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Kazhdan-Lusztig immanants III: Transition matrices between canonical bases of immanants

Brendon Rhoades and Mark Skandera

Abstract. We study two bases of the vector space of immanants of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$: the bitableaux basis of Désarménien-Kung-Rota, and a subset of the dual canonical basis called the basis of Kazhdan-Lusztig immanants. We show that the transition matrix between these bases is unitriangular, describe new vanishing results for the Kazhdan-Lusztig immanants, and relate both bases to other immanants defined in terms of characters of S_n.

Résumé. Nous étudions deux bases de l'espace des "immanants" dans $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$: la base de Désarménien-Kung-Rota, et la base de Kazhdan-Lusztig. Nous montrons que la maitrice de transition entre les deux bases est triangulaire, nous décrivons de nouveaux résultats de disparaition pour les immanants de Kazhdan-Lusztig, et nous rapportons les deux bases á d'autres immanants définis en termes de caractéres de S_n.

1. Introduction

The Kazhdan-Lusztig immanants (K-L immanants), named in [**RS06**] and first appearing perhaps in [**Du92**], are a collection of immanants indexed by permutations in S_n. The determinant is the K-L immanant indexed by the identity permutation. These immanants are defined using Kazhdan-Lusztig polynomials, which arose in the study of representations of the Hecke algebra $\mathcal{H}_n(q)$ [**KL79**]. These immanants posess various positivity properties, and results in [**RS05b**] provide combinatorial methods of testing other polynomials for these properties. (See [**LPP06**] for applications of these tests to problems in Schur positivity.)

The K-L immanants form a basis of the vector space of immanants, and are related unitriangularly to another basis which comes from the work of Désarménien, Kung, and Rota [**DKR78**]. The elements of this latter basis are called standard bitableaux and are indexed by ordered pairs of standard Young tableaux having the same shape. A fixed standard bitableau $(U{:}T)$ corresponds to a product of matrix minors, where the row and column sets involved are determined by the entries in the standard Young tableaux U and T. Désarménien, Kung, and Rota defined bitableaux in the context of combinatorial invariant theory.

Our main result in this paper is that the unitriangularity mentioned above is guaranteed by certain appropriate orderings of the basis elements. These appropriate orderings are realized as linear extensions of a partial order on S_n which is defined via the Schensted correspondence and an iterated version of the dominance order. We show that this result extends from the finite dimensional space of immanants to the full polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{nn}]$. In this context, our result gives a unitriangularity relation between a basis of $\mathbb{C}[x_{11}, \ldots, x_{nn}]$ indexed by pairs of semistandard Young tableaux and the dual canonical basis of the polynomial ring. We give some conditions satisfied by the off-diagonal elements of these transition matrices.

In [**RS05a**] K-L immanants and pattern avoidance are used to define a filtration of the vector space of immanants which is shown to be equivalent to a filtration which may be defined via products of matrix minors. In this paper we generalize the results of [**RS05a**] by defining a pair of collections of subspaces of immanants indexed by partitions. The inclusion partial orders on these collections are poset isomorphic to the dominance order. As before, one of these collections is defined using K-L immanants and the other is

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defined via bitableaux. Also as before, these collections of subspaces are shown to be equivalent. The proof of this equivalence relies strongly on the previously mentioned triangularity results. We give this collection of subspaces a graded structure corresponding to multiplication of immanants and determine where the irreducible character immanants fit in this collection. Finally, we give a result which characterizes when a K-L immanant factors as a product of smaller K-L immanants.

2. Background

For $n \in \mathbb{N}$, let $x = (x_{ij})_{1 \leq i,j, \leq n}$ be a matrix of n^2 variables. Denote by $\mathcal{I}_n(x)$ the subspace

$$\mathrm{span}_{\mathbb{C}}\{x_{1,w(1)} \cdots x_{n,w(n)} | w \in S_n\}$$

of the polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{nn}]$. $\mathcal{I}_n(x)$ is a vector space of dimension $n!$ and its elements are called immanants. Given any function $f : S_n \to \mathbb{C}$, define the immanant induced by f to be the immanant $\mathrm{Imm}_f(x)$ given by $\mathrm{Imm}_f(x) = \sum_{w \in S_n} f(w) x_{1,w(1)} \cdots x_{n,w(n)}$. Evidently, any immanant is $\mathrm{Imm}_f(x)$ for an appropriate choice of $f : S_n \to \mathbb{C}$.

Denote by $\mathrm{Par}(n)$ the set of all partitions of n. For any partition λ, let λ' denote its conjugate, whose diagram is obtained by reflecting the diagram of λ across its main diagonal. Given $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k) \in \mathrm{Par}(n)$, define the Young subgroup S_λ of S_n to be the set of permutations which respect the partition of the letters in $[n]$ given by $\{1, \ldots, \lambda_1\}$, $\ldots$, $\{n - \lambda_k + 1, \ldots, n\}$. The dominance order on $\mathrm{Par}(n)$ is defined by $\lambda \geq \mu$ if and only if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for every i (appending an infinite string of zeros to the tails of λ and μ so that these sums are always defined). The dominance order on $\mathrm{Par}(n)$ plays a key role in the representation theory of the symmetric group and will be critical in our results, as well. Write $\mathcal{P}_n$ for the set $\mathrm{Par}(n)$ endowed with dominance order.

Given a skew shape λ/μ, a λ/μ tableau is an assignment of a positive integer to every node of the diagram of λ/μ. A λ/μ tableau is normal if $\mu = \emptyset$. Given a tableau T, let $\mathrm{sh}(T) \in \mathrm{Par}(n)$ denote the shape of T. Using English notation, a tableau T is said to be (row) column (semistrict) strict if the entries of T (weakly) strictly increase (across rows) down columns. T is said to be semistandard if it is row semistrict and column strict. A normal tableau T with n nodes is said to be injective if every letter in $[n]$ appears exactly once in T. T is said to be standard if it is both semistandard and injective. Denote by $\mathcal{SYT}_n$ the set of all standard tableaux with n boxes. If T is a tableau and $S \subset \mathbb{N}$, let $T - S$ denote the diagram formed by deleting all of the entries of T which are in S. We record a classical result relating tableaux and the dominance order.

LEMMA 2.1. *(The Dominance Lemma) Suppose T and U are injective tableaux with n nodes. If* $\mathrm{sh}(T) \not\leq \mathrm{sh}(U)$ *in the dominance order, then there are at least two elements of the set $[n]$ which are in the same column in U but in different columns of T.*

A proof of the Dominance Lemma may be found, for example, in [**Sag01**].

A *bitableau* is an ordered pair $(U{:}T)$ of tableaux satisfying $\mathrm{sh}(U) = \mathrm{sh}(T)$. Given any property P on tableaux (such as injective, standard, ...), a bitableau is said to have property P if both of its entries have property P.

Désarménien, Kung, and Rota [**DKR78**] use bitableaux to get a basis of the polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{nn}]$ as follows. If I, J are subsets of $[n]$ satisfying $|I| = |J|$, define the I, J-minor of x, written $\Delta_{I,J}(x)$, to be the determinant of the submatrix of x with row set I and column set J. Given any column strict bitableau $(U{:}T)$ with entries drawn from $[n]$, define a polynomial $(U{:}T)(x)$ by

$$(U : T)(x) = \Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x),$$

where $I_1, \ldots, I_k$ are the columns of U and $J_1, \ldots, J_k$ are the columns of T. Abusing terminology, we will sometimes refer to the polynomials $(U{:}T)(x)$ as bitableaux. For example, if $n = 3$, we have that

$$\begin{pmatrix} 1 & 2 & 1 & & 2 & 1 & 3 \\ 3 & & & : & 3 & & \end{pmatrix}(x) = \begin{vmatrix} x_{12} & x_{13} \\ x_{32} & x_{33} \end{vmatrix} |x_{21}||x_{13}| = x_{12}x_{13}x_{21}x_{33} - x_{13}^2 x_{21} x_{32}.$$

As the above example shows, bitableaux are not in general immanants. It is easy to see, however, that the following holds.

OBSERVATION 2.2. A bitableau $(U{:}T)(x)$ is in $\mathcal{I}_n(x)$ if and only if $(U{:}T)$ is injective.

It is easy to see that the set of all bitableaux spans the polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{nn}]$. In fact, any monomial $x_{11}^{\alpha_{11}} \cdots x_{nn}^{\alpha_{nn}}$ may be expressed as a bitableau. The remarkable fact is that restricting to semistandard bitableaux leads to a basis of this ring.

THEOREM 2.3 ([**DKR78**]). *The set of all semistandard bitableaux is a basis of the polynomial ring* $\mathbb{C}[x_{11}, \ldots, x_{nn}]$.

The authors in [**DKR78**], in fact, give an explicit straightening algorithm by which one can expand any monomial as a $\mathbb{Z}$-linear combination of the above basis elements. Combining their result with our observation, we get a basis of $\mathcal{I}_n(x)$.

COROLLARY 2.1. *The set of all standard bitableaux is a basis of* $\mathcal{I}_n(x)$.

The Robinson-Schensted-Knuth (RSK) algorithm gives a bijection between the symmetric group S_n and the set of all standard bitableaux with n nodes. An introduction to this algorithm and its properties may be found, for example, in [**Sta99**]. In this paper,

$$w \mapsto (P(w) : Q(w))$$

will always mean that the permutation w maps to the bitableau $(P(w){:}Q(w))$ under *row* insertion. For example, taking $n = 5$,

$$52413 \mapsto \left(\begin{array}{ccccc} 1 & 3 & & 1 & 3 \\ 2 & 4 & : & 2 & 5 \\ 5 & & & 4 & \end{array} \right).$$

Two permutations w and v are said to belong to the same (right) left Knuth class if $(Q(w) = Q(v))$ $P(w) = P(v)$. Also, the *shape* of w, written $\mathrm{sh}(w)$, is defined to be the shape of either $P(w)$ or $Q(w)$.

Given any $i \in [n-1]$, let s_i denote the adjacent transposition in S_n whose left action on a permutation w, using one-line notation, has the effect of switching positions i and $(i+1)$ of w. Notice that this left action is the opposite of that used by Björner and Brenti in [**BB05**] - this will have consequences when we consider interpretations of Kazhdan-Lusztig cells as left or right Knuth classes. So, for example, $s_3 * 13245 = 13425$. For $1 \leq i \leq j \leq n$, let $S_{[i,j]}$ denote the subgroup of S_n which fixes all the letters not contained in the interval $[i, j]$.

Given any pair of permutations $u, v \in S_n$, we have a *Kazhdan-Lusztig polynomial* $P_{u,v}(q) \in \mathbb{N}[q]$. In [**KL79**], Kazhdan and Lusztig define these polynomials as the (appropriately normalized) coefficients of a basis of the Hecke algebra $\mathcal{H}_n(q)$ which is invariant under the bar involution. Specializing this basis to $q = 1$, we obtain a basis $\{C'_w(1) \,|\, w \in S_n\}$ of the symmetric group algebra $\mathbb{C}[S_n]$ which satisfies

$$C'_w(1) = \sum_{v \in S_n} P_{v,w}(1) v.$$

The basis $\{C'_w(1) \,|\, w \in S_n\}$ is referred to as the *Kazhdan-Lusztig basis*. We record some elementary facts about Kazhdan-Lusztig polynomials before continuing.

LEMMA 2.4. $P_{u,v}(q) \equiv 0$ *if and only if* $u \not\leq v$ *in the (strong) Bruhat order. Also,* $P_{u,u}(q) \equiv 1$ *always.*

Using the Kazhdan-Lusztig polynomials, the authors define the K-L immanants [**RS06**]. Given a permutation $w \in S_n$, define the *w-Kazhdan-Lusztig immanant* $\mathrm{Imm}_w(x) \in \mathcal{I}_n(x)$ by the formula

$$\mathrm{Imm}_w(x) = \sum_{v \in S_n} (-1)^{\ell(v) - \ell(w)} P_{w_o v, w_o w}(1) x_{1,v(1)} \cdots x_{n,v(n)}.$$

Here ℓ is the length function on S_n and w_o is the long element of S_n. We have that $\mathrm{Imm}_1(x) = \det(x)$, so K-L immanants are a generalization of the determinant. It follows from the above Lemma that the transition matrix between the set $\{\mathrm{Imm}_w(x) \,|\, w \in S_n\}$ of K-L immanants and the natural basis $\{x_{1,w(1)} \cdots x_{n,w(n)} \,|\, w \in S_n\}$ of $\mathcal{I}_n(x)$ may be taken to be triangular with 1's on the diagonal. Therefore, K-L immanants form a basis of $\mathcal{I}_n(x)$.

Melnikov [**Mel92**] defines and Taskin [**Taş06**] further studies a partial order on $\mathcal{SYT}_n$ which we shall find useful. Given a skew tableau T which is row and column strict, let $j(T)$ denote the normal tableau obtained by playing jeu de taquin on T. Given a standard Young tableau $T \in \mathcal{SYT}_n$ and an interval $1 \leq i \leq j \leq n$, let $T_{[i,j]}$ denote the element of $\mathcal{SYT}_{j-i+1}$ obtained by reducing all the entries of $j(T - ([1, i-1] \cup [j+1, n]))$

by $(i-1)$. For example, if $T = \begin{matrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{matrix}$, we have that $T_{[2,4]} = \begin{matrix} 1 & 2 \\ 3 & \end{matrix}$. Using this notation, define the chain order on $\mathcal{SYT}_n$ by setting $U \leq_{chain} T$ if and only if for every interval $[i,j]$ in $[n]$ we have that $\mathrm{sh}(U) \leq \mathrm{sh}(T)$ in the dominance order.

3. Vanishing Results

Given any square complex matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ and function $f : S_n \to \mathbb{C}$, we may consider applying the f-immanant $\mathrm{Imm}_f(x)$ to the matrix A via the formula

$$\mathrm{Imm}_f(A) = \sum_{w \in S_n} f(w) a_{1,w(1)} \cdots a_{n,w(n)},$$

where $A = (a_{ij})$. Our first step in studying the transition matrix between the bases of $\mathcal{I}_n(x)$ introduced in the previous section will be to give conditions on matrices A under which the immanants $\mathrm{Imm}_w(x)$ vanish on A for a given permutation w.

Our first lemma to this end relates restricted tableaux and RSK. For a permutation $w \in S_n$, let $w_{[i,j]}$ denote the permutation in S_{j-i+1} which is the normalized version of the sequence $w_i w_{i+1} \dots w_j$.

LEMMA 3.1. *Suppose that* $w \mapsto (P(w) : Q(w))$. *We have that* $\mathrm{sh}(Q(w)_{[i,j]}) = \mathrm{sh}(w_{[i,j]})$.

PROOF. Omitted.

$\square$

Next, we need a Lemma about the stability of certain subsets of $\mathbb{C}[S_n]$ under the right and left actions of certain parabolic subgroups of S_n.

LEMMA 3.2. *Let* $[i,j]$ *be an interval in* $[n]$ *and let* λ *be a partition of* $j-i+1$. *Consider* $\mathbb{C}[S_n]$ *as a* $(\mathbb{C}[S_{[i,j]}], \mathbb{C}[S_n])$*-bimodule. Define the linear subspace* W_λ *of* $\mathbb{C}[S_n]$ *by*

$$W_\lambda = \mathrm{span}\{C'_w(1) \,|\, \mathrm{sh}(Q(w)_{[i,j]}) \leq \lambda\},$$

where $\leq$ *is dominance order. Then,* W_λ *is in fact a sub-bimodule of* $\mathbb{C}[S_n]$.

PROOF. Define the two-sided, left, and right preorders $\leq_{LR}$, $\leq_R$, and $\leq_L$ on S_n as in [**Gec06**]. Identifying the cells of these preorders with two-sided and one-sided Knuth classes, denote the induced orders on $\mathcal{SYT}_n$ and $\mathrm{Par}(n)$ by $\leq_{LR}$, $\leq_R$, and $\leq_L$, as well.

Let $X_{[i,j]}$ be a set of distinguished Bruhat minimal *right* coset representatives for $S_{[i,j]}$ in S_n. Now any permutation $w \in S_n$ may be expressed uniquely as ux for some $u \in S_{[i,j]}$ and $x \in X_{[i,j]}$. Let Y denote the set $\{w \,|\, \mathrm{sh}(Q(w)_{[i,j]} \leq \lambda\} \subseteq S_n$. Given ux and vy in S_n factored as above with $ux \in Y$, suppose that $C'_{vy}(1)$ appears with nonzero coefficient in $zC'_{ux}(1)$ for some $z \in S_{[i,j]}$.

By the work of Geck on relative Kazhdan-Lusztig cells [**Gec06**], we also have that $u \leq_{LR} v$ when we consider $\leq_{LR}$ on the parabolic subgroup $S_{[i,j]}$ of S_n. By the Bruhat minimality of the coset representatives in $X_{[i,j]}$, it follows from Lemma 3.1 that $\mathrm{sh}(Q(ux)_{[i,j]}) = \mathrm{sh}(u)$ and $\mathrm{sh}(Q(vy)_{[i,j]}) = \mathrm{sh}(v)$.

Now, by the fact that the two-sided order $\leq_{LR}$ on $\mathrm{Par}(n)$ is the opposite of the dominance order, we have that $\mathrm{sh}(v) \leq \mathrm{sh}(u) \leq \lambda$ in dominance, since ux was assumed to be in Y. It follows that vy is also contained in Y, and therefore W_λ is stable under the left action of $\mathbb{C}[S_{[i,j]}]$.

To complete the proof we must show that W_λ is also stable under the right action of $\mathbb{C}[S_n]$. It can be shown that the opposite of the $\leq_R$ order on $\mathcal{SYT}_n$ is weaker than the chain order on $\mathcal{SYT}_n$. That is, $T \leq_R U$ implies that $T \leq_{chain} U$. The result now follows from the definitions of the chain order and the right Kazhdan-Lusztig order $\leq_R$.

$\square$

Next, we need a result about Bruhat maximal permutations in Young subgroups. For any subset $X \subseteq S_n$, let $[X]$ denote the formal sum $\sum_{x \in X} x$.

LEMMA 3.3. *Let* $\lambda \in Par(n)$ *and let* s_λ *denote the unique Bruhat maximal element in the Young subgroup* S_λ *of* S_n. *We have that* $\mathrm{sh}(s_\lambda) = \lambda'$. *Also, the Kazhdan-Lusztig basis element* $C'_{s_\lambda}(1)$ *is equal to* $[S_\lambda]$.

Proof. The assertion about the shape of s_λ is easy to verify from the definition of the Robinson-Schensted-Knuth algorithm.

Since s_λ avoids the patterns 4231 and 3412, the Schubert variety Γ_{s_λ} corresponding to s_λ is smooth and the Kazhdan-Lusztig polynomials $P_{u,s_\lambda}(q)$ are all identically 1 for any $u \in S_\lambda$.

It follows that

$$\begin{aligned} C'_{s_\lambda}(q) &= \sum_{u \in S_n} P_{u,s_\lambda}(q) u \\ &= \sum_{u \in S_\lambda} P_{u,s_\lambda}(q) u \\ &= \sum_{u \in S_\lambda} u \\ &= [S_\lambda], \end{aligned}$$

as desired. □

In analogy to the vanishing of the determinant on any matrix with two equal rows, we define a family functions which record repetition of rows in certain submatrices of a matrix. Specifically, for any interval $[i,j]$ in $[n]$, we define a function $\mu_{[i,j]} : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Par}(j-i+1)$ by letting $\mu_{[i,j]}(A)$ list the number of occurrences of each distinct row in rows i through j of A in weakly decreasing order. For example, if $n = 6$ and

$$A = \begin{pmatrix} 1 & 0 & 3 & 4 & 0 & 2 \\ 2 & 1 & 1 & 3 & 1 & 4 \\ 3 & 2 & 7 & 5 & 6 & 0 \\ 2 & 1 & 1 & 3 & 1 & 4 \\ 0 & -1 & 3 & 1 & 6 & 0 \\ 0 & -1 & 3 & 1 & 6 & 0 \end{pmatrix},$$

we have that $\mu_{[2,5]}(A) = (2,1,1)$.

We now state our vanishing result for K-L immanants.

Proposition 3.1. *Let A be an $n \times n$ complex matrix and let $[i,j]$ be an interval in $[n]$. Suppose $w \in S_n$ satisfies the condition $\mu_{[i,j]}(A) \not\leq \mathrm{sh}(Q(w)_{[i,j]})'$ in dominance. Then,* $\mathrm{Imm}_w(A) = 0$.

Proof. As before, we consider $\mathbb{C}[S_n]$ as a $(\mathbb{C}[S_{[i,j]}], \mathbb{C}[S_n])$-bimodule. Define the element $[A]$ by

$$[A] = \sum_{v \in S_n} a_{1,v(1)} \cdots a_{n,v(n)} v.$$

Let H denote the subgroup of $S_{[i,j]}$ which permutes the equal rows of A which occur in rows i through j. We have that

$$\begin{aligned} &= [H]g \\ &= x[S_{\mu_{[i,j]}(A)}]g', \end{aligned}$$

where $S_{\mu_{[i,j]}(A)}$ is a Young subgroup, g and g' are elements of $\mathbb{C}[S_n]$ which depend on A, and x is an element of $S_{[i,j]}$. By the duality of the Kazhdan- Lusztig basis of $\mathbb{C}[S_n]$ and the K-L immanants we may also express $[A]$ as

$$[A] = \sum_{v \in S_n} \mathrm{Imm}_v(A) C'_v(1).$$

In the notation of Lemma 3.2, mod out by $W_{\mu_{[i,j]}(A)'}$ and let $\theta : \mathbb{C}[S_n] \to \mathbb{C}[S_n]/W_{\mu_{[i,j]}(A)'}$ be the canonical projection of bimodules. By the Lemma 3.3, θ kills $[A]$. So, we have that

$$\begin{aligned} 0 &= \theta([A]) \\ &= \textstyle\sum_{v \in S_n} \mathrm{Imm}_v(A) \theta(C'_v(1)) \\ &= \textstyle\sum_v \mathrm{Imm}_v(A) \theta(C'_v(1)), \end{aligned}$$

where the last sum is over those permutations v in S_n which do not satisfy $\mathrm{sh}(Q(v)_{[i,j]}) \leq \mu_{[i,j]}(A)'$ in dominance. By the Lemma 3.2, the collection $\{\theta(C'_v(1))\}$ for all such permutations v is linearly independent, so $\mathrm{Imm}_v(A) = 0$ for each such permutation v.

□

While the previous proposition was formulated by considering equal rows of A, we could just as well have considered equal columns. Specifically, for any interval $[i,j]$ in $[n]$, define $\nu_{[i,j]} : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Par}(n)$ by letting $\nu_{[i,j]}(A)$ list the number of occurrences of distinct equal columns in columns i through j of A in decreasing order.

COROLLARY 3.1. *Suppose that $A \in Mat_n(\mathbb{C})$ and $w \in S_n$ satisfy $\nu_{[i,j]}(A) \not\leq \mathrm{sh}(P(w)_{[i,j]})'$ in dominance. Now,* $\mathrm{Imm}_w(A) = 0$.

PROOF. From [**RS06**] we get that $\mathrm{Imm}_w(A) = \mathrm{Imm}_{w^{-1}}(A^T)$. A standard property of the RSK correspondence is that w maps to $(P(w) : Q(w))$ if and only if w^{-1} maps to $(Q(w) : P(w))$. Combining these two facts with the above proposition yields the corollary. □

Specializing the above corollary somewhat, we recover an earlier result of the authors.

COROLLARY 3.2 ([**RS05a**]). *If A has k equal rows and w has no decreasing subsequence of length k, then* $\mathrm{Imm}_w(A) = 0$.

PROOF. Combine the previous Proposition with $i = 1$ and $j = n$ with the fact that the length of the first column of $\mathrm{sh}(w)$ is the length of the longest decreasing subsequence of w. □

A further specialization to the case $w = 1$ yields the familiar linear algebra fact that $\det(A) = 0$ whenever A has 2 equal rows. Extrapolating from linear algebra, one might suspect that a basis-free analogue of the above results would also hold. For example, $\mathrm{Imm}_w(A)$ might vanish if some condition on the matroid formed by considering the rows of A is satisfied. However, these basis-free conjectures typically fail due to the fact that the K-L immanants corresponding to permutations other than 1 are not basis independent, as can be readily checked.

An example of the failure of the obvious basis-free generalization of the preceding Corollary is that for $n = 4$

$$\mathrm{Imm}_{3412}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} = 2 - 1 + 0 - 0 = 1 \neq 0,$$

although the rowspace of the above matrix has dimension $2 \leq 4-2$, and 3412 has no decreasing subsequence of length 3.

4. Main Triangularity Results

With the vanishing results of the previous section in hand, we are equipped to prove that the transition matrices between the bases of $\mathcal{I}_n(x)$ developed in the introduction are unitriangular with respect to certain orderings of the basis elements. To realize these orderings, we define a partial order on S_n as follows.

We start by recalling the definition of a partial order on standard Young tableaux which we call *iterated dominance order* and denote by $\leq_{itdom}$. This order was used by Ehresmann (see [pp. 30-31][**Mat99**]) and occasionally appears elsewhere in the literature. For $U, T \in \mathcal{SYT}_n$, define $U \leq_{itdom} T$ if for every $i \in [n]$ we have that $\mathrm{sh}(U_{[1,i]}) \leq \mathrm{sh}(T_{[1,i]})$ in dominance. Observe that, since U and T are assumed to be standard, the restricted tableaux $U - [i+1, n]$ and $T - [i+1, n]$ are normal tableaux, so the games of jeu de taquin required to produce $U_{[1,i]}$ and $T_{[1,i]}$ are empty. Also notice that iterated dominance order is stronger than the chain order, i.e., $U \leq_{chain} T$ implies $U \leq_{itdom} T$.

Iterated dominance order induces a partial order on bitableaux by restricting the canonical induced order on the Cartesian product $\mathcal{SYT}_n \times \mathcal{SYT}_n$. Pulling back via RSK yields a partial order on S_n. Abusing terminology somewhat, we shall also call this partial order iterated dominance order and denote it by $\leq_{itdom}$.

We shall need an analogue of the vanishing results of the previous section which gives conditions for bitableaux vanishing. This task is much easier.

LEMMA 4.1. *Suppose A is an $n \times n$ matrix and $(U : T)(x)$ is a bitableau. If $\mu_{[1,j]}(A) \not\leq_{dom} \mathrm{sh}(U_{[1,j]})$, we have that* $(U : T)(A) = 0$.

PROOF. This immediately reduces to the case $j = n$, which is a consequence of Lemma 2.1. □

We record one final ingredient of our proof of triangularity. It is well known that the length of the first column of $\mathrm{sh}(w)$ is equal to the length of the longest decreasing subsequence of w. To get more information about w from its shape, recall that a k-decreasing subsequence of w is a disjoint union of k-decreasing subsequences of w. We now have the following result of Greene.

THEOREM 4.2. *(**[Gre74]**) (Greene's Theorem) The length of the longest k-decreasing subsequence of w is equal to the sum of the lengths of the first k columns of* $\mathrm{sh}(w)$.

THEOREM 4.3. *Let $(w_1 < \cdots < w_{n!})$ be any linear extension of the iterated dominance order on S_n. The transition matrix between the ordered bases*

$$\langle \mathrm{Imm}_{w_1}(x), \ldots, \mathrm{Imm}_{w_{n!}}(x) \rangle, \qquad \langle (Q(w_1)' : P(w_1)')(x), \ldots, (Q(w_{n!})' : P(w_{n!})'(x) \rangle$$

of $\mathcal{I}_n(x)$ is triangular with 1's on the diagonal.

PROOF. Fix a bitableau $(U : T)(x)$ and let $(U : T)(A)$ denote the result when it is applied to a complex matrix A. Since the K-L immanants form a basis of $\mathcal{I}_n(x)$, we can find numbers $d_w \in \mathbb{C}$ so that

$$(T' : U')(x) = \sum_{w \in S_n} d_w \mathrm{Imm}_w(x).$$

We show by induction that d_w is zero unless $w \leq_{itdom} v$, where v is the unique permutation which maps to $(U : T)$ under RSK.

Fix an integer i in $[n]$. Fix a positive integer k, as well and put

$$M = \mathrm{sh}(Q(v)_{[1,i]})'_1 + \cdots + \mathrm{sh}(Q(v)_{[1,i]})'_k.$$

For any positive integer c, define

$$T_c = \{w \in S_n | \mathrm{sh}(Q(w)_{[1,i]})'_1 + \cdots + \mathrm{sh}(Q(w)_{[1,i]})'_k = c\}.$$

Fix t such that $M < t \leq n$. Suppose by induction that for any c satisfying $t < c \leq n$, we have that $d_w = 0$ for each $w \in T_c$. Give the elements in T_t a total ordering $\{w_1 < \cdots < w_h\}$ which is a linear extension *of their Bruhat ordering* (not $\mathcal{C}_n$). Fix $r \in [h]$ and assume by induction that $d_{w_m} = 0$ for every m such that $1 \leq m < r$.

Now associate an $n \times n$ complex matrix B to the permutation w_r as follows. Since w_r is in T_t, by Greene's theorem and Lemma 3.1 we must have that the subsequence $w_{r_1} \ldots w_{r_i}$ must contain a k-decreasing subsequence σ of length t. Express σ as

$$\sigma = \sigma_1 \uplus \cdots \uplus \sigma_k$$

for some decreasing subsequences $\sigma_1, \ldots, \sigma_k$ of the first i letters of w_r. Without loss of generality we may assume that $|\sigma_1| \geq \cdots \geq |\sigma_k|$, where $|\bullet|$ denotes the length of a sequence. Let P be the permutation matrix of w_r. Let B be the matrix formed by replacing the submatrices of P with row sets $w_r^{-1}(\sigma_1), \ldots, w_r^{-1}(\sigma_k)$ and column sets $\sigma_1, \ldots, \sigma_k$ with matrices of 1's. It is clear that

$$\mu_{[1,i]}(B) = (|\sigma_1|, \ldots, |\sigma_k|, 1, \ldots, 1).$$

If $J < t$ and $w \in T_j$, we have that

$$\begin{aligned}
\mu_{[1,i]}(B)_1 + \cdots + \mu_{[1,i]}(B)_k &= |\sigma_1| + \cdots + |\sigma_k| \\
&= \mathrm{sh}(Q(w_r)_{[1,i]})'_1 + \cdots + \mathrm{sh}(Q(w_r)_{[1,i]})'_k \\
&= t \\
&> J \\
&= \mathrm{sh}(Q(w)_{[1,i]})'_1 + \cdots + \mathrm{sh}(Q(w)_{[1,i]})'_k.
\end{aligned}$$

By the vanishing result of Proposition 3.4, this implies that $\mathrm{Imm}_w(B) = 0$.

By the results on Kazhdan-Lusztig polynomials in Lemma 2.4 and the definition of our matrix B, we have that

$$\mathrm{Imm}_{w_{r+1}}(B) = \cdots = \mathrm{Imm}_{w_h}(B) = 0.$$

Also by Lemma 2.4 and the construction of B, we have that $\mathrm{Imm}_{w_r}(B) = 1$.

Finally, by a pigeonhole argument, it is easy to check that two of the equal rows of B fall into one of the minors in the product $(T' : U')(x)$, which implies that $(T' : U')(B) = 0$. Putting all of this together, we get that $d_{w_r} = 0$.

Since r, t, k, and $[1, i]$ were all arbitrary, we get that unless $Q(w) \leq_{itdom} Q(v)$, $d_w = 0$. Now, by giving an identical argument with Corollary 3.5 used in the place of Proposition 3.4, we may conclude that unless $P(w) \leq_{itdom} P(v)$, one has $d_w = 0$. Putting these facts together with the definition of the iterated dominance order yields that unless $w \leq_{itdom} v$, we have that $d_w = 0$.

This result implies that the transition matrix between any of the orders described in the claim is triangular. It remains to show that this matrix has 1's on the diagonal. This argument is far easier.

Let A be the $n! \times n!$ triangular matrix with respect to one of the orders in the claim whose columns list the Kazhdan-Lusztig basis elements expressed in the bitableaux basis. By [**DKR78**] we may conclude that the entries of A are all integers. It is easy to show via inverse Kazhdan-Lusztig polynomials that the entries of A^{-1} are all nonnegative integers. Since A and A^{-1} are both triangular matrices, these conditions force $\det(A) = \det(A^{-1}) = 1$ and all the diagonal entries of A to be 1.

□

Theorem 4.3 states that any linear extension of $\leq_{itdom}$ leads to a triangular transition matrix between the K-L immanant and standard bitableaux bases with 1's on the diagonal. It would be interesting to see if there are weaker partial orders on S_n (having possibly more linear extensions) under which this result goes through. In addition to the tightening effect this would have, such a result would also allow us to determine more of the entries of the transition matrix since all of the entries which are not compatible with any linear extension must be 0. The authors have checked that the no such weaker order exists for $n \leq 4$.

One way to get a partial order on S_n which is weaker than $\leq_{itdom}$ would be to define a partial order on $\mathcal{SYT}_n$ which is weaker than the iterated dominance order and induce a partial order on S_n as before. However, it is easy to check that even for $n = 3$ the chain order proves too weak for this task.

The authors do not know a complete description of the entries of the transition matrix in Theorem 4.3. However, it is possible to obtain some information about the expansion of row superstandard bitableaux in the K-L immanant basis. Recall that, given a shape λ in $\mathrm{Par}(n)$, the row superstandard tableau RSS_λ of shape λ is the tableaux obtained by filling the letters in $[n]$ in reading order into the shape λ. For example, $RSS_{(3,2)}$ is the tableau

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix} \quad .$$

PROPOSITION 4.1. *Let λ be in $\mathrm{Par}(n)$. Write the bitableaux $(RSS_\lambda{:}RSS_\lambda)(x)$ as*

$$(RSS_\lambda : RSS_\lambda)(x) = \sum_{w \in S_n} d_w \mathrm{Imm}_w(x)$$

for some numbers d_w. We have that $d_w \in \mathbb{P}$ for every w satisfying $\mathrm{sh}(w) \leq_{dom} \lambda$.

PROOF. Omitted. □

5. Extension to the Dual Canonical Basis

While our results so far appear to be tied to the finite dimensional vector space $\mathcal{I}_n(x)$, in this section we show that they may be extended with ease to the ambient polynomial ring $\mathbb{C}[x_{11}, \ldots, x_{nn}]$. Specifically, we naturally extend the bitableaux and K-L immanants bases of $\mathcal{I}_n(x)$ to (countably infinite) bases of the full polynomial ring. For the case of bitableaux, this extension is trivial in light of Theorem 2.2: we just require our bitableau to be merely semistandard (with entries drawn from $[n]$, of course). In the extension of the K-L immanant basis, however, something far more interesting happens.

Given two multisets M and N with entries drawn from $[n]$, such that $|M| = |N|$, we may define the *generalized submatrix* $x_{M,N}$ of x corresponding to (M, N) to be the $|M| \times |M|$ array formed by listing the rows of x in M and the columns of x in N in weakly increasing order. For example, if $n = 3$, $M = 1123$, and $N = 2233$, then we have that

$$x_{M,N} = \begin{pmatrix} x_{12} & x_{12} & x_{13} & x_{13} \\ x_{12} & x_{12} & x_{13} & x_{13} \\ x_{22} & x_{22} & x_{23} & x_{23} \\ x_{32} & x_{32} & x_{33} & x_{33} \end{pmatrix}$$

So, generalized submatrices of x are formed by repeating or deleting rows and columns of x. We may now consider the set $\{\mathrm{Imm}_w(x_{M,N})\}$ where M and N range over all weakly increasing multisets with the same size and entries from $[n]$ and w ranges over all permutations in the symmetric group $S_{|M|} = S_{|N|}$. It is easy to see that this set spans $\mathbb{C}[x_{11}, \ldots, x_{nn}]$. On the other hand, since $\det(x_{M,N}) = 0$ whenever any entry of $[n]$ occurs in M or N with multiplicity greater than 1, this set cannot be a basis. Using results of Du [**Du92**], it was shown in [**Ska06**] that the nonzero elements of this set are precisely the elements of a famous basis of $\mathbb{C}[x_{11}, \ldots, x_{nn}]$ called the Dual Canonical Basis. The Dual Canonical Basis had been defined more abstractly in the work of Lusztig, and this elementary characterization makes the following result trivial.

THEOREM 5.1. *The (countably infinite) transition matrix between the Dual Canonical Basis and the basis of semistandard bitableaux may be taken to be a direct sum of finite triangular matrices with 1's on the diagonal.*

PROOF. Apply Theorem 4.2 with repeated and deleted rows and columns. □

6. $\mathcal{P}_n$-filtrations

Given a vector space V and a poset P, a P-filtration is a collection $\mathcal{C}$ of subspaces of V such that the inclusion partial order on $\mathcal{C}$ is isomorphic to P as a poset. For example, if $V = \mathbb{C}^n$ and B_n is the Boolean lattice, then $\{\mathrm{span}\{\epsilon_s | s \in S\} \,|\, S \subseteq [n]\}$ is a B_n-filtration. In the case where P is a chain, P-filtrations reduce to ordinary filtrations.

We restrict ourselves to the case $V = \mathcal{I}_n(x)$ and $P = \mathcal{P}_n$ and define two $\mathcal{P}_n$-filtrations of $\mathcal{I}_n(x)$. The first of these is defined using products of matrix minors. Specifically, for a partition $\lambda \in \mathrm{Par}(n)$, put

$$\mathcal{U}_\lambda(x) = \mathrm{span}\{(U:T)(x)\},$$

where $(U:T)(x)$ ranges over all injective bitableaux $(U:T)$ such that $\mathrm{sh}(U) = \mathrm{sh}(T) \leq \lambda$ in $\mathcal{P}_n$. Since we are not requiring $(U:T)$ to be standard, the generators of $\mathcal{U}_\lambda(x)$ shown above do not in general form a basis for this space.

The second $\mathcal{P}_n$-filtration which we shall consider is defined using K-L immanants. Given a partition λ of n, set

$$\mathcal{V}_\lambda(x) = \mathrm{span}\{\mathrm{Imm}_w(x)\},$$

where w ranges all permutations in S_n satisfying $\mathrm{sh}(w) \geq \lambda'$ in the dominance order. It is easy to verify that the collections $\{\mathcal{U}_\lambda(x) \,|\, \lambda \in \mathrm{Par}(n)\}$ and $\{\mathcal{V}_\lambda(x) \,|\, \lambda \in \mathrm{Par}(n)\}$ are both $\mathcal{P}_n$-filtrations of the space $\mathcal{I}_n(x)$. Establishing their equivalence is our first goal.

PROPOSITION 6.1. *For any partition $\lambda \in Par(n)$ we have that $\mathcal{U}_\lambda(x) = \mathcal{V}_\lambda(x)$.*

PROOF. Given a partition λ, it is easy to check that the set

$$\{T \,|\, T \in \mathcal{SYT}_n, \mathrm{sh}(T) \leq_{dom} \lambda\}$$

is a lower order ideal in $\leq_{itdom}$. From Theorem 4.2 we may conclude that $\mathcal{V}_\lambda(x)$ is equal to the span of all *standard* bitableaux with shapes weakly dominated by λ. It follows that $\mathcal{V}_\lambda(x) \subseteq \mathcal{U}_\lambda(x)$.

To show the opposite inclusion, let $(U:T)$ be an injective bitableau of shape λ and expand $(U:T)(x)$ in the basis of K-L immanants:

$$(U:T)(x) = \sum_{w \in S_n} d_w \mathrm{Imm}_w(x)$$

for some $d_w \in \mathbb{C}$. Let μ be a maximal partition in the dominance order such that some $d_w \neq 0$ for some w of shape μ'. Suppose that it were not the case that $\mu \leq \lambda$ in dominance. This means that there is an integer k such that $\mu_1 + \cdots + \mu_k > \lambda_1 + \cdots + \lambda_k$. Without loss of generality assume that μ' maximizes the quantity $\mu_1 + \cdots + \mu_k$ for partitions whose associated permutations w do not all have $d_w = 0$. Choose w'

Bruhat minimal such that $\mathrm{sh}(w') = \mu'$ and $d_{w'} \neq 0$. So, by Greene's theorem, there exists a k-decreasing subsequence $\sigma_1 \uplus \cdots \uplus \sigma_k$ of w' such that $|\sigma_1| + \cdots + |\sigma_k|$ is equal to $|\mu_1| + \cdots + |\mu_k|$. Form a matrix B by replacing the submatrices of the permutation matrix of w' with row sets $w'^{-1}(\sigma_1), \ldots, w'^{-1}(\sigma_k)$ and column sets $\sigma_1, \ldots, \sigma_k$ with matrices of 1's. Plugging B into the above equation yields 0 on the left hand side and $d_{w'}$ on the right hand side, which is a contradiction. $\square$

In light of this result, from now on we use the notation $\mathcal{I}_\lambda(x)$ to denote either of the spaces $\mathcal{U}_\lambda(x)$ or $\mathcal{V}_\lambda(x)$. Our first task is to show that this $\mathcal{P}_n$-filtration equality implies the ordinary filtration equality which was proven in [**RS05a**].

For any $k \in \mathbb{N}$, define $S_{n,k}$ to be the set of permutations in S_n having no decreasing subsequence of length $k+1$. So, for example, $S_{3,2} = \{123, 213, 132, 231, 312\}$. It is obvious that $S_{n,1} \subset \cdots \subset S_{n,n} = S_n$. For any k, we let $\mathcal{V}_{n,k}(x)$ denote the subspace of $\mathcal{I}_n(x)$ given by $\mathrm{span}\{\mathrm{Imm}_w(x) \,|\, w \in S_{n,k}\}$. The above chain of inclusions gives rise to a filtration $\mathcal{V}_{n,1}(x) \subset \cdots \subset \mathcal{V}_{n,n}(x) = \mathcal{I}_n(x)$. Following [**RS05a**] we call this the $\mathcal{V}$ filtration; it is the analogue of the $\mathcal{P}_n$-filtration by the subspaces $\mathcal{V}_\lambda(x)$.

We now define an analogue of the $\mathcal{P}_n$-filtration by the subspaces $\mathcal{U}_\lambda(x)$. For any k, we set $\mathcal{U}_{n,k}(x)$ equal to the span of all bitableaux $(U:T)(x)$ such that $\mathrm{sh}(U)_1 = \mathrm{sh}(T)_1 \leq k$. That is, $\mathcal{U}_{n,k}(x)$ is the linear span of all products of k complementary matrix minors.

In [**RS05a**] it is shown somewhat laboriously that for any n and k, $\mathcal{U}_{n,k}(x) = \mathcal{V}_{n,k}(x)$, i.e., the $\mathcal{U}$ and $\mathcal{V}$ filtrations are equivalent. This is, however, an easy corollary of $\mathcal{P}_n$-filtration equality. To see this, one need only observe that $\mathcal{U}_{n,k}(x) = \mathcal{U}_{(k,1,\ldots,1)}(x)$ and $\mathcal{V}_{n,k}(x) = \mathcal{V}_{(k,1,\ldots,1)}(x)$.

For any immanant $\mathrm{Imm}_f(x)$ corresponding to a function $f : S_n \to \mathbb{C}$, we shall say that $\mathrm{Imm}_f(x)$ *fits in* λ if $\mathrm{Imm}_f(x) \in \mathcal{I}_\lambda(x)$ but for any $\mu <_{dom} \lambda$ we have $\mathrm{Imm}_f(x) \notin \mathcal{I}_\mu(x)$. Following [**RS05a**] further, it would be interesting to determine where the irreducible character immanants $\mathrm{Imm}_\lambda(x)$ fit in our $\mathcal{P}_n$-filtration. Although this problem is more easily analyzed using symmetric function theory (see [**MW85**]), we shall follow a different strategy here.

First we prove a lemma which gives a test to determine when a given immanant is contained in $\mathcal{I}_\lambda(x)$.

LEMMA 6.1. *Let $f : S_n \to \mathbb{C}$ be any function and* $\mathrm{Imm}_f(x)$ *be the corresponding immanant. Then,* $\mathrm{Imm}_f(x) \in \mathcal{I}_\lambda(x)$ *if and only if for every $n \times n$ matrix A satisfying $\mu_{[1,n]}(A) \not\leq \lambda$ in dominance, we have that* $\mathrm{Imm}_f(A) = 0$.

In the proof of this Lemma we shall find it useful to apply the $\mathcal{U}_\lambda(x)$ characterization of $\mathcal{I}_\lambda(x)$ in one direction and the $\mathcal{V}_\lambda(x)$ characterization in the other.

PROOF. The forward implication is easy: by the dominance Lemma, every generator $(U:T)(x)$ of $\mathcal{U}_\lambda(x)$ vanishes on any matrix A satisfying $\mu_{[1,n]}(A) \not\leq \lambda$.

For the other direction, suppose $\mathrm{Imm}_f(x)$ satisfies the latter condition and expand $\mathrm{Imm}_f(x)$ in Kazhdan-Lusztig immanants as $\mathrm{Imm}_f(x) = \sum_{z \in S_n} d_z \mathrm{Imm}_z(x)$. If $\mathrm{Imm}_f(x)$ were not in $\mathcal{V}_\lambda(x)$, we could choose some w Bruhat minimal such that d_w is nonzero and $\mathrm{sh}(w) \not\geq \lambda'$ in dominance. So, there is some number k such that $M \equiv \mathrm{sh}(w)'_1 + \cdots + \mathrm{sh}(w)'_k \geq \lambda_1 + \cdots \lambda_k$. Using Greene's theorem, choose a k-decreasing subsequence of w of length M. Form a matrix A by replacing the rows and columns of the permutation matrix of w corresponding to this subsequence with matrices of 1's. Plugging in A to the above equation yields 0 on the left hand side and d_w on the right hand side, which is a contradiction. $\square$

PROPOSITION 6.2. *The irreducible character immanant* $\mathrm{Imm}_\lambda(x)$ *fits in* λ.

PROOF. We first show that for any $\mu \in \mathrm{Par}(n)$ we have that $\sum_{w \in S_\mu} \chi^\lambda(w) = 0$ if and only if $\lambda \leq_{dom} \mu$. To see this, observe that

$$\begin{aligned}\sum_{w \in S_\mu} \chi^\lambda(w) &= \langle \chi^\lambda \downarrow^{S_n}_{S_\mu}, 1\rangle \\ &= \langle \chi^\lambda, 1 \uparrow^{S_n}_{S_\mu}\rangle \\ &= K_{\mu,\lambda},\end{aligned}$$

where $K_{\mu,\lambda}$ is a Kostka number. The first equality is the definition of a character inner product, the second is Frobenius reciprocity, and the third is a standard result which may be found, for example, in [**Sag01**]. The result follows. In particular, if $\mu <_{dom} \lambda$, one can use this to construct a 0,1-matrix A which by the preceding Lemma gives that $\mathrm{Imm}_\lambda(x) \notin \mathcal{I}_\mu(x)$.

We now extend this result to show that for any permutations $u, y \in S_n$ we have $\sum_{w \in S_\mu} \chi^\lambda(uwy) = 0$. In fact, we do this in a much more general setting. Given any finite group G with subgroup H, suppose that M is a finite dimensional G-module with character χ such that $\sum_{h \in H} \chi(h) = 0$. Consider the linear operator $\frac{1}{|H|} \sum_{h \in H} h$. It is routine to check that this operator is projection onto the submodule N of M which is fixed pointwise by every element of H. As such, its trace is equal to the dimension of N. This forces the dimension of N to be 0, so the above operator must be 0, as well. So, for any g and g' in G, the operator $\sum_{h \in H} ghg'$ is also 0 and thus has trace 0.

Now let $A = (a_{ij})$ be any $n \times n$ matrix such that $\mu_{[1,n]}(A) \not\leq_{dom} \lambda$. We have that

$$\begin{aligned} \mathrm{Imm}_\lambda(x) &= \sum_{w \in S_n} \chi^\lambda(w) a_{1,w(1)} \cdots a_{n,w(n)} \\ &= \sum_{w \in S_{\mu_{[1,n](A)}}} \chi^\lambda(uwy) M \\ &= 0, \end{aligned}$$

where u and y are appropriately chosen permutations in S_n and M is a quantity which depends on A. Since A was arbitrary, by the preceding Lemma we have that $\mathrm{Imm}_\lambda(A) \in \mathcal{I}_\lambda(x)$.

□

We now give our $\mathcal{P}_n$-filtration a graded structure. Call two square submatrices x_1 and x_2 of x complementary if their row sets and column sets each partition $[n]$. Given *complementary* square submatrices x_i of x of size n_i and functions $f_i : S_{n_i} \to \mathbb{C}$ for $i = 1, 2$, we obtain immanants $\mathrm{Imm}_{f_i}(x_i)$ on these smaller submatrices as before. We wish to obtain an immanant on the larger matrix x from these smaller immanants. This is easy to do: one checks that the polynomial $\mathrm{Imm}_{f_1}(x_1)\mathrm{Imm}_{f_2}(x_2)$ in fact belongs to the linear span of the monomials $\{x_{1,w(1)} \cdots x_{n,w(n)} \,|\, w \in S_n\}$, and is therefore contained in $\mathcal{I}_n(x)$. This gives rise to a linear map $\Phi : \mathcal{I}_{n_1}(x_1) \otimes \mathcal{I}_{n_2}(x_2) \to \mathcal{I}_n(x)$, where $\Phi(\mathrm{Imm}_{f_1}(x_1) \otimes \mathrm{Imm}_{f_2}(x_2)) = \mathrm{Imm}_{f_1}(x_1)\mathrm{Imm}_{f_2}(x_2)$. We consider how Φ behaves with respect to our $\mathcal{P}_n$-filtration.

Proposition 6.3. *The space $\mathcal{I}_n(x)$ is graded with respect to Φ in the sense that Φ maps $\mathcal{I}_{\lambda_1}(x_1) \otimes \mathcal{I}_{\lambda_2}(x_2)$ into $\mathcal{I}_{\lambda_1+\lambda_2}(x)$.*

Proof. Using the $\mathcal{U}$ description of our $\mathcal{P}_n$-filtration, it suffices to show that Φ maps $\mathcal{U}_{\lambda_1}(x_1) \otimes \mathcal{U}_{\lambda_2}(x_2)$ into $\mathcal{U}_{\lambda_1+\lambda_2})(x)$. Thus, it suffices to prove that, given generators $(U_i{:}T_i)(x_i)$ for $i = 1, 2$, the product $(U_1{:}T_1)(x_1)(U_2{:}T_2)(x_2)$ is contained in $\mathcal{U}_{\lambda_1+\lambda_2}(x)$. This latter statement holds because the product in question is in fact a generator for this space.

□

A direct proof of the previous result using the $\mathcal{V}$ description of our $\mathcal{P}_n$-filtration of $\mathcal{I}_n(x)$ would have involved solving a system of equations with coefficients given by products of various Kazhdan-Lusztig polynomials evaluated at 1. Such a proof would seem very difficult, in contrast to the simple proof presented above.

Having considered the effect of multiplying together K-L immanants of complementary submatrices of x, we now turn to the problem of factoring K-L immanants.

Proposition 6.4. *Let $w \in S_n$ and let x_1 and x_2 be complementary submatrices of x. Then,* $\mathrm{Imm}_w(x)$ *factors as* $\mathrm{Imm}_w(x) = \mathrm{Imm}_{w_1}(x_1)\mathrm{Imm}_{w_2}(x_2)$ *if and only if x_1 and x_2 are block antidiagonal submatrices of x and w has permutation matrix equal to the block antidiagonal matrix with the permutation matrices of w_1 and w_2 in the blocks corresponding to x_1 and x_2, respectively.*

Proof. Omitted. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA
E-mail address: `rhoad030@umn.edu`

DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY
E-mail address: `mas906@lehigh.edu`