part I

PLANE PARTITIONS: HOW MACMAHON'S DREAM HAS COME TRUE *)

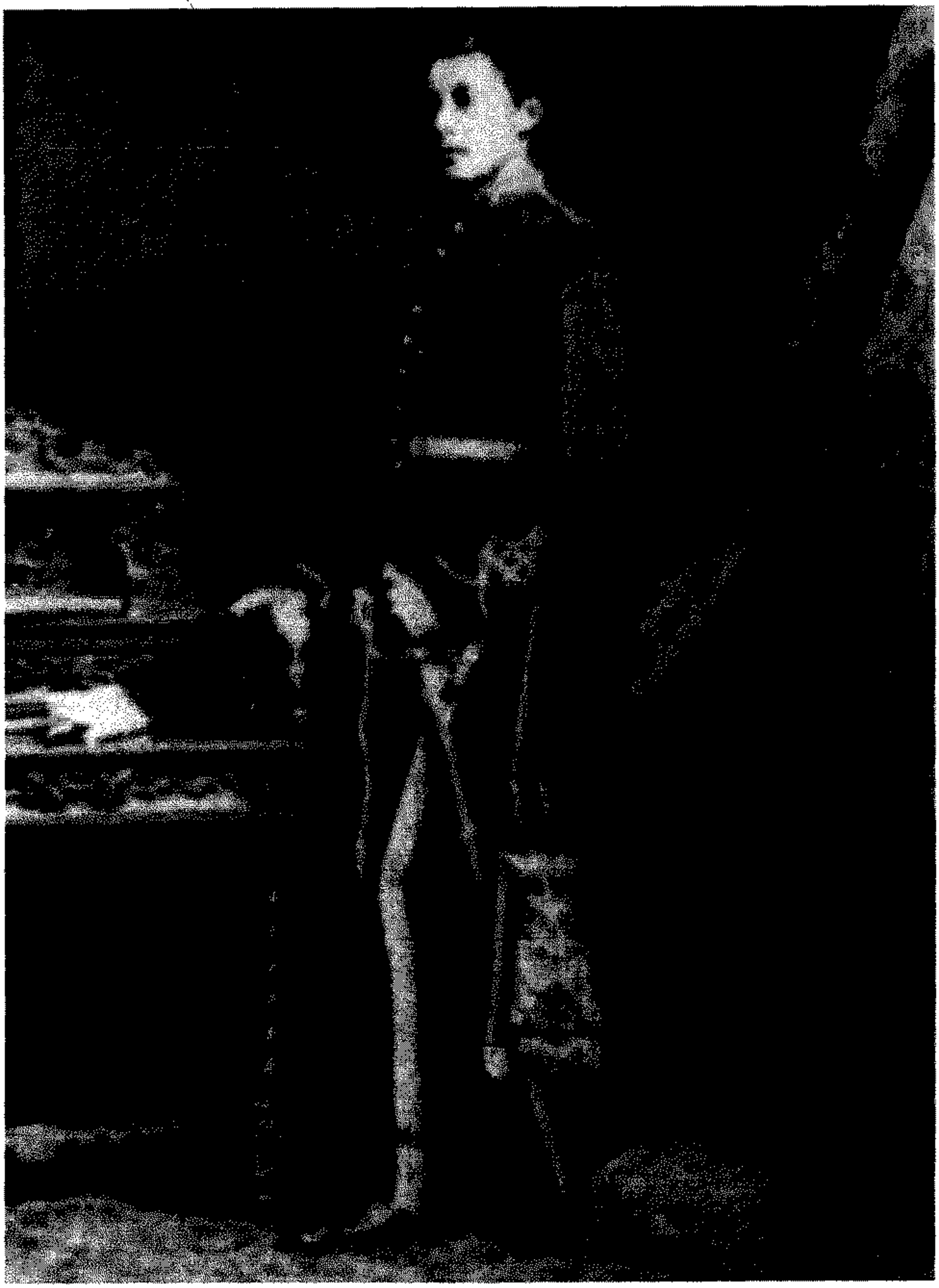
*) G.E. Andrews & PP: PA XII (J. of London Math. Soc.; to appear)

Percy Alexander MacMahon

(26.9.1854 – 25.12.1929)

http://www-history.mcs.st-andrews.ac.uk

"A good soldier spoiled" (Pat. Garcia)



http://www-history.mcs.st-andrews.ac.uk

Introduction

The story begins with the paper:

P.A. MacMahon, "Memoir on the Theory of Partitions of Numbers – Part I", Phil. Trans. 187 (1897), 619 – 673.

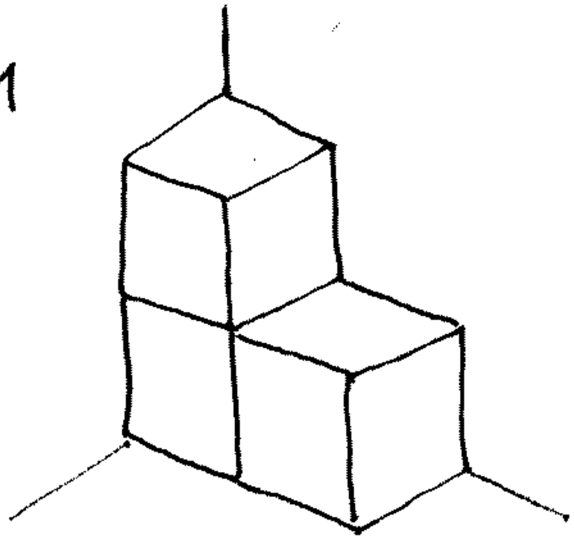
Note: Parts II – VII followed.

PLANE PARTITIONS :

Ex.: $3 = 2+1 = 1+1+1$

$$= \begin{matrix} 2 \\ + \\ 1 \end{matrix} \quad = \begin{matrix} 1 & +1 \\ + & \\ 1 & \end{matrix}$$

$$= \begin{matrix} 1 \\ + \\ 1 \\ + \\ 1 \end{matrix}$$



CONJECTURE (pp. 657 – 658)

"The enumeration of the three–dimensional graphs that can be formed with a given number of nodes, corresponding to the regularized partitions of multi–partite numbers of given content, is a weighty problem. I have verified to a high order that the generating function of the complete system is

$$(1-q)^{-1}(1-q^2)^{-2}(1-q^3)^{-3}(1-q^4)^{-4}\ldots\text{ad inf.},$$

and, so far as my investigations have proceeded, everything tends to confirm the truth of this conjecture."

$$= 1 + q + 3q^2 + 6q^3 + 13q^4 + \cdots$$

$$3,\ 21,\ 111,\ \begin{matrix}2\\1\end{matrix},\ \begin{matrix}1&1\\1&\end{matrix},\ \begin{matrix}1\\1\\1\end{matrix}$$

J.W.L. GLAISHER (Referee's report for the Philosophical Transactions of the Royal Society, June 8, 1896)[1]

"I don't fancy the paper very much, but it must be printed. I don't care much for a paper on very technical mathematics being published in the Phil. Trans. unless there is something very striking in it. However, it is one of a series, and they are in deep water now and cannot go on much farther. I have made my report because there is no more to be said that it should be published (though the interesting results are the conjectural ones!), the balance being on that side."

[[1] Printed with permission of the Royal Society]

Develop the method of
PARTITION ANALYSIS
to prove the conjecture.

BUT :

His efforts did not turn out as he had hoped, and he had to spend nearly 20 years finding an alternative treatment.

3 ROWS and 3 COLUMNS

$$\sum_{a_{ij} \geq 0} q^{a_{11}+a_{12}+a_{13}+\cdots+a_{31}+a_{32}+a_{33}}$$

s.t.

$$\begin{tikzcd}
a_{11} \arrow[r] \arrow[d] & a_{12} \arrow[r] \arrow[d] & a_{13} \arrow[d] \\
a_{21} \arrow[r] \arrow[d] & a_{22} \arrow[r] \arrow[d] & a_{23} \arrow[d] \\
a_{31} \arrow[r] & a_{32} \arrow[r] & a_{33}
\end{tikzcd}$$

where

$$\longrightarrow \cdots \; := \; \geqslant$$

PLANE PARTITIONS:
3 rows and 3 columns

In[19]:=

```
OSum[
  q^(a11+a12+a13+a21+a22+a23+a31+a32+a33)
  ,
  {a11 ≤ a12, a12 ≤ a13,
   a21 ≤ a22, a22 ≤ a23,
   a31 ≤ a32, a32 ≤ a33,
   a11 ≤ a21, a21 ≤ a31,
   a12 ≤ a22, a22 ≤ a32,
   a13 ≤ a23, a23 ≤ a33},
   λ]
```

$$\mathrm{OSum}\left[q^{a_{11}+a_{12}+a_{13}+a_{21}+a_{22}+a_{23}+a_{31}+a_{32}+a_{33}}, \{a_{11} \le a_{12}, a_{12} \le a_{13}, a_{21} \le a_{22}, a_{22} \le a_{23}, a_{31} \le a_{32}, a_{32} \le a_{33}, a_{11} \le a_{21}, a_{21} \le a_{31}, a_{12} \le a_{22}, a_{22} \le a_{32}, a_{13} \le a_{23}, a_{23} \le a_{33}\}, \lambda\right]$$

Assuming $a_{11} \geq 0$

Assuming $a_{12} \geq 0$

Assuming $a_{13} \geq 0$

Assuming $a_{21} \geq 0$

Assuming $a_{22} \geq 0$

Assuming $a_{23} \geq 0$

Assuming $a_{31} \geq 0$

Assuming $a_{32} \geq 0$

Assuming $a_{33} \geq 0$

Out[19]=

$$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\underset{\geq}{\Omega}} 1\Big/\left(\left(1-\frac{q}{\lambda_1\,\lambda_7}\right)\left(1-\frac{q\,\lambda_7}{\lambda_3\,\lambda_8}\right)\left(1-\frac{q\,\lambda_8}{\lambda_5}\right)\left(1-\frac{q\,\lambda_1}{\lambda_2\,\lambda_9}\right)\left(1-\frac{q\,\lambda_3\,\lambda_9}{\lambda_4\,\lambda_{10}}\right)\left(1-\frac{q\,\lambda_5\,\lambda_{10}}{\lambda_6}\right)\left(1-\frac{q\,\lambda_2}{\lambda_{11}}\right)\left(1-\frac{q\,\lambda_4\,\lambda_{11}}{\lambda_{12}}\right)(1-q\,\lambda_6\,\lambda_{12})\right)$$

In[20]:= **OR[%]**

```
Eliminating λ12...
Eliminating λ11...
Eliminating λ10...
Eliminating λ9...
Eliminating λ8...
Eliminating λ7...
Eliminating λ2...
Eliminating λ1...
Eliminating λ3...
Eliminating λ5...
Eliminating λ4...
Eliminating λ6...
```

Out[20]=

$$\frac{1}{(1-q)\ (1-q^2)^2\ (1-q^3)^3\ (1-q^4)^2\ (1-q^5)}$$

IX
6

PLANE PARTITIONS:
3 rows and 4 columns

In[7]:=

```
OSum[
  q^(a11+a12+a13+a21+a22+a23+a31+a32+a33)
   q^(a14+a24+a34),
  {a11 ≤ a12, a12 ≤ a13,
   a13 ≤ a14,
   a21 ≤ a22, a22 ≤ a23,
   a23 ≤ a24,
   a31 ≤ a32, a32 ≤ a33,
   a33 ≤ a34,
   a11 ≤ a21, a21 ≤ a31,
   a12 ≤ a22, a22 ≤ a32,
   a13 ≤ a23, a23 ≤ a33,
   a14 ≤ a24, a24 ≤ a34},
   λ];
```

$$\mathrm{OSum}\Big[q^{a_{11}+a_{12}+a_{13}+a_{21}+a_{22}+a_{23}+a_{31}+a_{32}+a_{33}}\,q^{a_{14}+a_{24}+a_{34}},\ \{a_{11}\le a_{12},\ a_{12}\le a_{13},\ a_{13}\le a_{14},\ a_{21}\le a_{22},\ a_{22}\le a_{23},\ a_{23}\le a_{24},\ a_{31}\le a_{32},\ a_{32}\le a_{33},\ a_{33}\le a_{34},\ a_{11}\le a_{21},\ a_{21}\le a_{31},\ a_{12}\le a_{22},\ a_{22}\le a_{32},\ a_{13}\le a_{23},\ a_{23}\le a_{33},\ a_{14}\le a_{24},\ a_{24}\le a_{34}\},\ \lambda\Big];$$

In[8]:= **OR[%]**

Out[8]= $$\frac{1}{(1-q)\ (1-q^2)^2\ (1-q^3)^3}$$

$$\frac{1}{(1-q^4)^3\ (1-q^5)^2\ (1-q^6)}$$

CONJECTURE (MacMahon): The generating function for

PLANE PARTITIONS

with at most

r ROWS and c COLUMNS

is

$$\prod_{i=1}^{r}\prod_{j=1}^{c}(1-q^{i+j-1})^{-1}.$$

$r=3$, $c=4$

$$\begin{array}{cccc} q & q^2 & q^3 & q^4 \\ q^2 & q^3 & q^4 & q^5 \\ q^3 & q^4 & q^5 & q^6 \end{array}$$

Def. $P_{r,c}(n) :=$ no. of plane ptns. of n with $\leq r$ rows and $\leq c$ columns.

Conjecture 1897 [MacMahon]

$$\boxed{\sum_{n=0}^{\infty} P_{\infty,\infty}(n)\, q^n = \prod_{n=1}^{\infty} (1-q^n)^{-n}}$$

$$= 1 + q + 3q^2 + \underbrace{6}_{3,\ 21,\ 111,\ \begin{smallmatrix}2\\1\end{smallmatrix},\ \begin{smallmatrix}1&1\\1\end{smallmatrix},\ \begin{smallmatrix}1\\1\\1\end{smallmatrix}} q^3 + 13 q^4 + \dots$$

BUT: MacMahon's Partition Analysis project failed. In his book [Vol. II, 1916, p. 187] he comments on Ω and on the gen. function

$$\boxed{\sum_{n=0}^{\infty} P_{r,c}(n)\, q^n = \prod_{i=1}^{r}\prod_{j=1}^{c} (1-q^{i+j-1})^{-1}}$$

as follows:

part 3

PLANE PARTITIONS:
3 rows and 3 columns
(full generating function)

In[4]:=

$$\text{Crude33} = \text{OSum}[x_{11}^{a_{11}}\, x_{12}^{a_{12}}\, x_{13}^{a_{13}}\, x_{21}^{a_{21}}\, x_{22}^{a_{22}}\, x_{23}^{a_{23}}\, x_{31}^{a_{31}}\, x_{32}^{a_{32}}\, x_{33}^{a_{33}},$$
$$\{a_{11} \le a_{12},\ a_{12} \le a_{13},\ a_{21} \le a_{22},\ a_{22} \le a_{23},\ a_{31} \le a_{32},\ a_{32} \le a_{33},$$
$$a_{11} \le a_{21},\ a_{21} \le a_{31},\ a_{12} \le a_{22},\ a_{22} \le a_{32},\ a_{13} \le a_{23},\ a_{23} \le a_{33}\},\ \lambda]$$

Out[4]=

$$\underset{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8,\lambda_9,\lambda_{10},\lambda_{11},\lambda_{12}}{\overset{\Omega}{\geq}} 1 \Big/ \left(\left(1 - \frac{\mathbf{x}_{11}}{\lambda_1\,\lambda_7}\right) \left(1 - \frac{\mathbf{x}_{21}\,\lambda_7}{\lambda_3\,\lambda_8}\right) \left(1 - \frac{\mathbf{x}_{31}\,\lambda_8}{\lambda_5}\right) \left(1 - \frac{\mathbf{x}_{12}\,\lambda_1}{\lambda_2\,\lambda_9}\right) \left(1 - \frac{\mathbf{x}_{22}\,\lambda_3\,\lambda_9}{\lambda_4\,\lambda_{10}}\right) \left(1 - \frac{\mathbf{x}_{32}\,\lambda_5\,\lambda_{10}}{\lambda_6}\right) \left(1 - \frac{\mathbf{x}_{13}\,\lambda_2}{\lambda_{11}}\right) \left(1 - \frac{\mathbf{x}_{23}\,\lambda_4\,\lambda_{11}}{\lambda_{12}}\right) (1 - \mathbf{x}_{33}\,\lambda_6\,\lambda_{12}) \right)$$

In[5]:= `Factor[OR[Crude33]]`

Out[5]=
$$-(1 - x_{23}\, x_{32}\, x_{33}^2 - x_{13}\, x_{23}\, x_{32}\, x_{33}^2 - x_{13}\, x_{23}^2\, x_{32}\, x_{33}^2 -$$
$$x_{13}\, x_{22}\, x_{23}^2\, x_{32}\, x_{33}^2 - x_{23}\, x_{31}\, x_{32}\, x_{33}^2 -$$
$$x_{13}\, x_{23}\, x_{31}\, x_{32}\, x_{33}^2 - x_{13}\, x_{23}^2\, x_{31}\, x_{32}\, x_{33}^2 -$$
$$x_{13}\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}\, x_{33}^2 - x_{13}\, x_{21}\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}\, x_{33}^2 -$$
$$x_{13}\, x_{22}\, x_{23}^2\, x_{32}^2\, x_{33}^2 - x_{23}\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}\, x_{23}\, x_{31}\, x_{32}^2\, x_{33}^2 - x_{22}\, x_{23}\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}\, x_{22}\, x_{23}\, x_{31}\, x_{32}^2\, x_{33}^2 - x_{12}\, x_{13}\, x_{22}\, x_{23}\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 - x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$3\, x_{13}\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 - x_{12}\, x_{13}\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}^2\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 - x_{12}\, x_{13}^2\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}\, x_{21}\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}\, x_{22}^2\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 - x_{12}\, x_{13}\, x_{22}^2\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{12}\, x_{13}^2\, x_{22}^2\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}\, x_{21}\, x_{22}^2\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 - x_{12}\, x_{13}\, x_{21}\, x_{22}^2\, x_{23}^2$$
$$x_{31}\, x_{32}^2\, x_{33}^2 - x_{12}\, x_{13}^2\, x_{21}\, x_{22}^2\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}\, x_{22}\, x_{23}^2\, x_{31}^2\, x_{32}^2\, x_{33}^2 - x_{13}\, x_{21}\, x_{22}\, x_{23}^2\, x_{31}^2\, x_{32}^2\, x_{33}^2 -$$
$$x_{13}\, x_{21}\, x_{22}^2\, x_{23}^2\, x_{31}^2\, x_{32}^2\, x_{33}^2 -$$
$$x_{12}\, x_{13}\, x_{21}\, x_{22}^2\, x_{23}^2\, x_{31}^2\, x_{32}^2\, x_{33}^2 -$$
$$x_{12}\, x_{13}^2\, x_{21}\, x_{22}^2\, x_{23}^2\, x_{31}^2\, x_{32}^2\, x_{33}^2 + x_{13}\, x_{23}^2\, x_{32}\, x_{33}^3 +$$
$$x_{13}\, x_{23}^2\, x_{31}\, x_{32}\, x_{33}^3 + x_{13}\, x_{23}^2\, x_{32}^2\, x_{33}^3 +$$
$$x_{13}\, x_{22}\, x_{23}^2\, x_{32}^2\, x_{33}^3 + x_{13}\, x_{22}\, x_{23}^3\, x_{32}^2\, x_{33}^3 +$$
$$x_{13}^2\, x_{22}\, x_{23}^3\, x_{32}^2\, x_{33}^3 + x_{23}\, x_{31}\, x_{32}^2\, x_{33}^3 +$$
$$x_{13}\, x_{23}\, x_{31}\, x_{32}^2\, x_{33}^3 + x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^3 +$$
$$4\, x_{13}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^3 + x_{13}^2\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^3 +$$
$$x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^3 + 4\, x_{13}\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^3 +$$
$$x_{12}\, x_{13}\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^3 + x_{13}^2\, x_{22}\, x_{23}^2\, x_{31}\, x_{32}^2\, x_{33}^3 +$$

In[5]:= **Factor[OR[Crude33]]**

Out[5]=

$$
\begin{aligned}
&\ldots . + x_{12}^{2}\, x_{13}^{6}\, x_{21}\, x_{22}^{6}\, x_{23}^{10}\, x_{31}^{6}\, x_{32}^{11}\, x_{33}^{12} - \\
&\quad x_{12}^{2}\, x_{13}^{6}\, x_{21}^{2}\, x_{22}^{6}\, x_{23}^{10}\, x_{31}^{6}\, x_{32}^{11}\, x_{33}^{12} - \\
&\quad x_{12}^{2}\, x_{13}^{6}\, x_{21}^{2}\, x_{22}^{7}\, x_{23}^{10}\, x_{31}^{6}\, x_{32}^{11}\, x_{33}^{12} - \\
&\quad x_{12}^{2}\, x_{13}^{6}\, x_{21}^{2}\, x_{22}^{7}\, x_{23}^{11}\, x_{31}^{6}\, x_{32}^{11}\, x_{33}^{12} - \\
&\quad x_{12}^{2}\, x_{13}^{7}\, x_{21}^{2}\, x_{22}^{7}\, x_{23}^{11}\, x_{31}^{6}\, x_{32}^{11}\, x_{33}^{12} - \\
&\quad x_{12}^{2}\, x_{13}^{6}\, x_{21}^{2}\, x_{22}^{6}\, x_{23}^{10}\, x_{31}^{7}\, x_{32}^{11}\, x_{33}^{12} - \\
&\quad x_{12}^{2}\, x_{13}^{6}\, x_{21}^{2}\, x_{22}^{7}\, x_{23}^{10}\, x_{31}^{7}\, x_{32}^{11}\, x_{33}^{12} - \\
&\quad x_{12}^{2}\, x_{13}^{6}\, x_{21}^{2}\, x_{22}^{7}\, x_{23}^{11}\, x_{31}^{7}\, x_{32}^{11}\, x_{33}^{12} - \\
&\quad x_{12}^{2}\, x_{13}^{7}\, x_{21}^{2}\, x_{22}^{7}\, x_{23}^{11}\, x_{31}^{7}\, x_{32}^{11}\, x_{33}^{12} + \\
&\quad x_{12}^{2}\, x_{13}^{7}\, x_{21}^{2}\, x_{22}^{7}\, x_{23}^{12}\, x_{31}^{7}\, x_{32}^{12}\, x_{33}^{14}) / \\
&((-1 + x_{33})\, (-1 + x_{23}\, x_{33})\, (-1 + x_{13}\, x_{23}\, x_{33}) \\
&\quad (-1 + x_{32}\, x_{33})\, (-1 + x_{23}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{13}\, x_{23}\, x_{32}\, x_{33})\, (-1 + x_{22}\, x_{23}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{13}\, x_{22}\, x_{23}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{12}\, x_{13}\, x_{22}\, x_{23}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{31}\, x_{32}\, x_{33})\, (-1 + x_{23}\, x_{31}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{13}\, x_{23}\, x_{31}\, x_{32}\, x_{33})\, (-1 + x_{22}\, x_{23}\, x_{31}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{13}\, x_{22}\, x_{23}\, x_{31}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{12}\, x_{13}\, x_{22}\, x_{23}\, x_{31}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{21}\, x_{22}\, x_{23}\, x_{31}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{13}\, x_{21}\, x_{22}\, x_{23}\, x_{31}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{12}\, x_{13}\, x_{21}\, x_{22}\, x_{23}\, x_{31}\, x_{32}\, x_{33}) \\
&\quad (-1 + x_{11}\, x_{12}\, x_{13}\, x_{21}\, x_{22}\, x_{23}\, x_{31}\, x_{32}\, x_{33}))
\end{aligned}
$$

```
In[20]:= Subst =
  {x11^p_. -> z0^p,
   x12^p_. -> z1^p,
   x13^p_. -> z2^p,
   x21^p_. -> z-1^p,
   x22^p_. -> z0^p,
   x23^p_. -> z1^p,
   x31^p_. -> z-2^p,
   x32^p_. -> z-1^p,
   x33^p_. -> z0^p }

In[22]:= Factor[%5 /. Subst]
```

Out[22]=

$$-\frac{1}{(-1+z_0)\,(-1+z_{-1}z_0)\,(-1+z_{-2}z_{-1}z_0)}$$

$$\frac{1}{(-1+z_0 z_1)\,(-1+z_{-1}z_0 z_1)}$$

$$\frac{1}{(-1+z_{-2}z_{-1}z_0 z_1)\,(-1+z_0 z_1 z_2)}$$

$$\frac{1}{(-1+z_{-1}z_0 z_1 z_2)\,(-1+z_{-2}z_{-1}z_0 z_1 z_2)}$$

This way we were led to a rediscovery of a theorem by Emden R. Gansner (1981). *)

["The enumeration of plane partititions via the Burge correspondence", Illinois J.Math. **25** (1981), 533–554]

*) and to a generalization!

SUMMARY:

The key observations for applying PARTITION ANALYSIS to complete MacMahon's project:

set $\boxed{x_{i,j} \to z_{j-i}}$

and decompose (inductively!) the corresp. crude gen. fn.

Ex $\boxed{r=c=4}$

$$(x_{i,j}) \to \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_{-1} & z_0 & z_1 & z_2 \\ z_{-2} & z_{-1} & z_0 & z_1 \\ z_{-3} & z_{-2} & z_{-1} & z_0 \end{pmatrix}$$

Stanley's trace theorem (1973):

$$\begin{pmatrix} z & q & q & q \\ q & z & q & q \\ q & q & z & q \\ q & q & q & z \end{pmatrix}$$

Def. $T_{r,c}(t;n) :=$ no. of plane ptns. of n with $\le r$ rows and $\le c$ columns and with trace t

III 17

III 15 6

Ex.

$$\begin{array}{cccc} 4 & 3 & 2 & 2 \\ 4 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & & \\ 1 & 1 & & \end{array}$$

has trace $t = 4+3+1 = 8$.

Thm. [R. Stanley, 1973]

$$\boxed{\sum_{n=0}^{\infty} \sum_{t=0}^{\infty} T_{r,c}(t;n)\, z^t q^n = \prod_{i=1}^{r} \prod_{j=1}^{c} \left(1 - z q^{i+j-1}\right)^{-1}}$$

In his book (Vol. II), Stanley gives an elegant proof using the RSK-algorithm together with the Conjugate Frobenius-Bender-Knuth bijection.

Ex.

$$\begin{array}{c|cccc}
 & t_0 & t_1 & t_2 & t_3 \\
\hline
 & 4 & 3 & 2 & 2 \\
t_{-1} & 4 & 3 & 1 & 1 \\
t_{-2} & 2 & 2 & 1 & 1 \\
t_{-3} & 1 & 1 & & \\
t_{-4} & 1 & 1 & &
\end{array}$$

$t_3 = 2 \qquad t_{-1} = 6$

$t_2 = 3 \qquad t_{-2} = 3$

$t_1 = 5 \qquad t_{-3} = 2$

$t_0 = 8 \qquad t_{-4} = 1$

$\square$

Def. $\mathcal{P}_{r,c}(z_{-r+1},\dots,z_{-1}; z_0,\dots,z_{c-1}; q) :=$

$$\sum_{n=0}^{\infty} \sum_{t_{-r+1}=0}^{\infty} \cdots \sum_{t_{c-1}=0}^{\infty} \boxed{T_{r,c}(t_{-r+1},\dots,t_{-1}; t_0,\dots,t_{c-1}; n)} \cdot z_{-r+1}^{t_{-r+1}} \cdots z_{-1}^{t_{-1}} z_0^{t_0} \cdots z_{c-1}^{t_{c-1}} \cdot q^n$$

:= no. of plane ptns. of n with $\le r$ rows and $\le c$ columns and with i-trace t_i

THEOREM: [Emden R. Gansner, 1981]

$$\boxed{\mathcal{P}_{r,c}(z_{-r+1},\dots,z_{-1}; z_0,\dots,z_{c-1}; q) = \prod_{i=1}^{r}\prod_{j=1}^{c}\left(1 - z_{-i+1} z_{-i+2} \cdots z_{j-1}\, q^{i+j-1}\right)^{-1}}$$

part 4

Res O

THE RESULTS SUMMARIZED

Def.: ~~Let~~ $\mathbb{X} = (x_{i,j})$:

$$p_{m,n}\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{pmatrix} = p_{m,n}(\mathbb{X})$$

$$:= \sum_{\boxed{(a_{i,j}) \in \mathbb{P}_{m,n}}} x_{1,1}^{a_{1,1}} \cdots x_{1,n}^{a_{1,n}} \, x_{2,1}^{a_{2,1}} \cdots x_{2,n}^{a_{2,n}} \times \cdots x_{m,1}^{a_{m,1}} \cdots x_{m,n}^{a_{m,n}}$$

$$\boxed{(a_{i,j}) = \left(\begin{array}{ccccccc} a_{1,1} & \to & a_{1,2} & \to & \cdots & \to & a_{1,n} \\ \downarrow & & \downarrow & & & & \downarrow \\ a_{2,1} & \to & a_{2,2} & \to & \cdots & \to & a_{2,n} \\ \downarrow & & \downarrow & & & & \downarrow \\ \vdots & & \vdots & & \ddots & & \vdots \\ \downarrow & & \downarrow & & & & \downarrow \\ a_{m,1} & \to & a_{m,2} & \to & \cdots & \to & a_{m,n} \end{array}\right) \in \mathbb{P}_{m,n}}$$

Thm.: Let $\mathbb{X}_0 := 1$, $\mathbb{X}_k := x_1 \cdots x_k$ $(k \geq 1)$.

For $m, n \geq 0$:

Ro 26

$$p_{m+1,n+1}\begin{pmatrix} x_n & x_{n-1} & x_{n-2} & \cdots & x_1 & z_0 \\ x_{n+1} & x_n & x_{n-1} & \cdots & x_2 & z_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{n+m} & x_{n+m-1} & x_{n+m-2} & \cdots & x_{m+1} & z_m \end{pmatrix}$$

$$= \prod_{k=0}^{n-1} \frac{1}{\left(1-\frac{\mathbb{X}_n}{\mathbb{X}_k}\right)\left(1-\frac{\mathbb{X}_{n+1}}{\mathbb{X}_k}\right)\cdots\left(1-\frac{\mathbb{X}_{n+m}}{\mathbb{X}_k}\right)}$$

$$\times\, Q^{\{\mathbb{X}_0, \ldots, \mathbb{X}_{m-1}\}}_{\{\mathbb{X}_n, \ldots, \mathbb{X}_{n+m}\}}\left(\frac{z_0}{\mathbb{X}_0}, \frac{z_1}{\mathbb{X}_1}, \ldots, \frac{z_m}{\mathbb{X}_m}\right)$$

Res 3a

$$\left.\begin{matrix} 0 \\ z_1 \\ \vdots \\ z_m \end{matrix}\right)$$

$$\times\, Q^{\{\overline{X}_0, \dots, \overline{X}_{m-1}\}}_{\{\overline{X}_n, \dots, \overline{X}_{n+m}\}}\left(\frac{0}{\overline{X}_0}, \frac{z_1}{\overline{X}_1}, \dots, \frac{z_m}{\overline{X}_m}\right)$$

NOTE.

Res 3b

$$Q^{\{\overline{X}_0, \dots, \overline{X}_{m-1}\}}_{\{A_0, \dots, A_m\}}(0, y_1, \dots, y_m) =$$

Cor. Let $Y_k := q^k x_{c-k} \cdots x_{c-1}$. For $r, c \geq 0$: Rs 4

$$P_{r+1,c+1}\begin{pmatrix} q x_0 & q x_1 & q x_2 & \cdots & q x_{c-1} & q x_c z_0 \\ q x_{-1} & q x_0 & q x_1 & \cdots & q x_{c-2} & q x_{c-1} z_1 \\ q x_{-2} & q x_{-1} & q x_0 & \cdots & q x_{c-3} & q x_{c-2} z_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ q x_{-r} & q x_{-r+1} & q x_{-r+2} & \cdots & q x_{c-r-1} & q x_{c-r} z_r \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} q^n \sum_{\substack{t_{-r}, \dots, t_c \geq 0 \\ a_0 \geq \cdots \geq a_r \geq 0}} T_{r+1,c+1}(t_{-r}, \dots, t_{-1}, t_0, \dots, t_c; a_0, \dots, a_r; n)$$

$$\times\, x_{-r}^{t_{-r}} \cdots x_c^{t_c} \cdot z_0^{a_0} z_1^{a_1} \cdots z_r^{a_r}$$

$\boxed{\text{no. of } \pi = (a_{ij}) \in P_{r+1,c+1} \text{ such that } |\pi| = n, \ \text{trace}_k(\pi) = t_k, \text{ and } a_{i,c+1} = a_{i-1}}$

$$= \prod_{i=1}^{r+1} \prod_{j=1}^{c} \frac{1}{1 - x_{-i+1} \cdots x_{j-1} q^{i+j-1}}$$

$$\times\, Q^{\{Y_0, \dots, Y_{r-1}\}}_{\{Y_c, \dots, Y_{c+r}\}}\left(q x_c z_0, \frac{z_1}{Y_0}, \frac{z_2}{Y_1}, \dots, \frac{z_r}{Y_{r-1}}\right)$$

part 5

THE RATIONAL FUNCTIONS $Q^{\mathbb{X}}/A$

Lagrange Symmetrization (e.g., A. Lascoux [CBMS-AMS LN, 2003])

(Ex.) Given: $\boxed{f = f_{\{A_0,A_1\}}(z)} \in K(A_0,A_1,z)$,

symmetric in A_0 and A_1.

Then: $(\mathbb{A} := \{A_0, A_1, A_2\})$

$$\boxed{L(f) := \sum_{i=0}^{2} \frac{f_{\mathbb{A}\setminus\{A_i\}}(A_i)}{\prod\limits_{A' \in \mathbb{A}\setminus\{A_i\}} (A_i - A')}} \in K(A_0,A_1,A_2)$$

$$= \frac{f_{\{A_2,A_1\}}(A_0)}{(A_0-A_1)(A_0-A_2)} + \frac{f_{\{A_0,A_2\}}(A_1)}{(A_1-A_0)(A_1-A_2)} + \frac{f_{\{A_0,A_1\}}(A_2)}{(A_2-A_0)(A_2-A_1)}$$

is symmetric in A_0, A_1, and A_2.

Definition of $Q_{\mathbb{A}}^{\mathbb{X}} \in \mathbb{Q}(\mathbb{A}, \mathbb{X}, z_0, \dots, z_n)$

$$\mathbb{A} = \{A_0, \dots, A_n\}, \quad \mathbb{X} = \{X_0, \dots, X_{n-1}\}$$

$$\boxed{n=0} \quad Q_{\mathbb{A}}^{\mathbb{X}}(z_0) := \frac{1}{1-A_0 z_0};$$

$$\boxed{n \geq 1} \quad Q_{\mathbb{A}}^{\mathbb{X}}(z_0, \dots, z_n) := \frac{(-1)^n A_0 \cdots A_n}{1 - A_0 \cdots A_n z_0 \cdots z_n} \cdot \mathcal{L}(f)$$

where for $i \in \{0, \dots, n\}$:

$$f_{\mathbb{A} \setminus \{A_i\}}(z) := \frac{1}{z} \prod_{j=0}^{n-1} \left(1 - \frac{z}{X_j}\right) \times Q_{\mathbb{A} \setminus \{A_i\}}^{\mathbb{X} \setminus \{X_{n-1}\}}(z_0, \dots, z_{n-1})$$

NOTE: Similarly we define rat. fcts.

$$R_{\mathbb{A}}^{\mathbb{X}}(w_0, \dots, w_n; z_0, \dots, z_n).$$

□

NOTE: Both $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$ are symmetric in the $\mathbb{A}$ variables.

□

A Crucial Relation

($\mathbb{A} = \{A_0,\dots,A_n\}$, $\mathbb{X} = \{X_0,\dots,X_{n-1}\}$,
additional variables $A_{n+1}, w_0,\dots,w_{n+1}, z_0,\dots,z_n$)

$$\boxed{\begin{aligned}&\underset{\geq}{\Omega}\, \frac{Q^{\mathbb{X}}_{\mathbb{A}}(w_0\lambda_0,\dots,w_n\lambda_n)}{1-A_0\cdots A_{n+1}w_0\cdots w_{n+1}\lambda_0\cdots\lambda_n} \prod_{k=0}^{n}\left(1-\frac{z_0\cdots z_k}{\lambda_0\cdots\lambda_k}\right)^{-1} \\ &= \frac{1}{1-A_{n+1}w_{n+1}}\Big[R^{\mathbb{X}}_{\mathbb{A}}(w_0,\dots,w_n;z_0,\dots,z_n) \\ &\qquad - A_{n+1}w_{n+1}\, R^{\mathbb{X}}_{\mathbb{A}}(w_0,\dots,w_{n-1},w_nw_{n+1}A_{n+1};z_0,\dots,z_n)\Big]\end{aligned}}$$

Proof. by elementary $\underset{\geq}{\Omega}$ elimination. $\square$

SKETCH OF THE PROOF

Basic Reduction Lemma $(m, n \geq 0)$

$$p_{m+1,n+1}\left(\begin{array}{ccc|c} x_{1,1} & \cdots & x_{1,n} & z_0 \\ \vdots & & \vdots & \vdots \\ x_{m+1,1} & \cdots & x_{m+1,n+1} & z_m \end{array}\right)$$

$$= \Big(1 - z_0 \cdots z_m \prod_{\substack{1 \leq i \leq m+1 \\ 1 \leq j \leq n}} x_{i,j}\Big)^{-1}$$

$$\times \underset{\geq}{\Omega}\, p_{m+1,n}\left(\begin{array}{ccc|c} x_{1,1} & \cdots & x_{1,n-1} & \lambda_0 x_{1,n} \\ \vdots & & \vdots & \vdots \\ x_{m,1} & \cdots & x_{m,n-1} & \lambda_{m-1} x_{m,n} \\ x_{m+1,1} & \cdots & x_{m+1,n-1} & x_{m+1,n} \end{array}\right)$$

$$\times \prod_{i=1}^{m} \left(1 - \frac{z_0 \cdots z_{i-1}}{\lambda_0 \cdots \lambda_{i-1}}\right)^{-1}$$

PROOF: Immediate from crude gen. fn.

$$p_{m+1,n+1} = \underset{\geq}{\Omega}\,(\ldots).$$

□

part 7.

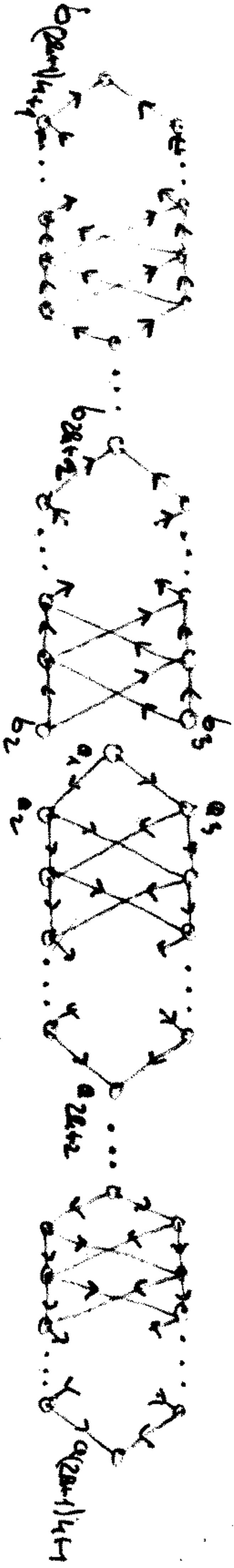
Conclusion

- Moll : algorithms can be combined to new methods for proving
- PA : — " — and also for mathematical discovery

Concluding Example (from G.E. Andrews & PP, "MacMahon's Partition Analysis XI : Broken Diamonds and Modular Forms", Acta Arithm. 126 (2007) ;

M4
C2

The broken k-diamond of length $2n$



$D_{n,k} :=$

$$\sum q^{a_1+\dots+a_{(2k+1)n+1}+b_1+\dots+b_{(2k+1)n+1}}$$

$$\boxed{D_{\infty,k} = \sum_{j=0}^{\infty} \Delta_k(j)\, q^j}$$

M5
C3

Theorem

$$D_{\infty,k} = \prod_{j=1}^{\infty} \frac{1+q^j}{(1-q^j)^2 (1+q^{(2k+1)j})}$$

$$= q^{-\frac{k}{12}} \frac{\eta(2\tau)\,\eta((2k+1)\tau)}{\eta(\tau)^3\,\eta((4k+2)\tau)}$$

where

$$q = e^{2\pi i \tau}$$

and

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{e=1}^{\infty} (1-q^e)$$

(Dedekind's η-function)

M6
C4a

Taylor series expansions of modular forms often have interesting arithmetical properties.

Thm.: $\boxed{\Delta_1(2m+1) \equiv 0 \pmod 3}$

Conj.1 $\Delta_2(10m+2) \equiv 0 \pmod 2$ *)

Conj.2 $\Delta_2(25m+14) \equiv 0 \pmod 5$ **)

Conj.3 For $m \geq 0$, if $3 \nmid n$ then:
$\Delta_2(625m+314) \equiv 0 \pmod{5^2}$

*) Settled by M. Hirschhorn & J. Sellers

**) Settled by Song Heng Chan

C4
6