q-Eulerian Polynomials: Excedance Number and Major Index

John Shareshian & Michelle Wachs

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$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq) - tq\exp_q(z)}$$

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Specialization of symmetric function identity

Permutation Statistics

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$A_3(t) = 1 + 4t + t^2$$

Permutation Statistics

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

Exponential generating function:

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t}$$

Permutation Statistics

q-analogs

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$1 + 2q + 2q^2 + q^3 = (1 + q + q^2)(1 + q)$$

q-Eulerian polynomials

$$A_n^{\mathrm{inv},\mathrm{des}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{des}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{inv},\mathrm{exc}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{exc}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

q-Eulerian polynomials

Theorem (Stanley 1976)

$$\sum_{n\geq 0} A_n^{\mathrm{inv},\mathrm{des}}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}$$

where

$$\mathrm{Exp}_q(z) := \sum_{n\geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}$$

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Theorem (Shareshian & MW 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}$$

where

$$\exp_q(z) := \sum_{n\geq 0} \frac{z^n}{[n]_q!}.$$

Symmetric Function Generalization

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} \quad = \quad \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}$$

Symmetric Function Generalization

$H(z) := \sum_{n\geq 0} h_n z^n$, where h_n is the nth complete homogeneous symmetric function in $x_1, x_2, \ldots$.

$$\frac{(1-t)H(z)}{H(zt)-tH(z)}$$

$$\Big\downarrow \quad \begin{aligned} x_i &:= q^{i-1} \\ z &:= z(1-q) \\ t &:= qt \end{aligned}$$

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq) - tq\exp_q(z)}$$

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Symmetric Function Generalization

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be obtained by placing bars above each excedance.

$$\bar{5}\bar{3}14\bar{6}2$$

View $\bar{\sigma}$ as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n\}.$$

Define

$$\mathrm{DEX}(\sigma) := \mathrm{DES}(\bar{\sigma})$$

$$\mathrm{DEX}(531462) = \mathrm{DES}(\bar{5}.\bar{3}14.\bar{6}2) = \{1, 4\}$$

$$\sum_{i \in \mathrm{DEX}(\sigma)} i = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$$

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Symmetric Function Generalization

For $T \subseteq [n-1]$, quasisymmetric function

$$F_T(x_1, x_2, \dots) := \sum_{\substack{s_1 \geq \cdots \geq s_n \\ i \in T \Rightarrow s_i > s_{i+1}}} x_{s_1} \dots x_{s_n}$$

From theory of quasisymmetric functions we have

$$F_T(1, q, q^2, \dots) = \frac{q^{\sum T}}{(1-q)(1-q^2)\dots(1-q^n)}$$

Hence

$$F_{\mathrm{DEX}(\sigma)}(1, q, q^2, \dots) = \frac{q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}}{(1-q)(1-q^2)\dots(1-q^n)}$$

Symmetric Function Generalization

By setting $x_i := q^{i-1}$ and $z := z(1-q)$ in

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} F_{\mathrm{DEX}(\sigma)} t^{\mathrm{exc}(\sigma)} z^n$$

we get

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} q^{maj(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} \frac{z^n}{[n]_q!}$$

Now set $t := qt$ to get

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!}$$

Symmetric Function Generalization

$H(z) := \sum_{n\geq 0} h_n z^n$, where h_n is the nth complete homogeneous symmetric function in $x_1, x_2, \ldots$.

$$\sum_{n\geq 0}\sum_{\sigma\in\mathfrak{S}_n} F_{\mathrm{DEX}(\sigma)} t^{\mathrm{exc}(\sigma)} r^{\mathrm{fix}(\sigma)} z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$$

$$\Big\downarrow \quad \begin{aligned} x_i &:= q^{i-1} \\ z &:= z(1-q) \\ t &:= qt \end{aligned} \qquad\qquad \Big\downarrow \quad \begin{aligned} x_i &:= q^{i-1} \\ z &:= z(1-q) \\ t &:= qt \end{aligned}$$

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc,fix}}(q,t,r)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(rz)}{\exp_q(ztq) - tq\exp_q(z)}$$

Symmetric Function Generalization

Theorem (Shareshian and MW, 2006)

Let

$$Q_{n,j,k} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ \mathrm{fix}(\sigma) = k}} F_{\mathrm{DEX}(\sigma)}$$

Then

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \sum_{k=0}^{n} Q_{n,j,k}\, t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt) - tH(z)}$$

Another specialization of $\sum Q_{n,j,k}\, t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

Two ways to specialize the quasisymmetric functions:

$$F_T(1, q, q^2, \dots) = \frac{q^{\sum T}}{(q;q)_n}$$

$$\sum_{m \geq 0} F_T(1, q, \dots, q^{m-1}) p^m = \frac{p^{|T|+1} q^{\sum T}}{(p;q)_{n+1}}.$$

where

$$(a;q)_n := \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & \text{if } n \geq 1. \end{cases}$$

Another specialization of $\sum Q_{n,j,k}\, t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

First specialization: $\sum \mathrm{DEX}(\sigma) = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$.

Theorem (Shareshian and MW, 2006)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc,fix}}(q,t,r)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(rz)}{\exp_q(ztq) - tq\exp_q(z)}.$$

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Second specialization:

$$|\mathrm{DEX}(\sigma)| = \begin{cases} \mathrm{des}(\sigma) & \text{if } \sigma(1) = 1 \\ \mathrm{des}(\sigma) - 1 & \text{otherwise} \end{cases}$$

Another specialization of $\sum Q_{n,j,k}\, t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

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Theorem (Foata-Han (2007), Gessel-Reutenauer (1993) t=1)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc,fix,des}}(q,t,r,p)\frac{z^n}{(p;q)_{n+1}} =$$

$$= \sum_{m\geq 0} p^m \frac{(1-tq)(z;q)_m(ztq;q)_m}{((z;q)_m - tq(ztq;q)_m)(zr;q)_{m+1}},$$

Steps of the Proof of $\sum Q_{n,j,k}\, t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

1. Modification of bijection of Gessel and Reutenauer, which takes compatible pairs to ornaments, is used to give alternative characterization of $Q_{n,j,k}$

2. Bijection from ornaments to banners, using Lyndon decomposition, is used to give another alternative characterization of $Q_{n,j,k}$

3. Generalization of bijection of Stembridge is used to show $Q_{n,j,k}$ satisfies recurrence relation, which yields generating function.

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Step 1 - Bicolored necklaces and ornaments

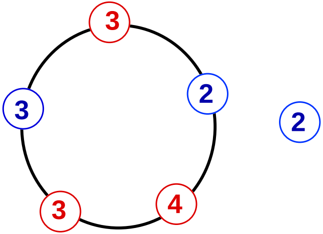
A bicolored necklace is a primitive circular word over alphabet

$$\{1, 1, 2, 2, \dots\}$$

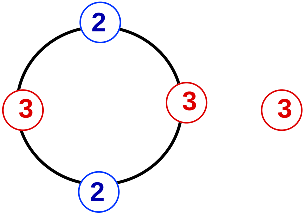
such that if size > 1

- a blue letter is followed by letter greater than or equal in value
- a red letter is followed by a letter less than or equal in value

Necklaces of size 1 are blue.



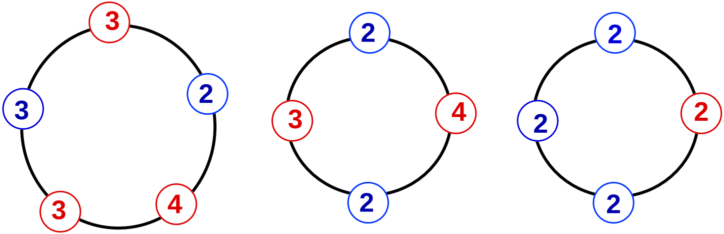
necklaces



not necklaces

Step 1 - Bicolored necklaces and ornaments

An ornament of type λ is a multiset of necklaces whose necklace sizes form partition λ



type $= (5, 4, 4)$ weight $= x_2^7 x_3^4 x_4^2$

Let $\mathcal{R}_{\lambda,j}$ = set of ornaments of type λ with j red letters.

Step 1 - Bicolored necklaces and ornaments

Theorem (Shareshian and MW)

For $\lambda \vdash n$, define

$$Q_{\lambda,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ \lambda(\sigma) = \lambda}} F_{\mathrm{DEX}(\sigma)}$$

Then

$$Q_{\lambda,j} = \sum_{R \in \mathcal{R}_{\lambda,j}} wt(R)$$

$$F_{\mathrm{DEX}(\sigma)} = \sum_{\substack{s_1 \geq \cdots \geq s_n \\ i \in \mathrm{DEX}(\sigma) \Rightarrow s_i > s_{i+1}}} x_{s_1} \ldots x_{s_n}$$

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Step 1 - Bicolored necklaces and ornaments

For $\lambda \vdash n$ and $j = 0, \ldots, n-1$, let

$$\mathrm{Com}_{\lambda,j} := \{(\sigma, s) : \sigma \in \mathfrak{S}_n,\ \lambda(\sigma) = \lambda, \mathrm{exc}(\sigma) = j,\ s \text{ is } \sigma\text{-compat}\}.$$

Bijection $\phi : \mathrm{Com}_{\lambda,j} \to \mathcal{R}_{\lambda,j}$

Let $\sigma = 45162387$ and $s = (7,7,7,5,5,4,2,2)$.

1. write σ in cycle form,

$$\sigma = (1,4,6,3)(2,5),(7,8).$$

2. color letters that are followed (cyclicly) by larger letters red and letters that are singletons or are followed by smaller letters blue,

$$(1,4,6,3)(2,5)(7,8).$$

3. replace each i by s_i, we have the ornament

$$(7,5,4,7)(7,5)(2,2).$$

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Let $\sigma = 45162387$ and $s = (7,7,7,5,5,4,2,2)$.

1. write σ in cycle form,
$$\sigma = (1,4,6,3)(2,5),(7,8).$$
2. color letters that are followed (cyclicly) by larger letters red and letters that are singletons or are followed by smaller letters blue,
$$({\color{red}1},{\color{red}4},{\color{blue}6},{\color{blue}3})({\color{red}2},{\color{blue}5})({\color{red}7},{\color{blue}8}).$$
3. replace each i by s_i, we have the ornament
$$({\color{red}7},{\color{red}5},{\color{blue}4},{\color{blue}7})({\color{red}7},{\color{blue}5})({\color{red}2},{\color{blue}2}).$$

Steps of the Proof of $\sum Q_{n,j,k}\, t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

1. Modification of bijection of Gessel and Reutenauer which takes compatible pairs to ornaments is used to give alternative characterization of $Q_{n,j,k}$

$\Longrightarrow$ 2. Bijection from ornaments to banners using Lyndon decomposition is used to give another alternative characterization of $Q_{n,j,k}$

3. Generalization of bijection of Stembridge is used to show $Q_{n,j,k}$ satisfies recurrence relation, which yields generating function.

Step 2 - Banners

A *banner* is a word over alphabet

$$\{1, 1, 2, 2, \ldots\}$$

such that

- blue letter is followed by letter greater than or equal in value or is last
- red letter is followed by a letter less than or equal in value

Example:

22757547

Step 2 - Banners

A Lyndon word over an ordered alphabet is a word that is strictly lexicographically larger than all its circular rearrangements.

The Lyndon factorization of a word over an ordered alphabet is a factorization into a weakly lexicographically increasing sequence of Lyndon words.

The Lyndon type $\lambda(w)$ of a word w is the partition whose parts are the lengths of the words in its Lyndon factorization.

Use the ordering $1 < 1 < 2 < 2 < \dots$ for our alphabet.
Example:

$$\lambda(22757547) = \lambda(22 \cdot 75 \cdot 7547) = 4, 2, 2$$

Step 2 - Banners

Theorem (Shareshian and MW)

Let $\mathcal{B}_{\lambda,j}$ = the set of banners of Lyndon type λ with j red letters. Then there is a weight-preserving bijection

$$\mathcal{B}_{\lambda,j} \to \mathcal{R}_{\lambda,j}.$$

Consequently

$$Q_{\lambda,j} = \sum_{b \in \mathcal{B}_{\lambda,j}} wt(b)$$

$$22757547 \mapsto 22 \cdot 75 \cdot 7547 \mapsto (2,2)(7,5)(7,5,4,7)$$

Steps of the Proof of $\sum Q_{n,j,k}\, t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$

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Step 3 - Recurrence Relation

Using ornaments one can easily show that the formula

$$\sum_{n\geq 0}\sum_{j=0}^{n-1}\sum_{k=0}^{n} Q_{n,j,k}\, t^j r^k z^n = \frac{(1-t)H(rz)}{H(zt)-tH(z)}$$

is equivalent to

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} Q_{n,j,0}\, t^j z^n = \frac{1-t}{H(zt)-tH(z)}$$

which is equivalent to recurrence relation

$$Q_{n,j,0} = \sum_{\substack{0\leq m\leq n-2\\ j+m-n<i<j}} Q_{m,i,0}\, h_{n-m}$$

Step 3 - Recurrence Relation

From Steps 1 and 2 we have

$$Q_{n,j,0} = \sum_{b \in \mathcal{B}_{n,j}} wt(b)$$

where

$$\mathcal{B}_{n,j} := \bigcup_{\substack{\lambda \vdash n \\ \lambda \text{ has no parts of size } 1}} \mathcal{B}_{\lambda,j}$$

Theorem (Shareshian and MW)

For all $n \geq 2$, there is a bijection

$$\gamma : \mathcal{B}_{n,j} \to \bigcup_{\substack{0 \leq m \leq n-2 \\ j+m-n < i < j}} \mathcal{B}_{m,i} \times \{(a_1 \leq \cdots \leq a_{n-m}) : a_i \in \mathbb{Z}^+\}$$

Connection with Toric Varieties

Let X_n be the toric variety associated with the Coxeter complex of $\mathfrak{S}_n$. The action of $\mathfrak{S}_n$ on X_n induces a representation of $\mathfrak{S}_n$ on $H^{2j}(X_n)$.

Theorem (Procesi, Stanley 1989)

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} \mathrm{ch} H^{2j}(X_n)\, t^j z^n = \frac{(1-t)H(z)}{H(zt)-tH(z)},$$

where ch *is the Frobenius characteristic,*

Theorem (Shareshian and MW)

$$\mathrm{ch} H^{2j}(X_n) = Q_{n,j} := \sum_{k=0}^{n} Q_{n,j,k}$$

Representation on Rees product

Action of $\mathfrak{S}_n$ on boolean algebra B_n induces an action of $\mathfrak{S}_n$ on maximal open intervals $I_{n,j} := (\hat{0}, ([n], j))$ of Rees product $(B_n - \{\hat{0}\}) * C_n$, which induces a representation of $\mathfrak{S}_n$ on $\tilde{H}_{n-2}(I_{n,j})$

Theorem (Shareshian and MW)

$$1 + \sum_{n\geq 1} \sum_{j=0}^{n-1} \mathrm{ch}\tilde{H}_{n-2}(I_{n,j+1})\, t^j z^n = \frac{(1-t)E(z)}{E(zt) - tE(z)},$$

where $E(z) = \sum_{n\geq 0} e_n z^n$ and e_n is the nth elementary symmetric function.

Corollary

$$\mathrm{ch}\tilde{H}_{n-2}(I_{n,j+1}) = \omega Q_{n,j}$$

and

$$H^{2j}(X_n) \cong_{\mathfrak{S}_n} \tilde{H}_{n-2}(I_{n,j+1}) \otimes \mathrm{sgn}$$