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Operations on posets and rational identities of type A

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Abstract. To each permutation σ, we associate a rational function $\psi_\sigma := \prod(x_{\sigma(i)} - x_{\sigma(i+1)})^{-1}$. The aim of this paper is to study the combinatorics of the sum Ψ_P of the ψ_w where the indices are taken in the linear extensions $\mathcal{L}(P)$ of a planar poset P. In particular, we describe different transformations on posets which will result in elementary operations on these functions.

Résumé. À chaque mot w, on associe la fonction rationnelle $\psi_\sigma := \prod(x_{\sigma(i)} - x_{\sigma(i+1)})^{-1}$. Le but de cet article est d'étudier la somme Ψ_P des fonctions ψ_w sur les extensions linéaires des posets. En particulier, nous décrivons différentes transformations sur les posets qui se traduisent par des opérations élémentaires sur ces fonctions.

1. Introduction

To each permutation σ, we associate a rational function

$$\psi_\sigma := \frac{1}{(x_{\sigma(1)} - x_{\sigma(2)}).(x_{\sigma(2)} - x_{\sigma(3)}) \dots (x_{\sigma(n-1)} - x_{\sigma(n)})}.$$

Summing this function on intervals of the permutohedron (*i.e.* for the weak order) gives remarkable properties. In particular, when permutations avoid some patterns, the sum can be simplified in a product of terms of type $\prod(x_i - x_j)^{-1}$.

The patterns appear in geometry, characterizing some families of Schubert varieties. Schubert varieties are indexed by permutations, and the varieties which are non singular are those whose indexing permutation does not contain the pattern 2143 nor the pattern 1324. In [**2**], Cortez has described geometrical properties of Schubert varieties for permutations avoiding the patterns 1324 and $\overline{2143}$. This was further clarified by Woo and Yong in [**6**], and Butler and Bousquet-Mélou in [**1**]. They use the fact that Hasse diagram naturally associated to a permutation avoiding 1324 and $\overline{2143}$ is acyclic.

Surprisingly, the same patterns will occur in the study of our rational functions. In fact our work is closely connected to a study of Greene [**3**] on rational identity related to Murnaghan-Nakayama formula for S_n (type A). Greene gives in [**3**] a closed expression for the sum Ψ_P of the ψ_w where the indices are taken in the linear extensions $\mathcal{L}(P)$ of a planar poset P,

$$\Psi_P = \sum_{w \in \mathcal{L}(P)} \psi_w.$$

He shows the equality

$$\Psi_P = \begin{cases} 0 & \text{if P is a disconnected graph,} \\ \prod_{y,z \in P} (x_y - x_z)^{\mu_P(y,z)} & \text{if P is a connected graph.} \end{cases}$$

where $\mu(x, y)$ denotes the Möbius function of the poset P. In the case of a permutation avoiding the patterns 1324 and $\overline{2143}$, the poset is planar, and the Möbius function takes only values 0 or -1. Therefore (Corollary 4.1) the function Ψ_P has numerator equal to 1 if and only if P has an acyclic Hasse diagram.

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The aim of this paper consists in pointing out the connexions between some operations on posets and rational identities involving the Ψ_P. These results are summarized in appendix A Table 1 and 2

To study the rational functions Ψ_P, we introduce some operations on posets in Section 2, and describe in Section 3 the identities on the rational functions that these operations induce. We finish with some explicit examples in Section 4: acyclic posets, 1-cycle posets and bipartite posets.

2. Operations on posets

2.1. Basic definitions. A *partially ordered set* (*poset*) P is a set endowed with a binary relation $\leq$ verifying:

1) for all $x \in P$, $x \leq x$ (reflexivity);
2) if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry);
3) if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

Let $\mathcal{R}(P)$ be the set of the pairs $(x, y) \in P \times P$ with $x \leq y$. A *linear extension* of a poset P is a total order compatible with P. We denote by $\mathcal{L}(P)$ the set of linear extensions of P.
Classically, the covering relation ($\preceq$) is defined by y *covers* x (or $y \preceq x$) if $x \leq y$ and if there is no $z \in P$ such that $x < z < y$. The *Hasse diagram* of a poset P, denoted by $H(P)$, is the oriented graph of the covering relation of P drawn in such a way that if $x \leq y$, then y is drawn at the right of x [1].
The Hasse diagram is the minimal set generating P by transitivity.
We denote by $Int(P)$ the subset of P composed by the elements which are neither minimal nor maximal and $Ext(P)$ denotes $P \setminus Int(P)$.

2.2. Permutations and posets. Let $[\sigma, \tau]$ be the interval whose lower bound is σ and upper bound is τ (*i.e.* the set of permutations greater than σ and lower than τ in the permutohedron).

For example, this is the interval $[123456, 132564]$ in $\mathfrak{S}_6$:

$$[123456, 132564] = \{132564, 123564, 132546, 123546, 132456, 123456\}.$$

To each permutation σ, we associate the poset P_σ whose linear extensions are the permutations in $[id, \sigma]$.

To obtain the Hasse diagram of P_σ, proceed as follows. The letters of the permutation σ are readen from right to left. While the values increase, we write them disposed on the same vertical line. If the values decrease, one writes them from the right to the left. Finally, we draw an edge between two vertices i and j if and only if $i < j$, the vertex j is to the right of i and there exists no vertex k such that $i < k < j$ and k is on the left of j and on the right of i.

Figure 1 shows an example of a Hasse diagram.

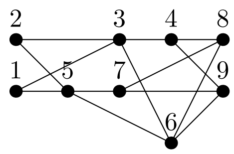


FIGURE 1. Hasse diagram of $P_{215736498}$

2.3. Collapses. Following the notation of [**9**], for a graph $\mathcal{G}$ we say that a vertex v of an edge e is *free* if v is not a vertex of any edge other than e. An *inner edge* is an edge such that none of its end-points are free vertices. Removing a non-inner edge e together with a free vertex of e is called an *elementary collapse*, and a sequence of elementary collapses is a *collapse*. We denote $Coll(\mathcal{G})$ the *collapsed graph* of $\mathcal{G}$, that is, the unique maximal subgraph of a graph $\mathcal{G}$ without free vertices.

[1]Usually, Hasse diagrams are drawn from bottom to top, but this representation takes more space and is less natural for our purposes.

2.4. Subposets and linear extensions. Let P be a poset and $X \subset P$. Let $w \in \mathcal{L}(X)$. We denote by $P_{(w)}$ the poset such that its linear extensions are linear extensions of P having w as subword. P_σ and $P_{(\sigma)}$ are not the same object, the second needs the definition of P to exist. P_σ is definied in Section 2.2.

In fact, $P_{(w)}$ is the poset P endowed with the transitive closure of $\mathcal{R}(w) \cup \mathcal{R}(P)$.
We define recursively $P_{(w_1,\ldots,w_k)} = \left(P_{(w_1,\ldots,w_{k-1})}\right)_{(w_k)}$.

Lemma 2.1. *Let P be a poset and $X_1, \ldots, X_k$ be k disjoint subsets of P, we have*

$$\mathcal{L}(P) = \bigsqcup_{l \in \mathcal{L}(X_1)\times\ldots\times\mathcal{L}(X_k)} \mathcal{L}(P_l)$$

where $\bigsqcup$ denotes the disjoint union.

For example, the set of linear extensions of the poset P in Figure 2 can be partitioned in disjoint subsets indexed by the linear extensions of X_1 and X_2.

$$\mathcal{L}(P) = \mathcal{L}(P_{(12,456)}) \sqcup \mathcal{L}(P_{(12,465)}) \sqcup \mathcal{L}(P_{(21,456)}) \sqcup \mathcal{L}(P_{(21,465)})$$

$$\mathcal{L}(P_{(12,456)}) = \{124356, 123456\}$$

$$\mathcal{L}(P_{(12,465)}) = \{124365, 123465\}$$

$$\mathcal{L}(P_{(21,456)}) = \{214356, 213456\}$$

$$\mathcal{L}(P_{(21,465)}) = \{214365, 123465\}$$

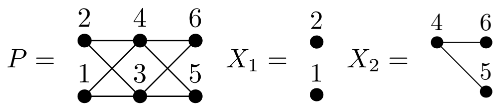


Figure 2. A poset P with two subposets X_1 and X_2 of P.

2.5. Contractions.

Proposition 2.1. *Let P be a poset and a, b be two elements of P. If $a \preceq b$ the relation $\leq_{a=b}$ defined by*

$$x \leq_{a=b} y \Leftrightarrow (x \leq y) \text{ or } ((x \leq a) \text{ or } (x \leq b)) \text{ and } ((a \leq y) \text{ or } (b \leq y))$$

is a partial order over $P \setminus \{b\}$.

Proof. The relation $\leq_{a=b}$ is obviously reflexive and transitive.
The antisymmetry follows from the fact that a covers b. Indeed, if ($a \leq x$ or $b \leq x$) and ($x \leq a$ or $x \leq b$) then $x = a$ or $x = b$. Hence, after looking at all the possibilities for x, y in $P \setminus \{b\}$ such that $x \leq_{a=b} y$ and $y \leq_{a=b} x$ we have that $x = y$. □

When a covers b, we denote by $P_{a=b}$ the *contraction of the edge* (a, b) (that is $P \setminus \{b\}$ endowed with $\leq_{a=b}$).

Proposition 2.2. *Let P be a poset and (a, b) be an edge in $H(P)$. Then $w'abw''$ is a linear extension of P if and only if $w'aw''$ is a linear extension of $P_{a=b}$.*

Proof. Suppose that $w'abw''$ is a linear extension of P and let $x, y \in P \setminus \{b\}$. For each $z \in P$, we denote by i_z the position of z in the word $w'abw''$. As $i_b = i_a + 1$ and $x \neq b$, $y \neq b$, the condition $((x \leq_P a)$ or $(x \leq_P b))$ and $((a \leq_P y)$ or $(b \leq_P y))$ implies $i_x \leq i_a$ and $i_y \geq ia$. Hence, if $x \leq_{P_{a=b}} y$ then $i_x \leq i_y$ which implies $w'aw'' \in \mathcal{L}(P)$.
Suppose now that $w'aw''$ is a linear extension of $P_{a=b}$. If $x \neq b$, $y \neq b$ then $x \leq_P y$ implies $x \leq_{P_{a=b}} y$ and $i_x \leq i_y$. If $x = b$ then $b \leq_P y$ implies $i_a \leq i_y$ and $i_b \leq i_y$. In the same way, $x \leq_P b$ implies $i_x \leq i_b$. Then in all the cases, $x \leq_P y$ implies $i_x \leq i_y$. Equivalently, $w'abw''$ is a linear extension of P. □

For example, the edge $(4,5)$ of the poset P in Figure 3 can be contracted. The linear extensions for $P_{4=5}$ are

$$\mathcal{L}(P) = \{12435, 21435, 12345, 21345\},$$

and, the set obtained by removing all words with no factor 45 has the same size than $\mathcal{L}(P_{4=5})$.

$$\{w'45w'' \in \mathcal{L}(P)\} = \{12345, 21345\}$$

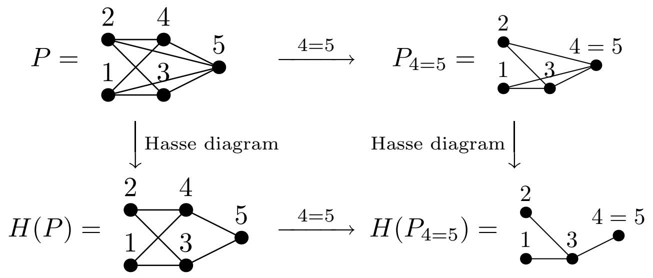


FIGURE 3. The posets P and $P_{4=5}$, their Hasse diagrams and their linear extensions.

2.6. Decontractions. A poset is *bipartite* if only has maximal and minimal elements.

PROPOSITION 2.3. *Each poset can be obtained from a bipartite poset by applying a succession of contractions.*

PROOF. Consider a poset P. We construct a new poset $\overline{P}$ by duplicating each vertex of $Int(P)$

$$\overline{P} = P \cup \{\overline{x} \mid x \in Int(P)\}.$$

The poset $\overline{P}$ is endowed with the relation $\leq'$ defined by $y \leq' z$ if and only if one of the following statements is true:

(1) $y = z$,
(2) $z = \overline{x}$ with $x \in Int(P)$, $y \in P$ and $y \preceq x$,
(3) $y \in P$, $z \in Ext(P)$ and $y \preceq z$,
(4) $z = \overline{y}$ with $y \in Int(P)$.

By construction, $\leq'$ is a partial order and each element of $\overline{P}$ is either minimal or maximal. Finally, when contracting the edges $\{(x, \overline{x}) \mid x \in Int(P)\}$, the poset P is recovered, see Figure 4 for an example of contraction. □

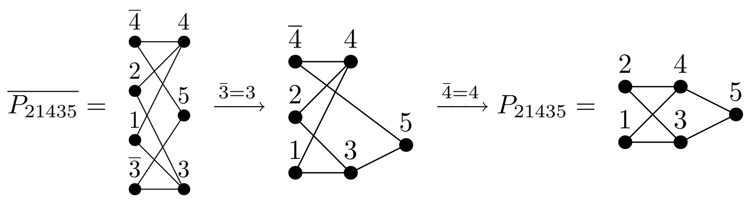


FIGURE 4. The poset P_{21435} is the contraction of the bipartite poset $\overline{P_{21435}}$.

2.7. Suppressions of extremal elements.

PROPOSITION 2.4. *Let P be a poset and m a maximal or a minimal element of P. Then mw is a linear extension of P if and only if w is a linear extension of $P \setminus \{m\}$.*

PROOF. Let us assume that m is minimal (the case where m is maximal being similar).

Suppose that mw is a linear extension of P and let x, y be two elements of P different of m. Hence, if $x \leq_{P\setminus\{m\}} y$ then $x \leq_w y$ which implies $w \in \mathcal{L}(P \setminus (\{m\}))$.

Conversely, suppose that w is a linear extension of $P \setminus \{m\}$. Let $x, y \in P$ be such that $x \leq_P y$. If $x \neq m$ then as m is minimal we have $y \neq m$ and $x \leq_w y$. So we have $x \leq_{mw} y$. If $x = m$, we have trivialy $x \leq_{mw} y$. We deduce that $mw \in \mathcal{L}(P)$. □

3. Operations on rational functions

3.1. Residues and contractions.

THEOREM 3.1. *Let a and b be two elements of a poset P. We have*

$$((x_a - x_b).\Psi_P)|_{x_a=x_b} = \begin{cases} \Psi_{P_{a=b}} & \text{if } (a,b) \text{ is an edge of the Hasse diagram of } P, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Applying the residue at $x_a = x_b$, we get

$$((x_a - x_b).\Psi_P)|_{x_a=x_b} = \sum_{\substack{w\in\mathcal{L}(P)\\ w=w'abw'' \text{ or } w=w'baw''}} ((x_a - x_b).\psi_w)|_{x_a=x_b} \tag{3.1}$$

We can consider three cases.

1) a and b are not comparable. Obviously, the word $w'abw''$ is a linear extension of P if and only if $w'baw''$ is also a linear exension of P. Hence, by considering the pairs $\psi_{w'abw''}$ and $\psi_{w'baw''}$ in (3.1), we obtain

$$((x_a - x_b).\Psi_P)|_{x_a=x_b} = \sum_{w'abw''\in\mathcal{L}(P)} ((x_a - x_b)[\psi_{w'abw''} + \psi_{w'baw''}])|_{x_a=x_b} = 0.$$

2) a and b are comparable but $a \not\preceq b$ and $b \not\preceq a$. Assuming that $a \leq b$ (the other case being similar), there is at least one element c such that $a \leq c \leq b$. Then $\mathcal{L}(P)$ contains no word having ab nor ba as a factor and the residue $((x_a - x_b).\Psi_P)|_{x_a=x_b}$ is equal to 0.
3) Now, if $a \preceq b$ (the case $b \preceq a$ is similar), by Proposition 2.2, we have

$$((x_a - x_b).\Psi_P)|_{x_a=x_b} = \Psi_{P_{a=b}}.$$

□

Theorem 3.1 and Proposition 2.3 show that the knowledge of the fraction Ψ for each bipartite poset is enough to compute any Ψ_P by applying a sequence of residues.

For example, the rational functions $\Psi_{\overline{P_{21435}}}$ and $\Psi_{\overline{P_{21435}}|_{\overline{3}=3,\overline{4}=4}}$ described in Figure 5 have the following numerators and denominators:

$$\begin{aligned} Num(\Psi_{\overline{P_{21435}}}) = {} & x_1.x_{\overline{4}} - x_1.x_4 + x_2.x_{\overline{3}} - x_2.x_3 + x_2.x_{\overline{4}} - x_2.x_4 + x_{\overline{3}}^2 - x_{\overline{3}}.x_3 + x_3.x_4 - x_{\overline{3}}.x_4 + x_1.x_{\overline{3}} - \\ & x_1.x_3 + x_{\overline{4}}^2 - x_{\overline{4}}.x_4 + x_1.x_2 + x_3.x_5 + x_4.x_5 - x_1.x_5 - x_2.x_5 - x_3.x_{\overline{4}} \end{aligned}$$

$$Den(\Psi_{\overline{P_{21435}}}) = (x_1 - x_3).(x_1 - x_4).(x_2 - x_3).(x_2 - x_4).(x_{\overline{3}} - x_3).(x_{\overline{3}} - x_5).(x_{\overline{4}} - x_4).(x_{\overline{4}} - x_5)$$

$$Den(\Psi_{P_{\overline{P_{21435}}|_{\overline{3}=3,\overline{4}=4}}}) = x_1.x_2 + x_3.x_5 + x_4.x_5 - x_1.x_5 - x_2.x_5 - x_3.x_4$$

$$Num(\Psi_{P_{\overline{P_{21435}}|_{\overline{3}=3,\overline{4}=4}}}) = (x_1 - x_3).(x_1 - x_4).(x_2 - x_3).(x_2 - x_4).(x_3 - x_5).(x_4 - x_5)$$

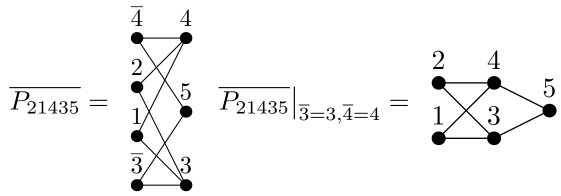


FIGURE 5. We obtain the fraction $\Phi_{P_{21435}}$ from $\Phi_{\overline{P_{21435}}}$.

3.2. Limits and suppressions of extremal elements.

THEOREM 3.2. *Let v be an element of a poset P.*

$$\lim_{x_v \to +\infty} (x_v.\Psi_P) = \begin{cases} -\Psi_{P\backslash v} & \text{if } v \text{ is maximal,} \\ \Psi_{P\backslash v} & \text{if } v \text{ is minimal,} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We have that

$$\lim_{x_v \to +\infty} (x_v.\Psi_P) = \sum_{\substack{w \in \mathcal{L}(P) \\ w = vw' \text{ or } w = w'v}} \lim_{x_v \to +\infty} (x_v.\Psi_{P_w}) \tag{3.2}$$

If v is not extremal then $\mathcal{L}(P)$ contains no word having v in his initial or last position. Hence, the limit $\lim_{x_v \to +\infty} (x_v.\Psi_P)$ is equal to 0.

Now, assume that v is minimal (the case with v maximal is similar). By Proposition 2.4, we have

$$\lim_{x_v \to +\infty} (x_v.\Psi_P) = \Psi_{P\backslash v}.$$

□

3.3. Connexity and annulation.

DEFINITION 3.3 (Greene [**3**]). A poset P is *planar* if his Hasse diagram may be ordered-imbedded in $\mathbb{R} \times \mathbb{R}$ without edge crossings, even when extra maximal and minimal elements are added.

See Figures 6 and 7 for examples of planar and non planar posets.

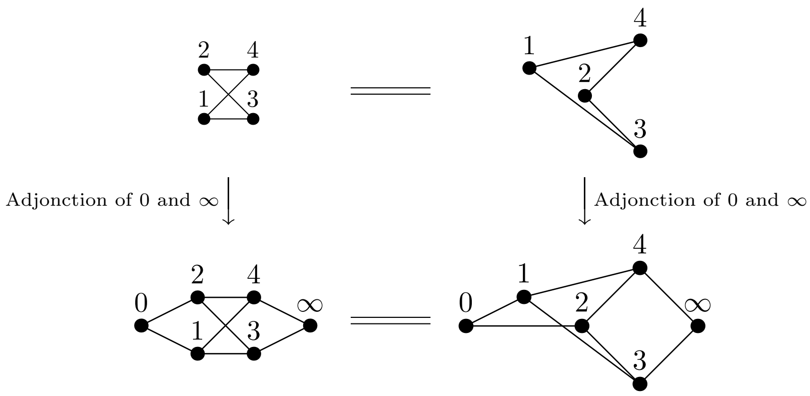


FIGURE 6. The poset P_{2143} is not planar.

THEOREM 3.4 (Greene [**3**]). *Let P be a planar poset, then*

$$\Psi_P = \begin{cases} 0 & \text{if } P \text{ is not connected,} \\ \prod_{y,z \in P} (x_y - x_z)^{\mu_P(y,z)} & \text{if } P \text{ is connected.} \end{cases}$$

where $\mu(x,y)$ denotes the Möbius function of the poset P.

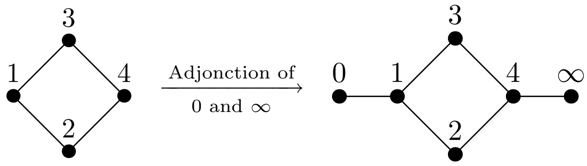


FIGURE 7. The poset P_{1324} is planar.

We have the following consequence.

COROLLARY 3.1. *Let P be a poset, the Hasse diagram of P is connected if and only if $\Psi_P \neq 0$.*

PROOF. Suppose first that P is connected and $\Psi_P = 0$. As contraction preserves the connexity, we can contract edges on P to obtain a new poset whith only two elements $a \leq b$. Using Theorem 3.1, we get Ψ_{a-b} from Ψ_P by applying a succession of residues. It follows that $\Psi_{a-b} = 0$. This is in contradiction with the direct computation $\Psi_{a-b} = \frac{1}{x_a - x_b}$. Hence, $\Psi_P \neq 0$.

Now, we consider the case of a disconnected poset P. Let $C_1, \ldots, C_k$ be the k connected components of P. By Lemma 2.1 we have :

$$\Psi_P = \sum_{w_1 \in \mathcal{L}(C_1), \ldots, w_k \in \mathcal{L}(C_k)} \Psi_{P_{(w_1, \ldots, w_k)}}.$$

However, the poset $P_{(w_1,\ldots,w_k)}$ is planar and disconnected. Applying the Greene theorem (Theorem 3.4), we obtain $\Psi_P = 0$. □

3.4. Reduced fractions and Hasse diagrams. We denote $Den(\psi_P)$ the denominator of the reduced fraction Ψ_P and $Num(\Psi_P)$ denotes its numerator.

COROLLARY 3.2. *Let P be a connected poset, then:*

$$Den(\psi_P) = \prod_{a \prec b} (x_a - x_b).$$

PROOF. Theorem 3.1 implies that $\prod_{a \prec b}(x_a - x_b)$ is a factor of $Den(\Psi_P)$. As contraction preserves the connexity (Corollary 3.1), we deduce that $Den(\Psi_P)$ is exactly $\prod_{a \prec b}(x_a - x_b)$. □

COROLLARY 3.3. *Let P be a connected poset, the degree of $Num(\Psi_P)$ is equal to the number of cycles in the non oriented Hasse diagram of P.*

PROOF. Let P be a connected poset with n elements. Let e be the number of edges in $H(P)$ and c the number of cycles. By Corollary 3.2 we deduce that the degree of the numerator of the reduced fraction is at most equal to $n - 1 - e$ and, from the Euler formula, it is equal to (see [**8**] or [**9**]) the number of cycles in $H(P)$. The polynomial $Num(P)$ being homogeneous, if it is non zero its degree is c. As P is connected, the Corollary 3.1 closes the proof. □

For example, the Hasse diagram of P_{132546} in Figure 8 has exactly 3 cycles. So the degree of his numerator is equal to 3.

$Num(\Psi_{P_{132546}}) = -x_1.x_2.x_3 + x_1.x_2.x_6 + x_2.x_3.x_4 + x_1.x_3.x_6 + x_1.x_4.x_5 + x_2.x_3.x_5 - x_1.x_4.x_6 - x_2.x_3.x_6 - x_2.x_4.x_5 - x_1.x_5.x_6 - x_3.x_4.x_5 + x_4.x_5.x_6)$

$Den(\Psi_{P_{132546}}) = (-x_1 + x_3).(-x_1 + x_2).(-x_5 + x_6).(-x_4 + x_6).(-x_3 + x_5).(-x_2 + x_5).(x_3 - x_4).(-x_2 + x_4)$

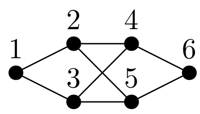


FIGURE 8. Numerator and denominator of the reduced fraction $\Psi_{P_{132546}}$.

3.5. Collapses and factorisations.

PROPOSITION 3.1. *Let v be an element of a connected poset P such that v is a free vertex in the Hasse diagram of P. Let s be the unique vertex such that $s \preceq v$ or $v \preceq s$, then*

$$\Psi_P = \begin{cases} \Psi_{P\setminus\{v\}} \cdot \frac{1}{x_v - x_s} & \textit{if } v \textit{ is minimal,} \\ \Psi_{P\setminus\{v\}} \cdot \frac{1}{x_s - x_v} & \textit{if } v \textit{ is maximal.} \end{cases}$$

PROOF. Let v be a free vertex in a Hasse diagram of a poset P. Let C_i be a family of polynomials whose degree in x_v is zero and such that

$$\Psi_P = \frac{\sum_i C_i.x_v^i}{\prod_{i \prec j}(x_i - x_j)}.$$

As v is a free vertex, v is maximal or minimal in P. Theorem 3.2 shows that

$$\lim_{x_v \to +\infty} (x_v.\Psi_P) = \begin{cases} -\Psi_{P\setminus v} & \text{if } v \text{ is maximal,} \\ \Psi_{P\setminus v} & \text{if } v \text{ is minimal.} \end{cases}$$

which implies that $C_0 \neq 0$ and for all $i \geq 1$, $C_i = 0$. Hence, $Num(\Psi_P) = C_0 = Num(\Psi_{P\setminus\{a\}})$. □

As a straightforward consequence, we have:

COROLLARY 3.4. $Num(\Psi_P) = Num(\Psi_{Coll(P)})$

4. Examples

4.1. Acyclic posets.

PROPOSITION 4.1. *The Hasse diagram $H(P)$ has no cycle if and only if $Num(P) = 1$.*

PROOF. This result is a direct application of Greene Theorem (Theorem 3.4) and the Corollary 3.3. □

A permutation σ avoids the patterns 1324 if there exists no integers $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ verifying $\sigma_{i_1} < \sigma_{i_3} < \sigma_{i_2} < \sigma_{i_4}$. A permutation σ has the pattern $\overline{2143}$ if for some indices $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ we have $\sigma_{i_2} \leq \sigma_{i_1} \leq \sigma_{i_4} \leq \sigma_{i_3}$ with the further restriction that there is no $i_1 \leq j \leq i_4$ such that $\sigma_{i_1} \leq \sigma_j \leq \sigma_{i_4}$.
Butler and Bousquet-Mélou have shown in [**1**] that the Hasse diagram of a poset associated to a permutation avoiding 1324 and $\overline{2143}$ has no cycle. As a consequence we have the following corollary.

COROLLARY 4.1. *The fraction Ψ_{P_σ} is completely simplifiable (ie $Num(\Psi_{P_\sigma}) = 1$) if and only if σ avoids the patterns* 1324 *and* $\overline{2143}$.

4.2. 1-cycle poset.

PROPOSITION 4.2. *Let P be an acyclic connected poset, then*

$$Num(P) = \sum_{i \in min(Coll(P))} x_i - \sum_{i \in max(Coll(P))} x_i.$$

PROOF. Let's consider the poset $P' = \overline{Coll(P)}$ obtained by the construction given in Proposition 2.3 on $Coll(P)$.

As P' is bipartite with only 1 cycle, by Corollary 3.3, we obtain $Num(\Psi_{P'}) = \sum_i c_i.x_i$, where $c_i \in \mathbb{Z}$. Let i be a minimal element in P'. As $P' \setminus \{i\}$ is acyclic and connected, $Num(\Psi_{P'\setminus\{i\}}) = 1$. Theorem 3.2 implies that $c_i = 1$ if i is maximal and $c_i = -1$ if i is minimal. So, we have:

$$Num(\Psi_{P'}) = \sum_{i \in min(P')} x_i - \sum_{i \in max(P')} x_i.$$

By Corollary 3.4 and Theorem 3.1 we have:

$$Num(\Psi_P) = \left(\prod_{e \in Int(Coll(P))} (x_{\overline{e}} - x_e).Num(\Psi_{P'}) \right) \Bigg|_{\substack{x_{\overline{u}} = x_u \\ u \in Int(Coll(P))}}.$$

Hence,

$$Num(\Psi_P) = \sum_{i\in min(Coll(P))} x_i - \sum_{i\in max(Coll(P)} x_i + \sum_{e\in Int(Coll(P))} (x_e - x_{\overline{e}})|_{x_{\overline{e}}=x_e}$$
$$= \sum_{i\in max(Coll(P))} x_i - \sum_{i\in max(Coll(P))} x_i.$$

□

For example, the numerator of the 1-cycle poset P in Figure 9 is

$$Num(\Psi_P) = x_1 + x_2 - x_4 - x_7.$$

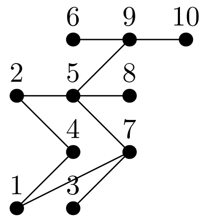
$P =$ 

FIGURE 9. 1-cycle poset.

4.3. Complete bipartite posets. We have seen (Subsection 3.1) that the bipartite posets are fundamental for the description of the functions Ψ_P. In this paragraph, we compute the special case of complete bipartite posets.

Consider a poset P constituted of two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ such that the elements of X are all smaller than the elements of Y, the elements of X (resp. Y) being incomparable. We construct a new poset P' by choosing a total order $y_1 < \cdots < y_m$ on Y. The rational functions Ψ_P and $\Psi_{P'}$ are related by the following identity:

$$\Psi_P = \Psi_{P'}.\sum_{\sigma\in\mathfrak{S}_m} \sigma^Y,$$

where σ^Y denotes the action of the permutation σ on the alphabet Y. The poset P' being planar, if we apply the Greene Theorem (Theorem 3.4), we obtain

$$\Psi'_P = \frac{1}{(x_1-y_1)\cdots(x_n-y_1)}\frac{1}{(y_1-y_2)\cdots(y_{m-1}-y_m)} = \frac{1}{R(X,Y)}\frac{R(X,Y\setminus y_1)}{\prod_{i=1}^{m-1} y_i - y_{i+1}}$$

where $R(X,Y)$ denotes the resultant of the alphabets X and Y. Hence,

$$\Psi_P = \left(\frac{1}{R(X,Y)}\frac{R(X,Y\setminus y_1)}{\prod_{i=1}^{m-1} y_i - y_{i+1}}\right).\sum_{\sigma\in\mathfrak{S}_m}\sigma^Y$$
$$= \left(\frac{1}{R(X,Y)}\frac{S_{(m-1)^n}(X-(Y\setminus y_1))}{\prod_{i=1}^{m-1} y_i - y_{i+1}}\right)\sum_{\sigma\in\mathfrak{S}_m}\sigma^Y$$
$$= \left(\frac{1}{R(X,Y)}\prod_{i<j-1}(y_i-y_j)S_{(m-1)^n}(X-(Y\setminus y_1))\right).\frac{1}{\Delta(Y)}\sum_{\sigma\in\mathfrak{S}_m}\sigma^Y$$

where S_λ is a Schur function (see for example [**7**]) and $\Delta(Y)$ is the Vandermonde determinant of the alphabet Y. We denote by ∂_i^Y the divided difference applied on the alphabet Y:

$$\partial_i^Y = \frac{1}{y_i - y_{i+1}}(1+s_i),$$

where s_i denotes the ith elementary transposition. The operator $\frac{1}{\Delta(Y)}\sum_{\sigma\in\mathfrak{S}_m}\sigma^Y$ admits an expression in term of divided differences (see prop. 7.6.2 in [**4**]):

$$\frac{1}{\Delta(Y)}\sum_{\sigma\in\mathfrak{S}_m}\sigma^Y=\partial_\omega^Y:=(\partial_{m-1}^Y)(\partial_{m-2}^Y\partial_{m-1}^Y)\cdots(\partial_1^Y\cdots\partial_{m-1}^Y).$$

The polynomial $S_{(m-1)^n}(X-(Y\setminus y_1))$ is symmetric in $y_2,\dots,y_m$, so we have

$$\prod_{i<j-1}(y_i-y_j)S_{(m-1)^n}(X-(Y\setminus y_1))\partial_\omega^Y=\prod_{i<j-1}(y_i-y_j)(\partial_{m-1}^Y)(\partial_{m-2}^Y\partial_{m-1}^Y)\cdots(\partial_2^Y\cdots\partial_{m-1}^Y)\times$$
$$\times\, S_{(m-1)^n}(X-(Y\setminus y_1))(\partial_1^Y\cdots\partial_{m-1}^Y).$$

$\prod_{i<j-1}(y_i-y_j)$ is a polynomial of degree $\binom{n-1}{2}$, so if we apply $(\partial_{m-1}^Y)(\partial_{m-2}^Y\partial_{m-1}^Y)\cdots(\partial_2^Y\cdots\partial_{m-1}^Y)$ we get a constant term. More precisely, the only monomial which has a non null contribution is $(-1)^{\binom{n-1}{2}}y_m^{m-2}y_{m-1}^{m-3}\cdots y_3$. Hence, a straightforward computation gives

$$\prod_{i<j-1}(y_i-y_j)(\partial_{m-1}^Y)(\partial_{m-2}^Y\partial_{m-1}^Y)\cdots(\partial_2^Y\cdots\partial_{m-1}^Y)=1.$$

We deduce a compact expression for Ψ_P:

$$\Psi_P=\frac{1}{R(X,Y)}S_{(m-1)^n}(X-(Y\setminus y_1))\partial_1^Y\cdots\partial_{m-1}^Y.$$

The polynomial $S_{(m-1)^n}(X-(Y\setminus y_1))\partial_1^Y\cdots\partial_{m-1}^Y$ is the Lagrange interpolation of $S_{(m-1)^n}(X-(Y\setminus y_1))$ considered as a function $f(y_1,Y\setminus y_1)$ (see [**4, 5**]). As $S_{(m-1)^n}(X-(Y\setminus y_1))$ is equal to the multi-Schur function [2]

$$S_{(m-1)^{n-1};m-1}(X-Y;X-(Y\setminus y_1)):=\begin{vmatrix} S_{m-1}(X-Y) & \cdots & S_{m-n}(X-Y) & S_{m-n-1}(X-(Y\setminus y_1))\\ \vdots & & \vdots & \vdots\\ S_{m+n-2}(X-Y) & \cdots & S_{m-3}(X-Y) & S_{m-2}(X-(Y\setminus y_1))\\ S_{m+n-1}(X-Y) & \cdots & S_{m-2}(X-Y) & S_{m-1}(X-(Y\setminus y_1))\end{vmatrix}$$

we get

$$S_{(m-1)^n}(X-(Y\setminus y_1))\partial_1^Y\cdots\partial_{m-1}^Y=S_{(m-1)^{n-1};m-1}(X-Y;X-(Y\setminus y_1))\partial_1^Y\cdots\partial_{m-1}^Y=S_{(m-1)^{n-1}}(X-Y).$$

The expression of Ψ_P follows:

PROPOSITION 4.3.

$$\Psi_P=\frac{1}{R(X,Y)}S_{(m-1)^{n-1}}(X-Y).$$

4.4. More examples. In general, we do not know formulas allowing to compute Ψ_P when the size of its expansion grows exponentially with the complexity of P. In some cases, we can express Ψ_P in terms of multi-Schur functions. For example, if P is the complete tri-partite poset composed by $X=\{x_1,\dots,x_n\}$, $Y=\{y_1,y_2\}$ and $Z=\{z_1,\dots,z_m\}$ such that $x<y<z$ for each $x\in X$, $y\in Y$ and $z\in Z$, then the numerator of Φ_P is the multi-Schur function $S_{n;m-1}(Y-X,Y-Z)$. Another example is given by the tri-partite poset composed by $X=\{x_1\}$, $Y=\{y_1,y_2,y_3\}$ and $Z=\{z_1,\dots,z_m\}$. In this case, the numerator can be evaluated as a sum of two multi-Schur functions: $S_{11;m-1}(Y-X,Y-Z)+S_{2;m-1}(Y-X,Y-Z)$. The proofs of these formulas are essentially the same than in the case of the bi-partite posets. Further examples are investigated in a forthcoming paper.

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[2]See Chapter 1 in [**4**] for a more general definition and a more extended list of properties.

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Appendix A. Table summarizing the operations and the properties of posets and their consequences on the rational functions Ψ_P

TABLE 1. Poset operators and rational function identities

Posets	Rational functions
Contraction of the edge (a,b) $P \to P_{a=b}$	$((x_a - x_b).\Psi_P)\|_{x_a=x_b} = \Psi_{P_{a=b}}$
Suppression of an extremal element v $P \to P \setminus \{v\}$	$\lim_{x_v \to +\infty} (x_v.\Psi_P) = \begin{cases} -\Psi_{P\setminus v} & \text{if } v \text{ is maximal,} \\ \Psi_{P\setminus v} & \text{if } v \text{ is minimal.} \end{cases}$
Collapse of an edge $s-v$ with a free vertex v $P \to P \setminus \{v\}$	$\Psi_P = \begin{cases} \Psi_{P\setminus\{v\}} \cdot \frac{1}{x_v - x_s} & \text{if } v \text{ is minimal} \\ \Psi_{P\setminus\{v\}} \cdot \frac{1}{x_s - x_v} & \text{if } v \text{ is maximal} \end{cases}$
$X_1, \ldots, X_k \subset P$, k disjoint subsets $P \to \{P_{(w_1,\ldots,w_k)} \| w_1 \in \mathcal{L}(X_1), \ldots, w_k \in \mathcal{L}(X_k)\}$	$\Psi_P = \sum_{w_1 \in \mathcal{L}(X_1),\ldots,w_k \in \mathcal{L}(X_k)} \Psi_{P_{(w_1,\ldots,w_k)}}$

TABLE 2. Poset properties and their expressions on rational functions

Posets	Rational functions
P connected	$\Psi_P \neq 0$ $Den(\psi_P) = \prod_{a \prec b} (x_a - x_b)$ The degree of the numerator is the number of cycles of P
P disconnected	$\Psi_P = 0$
P acyclic	$Num(\Psi_P) = 1$
P has 1 cycle	$Num(\Psi_P) = \sum_{i \in min(Call(P))} x_i - \sum_{i \in max(Call(P))} x_i$
P is complete bipartite	$Num(\Psi_P) = S_{(card(max(P))-1)^{(card(min(p))-1)}}(min(P), max(P))$

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