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Analysis of some exactly solvable diminishing urn models

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ABSTRACT. We study several exactly solvable Pólya-Eggenberger urn models with a *diminishing* character, namely, balls of a specified color, say x are completely drawn after a finite number of draws. The main quantity of interest here is the number of balls left when balls of color x are completely removed. We consider several diminishing urns studied previously in the literature such as the pills problem, the cannibal urns and the OK Corral problem, and derive exact and limiting distributions. Our approach is based on solving recurrences via generating functions and partial differential equations.

RÉSUMÉ. On se propose d'étudier plusieurs modèles d'urnes de Pólya-Eggenberger de nature "diminuante" ayant des solutions exactes, c'est-à-dire, les boules de couleur, disons x, sont toutes prises après un nombre fini de tirées. La quantité principale qui nous interesse est le nombre de boules qui restent dans l'urne au moment où il n'y a plus de boules de couleur x. Nous traitons, en particulier, plusieurs modèles d'urnes diminuantes proposés dans la litterature, comme le problème de pillules, le modèle d'urnes dit "cannibaliste" et le problème d'OK Corral, et obtenons des résultats exactes et asymptotiques. L'approche que nous utilisons est fondée sur le traitement de récurrences par voie de fonctions génératrices et équations aux dérivativés partielles.

1. Introduction

1.1. Diminishing urn models. We are concerned here with the so-called Pólya-Eggenberger urn models, which in the simplest case of two types of colors for the balls can be described as follows. At the beginning, the urn contains m black and n white balls. At every step, we choose a ball at random from the urn, examine its color and put it back into the urn and then add/remove balls according to its color by the following rules. If the ball is white, then we put a white and b black balls into the urn, while if the ball is black, then c white balls and d black balls are put into the urn. The values $a, b, c, d \in \mathbb{Z}$ are fixed integer values and the urn model is specified by the transition matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Urn models with r (≥ 2) types of colors can be described in an analogous way and are specified by an $r \times r$ transition matrix.

Urn models are simple, useful mathematical tools for describing many evolutionary processes in diverse fields of application such as analysis of algorithms and data structures, statistics and genetics. Due to their importance in applications, there is a huge literature on the stochastic behavior of urn models; see for example [**8, 11**]. Recently, a few different approaches have been proposed, which yield deep and far-reaching results for very general urn models; see [**2, 3, 7**].

Most papers in the literature impose the so-called *tenability* condition on the transition matrix, so that the process can be continued *ad infinitum* (or no balls of a given color being completely removed). However, in some applications (examples given below), there are urn models with a very different nature, which we will refer to as "diminishing urn models." For simplicity of presentation, we describe them in the case of balls with two types of colors, black and white. We consider Pólya-Eggenberger urn models specified by a transition matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and in addition there is a set of absorbing states $\mathcal{S} \subseteq \mathbb{N} \times \mathbb{N}$. The urn contains m black balls and n white balls at the beginning and evolves by successive draws at discrete instance according to the transition matrix until an absorbing state $s = (j, k) \in \mathcal{S}$ is reached, namely, the urn contains exactly j black balls and k white balls. Then the urn process stops. We only call an urn model "diminishing urn model" if it is guaranteed that from any initial state $(m, n) \in \mathbb{N} \times \mathbb{N}$ (starting with m black balls and n white balls) we will reach an absorbing state $s \in \mathcal{S}$ after a finite number of draws.

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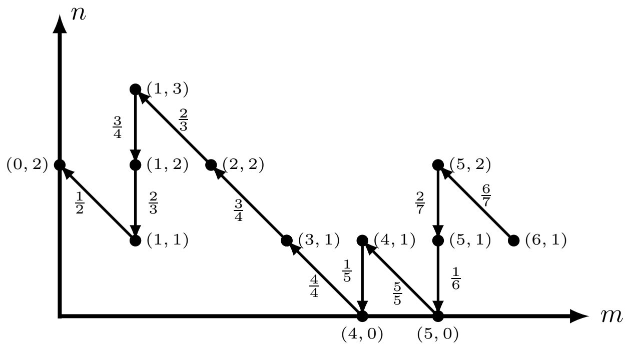

FIGURE 1. An example of a weighted path from $(6,1)$ to the absorbing state $(0,2)$ for the so called pills problem with transition matrix $M = [-1,0;1,-1]$ and the vertical absorbing axis $\mathcal{S} = \{(0,n) : n \geq 0\}$. The illustrated path has weight $\frac{6}{7}\frac{2}{7}\frac{1}{6}\frac{5}{5}\frac{1}{5}\frac{4}{4}\frac{3}{4}\frac{2}{4}\frac{3}{4}\frac{2}{3}\frac{1}{2} = \frac{3}{3920}$.

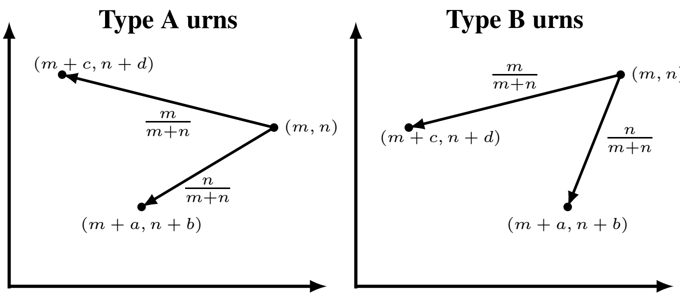

FIGURE 2. Type A and Type B urn models.

Diminishing urn models with more than two type of balls can be considered similarly; an example will be given below. For diminishing urns, the main questions are (i) starting at state (m,n), what is the probability of reaching the absorbing state $(j,k) \in \mathcal{S}$?, and (ii) what is then the number of balls left?.

Motivated by concrete applications, we distinguish the following two types of urns.

Type A: The entries of M satisfy $a, b \leq 0$, $(a,b) \neq (0,0)$, $d < 0$ and $c > 0$, and the set of absorbing states $\mathcal{S}$ consists of the vertical axis $m = 0$ (or a vertical wall $0 \leq m \leq M$, $M \geq 0$).

Type B: The entries of M satisfy $a, b, c, d \leq 0$, $(a,b) \neq (0,0)$ and $(c,d) \neq (0,0)$ and the set of absorbing states $\mathcal{S}$ consists of the vertical axis $m = 0$ (or a vertical wall $0 \leq m \leq M$, $M \geq 0$) and the horizontal axis $n = 0$ (or a horizontal wall $0 \leq n \leq N$, $N \geq 0$).

Note that the conditions on Type A urn models are in general not sufficient to guarantee that an absorbing state will be reached (if $b < -1$ then the urn process could reach states with $n < 0$), but this is the case for all models we consider here.

It is helpful to describe the evolution of the urn model by weighted lattice paths, which is described in the case of urns with two types of balls. If the urn contains m black balls and n white balls and we select a white ball (with probability $\frac{n}{m+n}$), then this corresponds to a step from (m,n) to $(m+a, n+b)$, to which the weight $\frac{n}{m+n}$ is associated; and if we select a black ball (with probability $\frac{m}{m+n}$), this corresponds to a step from (m,n) to $(m+c, n+d)$ (with weight $\frac{m}{m+n}$). The weight of a path after t successive draws consists of the product of the weight of every step. By this correspondence, the probability of starting at (m,n) and ending at (j,k) is equal to the sum of the weights of all possible paths starting at state (m,n) and ending at the absorbing state $(j,k) \in \mathcal{S}$ (which did not reach any absorbing state before). Unfortunately, the expressions so obtained for the probability are, although exact, less useful for large m or n. An example for the weighted path corresponding to the evolution of a diminishing urn is given in Figure 1.

The description of the urn model via weighted lattice paths then gives the following interpretation of the Type A and Type B urn models: in Type A we have a step lying in the lower-left quadrant and one step lying in the upper-left quadrant, whereas in Type B both steps are lying in the lower-left quadrant; see Figure 2.

1.2. Examples. We first describe a few motivating examples of diminishing urn models.

The pills problem. The transition matrix is given by $M = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and the absorbing axis is $\mathcal{S} = \{(0, n) : n \geq 0\}$. An interpretation is as follows. An urn has two types of pills in it, which are single-unit and double-unit pills, respectively. At every step, we pick a pill uniformly at random. If a single-unit pill is chosen, then we eat it up, and if the pill is of double unit, we break it into two halves—one half is eaten up and the other half is now considered of single unit and thrown back into the urn. The question is then, when starting with n single-unit pills and m double-unit pills, what is the probability that k single-unit pills remain in the urn when all double-unit pills are drawn?

This problem has been stated in [**12**], where the authors asked for a formula for the expected number of remaining single-unit pills, when there are no double-unit pills in the urn. The solution appeared in [**6**]. A more refined study is given recently in [**1**], where they derive exact formulæ for the variance and the third moment of the number of remaining single-unit pills; furthermore, a few generalizations are proposed.

A natural generalization is to consider r types of pills, which are of i units, $i = 1, \dots, r$, respectively. At every time step, a pill is chosen uniformly at random; if the pill is of single unit, it is eaten up, and if the pill is of i units, $i \geq 2$, it is broken into two parts, one of single unit and the other of $(i - 1)$ units. The piece of single unit is eaten up and the remaining piece is thrown back into the urn. We stop if there are no more pills of the largest units (r).

This problem corresponds to the diminishing urn model with the $r \times r$-transition matrix

$$M = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & -1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & -1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}$$

and the absorbing hyperplane $\mathcal{S} = \{(n_1, \dots, n_{r-1}, 0) : n_1, \dots, n_{r-1} \geq 0\}$. We will be interested in finding the probability that k pills of single unit remain in the urn when there are no more pills of r units, the starting configuration being n_i pills of i units.

A variant of the pills problem. To illustrate how a minor change in the entries of the transition matrix leads to very different behavior, we will also consider the transition matrix $M = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -2 \end{smallmatrix}\right)$ and the absorbing wall $\mathcal{S} = \{(0, n) : n \geq 0\} \cup \{(1, n) : n \geq 0\}$.

The cannibal urn. Introduced by R. F. Greene (unpublished) and analyzed in details by Pittel in [**13**], this urn model is a slight modification of the diminishing urn with $M = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -2 \end{smallmatrix}\right)$ and the vertical wall of absorbing states $\mathcal{S} = \{(0, n) : n \geq 0\} \cup \{(1, n) : n \geq 0\}$. In terms of weighted lattice paths, one starts at position (m, n), the weight (and thus the probability) of a step to $(m - 1, n)$ is $\frac{n}{m-1+n}$ (not $\frac{n}{m+n}$), and the weight to $(m - 2, n + 1)$ is $\frac{m-1}{m-1+n}$. The approach we use is also applicable to this modified urn model.

Such an urn was introduced to model the behavior of cannibals in biological population. It can be described as follows. A population consists of cannibals and non-cannibals. At every time step, a non-cannibal is selected as victim and removed; after that a member in the remaining population (cannibals and non-cannibals) is selected uniformly at random. If the selected individual is a cannibal it remains as a cannibal, but if the selected individual is a non-cannibal, it becomes then a cannibal. The question is, when starting with n cannibals and m non-cannibals, what is the number of resulting cannibals in the population at the moment when all non-cannibals are removed?

The OK Corral problem. This corresponds to the urn $M = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ with two absorbing axes: $\mathcal{S} = \{(0, n) : n \geq 0\} \cup \{(m, 0) : m \geq 0\}$. An interpretation is as follows. Two groups of gunmen, group A and group B (with n and m gunmen, respectively), face each other. At every discrete time step, one gunman is chosen uniformly at random who then shoots and kills exactly one gunman of the other group. The bloody gunfight ends when one group gets completely "eliminated". Two questions are of interest: (i) what is the probability that group A (group B) survives? and (ii) what is the probability that the gunfight ends with k survivors of group A (group B)?

This problem was introduced by Williams and McIlroy in [**15**] and studied recently by several authors using different approaches, leading to very interesting results; see [**2, 9, 10**]. Also the urn corresponding to the OK corral problem can be viewed as a basic model in the mathematical theory of warfare and conflicts; see [**10**].

Sampling without replacement. This is a toy example and corresponds to the urn $M = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ with two absorbing axes: $\mathcal{S} = \{(0, n) : n \geq 0\} \cup \{(m, 0) : m \geq 0\}$. In this classical model, balls are drawn one after another from an urn containing balls of two different colors and not replaced. What is the probability that k balls of one color remain when balls of the other color are all removed?

1.3. Recurrence. For diminishing urns, we study the position of the absorbing state. Probabilistically, we consider the pair of random variables $(X_{n,m}^{(1)}, X_{n,m}^{(2)})$, such that $\mathbb{P}\{(X_{n,m}^{(1)}, X_{n,m}^{(2)}) = (j,k)\}$ gives the probability that when starting at state (m,n) (with m black balls and n white balls), the urn process reaches the absorbing state (j,k), namely, the process terminates with j black balls and k white balls. For diminishing urns with a single vertical absorbing axis (or wall), we are only interested in the vertical position of the absorbing state; so we define $X_{n,m} := X_{n,m}^{(2)}$ and $\mathbb{P}\{X_{n,m} = k\}$ is then the probability that when starting with m black balls and n white balls, the urn process stops with k white balls remaining in the urn. We consider the probability generating function $h_{n,m}(v_1, v_2)$ or $h_{n,m}(v)$, respectively, defined by

$$h_{n,m}(v_1,v_2) := \sum_{j\ge 0}\sum_{k\ge 0}\mathbb{P}\{(X_{n,m}^{(1)}, X_{n,m}^{(2)}) = (j,k)\}v_1^j v_2^k, \quad h_{n,m}(v) := \sum_{k\ge 0}\mathbb{P}\{X_{n,m}=k\}v^k. \tag{1}$$

According to the outcome of the first draw of the urn process, we obtain the following recurrences for the probability generating functions

$$h_{n,m}(v_1,v_2) = \frac{n}{m+n}h_{n+a,m+b}(v_1,v_2) + \frac{m}{m+n}h_{n+c,m+d}(v_1,v_2), \tag{2a}$$

$$h_{n,m}(v) = \frac{n}{m+n}h_{n+a,m+b}(v) + \frac{m}{m+n}h_{n+c,m+d}(v), \tag{2b}$$

for $(m,n) \notin S$. The boundary values at the absorbing states $(m,n) \in S$ are given by $h_{n,m}(v_1,v_2) = v_1^m v_2^n$ and $h_{n,m}(v) = v^n$, respectively.

We solve such recurrences via generating functions for a few special cases below. For urn models of Type B we can always introduce generating functions[1]

$$H(z,w) := \sum_{(m,n)\notin S} h_{n,m}(v_1,v_2)z^n w^m,$$

and the recurrence (2a) can be translated into the following first order linear partial differential equation (PDE)

$$z(1-z^{-a}w^{-b})H_z(z,w) + w(1-z^{-c}w^{-d})H_w(z,w) + (az^{-a}w^{-b} + dz^{-c}w^{-d})H(z,w) = F(z,w),$$

(see [**3**]) where the inhomogeneous part $F(z,w)$ is fully determined by the boundary values. Such PDEs can be treated (at least in principle) by the method of characteristics, see, for example, [**14**]. For urn models of Type A, the situation becomes more involved. The same approach may still apply but the additional difficulty is the fact that the inhomogeneous part $F(z,w)$ involves evaluations of the function $H(z,w)$ or their partial derivatives at $z = 0$. Fortunately, for the cases we consider here, we can solve this problem by introducing an appropriate normalizing factor; the resulting generating function satisfies then a simpler PDE (with boundary values properly eliminated) that can be explicitly solved.

Another general difficulty in solving the recurrences (2) by solving the associated PDEs is how to adapt the general solution to the boundary values. By the method of characteristics, we see that the general solution is given by $H(z,w) = H^{[p]}(z,w) + f(z,w)C(\xi(z,w))$ with an arbitrary continuous function $C(x)$. Often it is not obvious how to find $C(x)$ such that $H(z,w)$ satisfies the boundary values. However, for the examples treated here, we can always solve this problem by using the analyticity of the function $H(z,w)$ in a neighborhood of $(z,w) = (0,0)$, by choosing a suitable curve $z = q(w)$ and by considering the limit $\lim_{w\to 0} H(q(w),w)$ (depending on $f(z,w)$ and $\xi(z,w)$).

As we show later, we obtain for all problems mentioned above closed-form solutions for $H(z,w)$. From such exact forms, we can easily derive the corresponding exact solutions for the underlying probability. Also we can apply general analytic tools such as singularity analysis and saddle-point method (see [**5**]) and obtain rather precise information on the asymptotic growth of the underlying probabilities. However, for problems in two or more variables as we are dealing with here, the treatment is generally more involved than in the univariate case.

2. The pills problem

2.1. The original problem. We start by considering Type A diminishing urn model with the transition matrix $M = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right)$ and the vertical absorbing axis $\mathcal{S} = \{(0,n) : n \ge 0\}$.

The recurrence (2b) for the probability generating function $h_{n,m}(v)$ now becomes

$$h_{n,m}(v) = \frac{n}{n+m}h_{n-1,m}(v) + \frac{m}{n+m}h_{n+1,m-1}(v), \tag{3}$$

[1]The generating function also depends on v_1 and v_2, but we avoid the heavier notation $H(z,w;v_1,v_2)$.

for $n \ge 0$ and $m \ge 1$, with the boundary values $h_{n,0}(v) = v^n$.

Instead of considering the generating function $\tilde{H}(z,w) := \sum_{n\ge 0}\sum_{m\ge 1} h_{n,m}(v) z^n w^m$, which will involve the unknown boundary values $h_{0,m}(v)$ (or $\tilde{H}(0,w)$) in the resulting PDE, we introduce the modified generating function

$$H(z,w) := \sum_{n\ge 0}\sum_{m\ge 1}\binom{m+n}{m} h_{n,m}(v) z^n w^m. \tag{4}$$

Then H satisfies, by recurrence (3), the first-order linear PDE

$$(z - z^2 - w)H_z(z,w) + w(1-z)H_w(z,w) - zH(z,w) = \frac{wv}{(1-vz)^2}, \tag{5}$$

with the initial condition $H(z,0) = 0$. We see that the unknown boundary values $h_{0,m}(v)$ nicely disappear.

To solve equation (5), we apply the method of characteristics. Thus we first consider the corresponding reduced PDE

$$(z - z^2 - w)H_z(z,w) + w(1-z)H_w(z,w) = 0, \tag{6}$$

and find the first integrals for the system of ordinary differential equations (the so-called system of characteristic differential equations)

$$\dot{z} = z - z^2 - w, \quad \dot{w} = w(1-z). \tag{7}$$

We regard here z and w as dependent variables of t, namely, $z = z(t)$, $w = w(t)$ and $\dot{z} := \frac{dz(t)}{dt}$, etc. By reducing (7) to a differential equation (DE) of Bernoulli type, we obtain the following first integral of (7)

$$\xi(z,w) := \frac{we^{z/w}}{1-z-w} = \text{const.}$$

Thus the general solution of the reduced PDE (6) is as follows.

$$H^{[r]}(z,w) = C\Big(\frac{we^{z/w}}{1-z-w}\Big),$$

where $C(x)$ is an arbitrary continuous function

Now consider the inhomogeneous PDE

$$(z - z^2 - w)H_z(z,w) + w(1-z)H_w(z,w) - zH(z,w) = F(z,w). \tag{8}$$

We use the following transformation from (z,w)-coordinates to (η,ξ)-coordinates: $\xi = \frac{we^{z/w}}{1-w-z}$ and $\eta = \frac{w}{1-w-z}$, or equivalently $z = z(\eta,\xi) = \frac{\eta\log(\xi/\eta)}{1+\eta+\eta\log(\xi/\eta)}$ and $w = w(\eta,\xi) = \frac{\eta}{1+\eta+\eta\log(\xi/\eta)}$, which leads to the DE

$$H_\eta(\eta,\xi) - \frac{\log(\xi/\eta)}{1+\eta+\eta\log(\xi/\eta)}H(\eta,\xi) = \frac{1}{\eta}F\big(z(\eta,\xi), w(\eta,\xi)\big). \tag{9}$$

The general solution of the corresponding homogeneous DE $H_\eta(\eta,\xi) - \log(\xi/\eta)H(\eta,\xi)/(1+\eta+\eta\log(\xi/\eta)) = 0$ can be obtained easily and is given by

$$H^{[h]}(z,w) = \frac{1}{1-w-z}C\Big(\frac{we^{\frac{z}{w}}}{1-w-z}\Big),$$

where we applied the inverse (η,ξ)-transform.

The inhomogeneous DE (9) can then be solved by using the method of variation of parameters. We obtain for the inhomogeneous part $F(z,w) = wv/(1-vz)^2$ the following particular solution

$$H^{[p]}(z,w) = vw\int_0^1 \frac{dq}{\big(1 - z(1+(v-1)q) - w(1-q-(v-1)q\log q)\big)^2}. \tag{10}$$

It turns out that the particular solution (10), which is analytic around $z = 0$ and $w = 0$, already satisfies the initial condition, so (10) is the required solution of the problem, $H(z,w) = H^{[p]}(z,w)$.

Extracting coefficients of z^n and w^m in (10) gives for $n \ge 0$ and $m \ge 1$ the following explicit form

$$h_{n,m}(v) = \frac{1}{\binom{n+m}{n}}[z^n w^m]H(z,w) = mv\int_0^1 (1+(v-1)q)^n(1-q-(v-1)q\log q)^{m-1}dq. \tag{11}$$

From this expression, the expectation $\mathbb{E}(X_{n,m}) = h'_{n,m}(1)$ can be easily derived and is given by

$$\mathbb{E}(X_{n,m}) = \frac{n}{m+1} + H_m; \tag{12}$$

cf. [**1, 6**].

Higher moments can be obtained similarly by taking higher derivatives from (11), but the expressions soon become very messy; see [**1**] for the second and the third moments. Instead, we can apply (11) to derive the limiting distribution of $X_{n,m}$, for all ranges of n and m satisfying $\max(m,n) \to \infty$. The idea is roughly as follows. We first compute asymptotic approximations to the r-th factorial moments $\mathbb{E}(X_{n,m}^{\underline{r}}) := \mathbb{E}\big(X_{n,m}(X_{n,m}-1)\cdots(X_{n,m}-r+1)\big)$ starting from the relation $\mathbb{E}(X_{n,m}^{\underline{r}}) = h_{n,m}^{(r)}(1)$ and then by evaluating asymptotically the integrals as derivatives of the Beta-function. The result is

$$\mathbb{E}\big(X_{n,m}^r\big) \sim \mathbb{E}(X_{n,m}^{\underline{r}}) = \begin{cases} r!\Big(\dfrac{n}{m} + \log m\Big)^r \big(1 + \mathcal{O}\big((\log m)^{-1}\big)\big), & \text{for } m \to \infty, \\ \dfrac{n^r}{\binom{m+r}{r}}\big(1 + \mathcal{O}(n^{-1})\big), & \text{for } m \text{ fixed and } n \to \infty. \end{cases}$$

We then obtain the limiting distributions of $X_{n,m}$ after proper normalization, justified by standard arguments (moment sequence uniquely characterizes the distribution).

We collect our results for the pills problem in the following theorem.

THEOREM 1. *Starting with m double-unit pills and n single-unit pills, the probability generating function $h_{n,m}(v) := \sum_{k\ge 0} \mathbb{P}\{X_{n,m} = k\} v^k$ of the number $X_{n,m}$ of the remaining single-unit pills in the urn when all double-unit pills are all taken is given by*

$$h_{n,m}(v) = mv \int_0^1 (1 + (v-1)q)^n (1 - q - (v-1) q \log q)^{m-1} dq.$$

If $m \to \infty$, then the random variable $X_{n,m}$ converges, after suitable scaling, in distribution to an exponentially distributed random variable X with parameter $\lambda = 1$, namely

$$\frac{X_{n,m}}{\frac{n}{m} + \log m} \xrightarrow{(d)} X,$$

where X has density $f(x) = e^{-x}$ for $x \ge 0$.

If m is fixed and $n \to \infty$, then the random variable $X_{n,m}$ converges, after suitable scaling, in distribution to a Beta random variable B_m; in symbol

$$\frac{X_{n,m}}{n} \xrightarrow{(d)} B_m \stackrel{(d)}{=} \mathrm{Beta}(1,m),$$

where B_m has density $m(1-x)^{m-1}$, $0 \le x \le 1$.

Details of the proofs will be given in the full version of this extended abstract.

2.2. A generalization to r pills. We consider the random variable $X_{n_1,\dots,n_r}$, which gives the number of single-unit pills when all pills of r units are all taken, starting with n_i pills of i units, $i = 1, \dots, r$. The probability generating function $h_{n_1,\dots,n_r}(v) := \sum_{k\ge 0} \mathbb{P}\{X_{n_1,\dots,n_r} = k\} v^k$ satisfies for $n_1, \dots, n_{r-1} \ge 0$, $n_r \ge 1$ the recurrence

$$h_{n_1,\dots,n_r}(v) = \frac{n_1}{n_1 + \cdots + n_r} h_{n_1-1,n_2,\dots,n_r}(v) + \sum_{j=2}^{r} \frac{n_j}{n_1 + \cdots + n_r} h_{n_1,\dots,n_{j-2},n_{j-1}+1,n_j-1,n_{j+1},\dots,n_r}(v), \tag{13}$$

with the boundary value $h_{n_1,\dots,n_{r-1},0}(v) = v^{n_1}$. Let

$$H(z_1,\dots,z_r) := \sum_{n_1\ge 0} \cdots \sum_{n_{r-1}\ge 0} \sum_{n_r \ge 1} \binom{n_1 + \cdots + n_r}{n_1, \dots, n_r} h_{n_1,\dots,n_r}(v) z_1^{n_1} \cdots z_r^{n_r}. \tag{14}$$

The recurrence (13) then translates into the first-order linear PDE

$$\begin{aligned} &\sum_{j=1}^{r-1} (z_j - z_1 z_j - z_{j+1}) H_{z_j}(z_1,\dots,z_r) + (z_r - z_1 z_r) H_{z_r}(z_1,\dots,z_r) - z_1 H(z_1,\dots,z_r) \\ &\qquad = \frac{z_r}{(1 - v z_1 - z_2 - \cdots - z_{r-1})^2}, \end{aligned} \tag{15}$$

for $r \geq 3$, with the boundary condition $H(z_1, \dots, z_{r-1}, 0) = 0$.

By the method of characteristics, we then consider the characteristic system of DEs

$$\dot{z}_1 = z_1 - z_1^2 - z_2, \quad \dot{z}_2 = z_2 - z_1 z_2 - z_3, \quad \dots, \quad \dot{z}_{r-1} = z_{r-1} - z_1 z_{r-1} - z_r, \quad \dot{z}_r = z_r - z_1 z_r. \tag{16}$$

We can show that the $r-1$ functions $\xi_1(z_1, \dots, z_r), \dots, \xi_{r-1}(z_1, \dots, z_r)$ given below, where $\xi_1, \dots, \xi_{r-2}$ are given implicitly as the solution of a linear system of equations, give $r-1$ independent first integrals of (16)

$$\begin{aligned}
\frac{z_{r-2}}{z_r} &= \frac{\left(\frac{z_{r-1}}{z_r}\right)^2}{2!} + \xi_{r-2}, \\
\frac{z_{r-3}}{z_r} &= \frac{\left(\frac{z_{r-1}}{z_r}\right)^3}{3!} + \xi_{r-2}\frac{\left(\frac{z_{r-1}}{z_r}\right)}{1!} + \xi_{r-3}, \\
\frac{z_{r-4}}{z_r} &= \frac{\left(\frac{z_{r-1}}{z_r}\right)^4}{4!} + \xi_{r-2}\frac{\left(\frac{z_{r-1}}{z_r}\right)^2}{2!} + \xi_{r-3}\frac{\left(\frac{z_{r-1}}{z_r}\right)}{1!} + \xi_{r-4}, \\
\vdots &= \vdots \\
\frac{z_1}{z_r} &= \frac{\left(\frac{z_{r-1}}{z_r}\right)^{r-1}}{(r-1)!} + \xi_{r-2}\frac{\left(\frac{z_{r-1}}{z_r}\right)^{r-3}}{(r-3)!} + \xi_{r-3}\frac{\left(\frac{z_{r-1}}{z_r}\right)^{r-4}}{(r-4)!} + \cdots + \xi_2\frac{\left(\frac{z_{r-1}}{z_r}\right)}{1!} + \xi_1, \\
\xi_{r-1} &= \frac{z_r}{1 - z_1 - \cdots - z_r} e^{\frac{z_{r-1}}{z_r}}.
\end{aligned}$$

We can solve the PDE (15) by introducing $\eta := \frac{z_r}{1-z_1-\cdots-z_r}$ and $\xi_1, \dots, \xi_{r-1}$ as above and applying a transform to the $(\eta, \xi_1, \dots, \xi_{r-1})$-coordinates. We obtain then the following explicit solution

$$H(z_1, \dots, z_r) = \\ z_r \int_0^1 \frac{dq}{\left(1 - \sum_{j=1}^{r-1}(1 - (-1)^{j-1}(1-v)q\frac{\log^{j-1} q}{(j-1)!})z_j - (1 - q - (-1)^{r-1}(1-v)q\frac{\log^{r-1} q}{(r-1)!})z_r\right)^2}. \tag{17}$$

Thus we obtain after extracting coefficients of (17) an explicit formula for $h_{n_1,\dots,n_r}(v)$, which is given by the following theorem.

THEOREM 2. *Starting with n_1 pills of size 1, $\dots$, n_r pills of size r, $r \geq 3$, the probability generating function $h_{n_1,\dots,n_r}(v)$ of the number $X_{n_1,\dots,n_r}$ of pills of single-unit pills remaining in the urn when all pills of r units are chosen is given by*

$$h_{n_1,\dots,n_r}(v) = n_r \int_0^1 \prod_{j=1}^{r-1} \Big(1 - (-1)^{j-1}(1-v)q\frac{\log^{j-1} q}{(j-1)!}\Big)^{n_j} \Big(1 - q - (-1)^{r-1}(1-v)q\frac{\log^{r-1} q}{(r-1)!}\Big)^{n_r - 1} dq.$$

3. A variant of the pills problem

We consider now the Type A diminishing urn model with the transition matrix $M = \left(\begin{smallmatrix} -1 & 0 \\ 1 & -2 \end{smallmatrix}\right)$ and the vertical absorbing wall $\mathcal{S} = \{(0, n) : n \geq 0\} \cup \{(1, n) : n \geq 0\}$. The recurrence (2b) for the probability generating function $\tilde{h}_{n,\tilde{m}}(v)$ now has the form

$$\tilde{h}_{n,\tilde{m}}(v) = \frac{n}{n+\tilde{m}}\tilde{h}_{n-1,\tilde{m}}(v) + \frac{\tilde{m}}{n+\tilde{m}}\tilde{h}_{n+1,\tilde{m}-2}(v), \tag{18}$$

for $n \geq 0$ and $\tilde{m} \geq 2$, with the boundary values $\tilde{h}_{n,0}(v) = v^n$ and $\tilde{h}_{n,1}(v) = v^n$. Although one could study the recurrence in general, it is more convenient to assume that $\tilde{m}$ is even and we consider only the case $\tilde{m} := 2m$ by introducing $h_{n,m} := \tilde{h}_{n,\tilde{m}}$.

Let

$$H(z, w) := \sum_{n \geq 0} \sum_{m \geq 1} \binom{n+2m}{n} h_{n,m}(v) z^n w^m. \tag{19}$$

By (18), we obtain the first-order linear PDE for $H(z, w)$

$$2w(1-z)H_w(z, w) - wH_z(z, w) + (z-1)H(z, w) = \frac{wv}{(1-vz)^2}, \tag{20}$$

with the boundary condition $H(z,0)=0$. The characteristic system of DEs corresponding to (20) is given by

$$\dot{w} = 2w(1-z), \quad \dot{z} = -w. \tag{21}$$

One easily obtains the first integral of (21)

$$\xi(z,w) := z^2 - 2z - w = \text{const}. \tag{22}$$

Thus the general solution of the reduced equation $2w(1-z)H_w(z,w) - wH_z(z,w) = 0$ is equal to $H^{[r]}(z,w) = C(z^2-2z-w)$, with some continuous function $C(x)$.

To solve the inhomogeneous DE

$$2w(1-z)H_w(z,w) - wH_z(z,w) + (z-1)H(z,w) = F(z,w), \tag{23}$$

we choose a transform of variables from the (z,w)-coordinates to (η,ξ)-coordinates via

$$\xi = z^2 - 2z - w, \quad \eta = z, \tag{24}$$

leading to the DE

$$H_\eta(\eta,\xi) - \frac{\eta-1}{\eta^2-2\eta-\xi}H(\eta,\xi) = -\frac{1}{\eta^2-2\eta-\xi}F\big(z(\eta,\xi),w(\eta,\xi)\big). \tag{25}$$

Solving the DE (25) with the inhomogeneous part $F(z,w) = \frac{wv}{(1-vz)^2}$ leads, after applying the inverse (η,ξ)-transform, to

$$\begin{aligned} H(z,w) = \frac{\sqrt{w}}{v}\Bigg(&-\frac{\sqrt{w}}{(\alpha-\beta^2)(\beta-(1-z))} + \frac{\sqrt{1-\alpha}}{(\alpha-\beta^2)(\beta-1)} + \frac{\beta\arctan\Big(\frac{\sqrt{\alpha-\beta^2}\sqrt{u}}{\alpha-\beta(1-z)}\Big) - \beta\arctan\Big(\frac{\sqrt{\alpha-\beta^2}\sqrt{1-\alpha}}{\alpha-\beta}\Big)}{(\alpha-\beta^2)^{\frac{3}{2}}}\Bigg) \\ &+ \sqrt{w}\,C(z^2-2z-w), \end{aligned} \tag{26}$$

where we use the abbreviations $\alpha := (1-z)^2 - w$ and $\beta := (v-1)/v$, and $C(x)$ denotes an arbitrary continuous function.

To identify the unknown function $C(x)$ in (26), we observe that due to the analyticity of the required solution $H(z,w)$ in a complex neighborhood of $z=0$ and $w=0$ and $H(z,0)=0$

$$\lim_{w\to 0}\frac{H(z,w)}{\sqrt{w}} = 0.$$

This implies that

$$C(x) = -\frac{\sqrt{-x}}{v(1+x-\beta^2)(\beta-1)} + \frac{\beta}{v(1+x-\beta^2)^{\frac{3}{2}}}\arctan\Big(\frac{\sqrt{1+x-\beta^2}\sqrt{-x}}{1+x-\beta}\Big),$$

which yields the solution to the PDE (23) with inhomogeneous part $F(z,w) = \frac{wv}{(1-vz)^2}$

$$H(z,w) = \frac{w}{v(-\beta^2+\alpha)(1-z-\beta)} + \frac{\beta\sqrt{w}}{v(\alpha-\beta^2)^{\frac{3}{2}}}\arctan\Big(\frac{\sqrt{w}\sqrt{\alpha-\beta^2}}{\alpha-\beta(1-z)}\Big). \tag{27}$$

It follows that

$$\mathbb{E}(X_{n,2m}) = \frac{1}{\binom{n+2m}{n}}[z^n w^m]\,\frac{\partial}{\partial v}H(z,w)\Big|_{v=1} = \frac{4^m}{(2m+1)\binom{2m}{m}}n + \frac{4^m}{\binom{2m}{m}} - 1. \tag{28}$$

By the same procedures we used for the pills problem, we can derive the limiting distributions of $X_{n,m}$.

THEOREM 3. *Consider the urn model with the transition matrix $M = \left(\begin{smallmatrix}-1 & 0\\ 1 & -2\end{smallmatrix}\right)$. Let $X_{n,2m}$ denote the number of white balls in the urn at the moment when black balls are all removed (starting with $2m$ black balls and n white balls).*

If $m\to\infty$, then the random variable $X_{n,2m}$ converges, after suitable scaling, in distribution to a Rayleigh random variable R

$$\frac{X_{n,2m}}{\frac{n}{\sqrt{m}} + 2\sqrt{m}} \xrightarrow{(d)} R,$$

where R has density $2xe^{-x^2}$, $x \geq 0$.

If m is fixed and $n \to \infty$, then the random variable $X_{n,2m}$ converges, after suitable scaling, in distribution to a r.v. B_{2m}, which is the square-root of a Beta random variable; in symbols

$$\frac{X_{n,2m}}{n} \xrightarrow{(d)} B_{2m} \stackrel{(d)}{=} \sqrt{\mathrm{Beta}(1,m)},$$

where B_{2m} has density $2mx(1-x^2)^{m-1}$, $0 \le x \le 1$.

4. The cannibal urn

As mentioned in Subsection 1.2, this model can be described as a diminishing urn with the transition matrix $M = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -2 \end{smallmatrix}\right)$ and one vertical absorbing wall $\mathcal{S} = \{(0,n) : n \ge 0\} \cup \{(1,n) : n \ge 0\}$, but with slightly modified weights for the steps. The probability generating function $h_{n,m}(v)$ satisfies the recurrence

$$h_{n,m}(v) = \frac{n}{n+m-1} h_{n,m-1}(v) + \frac{m-1}{n+m-1} h_{n+1,m-2}(v), \tag{29}$$

for $n \ge 0$ and $m \ge 2$, with the boundary values $h_{n,1}(v) = h_{n,0}(v) = v^n$.

Similarly as above, we introduce the modified generating function

$$H(z,w) := \sum_{n\ge 0} \sum_{m \ge 1} \frac{1}{m} \binom{n+m-1}{m-1} h_{n,m}(v) z^n w^m,$$

which leads to the first order linear PDE with initial condition $H(z,0) = 0$

$$H_w(z,w) - (z+w) H_z(z,w) = \frac{1+wv}{1-vz}. \tag{30}$$

The system of characteristic DEs corresponding to (30) is given by

$$\dot{w} = 1, \quad \dot{z} = -w - z, \tag{31}$$

which leads to the first integral

$$\xi(z,w) := \frac{e^{-w}}{1-z-w} = \text{const}.$$

Thus the general solution of the reduced PDE corresponding to (30) is given by $H^{[r]}(z,w) = C\big(\frac{e^{-w}}{1-z-w}\big)$ with a continuous function $C(x)$. Using the transformation $\xi = \frac{e^{-w}}{1-z-w}$ and $\eta = w$, we finally obtain the exact solution of (30)

$$H(z,w) = \log\Big(\frac{1-zv}{e^{-w} - (e^{-w} - 1 + w + z)v}\Big). \tag{32}$$

Thus the probability $\mathbb{P}\{X_{n,m} = k\}$ satisfies

$$\mathbb{P}\{X_{n,m} = k\} = \frac{m}{\binom{n+m-1}{m-1}} [z^n w^m v^k] \log\Big(\frac{1-zv}{e^{-w} - (e^{-w} - 1 + w + z)v}\Big), \tag{33}$$

for $n \ge 0$, $m \ge 1$ and $k \ge 0$. From equation (33) we obtain the following theorem.

THEOREM 4. *The random variable $X_{n,m}$ of the number of cannibals remaining when there are no more non-cannibals (starting with n cannibals and m non-cannibals) satisfies*

$$\mathbb{P}\{X_{n,m} = k\} = \frac{(k-1)!}{(n+m-1)!} \sum_j \frac{(-1)^j}{(k-n-j)!} \sum_\ell \binom{m}{\ell} (-1)^\ell \frac{(n+j)^{m-\ell}}{(j-\ell)!}.$$

Furthermore, if $\mathbb{V}(X_{n,m}) \to \infty$, then $(X_{n,m} - \mathbb{E}(X_{n,m}))/\sqrt{\mathbb{V}(X_{n,m})}$ tends asymptotically to the standard normal variable.

Pittel [**13**] established asymptotic normality of $X_{n,m}$ (as $n+m \to \infty$) for all values of n and m except for the range when $m = o(n)$. Our result covers also this range. More precise results, including the local limit theorem and a Poisson limit law when the variance of $X_{n,m}$ remains bounded will be given elsewhere.

Remark. In a similar way, our approach can be applied to the Type A diminishing urn model with the same transition matrix $M = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -2 \end{smallmatrix}\right)$ and the absorbing states $\mathcal{S} = \{(0,n) : n \ge 0\} \cup \{(1,n) : n \ge 0\}$ as the cannibal urn, but with unmodified transition probabilities, namely, the probability generating function $h_{n,m}(v)$ satisfies the recurrence

$$h_{n,m}(v) = \frac{n}{n+m} h_{n,m-1}(v) + \frac{m}{n+m} h_{n+1,m-2}(v),$$

for $n \geq 0$ and $m \geq 2$, with the boundary values $h_{n,1}(v) = h_{n,0}(v) = v^n$. In particular, we have the closed-form solution

$$H(z,w) = \frac{2vz-2-w}{2(1-vz)^2} + \int_0^1 \frac{(1+wq)dq}{1-v+vwq+v(1-w-z)e^{w(1-q)}}. \tag{34}$$

5. The OK corral

We now briefly consider the Type B diminishing urn model with the transition matrix $M = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ and the two absorbing axes $\mathcal{S} = \{(0,n) : n \geq 0\} \cup \{(m,0) : m \geq 0\}$.

The recurrence (2a) for the probability generating function $h_{n,m}(v_1, v_2)$ as defined by (1) now satisfies

$$h_{n,m}(v_1,v_2) = \frac{m}{n+m} h_{n-1,m}(v_1,v_2) + \frac{n}{n+m} h_{n,m-1}(v_1,v_2), \tag{35}$$

for $n \geq 1$ and $m \geq 1$, with the boundary values $h_{n,0}(v_1,v_2) = v_2^n$, $h_{0,m}(v_1,v_2) = v_1^m$.

Unlike Type A urn models, no additional normalizing factor is needed for this case and the generating function $H(z,w) := \sum_{n\geq 1}\sum_{m\geq 1} h_{n,m}(v_1,v_2) z^n w^m$ satisfies the first-order linear PDE

$$z(1-w)H_z(z,w) + w(1-z)H_w(z,w) = \frac{wzv_1}{(1-v_1z)^2} + \frac{wzv_2}{(1-v_2w)^2}, \tag{36}$$

with the boundary conditions $H(z,0) = v_2z/(1-v_2z)$ and $H(0,w) = v_1w/(1-v_1w)$.

We apply again the method of characteristics to solve equation (36). We easily obtain that one first integral of the characteristic system of DEs

$$\dot{z} = z(1-w), \quad \dot{w} = w(1-z) \tag{37}$$

is

$$\xi(z,w) := \frac{z}{w} e^{w-z} = \text{const.} \tag{38}$$

We then use a transformation from (z,w)-coordinates to (η,ξ)-coordinates via $\xi = ze^{w-z}/w$ and $\eta = z/w$, or equivalently $w = \log(\xi/\eta)/(1-\eta)$ and $z = \eta\log(\xi/\eta)/(1-\eta)$. This gives the solution

$$H_\eta(\eta,\xi) = -\frac{1}{\eta\log(\xi/\eta)} F\big(z(\eta,\xi), w(\eta,\xi)\big), \tag{39}$$

to the inhomogeneous DE

$$z(1-w)H_z(z,w) + w(1-z)H_w(z,w) = F(z,w). \tag{40}$$

Probability that all black balls are removed. This corresponds to an evaluation of $h_{n,m}(v_1,v_2)$ at $v_1 = 0$ and $v_2 = 1$ or, equivalently to a study of (40) with inhomogeneous part $F(z,w) = \frac{wz}{(1-z)^2}$. We obtain the general solution of (40)

$$H(z,w) = \frac{z(1+w-z)}{(1-z)(z-w)} + C\Big(\frac{z}{w}e^{w-z}\Big), \tag{41}$$

where $C(x)$ denotes an arbitrary continuous function.

By considering (41) with $z = xw$, $x \in \mathbb{C}$ and by the fact that

$$\lim_{w\to 0} H(wx,w) = 0, \quad \text{for } x \in \mathbb{C},$$

we have

$$0 = \lim_{w\to 0} \frac{xw(1+w-wx)}{(1-wx)w(x-1)} + \lim_{w\to 0} C\big(\frac{wx}{w}e^{w(1-x)}\big) = \frac{x}{x-1} + C(x).$$

Thus $C(x) = \frac{x}{1-x}$, which yields the solution of (40) with inhomogeneous part $F(z,w) = \frac{wz}{(1-z)^2}$

$$H(z,w) = \frac{z(1+w-z)}{(1-z)(z-w)} + \frac{ze^{w-z}}{w-ze^{w-z}}. \tag{42}$$

Extracting coefficients of z^n and w^m in $H(z,w)$ gives the probability $p_{n,m}$ that all black balls are removed

$$p_{n,m} := [z^n w^m]H(z,w) = \frac{1}{(n+m)!} \sum_{r=1}^{n} (-1)^{n-r} \binom{n+m}{n-r} r^{n+m}. \tag{43}$$

This is exactly the formula stated in [**2**].

Probability that all black balls are removed and k white balls remain. We can apply the same procedure to compute the probability $\mathbb{P}\{X_{n,m}^{(2)} = k\}$ that all black balls are removed and k white balls remain in the urn, or the group of white balls has k "survivors," (when starting at state (m,n)). This corresponds to the evaluation of our H at $v_1 = 0$ and $v := v_2$, which leads to the study of the PDE (40) with inhomogeneous part $F(z,w) = \frac{wzv}{(1-vz)^2}$. The general solution of (40) with this inhomogeneous part satisfies

$$H(z,w) = -v\int_0^1 \frac{zw(w-z-\log q)dq}{(w-zq-vzq(w-z-\log q))^2} + C\Big(\frac{z}{w}e^{w-z}\Big).$$

We can identify $C(x)$ as before and obtain

$$H(z,w) = -v\int_0^1 \frac{zw(w-z-\log q)dq}{(w-zq-vzq(w-z-\log q))^2} - v\int_0^1 \frac{zwe^{w-z}\log q dq}{(w-ze^{w-z}q+vze^{w-z}q\log q)^2}. \tag{44}$$

By (44) and $\mathbb{P}\{X_{n,m}^{(2)} = k\} = [z^n w^m v^k]H(z,w)$, we obtain

$$\mathbb{P}\{X_{n,m}^{(2)} = k\} = \frac{k!}{(n+m)!}\sum_{r=1}^{n}(-1)^{n-r}\binom{n+m}{n-r}\binom{r-1}{k-1}r^{n+m-k}, \tag{45}$$

also stated in [**2**].

We collect the results for the OK corral problem in the following theorem.

THEOREM 5 (stated in [**2**]). *The probability $p_{n,m}$ that all black balls are removed and the probability $\mathbb{P}\{X_{n,m}^{(2)} = k\}$ that exactly k white balls remain in the urn when all black balls are removed (starting with m black balls and n white balls) are for the OK corral urn given by the following exact formulæ ($m \geq 1$, $n \geq 1$, $1 \leq k \leq n$):*

$$p_{n,m} = \frac{1}{(n+m)!}\sum_{r=1}^{n}(-1)^{n-r}\binom{n+m}{n-r}r^{n+m},$$

$$\mathbb{P}\{X_{n,m}^{(2)} = k\} = \frac{k!}{(n+m)!}\sum_{r=1}^{n}(-1)^{n-r}\binom{n+m}{n-r}\binom{r-1}{k-1}r^{n+m-k}.$$

More refined results can be found in [**2**].

6. Sampling without replacement

As another illustrating example, we consider the Type B diminishing urn model with the transition matrix $M = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and the two absorbing axes $\mathcal{S} = \{(0,n) : n \geq 0\} \cup \{(m,0) : m \geq 0\}$. This is by far the simplest diminishing urn model we have considered.

The recurrence (2a) for the probability generating function $h_{n,m}(v_1, v_2)$ has the form

$$h_{n,m}(v_1,v_2) = \frac{m}{n+m}h_{n,m-1}(v_1,v_2) + \frac{n}{n+m}h_{n-1,m}(v_1,v_2), \tag{46}$$

with boundary values $h_{n,0}(v_1,v_2) = v_2^n$, $h_{0,m}(v_1,v_2) = v_1^m$. Recurrence (46) can be solved most easily by introducing the modified generating function

$$H(z,w) := \sum_{n\geq 1}\sum_{m\geq 1}\binom{n+m}{m}h_{n,m}(v_1,v_2)z^n w^m,$$

which leads to the solution

$$H(z,w) = \frac{1}{1-w-z}\Big(\frac{wzv_2}{1-v_2 z} + \frac{wzv_1}{1-v_1 w}\Big). \tag{47}$$

To get the probability $p_{n,m}$ that the black balls are all drawn (starting at state (m,n)), we set $v_2 = 1$ and $v_1 = 0$ and extract the corresponding coefficients

$$p_{n,m} = \frac{1}{\binom{n+m}{m}}[z^n w^m]\frac{wz}{(1-w-z)(1-z)} = \frac{n}{m+n}.$$

On the other hand, to get the probability $\mathbb{P}\{X_{n,m}^{(2)} = k\}$ that all black balls are drawn and k white balls remain in the urn, we evaluate H at $v_1 = 0$, and extract the corresponding coefficients ($v := v_2$)

$$\mathbb{P}\{X_{n,m}^{(2)} = k\} = \frac{1}{\binom{n+m}{m}}[z^n w^m v^k]\frac{wzv}{(1-w-z)(1-vz)} = \frac{\binom{m-1+n-k}{m-1}}{\binom{m+n}{m}}.$$

Of course, these results for sampling without replacement are well-known and can be obtained by many ways.

7. Concluding remarks

Motivated by concrete examples in the literature, we studied here a few exactly solvable diminishing urn models. Many questions remain to be further clarified. E.g., a main difficulty for Type A urn models is to get rid of the unknown boundary values, which could be done for the urn models presented by introducing a normalizing factor for the generating functions. Of course, it would be very interesting to attack directly the differential equations for the "ordinary generating functions", which contain then evaluations of the unknown function (and its partial derivatives) at $z = 0$. For Type B urn models these difficulties with the boundary values do not appear and our approach can be used to obtain generating functions solutions for a variety of urns, e.g., for generalizations of the OK Corral, but the main difficulty here is then to extract the limiting distribution behaviour from the generating functions.

Generating functions turned out to be a very useful tool in the study of urn models as has been demonstrated in particular in [**2, 3**], where Polya-Eggenberger urn models satisfying the tenability condition on the transition matrix have been studied leading to exact and asymptotic results for the distribution of the type of balls in the urn after t draws starting at a certain state.

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