Combinatorics of Horn hypergeometric series

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Tianjin, China
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joint work with:

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Laura Matusevich (Texas A&M)

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Erdélyi's example

$$m = 2$$

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$$m = 2 \qquad\qquad n = 4$$

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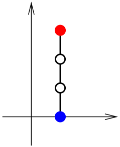

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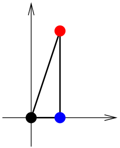

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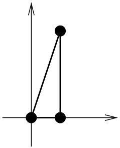

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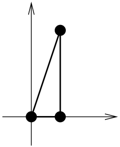

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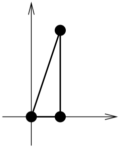

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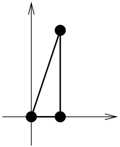

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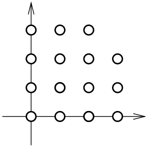

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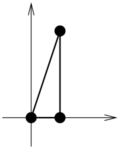

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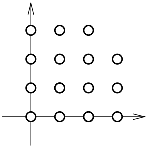

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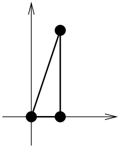

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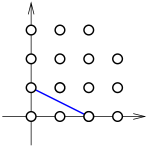

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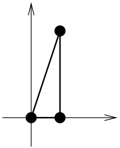

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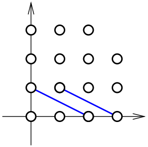

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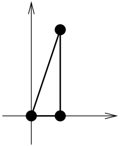

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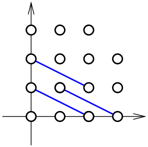

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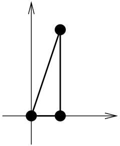

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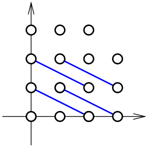

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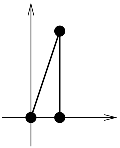

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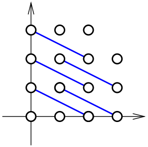

Erdélyi's example

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \quad \gamma = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$m = 2$, $n = 4$

$$\hat{\gamma} = \gamma - \frac{1}{3}\begin{bmatrix} 3a+2b+c \\ 0 \\ 0 \\ b+2c+3d \end{bmatrix} \in \mathrm{Span}_{\mathbb{C}} B \;\Rightarrow\; \hat{\gamma} = B \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \text{ with } \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \in \mathbb{C}^2$$

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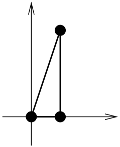

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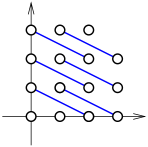

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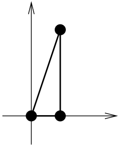

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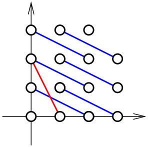

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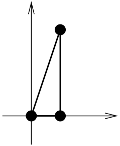

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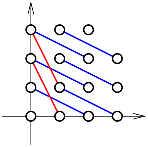

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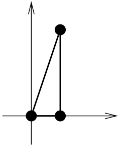

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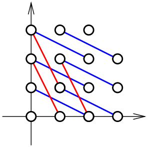

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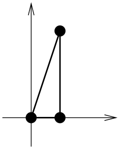

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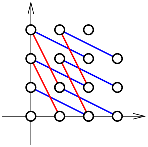

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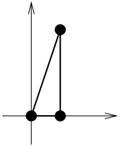

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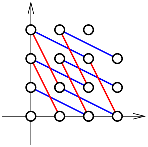

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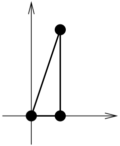

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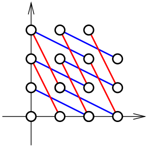

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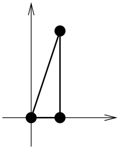

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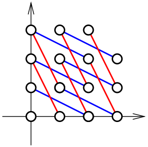

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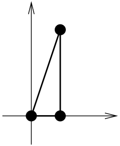

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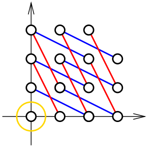

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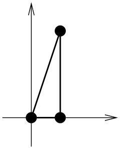

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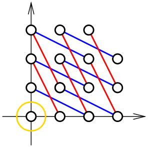

$$I_B = \left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3} - \left(\frac{\partial}{\partial z_2}\right)^2, \frac{\partial}{\partial z_2}\frac{\partial}{\partial z_4} - \left(\frac{\partial}{\partial z_3}\right)^2 \right)$$

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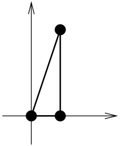

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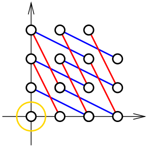

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$$z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} + z_4\frac{\partial}{\partial z_4} - \beta_1 \qquad z_2\frac{\partial}{\partial z_2} + 2z_3\frac{\partial}{\partial z_3} + 3z_4\frac{\partial}{\partial z_4} - \beta_2$$

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