ON EULER'S DIFFERENCE TABLE

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Abstract. In this paper we study Euler's difference table and its derivate over symmetric group. We will give some combinatorial interpretations to the relations defining them as well as their generating functions.
Résumé. Nous étudions dans ce papier la différence table d'Euler et sa dérivée sur le groupe groupe symétrique. Nous donnerons des interprétations combinatoires de leurs relations de récurrence ainsi que leurs fonctions génératrices.

1. Introduction

The classical derangement numbers (or the numbers of permutations without fixed points) are always treated as a special case of permutations with the statistic *fixed points*. Many authors studied in depth these numbers [2], [3], [5], [7], [8], [9], [10], [11], [13], [15]. We will study in this paper a new statistic called *k-succession* over the symmetric group that will generalise the derangement theory. This statistic will give a combinatorial interpretation to the coefficients of the *difference table* $(e_n^k)_{0\leq k\leq n}$ introduced by Euler [1], [4].

e_n^k						
	$k=0$	1	2	3	4	5
$n=0$	0!					
1	0	1!				
2	1	1	2!			
3	2	3	4	3!		
4	9	11	14	18	4!	
5	44	53	64	78	96	5!

The coefficients e_n^k of this table are defined by

$$e_n^n = n! \text{ and } e_n^{k-1} = e_n^k - e_{n-1}^{k-1}.$$

We will then study the numbers d_n^k which are obtained from the numbers e_n^k by dividing them by $k!$. We obtain then the following table for some first values of the numbers d_n^k

d_n^k						
	$k=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	1	1	1			
3	2	3	2	1		
4	9	11	7	3	1	
5	44	53	32	13	4	1

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By a simple computation, we can find that the numbers d_n^k satisfy the following recurrences

$$\begin{cases} d_k^k = 1 \\ d_n^k = (n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k \text{ for } n > k \geq 0. \end{cases}$$

We could find that not only the coefficients of the first column of these tables, that is the values of e_n^k and d_n^k for $k = 0$, are the derangement numbers d_n but the relation defining the coefficients of each column

$$c_n^k = (n-1)c_{n-1}^k + (n-k-1)c_{n-2}^k,$$

where the letter c can be replaced by d or e, generalises the well-known relation of derangement numbers $d_n = (n-1)(d_{n-1} + d_{n-2})$. We will give a combinatorial interpretation to each recurrence relation defining the number $e_{n+k,m}^k$ of permutations over $n+k$ objects having m numbers of k-successions as well as its generating function. Kreweras [6] studied the numbers s_n which are the numbers e_n^1 and we will generalise his result . Let us denote by $[n]$ the interval $\{1\,2\cdots n\}$, by σ a permutation of the symmetric group $\mathfrak{S}_n$. In this paper, we will use the linear notation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and the notation of the decomposition in a product of disjoint cycles to write a permutation σ.

Definition 1.1. We say that the integer i is a k-succession for a permutation σ of the symmetric group $\mathfrak{S}_n$ if $\sigma(i) = i + k$.

Remark 1.2. A fixed point of the permutation σ is a 0-succession of the permutation σ and a succession of the permutation σ is a 1-succession of the permutation σ.

We will denote by $e_{n,m}^k$ the number of permutations of the symmetric group $\mathfrak{S}_n$ having m numbers of k-successions and by e_n^k the number of permutations of the symmetric group $\mathfrak{S}_n$ without k-successions.

Example 1.3. For $n = 10$, the permutation $\sigma = (1253)(478)(69)(10)$ has 2 numbers of 1-successions which are 1 and 7, 3 of 3-successions which are $2, 4$ and 6 and has 1 of 0-succession (or a fixed point) which is 10. The followings are the first sets of permutations without 1-successions

$$E_1^1 = \{(1)\},$$
$$E_2^1 = \{(1)(2)\},$$
$$E_3^1 = \{(1)(2)(3),\ (13)(2),\ (132)\},$$
$$E_4^1 = \{(1)(2)(3)(4),\ (13)(2)(4),\ (14)(2)(3),\ (1)(24)(3),\ (132)(4),\ (13)(24),$$
$$(143)(2),\ (142)(3),\ (1324),\ (1432),\ (1)(243)\}.$$

Definition 1.4. We say that a permutation σ is a k-fixed-points-permutation if for all integers i in the interval $[k]$, $\sigma^p(i) \notin [k] \setminus \{i\}$ for all integer p and $Fix(\sigma) \subseteq [k]$.

We will denote by D_n^k the set of k-fixed-points-permutations of the symmetric group $\mathfrak{S}_n$.

Example 1.5. We have

$$D_1^0 = \{\},\ D_1^1 = \{1\},$$
$$D_2^0 = D_2^1 = \{21\},\ D_2^2 = \{12\}.$$
$$D_3^0 = \{231, 312\},\ D_3^1 = \{132, 231, 312\},\ D_3^2 = \{132,\ 312\},\ D_3^3 = \{123\}.$$

Remark 1.6. The permutation $12\cdots k$ is the only k-fixed-points-permutation of the symmetric group $\mathfrak{S}_k$.

2. k-SUCCESSION DISTRIBUTION OVER SYMMETRIC GROUP

Theorem 2.1. *For all integers $k \geq 0$ and $n \geq m \geq 0$, we have*

$$e^k_{n+k,m} = \binom{n}{m} e^k_{n+k-m}.$$

Proof. For each subset A of $[n] = \{1, \dots, n\}$ of order m, the number of permutations $\pi \in \mathfrak{S}_{n+k}$ with set of k-successions A is equal to the number of permutations $\sigma \in \mathfrak{S}_{n+k-m}$ (ie equal to e^k_{n+k-m}). To see this, remove the numbers $\pi(a) = a+k$, $a \in A$, from $\pi(1)\pi(2)\cdots\pi(n)$ and standardise to get $\sigma(1)\sigma(2)\cdots\sigma(n+k-m)$. More formally, let us take the order preserving bijections $\iota : [n+k] \setminus A \to [n+k-m]$ and $\nu : [n+k] \setminus (A+k) \to [n+k-m]$ where $A+k = \{a+k : a \in A\}$ and we define $\sigma(i) = \nu \circ \pi \circ \iota^{-1}(i)$ for all $i \in [n+k-m]$. Notice that the permutation σ has no k-successions: if an integer $i \in [n+k-m]$ were a k-succession for σ, that is $\sigma(i) = \nu \circ \pi \circ \iota^{-1}(i) = i+k$, then we would have $\pi \circ \iota^{-1}(i) = \nu^{-1}(i+k) = \iota^{-1}(i)+k$ the integer $\iota^{-1}(i)$ would be a k-succession for the permutation π which contradicts the fact that we remove all the k-successions of the permutation π. Notice also that the inverse map $\sigma \mapsto \pi$ is defined by

$$\begin{cases} \pi(i) = \nu^{-1} \circ \sigma \circ \iota(i) \text{ for all } i \in [n+k] \setminus A \\ \pi(a) = a+k \text{ otherwise.} \end{cases}$$

□

Corollary 2.2. *For all integers $k \geq 0$ and $n \geq m \geq 1$, we have*

$$n e^k_{n+k-1,m-1} = m e^k_{n+k,m}.$$

Proof. By the equations $e^k_{n+k,m} = \binom{n}{m} e^k_{n+k-m}$ and $e^k_{n+k-1,m-1} = \binom{n-1}{m-1} e^k_{n+k-m}$ from the previous theorem, we obtain the result.

□

Proposition 2.3. *For $n \geq k$ and $n-k \geq m \geq 0$, we have*

$$e^k_{n,m} = e^k_{n-1,m-1} + (n-1-m)e^k_{n-1,m} + (m+1)e^k_{n-1,m+1}.$$

Proof. Notice that all permutation σ' in the set $\mathfrak{S}_n$ is obtained from a permutation σ in $\mathfrak{S}_{n-1}$ by multiplying σ on the left by a transposition $(i \quad n)$ for an integer $i \in [n]$. Notice also that if the integer i is a k-succession for σ, then when we multiply σ by the transposition $(i \quad n)$ on the left, we delete a k-succession for σ'. Now let us look for the various cases for the integer i

(1) If $i = n-k$, then the permutation $\sigma' = (i \quad n)\sigma$ has a new k-succession which is the integer i itself.
(2) If the integer i is a k-succession of the permutation σ, then the permutation $\sigma' = (i \quad n)\sigma$ has one fewer k-successions than the permutation σ.
(3) If the integer i is not a k-succession of the permutation σ and $i \neq n-k$, then the permutation $\sigma' = (i \quad n)\sigma$ has the same number of k-successions as the permutation σ.

It follows straightforwardly that we obtain all the permutations in the set $\mathcal{E}^k_n$ having m numbers of k-successions by considering all the permutation σ indicated in the following three cases and by multiplying them by the appropriate transposition

(1) $\sigma \in \mathfrak{S}_{n-1}$ having $m-1$ numbers of k-successions and the only possibility for the choice of transposition by which multiply σ is the transposition $(n-k \quad n)$.
(2) $\sigma \in \mathfrak{S}_{n-1}$ having m numbers of k-successions and there exist $n-1-m$ possibilities for the choice of the transposition: the transpositions $(i \quad n)$ where the integer i is not a k-succession of the permutation σ and which is not equal to $n-k$.
(3) $\sigma \in \mathfrak{S}_{n-1}$ having $m+1$ numbers of k-successions and there exist $m+1$ possibilities for the choice of the transposition: the transpositions $(i \quad n)$ where the integer i is a k-succession of the permutation σ.

□

3. Numbers e^k_{n+k}

We will give in section the two different relations satisfied by the numbers e^k_n of permutations without k-successions.

3.1. **Recurrence relations.** Notice the set D_n of derangements or permutations without fixed points is equal to $E^0_{n,0}$. This first relation is a generalization of the well-known relation on derangement numbers $d_n = (n-1)(d_{n-1} + d_{n-2})$.

Theorem 3.1 (First relation). *The numbers e^k_n satisfy the following recurrence relation*

$$e^k_n = (n-1)e^k_{n-1} + (n-1-k)e^k_{n-2}, \quad n \geq k \geq 0.$$

To give a combinatorial proof of this theorem, we need the following definition and its property.

Definition 3.2. For a given nonnegative integer k, let us consider the *k-transformation* $\vartheta : [n-k-1] \times \mathfrak{S}_{n-2} \to \mathfrak{S}_n$ defined as below.
For each pair $(j, \sigma) \in [n-k-1] \times \mathfrak{S}_{n-2}$, we define the permutation $\sigma' = \vartheta(j, \sigma)$ such that

$$\sigma'(i) = \begin{cases} \sigma(i) & \text{if } i < j \text{ and } \sigma(i) < j+k \\ \sigma(i)+1 & \text{if } i < j \text{ and } \sigma(i) \geq j+k \\ n & \text{if } i = j \\ \sigma(i-1) & \text{if } i > j \text{ and } \sigma(i-1) < j+k \\ \sigma(i-1)+1 & \text{if } i > j \text{ and } \sigma(i-1) \geq j+k \\ j+k & \text{if } i = n \end{cases}$$

Proposition 3.3. *The k-transformation ϑ perserves the number of k-successions, that is, if the permutation σ has m numbers of k-successions, then the permutation $\sigma' = \vartheta(j, \sigma)$ has m numbers of k-successions also.*

Proof. Notice that if the integer i is a k-succession for the permutation σ, then $\sigma'(i) = i + k$ if $i < j$ or $\sigma'(i+1) = \sigma(i) + 1 = i + k + 1$ if $i \geq j$. □

Proof of the Theorem 3.1. Let $\mathcal{E}^k_n$ be the set of permutations over n objects without k-successions. We obtain all permutation σ' of the set $\mathcal{E}^k_n$

- either by multiplying a permutation σ of the set $\mathcal{E}^k_{n-1}$ on the left by a transposition $(i \quad n)$ for an integer $i \in [n]$ and $i \neq n-k$. There exist $n-1$ possibilities of choice of the transposition $(i \quad n)$.
- or from a pair (j, σ) of the set $[n-k-1] \times \mathcal{E}^k_{n-2}$ by the k-transformation ϑ.

We deduce then that the numbers e_n^k satisfy the relation

$$e_n^k = (n-1)e_{n-1}^k + (n-1-k)e_{n-2}^k, \quad n \geq k.$$

$\square$

Remark 3.4. For $0 \leq n \leq k$, all permutations of the set $\mathfrak{S}_n$ do not have a k-succession, that is $e_n^k = n!$ for $0 \leq n < k$.

The following relation is the way Euler [1], [4] defined the difference table.

Theorem 3.5 (Second Relation)**.** *For $n \geq k$, we have*

$$e_n^k = e_{n-1}^{k-1} + e_n^{k-1}.$$

To prove this theorem, we need the following definitions as well as their properties.

Definition 3.6 (k-Vertical translation.)**.** A *k-vertical translation* of the permutation σ of the symmetric group $\mathfrak{S}_{n-1}$ is the permutation σ' of the symmetric group $\mathfrak{S}_n$ defined as below

$$\sigma'(i) = \begin{cases} \sigma(i)+1 & \text{if} \quad i \neq \sigma^{-1}(k-1) \\ 1 & \text{if} \quad i = \sigma^{-1}(k-1) \\ k & \text{if} \quad i = n \end{cases}$$

Proposition 3.7. *For a given permutation σ' in the symmetric group $\mathfrak{S}_n$ such that $\sigma'(n) = k$, we define the permutation σ of the symmetric group $\mathfrak{S}_{n-1}$ which is the antecedant of the permutation σ' by the k-vertical translation in the following way*

$$\sigma(i) = \begin{cases} \sigma'(i)-1 & \textit{if} \quad i \neq \sigma'^{-1}(1) \\ k-1 & \textit{if} \quad i = \sigma'^{-1}(1) \end{cases}$$

Definition 3.8 (Horizontal translation)**.** A horizontal translation of the permutation σ of the symmetric group $\mathfrak{S}_n$ is the permutation σ' of the symmetric group $\mathfrak{S}_n$ defined as below

$$\sigma'(i) = \begin{cases} \sigma(i+1) & \text{for} \quad i = 1, \ldots, n-1 \\ \sigma(1) & \text{if} \quad i = n \end{cases}$$

Proposition 3.9. *For a given permutation σ' in the symmetric group $\mathfrak{S}_n$ such that $\sigma'(n) \neq k$, we define the permutation σ of the symmetric group $\mathfrak{S}_n$ which is the antecedant of the permutation σ' by the horizontal translation in the following way*

$$\sigma(i) = \begin{cases} \sigma'(i-1) & \textit{for} \quad i = 2, \ldots, n \\ \sigma'(n) & \textit{if} \quad i = 1 \end{cases}$$

Lemma 3.10. *The k-vertical translation of the permutation $\sigma \in \mathcal{E}_{n-1}^{k-1}$ is a permutation of the set $\mathcal{E}_n^k$.*

Proof. Let $\sigma \in \mathfrak{S}_{n-1}$ a permutation without $(k-1)$-successions, that is, for all integer $j \leq n-k$ then $\sigma(j) \neq j+k-1$. The permutation σ' which is the k-vertical translation of the permutation σ defined in Definition 3.6 has no k-succession, that is, for all integer $j \leq n-k$ then $\sigma'(j) \neq j+k$. $\square$

Lemma 3.11. *The horizontal translation of the permutation $\sigma \in \mathcal{E}_n^{k-1}$ is a permutation of the set $\mathcal{E}_n^k$.*

Proof. Let $\sigma \in \mathfrak{S}_n$ a permutation without $(k-1)$-successions, that is, for all integer $j \leq n-k$ then $\sigma(j) \neq j+k-1$. The permutation σ' which is the horizontal translation of the permutation σ defined in Definition 3.8 do not have a k-succession, that is, for all integer $j \leq n-k$ then $\sigma'(j) \neq j+k$. □

Proof of the Theorem 3.5. All permutation σ' in the set $\mathcal{E}_n^k$ is obtained by a permutation σ in the set $\mathcal{E}_{n-1}^{k-1} \cup \mathcal{E}_n^{k-1}$ by the following transformation

1. If the permutation σ is an element of the set $\mathcal{E}_{n-1}^{k-1}$, then the permutation σ' is the k-vertical translation of the permutation σ.
2. If the permutation σ is an element of the set $\mathcal{E}_n^{k-1}$, then the permutation σ' is the horizontal translation of the permutation σ.

For the inverse transformation, notice that if the permutation $\sigma' \in \mathcal{E}_n^k$ such that $\sigma'(n) = k$ has no k-successions, then the permutation σ is an element of the set $\mathcal{E}_{n-1}^{k-1}$ defined by Proposition 3.7 and has no $(k-1)$-successions by Lemma 3.10 and in other cases, the permutation σ is an element of the set $\mathcal{E}_n^{k-1}$ defined by Proposition 3.9 and has no $(k-1)$-successions by Lemma 3.11. □

4. Numbers d_n^k

4.1. First relation of the numbers d_n^k.

Theorem 4.1. *For $0 \leq k \leq n-1$, we have*

$$d_n^k = (n-1)d_{n-1}^k + (n-k-1)d_{n-2}^k.$$

Proof. Let us consider the map $\varphi : D_n^k \to [n-1] \times D_{n-1}^k \cup [n-k-1] \times D_{n-2}^k$ which associates to each permutation σ a pair $(m, \sigma') = \varphi(\sigma)$ defined as follows

1. If the integer n is in a cycle of length greater or equal to 3, or the length of the cycle which contains the integer n is equal to 2 and $\sigma(n) \leq k$, then the integer m is equal to $\sigma^{-1}(n)$ and the permutation σ' is obtained from the permutation σ by removing the integer n from his cycle. The permutation σ' is indeed an element of the set D_{n-1}^k.
2. If the length of the cycle which contains the integer n is equal to 2 and $\sigma(n) > k$, then the integer m is equal to $\sigma(n)$ and the permutation σ' is obtained from the permutation σ by removing the cycle $(\sigma(n), n)$ and then decreasing by 1 all integers between $\sigma(n)+1$ and $n-1$ in each cycle. The permutation σ' is indeed an element of the set D_{n-2}^k.

The map φ is bijective. Notice that a pair (m, σ') in the image $\varphi(D_n^k)$ is contained either in the set of all pairs of $[n-1] \times D_{n-1}^k$ if the integer n lies in a cycle of length greater than 2 or equal to 2 and $\delta(n) \leq k$, or in the set of all pairs of $[n-k-1] \times D_{n-2}^k$ if the integer n lies in a cycle of length equal to 2 and $\delta(n) > k$. So it remains to prove that there exists a map $\tilde{\varphi}$ that

- associates an element D_n^k where the integer n lies in a cycle of length greater than 2 or equal to 2 and the integer n lies in a cycle which contains an integer less or equal to k with every pair of $[n-1] \times D_{n-1}^k$.
- associates an element D_n^k where the integer n lies in a cycle of length equal to 2 and the integer n lies in a cycle which contains an integer greater than k with every pair of $[n-k-1] \times D_{n-2}^k$.
- is the inverse of φ.

We define the permutation $\sigma = \tilde{\varphi}(m, \sigma')$ of the set D_n^k

- either by inserting the integer n in a cycle of the permutation σ' after the integer $m \in [n-1]$ if σ' is an element of the set D^k_{n-1}. In such case, the integer n lies in a cycle of length greater to 2 or in a transposition and $\sigma(n) \le k$.
- or by creating the transposition (m, n) with $k < m \le n-2$ and then increasing by 1 all integers between m and $n-2$ in each cycle of the permutation σ' if the permutation σ' is an element of the set D^k_{n-2}. In such case, the integer n is in a transposition and $\sigma(n) > k$.

The map $\tilde{\varphi}$ is the inverse of the map φ. □

Corollary 4.2. *The number d^k_n equals the cardinality of the set of k-fixed-points-permutations in the symmetric group $\mathfrak{S}_n$.*

4.2. **Second relation of the numbers d^k_n.** We will give another relation satisfied by the numbers d^k_n which is easily deduced from the generating function, but we will give its combinatorial interpretation.

Definition 4.3. Let us consider the map $\vartheta : D^{k-1}_{n-1} \cup D^{k-1}_n \to [k] \times D^k_n$ which associates to a permutation σ a pair $(m, \sigma') = \vartheta(\sigma)$ defined as below

(1) If $\sigma \in D^{k-1}_{n-1}$, then the integer m is equal to k and the permutation σ' is obtained from the permutation σ by creating the cycle (k) and then by increasing by 1 all integers greater or equal to k in each cycle of the permutation σ.
(2) If $\sigma \in D^{k-1}_n$, then the integer m is equal to the smallest integer in the cycle which contains the integer k and the permutation σ' is obtained from the permutation σ by removing the word $k \quad \sigma(k) \cdots \sigma^{-1}(m)$ and then creating the cycle $(k \quad \sigma(k) \cdots \sigma^{-1}(m))$.

Proposition 4.4. *The map ϑ is a bijection.*

Proof. The map ϑ is injective. It suffices to show that ϑ is surjective. Let us look at various cases of the pair (m, σ').

(1) If $m = k$ and $\sigma'(k) = k$, then we define the permutation σ by deleting the cycle (k) and then decreasing by 1 all integers greater than k in each cycle. It follows straightforwardly that the permutation σ is an element of the set D^{k-1}_{n-1}.
(2) If $m = k$ and $\sigma'(k) \neq k$, then $\sigma = \sigma'$ and $\sigma \in D^{k-1}_n$.
(3) If $m \neq k$, then the permutation σ is obtained from the permutation σ' by removing the cycle which contains k and then inserting the word $k\sigma'(k)\sigma'^2(k)\cdots$ in the cycle which contains the integer m just before the integer $\sigma'^{-1}(m)$. The permutation σ is indeed an element of the set D^{k-1}_n.

It is impossible by construction of the map ϑ that $m = k$ and the integer k is in the same cycle as an integer smaller than k. □

Theorem 4.5. *For all integers $1 \le k \le n$, we have*

$$kd^k_n = d^{k-1}_{n-1} + d^{k-1}_n.$$

Proof. By the bijection ϑ, we have

$$\#D^{k-1}_{n-1} + \#D^{k-1}_n = \#[k] \times D^k_n,$$

that is,

$$kd^k_n = d^{k-1}_{n-1} + d^{k-1}_n.$$

□

4.3. **Third relation for the numbers d_n^k.** The following unexpected relation is a generalization of the famous relation on derangement numbers and a bijective proof will be given.

Theorem 4.6. *For all integers $0 \leq k \leq n-1$, one has*

$$nd_{n-1}^k = d_n^k + d_{n-2}^{k-1}.$$

Proof. Let us consider the map $\varsigma : [n] \times D_{n-1}^k \to D_n^k \cup D_{n-2}^{k-1}$ which associates to a pair (m, σ) a permutation $\sigma' = \varsigma((m, \sigma))$ defined in the following ways

(1) If $m < n$, then the permutation σ' is obtained from the permutation σ by inserting the integer n in the cycle which contains m just before the integer m itself. The permutation σ' is indeed an element of the set D_n^k.
(2) If $m = n$ and $\sigma(1) \neq 1$, then the permutation $\sigma' = \varsigma((n, \sigma))$ is obtained from the permutation σ by removing the integer $\sigma(1)$ and then creating the cycle $(n \quad \sigma(1))$. The permutation σ' is indeed an element of the set D_n^k and $\sigma'(n) > k$.
(3) If $m = n$ and $\sigma(1) = 1$, then the permutation $\sigma' = \varsigma((n, \sigma))$ is obtained from the permutation σ by removing the cycle (1) and then by decreasing by 1 all integers in each cycle. It follows straightforwardly that the permutation σ' is an element of the set D_{n-2}^k.

The map ς is a bijection. The map ς is injective. It suffices to show that ς is surjective. Let us look at various cases of the permutation σ'.

(1) If the permutation σ' is an element of the set D_n^k and the cycle which contains n is different of the transposition $(n \quad \sigma'(n))$ where $\sigma'(n) > k$, then the couple (m, σ) is defined by $m = \sigma'^{-1}(n)$ and the permutation σ is obtained by removing the integer n from the cycle containing it.
(2) If the permutation σ' is an element of the set D_n^k and the cycle which contains n is a transposition $(n \quad \sigma'(n))$ where $\sigma'(n) > k$, then the couple (m, σ) is defined by $m = n$ and the permutation σ is obtained by removing the cycle $(n \quad \sigma'(n))$ and inserting the integer $\sigma'(n)$ in the cycle which contains the integer 1 just after 1.
(3) If the permutation σ' is an element of the set D_{n-2}^{k-1}, then the couple (m, σ) is defined by $m = n$ and the permutation σ is obtained by increasing by 1 all the integers in each cycle of the permutation σ' and then creating the new cycle (1).

$\square$

Remark 4.7. If we set $d_{-1}^{-1} = 1$ and $d_{n-1}^{-1} + d_n^{-1} = 0d_n^0$, that is, $d_{n-1}^{-1} + d_n^{-1} = 0$, then we obtain

$$d_n^0 + d_{n-2}^{-1} = nd_{n-1}^0.$$

We will give a combinatorial interpretation of this relation directly on derangements in the following section.

5. The famous $d_n = nd_{n-1} + (-1)^n$

Notice that the set D_n of derangements or permutations without fixed points is equal to the set D_n^0.

Definition 5.1. Let us define the *critical derangement* $\Delta_n = (1\,2)(3\,4)\cdots(n-1 \quad n)$ if the integer n is even and the sets

- $E_n = \{\Delta_n\}$ if the integer n is even and $E_n = \emptyset$ otherwise,

- $F_n = \{(n, \Delta_{n-1})\}$ if the integer n is odd and $F_n = \emptyset$ otherwise.

Let $\tau : [n] \times D_{n-1} \setminus F_n \to D_n \setminus E_n$ be the map which associates to a pair (i, δ) a permutation $\delta' = \tau((i, \delta))$ defined as below:

(1) If the integer $i < n$, then the permutation $\delta' = \delta(i \quad n)$. In other words, the permutation δ' is obtained from the permutation δ by inserting the integer n in the cycle which contains the integer i just after the integer i.
(2) If the integer $i = n$, then let p be the smallest integer such that the transpositions $(1\,2), (3\,4), \ldots, (2p-1 \quad 2p)$ are cycles of the permutation δ and the transposition $(2p+1 \quad 2p+2)$ is not.
   (a) If $\delta(2p+1) = 2p+2$, then the permutation δ' is obtained from the permutation δ by removing the integer $2p+1$ from the cycle which contains it, and then creating the new cycle $(2p+1 \quad n)$.
   (b) If $\delta(2p+1) \neq 2p+2$, then we have to distinguish the following two cases:
      (i) If the length of the cycle which contains the integer $2p+1$ is equal to 2, then the permutation δ' is obtained from the permutation δ by removing the cycle $(2p+1\,\delta(2p+1))$, and then inserting the integer $2p+1$ in the cycle which contains the integer $2p+2$ just before the integer $2p+2$ and creating the new cycle $(\delta(2p+1) \quad n)$.
      (ii) If the length of the cycle which contains the integer $2p+1$ is greater than 2, then then the permutation δ' is obtained from the permutation δ by removing the integer $\delta(2p+1)$ and then creating the new cycle $(\delta(2p+1) \quad n)$.

Proposition 5.2. *The map τ is bijective.*

Proof. Notice that the only pair (i, δ) which is not defined by the map τ is the pair (n, Δ_{n-1}) if the integer $n-1$ is even. Notice also that the image $\tau([n-1] \times D_{n-1})$ is contained in the set of all derangements D_n where the integer n lies in a cycle of length greater or equal to 3 and the image $\tau(\{n\} \times D_{n-1} \setminus F_n)$ is contained in the set of all derangements D_n where the integer n lies in a cycle of length equal to 2. So we only need to show that there exists a map ζ that

- associates an element of $[n-1] \times D_{n-1}$ with every derangement of D_n in which the integer n lies in a cycle of length greater or equal to 3.
- associates an element of $\{n\} \times D_{n-1} \setminus F_n$ with every derangement of D_n in which the integer n lies in a cycle of length 2.
- is the inverse of τ.

It is straightforward to verify that the map ζ is defined as follows.

(1) If the integer n lies in a cycle of length greater or equal to 3, then $\zeta(\delta)$ is the pair (i, δ') where $i = \delta^{-1}(n)$ and the permutation δ' is obtained by removing the integer n from the derangement δ. The permutation δ' is a derangement of D_{n-1} and the integer i is smaller than n.
(2) If the integer n lies in a cycle of length equal to 2, then let p the smaller nonnegative integer such that $(1\,2), (3\,4), \ldots, (2p-1 \quad 2p)$ are cycles of the derangement δ and the transposition $(2p+1 \quad 2p+2)$ is not.
   (a) If $\delta(n) = 2p+1$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n \quad 2p+1)$ and then inserting the integer $2p+1$ in the cycle which contains the integer $2p+2$ just before the integer $2p+2$.

In other words, we have
$\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad n)(2p+2\ldots)\cdots$ and
$\delta' = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad 2p+2\ldots)\cdots$.

(b) If $\delta(2p+1) \neq n$, then we have to distinguish the following two cases:

(i) If $\delta(2p+1) \neq 2p+2$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n\,\delta(n))$ and then inserting the integer $\delta(n)$ in the cycle which contains the integer $2p+1$ just before the integer $2p+1$.
In other words, we have
$\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1\ldots)\cdots(\delta(n) \quad n)\cdots$ and
$\delta' = (12)(34)\cdots(2p-1 \quad 2p)(2p+1\ldots\delta(n))\cdots$.

(ii) If $\delta(2p+1) = 2p+2$, then $\zeta(\delta)$ is the pair (n, δ') where the permutation δ' is obtained from the derangement δ by deleting the cycle $(n \quad \delta(n))$ and the integer $2p+1$ and then creating the new cycle $(2p+1 \quad \delta(n))$.
In other words, we have
$\delta = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad 2p+2\ldots)\cdots(\delta(n) \quad n)\cdots$ and
$\delta' = (12)(34)\cdots(2p-1 \quad 2p)(2p+1 \quad \delta(n))(2p+2\ldots)\cdots$.

Notice that the derangement Δ_n, if the integer n is even, is the only derangement which is not defined by the map ζ. □

Corollary 5.3. *If the integer n is even, then we have*

$$d_n = nd_{n-1} + 1.$$

If the integer n is odd, then we have

$$d_n + 1 = nd_{n-1}.$$

6. Generating functions

6.1. Generating of the difference table and its derivate.

Proposition 6.1. *The generating function $E^{(k)}(u) = \sum_{n\geq 0} e^k_{n+k}\frac{u^n}{n!}$ of the numbers e^k_{n+k} for a fixed integer k satisfies the following differential equation*

$$(1-u)E^{(k)'} = (k+u)E^{(k)}$$

with the initial condition $E^{(k)}(0) = k!$.

Proof. This differential equation could be deduced from the recurrence relation

$$e^k_{n+k} = (n+k-1)e^k_{n-1+k} + (n-1)e^k_{n-2+k}.$$

The initial condition is due to the fact that $e^k_k = k!$. □

Theorem 6.2. *The generating function $E^{(k)}(u) = \sum_{n\geq 0} e^k_{n+k}\frac{u^n}{n!}$ of the numbers e^k_{n+k} for a fixed integer k has the closed form*

$$E^{(k)}(u) = k!\frac{\exp(-u)}{(1-u)^{k+1}}.$$

Proof. The function $k!\dfrac{\exp(-u)}{(1-u)^{k+1}}$ satisfy the differential equation in Proposition 6.1 as well as the initial condition. □

Corollary 6.3. *The generating function* $D^{(k)}(u) = \sum_{n\geq 0} d^k_{n+k}\dfrac{u^n}{n!}$ *of the numbers* d^k_{n+k} *for a fixed integer* k *has the closed form*

$$D^{(k)}(u) = \frac{\exp(-u)}{(1-u)^{k+1}}.$$

Theorem 6.4. *The generating function* $E(x,u) = \sum_{k\geq 0}\sum_{n\geq 0} e^k_{n+k}\dfrac{x^k}{k!}\dfrac{u^n}{n!}$ *of the numbers* e^k_{n+k} *has the closed form*

$$E(x,u) = \frac{\exp(-u)}{1-x-u}.$$

Proof. We already have

$$\sum_{n\geq 0} e^k_{n+k}\frac{u^n}{n!} = k!\frac{\exp(-u)}{(1-u)^{k+1}}$$

by Theorem 6.2, and then

$$\sum_{k\geq 0}\sum_{n\geq 0} e^k_{n+k}\frac{x^k}{k!}\frac{u^n}{n!} = \sum_{k\geq 0} x^k\frac{\exp(-u)}{(1-u)^{k+1}}$$

is easily computed and gives the result. □

Corollary 6.5. *The generating function* $D(x,u) = \sum_{k\geq 0}\sum_{n\geq 0} d^k_{n+k}x^k\dfrac{u^n}{n!}$ *of the numbers* d^k_{n+k} *has the closed form*

$$D(x,u) = \frac{\exp(-u)}{1-x-u}.$$

6.2. **Generating function of the k-succession distribution over symmetric group.** We will give in this section the generating function

$$\tilde{E}(t,x,u) = \sum_{k\geq 0}\sum_{n\geq 0}\sum_{n\geq m\geq 0} e^k_{n+k,m}t^m\frac{x^k}{k!}\frac{u^n}{n!}$$

of the numbers $e^k_{n+k,m}$.

Proposition 6.6. *The generating function*

$$\tilde{E}^{(k)}(t,u) = \sum_{n\geq 0}\sum_{n\geq m\geq 0} e^k_{n+k,m}t^m\frac{u^n}{n!}$$

of the numbers $e^k_{n+k,m}$ *for a fixed nonnegative integer* k *satisfies the following partial differential equation*

$$u\tilde{E}^{(k)} = \frac{\partial \tilde{E}^{(k)}}{\partial t}$$

with the initial condition

$$\tilde{E}^{(k)}(0,u) = E^{(k)}(u) = k!\frac{\exp(-u)}{(1-u)^{k+1}}.$$

Proof. This partial differential equation can easily be deduced from the relation in Theorem 2.2 and the initial condition is due to the fact that for $t = 0$, the function $\tilde{E}^{(k)}(0, u)$ is the generating function $E^{(k)}(u)$ of the numbers e^k_{n+k} which is given in Theorem 6.2. □

Theorem 6.7. *The generating function*

$$\tilde{E}^{(k)}(t, u) = \sum_{n\geq 0} \sum_{n\geq m\geq 0} e^k_{n+k,m} t^m \frac{u^n}{n!}$$

of the numbers $e^k_{n+k,m}$ for a fixed nonnegative integer k has the closed form

$$\tilde{E}^{(k)}(t, u) = k! \frac{\exp u(t-1)}{(1-u)^{(k+1)}}.$$

Proof. The partial diffirential equation in Proposition 6.6 is easily computed and has the solution

$$f(u) \exp(tu).$$

The initial condition defines the function $f(u)$ and gives the result. □

Theorem 6.8. *The generating function*

$$\tilde{E}(t, x, u) = \sum_{k\geq 0} \sum_{n\geq 0} \sum_{n\geq m\geq 0} e^k_{n+k,m} t^m \frac{x^k}{k!} \frac{u^n}{n!}$$

of the numbers $e^k_{n+k,m}$ has the closed form

$$\tilde{E}(t, x, u) = \frac{\exp u(t-1)}{1-x-u}.$$

Proof. We already have

$$\sum_{n\geq 0} \sum_{n\geq m\geq 0} e^k_{n+k,m} t^m \frac{u^n}{n!} = k! \frac{\exp u(t-1)}{(1-u)^{(k+1)}}$$

by Theorem 6.7. Then

$$\sum_{k\geq 0} \sum_{n\geq 0} \sum_{n\geq m\geq 0} e^k_{n+k,m} t^m \frac{x^k}{k!} \frac{u^n}{n!} = \sum_{k\geq 0} \frac{\exp u(t-1)}{(1-u)^{(k+1)}} x^k$$

is easily computed and gives the result. □

7. Explicit formula for the number $e^k_{n+k,m}$

We will generalise in this section the inclusion-exclusion relation of the derangement numbers [13] and the relation established by Kreweras for e^1_n [6].

Theorem 7.1. *The number e^k_{n+k} has the closed formula*

$$e^k_{n+k} = n! \sum_{p=0}^{n} \frac{(-1)^p (n+k-p)!}{p!(n-p)!}.$$

Proof. The Theorem 2.1 yields

$$(n+k)! = \sum_{m=0}^{n} e^k_{n+k,m} = \sum_{m=0}^{n} \binom{n}{m} e^k_{n+k-m}.$$

Using Inclusion-Exclusion, we get

$$e_{n+k}^{k} = \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} (p+k)!.$$

□

Corollary 7.2. *The number d_n^k has the closed formula*

$$d_n^k = \sum_{i=0}^{k} (-1)^i \binom{n-k}{i} \frac{(n-i)!}{k!}.$$

Remark 7.3. (1) For $k=0$, we obtain the well-known relation of the derangement numbers

$$d_n = n! \sum_{p=0}^{n} \frac{(-1)^p}{p!}.$$

(2) for $k=1$, we obtain the explicit formula for the number of permutations of the symmetric group $\mathfrak{S}_n$ without *successions* established by Kreweras [6]

$$s_n = (n-1)! \sum_{p=0}^{n} \frac{(-1)^p (n-p)}{p!}.$$

Theorem 7.4. *For $k \geq 0$ and $n \geq m \geq 0$, we have*

$$e_{n+k,m}^{k} = \frac{n!}{m!} \sum_{p=0}^{n-m} \frac{(-1)^p (n+k-m-p)!}{p!(n-m-p)!}.$$

Proof. Applying Theorem 7.1 and Theorem 2.1 again, we obtain

$$e_{n+k,m}^{k} = \binom{n}{m} e_{n-m+k}^{k} = \binom{n}{m} \sum_{p=0}^{n-m} (-1)^{n-m-p} \binom{n-m}{p} (p+k)!.$$

□

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