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# Involutions avoiding the class of permutations in $\mathfrak{S}_k$ with prefix 12

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Abstract. An involution $\pi$ is said to be $\tau$-avoiding if it does not contain any subsequence having all the same pairwise comparisons as $\tau$. This paper concerns the enumeration of involutions which avoid a set $\mathcal{A}_k$ of subsequences increasing both in number and in length at the same time. Let $\mathcal{A}_k$ be the set of all the permutations $12\pi_3\dots\pi_k$ of length $k$. For $k=3$ the only subsequence in $\mathcal{A}_k$ is 123 and the 123-avoiding involutions of length $n$ are enumerated by the central binomial coefficients $\binom{n}{\lfloor n/2\rfloor}$. For $k=4$ we give a combinatorial explanation that shows the number of involutions of length $n$ avoiding $\mathcal{A}_4$ is the same as the number of symmetric Schröder paths of length $n-1$. For each $k\geq 3$ we determine the generating function for the number of involutions avoiding the subsequences in $\mathcal{A}_k$, according to length, first entry and number of fixed points.

## 1. Introduction

Let $[d]$ denote a totally ordered alphabet on $d$ letters, and let $\mathfrak{S}_n$ denote the set of permutations of $[n]=\{1,\dots,n\}$, written in one-line notation, and suppose $\pi\in\mathfrak{S}_n$. We write $|\pi|$ to denote the length of $\pi$, namely $n$, and for all $i\in[n]$, we write $\pi_i$ to denote the $i$-th element of $\pi$. Let $\alpha=\alpha_1\dots\alpha_m\in[p_1]^m$, $\beta=\beta_1\dots\beta_m\in[p_2]^m$. We say that $\alpha$ is *order-isomorphic* to $\beta$ if for all $1\leq i<j\leq m$ one has $\alpha_i<\alpha_j$ if and only if $\beta_i<\beta_j$. For two permutations $\pi\in\mathfrak{S}_n$ and $\tau\in\mathfrak{S}_k$, an *occurrence* of $\tau$ in $\pi$ is a subsequence $1\leq i_1<i_2<\cdots<i_k\leq n$ such that $(\pi_{i_1},\dots,\pi_{i_k})$ is order-isomorphic to $\tau$; in such a context $\tau$ is usually called the *pattern*. We say that $\pi$ *avoids* $\tau$, or is $\tau$*-avoiding*, if there is no occurrence of $\tau$ in $\pi$. A natural generalization of single pattern avoidance is *subset avoidance*, that is, we say that $\pi\in\mathfrak{S}_n$ avoids a subset $T\subseteq\mathfrak{S}_k$ if $\pi$ avoids all $\tau\in T$. The set of all $\tau$-avoiding (resp. $T$-avoiding) permutations of length $n$ is denoted $\mathfrak{S}_n(\tau)$ (resp. $\mathfrak{S}_n(T)$).

Several authors have considered the case of general $k$ in which $T$ enjoys various algebraic properties. Barcucci et al. [**2**] treat the case of permutations avoiding the collection of permutations in $\mathfrak{S}_k$ that have suffix $(k-1)k$. Adin and Roichman [**1**] look at the case where $T$ is a Kazhdan–Lusztig cell of $\mathfrak{S}_k$, or, equivalently, a Knuth equivalence class (see [**12**, Vol. 2, Ch. A1]). Mansour and Vainshtein [**10**] consider the situation where $T$ is a maximal parabolic subgroup of $\mathfrak{S}_k$. In the current paper an analogous result is established for pattern-avoiding involutions. Simion and Schmidt [**11**] considered

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the first cases of pattern-avoiding involutions, which was continued in Gouyou-Beauchamps [**5**] and Gessel [**6**] for increasing patterns, and subsequently in Guibert's Ph.D. thesis [**7**].

Kremer [**8**, Corollary 9] has shown that $|\mathfrak{S}_n(1243, 2143)|$, the number of permutations in $\mathfrak{S}_n$ that avoid both 1243 and 2143, is given by $r_{n-1}$, for all $n \geq 0$, where $r_n$ is the $n$-th Schröder number, defined by $r_0 = 1$ and $r_n = r_{n-1} + \sum_{i=1}^{n} r_{i-1}r_{n-i}$ for $n \geq 1$. As a result, for all $n \geq 0$, the set $\mathfrak{S}_{n+1}(1243, 2143)$ is in bijection with the set of *Schröder paths* of length $n$. These are the lattice paths from $(0, 0)$ to $(n, n)$ which contain only east $E = (1, 0)$, north $N = (0, 1)$, and diagonal $D = (1, 1)$ steps and which do not pass below the line $y = x$. A *symmetric Schröder path* of length $n$ is a Schröder path of length $n$ which is symmetric about the line $x + y = n$.

We say $\pi$ is an *involution* whenever $\pi_{\pi_i} = i$ for all $i \in [n]$. We denote by $\mathcal{I}_n$ and $I_n$ the set of involutions in $S_n$ and its cardinality, respectively. We say that $i$ is a fixed point of a permutation $\pi$ if $\pi_i = i$. Define $J_n(p)$ to be the polynomial $\sum_{j=0}^{n} I_{n;j}p^j$, where $I_{n;j}$ is the number of involutions in $\mathcal{I}_n$ with $j$ fixed points. For example, $J_0(p) = 1$, $J_1(p) = p$, $J_2(p) = 1 + p^2$, and $J_3(p) = 3p + p^3$. It is not hard to see that the polynomial $J_n(p)$ satisfies the recurrence relation $J_n(p) = pJ_{n-1}(p) + (n-1)J_{n-2}(p)$, $n \geq 1$, with the initial condition $J_0(p) = 1$. The exponential generating function for the sequence $J_n(p)$ is given by $e^{px+x^2/2}$.

The main result of this paper can be formulated as follows. Let $\mathcal{A}_k$ be the class of permutations in $\mathfrak{S}_k$ with prefix 12, that is,

$$\mathcal{A}_k = \{\pi_1\pi_2 \ldots \pi_k \in \mathfrak{S}_k \mid \pi_1 = 1, \pi_2 = 2\}.$$

THEOREM 1.1. *Let $k \geq 2$. Then the generating function for the number of $\mathcal{A}_k$-avoiding involutions of length $n$ with $m$ fixed points is given by*

$$\begin{aligned}\sum_{n\geq 0}\sum_{\pi\in\mathcal{I}_n} x^n p^{\#\textit{fixed points in }\pi} &= \sum_{j=0}^{k-3} J_j(p)x^j - \frac{x^{k-3}}{2}J_{k-2}(p)\Big(p + (p(k-3)x^2 - 2x - p)u_0(x)\Big)\\ &\quad - \frac{x^{k-4}}{2}J_{k-3}(p)\Big(x + p - (x^3(k-3) - px^2(k-1) + x + p)u_0(x)\Big),\end{aligned}$$

*where $u_0(x) = 1/\sqrt{1 - 2(k-1)x^2 + (k-3)^2x^4}$.*

The proof is given in Section 3. Theorem 1.1 with $k = 3$ and $p = 1$ shows the generating function for the number 123-avoiding involutions of length $n$ to be $\frac{2x}{2x-1+\sqrt{1-4x^2}} = \sum_{n\geq 0}\binom{n}{\lfloor n/2\rfloor}x^n$ (see [**11**]). Moreover, Theorem 1.1 with $k = 3$ and $p = 0$ gives the number of 123-avoiding involutions of length $2n$ without fixed points to be $\frac{1}{2}\binom{n}{\lfloor n/2\rfloor}$. Also, Theorem 1.1 with $k = 4$ and $p = 1$ gives the generating function for the number $\{1234, 1243\}$-avoiding involutions of length $n$ to be $\frac{1-x}{2} + \frac{1+x}{2}\sqrt{\frac{1+2x-x^2}{1-2x-x^2}}$. In Section 2 we give a combinatorial explanation for this result and prove that the number $\{1234, 1243\}$-avoiding involutions of length $n$ is the same as the number of symmetric Schröder paths of length $n - 1$.

## 2. Symmetric Schröder paths and $\{1234, 1243\}$-avoiding involutions

In this section we give a combinatorial explanation for why the number of *symmetric Schröder paths* of length $n$ (for definition, see below) is the same as the number of $\{1234, 1243\}$-avoiding involutions on $n+1$ elements. This explanation is via the existence of a bijection between the two sets. The bijection is in fact the composition of three bijections which are explained in the following three subsections. Unfortunately the composition does not appear to reduce to one that has a simple description.

**2.1. Schröder paths and $\{1243, 2143\}$-avoiding involutions.** The involutions in $\mathcal{I}_n(1243, 2143)$ may be related to Schröder paths via the bijection $\varphi$ in Egge and Mansour [**4**], which we now describe.

Let $S_n$ be the set of all Schröder paths from $(0,0)$ to $(n,n)$ and $SS_n$ the subcollection which are symmetric about the line $x+y=n$. For such a path $p \in S_n$ let us select a sequence of diagonals (parallel to $y=x$ and passing through the points (0.5, 0.75) of the unit squares with integer coordinates) that are contained within (and on) the bounded region between $p$ and the line $y=x$ in the following manner;

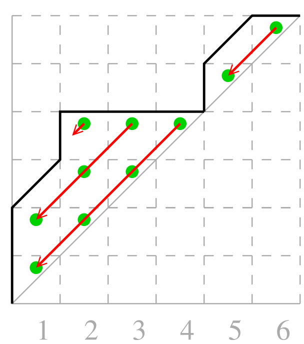
Let $p$ be a Schröder path $p$ from $(0,0)$ to $(n,n)$ and $s_i$ be the transposition $(i, i+1)$.

**Step 1:** If a square with integer coordinates $(i-1, m-1)$, $(i, m-1)$, $(i-1, m)$ and $(i, m)$ has interior point $(i-0.5, m-0.25)$ within the region between the path and the line $y=x$ then earmark this square and label it $s_i$. Let $j=1$.

**Step 2:** Choose the rightmost earmarked square that is not yet marked (with label $s_k$, say) and suppose one may see (to the south-west) as far as a square with label $s_l$, where $l \leq k$ of course. Mark both these squares and all those in-between. Let $\sigma_j = s_k s_{k-1} \ldots s_l$. If there are no further earmarked squares then go to step 3. Otherwise increase $j$ by 1 and repeat.

**Step 3:** Let $\varphi(p) = \sigma_j \ldots \sigma_2 \sigma_1 (n+1, n, \ldots, 1)$.

Example 2.1. *Consider the path $p \in S_6$ in the diagram.*




*The green points show those squares that have been earmarked. We have $\sigma_1 = s_6 s_5$, $\sigma_2 = s_4 s_3 s_2 s_1$, $\sigma_3 = s_3 s_2 s_1$ and $\sigma_4 = s_2$. So*

$$\begin{aligned} \varphi(p) &= \sigma_4 \sigma_3 \sigma_2 \sigma_1 (7,6,5,4,3,2,1) \\ &= s_2\ s_3 s_2 s_1\ s_4 s_3 s_2 s_1\ s_6 s_5 (7,6,5,4,3,2,1) \\ &= (5,2,4,6,7,1,3). \end{aligned}$$

We generalise our notation slightly. If $\pi \in \mathfrak{S}_n$ and $A = [a_1, a_2, \ldots, a_n]$ is an increasing sequence of numbers, then let $\pi \leftarrow A$ be the sequence $(b_{a_1}, \ldots, b_{a_n})$ in which $b_{a_i} = a_{\pi_i}$ for all $1 \leq i \leq n$ and all other positions are left empty. Note that $(\pi_1, \ldots, \pi_n) \leftarrow [1, \ldots, n] = \pi$. We use $\cup$ to denote the 'filling up' of the (possibly) empty spaces in such a sequence by transpositions. (The reason for this is that we wish to end up with a sequence which will be a permutation, and the $\leftarrow$ operation does not necessarily guarantee a permutation.)

Example 2.2. *We have* $(1,4,2,3,5,6) \leftarrow [1,2,4,5,6,7] = (1,5,\ ,2,4,6,7)$ *and so* $((1,4,2,3,5,6) \leftarrow [1,2,4,5,6,7]) \cup (3,8) = (1,5,8,2,4,6,7,3)$.

PROPOSITION 2.3.
**(i)** *For all* $n \geq 2$ *and* $p = Dp'D \in SS_n$,

$$\varphi(Dp'D) = (\varphi(p') \leftarrow [2, \ldots, n]) \cup (1, n+1).$$

**(ii)** *For all* $n \geq 1$ *and* $p = Np'E \in SS_n$ *where* $p' \in SS_n$, $\varphi(p) = (\varphi(p'), n+1)$.
**(iii)** *For all* $n \geq 0$ *and* $p = NqEp'Nq^{\perp}E \in SS_n$, *where* $q \in S_{m-2}$, $p \in SS_{n-2m}$ *and* $Nq^{\perp}E$ *is the reflection of* $NqE$ *from* $(n-m, n-m)$ *to* $(n,n)$, *we have*

$$\begin{aligned}\varphi(p) &= \sigma(qp'q^{\perp})\{1, \ldots, m, m+2, \ldots, n\} \cup (n+1, m+1) \\ &= (\varphi(qp'q^{\perp}) \leftarrow [1, \ldots, m, m+2, \ldots, n]) \cup (n+1, m+1).\end{aligned}$$

PROOF. **(i)** If the path begins and ends with $D$, then $s_1$ and $s_n$ will occur nowhere in the collections of $\sigma_j$'s. Thus the entries in positions 1 and $n+1$ in $(n+1, n, \ldots, 1)$ will remain unchanged.

**(ii)** In this case, $\sigma_1 = s_n s_{n-1} \cdots s_1$ and $s_n$ will occur nowhere in $\sigma_2, \ldots, \sigma_k$. So $\varphi(p) = \sigma_k \cdots \sigma_2(n, n-1, \ldots, 1, n+1) = \sigma_k \cdots \sigma_2(n, n-1, \ldots, 1), n+1 = (\varphi(p'), n+1)$.

**(iii)** Let us suppose that $p = NqEp'Nq^{\perp}E \in SS_n$ where $q \in S_{m-2}$, $p \in SS_{n-2m}$ and $q^{\perp} \in S_{m-2}$ is the reflection of $q$,where $m \leq \lfloor n/2 \rfloor$. Let $\sigma_1, \ldots, \sigma_j$ be the strings of $s_i$'s constructed from the $Nq^{\perp}E$ region of $p$, $\sigma_{j+1}, \ldots, \sigma_k$ the strings constructed from the $p'$ region and $\sigma_{k+1}, \ldots, \sigma_k + j$ the strings constructed from the $NqE$ region. (Symmetry guarantees this latter collection has the same number of $\sigma$'s as the first.)

Let $\sigma' = \sigma_j \cdots \sigma_1$, $\sigma'' = \sigma_k \cdots \sigma_{j+1}$ and $\sigma''' = \sigma_{k+j} \cdots \sigma_{k+1}$. It is clear that $\sigma_1 = s_n \cdots s_{n-m+1}$ and $\sigma_{k+1} = \sigma_m \cdots \sigma_1$. Since the path has been partitioned into 3 distinct non-overlapping regions, $\sigma'$ may operate only on positions $n-m+1$ to $n+1$ (inclusive) of a permutation, $\sigma''$ may operate only on positions $m+1$ to $n-m+1$ (inclusive) and $\sigma'''$ may operate only on positions 1 to $m+1$ (inclusive.) This gives

$$\begin{aligned}&\varphi(p) \\ &= \sigma'''\sigma''\sigma'(n+1, n, \ldots, 1) \\ &= \sigma'''\sigma''\sigma_j \cdots \sigma_2(n+1, \ldots, m+2, m, \ldots, 1, m+1) \\ &= \sigma'''\sigma''(n+1, \ldots, m+2, a_1, \ldots, a_m, m+1),\end{aligned}$$

where $a_i \in [1, m]$,

$$\begin{aligned}&= \sigma'''(n+1, \ldots, n-m+2, b_1, \ldots, b_{n-2m+1}, a_2, \ldots, a_m, m+1) \\ &= \sigma_{k+j} \cdots \sigma_{k+2}(n, \ldots, n-m+2, b_1, n+1, b_2 \ldots, b_{n-2m+1}, a_2, \ldots, a_m, m+1) \\ &= (c_1, \ldots, c_m, n+1, b_2 \ldots, b_{n-2m+1}, a_2, \ldots, a_m, m+1).\end{aligned}$$

It is routine to check that this is the same as

$$(\sigma(qp'q^{\perp}) \leftarrow [1, \ldots, m, m+2, \ldots, n]) \cup (n+1, m+1)$$

where $\cup(n+1, m+1)$ signifies inserting $n+1$ between positions $m$ and $m+1$ of the resulting sequences and inserting $m+1$ as a suffix (i.e. $(n+1, m+1)$ is a transposition in the resulting permutation.) □

PROPOSITION 2.4. *For all* $p \in SS_n$ *we have* $\varphi(p) \in \mathcal{I}_{n+1}(1243, 2143)$.

PROOF. In [**4**] the map $\varphi : S_n \to \mathfrak{S}_{n+1}(1243, 2143)$ was shown to be a bijection. Thus we need only show that if $p \in SS_n$ then $\varphi(p)$ is an involution. This is easily done via induction. For the base case, $\varphi(\emptyset) = (1)$, $\varphi(D) = (2,1)$ and $\varphi(NE) = (1,2)$ so that our proposition holds for all $p \in SS_0$ and $SS_1$. Suppose the proposition holds for all $p \in SS_i$ and $0 \leq i < n$. Fix $p \in SS_n$.

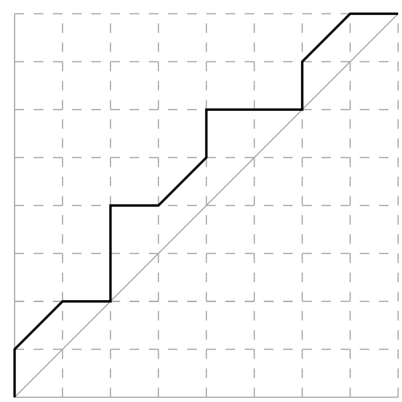
- If $p$ begins with a diagonal step then $p$ ends with a diagonal step so by Proposition 2.3(i) $\varphi(p) = (\varphi(p') \leftarrow [2,\ldots,n]) \cup (1, n+1)$. This will be an involution since $\varphi(p')$ is an involution.
- If $p = Np'E$ then by Proposition 2.3(ii), $\varphi(p) = \varphi(p'), n+1$. Since $p' \in SS_{n-1}$, $\varphi(p')$ is an involution and hence $p$ is also an involution.
- If $p = NqEp'Nq^{\perp}E$ then by Proposition 2.3(iii)

$$\varphi(p) = (\varphi(qp'q^{\perp}) \leftarrow [1,\ldots,m,m+2,\ldots,n]) \cup (n+1, m+1).$$

  Since $qp'q^{\perp} \in SS_{n-2}$ we have that $\varphi(qp'q^{\perp})$ is an involution on $[1,\ldots,n-2]$, and hence $\varphi(p)$ is also an involution.

$\square$

Example 2.5. *Consider $p \in SS_8$ in the diagram;*



*This path is of the form $p = NqEp'Nq^{\perp}E$ where $q = D \in S_1$ and $p' = NNEDNEE \in SS_4$. So $\varphi(p) = (\varphi(DNNEDNEED) \leftarrow [1,2,4,5,6,7,8]) \cup (9,3)$. The path deconstruction process described in the proof yields; $\varphi(DNNEDNEED) = (\varphi(NNEDNEE) \leftarrow [2,3,4,5,6]) \cup (1,7)$; $\varphi(NNEDNEE) = \varphi(NEDNE) \cup (5,5)$ and $\varphi(NEDNE) = (\varphi(D) \leftarrow [1,3]) \cup (4,2) = (3,4,1,2)$.*

*Beginning with the last expression and systematically replacing each in its preceding expression gives; $\varphi(NNEDNEE) = (3,4,1,2,5)$, $\varphi(DNNEDNEED) = ((3,4,1,2,5) \leftarrow [2,3,4,5,6]) \cup (1,7) = (7,4,5,2,3,6,1)$ and finally $\varphi(p) = ((7,4,5,2,3,6,1) \leftarrow [1,2,4,5,6,7,8]) \cup (9,3) = (8,5,9,6,2,4,7,1,3)$.*

Proposition 2.6. *For all $n \geq 0$, $\varphi : SS_n \rightarrow \mathcal{I}_{n+1}(1243, 2143)$ is a bijection.*

Proof. This is routine since $\varphi : S_n \rightarrow \mathfrak{S}_{n+1}(1243, 2143)$ is surjective. We need only show that to every involution $\pi$ in the image of $\varphi$, $\varphi^{-1}(\pi) \in SS_n$. If we have $\pi \in \mathcal{I}_{n+1}(1243, 2143)$ then we may construct the unique path $p \in SS_n$ recursively according the three separate cases:

- If $n+1$ is a fixed point of $\pi$, then $p = Dp'D$ for some $p' \in SS_{n-2}$. Repeat the procedure for the permutation $(\pi_1, \ldots, \pi_n)$ to determine $p'$.
- If $(1, n+1)$ is a transposition of $\pi$, then $\pi = (n+1)\pi'1$ and $p = Np'E$ where $p' \in SS_{n-1}$. Repeat the procedure for the permutation $(\pi_2 - 1, \pi_3 - 1, \ldots, \pi_n - 1)$ to determine $p'$.

- If $(m+1, n+1)$ is a transposition of $\pi$ for some $1 < m < n$, then $p = NqEp'Nq^{\perp}E$ where $q \in S_{m-1}$, $p' \in SS_{n-2m}$. Repeat the procedure for the permutation $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in I_{n-3}$ where
$$\alpha_i = \begin{cases} \pi_i & \text{if } i \leq m \text{ and } \pi_i < m+1, \\ \pi_i - 1 & \text{if } i \leq m \text{ and } \pi_i > m+1, \\ \pi_{i-1} & \text{if } i > m+1 \text{ and } \pi_i < m+1, \\ \pi_{i-1} - 1 & \text{if } i > m+1 \text{ and } \pi_i > m+1. \end{cases}$$
Note that once one has determined $r = P_1P_2\cdots$, where $P_i \in \{N, E, D\}$, such that $\varphi(r) = \alpha$ then $q$ is the suffix of $r$ which comprises of those steps ($\in \{N, E, D\}$) from $(0,0)$ to $(m-1, m-1)$, $p'$ is the subsequent sequence of steps from $(m-1, m-1)$ to $(n-m+1, n-m+1)$ and $q^{\perp}$ is the remaining collection.

□

**2.2. The bijection $\Phi_2 : \mathcal{I}_n(1243, 2143) \mapsto \mathcal{I}_n(2134, 2143)$.** Given $\pi \in \mathcal{I}_n(2134, 2143)$, let $\Phi_2(\pi) = (n+1-\pi_n, n+1-\pi_{n-1}, \ldots, n+1-\pi_1)$. Clearly $\Phi_2 : \mathcal{I}_n(1243, 2143) \to \mathcal{I}_n(2134, 2143)$. It is a bijection since $\Phi_2$ is the well known reverse-complement map and $\Phi_2^{-1} = \Phi_2$.

**2.3. The bijection $\Phi_1 : \mathcal{I}_n(1234, 1243) \mapsto \mathcal{I}_n(2134, 2143)$.** Given $\pi \in \mathcal{I}_n(1234, 1243)$ let us concentrate on occurrences of the patterns 2134,2143. Let $A_\pi = \emptyset$. If $(\pi_{i(1)}, \pi_{i(2)}, \pi_{i(3)}, \pi_{i(4)})$ is an occurrence of the pattern 2134 in $\pi$ then insert entries $i(1), i(2)$ into $A_\pi$. Do likewise for the pattern 2143. Let $A_\pi = \{i(1), i(2), \ldots, i(t)\}$ be the resulting collection of all such indices.

Lemma 2.7. *Given $\pi \in \mathcal{I}_n(1234, 1243)$ with sequence $A_\pi = \{i(1), i(2), \ldots, i(t)\}$ as outlined above, $\pi_{i(1)} > \pi_{i(2)} > \cdots > \pi_{i(t)}$.*

Proof. Without loss of generality let us hinge our argument around the pattern 1234. Assume there exists $a < b$ such that $\pi_{i(a)} < \pi_{i(b)}$. The value $i(b)$ is in the sequence $A_\pi$ for a reason: either there are indices $a', c, d$ such that $a' < i(b) < c < d$ and $\pi_{a'}, \pi_{i(b)}, \pi_c, \pi_d$ is an occurrence of 2134 or $i(b) < a' < b < c$ and $\pi_{i(b)}, \pi_{a'}, \pi_c, \pi_d$ is an occurrence of 2134.

If $a' < i(b) < c < d$ and $\pi_{a'}, \pi_{i(b)}, \pi_c, \pi_d$ is an occurrence of 2134 then since $a < b$, $\pi_{i(a)}, \pi_{i(b)}, \pi_c, \pi_d$ is an occurrence of 1234 in $\pi$. This cannot be the case since $\pi$ is 1234 avoiding.

So $i(b) < a' < b < c$ and $\pi_{i(b)}, \pi_{a'}, \pi_c, \pi_d$ is an occurrence of 2134. But since $a < b$, $i(a) < i(b)$ and $\pi_{i(a)}, \pi_{i(b)}, \pi_c, \pi_d$ will therefore be an occurrence of 1234. Again, this cannot be so since $\pi$ is 1234 avoiding, hence have $\pi_{i(a)} > \pi_{i(b)}$. □

Definition 2.8. *Let $\pi \in \mathcal{I}_n(1234, 1243)$. If $\pi$ contains no occurrence of the patterns 2134 and 2143 then let $\Phi_1(\pi) = \pi$. Otherwise let $A_\pi = \{i(1), \ldots, i(t)\}$ be the sequence associated with the 1 and 2's of occurrences of 2134 and 2143 in $\pi$ as mentioned above. Let $\Phi_1(\pi) = \alpha_1 \cdots \alpha_n$ where*

*(1) $\alpha_j = \pi_j$ for all $j \notin A_\pi$, and*

*(2) let $F$ be the largest Ferrers board (oriented as in Example 2.10) which is contained in the region strictly south west of every point of $\pi$ not in $A_\pi$. The points $\{(i(1), \alpha_{i(1)}), \ldots, (i(t), \alpha_{i(t)})\}$ are contained in $F$. In this board, recursively (in a lexicographic order) replace every occurrence of 21 by 12. Let $\{(i(1), \beta_{i(1)}), \ldots, (i(t), \beta_{i(t)})\}$ be the outcome. Finally let $\alpha_{i(j)} = \beta_{i(j)}$ for all $1 \leq j \leq t$.*

Note that in (2) an occurrence of 21 in $F$ in this case means the smallest rectangle containing both points of the 21 pattern must also be contained in the Ferrers board $F$. This definition is a particular instance of the bijective map $\phi^\star$ from Bousquet-Mélou and Steingrímsson [**3**]. The nature of the map is to iteratively replace occurrences of a monotone pattern in a Ferrers board so that upon termination there are no such occurrences.

PROPOSITION 2.9. *The map* $\Phi_1 : \mathcal{I}_n(1234, 1243) \to \mathcal{I}_n(2134, 2143)$ *is a bijection.*

PROOF. Let us first show the map $\Phi_1$ is well defined. Given $\pi \in \mathcal{I}_n(1234, 1243)$, suppose, without loss of generality, there is an occurrence of 2134 in $\Phi_1(\pi) = \alpha_1 \dots \alpha_n$. We assume $A_\pi$ is non-empty, for otherwise there would certainly be no occurrences of 2134 and 2143 in $\Phi_1(\pi)$ since $\Phi_1(\pi) = \pi$. Let us further suppose the lexicographically first occurrence of 2134 in $\Phi(\pi)$ is at $(a, b, c, d)$, i.e. $\alpha_a\alpha_b\alpha_c\alpha_d$ is such that $\alpha_b < \alpha_a < \alpha_c < \alpha_d$.

We note that, in this setting, at least one of $\{a, b, c, d\}$ must be contained in $A_\pi$, as otherwise one would have a 2134 pattern in $\pi$ whose existence was not recorded.

If $d \in A_\pi = \{i(1), \dots, i(t)\}$ then it follows that $\pi^{-1}(\alpha_a)$, $\pi^{-1}(\alpha_b)$, $\pi^{-1}(\alpha_c) \in A_\pi$. Furthermore, since $\pi_{i(1)} > \cdots > \pi_{i(t)}$, we must have that $\pi^{-1}(\alpha_a)$, $\pi^{-1}(\alpha_b)$, $\pi^{-1}(\alpha_c) > d$. Since operation (2) above was able to move the entries of $\alpha_a, \alpha_b$ and $\alpha_c$ in $\pi$ to the left of $\alpha_d$ in $\Phi_1(\pi)$, the rules of exchange on the Ferrers board would have replaced every occurrence of 21 by 12. This is not the case since $\alpha_a > \alpha_b$. Hence $d \notin A_\pi$. A similar argument shows that $c \notin A_\pi$.

The situation is that either (a) $a, b \in A_\pi$, (b) $a \in A_\pi$, $b \notin A_\pi$ or (c) $a \notin A_\pi$, $b \in A_\pi$.

For case (a), we have $\pi_a > \pi_b$ and the map $\Phi_1$ will make $\alpha_a \geq \pi_a$, $\alpha_b \geq \pi_b$, $\alpha_c = \pi_c$ and $\alpha_d = \pi_d$. Note that $\alpha_b < \alpha_a < \alpha_c < \alpha_d$. Since $\alpha_a\,\alpha_b$ is an occurrence of 21 in $\Phi_1(\pi)$, there is some $j \notin A_\pi$ such that $a < j < b$ and $\pi_a > \pi_j > \pi_b$. The action of $\Phi_1$ will ensure that $\alpha_a > \alpha_j > \alpha_b$. This means that $\alpha_a\,\alpha_j\,\alpha_c\,\alpha_d$ is an occurrence of 2134 in $\Phi_1(\pi)$ which is lexicographically smaller than $\alpha_a\,\alpha_b\,\alpha_c\,\alpha_d$. Hence case (a) is impossible.

For case (b), we have $a \in A_\pi$, $b, c, d \notin A_\pi$. Then $\alpha_a \geq \pi_a$ and $\alpha_a\pi_b\pi_c\pi_d$ is an occurrence of 2134 in $\Phi_1(\pi)$. But since $\alpha_a \geq \pi_a$ either $\pi_a > \pi_b$ or $\pi_a < \pi_b$. If $\pi_a > \pi_b$ then $\pi_a\,\pi_b\,\pi_c\,\pi_d$ is an occurrence of 2134 in $\pi$ which would mean that both $a$ and $b$ are in $A_\pi$. Since this is not the case we must have that $\pi_a < \pi_b$ but this implies $\pi_a\,\pi_b\,\pi_c\,\pi_d$ is an occurrence of 1234 in $\pi$. Hence case (b) is impossible.

A similar argument shows (c) cannot be true either. Hence $\Phi_1(\pi)$ does not contain the pattern 2134. Similar reasoning shows this to be the case for the pattern 2143 also.

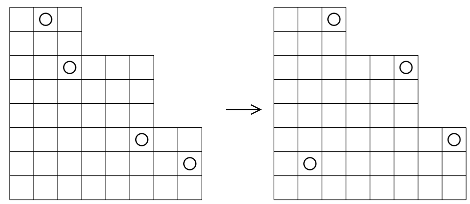
To show that $\Phi_1$ is injective, the inverse bijection $f : \mathcal{I}_n(2134, 2143) \to \mathcal{I}_n(1234, 1243)$ of $\Phi_1$ may be defined by: let $\pi \in \mathcal{I}_n(2134, 2143)$. If $\pi$ contains no occurrence of the patterns 1234 and 1243 then let $f(\pi) = \pi$. Otherwise let $B_\pi = \{i(1), \dots, i(t)\}$ be the sequence associated with the 1 and 2's of occurrences of 1234 and 1243 in $\pi$. Let $f(\pi) = \beta_1 \cdots \beta_n$ where

(1) $\beta_j = \pi_j$ if $j \notin B_\pi$ and

(2) let $F$ be the largest Ferrers board which is contained in the region strictly south west of every point of $\pi$ not in $B_\pi$. The points $\{(i(1), \beta_{i(1)}), \dots, (i(t), \beta_{i(t)})\}$ are contained in $F$. In this board, recursively (in a lexicographic order) replace every occurrence of 12 by 21. Let $\{(i(1), \gamma_{i(1)}), \dots, (i(t), \gamma_{i(t)})\}$ be the outcome. Now set $\beta_{i(j)} = \gamma_{i(j)}$ for all $1 \leq j \leq t$.

Surjectivity follows easily from Bousquet-Mélou and Steingrímsson [**3**, Theorem 1] since the sizes of $\mathcal{I}_n(1234, 1243)$ and $\mathcal{I}_n(2134, 2143)$ are equal. □

Example 2.10. *Let* $\pi = (11,8,6,12,10,3,7,2,9,5,1,4)$. *Then* $A_\pi = \{2,3,6,8\}$. *Now the Ferrers board containing the 21's of* $\pi$ *is transformed as:*



*This results in* $\Phi_1(\pi) = (11,2,8,12,10,6,7,3,9,5,1,4)$.

Theorem 2.11. *The map* $f = \Phi_1^{-1} \circ \Phi_2 \circ \varphi : SS_{n-1} \mapsto \mathcal{I}_n(1234,1243)$ *is a bijection.*

Proof. From Proposition 2.6, Section 2.2 and Proposition 2.9 we have bijections $\varphi : SS_{n-1} \mapsto \mathcal{I}_n(1243,2143)$, $\Phi_2 : \mathcal{I}_n(1243,2143) \mapsto \mathcal{I}_n(2134,2143)$ and $\Phi_1^{-1} : \mathcal{I}_n(2134,2143) \mapsto \mathcal{I}_n(1234,1243)$. The composition $f = \Phi_1^{-1} \circ \Phi_2 \circ \varphi$ is therefore a bijection $f : SS_{n-1} \mapsto \mathcal{I}_n(1234,1243)$. □

We note that it would be nice to have a direct bijection between the set $\mathcal{I}_{n+1}(1234,1243)$ of involutions of length $n$ avoiding both 1234 and 1243, and the set $SS_n$ of Schröder paths of length $n$, or even between the set of permutations $\mathfrak{S}_{n+1}(1234,1243)$ and the set $S_n$ of Schröder paths of length $n$.

## 3. Proof of Theorem 1.1

To present the proof of Theorem 1.1, we must first consider the enumeration problem for the number $\mathcal{F}_k$-avoiding involutions according to length and number of fixed points, where $\mathcal{F}_k$ is the set of all permutations $\sigma \in \mathfrak{S}_k$ with $\sigma_1 = 1$.

### 3.1. Involutions avoiding $\mathcal{F}_k$.

In this subsection we present an explicit formula for the number of involutions that avoid all the patterns in $\mathcal{F}_k$. To do so we require some new notation. Define $f_k(n)$ to be the number of involutions $\pi \in \mathcal{I}_n(\mathcal{F}_k)$. Given $t \in [n]$, we also define

$$f_{k;m}(n;t) = \#\{\pi \in \mathcal{I}_n(\mathcal{F}_k) \mid \pi_1 = t \text{ and } \pi \text{ contains } m \text{ fixed points}\}.$$

Let $f_k(n;t) = f_k(n,p;t)$ and $f_k(n) = f_k(n,p)$ be the polynomials $\sum_{m=0}^{n} f_{k;m}(n;t)p^m$ and $\sum_{t=1}^{n} f_k(n;t)$, respectively. We denote by $F_k(x,p)$ the generating function for the sequence $f_k(n,p)$, that is $F_k(x,p) = \sum_{n\geq 0} f_k(n,p)x^n$.

Theorem 3.1. *We have*

$$F_k(x,p) = \sum_{j=0}^{k-2} J_j(p)x^j + \frac{x^{k-1}}{1-(k-1)x^2}((k-1)J_{k-2}(p)x + J_{k-1}(p)).$$

*Moreover, the number of involutions of length* $k+2n$ *(resp.* $k+2n-1$*) that avoid all the patterns in* $\mathcal{F}_k$ *is given by* $(k-1)^{n+1}I_{k-2}$ *(resp.* $(k-1)^n I_{k-1}$*), for all* $n \geq 0$.

Proof. Let $\pi \in \mathfrak{S}_n$ be a permutation that avoids all patterns in $\mathcal{F}_k$. We have $\pi_1 \geq n+2-k$. Thus $\pi \in \mathcal{I}_n(\mathcal{F}_k)$ with $\pi_1 = t \geq n+2-k$ if and only if $\pi_2 \ldots \pi_{t-1}\pi_{t+1} \ldots \pi_n$ is an involution on the numbers $2,\ldots,t-1,t+1,\ldots,n$ that avoids all the patterns in $\mathcal{F}_k$. Hence, $f_k(n;j) = f_k(n-2)$ for all $j = n+2-k, n+3-k, \ldots, n$, and $f_k(n,j) = 0$ for all $j = 1,2,\ldots,n+1-k$, where $n \geq k$. Thus, for $n \geq k$,

$$f_k(n) = (k-1)f_k(n-2).$$

Using the initial conditions $f_k(j) = J_j(p)$, $j = 1,2,\ldots,k-1$, we find that $f_k(k+2j) = (k-1)^{j+1}J_{k-1}(p)$ and $f_k(k+2j-1) = (k-1)^j J_{k-2}(p)$ for all $j \geq 0$. Rewriting these formulas in terms of generating functions we obtain

$$F_k(x,p) = \sum_{j=0}^{k-2} J_j(p)x^j + \frac{x^{k-1}}{1-(k-1)x^2}((k-1)J_{k-2}(p)x + J_{k-1}(p)),$$

as claimed. □

**3.2. Involutions avoiding $\mathcal{A}_k$.** In this subsection we prove Theorem 1.1. In order to do this, define $g_k(n)$ to be the number of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ and given $t_1, t_2, \ldots, t_m \in \mathbb{N}$, we also define

$$g_k(n; t_1, t_2, \ldots, t_m) = \#\{\pi_1 \ldots \pi_n \in \mathcal{I}_n(\mathcal{A}_k) \mid \pi_1 \ldots \pi_m = t_1 \ldots t_m\}.$$

Lemma 3.2. *Let $k \geq 3$. For all $3 \leq t \leq n+1-k$,*

$$g_k(n;t) = (k-2)g_k(n-2;t-1) + \sum_{j=1}^{t-2} g_k(n-2;j),$$

*with the initial conditions $g_k(n;1) = f_{k-1}(n-1)$, $g_k(n;2) = f_{k-1}(n-2)$, and $g_k(n;t) = g_k(n-2)$ for all $t = n+2-k, n+3-k, \ldots, n$.*

Proof. Let $\pi$ be any involution of length $n$ that avoids all patterns in $\mathcal{A}_k$ with $\pi_1 = t$. Now let us consider all possible values of $t$. If $t = 1$ then $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ if and only if $(\pi_2 - 1)(\pi_3 - 1)\ldots(\pi_n - 1) \in \mathcal{I}_{n-1}(\mathcal{F}_{k-1})$. If $t = 2$ then $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ if and only if $(\pi_3 - 2)(\pi_4 - 2)\ldots(\pi_n - 2) \in \mathcal{I}_{n-2}(\mathcal{F}_{k-1})$. Now assume that $3 \leq t \leq n+1-k$, then from the above definitions

$$g_k(n;t) = g_k(n;t,1) + \ldots + g_k(n;t,t-1) + g_k(n;t,t+1) + \cdots + g_k(n;t,n).$$

But any involution $\pi$ satisfying $\pi_1 < \pi_2 \leq n+2-k$ contains a pattern from the set $\mathcal{A}_k$ (see the subsequence of the letters $\pi_1, \pi_2, n+3-k, n+4-k, \ldots, n$ in $\pi$). Thus $g_k(n;t,r) = 0$ for all $t < r \leq n+2-k$ and so

$$g_k(n;t) = g_k(n;t,1) + \ldots + g_k(n;t,t-1) + g_k(n;t,n+3-k) + \cdots + g_k(n;t,n).$$

Also, if $\pi$ is an involution in $\mathcal{I}_n$ with $\pi_1 = t$ and $\pi_2 = r \geq n+3-k$, then the entry $r$ does not appear in any occurrence of $\tau \in \mathcal{A}_k$ in $\pi$. Thus, there exists a bijection between the set of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ with $\pi_1 = t$ and $\pi_2 = r \geq n+3-k$ and the set of involutions $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$ with $\pi' = t-1$. Therefore $g_k(n;t,r) = g_k(n-2;t-1)$ which gives

$$g_k(n;t) = g_k(n;t,1) + \ldots + g_k(n;t,t-1) + (k-2)g_k(n-2;t-1).$$

Also, if $\pi$ is an involution in $\mathcal{I}_n$ with $\pi_1 = t$, $\pi_2 = r < t$ and if $ta_2 \ldots a_k$ is an occurrence of a pattern from the set $\mathcal{A}_k$ in $\pi$, then $ra_2 \ldots a_k$ is an occurrence of a pattern from the set $\mathcal{A}_k$ in $\pi$. Thus, there exists a bijection between the set of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ with $\pi_1 = t$ and $\pi_2 = r < t$ and the set of involutions $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$ with $\pi'_1 = r-1$. Therefore $g_k(n;t,r) = g_k(n-2;r-1)$ which gives

$$g_k(n;t) = (k-2)g_k(n-2;t-1) + \textstyle\sum_{j=1}^{t-2} g_k(n-2;j),$$

as required. Finally, if $\pi$ is an involution in $\mathcal{I}_n$ with $\pi_1 = t \geq n+2-k$, then the entry $t$ does not appear in any occurrence of $\tau \in \mathcal{A}_k$ in $\pi$. Thus, there exists a bijection between the set of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ with $\pi_1 = t \geq n+2-k$ and the set of involutions $\pi' \in \mathcal{I}_{n-2}(\mathcal{A}_k)$. Therefore $g_k(n;t) = g_k(n-2)$, as claimed. $\square$

Let $G_k(n;v)$ be the polynomial $\sum_{t=1}^{n} g_k(n;t)v^{t-1}$. Rewriting the above lemma in terms of the polynomials $G_k(n;v)$ we have the following recurrence relation.

Lemma 3.3. *Let $k \geq 3$. For all $n \geq k$,*

$$\begin{aligned} G_k(n;v) &= f_{k-1}(n-1) + vf_{k-1}(n-2) - v(k-2)f_{k-1}(n-3) \\ &\quad + \left(\tfrac{v^2}{1-v} + (k-2)v\right) G_k(n-2;v) - \tfrac{v^n}{1-v} G_k(n-2;1) + \tfrac{v^{n-1}}{1-v}\left(k-2+\tfrac{v-v^{3-k}}{1-v}\right) G_k(n-4;1), \end{aligned}$$

*where $G_k(n;v) = I_{n-1} + \frac{v-v^n}{1-v} I_{n-2}$ for all $n = 0,1,\ldots,k-1$.*

Proof. Lemma 3.2 gives

$$\begin{aligned} G_k(n;v) &= f_{k-1}(n-1) + vf_{k-1}(n-2) + \textstyle\sum_{t=2}^{n-k} v^t \left((k-2)G_k(n-2;t) + \sum_{j=1}^{t-1} G_k(n-2;j)\right) \\ &\quad + \left(\textstyle\sum_{t=n+1-k}^{n-1} v^t\right) G_k(n-2;1) \\ &= f_{k-1}(n-1) + vf_{k-1}(n-2) + \tfrac{v^2}{1-v}(G_k(n-2;v) - G_k(n-4;1)\textstyle\sum_{j=n-1-k}^{n-3} v^j) \\ &\quad - \tfrac{v^{n+1-k}}{1-v}(G_k(n-2;1) - (k-1)G_k(n-4;1)) + \tfrac{v^{n+1-k}-v^n}{1-v} G_k(n-2;1) \\ &\quad + (k-2)v\left(G_k(n-2;v) - f_{k-1}(n-3) - G_k(n-4;1)\textstyle\sum_{j=n-k}^{n-3} v^j\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} G_k(n;v) &= f_{k-1}(n-1) + vf_{k-1}(n-2) - v(k-2)f_{k-1}(n-3) \\ &\quad + \left(\tfrac{v^2}{1-v} + (k-2)v\right) G_k(n-2;v) - \tfrac{v^n}{1-v} G_k(n-2;1) + \tfrac{v^{n-1}}{1-v}\left(k-2+\tfrac{v-v^{3-k}}{1-v}\right) G_k(n-4;1). \end{aligned}$$

To find the value of $G_k(n;v)$ for $n \leq k-1$, let $\pi$ be any involution with $\pi_1 = t$. If $t = 1$ then there are $I_{n-1}$ involutions, whereas if $t > 1$ there are $I_{n-2}$ involutions, hence $G_k(n;v) = v^0 I_{n-1} + \sum_{t=2}^{n} v^{t-1} I_{n-2} = I_{n-1} + \frac{v-v^n}{1-v} I_{n-2}$, as required. $\square$

Lemma 3.3 can be generalised as follows; let $g_{k;m}(n;t)$ be the number of involutions $\pi \in \mathcal{I}_n(\mathcal{A}_k)$ such that $\pi_1 = t$ and $\pi$ contains exactly $m$ fixed points. Define $G_k(n;t;p) = \sum_{m=0}^{n} g_{k;m}(n;t)p^m$ and $G_k(n;v,p) = \sum_{t=1}^{n} G_k(n;t;p)v^{t-1}$. Using the same arguments as those in the proofs of Lemma 3.2 and Lemma 3.3, while carefully considering the number of fixed points, we have the following result.

Lemma 3.4. *Let $k \geq 3$. For all $n \geq k$,*

$$\begin{aligned} &G_k(n;v,p) \\ &= pf_{k-1}(n-1) + vf_{k-1}(n-2) - pv(k-2)f_{k-1}(n-3) + \left(\tfrac{v^2}{1-v} + (k-2)v\right) G_k(n-2;v,p) \\ &- \tfrac{v^n}{1-v} G_k(n-2;1,p) + \tfrac{v^{n-1}}{1-v}\left(k-2+\tfrac{v-v^{3-k}}{1-v}\right) G_k(n-4;1,p), \end{aligned}$$

*where $G_k(n;v,p) = pJ_{n-1}(p) + \frac{v-v^n}{1-v} J_{n-2}(p)$ for all $n = 0,1,\ldots,k-1$.*

Let $G_k(x,v,p) = \sum_{n\geq 0} G_k(n;v,p)x^n$ be the generating function for the sequence $G_k(n;v,p)$. Define $J_i(v,p)$ to be the polynomial $\sum d_{tr}v^t p^r$ where $d_{tr}$ is the number of involutions $\pi \in \mathcal{I}_i$ such that

$\pi_1 = t+1$ and $\pi$ contains exactly $r$ fixed points. Rewriting the recurrence relation in the statement of Lemma 3.4 in terms of generating functions we obtain

$$\begin{aligned}
&G_k(x,v,p)\\
&\quad=\sum_{j=0}^{k-1} J_j(v,p)x^j + px\left(F_{k-1}(x,p)-\sum_{j=0}^{k-2} J_j(p)x^j\right) + vx^2\left(F_{k-1}(x,p)-\sum_{j=0}^{k-3} J_j(p)x^j\right)\\
&\quad-(k-2)pvx^3\left(F_{k-1}(x,p)-\sum_{j=0}^{k-4} J_j(p)x^j\right) - \tfrac{v^2x^2}{1-v}\left(G_k(xv,1,p)-\sum_{j=0}^{k-3} J_j(p)(xv)^j\right)\\
&\quad+vx^2\left(\tfrac{v}{1-v}+k-2\right)\left(G_k(x,v,p)-\sum_{j=0}^{k-3} J_j(v,p)x^j\right)\\
&\quad+\tfrac{(k-2)v^3x^4}{1-v}\left(G_k(xv,1,p)-\sum_{j=0}^{k-5} J_j(p)(xv)^j\right) - \tfrac{x^4(1-v^{k-2})}{v^{k-6}(1-v)^2}\left(G_k(xv,1,p)-\sum_{j=0}^{k-5} J_j(p)(xv)^j\right),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\left(1-\tfrac{x^2}{1-v}-(k-2)\tfrac{x^2}{v}\right)G_k(x/v,v,p)\\
&\quad=-\tfrac{x^2}{1-v}\left(1-(k-2)\tfrac{x^2}{v}+\tfrac{x^2(1-v^{k-2})}{v^{k-2}(1-v)}\right)G_k(x,1,p)\\
&\quad+\sum_{j=0}^{k-1} J_j(v,p)\tfrac{x^j}{v^j}+\tfrac{px}{v}\left(F_{k-1}(x/v,p)-\sum_{j=0}^{k-2} J_j(p)\tfrac{x^j}{v^j}\right)+\tfrac{x^2}{v}\left(F_{k-1}(x/v,p)-\sum_{j=0}^{k-3} J_j(p)\tfrac{x^j}{v^j}\right)\\
&\quad-(k-2)p\tfrac{x^3}{v^2}\left(F_{k-1}(x/v,p)-\sum_{j=0}^{k-4} J_j(p)\tfrac{x^j}{v^j}\right)+\tfrac{x^2}{1-v}\sum_{j=0}^{k-3} J_j(p)x^j\\
&\quad-\tfrac{x^2}{v}\left(\tfrac{v}{1-v}+k-2\right)\sum_{j=0}^{k-3} J_j(v,p)\tfrac{x^j}{v^j}-\tfrac{(k-2)x^4}{v(1-v)}\sum_{j=0}^{k-5} J_j(p)x^j+\tfrac{x^4(1-v^{k-2})}{v^{k-2}(1-v)^2}\sum_{j=0}^{k-5} J_j(p)x^j.
\end{aligned}$$

To solve this functional equation, we substitute

$$v := v_0 = \frac{1}{2}\left(1+(k-3)x^2+\sqrt{1-2(k-1)x^2+(k-3)^2x^4}\right),$$

where $v_0$ is the root of the coefficient of $G_k(x/v,v,p)$ above, into the above functional equation, that is, $1-\frac{x^2}{1-v_0}-(k-2)\frac{x^2}{v_0}=0$. Since $J_j(v,p)=pJ_{j-1}(p)+\frac{v-v^j}{1-v}J_{j-2}(p)$ for all $j=1,2,\ldots,k-1$ and $J_0(v,p)=1$, it is routine to show (via some rather tedious algebraic manipulation) that we obtain Theorem 1.1.

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