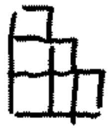
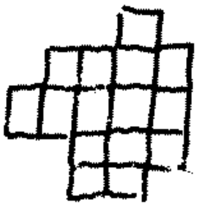
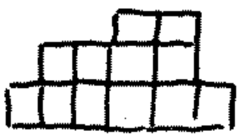
INCREASING & DECREASING SEQUENCES IN FILLINGS OF mOOn POLYOMINOES







UNIVERSITÄT WIEN

google MARTIN RUBEY

A few recent theorems:

(Backelin West Xin)

$\#$ permutations in S_n avoiding $(1,2,\cdots,t,\pi_1,\pi_2,\cdots,\pi_k)$

$= \#$ permutations in S_n avoiding $(t,t\text{-}1,\cdots,1,\pi_1,\pi_2,\cdots,\pi_k)$

(Bousquet-Mélou Steingrímsson)

$\#$ involutions in S_n avoiding $(1,2,\cdots,t,\pi_1,\pi_2,\cdots,\pi_k)$

$= \#$ involutions in S_n avoiding $(t,t\text{-}1,\cdots,1,\pi_1,\pi_2,\cdots,\pi_k)$

(Chen Deng Du Stanley Yan)

$\#$ k-noncrossing set partitions of $\{1,2,\cdots,n\}$

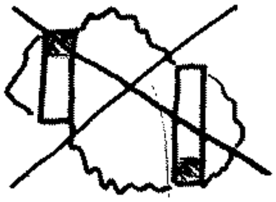
$= \#$ k-nonnesting set partitions of $\{1,2,\cdots,n\}$

We present an easy bijective proof of a common generalization, conjectured by Jonsson

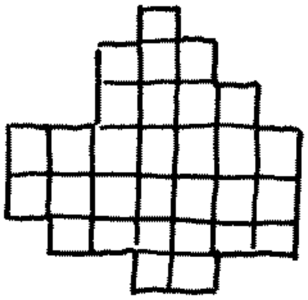
Moon Polyominoes

(= L-convex = 1-convex polyominoes)

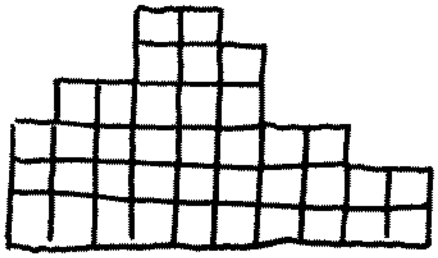
- column- and row convex : no holes
- intersection-free :



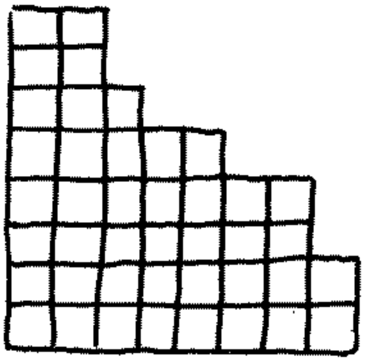
any two cells can be joined by a path changing direction at most once.



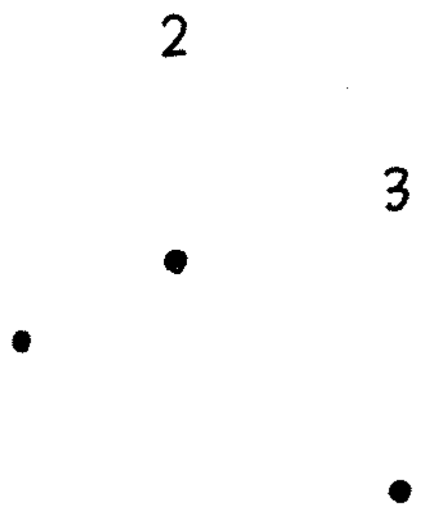
Moon Polyomino



Stack Polyomino



Ferrers Shape

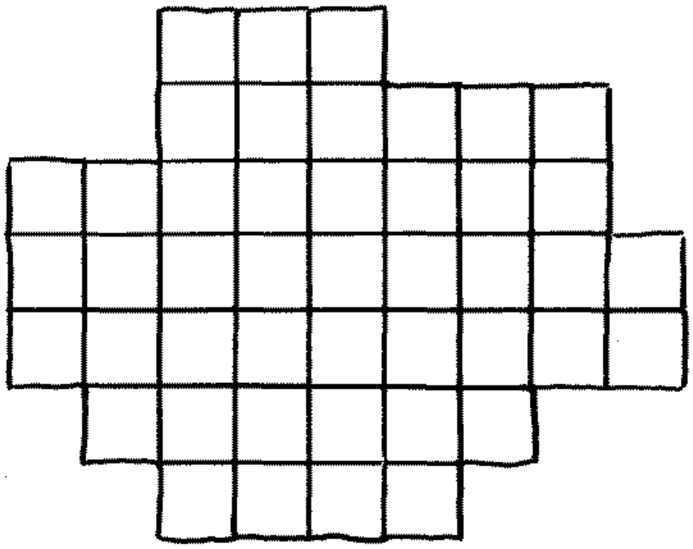


put balls into the boxes

several balls may share a box
(indicated by a number)

0-1-Filling : at most one ball per box

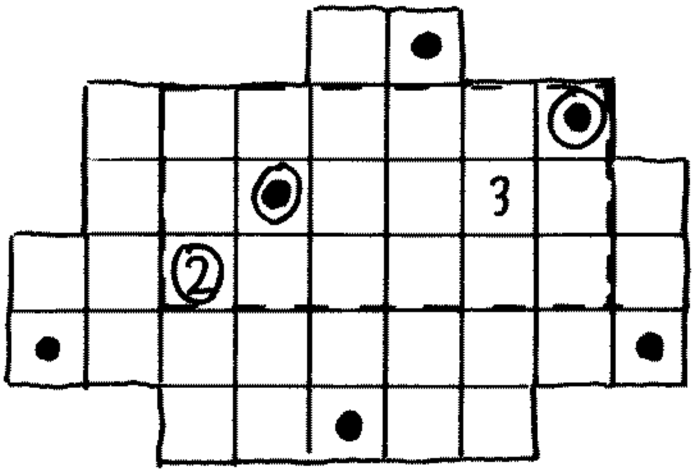
Fillings of Moon Polyominoes



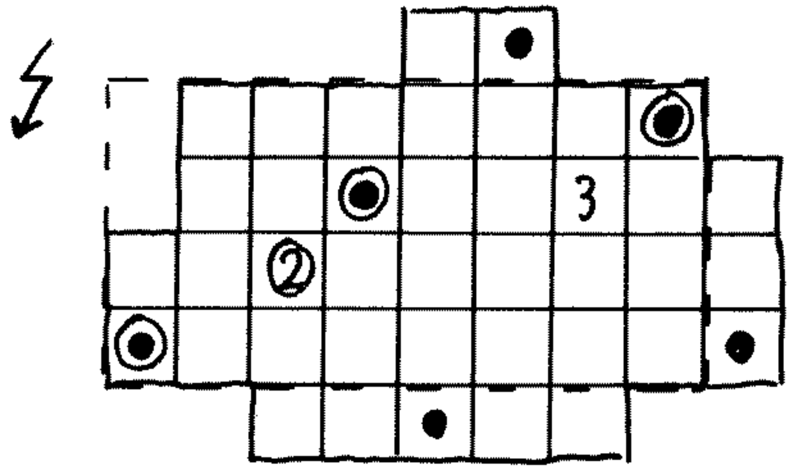
Chains in Moon Polyominoes

a **Chain** is a sequence of boxes containing balls

the smallest rectangle containing all elements
must be completely contained in the polyomino



a chain



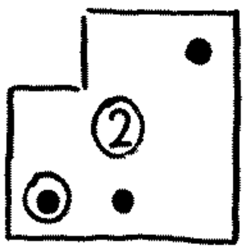
<u>not</u> a chain

Increasing and Decreasing Chains

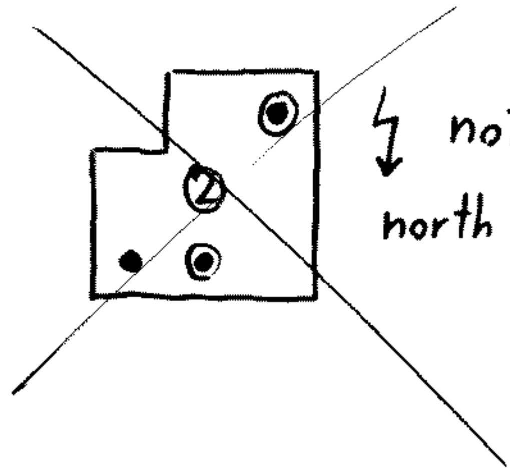
north-east (ne) chain:

every box is strictly north east of its predecessor

length = # boxes



length = 2

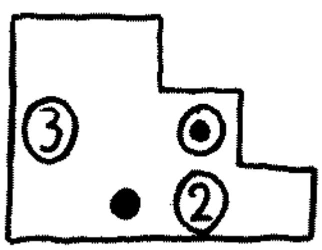


↯ not north east

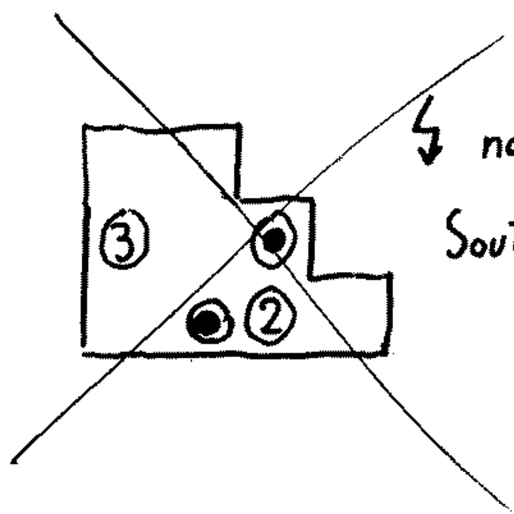
South-East (SE) chain:

every box is weakly south east of its predecessor

length = # balls



length = 6



↯ not South East

Fix a moon polyomino, and a number of balls.

Theorem 1 (conjectured by Jonsson)

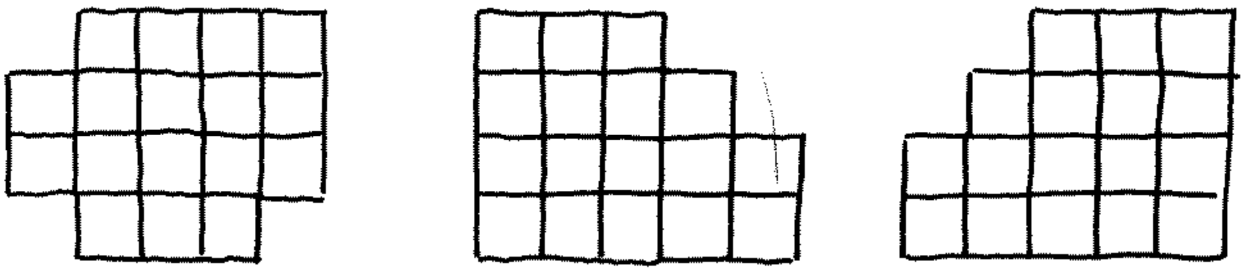
for 0-1 fillings, the number of fillings with length of longest ne chain given, depends only on the heights of the columns

Theorem 2:

for arbitrary fillings, the number of fillings with length of longest ne chain and length of longest SE chain given, depends only on the heights of the columns

These two theorems contain all of the previously mentioned as special cases (or easy consequences)

Example



all allow same number of

- 0-1 fillings with 5 Balls and

  longest ne chain of length 2

or • arbitrary fillings with 15 balls and

  longest ne chain of length 3 and

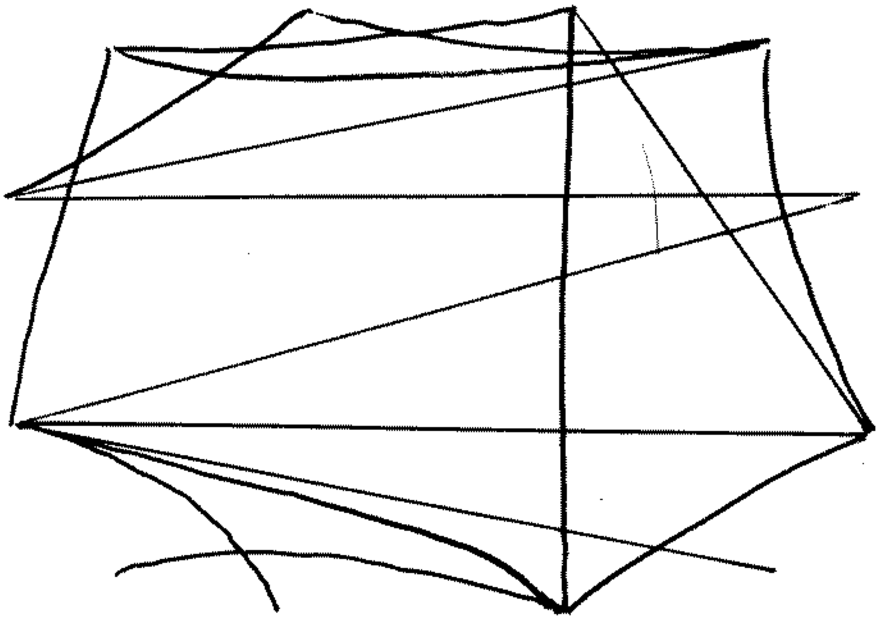
  longest SE chain of length 8

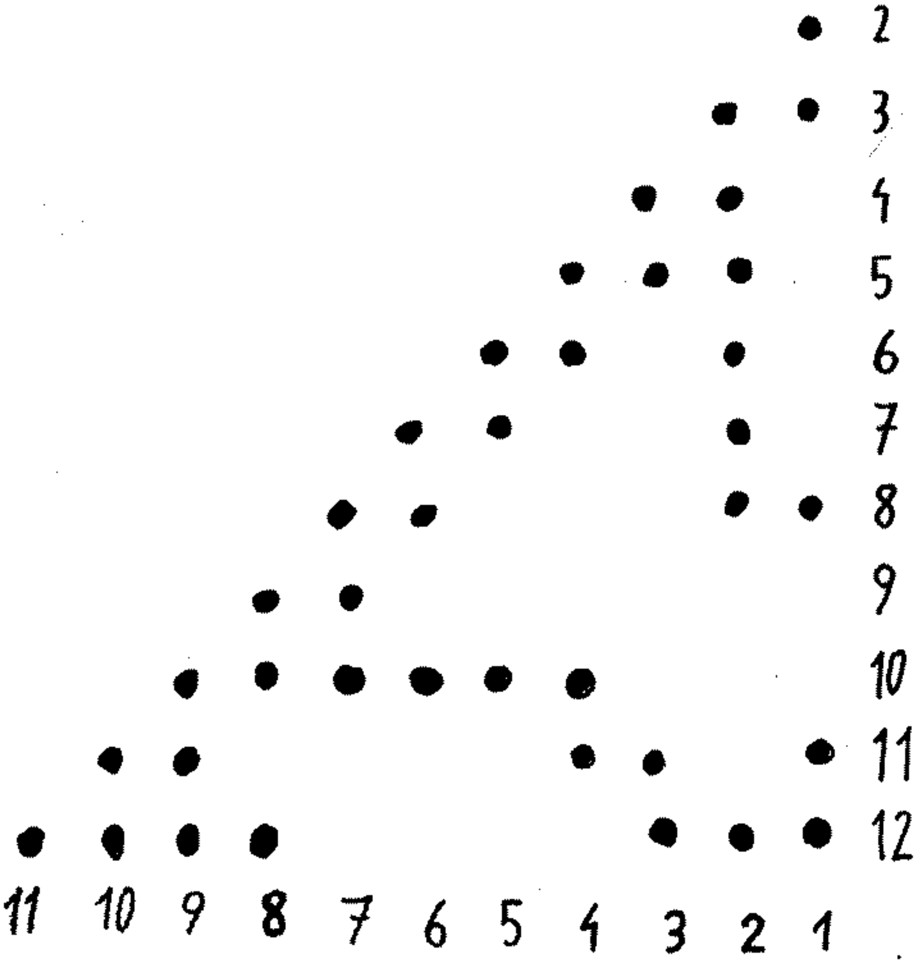
Another recent theorem:

(Jonsson) # k-triangulations of the n-gon

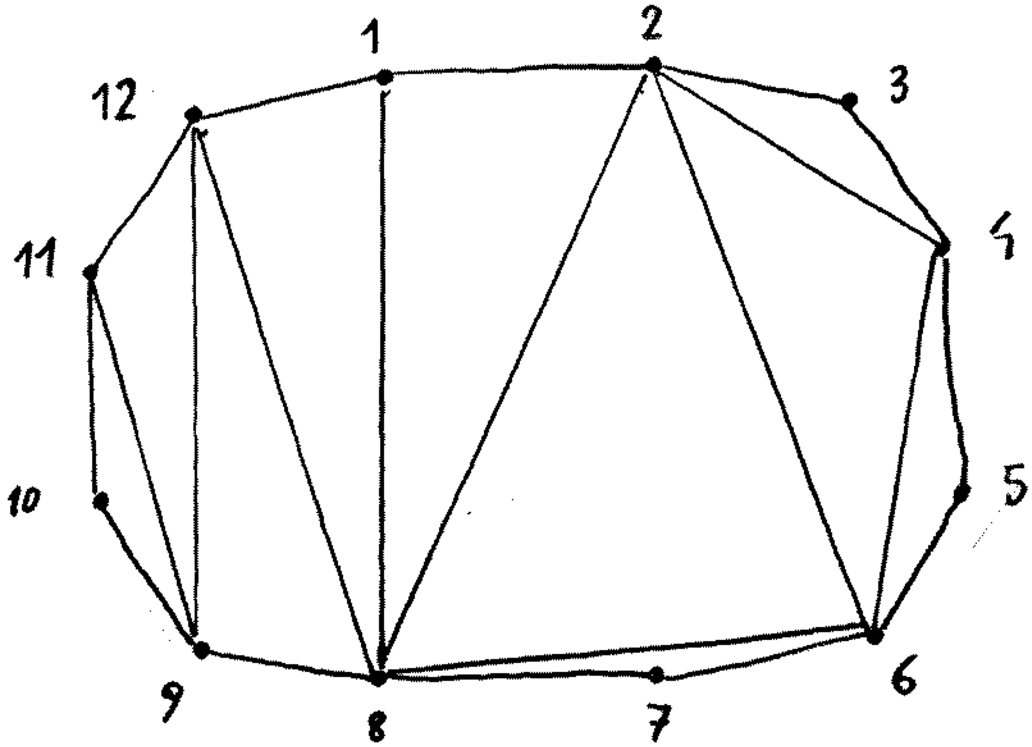
= # Fans of k Dyck Paths

2 -

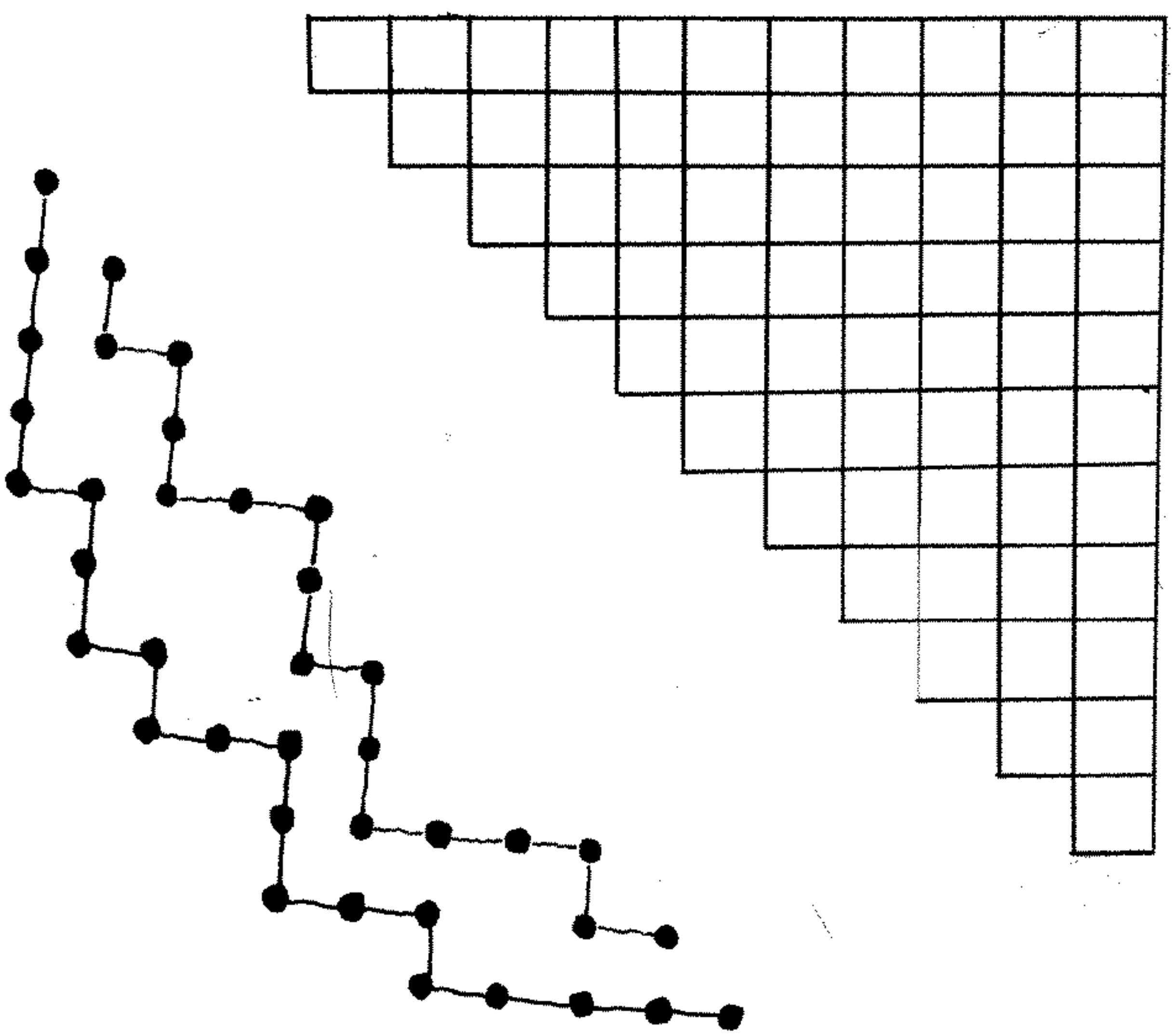




Triangulations



... of 2 Dyck Paths



We prove bijectively Theorem 2

(arbitrary fillings,

length of longest ne and longest SE chain invariant

Theorem 1

(0-1 fillings,

length of longest ne chain invariant)

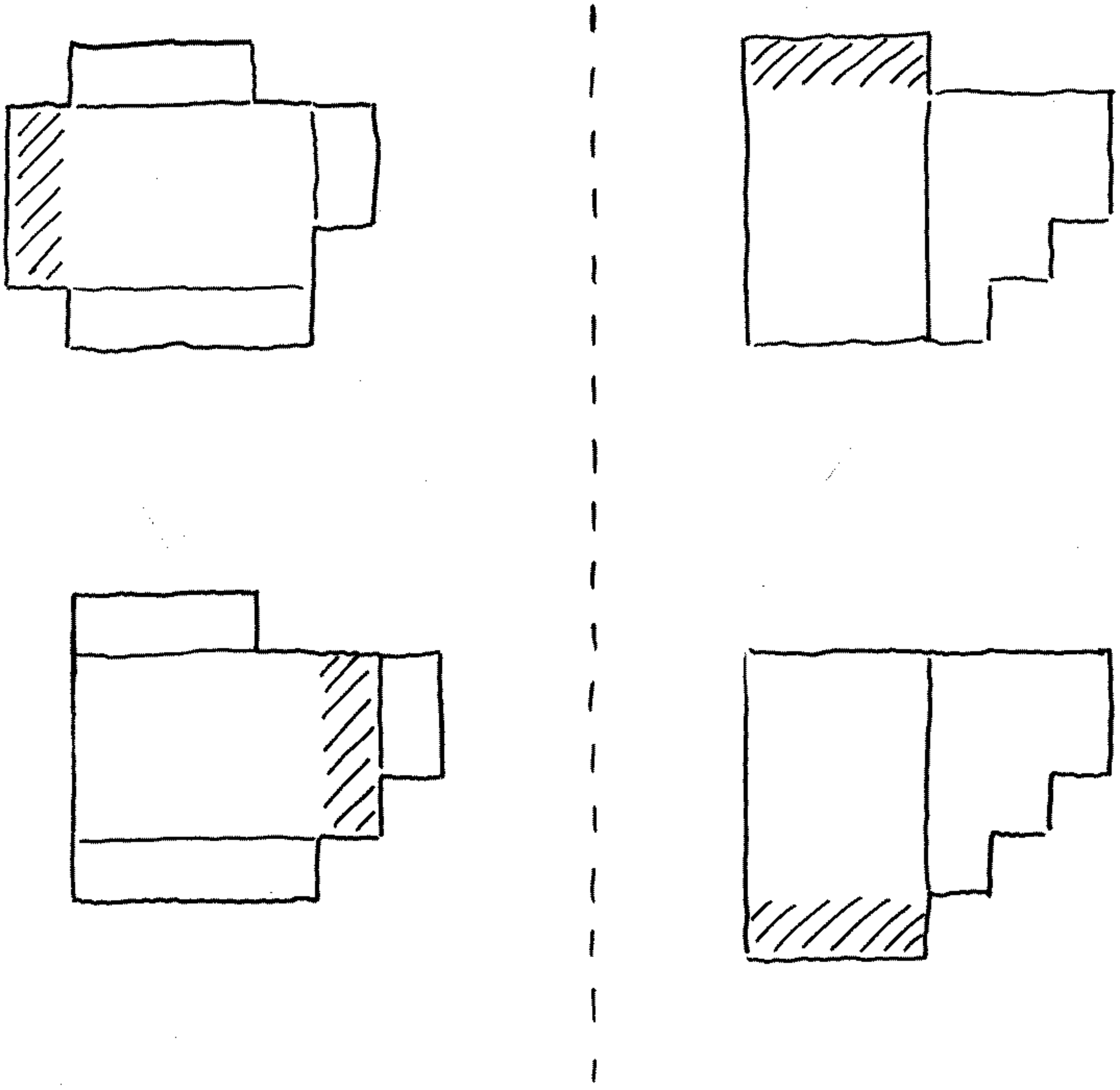
follows by inclusion - exclusion, or,

by involution principle

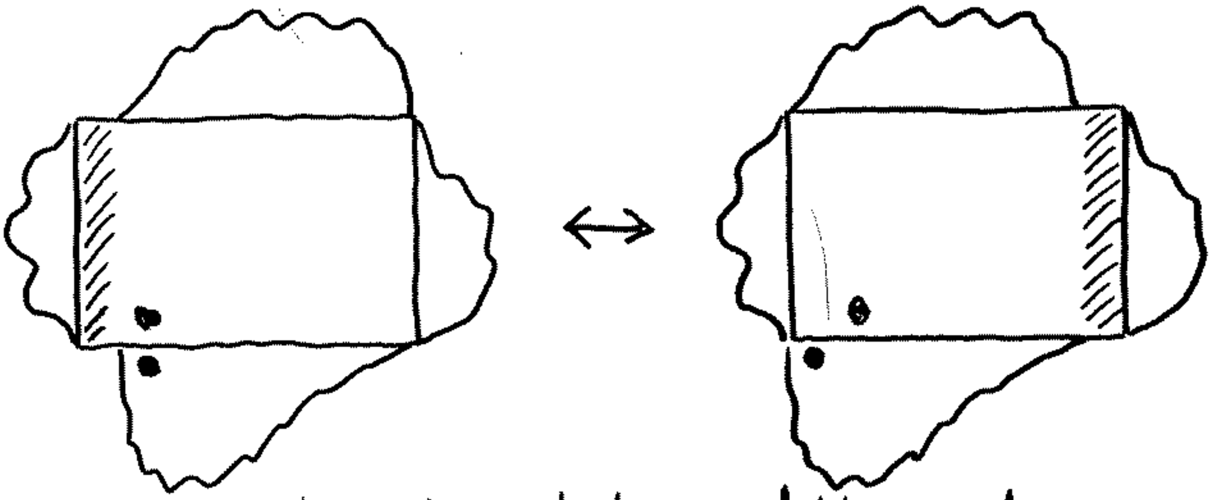
longest ne chain
has length 2

Observation

any two moon-polyominoes whose columns have the same heights can be transformed into each other by moving columns or rows to the other end:

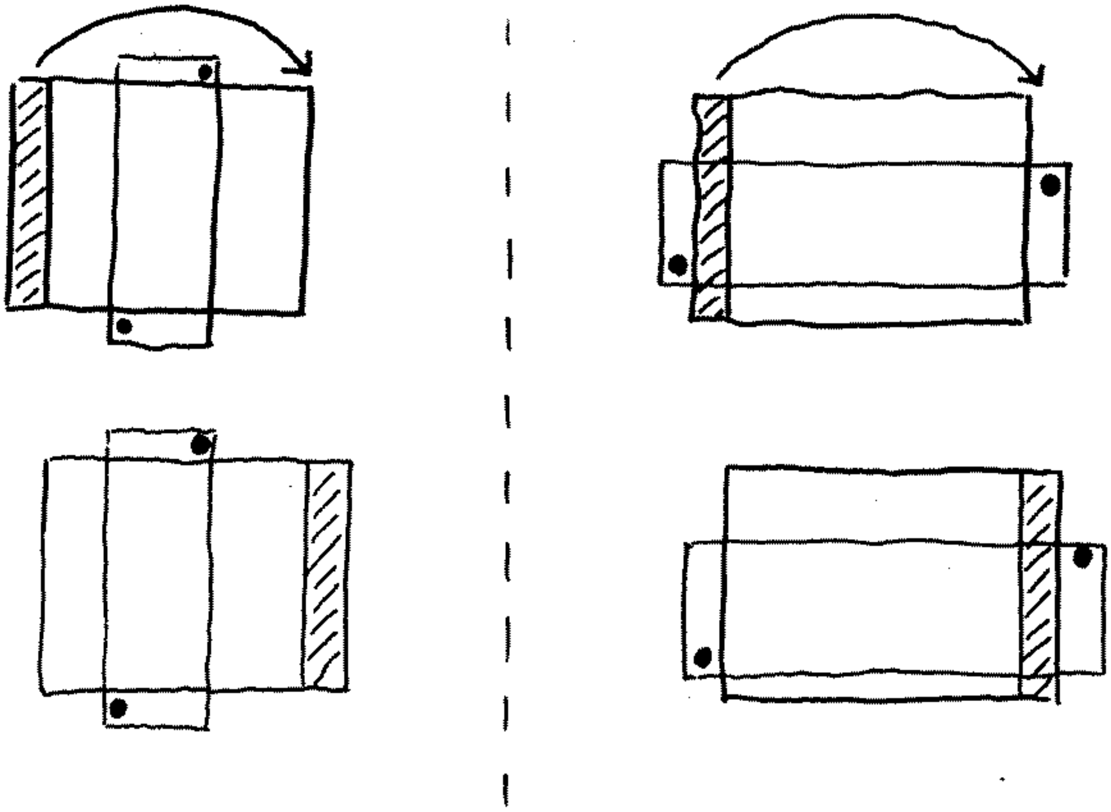


Idea



bijection between fillings of
the shape on the left and the shape on the right
that modifies only the filling inside the rectangle

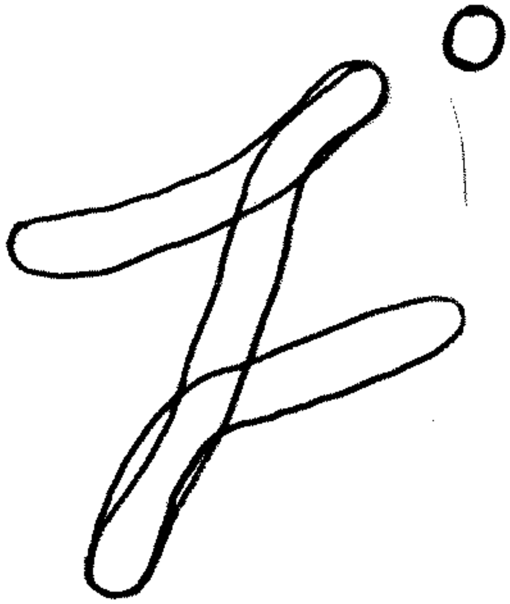
such that the length of a chain beginning or ending
outside of the rectangle remains unchanged



start with special case:

at most one ball in each column and row

Sergey Fomin's Growth Diagrams



for every corner, write

$$\lambda_1, \lambda_2, \lambda_3, \dots$$

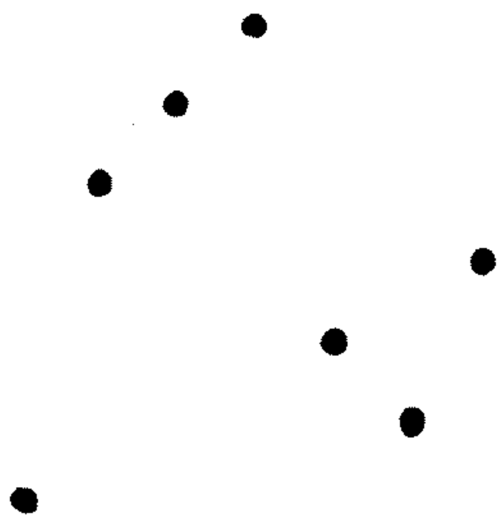
where

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k$$

is the maximum cardinality of a union

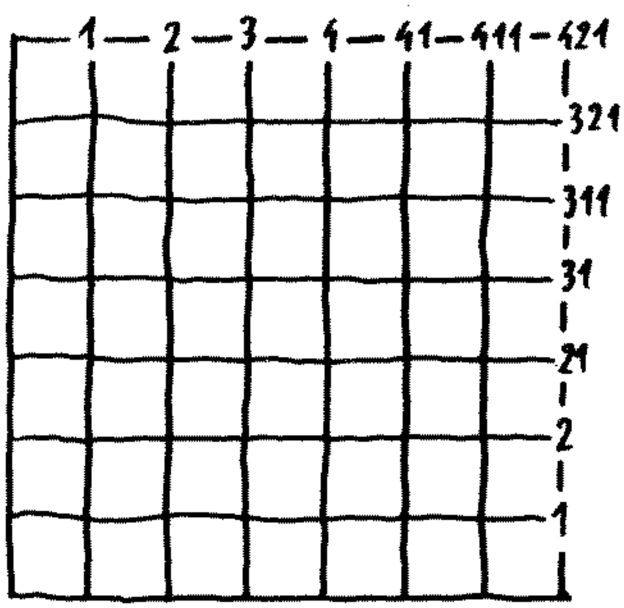
of k ne-chains below and to the left

of this corner



construct
filling
given the
labels

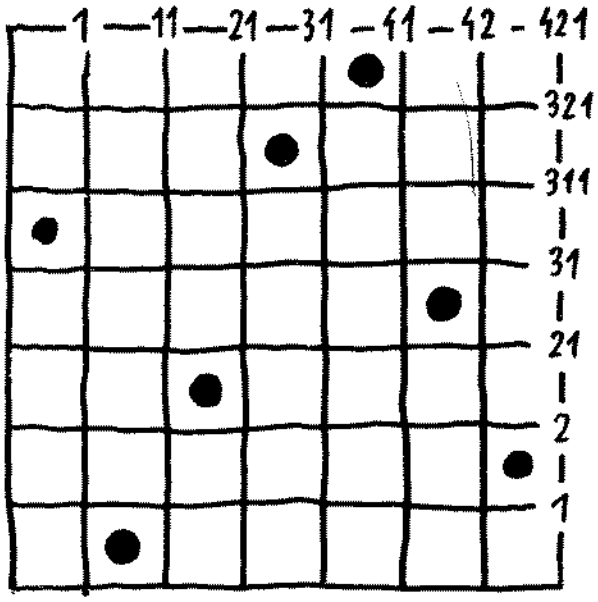
"jeu de taquin" for growth diagrams



write labels as before:

copy labels at the right

ignore first column to obtain top labels



1
11
21
31
41
42
421
321
311
31
21
2
1

general case:

several balls in one column or row

several balls in one box

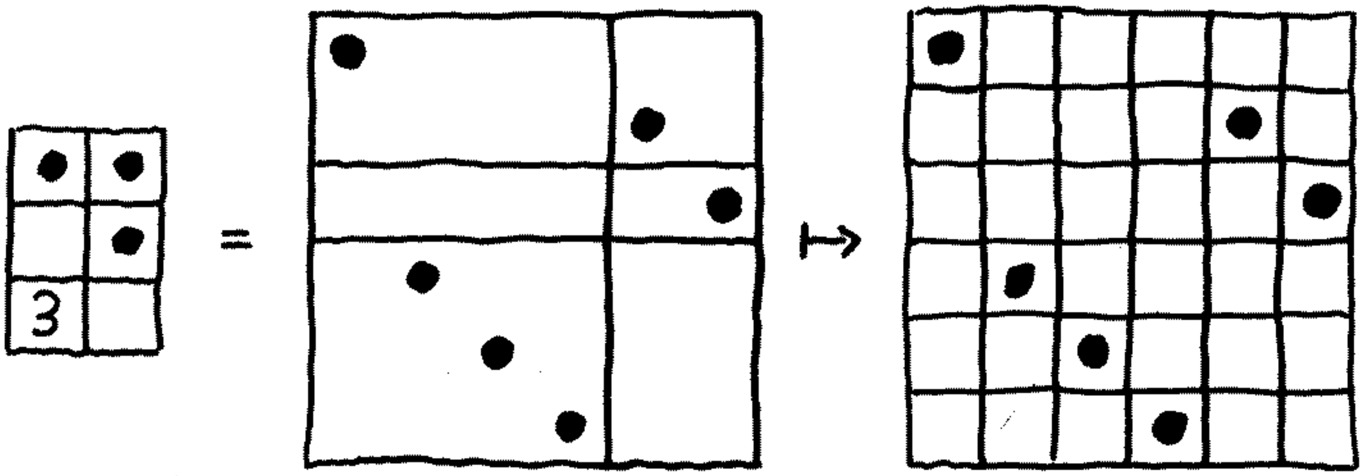
Separating balls

given an arbitrary filling,

construct a filling of a larger polyomino

such that there is at most one ball

per column and row:



balls in each row and each column

are arranged from north-west

to south-east

this construction behaves well under

jeu de taquin – ne and SE chains

are preserved

summary of bijection

- select column you want to move
- separate balls in corresponding rectangle
- apply growth diagram version of jeu de taquin
- shrink back rectangle

we obtain Theorem 2:

for a given moon polyomino,
a given number of balls,
the number of arbitrary fillings with
length of longest ne-chain and
length of longest SE-chain given,
depends only on the heights of the columns.

Remarks

- our bijection does not preserve

  0-1 fillings.

  Theorem 1 follows using the involution principl

- surprise :

  the order in which columns are moved

  is irrelevant ! (Proof extremely nasty)

New Corollaries

Theorem: # type B k-crossing set partitions

= # type B k-nesting set partitions

(conjectured by Armstrong)

Theorem: # type B k-triangulations

= # Fans of k Dyck paths

symmetric with respect to a vertical line

(conjectured by Fischer, Soll, Welker)

Outlook

- find relation to Backelin West Xin method

  Einar Steingrímsson Mireille Bousquet Mélou

  Anna de Mier

- generalize to arbitrary Coxeter groups

- bivariate symmetry

  $$\#(k\text{-crossing}, \ell\text{-nesting}) = \#(\ell\text{-crossing}, k\text{-nesting})$$

  Drew Armstrong