Aspects of Rational Universal Characters

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Schur Functions

Let $\Lambda(X)$ be the ring of symmetric functions in the variables X, and $s_\lambda(X)$ denote the Schur function corresponding to a partition λ

Jacobi–Trudi identity :

$$s_\lambda(X) = \det\left(h_{\lambda_i - i + j}(X)\right) = \det\left(e_{{}^t\lambda_i - i + j}(X)\right).$$

Cauchy identity :

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

Littlewood–Richardson coefficients :

$$s_\mu(X) \cdot s_\nu(X) = \sum_\lambda \mathrm{LR}_{\mu,\nu}^\lambda \, s_\lambda(X)$$

with non-negative integers $\mathrm{LR}_{\mu,\nu}^\lambda$.

Bi-determinant expression :

$$s_\lambda(x_1, \cdots, x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\le i,j\le n}}{\det\left(x_i^{n-j}\right)_{1\le i,j\le n}}.$$

Bernstein operators : If we define operators B_m by

$$\sum_{m\in\mathbb{Z}} B_m z^m = \exp\left(\sum_{k=1}^{\infty}\frac{p_k}{k}z^k\right)\exp\left(-\sum_{k=1}^{\infty}\frac{p_k^\perp}{k}z^{-k}\right),$$

where $p_k^\perp$ denotes the adjoint operator of the multiplication by p_k, then we have

$$s_\lambda(X) = B_{\lambda_1}B_{\lambda_2}\cdots B_{\lambda_{l(\lambda)}}(1).$$

Representation Theory of $\mathbf{GL}_N(\mathbb{C})$:

The Schur function $s_\lambda(x_1, \cdots, x_N)$ is the character of the irreducible polynomial representation of the general linear group $\mathbf{GL}_N(\mathbb{C})$ with highest weight

$$(\lambda_1, \cdots, \lambda_{l(\lambda)}, 0, \cdots, 0).$$

And

$$s_\lambda(1, q, q^2, \cdots, q^{N-1}) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{N+c(x)}}{1 - q^{h(x)}},$$

where $c(x)$ and $h(x)$ are the content and the hooklength respectively.

Remark : The (isomorphism classes of) finite dimensional irreducible representations of the algebraic group $\mathbf{GL}_N(\mathbb{C})$ are indexed by non-increasing sequence of integers

$$(\gamma_1, \cdots, \gamma_N) \quad (\gamma_1 \geq \cdots \geq \gamma_N).$$

Representation Theory of $\mathfrak{S}_n$:

The irreducible representations of the symmetric group $\mathfrak{S}_n$ of n letters are indexed by partitions of n.

$$\begin{array}{ccc} \mathrm{Irr}\,\mathfrak{S}_n & \longleftrightarrow & \mathcal{P}_n \\ S^\lambda & \longleftrightarrow & \lambda \end{array}$$

Let $K_0(\mathfrak{S}_n)$ be the Grothendieck group of the category of finite dimensional representations of $\mathfrak{S}_n$, and put

$$K_0(\mathfrak{S}_\bullet) = \bigoplus_{n\geq 0} K_0(\mathfrak{S}_n).$$

By using the natural embedding $\mathfrak{S}_m \times \mathfrak{S}_n \hookrightarrow \mathfrak{S}_{m+n}$, we can define a graded algebra structure on $K_0(\mathfrak{S}_\bullet)$. And, it follows from the Frobenius formula that, as graded algebras,

$$\begin{array}{ccc} K_0(\mathfrak{S}_\bullet) & \cong & \Lambda \\ [S^\lambda] & \longleftrightarrow & s_\lambda \end{array}$$

Rational Universal Characters

Definition of Rational Universal Characters (Koike)

For a pair of partitions (λ, μ) with $l(\lambda) \leq p$ and $l(\mu) \leq q$, put

$$s_{[\lambda,\mu]}(X,Y) = \det \left(\frac{\left(h_{\mu_{q+1-i}+i-j}(Y)\right)_{1\leq i\leq q, 1\leq j\leq p+q}}{\left(h_{\lambda_i-i+j-q}(X)\right)_{1\leq i\leq p, 1\leq j\leq p+q}} \right) \in \Lambda(X) \otimes \Lambda(Y),$$

and call it the rational universal character.

$$
\begin{aligned}
&s_{[\lambda,\mu]}(X,Y)\\
&= \det \left(\begin{array}{cccccccccc}
h_{\mu_q}(Y) & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & \ddots & & & & & & & \\
& & & h_{\mu_2}(Y) & h_{\mu_2-1}(Y) & h_{\mu_2-2}(Y) & & & & \\
& & & h_{\mu_1+1}(Y) & h_{\mu_1}(Y) & h_{\mu_1-1}(Y) & & & & \\
\hline
& & & & h_{\lambda_1-1}(X) & h_{\lambda_1}(X) & h_{\lambda_1+1}(X) & & & \\
& & & & h_{\lambda_2-2}(X) & h_{\lambda_2-1}(X) & h_{\lambda_2}(X) & & & \\
& & & & & & & \ddots & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & h_{\lambda_p}(X)
\end{array}\right).
\end{aligned}
$$

It is clear from this definition that

$$s_{[\lambda,\emptyset]}(X,Y) = s_\lambda(X),$$
$$s_{[\emptyset,\mu]}(X,Y) = s_\mu(Y).$$

Example :

$$s_{[(1),(1)]}(X,Y) = \det \begin{pmatrix} h_1(Y) & h_0(Y) \\ h_0(X) & h_1(X) \end{pmatrix} = h_1(X)h_1(Y) - 1,$$

$$s_{[(2,1),(1,1)]}(X,Y) = \det \begin{pmatrix} h_1(Y) & 1 & 0 & 0 \\ h_2(Y) & h_1(Y) & 1 & 0 \\ 1 & h_1(X) & h_2(X) & h_3(X) \\ 0 & 0 & 1 & h_1(X) \end{pmatrix}.$$

Cauchy identity for Rational Universal Characters

Theorem (Koike)

$$\sum_{\lambda,\mu} s_{[\lambda,\mu]}(X,Y)s_\lambda(U)s_\mu(V) = \frac{\prod_{j,k}(1-u_jv_k)}{\prod_{i,j}(1-x_iu_j)\prod_{i,k}(1-y_iv_k)}.$$

Proposition

$$s_{[\lambda,\mu]}(X,Y) = \sum_{\xi,\eta}\left(\sum_{\tau}(-1)^{|\tau|}\,\mathrm{LR}^{\lambda}_{\tau,\xi}\,\mathrm{LR}^{\mu}_{{}^t\tau,\eta}\right)s_\xi(X)s_\eta(Y),$$

$$s_\lambda(X)s_\mu(Y) = \sum_{\xi,\eta}\left(\sum_{\tau}\mathrm{LR}^{\lambda}_{\tau,\xi}\,\mathrm{LR}^{\mu}_{\tau,\eta}\right)s_{[\xi,\eta]}(X,Y).$$

Corollary $\{s_{[\lambda,\mu]}\}_{\lambda,\mu}$ forms a basis of $\Lambda(X)\otimes\Lambda(Y)$.

Structure Constants w.r.t. Rational Universal Characters

Proposition

$$s_{[\xi,\eta]}(X,Y) \cdot s_{[\sigma,\tau]}(X,Y) = \sum_{\lambda,\mu} M^{[\lambda,\mu]}_{[\xi,\eta],[\sigma,\tau]} s_{[\lambda,\mu]}(X,Y)$$

with

$$M^{[\lambda,\mu]}_{[\xi,\eta],[\sigma,\tau]} = \sum_{\alpha,\beta,\gamma,\delta} \left(\sum_{\kappa} \mathrm{LR}^{\xi}_{\kappa,\alpha} \, \mathrm{LR}^{\tau}_{\kappa,\beta} \right) \left(\sum_{\pi} \mathrm{LR}^{\eta}_{\pi,\gamma} \, \mathrm{LR}^{\sigma}_{\pi,\delta} \right) \mathrm{LR}^{\lambda}_{\alpha,\delta} \, \mathrm{LR}^{\mu}_{\beta,\gamma} \in \mathbb{N}.$$

Duality

Proposition Let $\tilde{\omega} : \Lambda(X) \otimes \Lambda(Y) \longrightarrow \Lambda(X) \otimes \Lambda(Y)$ be an algebra automorphism defined by

$$\tilde{\omega}(h_k(X)) = e_k(X), \quad \tilde{\omega}(h_k(Y)) = e_k(Y).$$

Then we have

$$\tilde{\omega}\left(s_{[\lambda,\mu]}(X,Y)\right) = s_{[{}^t\lambda,{}^t\mu]}(X,Y).$$

Corollary For a pair of partitions (λ, μ) with $l({}^t\lambda) \leq p$ and $l({}^t\mu) \leq q$, we have

$$s_{[\lambda,\mu]}(X,Y) = \det\left(\frac{\left(e_{{}^t\mu_{q+1-i}+i-j}(Y)\right)_{1\leq i\leq q, 1\leq j\leq p+q}}{\left(e_{{}^t\lambda_i-i+j-q}(X)\right)_{1\leq i\leq p, 1\leq j\leq p+q}}\right).$$

Another Determinant Expression

Theorem (cf. El Samra–King) For a pair of partitions (λ, μ), take an integer f satisfying $f \geq l(\lambda)$, $l({}^t\mu)$. Then we have

$$s_{[\lambda,\mu]}(X,Y) = (-1)^{f(f-1)/2} \det \left(s_{[(\lambda_i - i + f),(1^{{}^t\mu_j - j + f})]}(X,Y) \right)_{1 \leq i,j \leq f}.$$

Remark : For two non-negative integers a and b, we have

$$s_{[(a),(1^b)]}(X,Y) = \sum_k (-1)^k h_{a-k}(X) e_{b-k}(Y).$$

Vertex Operators

Theorem (Tsuda) If we define operators B_n^+, B_n^- $(n \in \mathbb{Z})$ by

$$\sum_{n\in\mathbb{Z}} B_n^+ z^n = \exp\left(\sum_{k\geq 1}\left(\frac{p_k(X)}{k} - \frac{p_k(Y)^\perp}{k}\right) z^k\right) \cdot \exp\left(-\sum_{k\geq 1}\frac{p_k(X)^\perp}{k} z^{-k}\right),$$

$$\sum_{n\in\mathbb{Z}} B_n^- z^{-n} = \exp\left(\sum_{k\geq 1}\left(\frac{p_y(X)}{k} - \frac{p_k(X)^\perp}{k}\right) z^{-k}\right) \cdot \exp\left(-\sum_{k\geq 1}\frac{p_k(Y)^\perp}{k} z^{k}\right),$$

then we have

$$s_{[\lambda,\mu]}(X,Y) = B^+_{\lambda_1} B^+_{\lambda_2} \cdots B^+_{\lambda_{l(\lambda)}} B^-_{\mu_1} B^-_{\mu_2} \cdots B^-_{\mu_{l(\mu)}}(1).$$

Remark : $[B^+_m, B^-_n] = 0.$

Tsuda uses these vertex operators to introduce a series of non-linear differential equations of infinite order, called the UC hierarchy, which can be regarded as an extension of the KP hierarchy. And he shows that the rational universal characters are solutions to the UC hierarchy.

Representation Theory of $\mathbf{GL}_N(\mathbb{C})$

If $l(\lambda) + l(\mu) \leq N$, then

$$s_{[\lambda,\mu]}(x_1, \cdots, x_N, 0, \cdots; x_1^{-1}, \cdots, x_N^{-1}, 0, \cdots)$$

is the character of the irreducible rational representation of $\mathbf{GL}_N(\mathbb{C})$ with highest weight

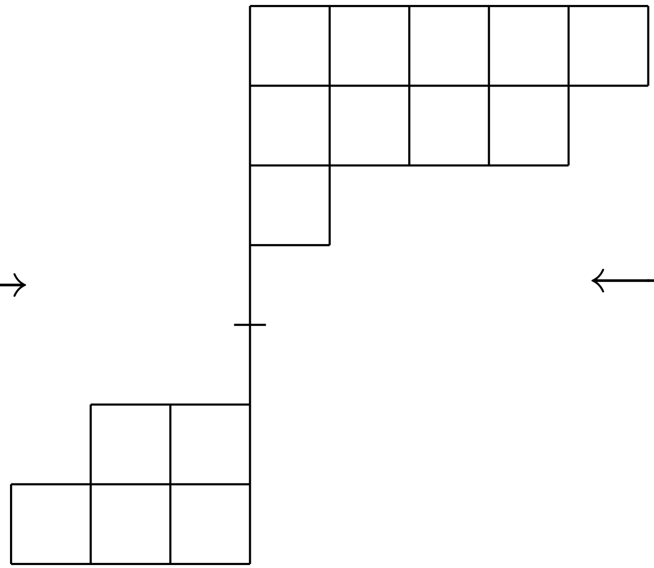
$$\gamma = (\lambda_1, \lambda_2, \cdots, \lambda_{l(\lambda)}, 0, \cdots, 0, -\mu_{l(\mu)}, \cdots, -\mu_2, -\mu_1).$$

Equivalently,

$$s_{[\lambda,\mu]}(x_1, \cdots, x_N; x_1^{-1}, \cdots, x_N^{-1}) = \frac{\det\left(x_i^{\gamma_j+n-j}\right)_{1\leq i,j\leq n}}{\det\left(x_i^{n-j}\right)_{1\leq i,j\leq n}}.$$

Example :

$$(5,4,1,0,0,-2,-3) \longleftrightarrow$$



$$\longleftrightarrow \quad \begin{gathered} \lambda = (5,4,1) \\ \mu = (3,2) \end{gathered}$$

Example : If $\lambda = \mu = \Box$, then

$$s_{[\Box,\Box]}(X, Y) = h_1(X)h_1(Y) - 1.$$

On the other hand, the irreducible rational representation of $\mathbf{GL}_N(\mathbb{C})$ with highest weight $(1, 0, \cdots, 0, -1)$ is the representation on

$$\mathfrak{sl}_N(\mathbb{C}) = \{A \in \mathbf{M}_N(\mathbb{C}) : \mathrm{tr}(A) = 0\},$$

where $\mathbf{GL}_N(\mathbb{C})$ acts on $\mathfrak{sl}_N(\mathbb{C})$ by $g \cdot A = gAg^{-1}$. And the character of this representation is

$$\sum_{i \neq j} x_i x_j^{-1} + N - 1 = \left(\sum_{i=1}^{N} x_i\right) \cdot \left(\sum_{j=1}^{N} x_j^{-1}\right) - 1.$$

$\boldsymbol{q}$-Dimenstion Formula

Theorem (El Samra–King) Suppose that $l(\lambda) + l(\mu) \leq N$. If we substitute

$$x_i = \begin{cases} q^{(N+1)/2-i} & (1 \leq i \leq N) \\ 0 & (i > N) \end{cases}, \quad y_i = \begin{cases} q^{-(N+1)/2+i} & (1 \leq i \leq N) \\ 0 & (i > N) \end{cases}$$

then we have

$$s_{[\lambda,\mu]} = \prod_{(i,j)\in\lambda} \frac{[N - {}^t\mu_i - {}^t\lambda_j + i + j - 1]}{[h_\lambda(i,j)]} \prod_{(k,l)\in\mu} \frac{[N + \mu_k + \lambda_l - k - l + 1]}{[h_\mu(k,l)]}.$$

where we use the notation

$$[k] = q^{k/2} - q^{-k/2}.$$

Representation Theory of Rational Brauer algebras

These algebras are defined and studied by Benkart–Chakrabarti–Halverson–Leduc–Lee–Stroomer.

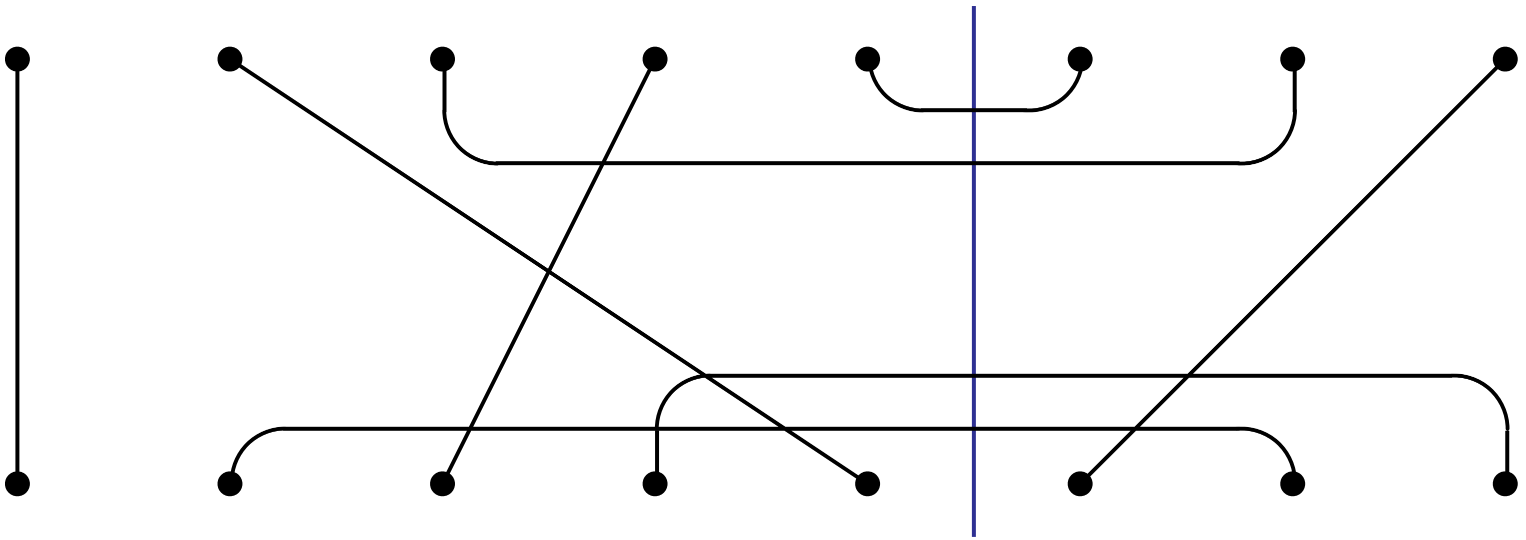
Let m, n be non-negative integers. A (m, n)-diagram is a graph with $2(m + n)$ vertices arranged in two rows of equal lengths and $(m + n)$ edges, satisfying the following three conditions:

- each vertex is incident to exactly one edge,
- each horizontal edge begins and ends on opposite side of the wall,
- no vertical edge crosses the wall,

where the wall is placed between the m-th and $(m + 1)$-th columns.

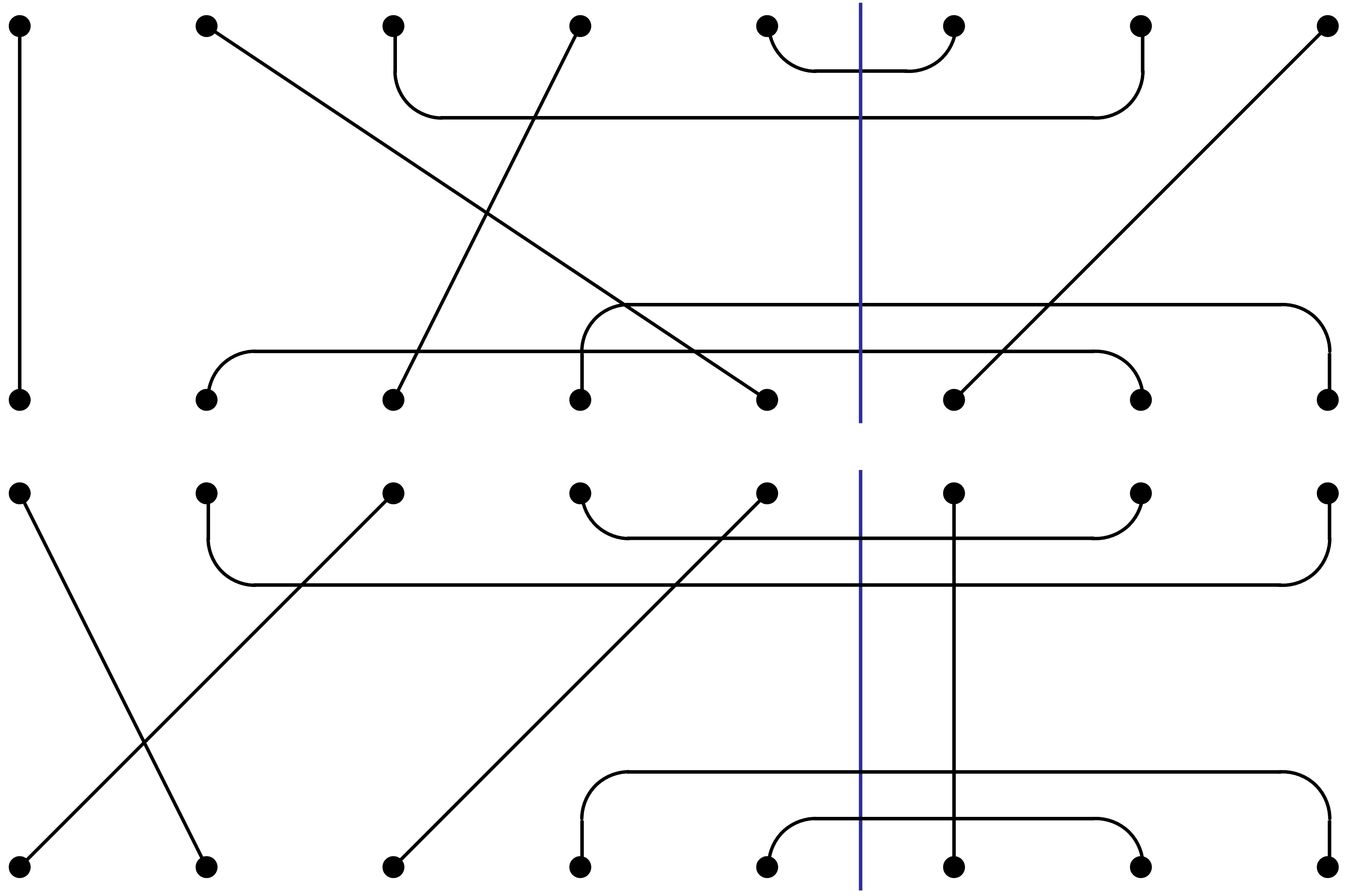
Let $D_{m,n}$ be the set of all (m, n)-diagrams.

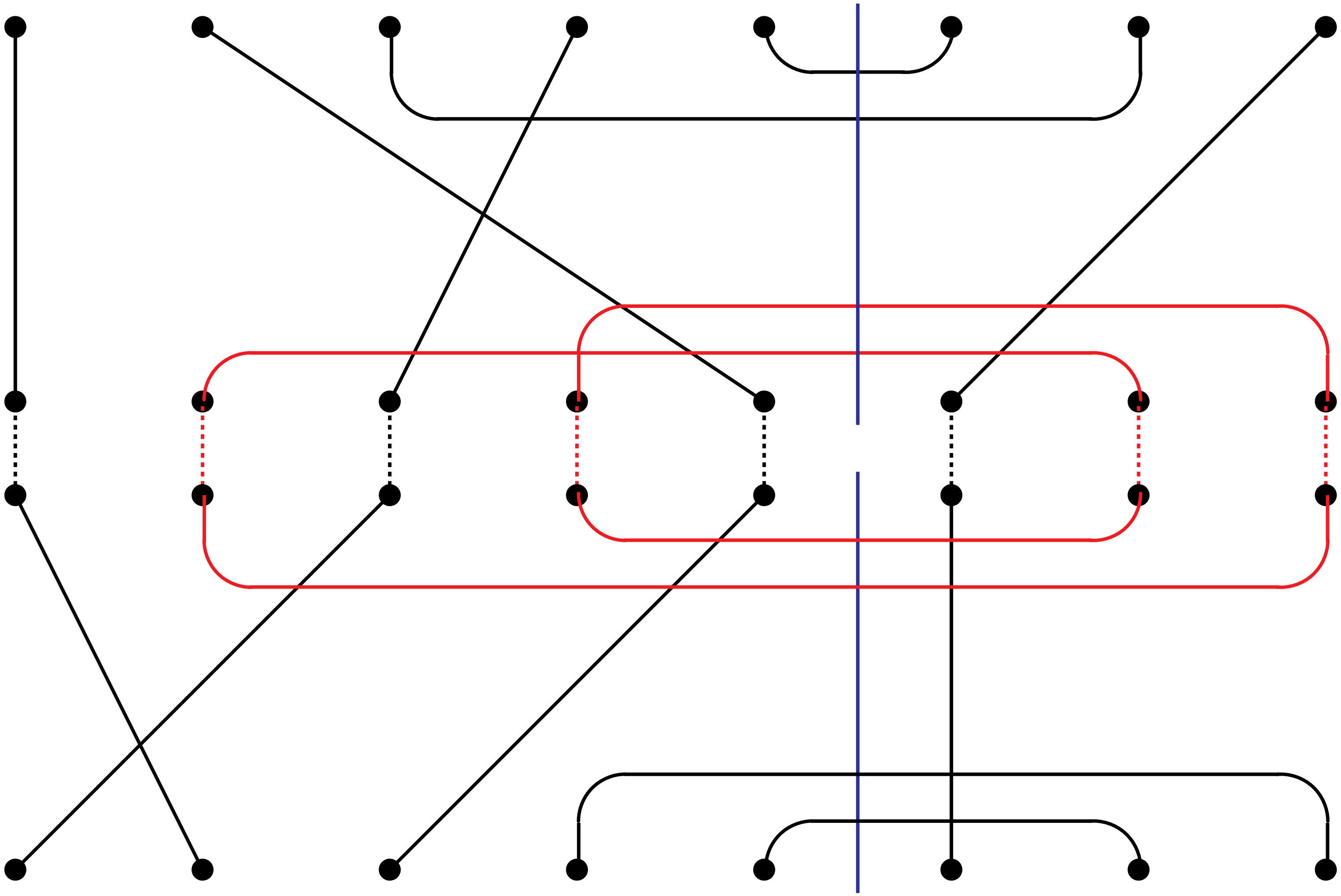
Example : $m = 5$, $n = 3$.

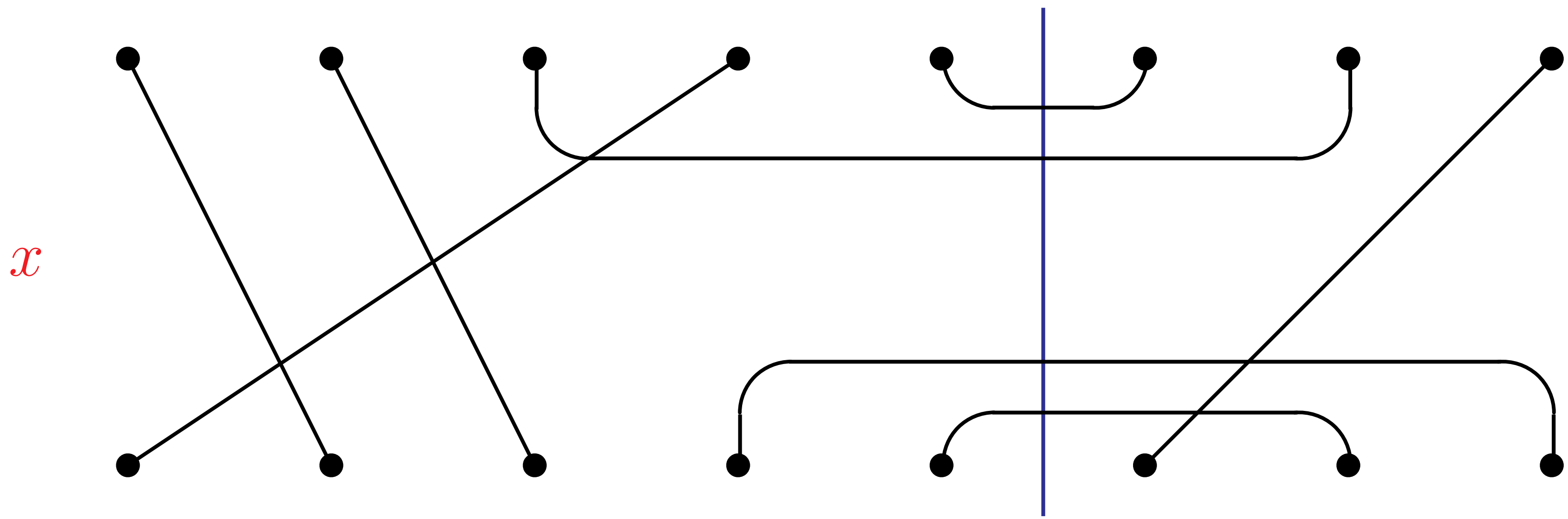


Let $\mathcal{B}^x_{m,n}$ be the $\mathbb{C}(x)$ vector space with basis $D_{m,n}$. Then we can define an associative algebra structure on $\mathcal{B}^x_{m,n}$ by concatenating two diagrams and replacing each loop by x. We call this algebra the rational Brauer algebra. (Similarly, we can define a $\mathbb{C}$-algebra $\mathcal{B}^\alpha_{m,n}$ for $\alpha \in \mathbb{C}$.)

Example :







x

Schur–Weyl type Duality

Let $V = \mathbb{C}^N$ be the vector representation of $\mathbf{GL}_N(\mathbb{C})$ and consider the mixed tensor representation

$$T^{m,n} = V^{\otimes m} \otimes (V^*)^{\otimes n}.$$

Theorem (BCHLLS) There exist an algebra homomorphism

$$\phi : \mathcal{B}_{m,n}^N \longrightarrow \mathrm{End}_{\mathbf{GL}_N(\mathbb{C})}(T^{m,n}),$$

such that $\mathcal{B}_{m,n}^N$ and $\mathbf{GL}_N(\mathbb{C})$ form a dual pair in $\mathrm{End}(T^{m,n})$, i.e.,

$$\mathrm{End}_{\mathbf{GL}_N(\mathbb{C})}(T^{m,n}) = \phi(\mathcal{B}_{m,n}^N),$$
$$\mathrm{End}_{\mathcal{B}_{m,n}^N}(T^{m,n}) = \langle \mathbf{GL}_N(\mathbb{C}) \rangle.$$

Moreover, if $N \geq m + n$, then ϕ is an isomorphism.

Theorem (BCHLLS) The rational Brauer algebra $\mathcal{B}^x_{m,n}$ is a semisimple $\mathbb{C}(x)$ algebra and the irreducible representations are indexed by triples (λ, μ, k) with $\lambda \vdash m-k$ and $\mu \vdash n-k$.

$$\begin{array}{ccc} \mathrm{Irr}\,\mathcal{B}^x_{m,n} & \longleftrightarrow & \{(\lambda,\mu,k) : \lambda \vdash m-k,\ \mu \vdash n-k\} \\ V^{\lambda,\mu,k} & \longleftrightarrow & (\lambda,\mu,k) \end{array}$$

Let $K_0(\mathcal{B}^x_{m,n})$ be the Grothendieck group of the category of finite dimensional $\mathcal{B}^x_{m,n}$-modules and put

$$K_0(\mathcal{B}^x_{\bullet,\bullet}) = \bigoplus_{m,n\geq 0} K_0(\mathcal{B}^x_{m,n}).$$

Then we can use the canonical embedding $D_{m,n} \times D_{p,q} \hookrightarrow D_{m+p,n+q}$ to define a bi-graded algebra structure on $K_0(\mathcal{B}^x_{\bullet,\bullet})$.

Theorem As bi-graded algebras,

$$\begin{array}{ccc} K_0(\mathcal{B}^x_{\bullet,\bullet}) & \cong & (\Lambda(X) \otimes \Lambda(Y))[t] \\ [V^{\lambda,\mu,k}] & \longleftrightarrow & s_{[\lambda,\mu]}(X,Y;t)t^k \end{array}$$

where the grading on the right-hand side is given by

$$\deg h_k(X) = (k,0), \quad \deg h_k(Y) = (0,k), \quad \deg t = (1,1).$$

Also $s_{[\lambda,\mu]}(X,Y;t)$ is defined by

$$\sum_{\lambda,\mu} s_{[\lambda,\mu]}(X,Y;t) s_\lambda(U) s_\mu(V) = \frac{\prod_{j,k}(1 - t u_j v_k)}{\prod_{i,j}(1 - x_i u_j) \prod_{i,k}(1 - y_i v_k)}.$$

Remark : $s_{[\lambda,\mu]}(X,Y;1)$ is the rational universal character.

Rational Universal Characters
and
Kawanaka's q-Cauchy Identity

Notation

For two partitions λ and μ, we define

$$h_{\lambda,\mu}(i,j) = \lambda_i + {}^t\mu_j - i - j + 1,$$
$$n(\lambda,\mu) = \sum_{(i,j)\in\lambda-\mu} ({}^t\lambda_j - i),$$

and

$$J_{\lambda,\mu}(t) = t^{n(\lambda,\mu)} \prod_{x\in\lambda} \frac{1+t^{h_{\lambda,\mu}(x)}}{1-t^{h_\lambda(x)}} \cdot t^{n(\mu,\lambda)} \prod_{x\in\mu} \frac{1+t^{h_{\mu,\lambda}(x)}}{1-t^{h_\mu(x)}}.$$

Note that the hook length is given by

$$h_\lambda(i,j) = h_{\lambda,\lambda}(i,j).$$

Kawanaka's q-Cauchy Identity

Theorem (Kawanaka)

$$\prod_{i=1}^{\infty} \frac{(-u_i q; q^2)_\infty (-v_i q; q^2)_\infty}{(u_i q; q^2)_\infty (v_i q; q^2)_\infty} \prod_{i,j=1}^{\infty} \frac{1}{1 - u_i v_j}$$
$$= \sum_{\lambda,\mu} q^{|\lambda-\mu|+|\mu-\lambda|} J_{\lambda,\mu}(q^2) s_\lambda(U) s_\mu(V),$$

where

$$(a;q)_\infty = \prod_{r \geq 0} (1 - aq^r).$$

If we put $q = 0$, then this reduces to the classical Cauchy identity

$$\prod_{i,j} \frac{1}{1 - u_i v_j} = \sum_\lambda s_\lambda(U) s_\lambda(V).$$

Kawanaka's motivation is an explicit computation of a ***q*-Frobenius–Schur indicator** for imprimitive complex reflection groups $G = G(m, p, n)$:

$$\Psi_G(\chi, q) = \frac{1}{\#G} \sum_{w \in G} \chi(w^2) \frac{\det(1 + q\rho(w))}{\det(1 - q\rho(w))},$$

where $\rho : G \to \mathbf{GL}_n(\mathbb{C})$ is the reflection representation and χ is an irreducible character of G.

Kawanaka's proof uses the induction on the number of the variables in X. Ishikawa–Wakayama provides another proof by giving a determinant expression of $q^{|\lambda - \mu| + |\mu - \lambda|} J_{\lambda,\mu}(q^2)$ and applying the Cauchy–Binet formula. However both of two proofs are complicated.

Proof : We give a proof of the dual form of the Kawanaka's q-Cauchy identity

$$\prod_{i=1}^{\infty} \frac{(-u_i q; q^2)_\infty (-v_i q; q^2)_\infty}{(u_i q; q^2)_\infty (v_i q; q^2)_\infty} \prod_{i,j=1}^{\infty} (1 + u_i v_j)$$
$$= \sum_{\lambda,\mu} q^{|{}^t\lambda - \mu| + |\mu - {}^t\lambda|} J_{{}^t\lambda,\mu}(q^2) s_\lambda(U) s_\mu(V).$$

Consider a homomorphism $\pi : \Lambda(X) \otimes \Lambda(Y) \longrightarrow \mathbb{Q}(a^{1/2}, q^{1/2})$ defined by

$$\pi(h_k(X)) = \pi(h_k(Y)) = \prod_{i=1}^{k} \frac{[a; i-1]}{[i]} \quad (k \geq 1),$$

where

$$[k] = q^{k/2} - q^{-k/2}, \quad \text{and} \quad [a; k] = a^{1/2} q^{k/2} - a^{-1/2} q^{-k/2}.$$

The q-binomial theorem gives

$$\pi\left(\prod_i \frac{1}{1-x_i u}\right) = \frac{(a^{1/2}q^{1/2}u;q)_\infty}{(a^{-1/2}q^{1/2}u;q^2)_\infty}.$$

And, by using El Samra–King's q-dimension formula, we see that

$$\pi(s_{[\lambda,\mu]}(X,Y)) \\ = \prod_{(i,j)\in\lambda} \frac{[a; -{}^t\mu_i - {}^t\lambda_j + i + j - 1]}{[h_\lambda(i,j)]} \prod_{(k,l)\in\mu} \frac{[a; \mu_k + \lambda_l - k - l + 1]}{[h_\mu(k,l)]}.$$

By applying π to the both hand sides of the Cauchy identity for rational universal characters

$$\sum_{\lambda,\mu} s_{[\lambda,\mu]}(X,Y) s_\lambda(U) s_\mu(V) = \frac{\prod_{j,k}(1-u_j v_k)}{\prod_{i,j}(1-x_i u_j)\prod_{i,k}(1-y_i v_k)},$$

we have

$$\sum_{\lambda,\mu}\left(\prod_{(i,j)\in\lambda}\frac{[a;-{}^t\mu_i-{}^t\lambda_j+i+j-1]}{[h_\lambda(i,j)]}\prod_{(k,l)\in\mu}\frac{[a;\mu_k+\lambda_l-k-l+1]}{[h_\mu(k,l)]}\right)$$
$$\times s_\lambda(U)s_\mu(V)$$
$$=\prod_{i=1}^{\infty}\frac{(a^{1/2}q^{1/2}u_i;q)_\infty(a^{1/2}q^{1/2}v_i;q)_\infty}{(a^{-1/2}q^{1/2}u_i;q)_\infty(a^{-1/2}q^{1/2}v_i;q)_\infty}\prod_{i,j}(1-u_iv_j).$$

By replacing

$$a^{1/2}\mapsto\sqrt{-1}a^{1/2},\quad u_i\mapsto\sqrt{-1}u_i,\quad v_i\mapsto\sqrt{-1}v_i.$$

and then substituting $a=1$, it turns out that the proof of Kawanaka's q-Cauchy identity is reduced to proof of the following lemma.

Lemma

$$(-1)^{|\lambda|+|\mu|} \prod_{(i,j)\in\lambda} \frac{[{}^t\mu_i + {}^t\lambda_j - i - j + 1]_+}{[h_\lambda(i,j)]} \prod_{(k,l)\in\mu} \frac{[\mu_k + \lambda_l - k - l + 1]_+}{[h_\mu(k,l)]}$$

$$= q^{(|{}^t\lambda-\mu|+|\mu-{}^t\lambda|)/2} J_{{}^t\lambda,\mu}(q^2).$$

where

$$[k]_+ = q^{k/2} + q^{-k/2}.$$

The proof of this Lemma is reduced to showing

$$\sum_{(i,j)\in\lambda} (\lambda_i - {}^t\mu_i) + \sum_{(k,l)\in\mu} ({}^t\mu_l - \lambda_l)$$

$$= |{}^t\lambda - \mu| + |\mu - {}^t\lambda| + 2n({}^t\lambda, \mu) + 2n(\mu, {}^t\lambda).$$

Rational Universal Characters and Painlevé-type equations

Painlevé equations

The Painlevé equations are non-linear ordinary differential equations of 2nd order, which were discovered by P. Painlevé around 1900 in his study of algebraic differential equations $y'' = R(t, y, y')$ without movable singularities (branching points).

Example : The fifth Painlevé equation P_{V} is

$$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{2t^2}\left(\kappa_\infty^2 y - \frac{\kappa_0^2}{y}\right) - (\theta+1)\frac{y}{t} - \frac{y(y+1)}{2(y-1)},$$

where κ_∞, κ_0 and θ are parameters.

It was known that the Painlevé equations admit algebraic or rational solutions for special values of parameters, which are obtained by specializing Schur functions. These specializations of Schur functions are called Yablonskii–Vorob'ev polynomials, Okamoto Polynomials, and Umemura polynomials. And these special polynomials are interesting from the combinatorial point of view.

In 2002, Masuda–Ohta–Kajiwara found a family of rational solutions of the fifth Painlevé equation P_{V}, which are described in terms of rational universal characters.

Theorem (Masuda–Ohta–Kajiwara) For non-negative integers m and n, we define $S_{m,n}(t,s)$ to be the specialization of

$$s_{[(n,n-1,\cdots,1),(m,m-1,\cdots,1)]}$$

obtained by substituting

$$p_k(X) \longmapsto -\frac{t}{2}k + (2s-m+n), \quad p_k(Y) \longmapsto \frac{t}{2}k + (2s-m+n).$$

Then

$$y = \frac{S_{m,n-1}(t,s)S_{m-1,n}(t,s)}{S_{m-1,n}(t,s-1)S_{m,n-1}(t,s+1)}$$

gives a rational solution of P_{V} with the parameters

$$\kappa_\infty = s, \quad \kappa_0 = s-m+n, \quad \theta = m+n-1.$$

Several Painlevé-type equations have algebraic or rational solutions obtained by specializing rational universal characters.

Algebraic solutions of q-P_{VI}

The sixth q-Painlevé equation q-P_{VI} is the following system of q-difference equations:

$$f \cdot \overline{f} = b_7 b_8 \frac{(g+b_5)(g+b_6)}{(g+b_7)(g+b_8)}, \quad g \cdot \underline{g} = b_3 b_4 \frac{(f+b_1)(f+b_2)}{(f+b_3)(f+b_4)},$$

where f and g are the unknown functions in variables $a_0, a_1, \cdots, a_5$ with $a_0 a_1 a_2^2 a_3^2 a_4 a_5 = q$, and

$$\overline{f} = f(\cdots, q a_2, q^{-1} a_3, \cdots), \quad \underline{g} = g(\cdots, q^{-1} a_2, q a_3, \cdots).$$

(The ratio a_2/a_3 plays the role of an independent variable and the other a_i's are parameters.) Also $b_1, \cdots, b_8$ are defined by

$$b_1 = a_3^2 a_4^{-1} a_5, \quad b_2 = a_3^2 a_4^3 a_5, \quad b_3 = a_3^{-2} a_4^{-1} a_5, \quad b_4 = a_3^{-2} a_4^{-1} a_5^{-3},$$
$$b_5 = a_0^{-1} a_1 a_2^{-2}, \quad b_6 = a_0^{-1} a_1^{-3} a_2^{-2}, \quad b_7 = a_0^{-1} a_1 a_2^2, \quad b_8 = a_0^3 a_1 a_2^2.$$

Let $R_{[\lambda,\mu]}(\xi,\eta,\zeta,q)$ be the specialization of the rational universal character $s_{[\lambda,\mu]}$ obtained by substituting

$$p_n(X) \longmapsto \frac{\eta^n + \xi^n - (-\zeta)^n - (-\zeta)^{-n}}{1-q^{2n}},$$
$$p_n(Y) \longmapsto \frac{\eta^{-n} + \xi^{-n} - (-\zeta)^n - (-\zeta)^{-n}}{1-q^{-2n}}.$$

Remark

$$\sum_{k\geq 0} R_{[(1^k),\emptyset]}(\xi,\eta,\zeta,q)t^k = \frac{(-\eta t, -\zeta t; q^2)_\infty}{(\xi t, \xi^{-1} t; q^2)_\infty},$$

so $R_{[(1^k),\emptyset]}(\xi,\eta,\zeta,q)$ is essentially the Al-Salam–Chihara polynomial.

Using $R_{[\lambda,\mu]}(\xi,\eta,\zeta,q)$, we define ρ_i $(i=0,1,2,3)$ as follows:

$$\begin{aligned}
\rho_0 &= R_{[\delta(m-1),\delta(n-1)]}(\xi,\eta,\zeta,q),\\
\rho_1 &= R_{[\delta(m),\delta(n-1)]}(\xi,\eta,q^{-2}\zeta,q),\\
\rho_2 &= R_{[\delta(m),\delta(n)]}(\xi,\eta,\zeta,q),\\
\rho_3 &= R_{[\delta(m-1),\delta(n)]}(\xi,\eta,q^{-2}\zeta,q),
\end{aligned}$$

where

$$\delta(k) = (k, k-1, \cdots, 1) \quad \text{if } k \geq 0,$$
$$\delta(k) = \delta(-k-1) \quad \text{if } k < 0.$$

Theorem (Tsuda–Masuda) The pair

$$f = \frac{\rho_1(\xi,\eta,\zeta,q)\cdot\rho_3(q^{-1}\xi,q^{-1}\zeta,q\eta,q)}{\rho_1(q^{-1}\xi,q^{-1}\zeta,q\eta,q)\cdot\rho_3(\xi,\eta,\zeta,q)},$$

$$g = \frac{\rho_0(\xi,\eta,\zeta,q)\cdot\rho_2(q^{-1}\xi,q^{-1}\zeta,q\eta,q)}{\rho_0(q^{-1}\xi,q^{-1}\zeta,q\eta,q)\cdot\rho_2(\xi,\eta,\zeta,q)}$$

give an algebraic solution of sixth q-Painlev'e equation q-P_{VI} with

$$\xi = a_2a_3, \quad \eta = \frac{a_2}{a_3}, \quad \zeta = \frac{a_0a_1a_2}{a_3a_4a_5}q,$$

and

$$\frac{a_0}{a_1} = q^{m-n}, \quad \frac{a_4}{a_5} = q^{m+n}.$$

Positivity of Tsuda–Masuda polynomials

Consider

$$
\begin{aligned}
P_{[\lambda,\mu]}(\xi,\eta,\zeta,q) \\
&= \xi^{|\lambda|+|\mu|}\eta^{|\mu|}\zeta^{|\mu|}q^{-2|\nu|}\prod_{b\in\lambda}(1-q^{2h_\lambda(b)})\prod_{c\in\mu}(q^{2h_\mu(c)}-1) \\
&\qquad \cdot R_{[\lambda,\mu]}(\xi,\eta,\zeta,q),
\end{aligned}
$$

where

$$
\nu_i = \max(0, {}^t\mu_i - \lambda_i)
$$

Conjecture (Tsuda) $P_{[\lambda,\mu]}(\xi,\eta,\zeta,q)$ is a polynomial in ξ, η, ζ and q^2 with non-negative integer coefficients :

$$
P_{[\lambda,\mu]}(\xi,\eta,\zeta,q) \in \mathbb{N}[\xi,\eta,\zeta,q^2].
$$

Theorem Conjecture is true if either λ or μ is the empty partition $\emptyset$.

$$P_{[\lambda,\emptyset]}, \quad \text{and} \quad P_{[\emptyset,\mu]} \in \mathbb{N}[\xi,\eta,\zeta,q^2].$$

More generally, let $\widetilde{R}_{[\lambda,\mu]}$ be the specialization of $s_{[\lambda,\mu]}$ obtained by substituting

$$p_n(X) \longmapsto \frac{\sum x_i^n - \sum(-y_j)^n}{1-q^n}$$

$$p_n(Y) \longmapsto \frac{\sum x_i^{-n} - \sum(-y_j)^{-n}}{1-q^{-n}}$$

Theorem

$$\prod_{b\in\lambda}(1-q^{h(b)}) \cdot \widetilde{R}_{[\lambda,\emptyset]} \in \mathbb{N}[x_1,\cdots,y_1,\cdots,q].$$