Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Extended quadratic algebra and a model of the equivariant cohomology ring of flag varieties

Anatol N. Kirillov and Toshiaki Maeno

Abstract. For the root system of type A we introduce and study a certain extension of the quadratic algebra invented by S. Fomin and the first author, to construct a model for the equivariant cohomology ring of the corresponding flag variety. As an application of our construction we describe a generalization of the equivariant Pieri rule for double Schubert polynomials. For a general finite Coxeter system we construct an extension of the corresponding Nichols-Woronowicz algebra. In the case of finite crystallographic Coxeter systems we present a construction of "extended Nichols-Woronowicz algebra model" for the equivariant cohomology of the corresponding flag variety.

Résumé. Pour le système de racines de type A nous introduisons et étudions une certaine extension de l'algèbre quadratique inventée par S. Fomin et le premier auteur, construire un modèle pour la cohomologie équivariante de la variété des drapeaux correspondante. Comme une application de notre construction nous décrivons une généralisation de la formule de Pieri équivariante pour les polynômes de Schubert doubles. Pour un système de Coxeter fini général nous construisons une extension de l'algèbre de Nichols-Woronowicz correspondante. Dans le cas de systèmes de Coxeter cristallographique fini nous présentons une construction de "modèle par l'algèbre de Nichols-Woronowicz étendu" pour la cohomologie équivariante de la variété des drapeaux correspondante.

1. Introduction

In the paper [**4**] S. Fomin and the first author have introduced and study a model for the cohomology ring of flag varieties of type A as a commutative subalgebra generated by the so-called Dunkl elements in a certain noncommutaive quadratic algebra $\mathscr{E}_n$. One of the main advantages of an approach developed in [**4**] is that it admits a simple generalization which is suitable for description of the quantun cohomology ring of flag varieties, as well as (quantum) Schubert polynomials. Constructions from the paper [**4**] have been generalized to other finite root systems by the authors in [**8**]. One of the main constituents of the above constructions is the Dunkl element. The basic properties of the Dunkl elements are: 1) they are pairwise commuting; 2) in the so-called Calogero-Moser representation [**4, 8**] they correspond to the *truncated* (i.e. without differential part) Dunkl operators [**3**]; 3) in the crystallographic case they correspond – after applying the so-called Bruhat representation [**4, 8**]– to the Monk formula in the cohomology ring of the flag variety in question; 4) in the crystallographic case, a subtraction-free expressions of Schubert polynomials calculated at the Dunkl elements in the algebra $\widetilde{\mathscr{BE}}(\Sigma)$, if exist, provide a combinatorial rule for describing the Schubert basis structural constants, i.e. the intersection numbers of Schubert classes.

In the case of classical root systems Σ, the first author [**6**] has defined a certain extension $\widetilde{\mathscr{BE}}(\Sigma)$ of the algebra $\mathscr{BE}(\Sigma)$ together with a pairwise commuting family of elements, called Dunkl elements, which after applying the Calogero-Moser representation exactly coincide with the rational Dunkl operators. One of the main objective of our paper is to study a commutative subalgebra generated by the Dunkl elements in the extended algebra $\widetilde{\mathscr{BE}}(\Sigma)$ in the case of type A root systems. Our main result in this direction is:

Key words and phrases. quadratic algebras, double Schubert polynomials, flag varieties.
This work is supported by Grant-in-Aid for Scientific Research.
2000 *Mathematical Subject Classification.* Primary 16W30; Secondary 14N15.

THEOREM 1.1. (Pieri formula in the algebra $\mathcal{E}_n\langle R\rangle[t]$)

$$e_k(\theta_1^{(n)},\dots,\theta_m^{(n)}) =$$

$$\sum_{p\geq 0}(-t)^p\binom{2p-1}{p}\Big\{\sum_{S,I,J} X_S \prod_{\substack{a=1\\ i_a\in I,\ j_a\in J}}^{|I|}[i_a,j_a]\Big\}.$$

See Section 2, Theorem 2.5, for a detailed explanation of conditions on sets I, J and S in the second summation, and those on indices $\{i_a, j_a\}_{a=1}^{|I|}$ in the product.

Important consequence of the above Theorem is that in the case $t = 0$, a commutative subalgebra generated by the Dunkl elements in the algebra $\mathcal{E}_n\langle R\rangle$ is canonically isomorphic to the T-equivariant cohomology ring of the type A flag variety Fl_n. Our formula, in the case $t = 0$, describes a multiplication rule for the Schubert polynomial corresponding to the $(m+1)$-cycle $c[k,m] := (k-m+1,\dots,k,k+1) \in \mathbb{S}_n$ and the double Schubert polynomial corresponding to a permutation in $\mathbb{S}_n$. An analog of Pieri's formula for multiplication of the Schubert class corresponding to special permutation $c[k,m]$ and the one corresponding to arbitrary permutation $w \in \mathbb{S}_n$ has been obtained by by S. Veigneau [**14**, Section 5.3, Proposition 5.7]. Here we treat Schubert and double Schubert polynomials as certain elements in the algebra $\mathcal{E}_{n-1}$, see Theorem 3.1.

In order to obtain the corresponding formula in the equivariant cohomology ring of a flag variety of type A, one needs to use the Bruhat representation of the algebra $\mathcal{E}_{n-1}$, see Section 3. The existance of the Bruhat representation of the algebra $\mathcal{E}_n\langle R\rangle[t]$ plays a crucial role in application to the equivariant Schubert calculus, and constitutes an important step in the proof of Corollary 2.2. Note that the Bruhat representation has a huge kernel, so that it's not evident why our formula (in the case $t = 0$) implies the known analogs of the equivariant Pieri formula due to B. Kostant and S. Kumar [**10**], S. Robinson [**13**], S. Veigneau [**14**] and vise versa, if so. We expect that for general t our formula describes an analog of the Pieri rule in the A-equivariant cohomology of (type A_{n-1}) flag variety for the group $A := T \times \mathbb{C}^\times$.

So far as we know, for the first time a Pieri-type formula in the equivariant cohomology ring of generalized flag varieties for Kac-Moody groups has been proved by B. Kostant and S. Kumar [**10**, Proposition 3.41, 3.42], see also [**10**], p.190, *Note added in proof.* In the case of flag varieties of type A_{n-1}, a significant simplification of Kostant and Kumar's Pieri-type formula has been obtained by S. Robinson [**13**]. Robinson's Pieri-type formula gives an answer in terms of a linear combination of the value of the Kostant polynomials on some special permutations, [**13**, Definition 4.2]. A compact expression for the value of a Kostant polynomial on an arbitrary permutation has been obtained by S. Billey [**2**]. It seems an interesting task to understand relationships between the equivariant Pieri-type formulas obtained in our paper with those obtained by S. Robinson and S. Veigneau.

Another objective of our paper is to construct the "Nichols-Woronowicz model" for the coinvariant algebra of a finite Coxeter group W. Recall that the Nichols-Woronowicz algebra model for the cohomology ring of flag varieties has been invented by Y. Bazlov [**1**]. In Section 4 we introduce a certain extension of the Nichols-Woronowicz algebra $\mathcal{B}_W$ and construct a commutative subalgebra in the extended Nichols-Woronowicz algebra. Our second main result states that for crystallographic root systems and $t = 0$, the commutative subalgebra in question is isomorphic to the T-equivariant cohomology ring of the corresponding flag variety.

2. Extension of the quadratic algebra

DEFINITION 2.1. The algebra $\mathcal{E}_n$ is an associative algebra generated by the symbols $[i,j]$, $1 \leq i,j \leq n$, $i \neq j$, subject to the relations:
(0) : $[i,j] = -[j,i]$
(1) : $[i,j]^2 = 0$,
(2) : $[i,j][k,l] = [k,l][i,j]$, if $\{i,j\}\cap\{k,l\} = \emptyset$,
(3) : $[i,j][j,k] + [j,k][k,i] + [k,i][i,j] = 0$.

Let us consider the extension $\mathcal{E}_n\langle R\rangle[t]$ of the quadratic algebra $\mathcal{E}_n$ by the polynomial ring $R[t] = \mathbb{Z}[x_1,\dots,x_n][t]$ defined by the commutation relations:
(A): $[i,j]x_k = x_k[i,j]$, for $k \neq i,j$,

(B): $[i,j]x_i = x_j[i,j] + t$, $[i,j]x_j = x_i[i,j] - t$, for $i < j$,
(C): $[i,j]t = t[i,j]$.

Note that the $\mathbb{S}_n$-invariant subalgebra $R^{\mathbb{S}_n}[t]$ of $R[t]$ is contained in the center of the algebra $\mathcal{E}_n\langle R\rangle[t]$.

DEFINITION 2.2. (1) We define the $R[t]$-algebra $\widetilde{\mathcal{E}}_n[t]$ by

$$\widetilde{\mathcal{E}}_n[t] = \mathcal{E}_n\langle R\rangle[t] \otimes_{R^{\mathbb{S}_n}} R.$$

More explicitly, $\widetilde{\mathcal{E}}_n[t]$ is an algebra over the polynomial ring $\mathbb{Z}[y_1, \ldots, y_n]$ generated by the symbols $[i,j]$, $1 \le i,j \le n$, $i \ne j$, and $x_1, \ldots, x_n, t$ satisfying the relations in the definition of the algebra $\mathcal{E}_n\langle R\rangle[t]$ together with the identification $e_i(x_1, \ldots, x_n) = e_i(y_1, \ldots, y_n)$, for $i = 1, \ldots, n$. Denote by $\widetilde{\mathcal{E}}_{n,t_0}$ the specialization of $\widetilde{\mathcal{E}}_n[t]$ at $t = t_0$.

(2) The Dunkl elements $\theta_i \in \widetilde{\mathcal{E}}_n[t]$, $i = 1, \ldots, n$, are defined by the formula

$$\theta_i = \theta_i^{(n)} = x_i + \sum_{j \ne i} [i,j].$$

REMARK 2.3. Note that x_i's do not commute with the Dunkl elements, but only symmetric polynomials in x_i's do. By this reason we need the second copy of $R = \mathbb{Z}[y_1, \ldots, y_n]$, where y_i's are in the center of the algebra $\widetilde{\mathcal{E}}_n[t]$, and $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ for any symmetric polynomial f.

LEMMA 2.4. *The Dunkl elements commutes each other.*

Proof. This follows from the fact that

$$(x_i + x_j)[i,j] = [i,j](x_i + x_j).$$

Let $e_k(x_1, \ldots, x_n)$, $1 \le k \le n$, stand for the elementary symmetric polynomial of degree k in the variables $x_1, \ldots, x_n$. We put by definition, $e_0(x_1, \ldots, x_n) = 1$, and $e_k(x_1, \ldots, x_n) = 0$, if $k < 0$.

THEOREM 2.5. (Pieri formula in the algebra $\mathcal{E}_n\langle R\rangle[t]$)

$$e_k(\theta_1^{(n)}, \ldots, \theta_m^{(n)}) =$$

$$\sum_{p \ge 0} (-t)^p \binom{2p-1}{p} \left(e_{k-2p}(x_1, \ldots, x_m) + \sum_{r \ge 1} \sum_{\substack{S \subset [1,\ldots,m] \\ |S| = k-r-2p}} X_S \sum_{I,J} \prod_{i_a \in I, j_a \in J} [i_a, j_a] \right),$$

where $X_S := \prod_{s \in S} x_s$; *$I$ and J are subsets of the same cardinality r in the set $[1, \ldots, n] \setminus S$; the product is taken over pairs $\{i_a, j_a\}_{a=1}^{r=|I|}$ such that $1 \le i_a \le m < j_a \le n$ and the indices $i_1, \ldots, i_{|I|}$ are all distinct.*

Sketch of Proof. Since the defining relations for the algebra $\mathcal{E}_n\langle R\rangle[t]$ are invariant with respect to the action of the symmetric group $\mathbb{S}_n$, we have

$$k!(m-k)! e_k(\theta_1^{(n)}, \ldots, \theta_m^{(n)}) = \sum_{w \in \mathbb{S}_m} w(e_k(\theta_1^{(n)}, \ldots, \theta_k^{(n)})),$$

where we regard $\mathbb{S}_m$ as the subgroup of $\mathbb{S}_n$ which interchanges only indices $1, \ldots, m$. So, the first step of our proof is to show that it is enough to prove the statement of Theorem 2.5 in the case $m = k$ only. In the case $m = k$, we compute the product $\prod_{j=1}^{k} \theta_j^{(n)}$ by induction on k taking into account an observation that all computations have to be invariant with respect to the action of the symmetric group $\mathbb{S}_k$.

COROLLARY 2.1. The list of relations in the algebra $\widetilde{\mathcal{E}}_n[t]$

$$e_k(\theta_1^{(n)}, \ldots, \theta_n^{(n)}) =$$

$$\sum_{p \ge 0} (-t)^p \binom{2p-1}{p} e_{k-2p}(y_1, \ldots, y_n), \ 1 \le k \le n,$$

describes the complete set of relations among the Dunkl elements $\theta_1^{(n)}, \ldots, \theta_n^{(n)}$.

COROLLARY 2.2. For $t = 0$, the sublagebra of $\widetilde{\mathcal{E}}_{n,0}$ generated by the Dunkl elements $\theta_1, \ldots, \theta_n$ over $\mathbb{Z}[y_1, \ldots, y_n]$ is isomorphic to the T-equivariant cohomology ring $H_T^*(Fl_n)$.

Proof. First of all it follows from Theorem 2.5 that the natural map $\theta_i \mapsto z_i = -c_1(U_i/U_{i-1})$, $y_i \mapsto y_i$ defines a well-defined homomorphism

$$\mathbb{Z}[y_1, \dots, y_n][\theta_1, \dots, \theta_n] \longrightarrow H_T^*(Fl_n), \tag{2.1}$$

where $(0 = U_0 \subset U_1 \subset \cdots \subset U_n)$ is the universal flag over Fl_n.

On the other hand, it follows from the definitions that the image of Dunkl's element θ_i in the Bruhat representation (see Section 3) acts according to the rule:

$$\theta_i \ \underline{w} = y_{w(i)}\underline{w} + \sum_{\substack{j>i \\ l(wt_{ij})=l(w)+1}} \underline{wt_{ij}} - \sum_{\substack{j<i \\ l(wt_{ij})=l(w)+1}} \underline{wt_{ij}}.$$

This rule exactly corresponds to the Monk formula for double Schubert polynomials, see e.g. [**11**, Exercise 2.7.2]. Therefore the images of the coset 1 under the action of the commutative subalgebra generated by the Dunkl elements span the entire quotient $\mathbb{Z}[y_1, \dots, y_n, z_1, \dots, z_n]/J_n$, where J_n denotes the ideal generated by the elements $e_k(z_1, \dots, z_n) - e_k(y_1, \dots, y_n)$, $1 \le k \le n$. This exactly means that the homomorphism (2.1) is an isomorphism.

Theorem 2.6. *The subalgebra generated by the elements* $g_1 := [1,2], g_2 := [2,3], \cdots, g_{n-1} := [n-1,n]$ *in the algebra* $\mathcal{E}_n\langle R\rangle[t]$ *is isomorphic to the nil degenerate affine Hecke algebra of type* $A_{n-1}^{(1)}$, *i.e. the algebra given by two sets of generators* $g_1, \cdots, g_{n-1}$ *and* $x_1, \cdots, x_n$ *subject to the set of defining relations:*

$$g_i^2 = 0, \quad g_ig_j = g_jg_i, \quad \text{if } |i-j| > 1, \quad g_ig_jg_i = g_jg_ig_j, \quad \text{if } |i-j| = 1,$$
$$g_ix_i - x_{i+1}g_i = t.$$

3. Bruhat representation

Let us recall the definition of the Bruhat representation of the algebra $\mathcal{E}_n$ on the group ring of the symmetric group $\mathbb{Z}\langle\mathbb{S}_n\rangle = \oplus_{w \in \mathbb{S}_n}\mathbb{Z}\cdot\underline{w}$. The operator σ_{ij}, $i<j$, is defined as follows:

$$\sigma_{ij}(\underline{w}) = \begin{cases} \underline{wt_{ij}}, & \text{if } l(wt_{ij}) = l(w)+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the Bruhat representation of $\mathcal{E}_n$ is defined by $[i,j].\underline{w} := \sigma_{ij}(\underline{w})$.

Now we extend the Bruhat representation to that of the algebra $\mathcal{E}_n\langle R\rangle[t]$ defined on

$$R[t]\langle\mathbb{S}_n\rangle = \oplus_{w\in\mathbb{S}_n}\mathbb{Z}[y_1, \dots, y_n][t]\cdot\underline{w}.$$

For $f(y) \in \mathbb{Z}[y_1, \dots, y_n][t]$ and $w \in \mathbb{S}_n$, we define the $\mathbb{Z}[t]$-linear operators $\tilde{\sigma}_{ij}$, $i<j$, and ξ_k as follows:

$$\tilde{\sigma}_{ij}(f(y)\underline{w}) = \begin{cases} t(\partial_{w(i)w(j)}f(y))\underline{w} + f(y)\underline{wt_{ij}}, & \text{if } l(wt_{ij}) = l(w)+1, \\ t(\partial_{w(i)w(j)}f(y))\underline{w}, & \text{otherwise,} \end{cases}$$

$$\xi_k(f(y)\underline{w}) = (y_{w(k)}f(y))\underline{w}.$$

Proposition 3.1. *The algebra* $\mathcal{E}_n\langle R\rangle[t]$ *acts* $\mathbb{Z}[t]$*-linearly on* $\mathbb{Z}[y][t]\langle\mathbb{S}_n\rangle$ *via* $[ij] \mapsto \tilde{\sigma}_{ij}$ *and* $x_k \mapsto \xi_k$.

Proof. Let us check the compatibility with the defining relations of the algebra $\widetilde{\mathcal{E}}_n[t]$. We show the compatibility only with the relations (1), (3) and (B). The rest are easy to check.

Let us start with the relation (1). We have

$$\tilde{\sigma}_{ij}^2(f(y)\underline{w}) = \tilde{\sigma}_{ij}\left(t(\partial_{w(i)w(j)}f(y))\underline{w} + f(y)\sigma_{ij}(\underline{w})\right)$$
$$= t^2(\partial_{w(i)w(j)}^2 f(y))\underline{w} + t(\partial_{w(i)w(j)}f(y))\sigma_{ij}(\underline{w})$$
$$+ t(\partial_{w(j)w(i)}f(y))\sigma_{ij}(\underline{w}) + f(y)\sigma_{ij}^2(\underline{w}).$$

Since $\partial_{w(i)w(j)}^2 = 0$, $\sigma_{ij}^2 = 0$ and $\partial_{w(i)w(j)} = -\partial_{w(j)w(i)}$, we get $\tilde{\sigma}_{ij}^2 = 0$.

For the relation (3), we have

$$\tilde{\sigma}_{ij}\tilde{\sigma}_{jk}(f(y)\underline{w}) = \tilde{\sigma}_{ij}\left(t(\partial_{w(j)w(k)}f(y))\underline{w} + f(y)\sigma_{jk}(\underline{w})\right)$$
$$= t^2(\partial_{w(i)w(j)}\partial_{w(j)w(k)}f(y))\underline{w} + t(\partial_{w(j)w(k)}f(y))\sigma_{ij}(\underline{w})$$

$$+t(\partial_{w(i)w(k)}f(y))\sigma_{jk}(\underline{w}) + f(y)\sigma_{ij}\sigma_{jk}(\underline{w}).$$

We also obtain $\tilde{\sigma}_{jk}\tilde{\sigma}_{ki}(f(y)\underline{w})$ and $\tilde{\sigma}_{ki}\tilde{\sigma}_{ij}(f(y)\underline{w})$ by the cyclic permutation of i, j, k. The 3-term relations

$$\partial_{w(i)w(j)}\partial_{w(j)w(k)} + \partial_{w(j)w(k)}\partial_{w(k)w(i)} + \partial_{w(k)w(i)}\partial_{w(i)w(j)} = 0$$

and

$$\sigma_{ij}\sigma_{jk} + \sigma_{jk}\sigma_{ki} + \sigma_{ki}\sigma_{ij} = 0$$

show the desired equality

$$\tilde{\sigma}_{ij}\tilde{\sigma}_{jk} + \tilde{\sigma}_{jk}\tilde{\sigma}_{ki} + \tilde{\sigma}_{ki}\tilde{\sigma}_{ij} = 0.$$

Finally, we check the relation (B). We have

$$\tilde{\sigma}_{ij}\xi_i(f(y)\underline{w}) = \tilde{\sigma}_{ij}(y_{w(i)}f(y)\underline{w})$$

$$= t\partial_{w(i)w(j)}(y_{w(i)}f(y))\underline{w} + (y_{w(i)}f(y))\sigma_{ij}(\underline{w})$$

$$= t(f(y)\underline{w}) + t(y_{w(j)}\partial_{w(i)w(j)}f(y))\underline{w} + y_{wt_{ij}(j)}\sigma_{ij}(\underline{w})$$

$$= \xi_j\tilde{\sigma}_{ij}(f(y)\underline{w}) + t(f(y)\underline{w}).$$

Denote by L_n the submodule of $\mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]$ generated by the monomials $x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} y_1^{j_1} \cdots y_{n-1}^{j_{n-1}}$, $0 \le i_k \le n-k$, $0 \le j_k \le n-k$.

THEOREM 3.1. *Let $\mathfrak{S}_w(x,y)$ be the double Schubert polynomial corresponding to $w \in \mathbb{S}_n$. Then, we have*

$$\mathfrak{S}_w(\theta, y)(\underline{\mathrm{id.}}) = \underline{w}.$$

Conversely, if a polynomial $F(x,y) \in L_n$ satisfies the condition

$$F(\theta, y)(\underline{\mathrm{id.}}) = \underline{w},$$

then $F(x,y) = \mathfrak{S}_w(x,y)$.

Proof. The first statement follows from the Monk formula for the double Schubert polynomials and

$$(\theta_i - y_{w(i)})(\underline{w}) = \xi_i(\underline{w}) + \sum_{j \ne i} \sigma_{ij}(\underline{w}) - y_{w(i)}\underline{w}$$

$$= \sum_{j<i, l(wt_{ij})=l(w)+1} \underline{wt_{ij}} - \sum_{j>i, l(wt_{ij})=l(w)+1} \underline{wt_{ij}}.$$

The second statement is a consequence from the fact that the double Schubert polynomials are the unique solution in L_n for the system of equations coming from the Monk formula.

REMARK 3.2. Only when $t = 0$, one can extend $\mathbb{Z}[y][t]$-linearly the Bruhat representation of the algebra $\mathscr{E}_n\langle R\rangle[t]$ to that of the algebra $\widetilde{\mathscr{E}}_n[t]$. In fact, Theorem 3.1 describes the multiplicative structure of the $\mathbb{Z}[y]$-subalgebra generated by the Dunkl elements in $\widetilde{\mathscr{E}}_{n,0}$, which is isomorphic to $H_T^*(Fl_n)$. Nevertheless, the characterization of the double Schubert polynomials in Theorem 3.1 holds for arbitrary t.

4. Nichols-Woronowicz model

The model of the equivariant cohomology ring $H_T^*(Fl_n)$ in the algebra $\widetilde{\mathcal{E}}_n$ has a natural interpretation in terms of the Nichols-Woronowicz algebra. The Nichols-Woronowicz approach leads us to the uniform construction for arbitrary root systems.

We denote by $\mathcal{B}_W$ the Nichols-Woronowicz algebra associated to the Yetter-Drinfeld module

$$V = \bigoplus_{\alpha \in \Delta} \mathbb{R}[\alpha]/([\alpha]+[-\alpha])$$

over the finite Coxeter group W of the root system Δ. Let $\mathfrak{h}$ be the reflection representation of W and $R = \mathrm{Sym}\mathfrak{h}^*$ the ring of polynomial functions on $\mathfrak{h}$. Let us consider the extension $\mathcal{B}_W\langle R\rangle[t]$ of the algebra $\mathcal{B}_W$ by the polynomial ring $R[t]$ defined by the commutation relation

$$[\alpha]x = s_\alpha(x)[\alpha] + t(x,\alpha) \quad \text{for } x \in \mathfrak{h}^*.$$

Definition 4.1. We define the R-algebra $\widetilde{\mathcal{B}}_W$ by

$$\widetilde{\mathcal{B}}_W = \mathcal{B}_W\langle R\rangle[t] \otimes_{R^W} R.$$

Choose a W-invariant constants $(c_\alpha)_\alpha$. Let us consider a linear map $\mu : \mathfrak{h}^* \to \widetilde{\mathcal{B}}_W$ defined as

$$\mu(x) = x + \sum_{\alpha \in \Delta_+} c_\alpha (x,\alpha)[\alpha]$$

for $x \in \mathfrak{h}^*$.

Proposition 4.1. $[\mu(x),\mu(y)] = 0$, $x,y \in \mathfrak{h}^*$.

The linear map μ extends to a homomorphism of algebras

$$\mu : R \to \mathcal{B}_W\langle R\rangle[t].$$

Denote by $\widetilde{\mu}$ the composite of the homomorphisms

$$R \otimes_{\mathbb{Z}} R \stackrel{\mu\otimes 1}{\to} \mathcal{B}_W\langle R\rangle[t] \otimes_{\mathbb{Z}} R \to \widetilde{\mathcal{B}}_W.$$

Theorem 4.2. *If $t = 0$ and the constants $(c_\alpha)_\alpha$ are generic, the image of the homomorphism $\widetilde{\mu}$ is isomorphic to the algebra $R \otimes_{R^W} R$. In particular, when W is the Weyl group, it is isomorphic to the T-equivariant cohomology ring $H_T^*(G/B)$ of the corresponding flag variety G/B.*

The proof is based on the correspondence between the twisted derivation D_α and the divided difference operator ∂_α. We define the operator D_α as the twisted derivation on $\widetilde{\mathcal{B}}_W$ determined by the conditions:
(1): $D_\alpha(x) = 0$, for $x \in R$,
(2): $D_\alpha([\beta]) = \delta_{\alpha,\beta}$, for $\alpha, \beta \in \Delta_+$,
(3): $D_\alpha(fg) = D_\alpha(f)g + s_\alpha(f)D_\alpha(g)$.
The operator D_α is linear with respect to R on the second component.

Proposition 4.2.

$$\cap_{\alpha\in\Delta_+}\mathrm{Ker}(D_\alpha) = R[t] \otimes_{R^W} R$$

Proof. Any element $\omega \in \mathcal{B}_W\langle R\rangle[t]$ can be written as

$$\omega = f_1\varphi_1 + \cdots + f_k\varphi_k,$$

where $f_1,\ldots,f_k \in R[t]$ are linearly independent, and $\varphi_1,\ldots,\varphi_k \in \mathcal{B}_W$. We have

$$D_\alpha(\omega) = s_\alpha(f_1)D_\alpha(\varphi_1) + \cdots + s_\alpha(f_k)D_\alpha(\varphi_k)$$

from the twisted Leibniz rule. If $D_\alpha(\omega) = 0$, we have $D_\alpha(\varphi_1) = \cdots = D_\alpha(\varphi_k) = 0$. Hence, $\omega \in \cap_{\alpha\in\Delta_+}\mathrm{Ker}(D_\alpha)$ implies that $\varphi_i \in \mathcal{B}_W^0 = \mathbb{R}$ for $i = 1,\ldots k$. This means $\omega \in R[t]$.

Proposition 4.3.

$$D_\alpha(\widetilde{\mu}(x)) = c_\alpha\widetilde{\mu}(\partial_\alpha(x))$$

for $x \in R \otimes_{\mathbb{Z}} R$.

Proof. When $x = \beta \otimes 1$, $\beta \in \Delta$, we can check that
$$D_\alpha(\widetilde{\mu}(\beta \otimes 1)) = c_\alpha(\beta, \alpha) = c_\alpha \widetilde{\mu}(\partial_\alpha(\beta)).$$
Hence, we have $D_\alpha(\widetilde{\mu}(x)) = c_\alpha \widetilde{\mu}(\partial_\alpha(x))$ for $x \in \mathfrak{h}^* \otimes R$. On the other hand, the both-hands sides satisfy the same twisted Leibniz rule, so it follows that $D_\alpha(\widetilde{\mu}(x)) = c_\alpha \widetilde{\mu}(\partial_\alpha(x))$ for $x \in R \otimes R$.

(*Proof of Theorem 4.2*) If $x \in R^W \otimes_{\mathbb{Z}} R$, we have $D_\alpha(\widetilde{\mu}(x)) = 0$ for every $\alpha \in \Delta_+$ from Proposition 4.3. This implies from Proposition 4.2 that $\widetilde{\mu}(x) \in R^W \otimes_{R^W} R$. When $t = 0$, $\widetilde{\mu}(x)$ coincides with the element of R which is obtained by replacing all the symbols $[\alpha]$ by zero in $\widetilde{\mu}(x)$. Hence, the homomorphism $\widetilde{\mu}$ factors through $R \otimes_{R^W} R \to \widetilde{\mathscr{B}}_W$. Since a linear basis of the coinvariant algebra of W gives an R^W-basis of R, it is easy to see that $R \otimes_{R^W} R \to \widetilde{\mathscr{B}}_W$ is injective.

5. Multiparameter quantum deformation $\mathscr{E}_n^q[t]$ of the algebra $\mathscr{E}_n[t]$.

Let $q := \{q_{ij} | 1 \le i < j \le m\}$ be a set of parameters. Replace the relation (1) $[ij]^2 = 0$ in Definition 2.1 by $[ij]^2 = q_{ij}$, $1 \le i < j \le n$. Denote by $e_k(x_1, \dots, x_n \mid q)$ the multiparameter quantum deformation of the elementary symmetric functions [**4**, (15.2)]:
$$e_k(x_1, \dots, x_n \mid q) = \sum_p \sum_{\substack{1 \le i_1 < \dots < i_p \le n \\ i_1 < j_1, \dots, i_p < j_p}} e_{k-2p}(X_{\overline{I \cup J}}) \prod_{a=1}^{p} q_{i_a j_a},$$
where $i_1, \dots, i_p$ and $j_1, \dots, j_p$ should be distinct elements of the set $\{1, \dots, n\}$ and $X_{\overline{I \cup J}}$ denotes the set of variables x_a for which the subscript a is neither one of i_k nor one of j_k. We define the Dunkl elements θ_i^q by the same formula as for θ_i.

THEOREM 5.1. (Pieri formula in the algebra $\mathscr{E}_n^q[t]$)
$$e_k(x_1, \dots, x_n \mid q) = \textit{RHS of the formula in Theorem 2.5}$$

CONJECTURE 5.2. The list of relations in the algebra $\mathscr{E}_n^q[t]$
$$e_k(\theta_1^q, \dots, \theta_n^q \mid q) = \sum_{p \ge 0} (-t)^p \binom{2p-1}{p} e_{k-2p}(y_1, \dots, y_n), \ 1 \le k \le n,$$
describes the complete set of relations among the Dunkl elements $\theta_1^q, \dots, \theta_n^q$.

Now we assume thar $q_{ij} = 0$ except the case $j = i + 1$.

CONJECTURE 5.3. For $t = 0$ the sublagebra of $\mathscr{E}_{n,0}^q$ generated by the Dunkl elements $\theta_1^q, \dots, \theta_n^q$ over $\mathbb{Z}[y_1, \dots, y_n]$ is isomorphic to the T-equivariant quantum cohomology ring $QH_T^*(Fl_n)$.

A construction of the equivariant small quantum cohomology of flag varieties has been done by A. Givental and B. Kim [**5**], and more recently by L. Mihalcea [**12**].

References

[1] Y. Bazlov, *Nichols-Woronowicz algebra model for Schubert calculus on Coxeter groups,* J. Algebra **297** (2006), no. 2, 372–399.

[2] S. Billey, *Kostant polynomials and the cohomology ring for G/B,* Duke Math. J. **96** (1999), 205–224.

[3] C. Dunkl, *Harmonic polynomials and peak sets of reflection groups,* Geom. Dedicata **32** (1989), 157–171.

[4] S. Fomin and A. N. Kirillov, *Quadratic algebras, Dunkl elements and Schubert calculus,* Advances in Geometry, (J.-L. Brylinski, R. Brylinski, V. Nistor, B. Tsygan and P. Xu, eds.) Progress in Math. **172**, Birkhäuser, (1995), 147–182.

[5] A. Givental and B. Kim *Quantum cohomology of flag manifolds and Toda lattices,* Comm. Math. Phys. **168** (1995), 609-641.

[6] A. N. Kirillov, *On some quadratic algebras II,* preprint.

[7] A. N. Kirillov and T. Maeno, *Noncommutative algebras related with Schubert calculus on Coxeter groups,* European J. of Combin. **25** (2004), 1301–1325.

[8] A. N. Kirillov and T. Maeno, *A note on quantization operators on Nichols algebra model for Schubert calculus on Weyl groups,* Lett. Math. Phys. **72** (2005), no. 3, 233–241.

[9] A. N. Kirillov and T. Maeno, *On some noncommutative algebras related to K-theory of flag varieties, Part I,* Int. Math. Res. Not. **2005**, no. 60, 3753–3789.

[10] B. Kostant and S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Adv. Math. **62** (1986), 187–237.

[11] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci,* SMF/AMS Texts and Monographs vol. **6**, 2001.
[12] L. Mihalcea, *Equivariant quantum cohomology of homogeneous spaces,* math. AG/0501213.
[13] S. Robinson, *A Pieri-type formula for* $H_T^*(SL_n(\mathbb{C})/B)$, J. Algebra **249** (2002), 38–58.
[14] S. Veigneau, *Calcul Symbolique et Calcul Distribué en Combinatoire Algébrique,* Ph. D. Thesis, 1996, `http://igm.univ-mlv.fr/~veigneau/`

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `kirillov@kurims.kyoto-u.ac.jp`
URL: `http://www.kurims.kyoto-u.ac.jp/~kirillov`

Department of Electrical Engineering, Kyoto University, Kyoto 606-8501, Japan
E-mail address: `maeno@kuee.kyoto-u.ac.jp`