Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Grid polygons from permutations and their enumeration by the kernel method

Toufik Mansour and Simone Severini

Abstract. A *grid polygon* is a polygon whose vertices are points of a grid. We define an injective map between permutations of length n and a subset of grid polygons on n vertices, which we call *consecutive-minima polygons*. By the *kernel method*, we enumerate sets of permutations whose consecutive-minima polygons satisfy specific geometric conditions. We deal with 2-variate and 3-variate generating functions.

1. Introduction

The enumeration of permutations that satisfy certain constraints has recently attracted interest (see *e.g.* [**1, 3, 4, 7, 8, 12, 13, 14, 15**]). In particular, permutation patterns have been extensively studied over the last decade (see for instance [**4**] and references therein). The tools involved in these works include generating trees (with either one or two labels), combinatorial approaches, recurrences relations, enumeration schemes, scanning elements algorithms, *etc.*

Permutations are traditionally associated to a number of combinatorial and algebraic objects, like matrices, trees, posets, graphs, *etc.* (see *e.g.* [**5, 11**]). The purpose of this work is to begin a study of the interplay between permutations and polygons. A practical motivation comes from computational geometry, where the complexity of algorithms for polygons is an important subject [**6**]. Of course, it is intuitive that imposing combinatorial constraints on geometric objects commonly reduces generality; on the other side, techniques from combinatorics may provide a fertile background for the design of algorithms, even if the analysis is restricted to toy-cases.

Here we associate permutations to a subset of the grid polygons and enumerate sets of permutations whose polygons satisfy specific geometric conditions. Clearly, there are many potential ways to associate permutations to polygons. Each way presumably has a special feature which helps to underline some particular property of the permutations. If we want to keep a one-to-one correspondence, this arbitrariness is materialized in two points:

- Of all possible polygons associated to a given permutation, we choose the one with a fixed extremal property, for example, the polygon with minimum area or perimeter.
- We decide how to construct a polygon according to some chosen rule. The rule should guarantee the association of each permutation to a single polygon, unequivocally.

2000 *Mathematics Subject Classification.* 05A05, 05A15.
Key words and phrases. Grid polygons, kernel method, permutations.

We opt here for the second approach, as it is formalized in what follows. A *grid of side* n is an $n \times n$ array containing n^2 points, n in each row and each column. The distance between two closest points in the same row or in the same column is usually taken to be 1 unit. A *permutation of length* n is a complete ordering of the elements of the set $[n] = \{1, ..., n\}$. We associate a grid of side n, denoted by L_π, to a permutation π of length n. If the permutation takes i to $j = \pi_i$, we mark the point (i, j) of the grid, that is the point in the row i and the column j. For example, the grid L_π represented in Figure 1 is associated to the permutation $\pi = 4523176$.

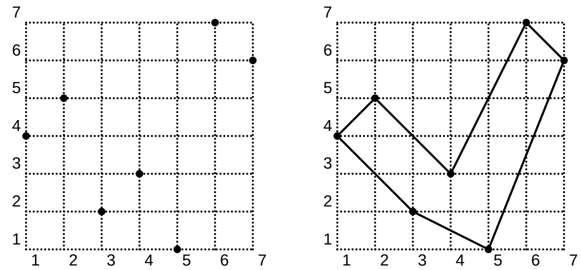

FIGURE 1. The grid $L_{4523176}$ and the consecutive-minima polygon $P_{4523176}$.

A *grid polygon* on n vertices is a polygon whose vertices are n points of a grid. A *permutation polygon* on n vertices is a grid polygon with the following properties: the side of the grid is n; in every row and every column of the grid there is one and only one vertex of the polygon. It is intuitive to observe that a permutation can be associated to more than one polygon depending on how we connect the marked points of the grid. Let L_π be a grid of side n of a permutation π.

- A point (i, j) is said to be a *left-right minimum* of L_π if there is no point (i', j') of L_π such that $i' < i$ and $j' < j$.
- A point (i, j) is said to be a *right-left minimum* of L_π if there is no point (i', j') of L_π such that $i' > i$ and $j' < j$.
- A point (i, j) is said to be a *source* of L_π if either $i = 1$, $i = n$, or (i, j) is not a left-right minimum or a right-left-minimum.

We say that two points (i, j) and (i', j') of the grid L_π (resp. of the set of left-right minima, right-left minima, sources) are *consecutive* if there is no point (a, b) in the set of (resp. left-right minima's, right-left minima's, sources) such that $i < a < i'$ or $i' < a < i$. For example, the left-right-minima of $L_{4523176}$ are $(1, 4)$, $(3, 2)$ and $(5, 1)$; the right-left-minima are $(7, 6)$ and $(5, 1)$; the sources are $(1, 4)$, $(2, 5)$, $(4, 3)$, $(6, 7)$, and $(7, 6)$ (see Figure 1).

DEFINITION 1.1. *A* consecutive-minima polygon *(in what follows just* polygon*) of a permutation π, denoted by P_π, is a permutation polygon in which two vertices $a = (i, j)$ and $b = (i', j')$, $i < i'$, are connected if one of the following conditions is satisfied:*

- *a and b are consecutive left-right minima of L_π;*
- *a and b are consecutive right-left-minima of L_π;*
- *a and b are consecutive sources of L_π.*

In such a context (a, b) is called an edge *of P_π.*

For example, the polygon P_π for all $\pi \in S_4$ is represented in Figure 2. In the next sections we will deal with several questions about the number of polygons on n vertices that satisfy a certain set of conditions. In order to do so, we first need to give some further definition. Let P be a

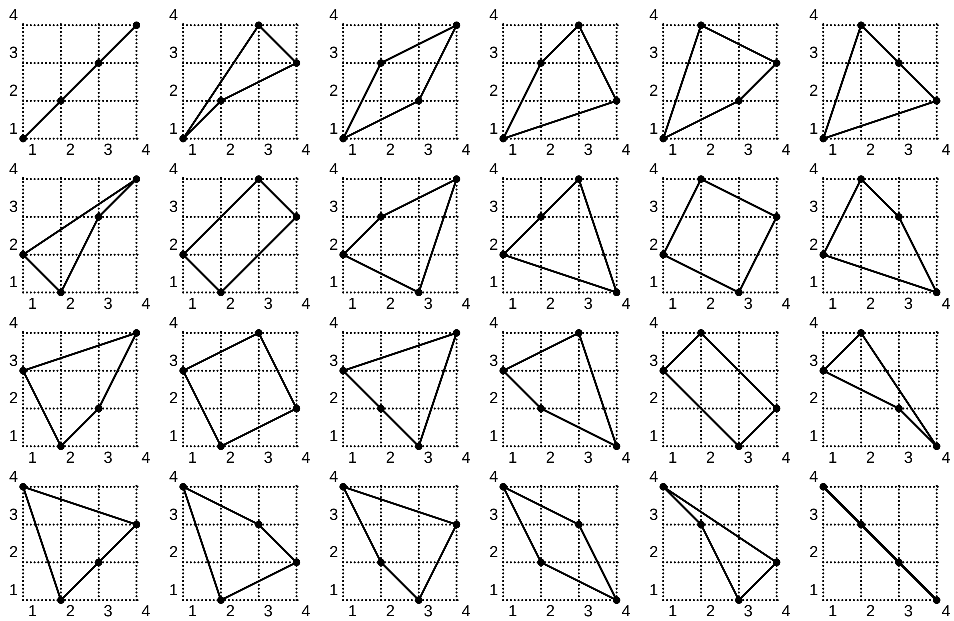

FIGURE 2. The consecutive-minima polygons on 4 vertices.

polygon, an edge $((i,j),(i',j'))$, $i < i'$, of P is said to be *increasing* (resp. *decreasing*) if $j < j'$ (resp. $j > j'$). A *path* of P is a sequence $(a_0,a_1),(a_1,a_2),\ldots,(a_{s-1},a_s)$ of edges of P. A *face* of P is either a maximal path of increasing edges or a maximal path of decreasing edges. For example, there are exactly 4, 16, 4 polygons on 4 vertices of exactly two, three, four faces, respectively. This can be observed in Figure 2.

A polygon is said to be *k-faces* if it has exactly k faces. We will present an explicit formula for the number of k-faces polygons on n, where $k = 2,3,4$. It seems to be a challenging question to find an explicit formula for any k.

The technique considered in this paper makes use of generating functions to convert recurrence relations to functional equations which are then solved by the *kernel method* as described in [**2**]. It may be interesting to remark that the kernel method is a routine approach when dealing with 2-variate generating functions. However, for functional equations with more than two variables there is no systematic approach. Bousquet-Mélou [**3**] enumerates four different pattern avoiding classes of permutations by using the kernel method with 3-variate generating functions. We suggest here another class, namely the square permutations, see Section 5. To these permutations corresponds a functional equation defining 3-variate generating functions. Interestingly, such permutations are not immediately related to pattern avoidance.

The remainder of the paper is composed of five sections. In Section 2, we make some general observations about consecutive-minima polygons. We characterize convex polygons and enumerate polygons on n vertices with maximum number of faces. In Section 3, 4 and 5, we enumerate 2-faces, 3-faces and 4-faces polygons, respectively. Section 6 is a list of open problems.

2. Some general observations

2.1. Convexity. A polygon is *convex* if the internal angle formed at each vertex is smaller than 180°. Give a sequence $a_1, a_2, \dots, a_n$, we say that the subsequence $a_{i_1}, \dots, a_{i_m}$ with $i_1 < i_2 < \dots < i_m$ is *fast-growing* if

$$\frac{a_{i_j} - a_{i_{j+1}}}{i_j - i_{j+1}} < \frac{a_{i_{j+1}} - a_{i_{j+2}}}{i_{j+1} - i_{j+2}},$$

for any $j = 1, 2, \dots, m-2$, and *slow-growing* if

$$\frac{a_{i_j} - a_{i_{j+1}}}{i_j - i_{j+1}} > \frac{a_{i_{j+1}} - a_{i_{j+2}}}{i_{j+1} - i_{j+2}},$$

for any $j = 1, 2, \dots, m-2$.

The consecutive-minima polygon P_π is convex if and only if

- the subsequence of left-right-minima of π is fast-growing;
- the subsequence of right-left-maxima of π is fast-growing;
- the subsequence L_π of the sources of π is slow-growing.

2.2. Number of faces. The number of 1-face polygons on n vertices is exactly 2, that is the polygons corresponding to the two permutations $12\dots n$ and $n\dots 1$. The number of different consecutive-minima polygon on n vertices is exactly n. To clarify this observation, let P be any consecutive-minima polygon on n vertices, such that k is maximal if each face is a segment connecting two vertices, and thus $k \le n$. It is not difficult to show that there exists at least one k-face consecutive-minima polygon for any $k = 1, 2, \dots, n$. Let

$$\phi^{2k} = 2436587\dots(2k-2)(2k-3)(2k)(2k+1)(2k+2)\dots n1(2k-1)$$

and

$$\phi^{2k+1} = 2436587\dots(2k)(2k-1)1(2k+1)(2k+2)\dots n$$

be two permutations of length n for all $k \ge 2$, $\phi^1 = 123\dots n$, $\phi^2 = 13245\dots n$, and $\phi^3 = 2134\dots n$, then we can see that $P_{\phi^{k'}}$ has exactly k'-faces. Hence, for any $n \ge 4$ we have n different consecutive-minima polygon on n vertices.

What about permutations with maximum number of faces? Let π be a permutation of length $n \ge 3$. Since the maximum number of faces of P_π is n, one of the following holds:

(1) π is an alternating permutation (π is said to be *alternating* if either $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots \pi_n$ or $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots \pi_n$) such that $\pi_1 = 1$ and $\pi_n = 2$;
(2) π is an alternating permutation such that $\pi_1 = 2$ and $\pi_n = 1$;
(3) Removing the letter $\pi_i = 1$, $2 \le i \le n-1$, from π then

$$(\pi_1 - 1)\dots(\pi_{i-1} - 1)(\pi_{i+1} - 1)\dots(\pi_n - 1)$$

is a permutation satisfying either (1) or (2).

Let E_n be the the number of alternating permutations of length n (see [**10**, A000111] and references therein). Thus the number of permutations satisfying either (1) or (2) is exactly E_{n-2} if n is odd, otherwise it is 0. Hence, we can state the following result.

PROPOSITION 2.1. *The number of polygons on n vertices with maximum number of faces (n faces) is given by $2E_{n-2}$ if n odd, and $2(n-2)E_{n-3}$ if n even.*

3. Enumeration of two faces polygons

A permutation is said to be *parallel* if its polygon has exactly two faces. For instance, there are 4 parallel permutations of length 4, namely 1234, 1324, 4231 and 4321. We denote the set of all parallel permutations of length n by $\mathcal{P}_n$.

We say that a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ *avoids* $\tau = \tau_1\tau_2\ldots\tau_k \in S_k$ (or τ-*avoiding*) if there no subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_p} < \pi_{i_q}$ if and only if $\tau_p < \tau_q$ for all $1 \leq p < q \leq k$. For example, the permutation 4132 avoids 231 but it does not avoid 123. For any pattern τ we write $S_n(\tau)$ to denote the permutations of S_n which avoid τ.

We will count the number of parallel permutations $\pi \in S_n$ whose faces are both increasing, that is $\pi_1 = 1$ and $\pi_n = n$. This number can then be doubled to find the total number of parallel permutations in S_n. On the other hand, it is easy to check that $\pi = \pi_1\pi_2\cdots\pi_n$ is a parallel permutation with two increasing faces if and only if π avoids 321, $\pi_1 = 1$ and $\pi_n = n$. Using [**11**, Exercise 6.19(ee)], we obtain the following result.

Theorem 3.1. *The number of parallel permutations π of length n is $\frac{2}{n-1}\binom{2n-4}{n-2}$, for all $n \geq 2$.*

4. Enumeration of three faces polygons

A permutation π is said to be *triangular* if it begins at letter 1 and its polygon P_π has at most 3 faces. For example, there exist $1, 1, 2, 6, 20$ triangular permutations of length $1, 2, 3, 4, 5$, respectively. We denote the set of all triangular permutations of length n by $\mathcal{T}_n$. Given $a_1, a_2, \ldots, a_d \in \mathbb{N}$, we define

$$t_{n;a_1,a_2,\ldots,a_d} = \#\{\pi_1\pi_2\ldots\pi_n \in_n | \ \pi_1\pi_2\ldots\pi_d = a_1a_2\ldots a_d\},$$

The cardinality of the set $\mathcal{T}_n$ is denoted by t_n.

Theorem 4.1. *The number of triangular permutations of length $n+2$ is $\binom{2n}{n}$. Moreover, the ordinary generating function $t(v;x) = \sum_{n\geq 2}\sum_{a=2}^{n} t_{n;1,a}v^{a-2}x^n$ is given by*

$$\frac{x^2(1-v)(1-xv)^2}{(1-2xv)(1-v+xv^2)} + \frac{x^3}{1-v+xv^2}\cdot\frac{1}{\sqrt{1-4x}}.$$

Proof. From the definitions, we have that

$$t_n = t_{n;1} = t_{n;1,2} + t_{n;1,3} + \cdots + t_{n;1,n}.$$

For all $a = 3, 4, \ldots, n-1$,

$$t_{n;1,a} = t_{n;1,a,2} + \sum_{j=a+1}^{n} t_{n;1,a,j} = t_{n-1;1,a-1} + \sum_{j=a+1}^{n} t_{n-1;1,j-1} = \sum_{j=a-1}^{n-1} t_{n-1;1,j}, \tag{4.1}$$

with the initial conditions $t_{n;1,2} = t_{n-1;1}$ and $t_{n;1,n} = 2^{n-3}$. To see that

$$t_{n;1,n} = 2^{n-3} \tag{4.2}$$

we consider the following equation $t_{n;1,n} = t_{n;1,n,2} + t_{n;1,n,n-1} = t_{n-1;1,n-1} + t_{n-1;1,n-1} = 2t_{n-1;1,n-1}$ for all $n \geq 4$, and $t_{3;1,3} = 1$ which implies that $t_{n;1,n} = 2^{n-3}$ as claimed. To see (4.1), let $\pi = \pi_1\pi_2\cdots\pi_n$ be any triangular permutation such that $\pi_1 = 1$ and $\pi_2 = a$, and let us consider the possibly values of π_3. Since π has at most three faces we obtain that either $\pi_3 = 2$ or $\pi_3 \geq \pi_2$:

- In the case $\pi_3 = 2$ we have that π is a triangular permutation if and only if

  $$\pi' = 1(a-1)(\pi_4 - 1)\cdots(\pi_n - 1)$$

  is a triangular permutation.

- In the case $\pi_3 = j > a$ we have that π is a triangular permutation if and only if
$$\pi' = 1(j-1)\pi'_4 \cdots \pi'_n$$
is a triangular permutation, where $\pi'_i = \pi_i$ if $\pi_i \le a-1$, otherwise $\pi'_i = \pi_i - 1$.

Summing over all possibly values of π_3 we obtain (4.1). To solve (4.1), we need to define $t_n(v) = \sum_{a=2}^{n} t_{n;1,a} v^{a-2}$. Thus, multiplying the above recurrence relation by v^{a-2} and summing over $a = 3, 4, \ldots, n-1$ we obtain that

$$t_n(v) = t_{n-1}(1) + 2^{n-3}v^{n-2} + \sum_{a=3}^{n-1}\sum_{j=a-1}^{n-1} t_{n-1;1,j}v^{a-2},$$

which is equivalent to

$$t_n(v) = t_{n-1}(1) + 2^{n-4}v^{n-2} + \frac{v}{1-v}(t_{n-1}(1) - vt_{n-1}(v)),$$

for $n \ge 4$. Let $t(v;x) = \sum_{n\ge 2} t_n(v)x^n$. Multiplying the above recurrence relation with x^n, summing over all possibly $n \ge 4$, and using the initial conditions $t_2(v) = 1$ and $t_3(v) = 1 + v$, we obtain the following functional equation

$$t(v;x) = \frac{xv}{1-v}(t(1;x) - vt(v;x)) + xt(1;x) - \frac{(1-xv)^2x^2}{1-2xv},$$

which is equivalent to

$$\left(1 + \frac{xv^2}{1-v}\right) t(v;x) = \frac{x}{1-v}t(1;x) - \frac{(1-xv)^2x^2}{1-2xv}.$$

This type of equation can be solved using the kernel method. Substitute $v = \frac{1-\sqrt{1-4x}}{2x}$ in the above functional equation to get $t(1;x) = \frac{x^2}{\sqrt{1-4x}}$, that is, the number of triangular permutations of length n is exactly $\binom{2n-4}{n-2}$, as required. Moreover, substituting the expression of $t(1;x)$ in the functional equation, we get an explicit formula for $t(v;x)$, as claimed. □

As a corollary of Theorem 3.1 and Theorem 4.1 we get the following.

COROLLARY 4.2. *The number of polygons on n vertices with exactly three faces is*

$$\frac{4(n-2)}{n-1}\binom{2n-4}{n-2},$$

for all $n \ge 2$.

PROOF. Theorem 3.1 and Theorem 4.1 give that the number of permutations π of length n that begin at letter 1 and its polygon P_π has exactly three faces is $\frac{n-2}{n-1}\binom{2n-4}{n-2}$. If P_π has exactly three faces then also $P_{\pi'}$ and $P_{\pi''}$ have exactly three faces, where π' is the complement of π and π'' is the reversal of π. (Recall that the *reversal* of a permutation $\pi_1\pi_2\ldots\pi_n$ is $\pi_n\ldots\pi_2\pi_1$; the *complement* of is the permutation $(n+1-\pi_1)(n+1-\pi_2)\ldots(n+1-\pi_n)$). From this fact, we obtain that the number of polygons on n vertices with exactly three faces is four times the number of permutations π of length n that begin at letter 1 and whose polygon P_π has exactly three faces. This number is $\frac{4(n-2)}{n-1}\binom{2n-4}{n-2}$, as required. □

5. Enumeration of four faces polygons

A permutation π is said to be *square* if the subsequence of the sources of L_π lies on at most two faces of P_π. For example, there exists $1, 2, 6, 24, 104, 464, 2088$ square permutations of length $1, 2, 3, 4, 5, 6, 7$, respectively. We denote the set of all square permutations of length n by $\mathcal{Q}_n$. Given $a_1, a_2, \dots, a_d \in \mathbb{N}$, we define

$$q_{n;a_1,a_2,\dots,a_d} = \#\{\pi_1\pi_2\dots\pi_n \in \mathcal{Q}_n \mid \pi_1\pi_2\dots\pi_d = a_1a_2\dots a_d\},$$

The cardinality of the set set $\mathcal{Q}_n$ by q_n. Clearly, a triangular permutation is a square permutation. We derive an explicit formula for the number of square permutations of length n as follows.

Theorem 5.1. *The ordinary generating function for the number of square permutations of length n is given by*

$$1 + x + \frac{2(1-3x)x^2}{(1-4x)^2} - \frac{4x^3}{(1-4x)^{3/2}}.$$

Moreover, the number of square permutations of length n is

$$2(n+2)4^{n-3} - 4(2n-5)\binom{2n-6}{n-3},$$

for all $n \geq 3$.

Proof. From the symmetry arising in the construction of square permutations we have that for all $n \geq a > b \geq 1$,

$$q_{n;a,b} = q_{n;n+1-a,n+1-b} \text{ and } q_{n;a,b} = q_{n;b,a}. \tag{5.1}$$

Define $Q_n(u,v) = \sum_{a=1}^n \sum_{b=1}^n q_{n;a,b} v^{a-1} u^{b-1}$, for all $n \geq 2$, and $Q(u,v;x) = \sum_{n\geq 0} Q_n(v,u)x^n$ to be the ordinary generating function for the sequence $Q_n(u,v)$. Thus, (5.1) gives

$$Q(v,u;x) = Q'(v,u;x) + Q'(u,v;x), \tag{5.2}$$

where

$$Q'(u,v;x) = \sum_{n\geq 2} Q'_n(v,u)x^n = \sum_{n\geq 2} x^n \sum_{a=2}^{n}\sum_{b=1}^{a-1} q_{n;a,b} v^{a-1} u^{b-1}.$$

To find an explicit formula for $Q'(1,1;x)$, which leads us to explicit formula for $Q(1,1,;x)$, the ordinary generating function for the number of square permutations of length n, we need to divide the generating function $Q'(u,v;x)$ into three parts. For all $n \geq a > b \geq 1$, define

$$\begin{aligned}
A(v;x) = \textstyle\sum_{n\geq 2} A_n(v)x^n &= \textstyle\sum_{n\geq 2} x^n \sum_{a=2}^n q_{n;a,1} v^{a-1},\\
B(v;x) = \textstyle\sum_{n\geq 2} B_n(v)x^n &= \textstyle\sum_{n\geq 3} x^n \sum_{b=2}^{n-1} q_{n;n,b} v^{b-1},\\
C(v,u;x) = \textstyle\sum_{n\geq 2} C_n(u,v)x^n &= \textstyle\sum_{n\geq 4} x^n \sum_{a=3}^{n-1}\sum_{b=2}^{i-1} q_{n;a,b} v^{a-1}u^{b-1}.
\end{aligned}$$

Clearly, for all $n \geq 2$, $Q'_n(v,u) = C_n(v,u) + v^{n-1}B_n(u) + A_n(v)$ and then

$$Q'(u,v;x) = C(v,u;x) + \frac{1}{v}B(u;xv) + A(v;x). \tag{5.3}$$

Expression for $A(v;x)$: First, we find an explicit formula for the ordinary generating function $A(v;x)$. From the definitions and (5.1), we have that

$$\begin{aligned}
&q_{n;2,1} = q_{n-1;1} = \textstyle\sum_{b=2}^{n-1} q_{n-1;1,b} = \sum_{b=2}^{n-1} q_{n-1;b,1} = A_{n-1}(1),\\
&q_{n;a,1} = q_{n;a,1,2} + \textstyle\sum_{b=a+1}^{n} q_{n;a,1,b} = q_{n-1;a-1,1} + \sum_{b=a+1}^{n} q_{n-1;1,b-1} = \sum_{b=a-1}^{n-1} q_{n-1;b,1},\\
&q_{n;n,1} = q_{n;n,1,2} + q_{n;n,1,n-1} = 2q_{n-1;n-1,1}.
\end{aligned}$$

Using $q_{3;3,1}=1$ and the recurrence relation for the sequence $q_{n;n,1}$, we obtain that, for all $n\geq 3$,

$$q_{n;n,1}=2^{n-3}. \tag{5.4}$$

Multiplying by v^{a-1} and summing over all $a=3,4,\dots,n-1$, we obtain that

$$\begin{aligned} A_n(v) &= vA_{n-1}(1)+\textstyle\sum_{a=3}^{n-1} v^{a-1}\sum_{j=a-1}^{n-1} q_{n-1;j,1}+q_{n;n,1}v^{n-1},\\ &= vA_{n-1}(1)+\textstyle\sum_{a=2}^{n-2} q_{n-1;a,1}\frac{v^2-v^{i+1}}{1-v}+\frac{v^2-v^{n-1}}{1-v}q_{n-1;n-1,1}+q_{n;n,1}v^{n-1}. \end{aligned}$$

Then (5.4) leads us to

$$A_n(v)=vA_{n-1}(1)+\frac{v^2}{1-v}(A_{n-1}(1)-2^{n-4}-A_{n-1}(v)+2^{n-4}v^{n-2})+2^{n-4}\frac{v^2-v^{n-1}}{1-v}+2^{n-3}v^{n-1},$$

which is equivalent to

$$A_n(v)=vA_{n-1}(v)+\frac{v^2}{1-v}(A_{n-1}(1)-A_{n-1}(v))+2^{n-4}v^{n-4},$$

for all $n\geq 4$, with initial conditions $A_2(v)=v$ and $A_3(v)=v+v^2$. Writing the above recurrence relation in terms of generating functions,

$$A(v;x)-(v+v^2)x^3-vx^2=vx(A(1;x)-x^2)+\frac{xv^2}{1-v}(A(1;x)-x^2-A(v;x)+vx^2)+\frac{v^3x^4}{1-2vx}.$$

Equivalently,

$$\left(1+\frac{v^2x}{1-v}\right)A(v;x)=vx^2+\frac{v^3x^4}{1-2vx}+\frac{vx}{1-v}A(1;x).$$

This type of equation can be solved systematically using the kernel method. We substitute $v=\frac{1-\sqrt{1-4x}}{2x}$ in the above functional equation to get $A(1;x)=\frac{x^2}{\sqrt{1-4x}}$ and then

$$A(v;x)=\frac{1}{1-v+v^2x}\left(v(1-v)x^2+\frac{v^3(1-v)x^4}{1-2vx}+\frac{vx^3}{\sqrt{1-4x}}\right). \tag{5.5}$$

Expression for $B(v;x)$: Using the symmetry on the set of square permutations, see (5.1), we obtain that

$$B(v;x)=\sum_{n\geq3}x^n\sum_{j=2}^{n-1}q_{n;j,n}v^{j-1}=\sum_{n\geq3}x^n\sum_{j=2}^{n-1}q_{n;n+1-j,1}v^{j-1}=\sum_{n\geq3}x^n\sum_{j=2}^{n-1}q_{n;j,1}v^{n-j},$$

and from the definition of the generating function $A(v;x)$ together with (5.4),

$$B(v;x)=\frac{1}{v}A(1/v;vx)-\frac{x^2(1-x)}{1-2x}.$$

It follows that

$$B(v;x)=\frac{vx^3(1-x)}{(1-2x)(1-v-x)}-\frac{x^3v^2}{(1-v-x)\sqrt{1-4vx}}. \tag{5.6}$$

Expression for $C(v,u;x)$: From the definitions and (5.1), for all $n-1\geq a>b\geq 2$,

$$\begin{aligned} q_{n;a,b} &= \sum_{j=1}^{b-1} q_{n;a,b,j} + \sum_{j=a+1}^{n} q_{n;a,b,j} \\ &= \sum_{j=1}^{b-1} q_{n-1;a-1,j} + \sum_{j=a+1}^{n} q_{n-1;j-1,b} \\ &= \sum_{j=1}^{b-1} q_{n-1;a-1,j} + \sum_{j=a}^{n-1} q_{n-1;j,b}. \end{aligned}$$

Thus, for all $n\geq 5$,

$$\begin{aligned} C_n(v,u) &= \sum_{a=3}^{n-1}\sum_{b=2}^{a-1}\left(\sum_{j=1}^{b-1} q_{n-1;a-1,j} + \sum_{j=a}^{n-1} q_{n-1;j,b}\right) v^{a-1}u^{b-1} \\ &= \sum_{a=2}^{n-2}\sum_{b=1}^{a-1} q_{n-1;a,b}\frac{u^j-u^i}{1-u}v^i + \sum_{a=3}^{n-1}\sum_{b=2}^{a-1} q_{n-1;a,b}\frac{v^j-v^i}{1-v}u^{j-1}. \end{aligned}$$

Therefore, by the definition of the sequences $A_n(v)$, $B_n(v)$ and $C_n(n,u)$ together with (5.1), for all $n\geq 5$,

$$\begin{aligned} &C_n(v,u) \\ &= \tfrac{vu}{1-u}(C_{n-1}(v,u)-C_{n-1}(vu,1)) + \tfrac{v}{1-v}(C_{n-1}(1,vu)-C_{n-1}(v,u)) \\ &+\tfrac{v}{1-v}(B_{n-1}(vu)-v^{n-2}B_{n-1}(u)) + \tfrac{uv}{1-u}(A_{n-1}(v)-A_{n-1}(vu)) - 2^{n-4}\tfrac{uv^{n-1}(1-u^{n-2})}{1-u}. \end{aligned}$$

By converting the above recurrence relation in terms of generating functions with the use of the initial condition $C_4(v,u)=2uv^2$ (this holds immediately from the definitions), we can write

$$\begin{aligned} C(v,u;x) &= 2uv^2x^4 + \tfrac{vux}{1-u}(C(v,u;x)-C(vu,1;x)) + \tfrac{vx}{1-v}(C(1,vu;x)-C(v,u;x)) \\ &+\tfrac{vx}{1-v}(B(vu;x)-vux^3) - \tfrac{v}{1-v}(B(u;vx)-v^3ux^3) \\ &+\tfrac{vux}{1-u}(A(v;x)-vx^2-v(1+v)x^3) - \tfrac{vux}{1-u}(A(vu;x)-vux^2-vu(1+vu)x^3) \\ &-\tfrac{2v^4ux^5}{(1-2vx)(1-u)} + \tfrac{2v^4u^4x^5}{(1-2vux)(1-u)}. \end{aligned}$$

It is well known that this type of functional equations with several variables are in general very hard to solve (see *e.g.* [**3**]). However, in our case, we are able to find an explicit formula for the ordinary generating function $C(1,1;x)$, as it is described below.

Explicit formula for $C(1,1;x)$: Substituting $u=v^{-1}$ in the above functional equation gives

$$\begin{aligned} &C(v,v^{-1};x) \\ &= 2vx^4 - \tfrac{vx}{1-v}(C(v,v^{-1};x)-C(1,1;x)) + \tfrac{vx}{1-v}(C(1,1;x)-C(v,v^{-1};x)) \\ &+\tfrac{vx}{1-v}(B(1;x)-x^3) - \tfrac{x}{1-v}(B(v^{-1};vx)-v^2x^3) - \tfrac{vx}{1-v}(A(v;x)-vx^2-v(1+v)x^3) \\ &+\tfrac{vx}{1-v}(A(1;x)-x^2-2x^3) + \tfrac{2v^4x^5}{(1-2vx)(1-v)} - \tfrac{2vx^5}{(1-2x)(1-v)}. \end{aligned}$$

This is equivalent to

$$\begin{aligned} &\left(1+\tfrac{2vx}{1-v}\right)C(v,v^{-1};x) \\ &\quad = -(1+x+vx)vx^3 + \tfrac{2vx}{1-v}C(1,1;x) + \tfrac{vx}{1-v}B(1;x) - \tfrac{x}{1-v}B(v^{-1};vx) \\ &\quad -\tfrac{vx}{1-v}A(v;x) + \tfrac{vx}{1-v}A(1;x) + \tfrac{2v^4x^5}{(1-2vx)(1-v)} - \tfrac{2vx^5}{(1-2x)(1-v)}, \end{aligned}$$

By taking $v = \frac{1}{1-2x}$ and using (5.5) and (5.6),

$$C(1,1;x) = \frac{2(3x-1)x^2}{(1-4x)^{3/2}} + \frac{2x^2(1-7x+15x^2-8x^3)}{(1-2x)/(1-4x)^2}.$$

Explicit formula for $Q(1,1;x)$: Equations (5.3), (5.5) and (5.6) give an explicit formula for $Q'(1,1;x)$, namely $Q'(1,1;x) = \frac{(1-3x)x^2}{(1-4x)^2} - \frac{2x^3}{(1-4x)^{3/2}}$. Hence, by (5.2), we obtain that $Q(1,1;x) = 2Q'(1,1;x)$ and the ordinary generating function for the number of square permutations of length n is given by $1 + x + 2Q'(1,1;x)$ (1 for the empty permutation and x for the permutation of length 1), as required. □

COROLLARY 5.2. *The number of polygons on n vertices with four faces such that the sources of the polygon lies on exactly two faces is given by*

$$2(n+2)4^{n-3} - 2(n+1)\binom{2n-4}{n-3}.$$

PROOF. The formula is obtained directly from Theorem 5.1 and Corollary 4.2. □

6. Open problems

In this paper we have used a technique based on the kernel method to solve functional equations for enumerating k-faces polygons on n vertices, where $k = 2, 3, 4$. The results suggest the following problems:

- The most important question in our context is to find an explicit formula for the number of k-faces polygons on n vertices for any k.
- Can we find a combinatorial interpretations for the formula

$$2(n+2)4^{n-3} - 2(n+1)\binom{2n-4}{n-3},$$

  the number of square permutations of length n.
- All questions about geometric properties of consecutive-minima polygons remain open (for example, maximal perimeter, maximal area, number of different polygons up to symmetries, *etc.*).

Acknowledgments. Part of this work has been carried out while the second author was visiting the Institut H. Poincaré (IHP) during the programme Quantum Information, Computation and Complexity (January 2006 - April 2006).

References

[1] M. H. Albert and M. D. Atkinson, The enumeration of simple permutations, *J. Integer Sequences* **6** (2003), Article 03.4.4.

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps, Generating functions for generating trees, *Discr. Math.* **246:1-3**, 2002, 29–55.

[3] M. Bousquet-Mélou, Four classes of pattern-avoiding permutations under one roof: generating trees with two labels, *Elect. J. Combin.* **9:2**, 2003, #R19.

[4] F. Ghassan and T. Mansour, Three-letter-pattern-avoiding permutations and functional equations, *Elect. J. Combin.* **13** (2006) #R51.

[5] M. C. Golumbic, *Algorithmic graph theory and perfect graphs*. Second edition. With a foreword by Claude Berge. Annals of Discrete Mathematics, 57. Elsevier Science B.V., Amsterdam, 2004.

[6] J. E. Goodman, J. Pach and E. Welzl (Eds.), *Combinatorial and computational geometry*. Mathematical Sciences Research Institute Publications, 52. Cambridge University Press, Cambridge, 2005.

[7] T. Mansour, The enumeration of permutations whose posets have a maximal element, *Adv. Appl. Math.* **37:4** (2007) 434–442.

[8] T. Mansour, On an open problem of Green and Losonczy: exact enumeration of freely braided permutations, *Discr. Math. and Theor. Comp. Sci.* **6:2** (2004), 461–470.

[9] T. Mansour, Combinatorial methods and recurrence relations with two indices, *J. Diff. Eq. and Appl.*, 2006.

[10] N. J. A. Sloane and S. Plouffe, The Encyclopedia of Integer Sequences, Academic Press, New York (1995).

[11] R. P. Stanley, *Enumerative combinatorics*. Volumes 1-2. With a foreword by Gian-Carlo Rota. Corrected reprint of the 1986 original. Cambridge Studies in Advanced Mathematics, 49. Cambridge University Press, Cambridge, 1997-1999.

[12] V. Vatter, Enumeration schemes, *Combin., Prob. and Computing*, to appear, http://math.rutgers.edu/∼ vatter/publications/wilfplus.

[13] J. West, Generating trees and the Catalan and Schröder numbers, *Discr. Math.* **146** (1995) 247–262.

[14] D. Zeilberger, A proof of Julian West's conjecture that the number of two stack-sortable permutations of length n is $2(3n)!/((n+1)!(2n+1)!)$, *Discr. Math.* **102** (1992) 85–93.

[15] D. Zeilberger, Enumeration schemes, and more importantly, their automatic generation, *Annals of Combin.* **2** (1998) 185–195.

Department of Mathematics, University of Haifa, Haifa 31905, Israel
E-mail address: toufik@math.haifa.ac.il

Institute for Quantum Computing and Department of Combinatorics and Optimization, University of Waterloo, Waterloo N2L 3G1, Canada
E-mail address: simoseve@gmail.com