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Quiver coefficients

Anders Buch (Rutgers)

http://math.rutgers.edu/∼asbuch/papers/quivcoef.pdf

Collaborators on subject:

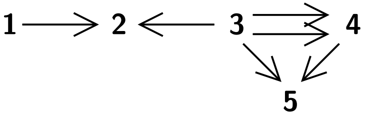
Fehér , Fomin , Fulton , Kresch , Rimányi ,

Shimozono , Sottile , Tamvakis , Yong

FPSAC 2007, Nankai University, Tianjin, China

Classes of quiver cycles

Q quiver (finite directed graph)



Vertex set: $[n] = \{1, 2, \ldots, n\}$

Dimension vector: $e = (e_1, \ldots, e_n) \in \mathbb{N}^n$

Set $E_i = \mathbb{F}^{e_i}$, where $\mathbb{F}$ is a field.

Representation space: $X = \bigoplus_{i \to j} \mathrm{Mat}(e_j \times e_i) = \bigoplus_{i \to j} \mathrm{Hom}_{\mathbb{F}}(E_i, E_j)$

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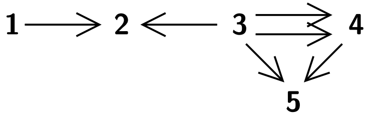

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Representation space: $X = \bigoplus_{i \to j} \mathrm{Mat}(e_j \times e_i) = \bigoplus_{i \to j} \mathrm{Hom}_{\mathbb{F}}(E_i, E_j)$

Group action: $G = \mathrm{GL}(E_1) \times \cdots \times \mathrm{GL}(E_n)$ acts on X by

$$(g_1, \ldots, g_n).(\phi_{i \to j}) = (g_j \, \phi_{i \to j} \, g_i^{-1})$$

Def: A **quiver cycle** is a G-stable closed subvariety $\Omega \subset X$

Problem: Describe the cohomology class $[\Omega]$ and Grothendieck class $[\mathcal{O}_\Omega]$

Write $E_i = E_{i,1} \oplus \cdots \oplus E_{i,e_i}$ where $E_{i,s} \cong \mathbb{F}$

$T(E_i) \; := \; \mathrm{GL}(E_{i,1}) \times \cdots \times \mathrm{GL}(E_{i,e_i}) \; \subset \mathrm{GL}(E_i)$ diagonal matrices

Set $T \; = \; T(E_1) \times \cdots \times T(E_n) \; \subset G$

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$\mathbb{F}[X] = \{$ polynomial functions $p : X \to \mathbb{F}$ $\}$

T acts on $\mathbb{F}[X]$ by $(t.p)(\phi) = p(t^{-1}.\phi)$

$\mathcal{O}_\Omega = \mathbb{F}[X]/\mathcal{I}(\Omega)$ where $\mathcal{I}(\Omega) = \{p \in \mathbb{F}[X] : p(\phi) = 0 \ \forall \phi \in \Omega\}$

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T-equivariant resolution: $0 \to F_k \to \cdots \to F_0 \to \mathcal{O}_\Omega \to 0$ where F_i is a free $\mathbb{F}[X]$-module with linear T-action, s.t. $t.(p\, m) = (t.p)\,(t.m)$ for all $m \in F_i$

Note: $F_i/\mathfrak{m}F_i$ is a T-representation, $\mathfrak{m} = \mathcal{I}(\{0\}) \subset \mathbb{F}[X]$

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Def: $K_T(X)$ = " ring of virtual representations of T " $= \mathbb{Z}\left[E_{i,s}^{\pm 1}\right]$

Grothendieck class: $[\mathcal{O}_\Omega] = \sum_{i \geq 0} (-1)^i [F_i/\mathfrak{m}F_i] \in K_T(X)$

Note: $K_T(X) \subset \mathbb{Z}[[x_{i,s}]]$ where $x_{i,s} = 1 - E_{i,s}^{-1}$ **Chern roots**

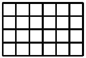
$H_T(X) := \mathbb{Z}[x_{i,s}]$; $[\Omega]$ = leading term of $[\mathcal{O}_\Omega]$

Example: $Q = \{ \mathbf{1} \to \mathbf{2} \}$

$GL(E_1) \times GL(E_2)$ acts on $X = \mathrm{Hom}_{\mathbb{F}}(E_1, E_2)$ by $(g_1, g_2).\phi = g_2 \, \phi \, g_1^{-1}$

$\Omega = \Omega_r = \{\phi \in X \mid \mathrm{rank}(\phi) \leq r\}$

Thom–Porteous : $[\Omega] = S_\lambda(E_2 - E_1) = S_\lambda(x_{2,1}, \ldots, x_{2,e_2}; x_{1,1}, \ldots, x_{1,e_1})$

where $\lambda = (e_1 - r)^{e_2 - r} =$  $e_2 - r$

$e_1 - r$

Theorem (B) $[\mathcal{O}_\Omega] = \mathcal{G}_\lambda(E_2 - E_1) \in K_T(X)$

where $\mathcal{G}_\lambda$ = stable Grothendieck polynomial for λ

Set-valued tableaux (B)

$\mathcal{T} =$

1,3	3	4,5	5,6,8
4	5,8		
6,7			

Set-valued tableaux (B)

$$\mathcal{T} = \begin{array}{|c|c|c|c|}\hline 1,3 & 3 & 4,5 & 5,6,8 \\ \hline 4 & 5,8 & & \\ \cline{1-2} 6,7 & & & \\ \cline{1-1} \end{array}$$

$$x^{\mathcal{T}} = \prod_{i \geq 1} x_i^{\#\text{ boxes } \ni i} = x_1\, x_3^2\, x_4^2\, x_5^3\, x_6^2\, x_7\, x_8^2$$

$$|\mathcal{T}| = \deg(x^{\mathcal{T}}) = 13$$

Define: $\mathcal{G}_\lambda(x) = \sum_{\text{shape}(\mathcal{T})=\lambda} (-1)^{|\mathcal{T}|-|\lambda|}\, x^{\mathcal{T}}$

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Bialgebra of stable Groth. polys. $\Gamma = \bigoplus_\lambda \mathbb{Z} \cdot \mathcal{G}_\lambda \subset \mathbb{Z}[[x_1, x_2, \ldots]]$

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LR-rule (B) : $\mathcal{G}_\lambda \cdot \mathcal{G}_\mu = \sum_\nu c_{\lambda\mu}^\nu \, \mathcal{G}_\nu$ where $|\nu| \geq |\lambda| + |\mu|$ and

$c_{\lambda\mu}^\nu = (-1)^{|\nu|-|\lambda|-|\mu|} \cdot \#$ certain set-valued $\mathcal{T}$

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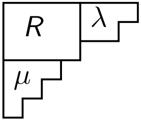
Coproduct (B) : $\Delta : \Gamma \to \Gamma \otimes \Gamma$

$\Delta(\mathcal{G}_\nu) = \sum_{\lambda,\mu} d_{\lambda\mu}^\nu\, \mathcal{G}_\lambda \otimes \mathcal{G}_\mu$; $d_{\lambda\mu}^\nu = c_{R,\nu}^\rho$ where $\rho =$



Notation:

Given a rational T-rep. U , write $U = U_1 \oplus \cdots \oplus U_p$, $\dim U_i = 1$

Set $\mathcal{G}_\lambda(U) = \mathcal{G}_\lambda(1 - U_1^{-1}, \ldots, 1 - U_p^{-1}, 0, 0, \ldots) \in K_T(X)$

Definition Given two rational T-representations U and V, set

$$\mathcal{G}_\nu(U - V) = \sum_{\lambda,\mu} d_{\lambda\mu}^{\nu}\, \mathcal{G}_\lambda(U) \cdot \mathcal{G}_{\mu'}(V^*) \in K_T(X)$$

Note: $S_\nu(U - V) \in H_T(X)$ is the leading term of $\mathcal{G}_\nu(U - V)$

Equioriented quiver of type A: $Q = \{ \mathbf{1} \to \mathbf{2} \to \cdots \to \mathbf{n} \}$

$X = \{(\phi_1, \dots, \phi_{n-1}) : E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} \cdots \to E_{n-1} \xrightarrow{\phi_{n-1}} E_n\}$

Any quiver cycle is given by rank conditions $r = \{r_{i\,j}\}$, $1 \leq i < j \leq n$

$\Omega_r = \{\phi \in X \mid \mathrm{rank}(E_i \xrightarrow{\phi_{j-1}\dots\phi_i} E_j) \leq r_{i\,j} \ \forall i < j\}$

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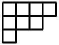
Thm (B–Fulton)**:** Formula for $[\Omega_r] \in H_T^*(X)$

Thm (B) $[\mathcal{O}_{\Omega_r}] = \sum c_\mu(r)\, \mathcal{G}_{\mu_1}(E_2 - E_1)\, \mathcal{G}_{\mu_2}(E_3 - E_2) \cdots \mathcal{G}_{\mu_{n-1}}(E_n - E_{n-1})$

sum over sequences $\mu = (\mu_1, \ldots, \mu_{n-1})$ of partitions $\mu_i =$ 
for which $\sum |\mu_i| \geq \mathrm{codim}(\Omega_r)$

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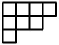
sum over sequences $\mu = (\mu_1, \ldots, \mu_{n-1})$ of partitions $\mu_i =$ 
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Conjecture (B–Fulton) , **Theorem** (Knutson–Miller–Shimozono, B, Miller)

Quiver coefs. have alternating signs: $(-1)^{\sum |\mu_i| - \operatorname{codim}(\Omega_r)}\, c_\mu(r) \geq 0$

Stable Grothendieck polynomials

Fomin–Kirillov: Defined $\mathcal{G}_w(x)$ for any permutation $w \in S_N$

B: $\mathcal{G}_w = \sum a_{w,\lambda} \mathcal{G}_\lambda$, $a_{w,\lambda} =$ quiver coefficient

Lascoux: $(-1)^{|\lambda|-\ell(w)} \cdot a_{w,\lambda} \geq 0$

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Def. (BKSTY) A **decreasing tableau** is a Young tableau with strictly decreasing rows and columns.

$$\mathcal{T} = \begin{array}{|c|c|c|c|} \hline 5 & 4 & 3 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array}$$

Define $w(\mathcal{T}) := s_2 \cdot s_3 \cdot s_1 \cdot s_5 \cdot s_4 \cdot s_3 \cdot s_1$ where $s_i = (i, i+1)$, using

Hecke product of permutations: $w \cdot s_i \ := \begin{cases} w\, s_i & \text{if } \ell(w\, s_i) > \ell(w) \\ w & \text{otherwise} \end{cases}$

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Thm (B–Kresch–Shimozono–Tamvakis–Yong)

$$a_{w,\lambda} = (-1)^{|\lambda|-\ell(w)} \cdot \#\text{ decreasing } \mathcal{T} \text{ of shape } \lambda \text{ such that } w(\mathcal{T}) = w.$$

Example: $w = 2\,1\,4\,3 = s_1 \cdot s_3$

$$\mathcal{G}_w \;=\; \mathcal{G}_{(1,1)} + \mathcal{G}_{(2)} - \mathcal{G}_{(2,1)}$$

$$\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Thm (Knutson–Miller–Shimozono , Rimányi)

Component formula for $[\Omega_r] \in H_T^*(X)$ in terms of minimal lace diagrams.

Implies positivity of cohomological quiver coefficients.

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Permutation diagram: $(n = 4)$

$$\begin{matrix} W_{12} & & W_{23} & & W_{34} \\ & W_{13} & & W_{24} & \\ & & W_{14} & & \end{matrix}$$

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Factorize:

$W_{14} = \alpha \cdot \beta$

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Thm (Knutson–Miller–Shimozono , Rimányi)

Component formula for $[\Omega_r] \in H_T^*(X)$ in terms of minimal lace diagrams.

Implies positivity of cohomological quiver coefficients.

Generalization: For $1 \le i < j \le n$, define permutation $W_{i\,j}$ by

$$W_{i\,j}(p) = \begin{cases} p + r_{i,j-1} - r_{i,j} & \text{if } r_{i,j} < p \le r_{i+1,j} \\ p - r_{i+1,j} + r_{i,j} & \text{if } r_{i+1,j} < p \le r_{i+1,j} + r_{i,j-1} - r_{i,j} \\ p & \text{otherwise} \end{cases}$$

Permutation diagram: $(n = 4)$

W_{12} $\quad$ W_{23} $\quad$ W_{34}

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Def (B) A KMS-factorization is a sequence of permutations of the form $(W_{12} \cdot \sigma_2 \,,\; \tau_2 \cdot W_{23} \cdot \sigma_3 \,,\; \tau_3 \cdot W_{3,4})$

Theorem (B , Miller)

$$[\mathcal{O}_{\Omega_r}] = \sum (-1)^{\sum \ell(w_i) - \mathrm{codim}(\Omega_r)} \mathcal{G}_{w_1}(E_2 - E_1) \cdots \mathcal{G}_{w_{n-1}}(E_n - E_{n-1})$$

sum over all KMS-factorizations $(w_1, \ldots, w_{n-1})$.

Corollary Equioriented quiver coefficients have alternating signs.

Formula (B–Kresch–Shimozono–Tamvakis–Yong)

$c_\mu(r) = \pm \#$ sequences $(\mathcal{T}_1, \ldots, \mathcal{T}_{n-1})$ of decreasing tableaux of shapes μ for which $(w(\mathcal{T}_1), \ldots, w(\mathcal{T}_{n-1}))$ is a KMS-factorization.

Non-equioriented quiver of type A: $Q = \{\, \mathbf{1} \to \mathbf{2} \leftarrow \mathbf{3} \leftarrow \mathbf{4} \to \mathbf{5} \to \mathbf{6} \,\}$

B–Rimányi**:** Positive formula for $[\Omega]$

Alternating conjecture for $[\mathcal{O}_\Omega]$

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B–Rimányi : Positive formula for $[\Omega]$

Alternating conjecture for $[\mathcal{O}_\Omega]$

Quiver of Dynkin type : $$Q = \left\{ \begin{array}{lll} & & \ \mathbf{5} \\ \mathbf{1} \longleftarrow \mathbf{2} \longrightarrow \mathbf{3} \longrightarrow \mathbf{4} & \begin{array}{l} \swarrow \\ \searrow \end{array} & \\ & & \ \mathbf{6} \end{array} \right\}$$

Gabriel : Classification of orbit closures.

Fehér–Rimányi : Equations for $[\Omega] \in H_T^*(X)$

Bobiński–Zwara : Rational singularities for quivers of types A and D

Reineke : Explicit desingularization of Ω

Knutson–Shimozono : Formula for $[\mathcal{O}_\Omega]$ based on Demazure operators

B : Formula for $[\mathcal{O}_\Omega]$ based on quiver coefficients

Generalized Quiver coefficients

Q quiver without oriented loops. Let $\Omega \subset X$ be a quiver cycle.

For $i \in [n]$ set $M_i = \bigoplus_{j \to i} E_j$

Example: $Q = \{\, \mathbf{1} \rightrightarrows \mathbf{2} \leftarrow \mathbf{3} \,\}$ gives $M_2 = E_1 \oplus E_1 \oplus E_3$

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Short-hand: $[\mathcal{O}_\Omega] \ = \ \sum_\mu c_\mu(\Omega)\ \mathcal{G}_{\mu_1} \otimes \mathcal{G}_{\mu_2} \otimes \cdots \otimes \mathcal{G}_{\mu_n}$

Note: sum may be infinite !!

Example

$Q = \{ \mathbf{1} \rightrightarrows \mathbf{2} \}$, dimension vector $e = (3,3)$

$\Omega = \overline{G.\phi} \subset X$; $\phi = \left(\begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \end{bmatrix}, \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix} \right)$

Zwara : Ω has **bad singularities**.

With help from Macaulay 2 **:**

$$[\Omega] = 3 \otimes S_{3,1} + 4\, S_1 \otimes S_3 + 1 \otimes S_{2,2} + 2\, S_1 \otimes S_{2,1} + 3\, S_2 \otimes S_2 + S_2 \otimes S_{1,1} \\ +2\, S_3 \otimes S_1 + S_4 \otimes 1$$

Grothendieck class:

$$[\mathcal{O}_\Omega] = 3 \otimes \mathcal{G}_{3,1} + 4\,\mathcal{G}_1 \otimes \mathcal{G}_3 + 1 \otimes \mathcal{G}_{2,2} + 2\,\mathcal{G}_1 \otimes \mathcal{G}_{2,1} + 3\,\mathcal{G}_2 \otimes \mathcal{G}_2 + \mathcal{G}_2 \otimes \mathcal{G}_{1,1} + 2\,\mathcal{G}_3 \otimes \mathcal{G}_1 + \mathcal{G}_4 \otimes 1$$

$$-3 \otimes \mathcal{G}_{3,2} - 8\,\mathcal{G}_1 \otimes \mathcal{G}_{3,1} - 6\,\mathcal{G}_2 \otimes \mathcal{G}_3 - 2\,\mathcal{G}_1 \otimes \mathcal{G}_{2,2} - 5\,\mathcal{G}_2 \otimes \mathcal{G}_{2,1} - 4\,\mathcal{G}_3 \otimes \mathcal{G}_2 - 2\,\mathcal{G}_3 \otimes \mathcal{G}_{1,1} - 2\,\mathcal{G}_4 \otimes \mathcal{G}_1$$

$$-1 \otimes \mathcal{G}_{4,2} - 3 \otimes \mathcal{G}_{4,1,1} - 6\,\mathcal{G}_1 \otimes \mathcal{G}_{4,1} - 3\,\mathcal{G}_2 \otimes \mathcal{G}_4 - 6\,\mathcal{G}_{1,1} \otimes \mathcal{G}_4 + 4\,\mathcal{G}_1 \otimes \mathcal{G}_{3,2} + 7\,\mathcal{G}_2 \otimes \mathcal{G}_{3,1} + 2\,\mathcal{G}_3 \otimes \mathcal{G}_3 + \mathcal{G}_2 \otimes \mathcal{G}_{2,2} + 4\,\mathcal{G}_3 \otimes \mathcal{G}_{2,1} + \mathcal{G}_4 \otimes \mathcal{G}_2 + \mathcal{G}_4 \otimes \mathcal{G}_{1,1}$$

$$+1 \otimes \mathcal{G}_{4,3} + 5 \otimes \mathcal{G}_{4,2,1} + 10\,\mathcal{G}_1 \otimes \mathcal{G}_{4,2} + 10\,\mathcal{G}_1 \otimes \mathcal{G}_{4,1,1} + 14\,\mathcal{G}_2 \otimes \mathcal{G}_{4,1} + 15\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,1} + 4\,\mathcal{G}_3 \otimes \mathcal{G}_4 + 12\,\mathcal{G}_{2,1} \otimes \mathcal{G}_4 - \mathcal{G}_2 \otimes \mathcal{G}_{3,2} - 2\,\mathcal{G}_3 \otimes \mathcal{G}_{3,1} - \mathcal{G}_4 \otimes \mathcal{G}_{2,1}$$

$$-2 \otimes \mathcal{G}_{4,3,1} - 4\,\mathcal{G}_1 \otimes \mathcal{G}_{4,3} - 1 \otimes \mathcal{G}_{4,2,2} - 16\,\mathcal{G}_1 \otimes \mathcal{G}_{4,2,1} - 16\,\mathcal{G}_2 \otimes \mathcal{G}_{4,2} - 12\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,2} - 12\,\mathcal{G}_2 \otimes \mathcal{G}_{4,1,1} - 10\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,1,1} - 10\,\mathcal{G}_3 \otimes \mathcal{G}_{4,1} - 29\,\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,1} - \mathcal{G}_4 \otimes \mathcal{G}_4 - 7\,\mathcal{G}_{3,1} \otimes \mathcal{G}_4 - 3\,\mathcal{G}_{2,2} \otimes \mathcal{G}_4$$

$$+ \cdots$$

$$-\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,3,2} - 2\,\mathcal{G}_{3,1} \otimes \mathcal{G}_{4,3,1} - \mathcal{G}_{4,1} \otimes \mathcal{G}_{4,2,1} - 3\,\mathcal{G}_{3,2} \otimes \mathcal{G}_{4,2,1}$$

Conjecture:

(1) There are finitely many non-zero quiver coefficients $c_\mu(\Omega)$ for each Ω.

(2) Cohomological quiver coefficients are non-negative, i.e.
$\sum |\mu_i| = \operatorname{codim}(\Omega) \;\Rightarrow\; c_\mu(\Omega) \geq 0$

(3) If Ω has rational singularities, then the quiver coefficients for Ω have alternating signs, i.e. $(-1)^{\sum |\mu_i| - \operatorname{codim}(\Omega)} c_\mu(\Omega) \geq 0$

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Analogy: $Y = G/P$ flag variety, $\Omega \subset Y$ closed subvariety.

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Status of Conjecture:

- True if Q is equioriented of type A
- True if Q is any quiver of type A_3
- (1) is true if Q is of Dynkin type and Ω has rational singularities.

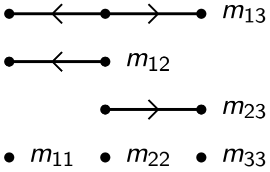
Outbound quiver of type A_3: $Q = \{ \mathbf{1} \leftarrow \mathbf{2} \rightarrow \mathbf{3} \}$

$X = \mathrm{Hom}(E_2, E_1) \oplus \mathrm{Hom}(E_2, E_3)$

Orbit closures $\Omega \subset X$ correspond to vectors

$$(m_{11}, m_{22}, m_{33}, m_{12}, m_{23}, m_{13}) \in \mathbb{N}^6$$

for which $e_k = \sum_{i \leq k \leq j} m_{i\,j}$ for $k = 1, 2, 3$.



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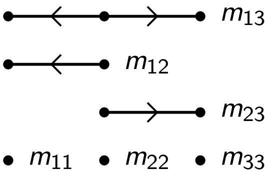

$$\Omega = \{ (\phi_1, \phi_3) \in X \mid \mathrm{rank}(\phi_1) \le m_{12} + m_{13} \;,\; \mathrm{rank}(\phi_3) \le m_{23} + m_{13} \;,\; \mathrm{rank}(\phi_1 \oplus \phi_3 : E_2 \to E_1 \oplus E_3) \le m_{12} + m_{23} + m_{13} \}$$

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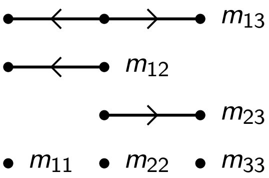

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Set $R = (m_{22})^{m_{13}}$. Write $\; \Delta^2(\mathcal{G}_R) = \sum_{\lambda,\mu,\nu} d^R_{\lambda,\mu,\nu}\, \mathcal{G}_\lambda \otimes \mathcal{G}_\mu \otimes \mathcal{G}_\nu$

Thm: $\; [\mathcal{O}_\Omega] = \sum_{\lambda,\mu,\nu} d^R_{\lambda,\mu,\nu}\; \mathcal{G}_{(m_{22}+m_{23})^{m_{11}},\lambda} \otimes \mathcal{G}_\mu \otimes \mathcal{G}_{(m_{22}+m_{12})^{m_{33}},\nu}$

Inbound quiver of type A_3: $Q = \{\mathbf{1} \to \mathbf{2} \leftarrow \mathbf{3}\}$

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Define: $c_{\lambda,\mu,\nu} = \sum_{\sigma,\tau} d_{\lambda,\sigma}^{(m_{33})^{m_{12}}} \, d_{\tau,\mu}^{(m_{11})^{m_{23}}} \, c_{\sigma\tau}^{\mu}$

Thm: $[\mathcal{O}_\Omega] = \sum_{\lambda,\mu,\nu} c_{\lambda,\mu,\nu} \, \mathcal{G}_\lambda \otimes \mathcal{G}_{(m_{11}+m_{13}+m_{33})^{m_{22}},\mu} \otimes \mathcal{G}_\nu$

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Open problem: Show combinatorially that the inbound and outbound formulas are equivalent, i.e.

$$\sum_{\lambda,\mu,\nu} d^{(m_{13})^{m_{22}}}_{\lambda,\mu,\nu} \, \mathcal{G}_{(m_{11})^{(m_{22}+m_{23})}+\lambda}(E_2 - E_1) \cdot \mathcal{G}_\mu(-E_2) \cdot \mathcal{G}_{(m_{33})^{(m_{22}+m_{12})}+\nu}(E_2 - E_3)$$

$$= \sum_{\lambda,\mu,\nu} c_{\lambda,\mu,\nu} \, \mathcal{G}_\lambda(E_1) \cdot \mathcal{G}_{(m_{11}+m_{13}+m_{33})^{m_{22}},\mu}(E_2 - E_1 \oplus E_3) \cdot \mathcal{G}_\nu(E_3)$$

Quiver coefficients of Dynkin type

Def: For $j \in [n]$, let $\psi_j : \Gamma^{\otimes n+1} \to \Gamma^{\otimes n+1}$ be the linear map

$$\psi_j(\mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_j} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes \mathcal{G}_\lambda) =$$

$$\sum_{\sigma,\nu} \left(\sum_\tau d^{\mu_j}_{\sigma,\tau} \, c^\nu_{\tau,\lambda} \right) \mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_{j-1}} \otimes \mathcal{G}_\sigma \otimes \mathcal{G}_{\mu_{j+1}} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes \mathcal{G}_\nu$$

- apply coproduct Δ to $\mathcal{G}_{\mu_j}$
- then multiply one of the factors to $\mathcal{G}_\lambda$

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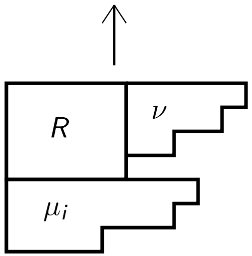
Def: For $i \in [n]$ and R rectangle, $\mathcal{A}_{i,R} : \Gamma^{\otimes n+1} \to \Gamma^{\otimes n+1}$ is the linear map

$$\mathcal{A}_{i,R}(\mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes \mathcal{G}_\nu) = \mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_{i-1}} \otimes \mathcal{G}_{R+\nu,\mu_i} \otimes \mathcal{G}_{\mu_{i+1}} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes 1$$



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Theorem

Let Q quiver of Dynkin type , $\Omega \subset X$ quiver cycle with rational sings.

Then $[\mathcal{O}_\Omega] \otimes 1 = \sum c_\mu(\Omega) \, \mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes 1$

is obtained by applying an explicitly determined sequence of operators ψ_j and $\mathcal{A}_{i,R}$ to $1 \otimes \cdots \otimes 1 \in \Gamma^{\otimes n+1}$

Note: Choice of operators and proof is based Reineke's desingularization.

Open questions:

1) Q quiver of Dynkin type, $\Omega \subset X$ orbit closure.

Find reduced equations for Ω, i.e. generators for $\mathcal{I}(\Omega) \subset \mathbb{F}[X]$

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2) Q general quiver (with loops), $\Omega \subset X$ quiver cycle, $M_i = \bigoplus_{j \to i} E_j$

Can we write $[\mathcal{O}_\Omega] = \sum_\mu c_\mu(\Omega)\, \mathcal{G}_{\mu_1}(E_1 - M_1) \cdots \mathcal{G}_{\mu_n}(E_n - M_n)$?

Can it be done with alternating signs ??

Note: Coefficients $c_\mu(\Omega)$ are not unique.

Examples suggest affirmative answer, e.g. equioriented cyclic quivers.