The combinatorics of associated Hermite polynomials

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Outline

1. Model for the associated Hermites and their moments
2. Proof of orthogonality & L^2 norm
3. Other models for the moments

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2. Proof of orthogonality & L^2 norm

3. Other models for the moments

Hermite polynomials: definition

The Hermite polynomials satisfy

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x),$$

$H_{-1}(x) = 0$, $H_0(x) = 1$. They're orthogonal with respect to a certain weight:

$$\int H_n(x)H_m(x)e^{-x^2/2}dx = \begin{cases} 0 & n \neq m \\ n! & n = m. \end{cases}$$

The quantities

$$\mu_n := \int x^n e^{-x^2/2}dx$$

are called the *moments*.

Model for usual Hermites & moments

Recursion suggests a combinatorial model for $H_{n+1}(x)$ as incomplete matchings on $[n+1]$:

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x),$$

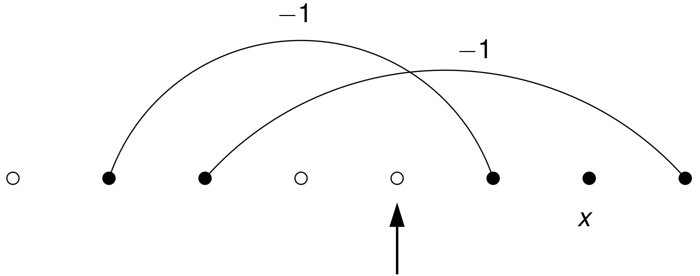
Either:

- vertex $n+1$ is fixed, with weight x, or
- we connect vertex $n+1$ to any of the n vertices to the left with an edge of weight -1,

then recursively fill in the remaining n (respectively, $n-1$) vertices with the same sort of weighted matching.

Example: building an incomplete matching

Part of a matching that will contribute to $H_8(x)$:



Vertex 5 might be left fixed (weight x) or be connected to one of the two vertices to its left (weight -1).

Moments for usual Hermites

The nth moment μ_n is the number of complete matchings on n vertices, so $\mu_{2n+1} = 0$ and $\mu_{2n} = (2n-1)(2n-3)\cdots 3\cdot 1$.

Can be found via Viennot's general combinatorial theory of OP (moments are weighted Motzkin paths) or by uniqueness of moments.

Associated Hermites: definition and model

The *associated Hermite polynomials* are defined by a small shift in the recursion:

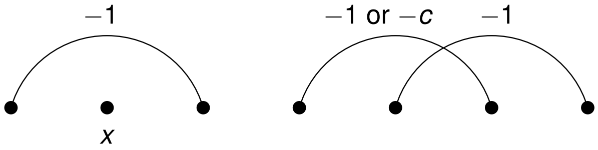
$$H_{n+1}(x;c) = xH_n(x;c) - (n+c)H_{n-1}(x;c)$$

with the same initial conditions.

We interpret the "$+c$" as meaning that a special choice of the edge may have weight -1 or $-c$. We'll use the rightmost available vertex.

Doing so means that edges that nest no fixed points or edges and have no left crossings may have weight -1 or $-c$; other edges have weight -1.

Example: weighted matching for $H_7(x; c)$



A matching that contributes $(-1)^3 x$ and $(-1)^2(-c)x$ to $H_7(x; c)$, which equals

$$x^7 + (6c - 21)x^5 + (10c^2 - 70c + 105)x^3 + (4c^3 - 42c^2 + 128c - 105)x$$

Moments of associated Hermites

The moment $\mu_n(c)$ for the associated Hermite polynomials are generating function for certain weighted complete matchings.

Using a recursive building process like before, and interpreting the $+c$ as being the *rightmost* available vertex, we have:

$\mu_n(c)$ is the g.f. for complete matchings on n vertices, where edges with no right crossings may have weight 1 or c.

Outline

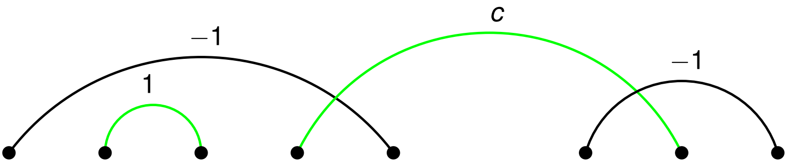
How to prove orthogonality?

$H_n(x;c)H_m(x;c)$	pairs of weighted partial matchings on $[n] \sqcup [m]$ vertices, with black edges weight -1 or $-c$
$\int H_n(x;c)H_m(x;c)d\mu$	complete matchings, with black edges weight -1 or $-c$ and fixed points replaced by complete matching from the moments, with green edges weight $+1$ or $+c$

Call those objects *paired matchings*. Want an involution on paired matchings with no fixed points when $n \neq m$ to prove orthogonality.

Example: paired matching

One paired matching contributing to $\int H_5(x; c) H_3(x; c) d\mu$:



An involution to prove orthogonality

In a paired matching on $[n] \sqcup [m]$, assume $n \geq m$, put the n vertices to the left of the m vertices.

Orthogonality involution

Find the leftmost edge that nests no other edges and flip its color.

If $n \neq m$, there's always such an edge. If $n = m$, all paired matchings cancel except those in which there's no "homogeneous" edges.

L^2 norm is $(c+1)_n$

Above involution shows $\int H_n(x;c)^2 d\mu$ is the generating function for paired matchings with only green edges going from left set of vertices to right set.

This is counting permutations by *left-to-right maxima*, whose generating function is $(c+1)_n = (c+1)(c+2)\cdots(c+n)$.

Outline

Other models for the moments

Take the moments of the associated Hermites and set $c = 1$.

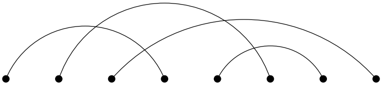
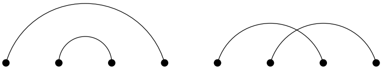
$\mu_0 = 1$	1
$\mu_2 = 1 + c$	2
$\mu_4 = 3 + 5c + 2c^2$	10
$\mu_6 = 15 + 32c + 22c^2 + 5c^3$	74
$\mu_8 = 105 + 260c + 234c^2 + 93c^3 + 14c^4$	706
$\mu_{10} = 945 + 2589c + 2750c^2 + 1450c^3 + 386c^4 + 42c^5$	8162

The sequence on the right is OEIS A000698.

Connected matchings

For rest of talk: weight moments with “leftmost available” weighting: nonnested edges eligible for weight 1 or c.

A *connected matching* is a (complete) matching in which every vertex (except the first and last) is nested by some edge.

Yes:  No: 

Theorem

$\mu_n(c)$ is also the g.f. for connected matchings on $n + 2$ points, with nonnested edges weight c *except* the edge containing the leftmost vertex.

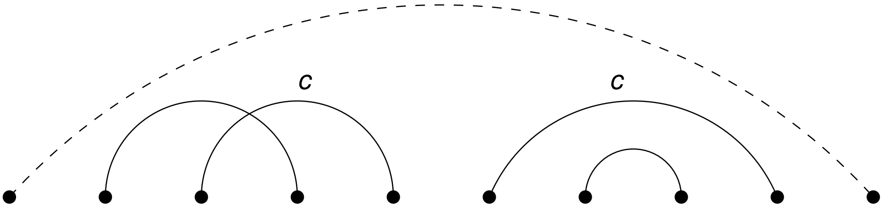
Connected matchings $\leftrightarrow$ complete matchings

To a matching on n vertices (with "nonnested 1 or c" weighting), add vertices 0 and $n + 1$ and connect them with an edge of weight 1. Then tailswap right to left with edges of weight c.

This is a weight-preserving bijection.

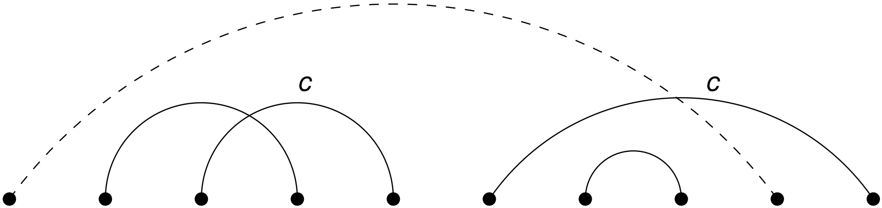
Connected matchings $\leftrightarrow$ complete matchings

Step 1:



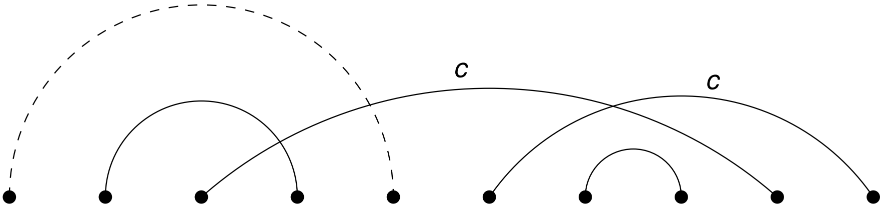
Connected matchings $\leftrightarrow$ complete matchings

Step 2:



Connected matchings ↔ complete matchings

Step 3:



Rooted maps

A *map* is a connected graph along with an embedding into an orientable surface. Loops and multiple edges are allowed. A *rooted map* is a map in which one edge has been directed.

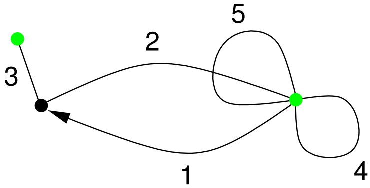
We'll give every vertex in a rooted map weight c *except* the vertex adjacent to the root edge.

Theorem

$\mu_n(c)$ is the generating function for rooted maps with n edges, weighted as above.

Rooted maps $\leftrightarrow$ connected matchings

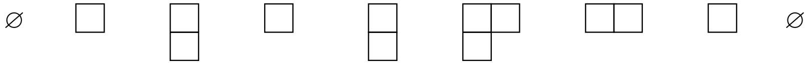
A bijection from rooted maps to connected matchings from Ossona de Mendez and Rosenstiehl happens to preserve weight!



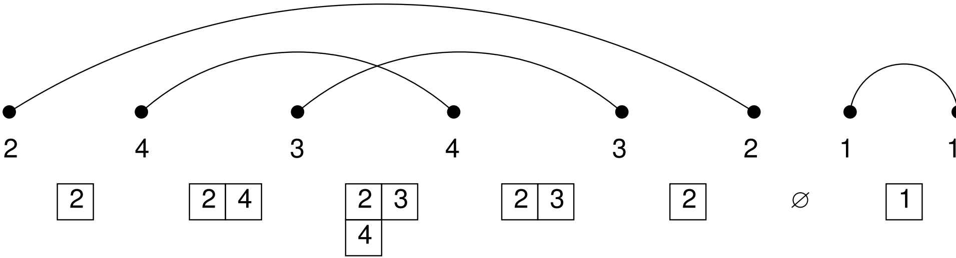
corresponds to $(1,5)(2,11)(3,9)(4,12)(6,7)(8,10)$.

Oscillating tableaux

An *oscillating tableau* is a walk in the Hasse diagram of the Young lattice, starting and ending at the empty partition.



Oscillating tableaux ↔ complete matchings



Weight an oscillating tableau by: entries that appear in the first columns of a shape may have weight 1 or c. Then:

Theorem

$\mu_n(c)$ is the generating function for weighted oscillating tableaux of length n.

Summary

- combinatorial model for the associated Hermite polynomials which generalizes that of the ordinary Hermite polynomials in a nice way;
- involution that proves orthogonality and L^2 norm;
- weight-preserving bijections between complete matchings on n vertices and...
  - connected matchings on $n + 2$ vertices,
  - rooted maps with n edges,
  - oscillating tableaux of length n.

Thank you

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Slides and abstract available from

www.math.umn.edu/~drake/fpsac