximal.

- Prove that a triangulation has a unique realizer if and only if it is stack.

Thanks.