Tableaux combinatorics of the asymmetric exclusion process

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Program

1. Background on a model from statistical mechanics: the asymmetric exclusion process (PASEP)

2. Background on the combinatorics: ⅃-diagrams and permutation tableaux

3. Main result: the relationship between 1 and 2.

4. The PASEP can be lifted to a Markov chain on permutations

5. Applications and connections to other things ...

PASEP model

The partially asymmetric exclusion process is a model for a system of interacting particles hopping left and right on a one-dimensional lattice of n sites.

New particles can enter the lattice from the left, and particles can exit from the right.

• The model is *partially asymmetric* in the sense that the probability of a particle jumping left is q times the probability of jumping right.

• *Exclusion:* at most one particle on each site

We'll depict particles as $\bullet$ or 1 and empty sites as $\circ$ or 0.

PASEP model

- Introduced by Spitzer in 1970
- A model for diffusion of particles, traffic jams, queuing problems
- Studied primarily by mathematical physicists (more than 250 papers on arXiv) but more recently by combinatorialists (Shapiro, Zeilberger, Brak, Corteel, Essam, Rechnitzer, Duchi, Schaeffer,...)
- Later today: related talk by Viennot.

PASEP model

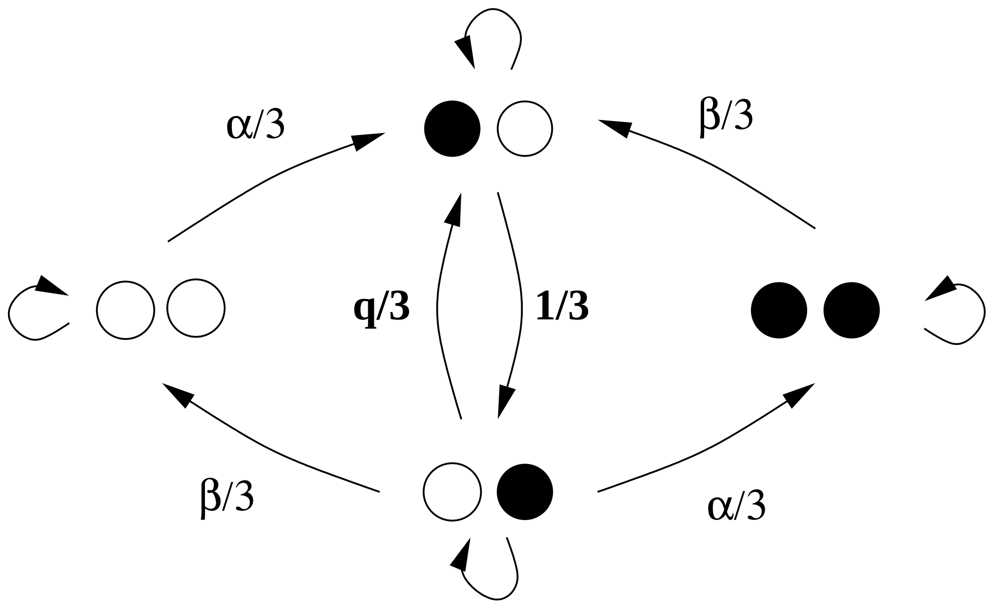
Let B_n be the set of all 2^n words in the language $\{\circ, \bullet\}^*$.

The PASEP is the Markov process on B_n with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{1}{n+1}$ and $P_{Y,X} = \frac{q}{n+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{n+1}$.
- If $X = B \bullet$ and $Y = B \circ$ then $P_{X,Y} = \frac{\beta}{n+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

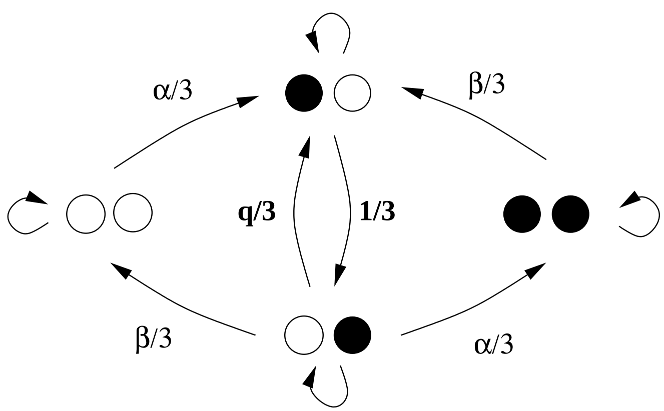
PASEP model

The state diagram of the PASEP model for $n = 2$.



Stationary Distribution of the ASEP model

The ASEP has a unique stationary distribution – that is, it has a unique left eigenvector of the transition matrix associated with eigenvalue 1. This is called the steady state.



(Solve for prob.'s, say when $\alpha = \beta = 1$.)

⅃-diagrams and permutation tableaux

Definition: A ⅃-diagram is a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ (where $\lambda_i \geq 0$) together with a filling with 0's and 1's such that:

• There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

1	**1**	**0**	**0**
0	**0**	**1**	**0**
1	**1**	**1**	**1**
0	**0**	**1**	

⅃-diagrams and permutation tableaux

1	1	0	0
0	0	1	0
1	1	1	1
0	0	1	

• ⅃-diagrams were introduced by Postnikov and shown to correspond to cells in the totally nonnegative part of the Grassmannian.

• Also, these objects were simultaneously introduced by Cauchon (*Cauchon diagrams*) for the study of primes in quantized coordinate rings of square matrices

• As we will see, they are related to the asymmetric exclusion process.

Permutation tableaux

Definition: We say that a ⅃-diagram is a *permutation tableau* if:

• Each column of the rectangle contains at least one 1.

Bijection from perm-tableaux to permutations (Postnikov, Steingrimsson-Williams):

1	**1**	**0**	**0**
0	**0**	**1**	**0**
1	**1**	**1**	**1**
0	**0**	**1**	

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q-Eulerian numbers

Define the *weight* $wt(\mathcal{T})$ of a permutation tableau $\mathcal{T}$ to be the number of 1's minus the number of columns.

Define $\hat{E}_{k,n}(q) = \sum_{\mathcal{T}} q^{wt(\mathcal{T})}$, summing over all perm-tableaux $\mathcal{T}$ with k rows and $n-k$ columns.

Theorem(W.):

$$\hat{E}_{k,n}(q) = q^{k-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]_q{}^n q^{ki-k} \left(\binom{n}{i} q^{k-i} + \binom{n}{i-1} \right).$$

Additionally, $\hat{E}_{k,n}(q)$ specializes at $q = -1, 0, 1$ to binomial numbers, Narayana numbers, and Eulerian numbers.

Note: there are also two interpretations for this polynomial in terms of permutations.

Corteel's result

Theorem (Corteel): Let $\alpha = \beta = 1$. In the steady state, the probability that the PASEP model with n sites is in a configuration with precisely k particles is:

$$\frac{\hat{E}_{k+1,n+1}(q)}{Z_n}$$

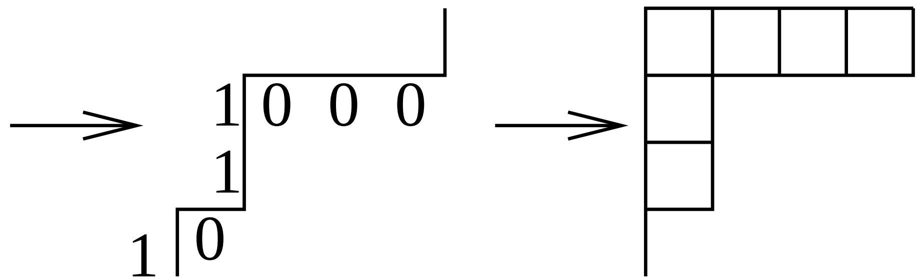
Here, Z_n is the *partition function* for the model – the sum of the probabilities of all possible states.

Question: Corteel's result doesn't say anything about the *location* of the particles. How can we refine this result?

Refinement

There is a easy bijection between words τ in $\{0,1\}^n$ and partitions of semiperimeter $n+1$ (where each column has length at least one):

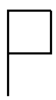
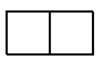
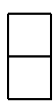
0 0 0 1 1 0 1



This associates the partition $\lambda(\tau)$ to τ.

Theorem (Corteel, W). In the steady state, the probability that the PASEP is in configuration τ is: $\frac{\sum_{\mathcal{T}} q^{wt(\mathcal{T})}}{Z_n}$ where the sum is over all permutation tableaux of shape $\lambda(\tau)$.

Example

State	Partition	Perm Tableaux	Weight	Probability
● ●			$q^0 = 1$	1/(q+5)
○ ●		$\begin{array}{|c|}\hline 1 \\ \hline \end{array}$	$q^0 = 1$	1/(q+5)
○ ○		$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$	$q^0 = 1$	1/(q+5)
● ○		$\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array}$ $\begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline \end{array}$ $\begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline \end{array}$	q +2	(q+2)/(q+5)

Further Refinement

Now want α and β to be general. Two more definitions ...

Given a permutation tableaux $\mathcal{T}$, let $f(\mathcal{T})$ be the number of 1's in the first row of $\mathcal{T}$.

We say that a 0 of $\mathcal{T}$ is *restricted* if it lies below some 1. And we say that a row is *unrestricted* if it does not contain a restricted 0. Let $u(\mathcal{T})$ be the number of unrestricted rows of $\mathcal{T}$ minus 1.

1	**1**	**0**	**0**
0	**0**	**1**	**0**
1	**1**	**1**	**1**
0	**0**	**1**	

Above, $f(\mathcal{T}) = 2$ and $u(\mathcal{T}) = 1$.

Refined Theorem

Theorem (Corteel, W). In the steady state, the probability that the PASEP is in configuration τ is $\frac{\sum_{\mathcal{T}} q^{wt(\mathcal{T})} \alpha^{-f(\mathcal{T})} \beta^{-u(\mathcal{T})}}{Z_n}$, where the sum ranges over all permutation tableaux $\mathcal{T}$ of shape λ.

How to prove this: the matrix ansatz

Let $P_n(\tau_1, \ldots, \tau_n)$ denote the probability that in the steady state, the PASEP model is in configuration τ. Define unnormalized weights $f_n(\tau_1, \ldots, \tau_n)$, which are equal to the $P_n(\tau_1, \ldots, \tau_n)$ up to a constant:

$$P_n(\tau_1, \ldots, \tau_n) = f_n(\tau_1, \ldots, \tau_n)/Z_n,$$

where Z_n is the *partition function* $\sum_\tau f_n(\tau_1, \ldots, \tau_n)$.

Theorem: (Derrida, Evans, Hakim, Pasquier) Suppose that D and E are matrices, V is a column vector, and W is a row vector, such that:

$$DE - qED = D + E$$

$$DV = \frac{1}{\beta} V$$

$$WE = \frac{1}{\alpha} W$$

Then

$$f_n(\tau_1, \ldots, \tau_n) = W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V.$$

Proof of main result – when $\alpha = \beta = 1$

(General proof uses same idea, just more variables.)

Idea: find D, E, V, W satisfying the relations of the ansatz such that products of D's and E's enumerate permutation tableaux.

Let D be the (infinite) upper triangular matrix (d_{ij}) such that $d_{i,i+1} = 1$ and $d_{i,j} = 0$ for $j \neq i+1$.

That is, D is the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let E be the (infinite) lower triangular matrix (e_{ij}) such that for $j \leq i$, $e_{ij} = \sum_{r=0}^{j-1} \binom{i-j+r}{r} q^r = \frac{[i]^{(i-j)}}{(i-j)!}$. (Otherwise $e_{ij} = 0$). Here, $[i]^{(k)}$ represents the kth derivative of $[i]$ with respect to q.

That is, E is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & [2] & 0 & 0 & 0 & \dots \\ 1 & [3]' & [3] & 0 & 0 & \dots \\ 1 & \frac{[4]''}{2} & [4]' & [4] & 0 & \dots \\ 1 & \frac{[5]'''}{6} & \frac{[5]''}{2} & [5]' & [5] & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

Let W be the (row) vector $(1, 0, 0, \dots)$ and V be the (column) vector $(1, 1, 1, \dots)$.

Then $DE - qED = D + E, DV = V, WE = W.$ Furthermore, $W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V$ enumerates permutation tableaux of shape $\lambda(\tau)$.

So by the result of Derrida et al, the generating function for permutation tableaux of a given shape $\lambda(\tau)$ corresponds exactly to the probability that in the steady state, the PASEP is in configuration τ.

Comments

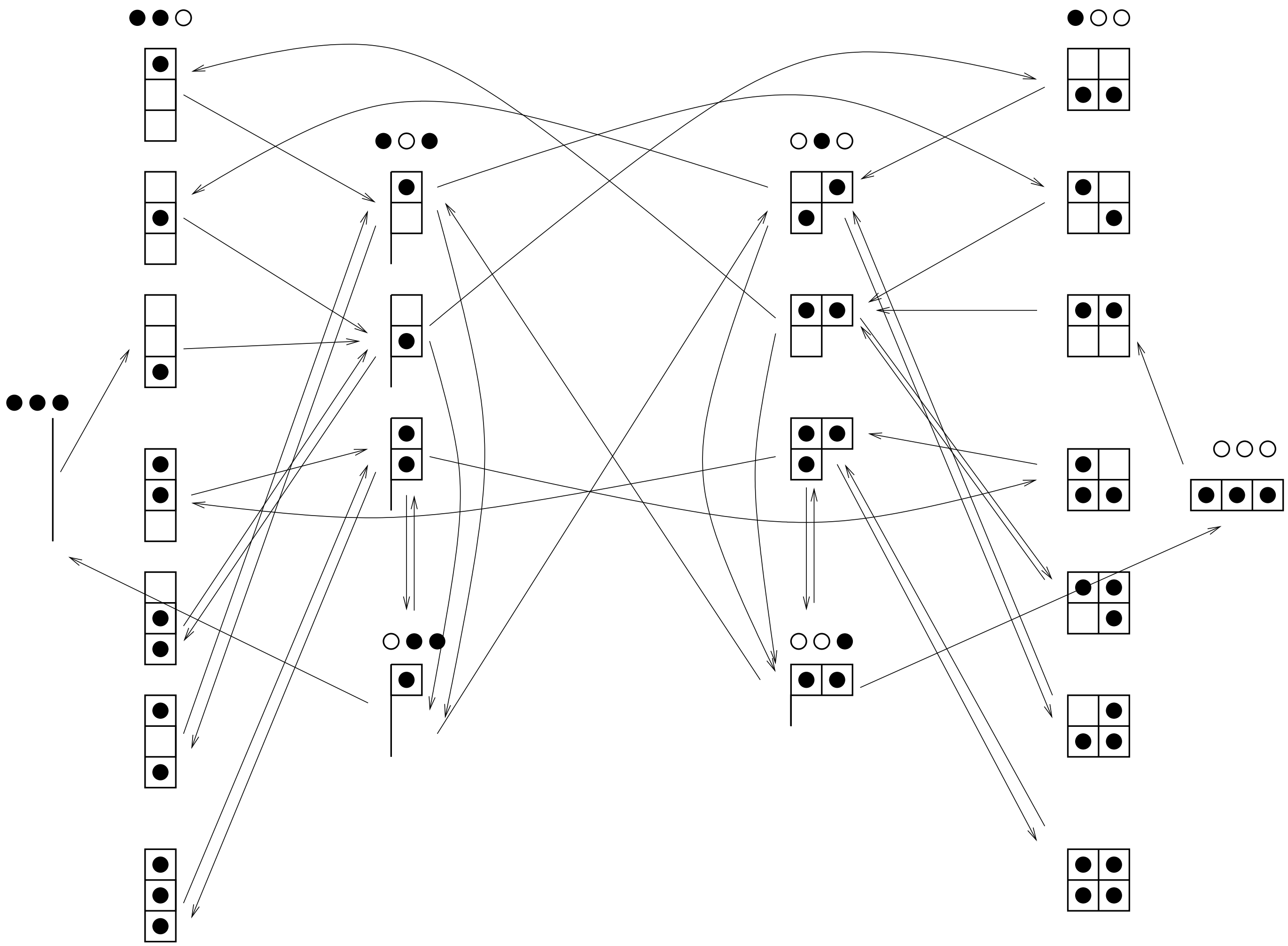
Our first proof of the theorem was algebraic and used the "matrix ansatz." It left us wondering ...

Question: what is the role of the permutation tableaux in this model and can one "lift" the PASEP to a chain on permutation tableaux – in such a way that the steady state probability of being at a particular tableaux $\mathcal{T}$ is

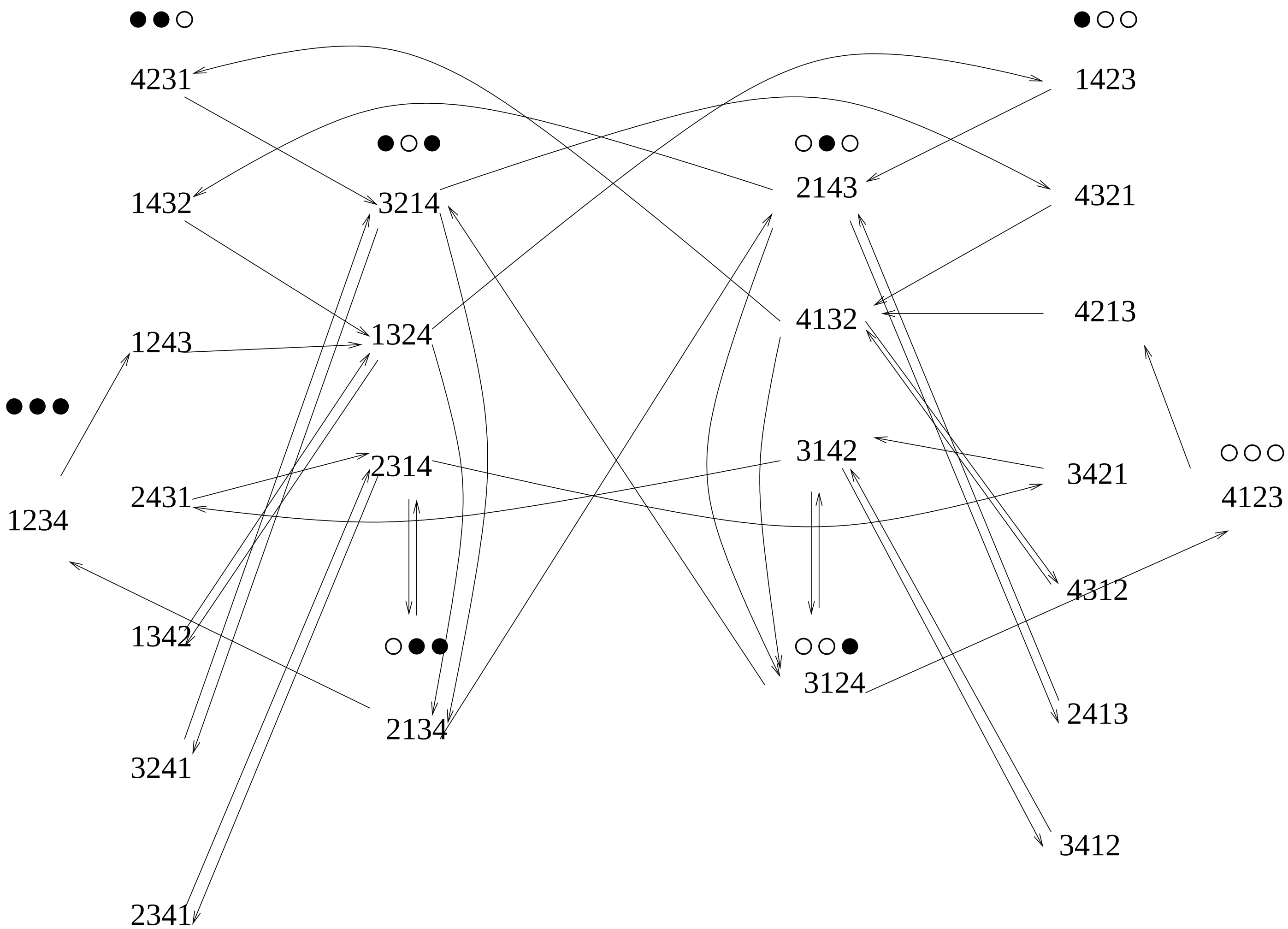
$$\frac{q^{wt(\mathcal{T})}\alpha^{-f(\mathcal{T})}\beta^{-u(\mathcal{T})}}{Z_n}?$$

Inspiration from work of Duchi and Schaeffer, who proved a similar statement in the $q = 0$ case for the Catalan-enumerated "complete configurations."

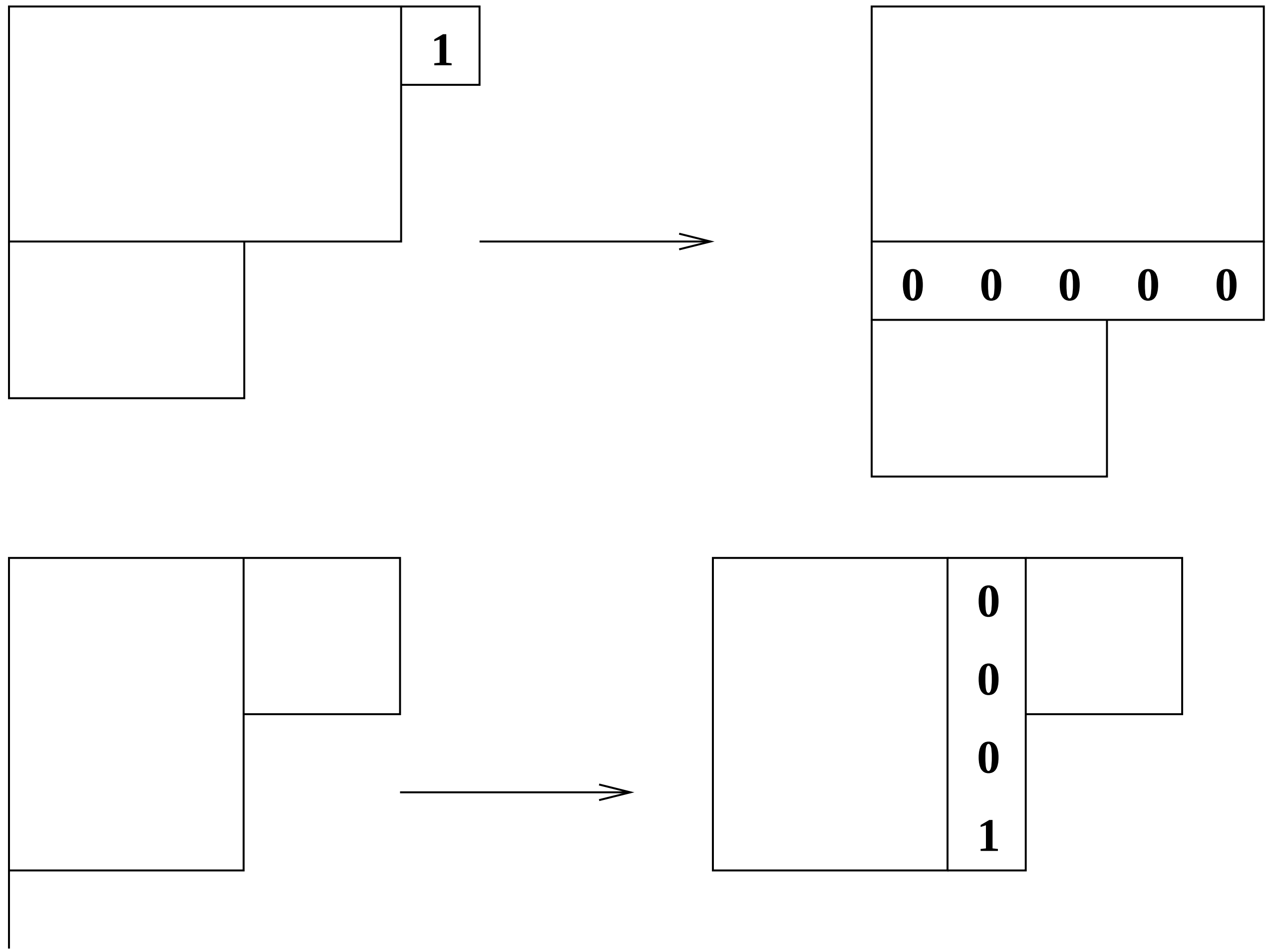
Yes! This is the "PT chain"



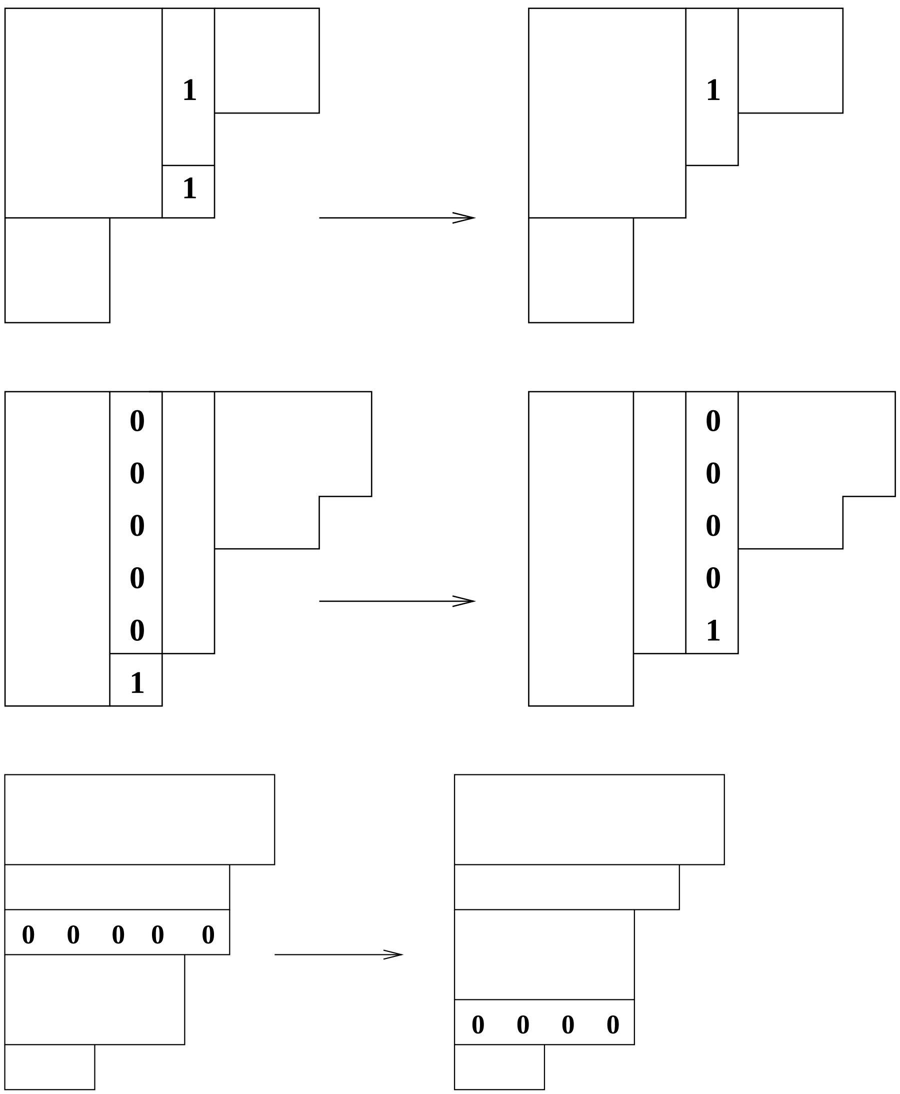
The "PT chain" on permutations



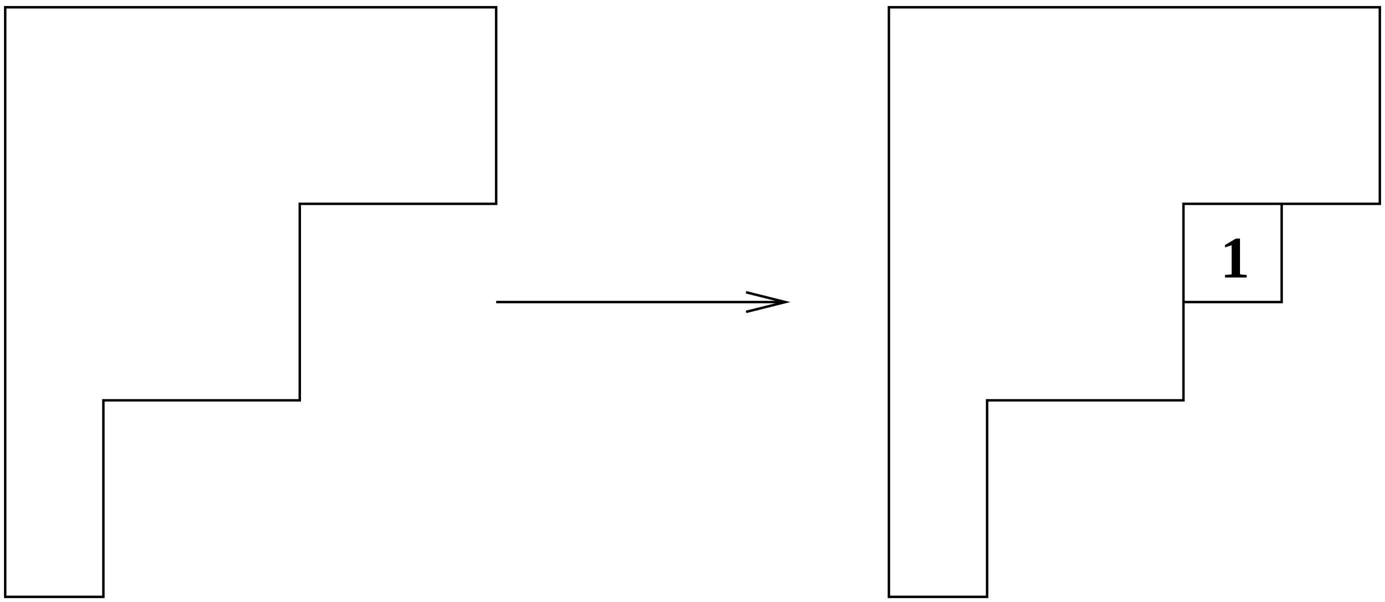
Transitions in the "PT chain": enter and exit



Transitions in the "PT chain": hop right



Transitions in the "PT chain": hop left

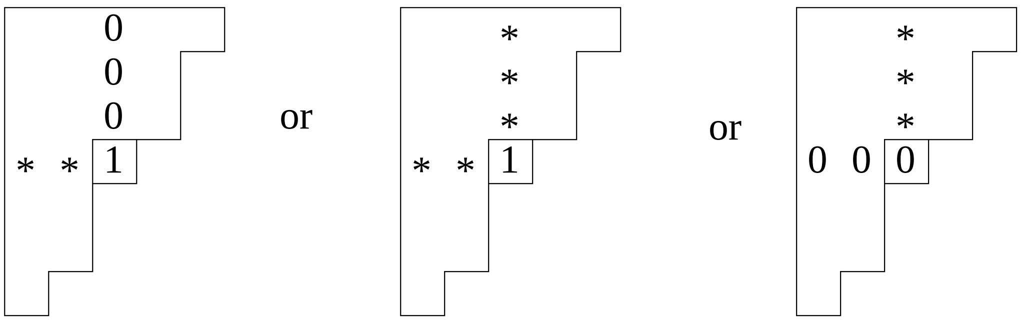


Applications: recurrences for ASEP ...

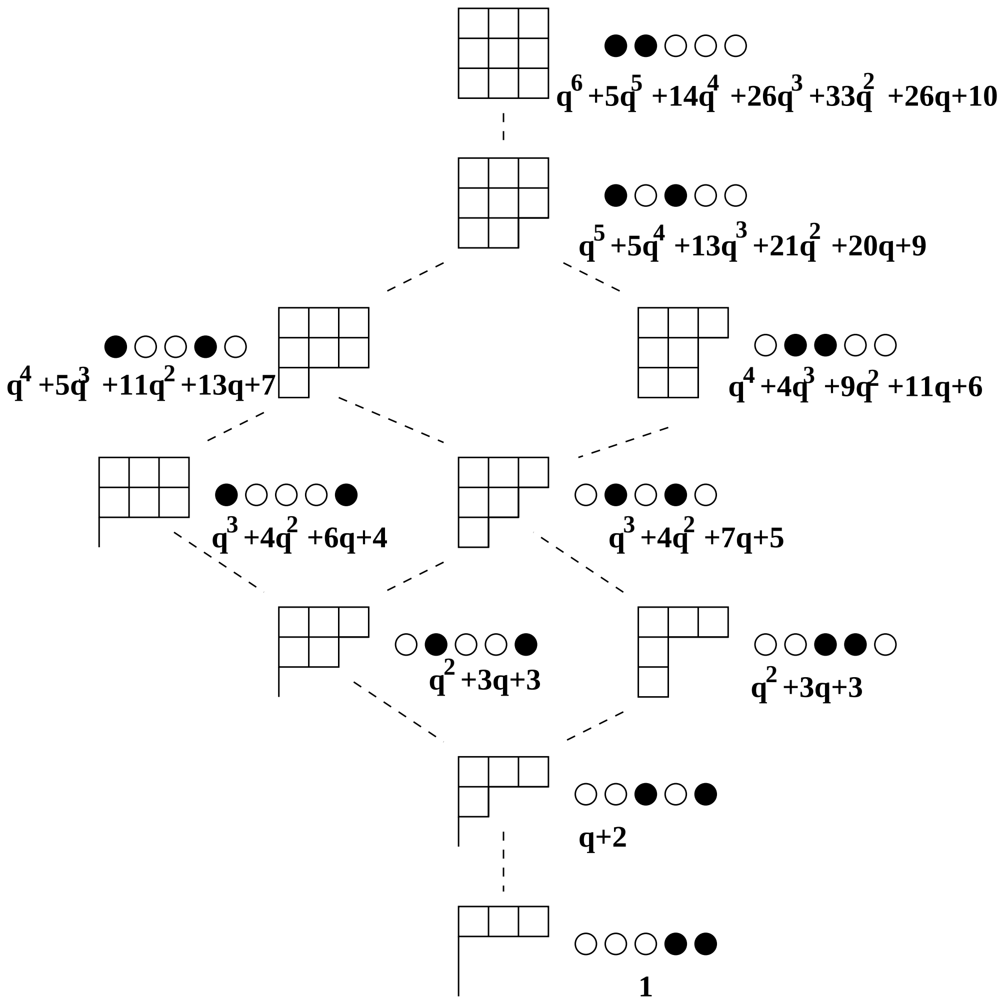
Theorem (Brak, Corteel, Rechnitzer, Essam): Steady state probabilities of the PASEP obey the following recurrence:

$$\begin{aligned} f_n(\tau_1, \tau_2, \ldots, \tau_{j-1}, \bullet, \circ, \tau_{j+2}, \ldots, \tau_n) = \\ f_{n-1}(\tau_1, \tau_2, \ldots, \tau_{j-1}, \bullet, \tau_{j+2}, \ldots, \tau_n) + \\ q f_n(\tau_1, \tau_2, \ldots, \tau_{j-1}, \circ, \bullet, \tau_{j+2}, \ldots, \tau_n) + \\ f_{n-1}(\tau_1, \ldots, \tau_{j-1}, \circ, \tau_{j+2}, \ldots, \tau_n). \end{aligned}$$

We get a new and much simpler picture proof of this result:

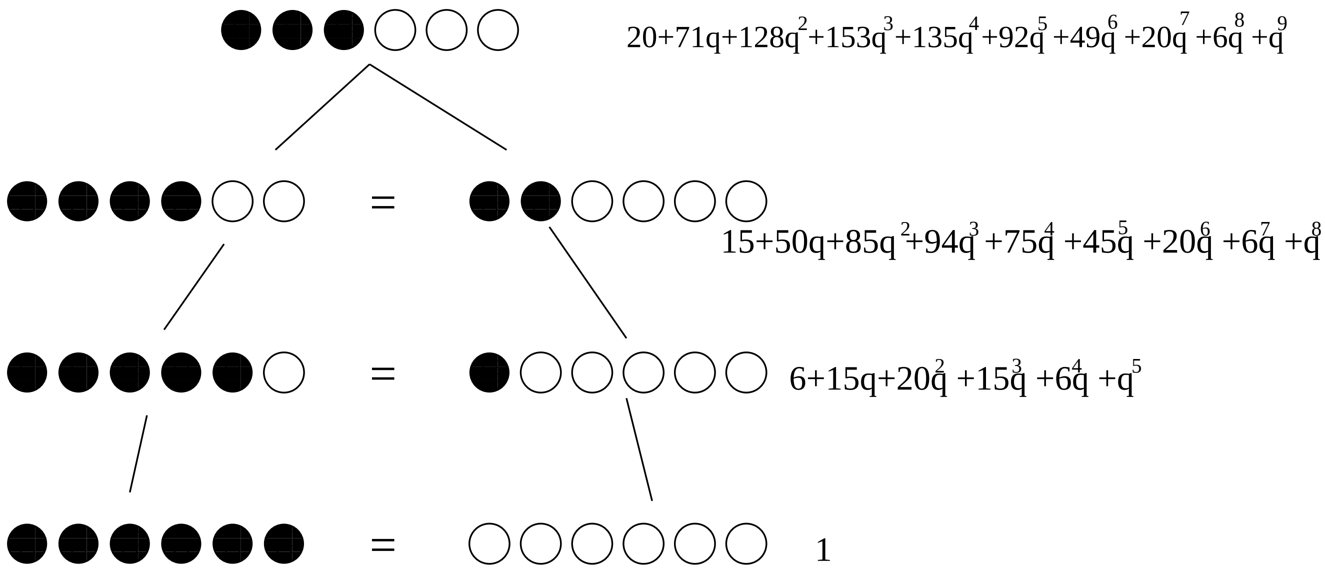


Applications: monotonicity results ($\alpha = \beta = 1$)...



Applications: monotonicity results ($\alpha = \beta = 1$)...

(Steingrimsson, W.)



Applications: monotonicity results ...

Def: Let $\tau, \tau' \in \{0,1\}^n$ be two states of the PASEP which contain exactly k particles. We define the partial order $\prec$ by $\tau \prec \tau'$ if and only if $\lambda(\tau) \subset \lambda(\tau')$.

Proposition (Corteel, W). Let $\alpha = \beta = 1$. Suppose that $\tau \prec \tau'$, and let $d := |\lambda(\tau')| - |\lambda(\tau)|$. Then $f_n(\tau') - f_n(\tau)$ is a non-negative polynomial. In other words, as one moves up the partial order $\prec$, the coefficients of $f_n(\tau)$ monotonically increase.

Proposition (Steingrimsson, W). Let $\alpha = \beta = 1$. If $d < n/2$, then $f_n(\bullet^{d+1}\circ^{n-d-1}) - f_n(\bullet^{d}\circ^{n-d})$ is a non-negative polynomial. As a corollary, the most probable state of the PASEP is $\bullet^{n/2}\circ^{n/2}$.

Questions/ Future directions

- Is there a connection with total positivity?
- Can one exploit connection with totally non-negative Grassmannian?
- Generalizations to several types of particles?
- Link with orthogonal polynomials?