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# Compound basis arising from the basic $A_1^{(1)}$-module

Kazuya Aokage, Hiroshi Mizukawa

Résumé. A new basis for the polynomial ring of infinitely many variables is constructed which consists of products of Schur functions and $Q$-functions. The transition matrix from the natural Schur function basis is investigated.

Résumé. On construit une nouvelle base pour l'anneau des polynômes d'une infinité de variables. Cette base consiste en produits de fonctions de Schur et Q-fonctions, dont la matrice de passage à la base naturelle de fonctions de Schur est aussi étudiée.

## 1. Introduction

We are interested in realizations of the basic representation of the simplest affine Lie algebra $A_1^{(1)}$(cf. [**4**]). The most well-known realization is $PU$, principal, untwisted, which is on the space

$$V^{PU} = \mathbb{C}[t_j; j \geq 1, \mathrm{odd}].$$

In the context of nonlinear integrable systems, this space appears as that of the KdV hierarchy. The second one is $HU$, homogeneous, untwisted, which is on

$$V^{HU} = \bigoplus_{m\in\mathbb{Z}} V^{(m)};\ V^{(m)} = \mathbb{C}[t_j; j \geq 1] \otimes q^m.$$

This space is for the NLS (nonlinear Schrödinger) hierarchy and for the Fock representation of the Virasoro algebra(cf. [**3**]). The third one is $PT$, principal, twisted, on $V^{PT}$ which coincides with $V^{PU}$. And the forth one is $HT$, homogenous, twisted, on $V^{HT}$ which is the same as $V^{HU}$. The Lie algebra of type $A_1^{(1)}$ is isomorphic to that of type $D_2^{(2)}$. One can discuss twisted version of $A_1^{(1)}$-modules via this isomorphism.

The purpose of this note is to give a weight basis for $V^{HT}$ and compare it with a standard Schur function basis for $V^{HU}$. We will show that the transition matrix has several interesting combinatorial properties.

## 2. A quick review of realizations

Let us first consider the principal untwisted version on $V^{PU} = \mathbb{C}[t_j; j \geq 1, \mathrm{odd}]$. To describe a weight basis for this space we need Schur functions and Schur's $Q$-functions in our setting. Let $P_n$ be the set of all partitions of $n$ and put $P = \bigcup_{n\geq 0} P_n$. For $\lambda \in P_n$, the Schur function $S_\lambda(t)$ is defined by

$$S_\lambda(t) = \sum_{\rho=(1^{m_1}2^{m_2}\ldots)\in P_n} \chi_\rho^\lambda \frac{t_1^{m_1} t_2^{m_2} \cdots}{m_1! m_2! \cdots},$$

where the summation runs over all partitions $\rho = (1^{m_1}2^{m_2}\ldots)$ of $n$ and $\chi_\rho^\lambda$ is the irreducible character of the symmetric group $\mathfrak{S}_n$, indexed by $\lambda$ and evaluated at the conjugacy class $\rho$. The Schur functions are the irreducible characters of the general linear groups. If the group element $g$ has eigenvalues $x_1, x_2, \ldots$, then the original irreducible character is recovered by putting $t_j := p_j/j$ $(j \geq 1)$, where $p_j = \sum_{i\geq 1} x_i^j$ is the $j$-th power sum.

The 2-reduction of a polynomial $f(t)$ is to "kill" the even numbered variables $t_2, t_4, \ldots$, i.e. ,

$$f^{(2)}(t) = f(t)|_{t_2=t_4=\ldots=0} \in V^{PU}.$$

The 2-reduced Schur functions are linearly dependent in general. However all linear relations among them are known and one can choose certain set $P' \subset P$ so that $\left\{S_\lambda^{(2)}; \lambda \in P'\right\}$ forms a basis for $V^{PU}$ (cf. [**1**]).

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The space $V^{PU}$ also affords the principal twisted realization. A weight basis is best described by Schur's $Q$-functions. Let $SP_n$ (resp. $OP_n$) be the set of all strict (resp. odd) partitions of $n$ and put $SP = \bigcup_{n\geq 0} SP_n$, $OP = \bigcup_{n\geq 0} OP_n$. For $\lambda \in SP_n$ the $Q$-function $Q_\lambda(t)$ is defined by

$$Q_\lambda(t) = \sum_{\rho=(1^{m_1}3^{m_3}\ldots)\in OP_n} 2^{\frac{\ell(\lambda)-\ell(\rho)+\epsilon}{2}} \zeta^\lambda_\rho \frac{t_1^{m_1}t_3^{m_3}\cdots}{m_1!m_3!\cdots},$$

where the summation runs over all odd, partitions $\rho = (1^{m_1}3^{m_3}\ldots)$ of $n$, $\epsilon = 0$ or $1$ according to that $\ell(\lambda) - \ell(\rho)$ is even or odd and $\zeta^\lambda_\rho$ is the irreducible spin character of $\mathfrak{S}_n$, indexed by $\lambda$ and evaluated at the conjugacy class $\rho$. For the $Q$-functions, we set $t_j = 2p_j/j$ $(j \geq 1, \text{odd})$ as the relation with the "eigenvalues". A more detailed account is found in [**7**].

In order to give the homogeneous, twisted realization we employ a combinatorics of strict partitions. We introduce the following h-abacus. For example, the h-abacus of $\lambda = (11, 10, 5, 3, 2)$ is shown below.

$$\begin{array}{ccc}
 & 1 & \textcircled{3} \\
\textcircled{2} & & \\
4 & \textcircled{5} & 7 \\
6 & & \\
8 & 9 & \textcircled{11} \\
\textcircled{10} & & \\
12 & 13 & 15 \\
\vdots & \vdots & \vdots
\end{array}$$

From this h-abacus of $\lambda$ we read off a triplet $(\lambda^{hc}; \lambda^h[0], \lambda^h[1])$ of partitions. Firstly $\lambda^h[0] = (5, 1)$, obtained just by taking halves of the circled positions of the leftmost column.

For obtaining $\lambda^h[1]$, we need the following process:

(1) For the third column, the circled positions correspond to the vacancies "$\circ$".
(2) For the second column, the circled positions correspond to being occupied "$\bullet$".
(3) Read the third column from infinity to the position 3 and consequently the second column from the position 1 to infinity, and make the Maya diagram

$$\begin{array}{ccccccccc}
\ldots & 15 & 11 & 7 & 3 & 1 & 5 & 9 & \ldots. \\
 & \underline{\bullet} & \circ & \bullet & \circ & \circ & \bullet & \underline{\circ} &
\end{array}$$

(4) For each $\bullet$, count the number of vacancies which are on the left of that $\bullet$, and get a partition

$$\lambda^h[1] = (3, 1).$$

Next the h-core $\lambda^{hc}$ is obtained by the following moving and removing:

(1) Remove all circles on the leftmost column.
(2) Move a circle one position up along the second or the third column.
(3) Remove the two circles at the positions 1 and 3 simultaneously.
(4) The "stalemate" determines the partition

$$\lambda^{hc} = (3).$$

Note that $\lambda^{hc}$ is always of the form

$$\Delta^h(m) = (4m-3, 4m-7, \ldots, 5, 1) \text{ or } \Delta^h(-m) = (4m-1, 4m-5, \ldots, 7, 3)$$

for some $m \in \mathbb{N}$ $(\Delta^h(0) = \emptyset)$. Let $HC$ be the set of all $\lambda^{hc}$'s. In this way we have a one-to-one correspondence between $\lambda \in SP$ and $(\lambda^{hc}; \lambda^h[0], \lambda^h[1]) \in HC \times SP \times P$ with the condition

$$|\lambda| = |\lambda^{hc}| + 2(|\lambda^h[0]| + 2|\lambda^h[1]|).$$

By making use of this one-to-one correspondence, we define the linear map $\eta : V^{PT} \to V^{HT}$ by

$$\eta(Q_\lambda(t)) = Q_{\lambda^h[0]}(t) S_{\lambda^h[1]}(t') \otimes q^{m(\lambda)}.$$

Here

$$m(\lambda) = (\text{number of circles on the second column}) - (\text{number of circles on the third column})$$

and $S_\nu(t') = S_\nu(t)|_{t_j \mapsto t_{2j}}$ for any $j \geq 1$. For any integer $m$, the set

$$\{\eta(Q_\lambda); \lambda \in SP, m(\lambda) = m\}$$

forms a basis for $V^{(m)} = \mathbb{C}[t_j; j \geq 1] \otimes q^m$ (cf. [**2**]). Under the condition $m = 0$, there is a one-to-one correspondence between the following two sets for any $n \geq 0$:

(i) $\{\lambda \in SP_{2n}; \lambda^{hc} = \emptyset\}$.
(ii) $\{(\mu, \nu) \in SP_{n_0} \times P_{n_1}; n_0 + 2n_1 = n\}$.

## 3. Compound basis

We begin with some bijections between sets of partitions. The first one is

$$\phi : P_n \longrightarrow \bigcup_{n_0+2n_1=n} SP_{n_0} \times P_{n_1}$$

defined by $\lambda \mapsto (\lambda^s, \lambda^p)$. Here the multiplicities $m_i(\lambda^s)$ and $m_i(\lambda^p)$ are given respectively by

$$m_i(\lambda^s) = \begin{cases} 1 & m_i(\lambda) \equiv 1 \pmod 2 \\ 0 & m_i(\lambda) \equiv 0 \pmod 2, \end{cases}$$

and

$$m_i(\lambda^p) = \begin{cases} \frac{1}{2}(m_i(\lambda) - 1) & m_i(\lambda) \equiv 1 \pmod 2 \\ \frac{1}{2}(m_i(\lambda)) & m_i(\lambda) \equiv 0 \pmod 2. \end{cases}$$

For example, if $\lambda = (5^3 4^4 2^7 1)$ then $\lambda^s = (521)$ and $\lambda^p = (54^2 2^3)$. We set

$$P_{n_0,n_1} = \phi^{-1}(SP_{n_0} \times P_{n_1}).$$

The second bijection is

$$\psi : P_n \longrightarrow \bigcup_{n_1+2n_2=n} OP_{n_1} \times P_{n_2}$$

defined by $\psi(\lambda) = (\lambda^o, \lambda^e)$. Here $\lambda^o$ is obtained by picking up the odd parts of $\lambda$, while $\lambda^e$ is obtained by taking halves of the even parts. For example, if $\lambda = (5^3 4^4 2^7 1)$, then $\lambda^o = (5^3 1)$ and $\lambda^e = (2^4 1^7)$.

The third bijection is called the Glaisher map. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a strict partition of $n$. Suppose that $\lambda_i = 2^{p_i} q_i$ $(i = 1, 2, \ldots)$, where $q_i$ is odd. Then an odd partition $\tilde{\lambda}$ of $n$ is defined by

$$m_{2j-1}(\tilde{\lambda}) = \sum_{q_i = 2j-1, i \geq 1} 2^{p_i}.$$

For example, if $\lambda = (8, 6, 4, 3, 1)$ then $\tilde{\lambda} = (3^3, 1^{13})$. This gives a bijection between $SP_n$ and $OP_n$.

Definition 3.1. Define a bijection $\pi$ on $P_n$ by

$$\pi(\lambda) = \psi^{-1}(\widetilde{\phi(\lambda)}) \ (\lambda \in P_n).$$

Here we remark that $\pi$ gives a bijection between $\phi^{-1}(SP_{n_1} \times P_{n_2})$ and $\psi(OP_{n_1} \times P_{n_2})$.

Proposition 3.2. *Let $(n_0, n_1)$ be fixed. Then we have*

$$\sum_{\lambda \in P_n} \ell(\lambda) = \sum_{\lambda \in P_n} (\ell(\lambda^s) + 2\ell(\lambda^p)) = \sum_{\lambda \in P_n} (\ell(\lambda^o) + \ell(\lambda^e)) = \sum_{\lambda \in P_n} (\ell(\tilde{\lambda^s}) + \ell(\lambda^e)),$$

$$\sum_{\lambda \in P_{n_0,n_1}} \ell(\lambda) = \sum_{\lambda \in P_{n_0,n_1}} (\ell(\lambda^s) + 2\ell(\lambda^p)) = \sum_{\lambda \in P_{n_0,n_1}} (\ell(\lambda^o) + \ell(\lambda^e)),$$

$$\sum_{\lambda \in P_n} 2\ell(\lambda^p) = \sum_{\lambda \in P_n} 2\ell(\lambda^e) = \sum_{\lambda \in P_n} (\ell(\lambda^o) + \ell(\lambda^e) - \ell(\lambda^s)) = \sum_{\lambda \in P_n} (\ell(\tilde{\lambda^s}) + \ell(\lambda^e) - \ell(\lambda^s)),$$

*and*

$$\sum_{\lambda \in P_{n_0,n_1}} 2\ell(\lambda^p) = \sum_{\lambda \in P_{n_0,n_1}} (\ell(\lambda^o) + \ell(\lambda^e) - \ell(\lambda^s)).$$

Looking at the representation spaces $V^{HU}$ and $V^{HT}$, we have the following two natural bases for the space

$$V_n^{(0)} = \mathbb{C}[t_j; j \geq 1]_n \text{ (the homogenous component of degree } n).$$

Namely we have

(i) $\{S_\lambda(t); \lambda \in P_n\}$,
(ii) $\{Q_{\lambda^s}(t)S_{\lambda^p}(t'); \lambda \in P_n\}$.

For simplicity we write

$$W_\lambda(t) = Q_{\lambda^s}(t)S_{\lambda^p}(t')$$

for $\lambda \in P_n$ and call the set (ii) the compound basis for $V_n^{(0)}$.

Our problem is to determine the transition matrix between these two bases. Let $A_n = (a_{\lambda\mu})$ be defined by

$$S_\lambda(t) = \sum_{\mu \in P_n} a_{\lambda\mu} W_\mu(t) \tag{1}$$

for $\lambda \in P_n$.

Here we remark the relation between our basis and the $Q^{'}$-functions. Lascoux, Leclerc and Thibon (cf. [**5**]) introduced the $Q^{'}$-functions as the basis for $V_n^{(0)}$ dual to $P$-functions with respect to the inner product

$$\langle F(t), G(t)\rangle_0 := F(\tilde{\partial})\overline{G(t)}|_{t=0},$$

where $\tilde{\partial} = (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots)$. For a strict partition $\mu$ we see that $Q^{'}_\mu(t) = Q_\mu(2t)$. Also, for a partition $\lambda$, we see that

$$Q^{'}_\lambda(t) = Q^{'}_{\lambda^s}(t)h_{\lambda^p}(t^{'})$$

where $h_{\lambda^p}$ is the complete symmetric function indexed by $\lambda^p$. Therefore the transition from $W_\lambda$ to $Q^{'}_\mu$ is essentially given by the Kostka numbers.

## 4. transition matrices

In the previous section, functions are expressed in terms of the "time variables" $t = (t_1, t_2, \ldots)$ of the soliton equations. However, for the description and the proof of our formula, it is more convenient to use the "original" variables of the symmetric functions, i.e., the eigenvalues $x = (x_1, x_2, \ldots)$.

The definition (1) of $a_{\lambda\mu}$ is rewritten as

$$S_\lambda(x, x) = \sum_{\mu \in P_n} a_{\lambda\mu} Q_{\mu^s}(x)S_{\mu^p}(x^2),$$

where $(x, x) = (x_1, x_1, x_2, x_2, \ldots)$ and $x^2 = (x_1^2, x_2^2, \ldots)$. Hereafter we will denote

$$W_\lambda(x) = Q_{\lambda^s}(x)S_{\lambda^p}(x^2),\ V_\lambda(x) = P_{\lambda^s}(x)S_{\lambda^p}(x^2),$$

where $P_{\lambda^s}(x) = 2^{-\ell(\lambda^s)}Q_{\lambda^s}(x)$. Also we will set the following spaces of symmetric functions

$$\Lambda = \mathbb{C}[p_r(x); r \geq 1],\ \Gamma = \mathbb{C}[p_r(x); r \equiv 1 \pmod 2],$$

and

$$\Gamma^{'} = \mathbb{C}[p_r(x); r \equiv 0 \pmod 2]$$

so that

$$\Lambda \cong \Gamma \otimes \Gamma^{'}.$$

We have two bases for $\Lambda$:

$$W = (W_\lambda(x))_\lambda \text{ and } V = (V_\lambda(x))_\lambda.$$

First we notice the following Cauchy identity.

PROPOSITION 4.1.

$$\prod_{i,j} \frac{1}{(1 - x_i y_j)^2} = \sum_{\lambda \in P} W_\lambda(x)V_\lambda(y).$$

By a standard argument, we have

Corollary 4.2.

$$\langle W_\lambda(x), V_\mu(x)\rangle_{-1} = \delta_{\lambda\mu},$$

*where we define the inner product* $\langle\ ,\ \rangle_{-1}$ *on* $\Lambda$ *by requiring that* $\langle p_\rho, p_\sigma\rangle_{-1} = 2^{-\ell(\rho)} z_\rho \delta_{\rho\sigma}$.

We will use another inner product $\langle\ ,\ \rangle_0$ on $\Lambda$ which is defined by $\langle p_\rho, p_\sigma\rangle_0 = z_\rho\delta_{\rho\sigma}$.

Theorem 4.3. *The matrix* $A_n$ *is integral.*

Example 4.4.

$$A_3 = \begin{array}{c|ccc} & (3,\emptyset) & (21,\emptyset) & (1,1) \\ \hline (3) & 1 & 0 & 1 \\ (21) & 1 & 1 & 0 \\ (1^3) & 1 & 0 & -1 \end{array}$$

$$A_4 = \begin{array}{c|ccccc} & (4,\emptyset) & (31,\emptyset) & (\emptyset,2) & (\emptyset,1^2) & (2,1) \\ \hline (4) & 1 & 0 & 1 & 0 & 1 \\ (31) & 1 & 1 & -1 & 0 & 1 \\ (2^2) & 0 & 1 & 1 & 1 & 0 \\ (1^4) & 1 & 0 & 0 & 1 & -1 \\ (21^2) & 1 & 1 & 0 & -1 & -1 \end{array}.$$

Theorem 4.5.

$$|\det A_n| = 2^{k_n},$$

*where* $k_n = \sum_{\lambda\in P_n} \ell(\lambda^e) = \sum_{\lambda\in P_n} (\ell(\tilde{\lambda^s}) - \ell(\lambda^s))$.

Example 4.6.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $k_n$ | 0 | 1 | 1 | 4 | 5 | 11 | 15 | 28 | $\cdots$ |

For simplicity we define the Green function $X_\sigma^\lambda$ by

$$Q_\lambda(x) = \sum_\sigma 2^{\ell(\sigma)} z_\sigma^{-1} X_\sigma^\lambda p_\sigma$$

which is easily computed from the spin character $\zeta_\sigma^\lambda$. Our Frobenius formula for $W$ reads

$$p_\sigma p_{2\rho} = \sum_{\lambda\in P_{n_0,n_1}} 2^{-\ell(\lambda^s)} X_\sigma^{\lambda^s} \chi_\rho^{\lambda^p} W_\lambda(x)$$

for $\sigma \in OP_{n_0}$ and $\rho \in P_{n_1}$. This gives that the transition matrix $M(p, W)$ is decomposed into diagonal blocks according to $(n_0, n_1)$. Hence

$$\begin{aligned} {}^t A_n A_n &= {}^t M(p,W)\, {}^t M(\tilde{S},p) M(\tilde{S},p) M(p,W) \\ &= {}^t M(p,W) D_n^2 Z_n^{-1} M(p,W). \end{aligned}$$

Since $D_n^2 Z_n^{-1}$ is diagonal, ${}^t A_n A_n$ is block diagonal.

Example 4.7.

$${}^t A_3 A_3 = \begin{array}{c} \\ (3,\emptyset) \\ (21,\emptyset) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{ccc} (3,\emptyset) & (21,\emptyset) & (1,1) \end{array} \\ \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \end{array}$$

$${}^t A_4 A_4 = \begin{array}{c} \\ (4,\emptyset) \\ (31,\emptyset) \\ (\emptyset,2) \\ (\emptyset,1^2) \\ (2,1) \end{array} \begin{array}{c} \begin{array}{ccccc} (4,\emptyset) & (31,\emptyset) & (\emptyset,2) & (\emptyset,1^2) & (2,1) \end{array} \\ \begin{pmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

For a pair $(n_0, n_1)$, let denote by $B_{n_0,n_1}$ the corresponding block in ${}^t A_n A_n$.

THEOREM 4.8.

$$|\det B_{n_0,n_1}| = 2^{\sum_{\lambda\in P_{n_0,n_1}}(\ell(\tilde{\lambda^s})+\ell(\lambda^p)-\ell(\lambda^s))}.$$

For the "principal" block $B_{n,0}$, we have

$$|\det B_{n,0}| = 2^{\sum_{\lambda\in SP_n}(\ell(\tilde{\lambda})-\ell(\lambda))}.$$

In this case, for each $\lambda \in SP_n$, $2^{\ell(\tilde{\lambda})-\ell(\lambda)}$ gives an elementary divisor of $B_{n,0}$. It is interesting that these elementary divisors coincide with those of the Cartan matrix for $\mathfrak{S}_n$ at characteristic $p = 2$ (cf. [**8**]). We conclude this note with an inner product expression of ${}^tA_nA_n$.

PROPOSITION 4.9.

$${}^tA_nA_n = \Big(\langle P_{\lambda^s}(x), P_{\mu^s}(x)\rangle_0 \langle S_{\lambda^p}(x^2), S_{\mu^p}(x^2)\rangle_0\Big)_{\lambda,\mu}.$$

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KAZUYA AOKAGE, DEPARTMENT OF MATHEMATICS, OKAYAMA UNIVERSITY, OKAYAMA 700-8530, JAPAN
*E-mail address*: `k.aokage@math.okayama-u.ac.jp`

HIROSHI MIZUKAWA, DEPARTMENT OF MATHEMATICS, NATIONAL DEFENSE ACADEMY IN JAPAN, YOKOSUKA 239-8686, JAPAN
*E-mail address*: `mzh@nda.ac.jp`