Algebraic constructions on set partitions

Maxime Rey

Laboratoire d'Informatique de l'Institut Gaspard-Monge
Université Paris-Est

2007 July 6

Preamble

Several works mixed

- combinatorial algorithms,
- Hopf algebras,
- partial orders.

- Goal : Similar construction over set partitions.

Preamble

Several works mixed

- combinatorial algorithms,
- Hopf algebras,
- partial orders.

permutations	**FQSym**	permutohedron
binary trees	**PBT**	associahedron
Std. Young tableaux	**FSym**	weak order on SYT
compositions	**NCSF**	hypercube
ordered set partitions	**WQSym**	pseudo-permutohedron
plane trees	TD	quotient of pp
segmented compositions	TC	quotient of pp

- Goal : Similar construction over set partitions.

Preamble

- combinatorial algorithms :
- Hopf algebra :
- partial order :

permutations	**FQSym**	permutohedron
binary trees	**PBT**	associahedron
Std. Young tableaux	**FSym**	weak order on SYT
compositions	**NCSF**	hypercube
ordered set partitions	**WQSym**	pseudo-permutohedron
plane trees	TD	quotient of pp
segmented compositions	TC	quotient of pp

- Goal : Similar construction over set partitions.

Preamble

- combinatorial algorithms : Robinson-Schensted correspondence and jeu de taquin,
- Hopf algebra :
- partial order :

permutations	**FQSym**	permutohedron
binary trees	**PBT**	associahedron
Std. Young tableaux	**FSym**	weak order on SYT
compositions	**NCSF**	hypercube
ordered set partitions	**WQSym**	pseudo-permutohedron
plane trees	TD	quotient of pp
segmented compositions	TC	quotient of pp

- Goal : Similar construction over set partitions.

Preamble

- combinatorial algorithms : Robinson-Schensted correspondence and jeu de taquin,
- Hopf algebra : **FSym** (Poirier-Reutenauer Hopf algebra),
- partial order :

permutations	**FQSym**	permutohedron
binary trees	**PBT**	associahedron
Std. Young tableaux	**FSym**	weak order on SYT
compositions	**NCSF**	hypercube
ordered set partitions	**WQSym**	pseudo-permutohedron
plane trees	TD	quotient of pp
segmented compositions	TC	quotient of pp

- Goal : Similar construction over set partitions.

Preamble

- combinatorial algorithms : Robinson-Schensted correspondence and jeu de taquin,
- Hopf algebra : **FSym** (Poirier-Reutenauer Hopf algebra),
- partial order : weak order on SYT.

permutations	**FQSym**	permutohedron
binary trees	**PBT**	associahedron
Std. Young tableaux	**FSym**	weak order on SYT
compositions	**NCSF**	hypercube
ordered set partitions	**WQSym**	pseudo-permutohedron
plane trees	TD	quotient of pp
segmented compositions	TC	quotient of pp

- Goal : Similar construction over set partitions.

Preamble

- combinatorial algorithms : Robinson-Schensted correspondence and jeu de taquin,
- Hopf algebra : **FSym** (Poirier-Reutenauer Hopf algebra),
- partial order : weak order on SYT.

permutations	**FQSym**	permutohedron
binary trees	**PBT**	associahedron
Std. Young tableaux	**FSym**	weak order on SYT
compositions	**NCSF**	hypercube
ordered set partitions	**WQSym**	pseudo-permutohedron
plane trees	TD	quotient of pp
segmented compositions	TC	quotient of pp

- Goal : Similar construction over set partitions.

Preamble

- combinatorial algorithms : Robinson-Schensted correspondence and jeu de taquin,
- Hopf algebra : **FSym** (Poirier-Reutenauer Hopf algebra),
- partial order : weak order on SYT.

permutations	**FQSym**	permutohedron
binary trees	**PBT**	associahedron
Std. Young tableaux	**FSym**	weak order on SYT
compositions	**NCSF**	hypercube
ordered set partitions	**WQSym**	pseudo-permutohedron
plane trees	TD	quotient of pp
segmented compositions	TC	quotient of pp

- Goal : Similar construction over set partitions.

Related works

- M. Rosas, B. Sagan, *Symmetric Functions in Noncommuting Variables*, Transactions of the American Mathematical Society, to appear.
- N. Bergeron and M. Zabrocki, *The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree* preprint math.CO/0509265.
- $NCSym$ is a non-commutative and co-commutative Hopf algebra.

Results

- $\mathfrak{P}$, a self-dual Hopf algebra on set partitions
  - Bidendriform $\Rightarrow$ free, co-free and self-dual
  - Multiplicative bases
  - **PBT** $\hookrightarrow \mathfrak{P} \hookrightarrow$ **FQSym**
- Bell order, a partial order on set partitions
  - Intervals of the Bell order describe the product of $\mathfrak{P}$
- Scrolling, a jeu de taquin-like on set partitions
- Robinson-Schensted analogue on set partitions
- Plactic-like monoid structure

Results

- $\mathfrak{P}$, a self-dual Hopf algebra on set partitions
  - Bidendriform $\Rightarrow$ free, co-free and self-dual
  - Multiplicative bases
  - **PBT** $\hookrightarrow \mathfrak{P} \hookrightarrow$ **FQSym**
- Bell order, a partial order on set partitions
  - Intervals of the Bell order describe the product of $\mathfrak{P}$
- Scrolling, a jeu de taquin-like on set partitions
- Robinson-Schensted analogue on set partitions
- Plactic-like monoid structure

Results

- $\mathfrak{P}$, a self-dual Hopf algebra on set partitions
  - Bidendriform $\Rightarrow$ free, co-free and self-dual
  - Multiplicative bases
  - **PBT** $\hookrightarrow \mathfrak{P} \hookrightarrow$ **FQSym**
- Bell order, a partial order on set partitions
  - Intervals of the Bell order describe the product of $\mathfrak{P}$
- Scrolling, a jeu de taquin-like on set partitions
- Robinson-Schensted analogue on set partitions
- Plactic-like monoid structure

Results

- $\mathfrak{P}$, a self-dual Hopf algebra on set partitions
  - Bidendriform $\Rightarrow$ free, co-free and self-dual
  - Multiplicative bases
  - **PBT** $\hookrightarrow \mathfrak{P} \hookrightarrow$ **FQSym**
- Bell order, a partial order on set partitions
  - Intervals of the Bell order describe the product of $\mathfrak{P}$
- Scrolling, a jeu de taquin-like on set partitions
- Robinson-Schensted analogue on set partitions
- Plactic-like monoid structure

A. Burstein and I. Lankham *"Combinatorics of Patience Sorting Piles"*, SLC54A.

Results

- $\mathfrak{P}$, a self-dual Hopf algebra on set partitions
  - Bidendriform $\Rightarrow$ free, co-free and self-dual
  - Multiplicative bases
  - **PBT** $\hookrightarrow \mathfrak{P} \hookrightarrow$ **FQSym**
- Bell order, a partial order on set partitions
  - Intervals of the Bell order describe the product of $\mathfrak{P}$
- Scrolling, a jeu de taquin-like on set partitions
- Robinson-Schensted analogue on set partitions
- Plactic-like monoid structure

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

$$\emptyset$$

A. Burstein and I. Lankham *"Combinatorics of Patience Sorting Piles"*, SLC54A.

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

$$\emptyset \qquad \longleftarrow \boxed{5}$$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

$$\boxed{5}$$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

$\boxed{5}$ $\longleftarrow$ $\boxed{8}$

$5 < 8$

Combinatorial algorithm

Robinson-Schensted analogue

$\sigma = 586714923$

5	8

5 < 8

Combinatorial algorithm
Robinson-Schensted analogue

$$\sigma = 586714923$$

5	8

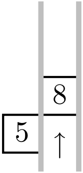
$\longleftarrow$ | 6 |

$5 < 6 < 8$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

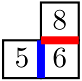


$$5 < 6 < 8$$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$



$5 < 6 < 8$

Combinatorial algorithm

Robinson-Schensted analogue

$\sigma = 586714923$

$6 < 7$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

$1 <$ 5

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

5	8	
1	6	7

$1 < 5$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

$$1 < 4 < 6$$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

$7 < 9$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

Combinatorial algorithm

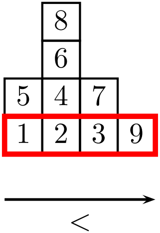
Robinson-Schensted analogue

$$\sigma = 586714923$$

Combinatorial algorithm

Robinson-Schensted analogue

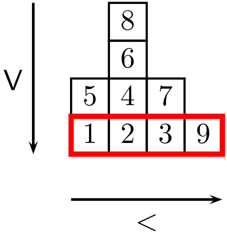
$$\sigma = 586714923$$



Combinatorial algorithm

Robinson-Schensted analogue

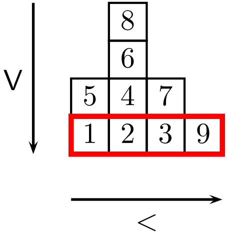
$$\sigma = 586714923$$



Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$

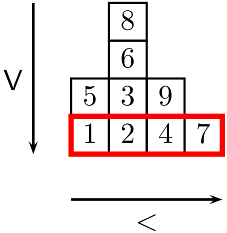


$$\longleftrightarrow \quad P(\sigma) = \{\{5,1\},\{8,6,4,2\},\{7,3\},\{9\}\}$$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$



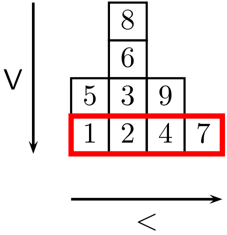
$$\longleftrightarrow$$

$$Q(\sigma) = P(\sigma^{-1}) = \{\{5,1\},\{8,6,3,2\},\{9,4\},\{7\}\}$$

Combinatorial algorithm

Robinson-Schensted analogue

$$\sigma = 586714923$$



$$\longleftrightarrow \quad Q(\sigma) = P(\sigma^{-1}) = \{\{5,1\},\{8,6,3,2\},\{9,4\},\{7\}\}$$

$\sigma \longmapsto (P(\sigma), Q(\sigma))$ is **not** a surjection.

Combinatorial algorithm

Robinson-Schensted analogue

There is no permutation σ such that:

$$P(\sigma) = \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad Q(\sigma) = \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Combinatorial algorithm

Robinson-Schensted analogue

Definition

For $\sigma, \pi \in \mathfrak{S}$, we write

$$\sigma \equiv \pi\,,$$

if $P(\sigma) = P(\pi)$. Then, σ and π belong to the same *Bell class*.

Hopf algebra on set partitions

Definition

We consider the following elements of **FQSym** :

$$\mathbf{P}_\delta := \sum_{P(\sigma)=\delta} \mathbf{F}_\sigma \,,$$

for every set partition δ.

Hopf algebra on set partitions
Associative algebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a subalgebra of* **FQSym** .

Hopf algebra on set partitions
Associative algebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a subalgebra of* **FQSym** .

$$12 \Cup 21 = 1243 + 1423 + 4123 + 1432 + 4132 + 4312$$

Hopf algebra on set partitions

Associative algebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a subalgebra of* **FQSym** .

$$12 \Cup 21 = 1243 + 1423 + 4123 + 1432 + 4132 + 4312$$

Hopf algebra on set partitions
Associative algebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta\in SP}$ *form a subalgebra of* **FQSym** .

$$12 \Cup 21 = 1243 + 1423 + 4123 + 1432 + 4132 + 4312$$

Hopf algebra on set partitions
Associative algebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a subalgebra of* **FQSym** .

$$1 \Cup (213+231) = (1324+1342)+(3124)+(3214+3241+3421)+(3142+3412)$$

$$\downarrow$$

$$\{1\} \times \{21|3\} = \{1|32|4\} + \{31|2|4\} + \{321|4\} + \{31|42\}$$

Hopf algebra on set partitions
Associative algebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a subalgebra of* **FQSym** .

$$1 \Cup (213+231) = (1324+1342)+(3124)+(3214+3241+3421)+(3142+3412)$$

$$\downarrow$$

$$\{1\} \times \{21|3\} = \{1|32|4\} + \{31|2|4\} + \{321|4\} + \{31|42\}$$

Proposition (Restriction to intervals)

Let $\sigma, \pi \in \mathfrak{S}$. *If* $\sigma \equiv \pi$, *then* $Std(\sigma|_I) \equiv Std(\pi|_I)$.

Hopf algebra on set partitions
Coalgebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a subcoalgebra of* **FQSym** .

Proposition

Let u, v, u', v' *be words. If* $u \equiv v$ *and* $u' \equiv v'$, *then* $u \cdot v \equiv u' \cdot v'$.

Hopf algebra on set partitions
Coalgebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a subcoalgebra of* **FQSym** .

$$\bar{\Delta}((4213+4231)) = 1 \otimes (213+231) + 21 \otimes 12 + 21 \otimes 21 + 321 \otimes 1 + 312 \otimes 1$$

$$\downarrow$$

$$\bar{\Delta}(\{421|3\}) = \{1\} \otimes \{21|3\} + \{21\} \otimes (\{1|2\} + \{21\}) + (\{321\} + \{31|2\}) \otimes \{1\}$$

Proposition

Let u, v, u', v' *be words. If* $u \equiv v$ *and* $u' \equiv v'$, *then* $u \cdot v \equiv u' \cdot v'$.

Hopf algebra on set partitions
Coalgebra

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a subcoalgebra of* **FQSym** .

$$\bar{\Delta}((4213+4231)) = 1 \otimes (213+231) + 21 \otimes 12 + 21 \otimes 21 + 321 \otimes 1 + 312 \otimes 1$$

$$\downarrow$$

$$\bar{\Delta}(\{421|3\}) = \{1\} \otimes \{21|3\} + \{21\} \otimes (\{1|2\} + \{21\}) + (\{321\} + \{31|2\}) \otimes \{1\}$$

Proposition

Let u, v, u', v' *be words. If* $u \equiv v$ *and* $u' \equiv v'$, *then* $u \cdot v \equiv u' \cdot v'$.

Hopf algebra on set partitions

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a Hopf subalgebra of* **FQSym** .

Proposition

$\mathcal{B}$ *is a bidendriform bialgebra. Hence,* $\mathcal{B}$ *is free, cofree and self-dual.*

Hopf algebra on set partitions

Theorem

The elements $(\mathbf{P}_\delta)_{\delta \in SP}$ *form a Hopf subalgebra of* **FQSym** .

Proposition

$\mathcal{B}$ *is a bidendriform bialgebra. Hence,* $\mathcal{B}$ *is free, cofree and self-dual.*

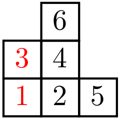
Bell classes

Canonical permutation

	6	
3	4	
1	2	5

- Column reading.

Bell classes
Canonical permutation



31

- Column reading.

Bell classes
Canonical permutation

	6	
3	4	
1	2	5

$31\,642$

- Column reading.

Bell classes

Canonical permutation

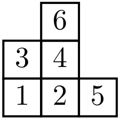
	6	
3	4	
1	2	5

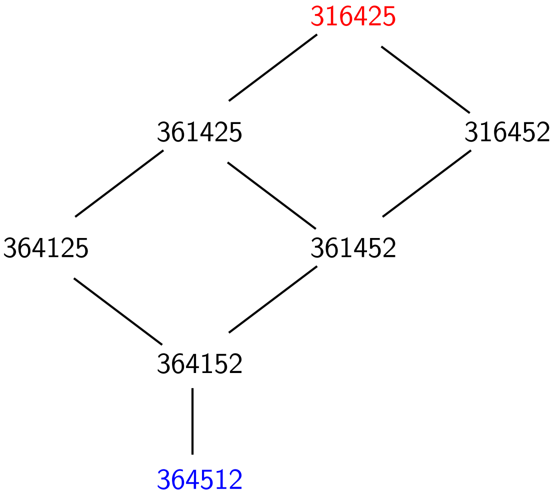
$31\,642\,5$

- Column reading.

Bell classes

Canonical permutation





Bell classes
Canonical permutation

Proposition

In every Bell class,

1. *the canonical element is the unique permutation of minimal length,*
2. *there is a unique element of maximal length.*

Bell classes
Intervals

Definition

Rewriting rule:

$$..b\underbrace{\ldots}_{>b}ac.. \longrightarrow ..b\underbrace{\ldots}_{>b}ca..$$

Theorem

$\forall \sigma \in \mathfrak{S},\ P(\sigma) \xrightarrow{*} \sigma.$

Bell classes
Intervals

Definition

Rewriting rule:

$$..b\underbrace{\ldots}_{>b}ac.. \longrightarrow ..b\underbrace{\ldots}_{>b}ca..$$

Theorem

$\forall \sigma \in \mathfrak{S},\ P(\sigma) \stackrel{*}{\longrightarrow} \sigma.$

Bell classes

Intervals

316425

Bell classes

Intervals

316425

Bell classes

Intervals

316425

↙

361425

Bell classes
Intervals

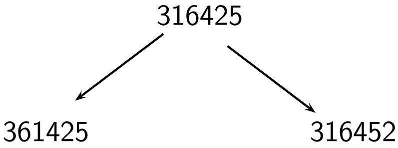
316425

↙

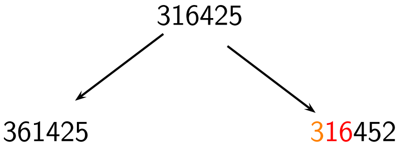
361425

Bell classes

Intervals

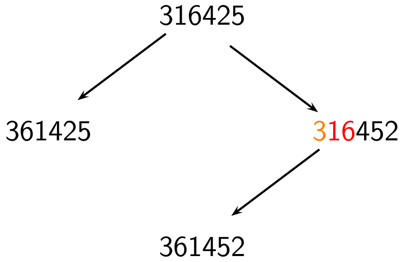


Bell classes
Intervals

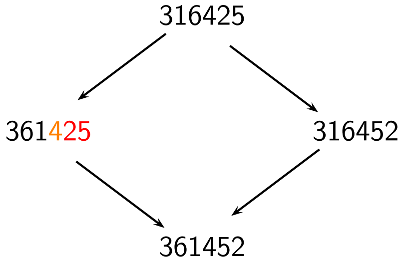


Bell classes

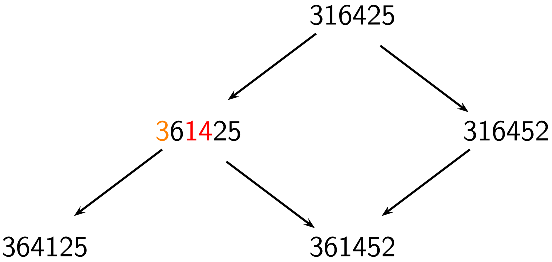
Intervals



Bell classes
Intervals

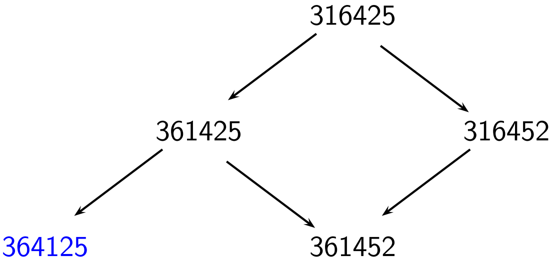


Bell classes
Intervals



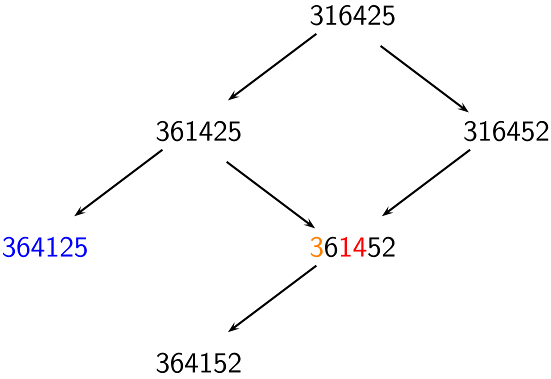
Bell classes

Intervals



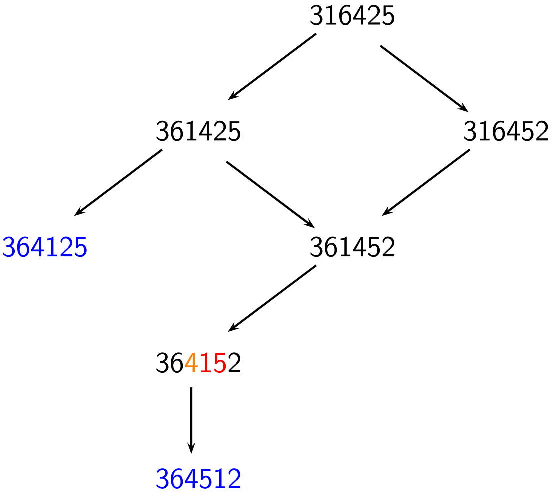
Bell classes

Intervals



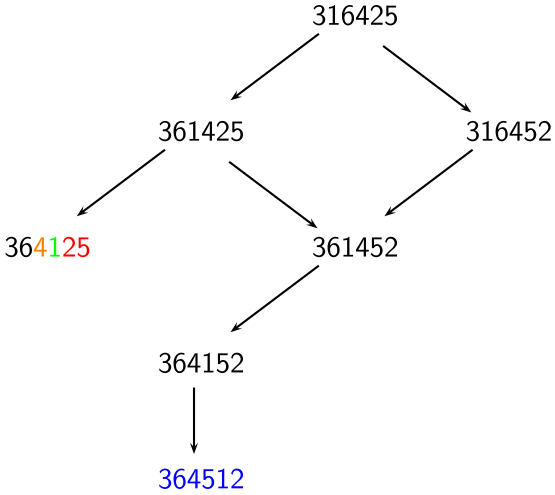
Bell classes

Intervals



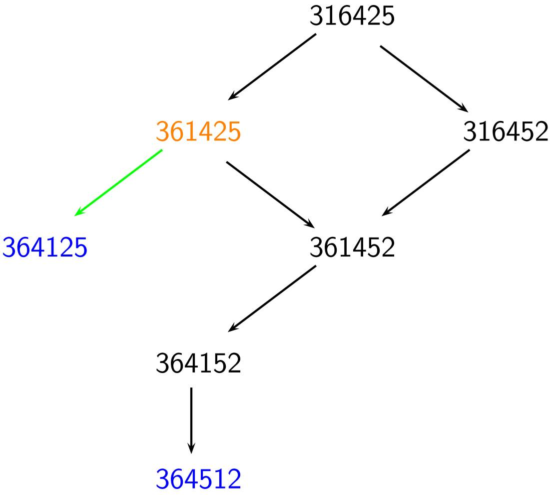
Bell classes

Intervals



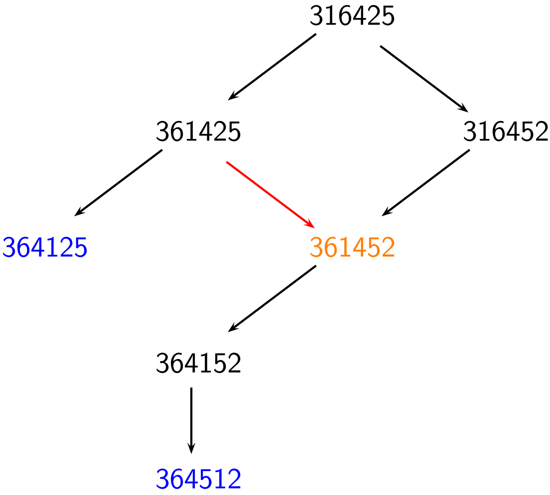
Bell classes

Intervals



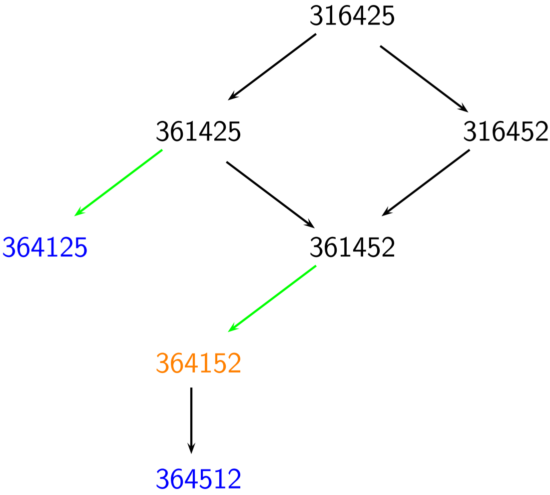
Bell classes

Intervals



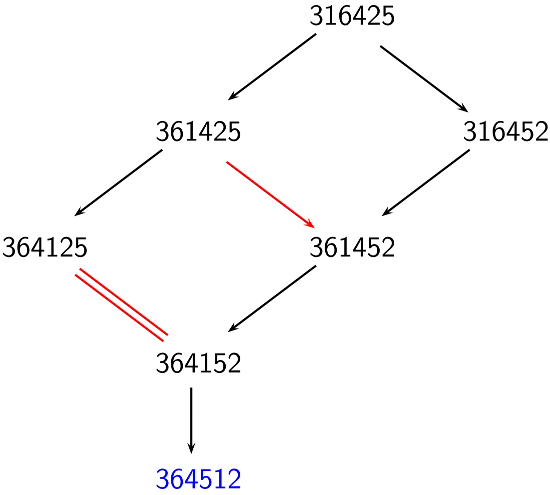
Bell classes

Intervals



Bell classes

Intervals

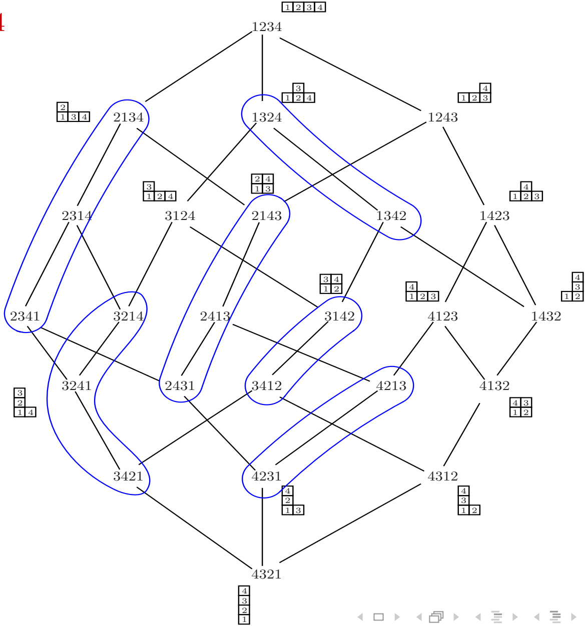


Bell classes
Intervals

Theorem

Bell classes are intervals of the permutohedron.

Intervals on $\mathfrak{S}_4$



Bell order
Cover relation

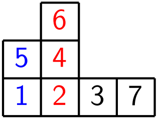
	6		
5	4		
1	2	3	7

Bell order
Cover relation

	6		
5	4		
1	2	3	7

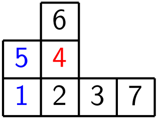
Bell order

Cover relation

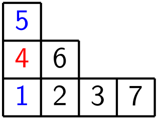


Bell order

Cover relation

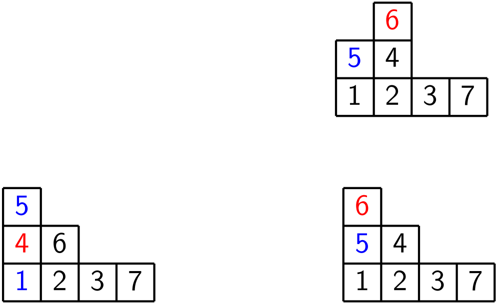


$1 < 2 < 4 < 5$



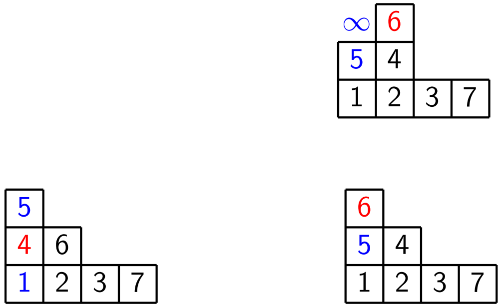
Bell order

Cover relation



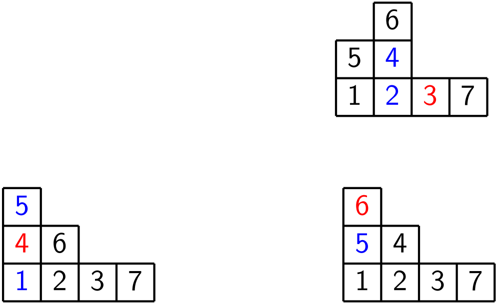
Bell order

Cover relation



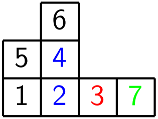
Bell order

Cover relation



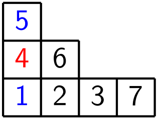
Bell order

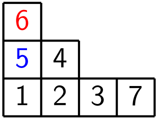
Cover relation



3 alone in its column

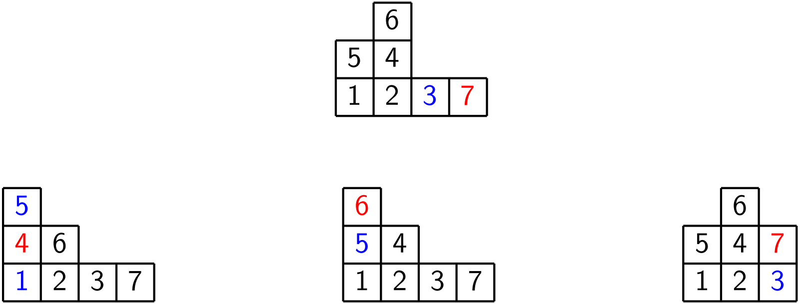
7 greater than 4





Bell order

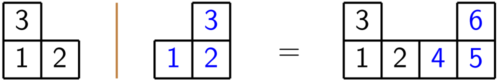
Cover relation



Poset and Hopf algebra

Intervals & Product

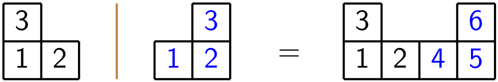
- $\{31\,|\,2\} \mid \{1\,|\,32\} = \{31\,|\,2\,|\,4\,|\,65\}$



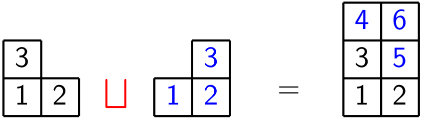
Poset and Hopf algebra

Intervals & Product

- $\{31\,|\,2\} \,|\, \{1\,|\,32\} = \{31\,|\,2\,|\,4\,|\,65\}$



- $\{31\,|\,2\} \sqcup \{1\,|\,32\} = \{431\,|\,652\}$



Poset and Hopf algebra

Intervals & Product

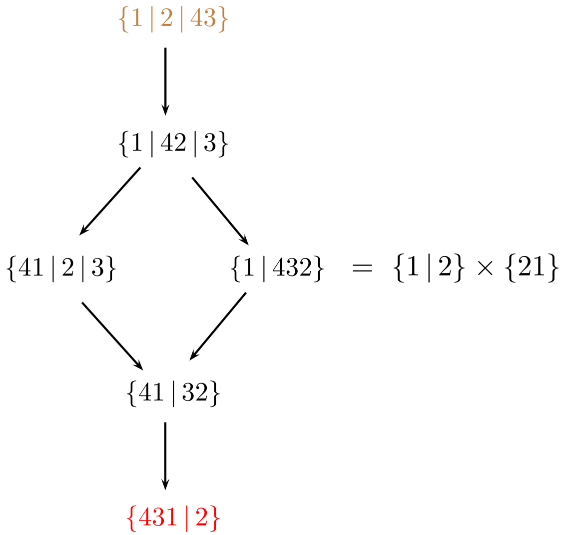
Theorem

For every set partitions α, β,

$$\mathbf{P}_\alpha \times \mathbf{P}_\beta = \sum_{(\alpha|\beta)\leq\delta\leq(\alpha\sqcup\beta)} \mathbf{P}_\delta \, .$$

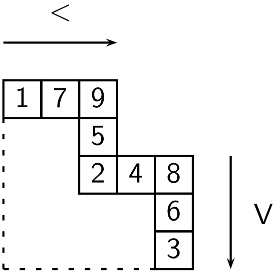
Poset and Hopf algebra

Intervals & Product



Scrolling
Jeu de taquin-like

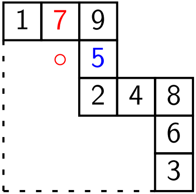
$\sigma = 179524863$



Scrolling

Jeu de taquin-like

$\sigma = 179524863$

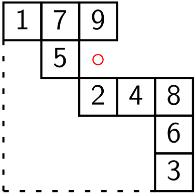


$5 < 7$

Scrolling

Jeu de taquin-like

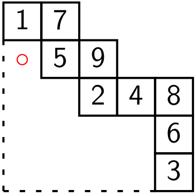
$$\sigma = 179524863$$



Scrolling

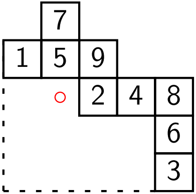
Jeu de taquin-like

$$\sigma = 179524863$$



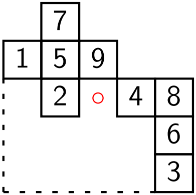
Scrolling
Jeu de taquin-like

$$\sigma = 179524863$$



Scrolling
Jeu de taquin-like

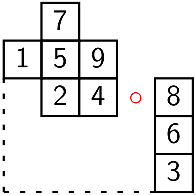
$\sigma = 179524863$



Scrolling

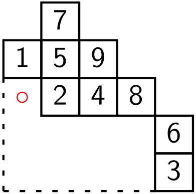
Jeu de taquin-like

$$\sigma = 179524863$$



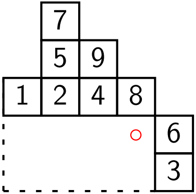
Scrolling
Jeu de taquin-like

$\sigma = 179524863$



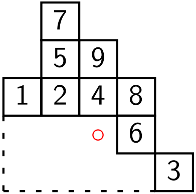
Scrolling
Jeu de taquin-like

$$\sigma = 179524863$$



Scrolling
Jeu de taquin-like

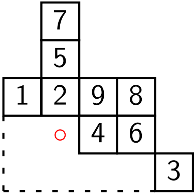
$\sigma = 179524863$



Scrolling

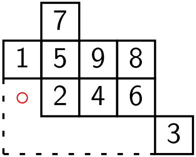
Jeu de taquin-like

$\sigma = 179524863$



Scrolling
Jeu de taquin-like

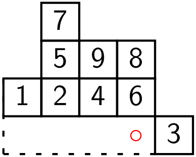
$\sigma = 179524863$



Scrolling

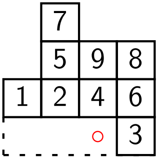
Jeu de taquin-like

$$\sigma = 179524863$$



Scrolling
Jeu de taquin-like

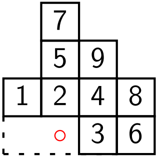
$$\sigma = 179524863$$



Scrolling

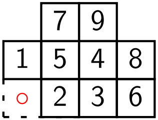
Jeu de taquin-like

$$\sigma = 179524863$$



Scrolling
Jeu de taquin-like

$$\sigma = 179524863$$



Scrolling
Jeu de taquin-like

$\sigma = 179524863$

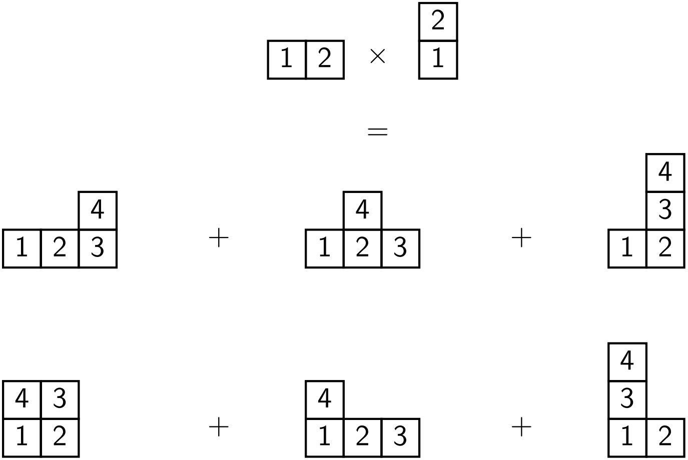
$$S(\sigma) = \begin{array}{cccc} & 7 & 9 & \\ & 5 & 4 & 8 \\ 1 & 2 & 3 & 6 \end{array}$$

Scrolling
Jeu de taquin-like

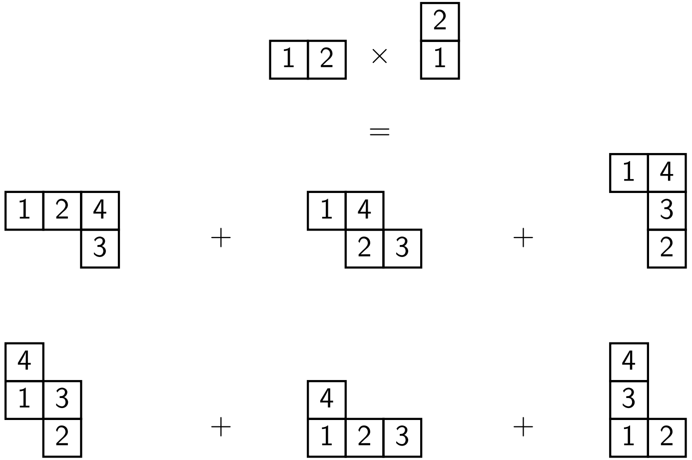
$$\sigma = 179524863$$

$$S(\sigma) = \begin{array}{|c|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 7 & 9 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 5 & 4 & 8 \\ \hline 1 & 2 & 3 & 6 \\ \hline \end{array} = P(\sigma)$$

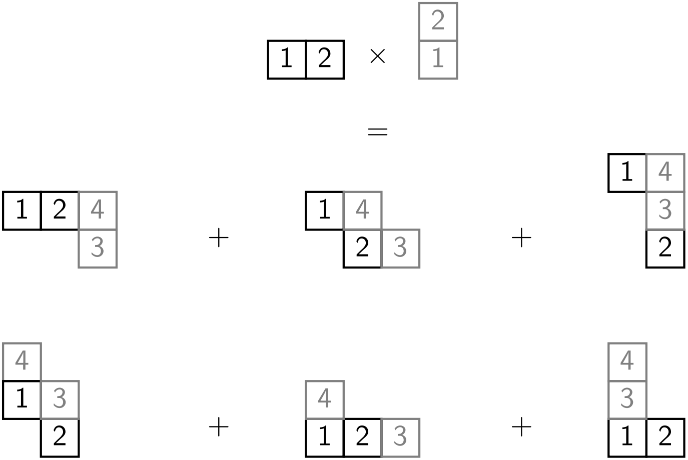
Scrolling and Hopf algebra



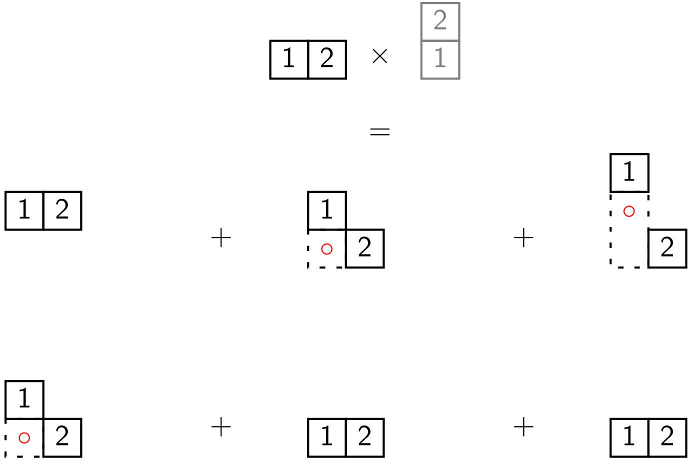
Scrolling and Hopf algebra



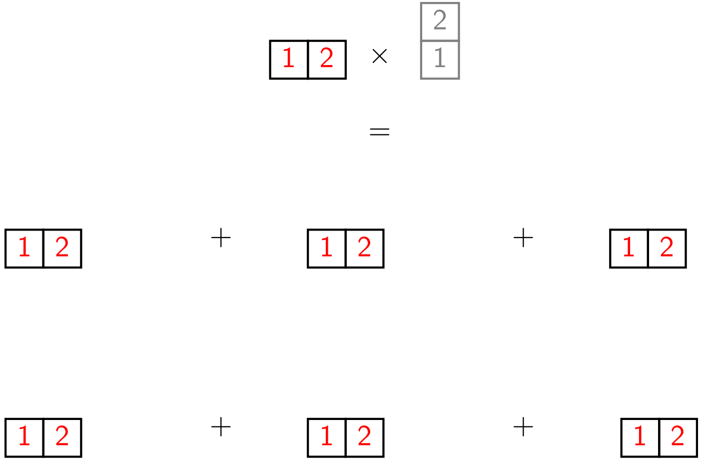
Scrolling and Hopf algebra



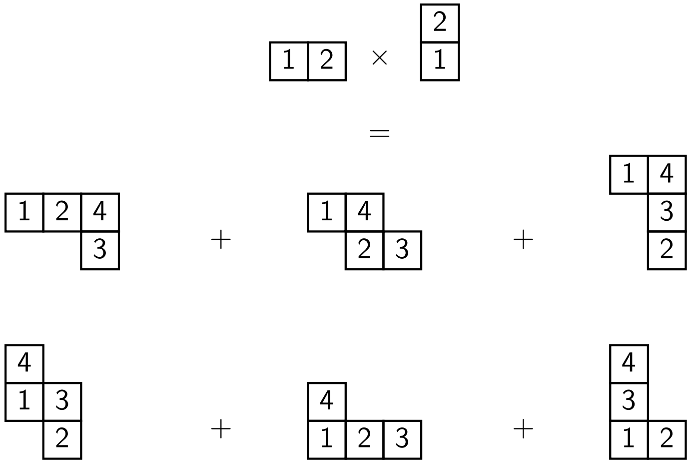
Scrolling and Hopf algebra



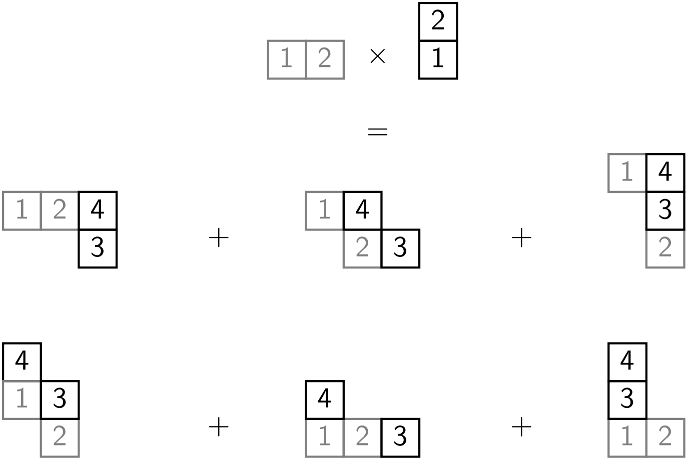
Scrolling and Hopf algebra



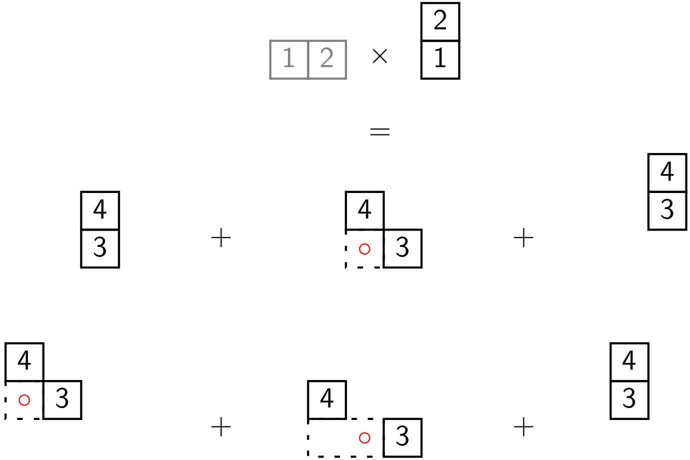
Scrolling and Hopf algebra



Scrolling and Hopf algebra



Scrolling and Hopf algebra



Scrolling and Hopf algebra

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}$$

$$=$$

$$\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline \end{array}$$

Scrolling and Hopf algebra