Representing Tropical Linear Spaces by Circuits

Josephine Yu

UC Berkeley / MIT

jyu@math.mit.edu

joint work with:

Debbie Yuster

Columbia / DIMACS

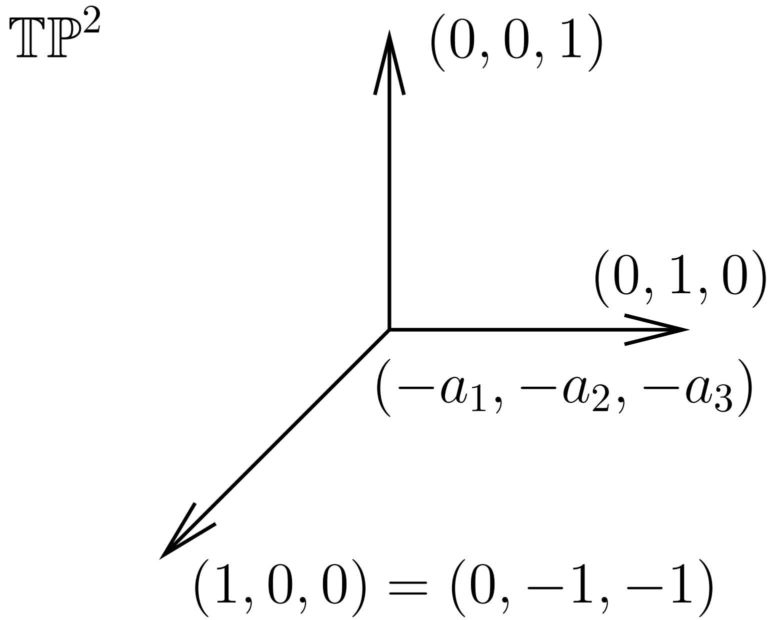
Tropical semiring : $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where $\oplus = \max$, $\odot = +$.

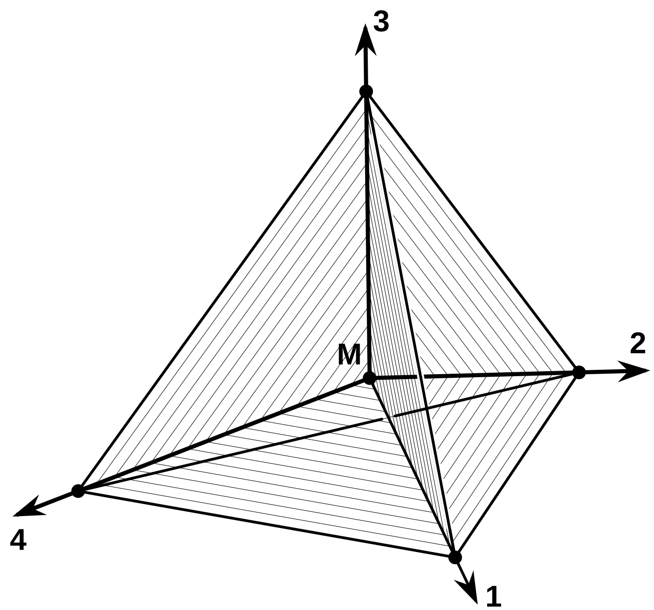
Tropical linear form (homogeneous): $(a_1 \odot x_1) \oplus \cdots \oplus (a_n \odot x_n)$

Tropical hyperplane:

$\{\mathbf{x} \in \mathbb{R}^n$: minimum in $(a_1 \odot x_1) \oplus \cdots \oplus (a_n \odot x_n)$ attained at least twice$\}$

Tropical projective space: $\mathbb{TP}^{n-1} = (\mathbb{T}^n \backslash (\infty, \ldots, \infty))/(1, \ldots, 1)$.

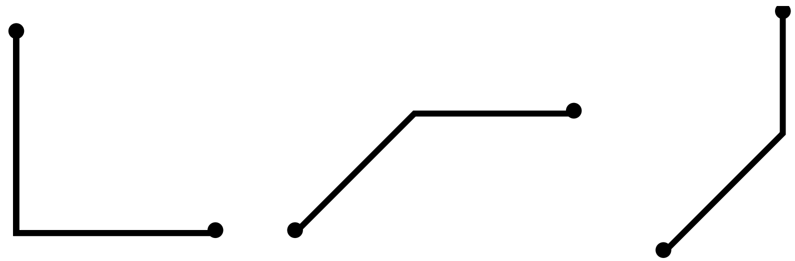




What is a tropical linear space?

First attempt: the image of a tropical linear map.

The image of a tropical linear map given by a matrix A is the tropical convex hull of the columns of A.



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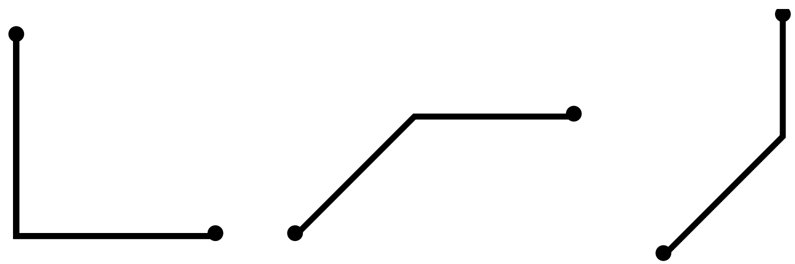
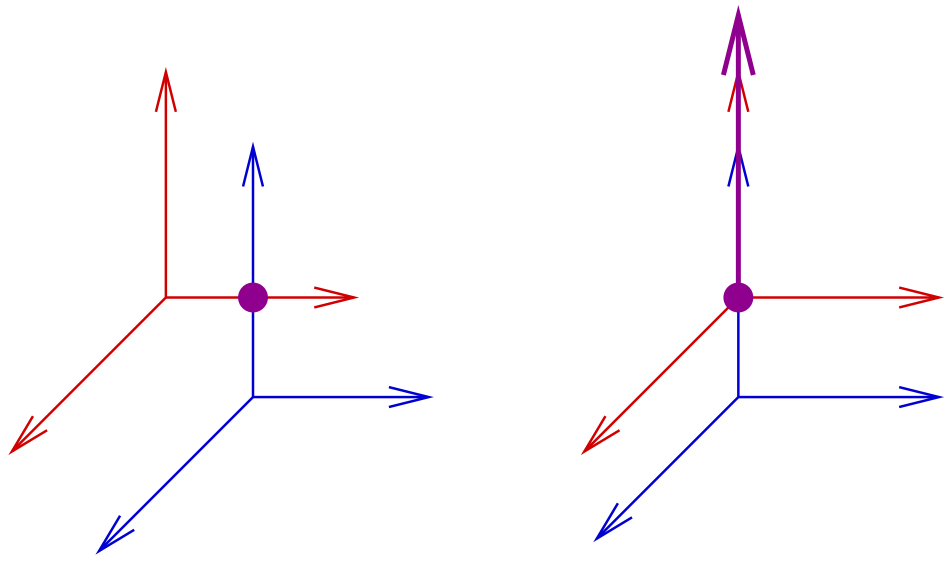


Second attempt: the intersection of some tropical hyperplanes.

Non-transverse intersections cause problems.



Valuated matroids

Fix $d < n$. A point $p \in \mathbb{TP}^{\binom{[n]}{d}-1}$ is a called a valuated matroid (or a tropical Plücker vector) if for any $d-2$ subset S of $[n]$ and distinct elements $i, j, k, l \in [n]\backslash S$, the minimum in

$$p(Sij) \odot p(Skl) \quad \oplus \quad p(Sik) \odot p(Sjl) \quad \oplus \quad p(Sil) \odot p(Sjl)$$

is attained at least twice.

Circuits of p:
$(p(\tau\backslash 1), \ldots, p(\tau\backslash n)) \in \mathbb{TP}^{n-1}$ for $d+1$-subsets $\tau \subset [n]$.

Cocircuits of p:
$(p(\sigma \cup 1), \ldots, p(\sigma \cup n))$ for $d-1$-subsets $\sigma \subset [n]$.

Notation: $p(\omega) = 0$ if $|\omega| \neq d$.

Tropical linear spaces

Definition (Speyer).
The tropical linear space L_p associated to a valuated matroid p is the intersection of all the hyperplanes corresponding to the circuits of p.

It is a codim-d polyhedral complex in $\mathbb{T}^n$.

Question : Can we represent tropical linear spaces as images and "kernels" of tropical linear maps?

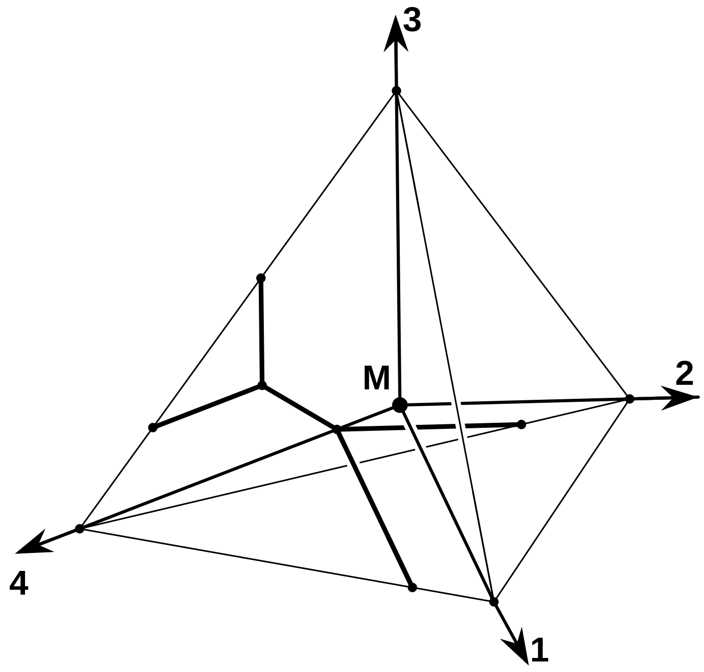


Image of a tropical linear map

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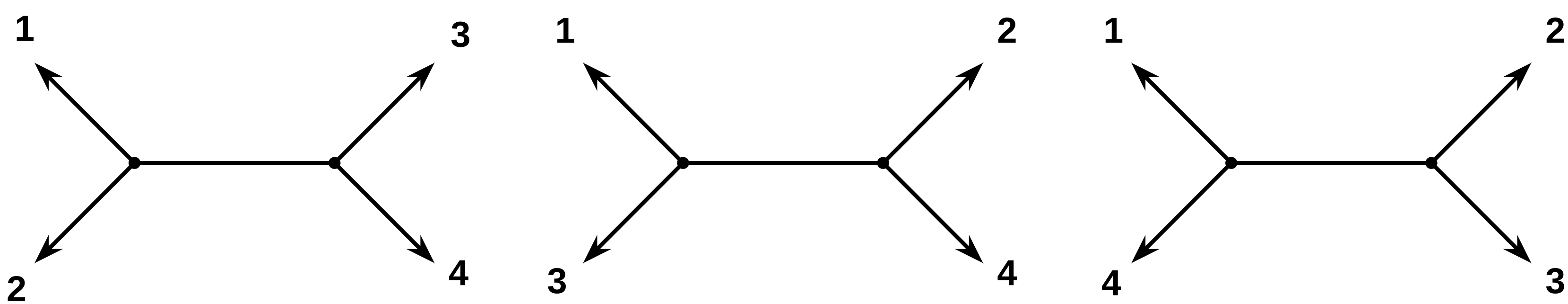
Theorem(Y.–Yuster).
$L_p = \text{image}(A)$ iff the columns of A contain all the cocircuits of p.

In particular, a tropical linear space is the tropical convex hull of cocircuits. This gives a new way to compute tropical linear spaces. (cf. Feichtner's talk yesterday).



Tropical linear space as a “tropical kernel"

A set of tropical linear forms that cut out the tropical linear space L_p is called a tropical basis of L_p.

Question : Which sets of circuits form a tropical basis?

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Constant coefficient case

- values of p are in $\{0, \infty\}$
- d-subsets of $[n]$ with non-infinity values form a matroid M
- the tropical linear space (also called the Bergman fan) is a subfan of the normal fan of the matroid polytope of M, consisiting of cones dual to loop-free faces.

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Non-constant coefficient case

- a subcomplex of the dual of a matroidal subdivision of the hypersimplex $\Delta_{d,n}$
- locally like constant coefficient
- related to the rank of a tropical matrix

Constant coefficient case (matroids)

The Bergman fan of a matroid on the ground set $[n]$ consists of all the weights $\omega \in \mathbb{R}^n$ such that the minimum weight is attained at least twice on each circuit.

Only look at matroids without loops or parallel elements.

Lemma (Ardila–Klivans).
The Bergman fan is determined by $0/1$ points in it.

Theorem(Y.–Yuster).

- The unique minimal tropical basis of a graphic matroid consists of the induced cycles.
- The unique minimal tropical basis of a cographic matroid consists of the edge cuts that split the graph into two 2-edge-connected subgraphs.

In general, minimal tropical bases are not unique.
They may even have different cardinalities.

Non-constant coefficient case

Question: Which sets of circuits form a tropical basis?

The tropical rank of a matrix is the size of the largest tropically non-singular submatrix, i.e. a submatrix where the minimum in its tropical determinant is not unique.

Theorem (Y.–Yuster)
Suppose that the valuated matroid p takes on only non-infinity values and that A is a matrix whose rows consist of some circuits forming a tropical basis. Then every d columns of A has tropical rank d.

Conjecture
The converse holds. That is, if every d columns of A has tropical rank d, then rows of A form a tropical basis.

Remark
When p takes on ∞-values, the result does not apply (at least not in the obvious way).

References

- Federico Ardila and Caroline Klivans, “The Bergman complex of a matroid and phylogenetic trees,” J. Combin. Theory Ser. B **96** (2006) 1, 38–49. (FPSAC 2004)
- Mike Develin, Francisco Santos, and Bernd Sturmfels. “On the rank of a tropical matrix,” *Combinatorial and computational geometry*, 213–242, Math. Sci. Res. Inst. Publ., **52**, Cambridge Univ. Press, Cambridge, 2005.
- Jürgen Richter-Gebert, Bernd Sturmfels, and Thorsten Theobald. “First steps in tropical geometry,” *Idempotent mathematics and mathematical physics*, 289–317, *Contemp. Math.*, **377**, Amer. Math. Soc., Providence, RI, 2005 (picture credits)
- David Speyer, “Tropical linear spaces,” `arXiv:math.CO/0410455`.
- Josephine Yu and Debbie S. Yuster, “Representing tropical linear spaces by circuits,” `arXiv:math.CO/0611579`.

Thank you!