Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Complexes of Directed Trees and Independence Complexes

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ABSTRACT. The theory of complexes of directed trees was initiated by Kozlov to answer a question by Stanley, and later on, results from the theory were used by Babson and Kozlov in their proof of the Lovász conjecture. We develop the theory and prove that complexes on directed acyclic graphs are shellable.

A related concept is that of independence complexes: construct a simplicial complex on the vertex set of a graph, by including each independent set of vertices as a simplex. Two theorems used for breaking and gluing such complexes are proved and applied to generalize results by Kozlov.

A fruitful restriction is anti-Rips complexes: a subset P of a metric space is the vertex set of the complex, and we include as a simplex each subset of P with no pair of points within distance r. For any finite subset P of $\mathbb{R}$ the homotopy type of the anti-Rips complex is determined.

1. Introduction

The theory of complexes of directed trees was initiated by Kozlov [**13**] and Babson and Kozlov used results from this theory in their proof of the Lovász conjecture [**1**]. In this paper we study three ways to connect topology with combinatorics by constructing simplicial complexes:

Name	Base object	Restriction on simplices
$\mathrm{DT}(G)$	Directed graph G	They are directed forests of G.
$\mathrm{Ind}(G)$	Undirected graph G	No vertices are adjacent in G.
$\mathrm{AR}_r(P)$	Point set P in metric space	No two vertices within distance r.

To decide the homotopy type of complexes on directed graphs, Kozlov [**13**] constructed complexes on undirected graphs. In the second part of this paper we show that by moving the problems into complexes on point sets of metric spaces, the homotopy type can be determined for several classes, and naturaly generalized. The first section treats complexes on directed trees. It is shown that certain complexes are shellable, for example those on directed acyclic graphs.

2. Complexes of directed trees

In this section all graphs are directed. A *directed forest* is an acyclic graph with at most one edge directed to each vertex. Equivalently, a directed forest is a collection of disjoint directed trees with all edges oriented away from the root.

DEFINITION 2.1. Let G be a directed graph. The complex of directed trees, $\mathrm{DT}(G)$ have the edges of G as vertex set. A set of edges is a simplex of $\mathrm{DT}(G)$ if the edges viewed as a graph is a directed forest.

Constructing simplicial complexes from graphs can, in principal, be done in two ways: For graphs that satisfy certain properties either their edges or their vertices form a simplex. For complexes of directed trees it is the edges, but later on complexes of the other kind are used.

A directed forest $H \subseteq G$ is *maximal* if H' is not a directed forest for any $H \subset H' \subseteq G$. The *roots* of a directed forest are the roots of the trees in the forest. A maximal face of $\mathrm{DT}(G)$ is the edge set of a maximal directed forest in G, and the other way around.

2000 *Mathematics Subject Classification.* 05C05,57M15.
Key words and phrases. Complexes of directed trees, independence complexes, anti-Rips complexes, graph complexes.
Research supported by ETH Zürich and Swiss National Science Foundation Grant PP002-102738/1.

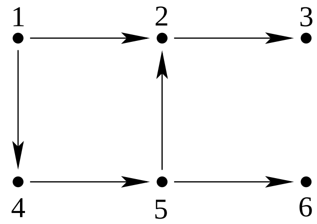

FIGURE 1. Graph G.

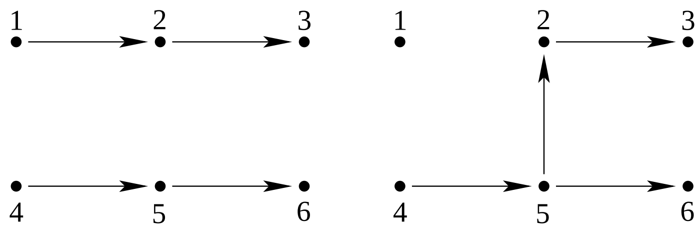

FIGURE 2. Maximal faces F_1 and F_2 of $\mathrm{DT}_R(G)$.

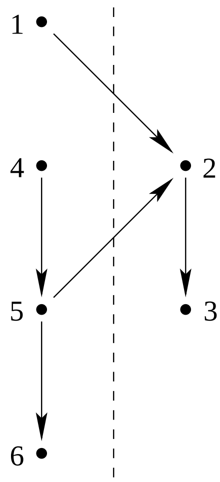

FIGURE 3. The union of F_1 and F_2 with their vertices partitioned.

DEFINITION 2.2. Let $R \subseteq V(G)$. The simplicial complex $\mathrm{DT}_R(G) \subseteq \mathrm{DT}(G)$ is generated by the faces of $\mathrm{DT}(G)$ which are edge sets of directed forests with R as roots.

The forests in $\mathrm{DT}(G)$ with roots R are the maximal faces of $\mathrm{DT}_R(G)$.

DEFINITION 2.3. An edge $(x \to y)$ of G is *nice* in a subcomplex Δ of $\mathrm{DT}(G)$ if

(i) there is an edge $(z \to y)$ in Δ such that $z \neq x$;
(ii) any forest $F \in \Delta$ without an edge directed to y can be extended with $(x \to y)$, and $F \cup \{(x \to y)\} \in \Delta$.

EXAMPLE 2.4. Let G be the directed graph in Figure 1. Let us find some nice edges in $\mathrm{DT}_R(G)$, where $R = \{1, 4\}$. The maximal faces are drawn in Figure 2. We partition the vertices into left and right side of a dotted line: A vertex v is on the left side if the tree which v is in have the same root r in all maximal faces of $\mathrm{DT}_R(G)$, and the path from r to v is the same in all maximal faces of $\mathrm{DT}_R(G)$. All other vertices are on the right side. The union of F_1 and F_2 with their vertices partitioned is depicted in Figure 3. In this example it is not hard to see that the edges crossing the dotted line are nice. If all maximal faces have the same roots, this is true in general.

PROPOSITION 2.1. *All edges crossing the dotted line in a construction as in Example 2.4 are nice if the maximal faces have the same roots.*

PROOF. First we need to show that an edge $(z \to w)$ can only cross the dotted line from left to right. Assume the contrary, i.e. that $(z \to w)$ is from right to left. Since w is on the left, the tree which w is in has the same root r in all maximal faces, and the path from r to w is the same. The vertex before w in that path is always the same, and $(z \to w)$ is in a maximal face, so the vertex is z. But then the path from r to z is the same in all maximal faces, and z should be on the left side, which is a contradiction.

Assume that $(x \to y)$ is an edge crossing the dotted line and that the maximal faces are $\{F_i\}_{i\in I}$.

The vertex y is not a root, hence there is an edge to y in all maximal faces. If $(x \to y)$ was the only edge to y in $\cup_{i\in I} F_i$, it would be in all maximal faces. But then the path to y is the same in all maximal faces, and y would be on the left side. Since y is on the right side, there is an edge $(z \to y)$ such $z \neq x$. Therefore condition (i) of the definition of nice edges is satisfied.

If $(x \to y)$ does not induce a directed cycle when added to a face, then condition (ii) is fulfilled. A directed cycle which contains $(x \to y)$ as an edge would cross the dotted line at least twice. But all edges crossing the dotted line go from left to right, hence $(x \to y)$ cannot induce a directed cycle. □

LEMMA 2.5. *If $R \subseteq V(G)$ is nonempty, and $\mathrm{DT}_R(G)$ has more than one maximal face, then there is an edge $(x \to y) \in E(G)$ which is nice in $\mathrm{DT}_R(G)$.*

PROOF. Construct the left/right-partition of the vertices as in Example 2.4. If there are no edges crossing the dotted line, all edges are on the left side, and there is only one maximal face. Hence there is an edge crossing the dotted line, and by Proposition 2.1 that edge is nice. □

DEFINITION 2.6. A simplicial complex Δ is *shellable* if its maximal faces can be ordered $F_1, F_2, \ldots, F_n$ such that for all $1 \leq i < k \leq n$, there are $1 \leq j < k$ and $e \in F_k$, such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{e\}$.

LEMMA 2.7. *Let F_i and F_k be maximal faces of $\mathrm{DT}_R(G)$, and $(x \to y) \in F_i \setminus F_k$ a nice edge in $\mathrm{DT}_R(G)$. Then there is a maximal face F_j of $\mathrm{DT}_R(G)$, and an $e \in F_k$, such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{e\}$.*

PROOF. There is an edge $(z \to y)$ in F_k, since it is maximal and $y \notin R$. Replace it with $(x \to y)$ to construct F_j. From the definition of nice edges we have that $F_j \in \mathrm{DT}_R(G)$. Now $F_j \cap F_k = F_k \setminus \{(z \to y)\}$. Since $(x \to y) \in F_i$ and $(z \to y) \notin F_i$, we conclude that $F_i \cap F_k = F_i \cap (F_k \setminus \{(z \to y)\}) \subseteq F_k \setminus \{(z \to y)\}$ $= F_j \cap F_k$. □

THEOREM 2.8. $\mathrm{DT}_R(G)$ *is shellable.*

PROOF. The proof is by induction over the number of maximal faces. If $\mathrm{DT}_R(G)$ has one maximal face, it is a simplex, and thus shellable. Assume that $\mathrm{DT}_R(G)$ has more that one maximal face. By Lemma 2.5 there is a nice edge $(x \to y)$. Define $G', G'' \subset G$ as follows:

$$E(G') = E(G) \setminus \{(z \to y) \mid z \neq x\} \quad E(G'') = E(G) \setminus \{(x \to y)\}.$$

A maximal face of $\mathrm{DT}_R(G)$ has exactly one edge to y, thus the set of maximal faces of $\mathrm{DT}_R(G)$ is the disjoint union of the sets of maximal faces of $\mathrm{DT}_R(G')$ and $\mathrm{DT}_R(G'')$. Since $(x \to y)$ is nice, it is in some, but not in all maximal faces of $\mathrm{DT}_R(G)$. Both $\mathrm{DT}_R(G')$ and $\mathrm{DT}_R(G'')$ have smaller numbers of maximal faces than $\mathrm{DT}_R(G)$, so by induction they are shellable. Order the maximal faces of $\mathrm{DT}_R(G)$:

$$\underbrace{F_1, F_2, \ldots, F_t,}_{\text{Shelling order of } \mathrm{DT}_R(G'),\ (x\to y)\in F_l.} \qquad \underbrace{F_{t+1}, F_{t+2}, \ldots, F_{t+s}}_{\text{Shelling order of } \mathrm{DT}_R(G''),\ (x\to y)\notin F_l.}$$

If for all $1 \leq i < k \leq s+t$, there are $1 \leq j < k$ and $(z \to w) \in F_k$ such that $F_i \cap F_k \subseteq F_j \cap F_k = F_k \setminus \{(z \to w)\}$, then $\mathrm{DT}_R(G)$ is shellable. If $1 \leq i < k \leq t$ or $t < i < k \leq s+t$ we are done. Assume $1 \leq i \leq t < k \leq s+t$.

The nice edge $(x \to y)$ is in F_i, but not in F_k. Construct F_j as described in Lemma 2.7. The edge $(x \to y)$ is in F_j, so $j \leq t < k$, and we have a shelling order. □

A *directed acyclic graph* is a directed graph without directed cycles. It is not hard to see that the vertices of a directed acyclic graph G can be ordered so that if $(x \to y) \in E(G)$, then x is before y. This is usually called the topological order. Let R be the set of vertices of G with no edges directed to them. The first vertex in the order is in R, so the set is not empty. Since G is a directed acyclic graph, so are all its subgraphs. Hence the maximal subgraphs, with at most one edge to each vertex, are the maximal forests. All maximal forests contain an edge to each vertex in $V(G) \setminus R$. Hence the roots of all maximal forests are R, and $\mathrm{DT}(G) = \mathrm{DT}_R(G)$.

COROLLARY 2.1. *If G is a directed acyclic graph, then $\mathrm{DT}(G)$ is shellable.*

A pure shellable simplicial complex Δ is homotopy equivalent to a wedge of spheres of the same dimension as Δ, or it is contractible. Thus, by calculating the Euler characteristic of Δ, its homotopy type can be determined. See Björner and Wachs, [**6**], for a proof of that, and further extensions.

Denote with $d^-(v)$ the number of edges directed to a vertex v.

LEMMA 2.9. *If G is a directed acyclic graph with at least one edge, then*

$$\tilde{\chi}(\mathrm{DT}(G)) = - \prod_{v \in V(G) \setminus R} (1 - d^-(v)),$$

where R is the set of vertices without edges directed to them.

PROOF. The proof is by induction over the number of vertices not in R. If there is only one vertex not in R, then the complex $\mathrm{DT}(G)$ is homotopy equivalent to $d^-(v)$ disjoint points, and the formula is true.

If $V(G) \setminus R$ has more than two vertices, order the vertices so that if $(x \to y) \in E(G)$, then x is before y, and so that all vertices of R come before the other ones. Denote the last vertex in this order with w. Let G' be the induced subgraph of G with vertex set $V(G) \setminus \{w\}$. Since w is the last one ordered, there are no edges from w, and $E(G) \setminus E(G')$ are the $d^-(w)$ edges to w.

Let $\alpha(i)$ be the number of subgraphs of G which are forests with i edges, and $\alpha'(i)$ similarly for G'. A forest with i edges in G either has no edge to w, or one of the $d^-(w)$ edges to w, hence for $i > 0$

$$\alpha(i) = \alpha'(i) + d^-(w)\alpha'(i-1).$$

Clearly $\alpha(0) = \alpha'(0) = 1$. The reduced Euler characteristic of $\mathrm{DT}(G)$ is

$$\begin{aligned}
\tilde{\chi}(\mathrm{DT}(G)) &= \sum_{i \geq 0} (-1)^{i+1} \alpha(i) \\
&= -1 + \sum_{i \geq 1} (-1)^{i+1} \big(\alpha'(i) + d^-(w)\alpha'(i-1)\big) \\
&= -1 + \sum_{i \geq 1} (-1)^{i+1} \alpha'(i) + d^-(w) \sum_{i \geq 1} (-1)^{i+1} \alpha'(i-1) \\
&= \sum_{i \geq 0} (-1)^{i+1} \alpha'(i) - d^-(w) \sum_{i \geq 0} (-1)^{i+1} \alpha'(i) \\
&= (1 - d^-(w)) \sum_{i \geq 0} (-1)^{i+1} \alpha'(i) \\
&= (1 - d^-(w)) \tilde{\chi}(\mathrm{DT}(G'))
\end{aligned}$$

Substituting the formula for $\tilde{\chi}(\mathrm{DT}(G'))$ concludes the proof. □

THEOREM 2.10. *If G is a directed acyclic graph, then $\mathrm{DT}(G)$ is homotopy equivalent to a wedge of $\prod_{v \in V(G) \setminus R} (d^-(v) - 1)$ spheres of dimension $\#V(G) - \#R - 1$, where R is the set of vertices without edges directed to them.*

PROOF. The complex $\mathrm{DT}(G)$ is shellable by Corollary 2.1, and the reduced Euler characteristic is $\pm \prod_{v \in V(G) \setminus R} (d^-(v) - 1)$ by Lemma 2.9. The maximal faces of $\mathrm{DT}(G)$ are the maximal forests of G. Since the forests have edges exactly to the vertices with non-zero in-degree, there are $\#V(G) - \#R$ edges in a maximal forest, and the dimension of a maximal face is $\#V(G) - \#R - 1$. □

COROLLARY 2.2. *If G is a directed acyclic graph, then the following statements are equivalent:*

- $\mathrm{DT}(G)$ *is a cone with one of the edges of G as apex.*
- *There is an edge of G which is in all maximal forests.*

- *There is a vertex in* G *with in-degree 1.*
- *The product of all* $(d^-(v) - 1)$ *for* $v \in V(G)$ *is zero.*
- $\mathrm{DT}(G)$ *is contractible.*

Proof. Follows directly from Theorem 2.10 and its proof. □

Several of the results in this section, in particular Theorem 2.8, can be put into the context of greedoids and proved with methods from [**4**].

3. Independence complexes

In the previous section the graphs were directed, but from now on all graphs are assumed to be undirected. A subset of the vertex set of a graph is *independent* if no two vertices in it are adjacent. In a graph G, the *neighborhood* of a vertex v, $N_G(v)$ is the set of vertices which are adjacent to v. If it is clear which G is meant, we just write $N(v)$.

If $W \subseteq V(G)$ then $G[W]$ is the induced subgraph with vertex set W, and $G \backslash W = G[V(G) \backslash W]$. Similarly for simplicial complexes, if $W \subseteq \Delta^{(0)}$ then $\Delta[W]$ is the induced subcomplex, and $\Delta \setminus W = \Delta[\Delta^{(0)} \setminus W]$.

Definition 3.1. Let G be an undirected graph. The *independence complex* of G, denoted $\mathrm{Ind}(G)$, is a simplicial complex with vertex set $V(G)$, and $\sigma \in \mathrm{Ind}(G)$ if σ is an independent set of G.

Proving the following standard facts is a good excercise to get acquainted with independence complexes.

- If $A \subseteq V(G)$ then $\mathrm{Ind}(G[A]) = \mathrm{Ind}(G)[A]$.
- If $A, B \subseteq V(G)$ then $\mathrm{Ind}(G[A]) \cap \mathrm{Ind}(G[B]) = \mathrm{Ind}(G[A \cap B])$.
- If $v \in V(G)$ then $\mathrm{lk}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G \setminus (N(v) \cup \{v\}))$, and $\mathrm{st}_{\mathrm{Ind}(G)}(v) = \mathrm{Ind}(G \setminus N(v)) = v * \mathrm{lk}_{\mathrm{Ind}(G)}(v)$.
- If $\mathrm{Ind}(G \setminus (N(v) \cup \{v\}))$ is contractible, then $\mathrm{Ind}(G) \simeq \mathrm{Ind}(G \setminus v)$
- If $v \in V(G)$ then $\mathrm{Ind}(G)$ is the union of $\mathrm{st}_{\mathrm{Ind}(G)}(v)$ and $\cup_{w \in N(v)} \mathrm{st}_{\mathrm{Ind}(G)}(w)$.
- If $v, w \in V(G)$ are adjacent and $N(v) \setminus \{w\} \subseteq N(w) \setminus \{v\}$, then $\mathrm{st}_{\mathrm{Ind}(G)}(v) \supseteq \mathrm{lk}_{\mathrm{Ind}(G)}(w)$.

If the neighborhood of a vertex v is included in the neighborhood of another vertex, the removal of v from the graph is called a fold. In the theory of `Hom`–complexes folds is a fundamental tool for reducing the size of the input graphs while preserving the simple homotopy type, see [**12**]. But in contrast with folds for `Hom`–complexes, the vertex with the larger neighborhood is removed to preserve the simple homotopy type of an independence complex.

Lemma 3.2. *If* $N(v) \subseteq N(w)$ *then* $\mathrm{Ind}(G)$ *collapses onto* $\mathrm{Ind}(G \setminus \{w\})$.

Proof. Match each maximal σ, such that $w \in \sigma$ and $v \notin \sigma$, with $\sigma \cup \{v\}$, and remove them by an elementary collapse step. Repeat this until all σ such that $w \in \sigma$ are gone. □

In particular, if $N(u) = \{v\}$ and $w \in N(v) \setminus \{u\}$, then $\mathrm{Ind}(G)$ collapses onto $\mathrm{Ind}(G \setminus \{w\})$.

Proposition 3.1 ([**8**]). *If* G *is a forest then* $\mathrm{Ind}(G)$ *is either contractible or homotopy equivalent to a sphere.*

Proof. It is sufficient to show that successive use of Lemma 3.2 starting with G gives a graph H with no adjacent edges, since $\mathrm{Ind}(H)$ is either contractible or homotopy equivalent to a sphere, and each use of the lemma provides a collapse.

The proof is by induction on the number of edges. If there are no adjacent edges we are done. If G is a forest with some adjacent edges, then there is a vertex u with only one neighbor v, such that there is a vertex w in $N(v) \setminus \{u\}$. By Lemma 3.2, we can remove w from G, and by induction $G \setminus \{w\}$ can be reduced to a graph without adjacent edges. □

Proposition 3.2. *Let* G *be a graph,* v, w *distinct non-adjacent vertices of* G*, and* G' *the graph* G *extended with an edge between* v *and* w*. If* $\mathrm{Ind}(G')$ *is* k*–connected and* $\mathrm{Ind}(G' \setminus (N_G(v) \cup N_G(w)))$ *is* $(k-1)$*–connected, then* $\mathrm{Ind}(G)$ *is* k*–connected.*

Proof. Recall the gluing lemma [**2**, 10.3(ii)]; if Δ_1 and Δ_2 are k–connected, and $\Delta_1 \cap \Delta_2$ is $(k-1)$–connected, then $\Delta_1 \cup \Delta_2$ is k–connected. Let $\Delta_1 = \mathrm{Ind}(G')$ and $\Delta_2 = \mathrm{Ind}(G \setminus (N_G(v) \cup N_G(w)))$. The complex Δ_1 is k–connected by assumption, and Δ_2 is k–connected since it is a cone. Using $\Delta_1 = \{\sigma \in$

$\mathrm{Ind}(G) \mid \{v,w\} \not\subseteq \sigma\}$ and $\Delta_2 = \{\sigma \in \mathrm{Ind}(G) \mid \sigma \cup \{v,w\} \in \mathrm{Ind}(G)\}$, we get that $\Delta_1 \cup \Delta_2 = \mathrm{Ind}(G)$, and $\Delta_1 \cap \Delta_2 = \mathrm{Ind}(G' \setminus (N_G(v) \cup N_G(w)))$ which is $(k-1)$–connected. The result follows from the gluing lemma. □

DEFINITION 3.3. A set of maximal simplices from a simplicial complex Δ are *generating simplices* if the removal of their interiors makes Δ contractible. Analogously, if $\mathcal{G}$ is a set of maximal faces of Δ such that $\Delta \setminus \mathcal{G}$ is contractible, then $\mathcal{G}$ are *generating faces* of Δ.

Note that if $\mathcal{G}$ are generating faces of Δ, then $\Delta \simeq \vee_{\sigma\in\mathcal{G}} S^{\dim\sigma}$. A shellable complex has generating faces, they are exactly the ones glued over their whole boundary when added. It is not hard to find complexes with generating faces that are not shellable, or to find complexes without generating faces. If $\mathcal{G}$ are generating faces of Δ', Δ collapses onto Δ', and all $\sigma \in \mathcal{G}$ are maximal in Δ, then $\mathcal{G}$ are generating faces of Δ.

To calculate the homotopy type of complexes in this section, a suitable subcomplex is found and contracted. Of course one can smash any contractible subcomplex, but the resulting identifications can be ugly. Our main vehicle is this lemma by Björner.

LEMMA 3.4. [**2**, Theorem 10.4(ii)] *Let $\Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_n$ be a simplicial complex with subcomplexes Δ_i. If $\Delta_i \cap \Delta_j \subseteq \Delta_0$ for all $1 \le i < j \le n$, and Δ_i is contractible for all $0 \le i \le n$, then*

$$\Delta \simeq \bigvee_{1\le i\le n} \mathrm{susp}(\Delta_0 \cap \Delta_i).$$

Lemma 3.4 is a special case of [**3**, Theorem 2.1], which can be used to generalize both Theorem 3.5 and 3.7. However, the amount of technicalities do not match the increased number of applications at this point.

Two degenerate cases working well with Lemma 3.4 are that $\mathrm{susp}(\emptyset) = S^0$, and that the wedge of nothing is a point.

THEOREM 3.5. *If all vertices in the neighborhood of $u \in V(G)$ are adjacent then*

$$\mathrm{Ind}(G) \simeq \bigvee_{v\in N(u)} \mathrm{susp}\, \mathrm{Ind}(G \setminus (N(u) \cup N(v))),$$

and the union of

$$\bigcup_{\substack{v\in N(u)\\ G\setminus(N(u)\cup N(v))=\emptyset}} \{\{v\}\} \qquad \text{and} \qquad \bigcup_{\substack{v\in N(v)\\ G\setminus(N(u)\cup N(v))\neq\emptyset}} \{\{v\}\cup\sigma \mid \sigma \in \mathcal{G}_v\}$$

are generating faces of $\mathrm{Ind}(G)$, *if $\mathcal{G}_v$ are generating faces of* $\mathrm{Ind}(G \setminus (N(u) \cup N(v)))$.

PROOF. We prove this by smashing the star of u with Lemma 3.4 in two ways. First let $\Delta_u = \mathrm{st}_{\mathrm{Ind}(G)}(u)$, and $\Delta_v = \mathrm{st}_{\mathrm{Ind}(G)}(v)$ for all $v \in N(u)$. Clearly their union is $\mathrm{Ind}(G)$, and they are all contractible. If v and w are different vertices in $N(u)$, then $\Delta_v \cap \Delta_w = \mathrm{st}_{\mathrm{Ind}(G)}(v) \cap \mathrm{st}_{\mathrm{Ind}(G)}(w) = v * \mathrm{lk}_{\mathrm{Ind}(G)}(v) \cap w * \mathrm{lk}_{\mathrm{Ind}(G)}(w) = \mathrm{lk}_{\mathrm{Ind}(G)}(v) \cap \mathrm{lk}_{\mathrm{Ind}(G)}(w)$ since v and w are adjacent. Using that $N(u) \setminus \{v\} \subseteq N(v) \setminus \{u\}$ and $N(u) \setminus \{w\} \subseteq N(w) \setminus \{u\}$, we get that $\mathrm{lk}_{\mathrm{Ind}(G)}(v) \cap \mathrm{lk}_{\mathrm{Ind}(G)}(w) \subseteq \mathrm{st}_{\mathrm{Ind}(G)}(u) = \Delta_u$. The conditions of Lemma 3.4 are satisfied, and therefore

$$\begin{aligned} \mathrm{Ind}(G) &\simeq \textstyle\bigvee_{v\in N(u)} \mathrm{susp}(\Delta_u \cap \Delta_v) \\ &= \textstyle\bigvee_{v\in N(u)} \mathrm{susp}\, \mathrm{Ind}(G \setminus (N(u) \cup N(v))). \end{aligned}$$

Now to the second part of the theorem. Let $V = \{v \in N(u) \mid G \setminus (N(u) \cup N(v)) \neq \emptyset\}$. Assume that $\mathcal{G}_v$ are generating faces of $\mathrm{Ind}(G \setminus (N(u) \cup N(v)))$ for $v \in V$. Let $\mathcal{G}$ be the union of

$$\bigcup_{v\in N(u)\setminus V} \{\{v\}\} \quad \text{and} \quad \bigcup_{v\in V} \{\{v\}\cup\sigma \mid \sigma\in\mathcal{G}_v\}.$$

To begin with, we need to prove that each element of $\mathcal{G}$ is a maximal face of $\mathrm{Ind}(G)$. Let σ be a maximal face of $\mathrm{Ind}(G \setminus (N(u) \cup N(v)))$. Then $\{v\} \cup \sigma$ is maximal in $\mathrm{Ind}(G \setminus (N(u) \setminus \{v\}))$, and also in $\mathrm{Ind}(G)$, since all vertices in $N(u) \setminus \{v\}$ are adjacent to v. If $G \setminus (N(u) \cup N(v)) = \emptyset$ for a $v \in N(u)$, then $N(v) = V(G) \setminus \{v\}$, and $\{v\}$ is maximal in $\mathrm{Ind}(G)$.

To conclude that $\mathcal{G}$ are generating faces of $\mathrm{Ind}(G)$ we also need to prove that $\Delta' = \Delta \setminus \mathcal{G}$ is contractible. For each $v \in V$ let $\Delta'_v = v * (\mathrm{Ind}(G \setminus (N(u) \cup N(v))) \setminus \mathcal{G}_v)$. It is not hard to see that Δ' is the union of Δ_u and $\cup_{v\in V}\Delta'_v$. All Δ'_v are contractible, and for different $v, w \in V$, $\Delta'_v \cap \Delta'_w \subseteq \Delta_v \cap \Delta_w \subseteq \Delta_u$. By Lemma 3.4

$$\begin{array}{rcl}\Delta' & \simeq & \bigvee_{v\in V} \mathrm{susp}(\Delta_u \cap \Delta'_v)\\ & = & \bigvee_{v\in V} \mathrm{susp}(\mathrm{st}_{\mathrm{Ind}(G)}(u) \cap v * (\mathrm{Ind}(G \setminus (N(u) \cup N(v))) \setminus \mathcal{G}_v))\\ & = & \bigvee_{v\in V} \mathrm{susp}(\mathrm{Ind}(G \setminus (N(u) \cup N(v))) \setminus \mathcal{G}_v)\\ & \simeq & \bigvee_{v\in V} \mathrm{susp}(\mathrm{point})\\ & \simeq & \bigvee_{v\in V} \mathrm{point}\\ & \simeq & \mathrm{point}\end{array}$$

Thus $\mathcal{G}$ are generating faces of $\mathrm{Ind}(G)$. □

In [**13**] the complex $\mathcal{L}_n^k$ was defined as the independence complex of the graph with vertex set $\{1, 2, \ldots n\}$, and two vertices $i < j$ are adjacent if $j - i < k$. The homotopy type $\mathcal{L}_n^k$, as well as its generating faces, was calculated for $k = 2$, and for $k > 2$ stated as an open question. For convenience define $\mathcal{L}_n^k = \emptyset$ if $n \leq 0$.

Corollary 3.1. *For all $k \geq 2$ and $n > 1$*

$$\mathcal{L}_n^k \simeq \bigvee_{1\leq i<\min(k,n)} \mathrm{susp}\left(\mathcal{L}_{n-k-i}^k\right)$$

Proof. The neighborhood of 1 is $\{2, 3, \ldots, \min(k, n)\}$, and any two different vertices of it are adjacent.

$$\begin{array}{rcl}\mathcal{L}_n^k & \simeq & \bigvee_{i\in N(1)} \mathrm{susp}(\mathrm{Ind}(G \setminus (N(1) \cup N(i))))\\ & = & \bigvee_{i=2}^{\min(k,n)} \mathrm{susp}(\mathrm{Ind}(G[\{j \mid k + i \leq j \leq n\}]))\\ & \simeq & \bigvee_{i=2}^{\min(k,n)} \mathrm{susp}(\mathrm{Ind}(G[\{j \mid 1 \leq j \leq n - k - i + 1\}]))\\ & = & \bigvee_{i=2}^{\min(k,n)} \mathrm{susp}\left(\mathcal{L}_{n-k-i+1}^k\right)\\ & = & \bigvee_{1\leq i<\min(k,n)} \mathrm{susp}\left(\mathcal{L}_{n-k-i}^k\right)\end{array}$$

□

Example 3.6. The generating faces of $\mathcal{L}_n^3$ produced by recursive use of Theorem 3.5, with $u = 1$, are

n	g.f.				
1	$\emptyset$	5	$\{3\}$	9	$\{2,7\},\{3,7\},\{3,8\}$
2	$\{2\}$	6	$\{2,6\}$	10	$\{3,8\},\{2,6,10\}$
3	$\{2\},\{3\}$	7	$\{2,6\},\{2,7\},\{3,7\}$	11	$\{2,6,10\},\{2,6,11\},\{2,7,11\},\{3,7,11\}$
		4	$\{2\},\{3\}$	8	$\{2,6\},\{2,7\},\{3,7\},\{3,8\}$

Corollary 3.2. *Let G be a graph with three distinct vertices u, v and w, such that $N(u) = \{v, w\}$, and $\{v, w\} \notin E(G)$. If $\mathrm{Ind}(G \setminus (N(u) \cup N(v)))$ and $\mathrm{Ind}(G \setminus (N(u) \cup N(w)))$ are $(k-1)$–connected, and $\mathrm{Ind}(G \setminus (N(u) \cup N(v) \cup N(w)))$ is $(k-2)$–connected, then $\mathrm{Ind}(G)$ is k–connected.*

Proof. Let G' be the graph G extended with an edge between v and w. By Proposition 3.2 it suffices to prove that $\mathrm{Ind}(G')$ is k–connected and $\mathrm{Ind}(G' \setminus (N_G(v) \cup N_G(w)))$ is $(k-1)$–connected. The neighborhood of u in G' is a complete graph, so by Theorem 3.5,

$$\begin{array}{rcl}\mathrm{Ind}(G') & \simeq & \mathrm{susp}(\mathrm{Ind}(G' \setminus (N_G(u) \cup N_G(v)))) \vee \mathrm{susp}(\mathrm{Ind}(G' \setminus (N_G(u) \cup N_G(w))))\\ & = & \mathrm{susp}(\mathrm{Ind}(G \setminus (N_G(u) \cup N_G(v)))) \vee \mathrm{susp}(\mathrm{Ind}(G \setminus (N_G(u) \cup N_G(w)))).\end{array}$$

The suspension of a $(k-1)$–connected complex is k–connected, and the wedge of k–connected complexes is k–connected, thus $\mathrm{Ind}(G')$ is k–connected. The neighborhood of v in $G' \setminus (N_G(v) \cup N_G(w))$ is a complete graph, so once again by Theorem 3.5,

$$\begin{array}{rcl}\mathrm{Ind}(G' \setminus (N_G(v) \cup N_G(w))) & \simeq & \mathrm{susp}(\mathrm{Ind}((G' \setminus (N_G(v) \cup N_G(w))) \setminus \{v, w\}))\\ & = & \mathrm{susp}(\mathrm{Ind}(G \setminus (N_G(u) \cup N_G(v) \cup N_G(w)))).\end{array}$$

The suspension of a $(k-2)$–connected complex is $(k-1)$–connected, hence $\mathrm{Ind}(G' \setminus (N_G(u) \cup N_G(v)))$ is $(k-1)$–connnected. □

The previous theorem can be used when we find a complete subgraph of G with a vertex without neighbours outside the subgraph. Removing the condition of the special vertex forces other conditions.

THEOREM 3.7. *Let K be a subset of $V(G)$ such that $G[K]$ is a complete graph, and $\mathcal{G}$ are generating faces of* $\text{Ind}(G \setminus K)$*, such that for each $k \in K$ and $\sigma \in \mathcal{G}$, one of vertices in σ is adjacent to k. Then*

$$\text{Ind}(G) \simeq \text{Ind}(G \setminus K) \vee \bigvee_{k \in K} \text{susp}\, \text{Ind}(G \setminus (K \cup N(k))).$$

Let $K' = \{k \in K \mid G \setminus (K \cup N(k)) \neq \emptyset\}$. If $\mathcal{G}_k$ are generating faces of $\text{Ind}(G \setminus (K \cup N(k)))$ *for each $k \in K'$, then the union of*

$$\mathcal{G}, \quad \bigcup_{k \in K \setminus K'} \{\{k\}\}, \quad \textit{and} \quad \bigcup_{k \in K} \{\{k\} \cup \sigma | \sigma \in \mathcal{G}_k\}$$

are generating faces of $\text{Ind}(G)$*.*

PROOF. The proof is in the same spirit as that of Theorem 3.5. The subcomplex $\Delta_0 = \text{Ind}(G \setminus K) \setminus \mathcal{G}$ will be contracted. Let $\Delta_k = \text{Ind}(G \setminus N(k))$ for each $k \in K$, and $\Delta_\tau = \{\sigma \mid \emptyset \neq \sigma \subseteq \tau\}$ for each $\tau \in \mathcal{G}$.

If $\sigma \in \text{Ind}(G)$ does not contain any vertex from K, then $\sigma \in \Delta_\sigma$ if $\sigma \in \mathcal{G}$, and $\sigma \in \Delta_0$ if $\sigma \notin \mathcal{G}$. If $\sigma \in \text{Ind}(G)$ and $k \in \sigma$ for a $k \in K$, then $\sigma \in \Delta_k$. Hence the union of these subcomplexes is Δ.

Now we check that the required intersections are subcomplexes of Δ_0. Note that if $\sigma \in \mathcal{G}$ and $k \in K$, then $\sigma \notin \Delta_k$ since by assumption there is a vertex in σ adjacent to k. If k_1 and k_2 are two different elements of K, then $\Delta_{k_1} \cap \Delta_{k_2} \subseteq \text{Ind}(G \setminus K)$ since k_1 and k_2 are adjacent. Since $\sigma \notin \Delta_{k_1}$ for any $\sigma \in \mathcal{G}$, $\Delta_{k_1} \cap \Delta_{k_2} \subseteq \text{Ind}(G \setminus K) \setminus \mathcal{G} = \Delta_0$. If $k \in K$ and $\sigma \in \mathcal{G}$, then $\Delta_k \cap \Delta_\sigma \subseteq \text{Ind}(G \setminus K)$ since $\sigma \in \text{Ind}(G \setminus K)$, and $\Delta_k \cap \Delta_\sigma \subseteq \text{Ind}(G \setminus K) \setminus \mathcal{G} = \Delta_0$ since $\tau \notin \Delta_0$ for all $\tau \in \mathcal{G}$. If σ_1 and σ_2 are different elements of $\mathcal{G}$, then $\Delta_{\sigma_1} \cap \Delta_{\sigma_2} \subseteq \Delta_0$ since $\sigma_1 \notin \Delta_{\sigma_2}$ and $\sigma_2 \notin \Delta_{\sigma_1}$.

By Lemma 3.4

$$\text{Ind}(G) \simeq \left(\bigvee_{k \in K} \text{susp}(\Delta_0 \cap \Delta_k) \right) \bigvee \left(\bigvee_{\sigma \in \mathcal{G}} \text{susp}(\Delta_0 \cap \Delta_\sigma) \right).$$

For $\sigma \in \mathcal{G}$, $\Delta_0 \cap \Delta_\sigma = \Delta_\sigma \setminus \{\sigma\} \simeq S^{\dim \sigma - 1}$. Hence

$$\bigvee_{\sigma \in \mathcal{G}} \text{susp}(\Delta_0 \cap \Delta_\sigma) \simeq \bigvee_{\sigma \in \mathcal{G}} \text{susp}\, S^{\dim \sigma - 1} \simeq \bigvee_{\sigma \in \mathcal{G}} S^{\dim \sigma} \simeq \text{Ind}(G \setminus K).$$

For all $k \in K$, $\Delta_0 \cap \Delta_k = (\text{Ind}(G \setminus K) \setminus \mathcal{G}) \cap \text{Ind}(G \setminus N(k)) = \text{Ind}(G \setminus K) \cap \text{Ind}(G \setminus N(k)) = \text{Ind}(G \setminus (K \cup N(k)))$ since for any $\sigma \in \mathcal{G}$ there is a $v \in \sigma$ adjacent to k, which implies that $\sigma \notin \text{Ind}(G \setminus N(k))$. Inserting this in the conclusion of the lemma proves the first part of the theorem.

Now the second part. Let $\mathcal{H}_1 = \mathcal{G}$, $\mathcal{H}_2 = \cup_{k \in K \setminus K'} \{\{k\}\}$, and $\mathcal{H}_3 = \cup_{k \in K} \{\{k\} \cup \sigma | \sigma \in \mathcal{G}_k\}$. To show that $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ are generating faces of $\text{Ind}(G)$, we need that all $\sigma \in \mathcal{H}$ are maximal faces of $\text{Ind}(G)$, and that $\text{Ind}(G) \setminus \mathcal{H}$ is contractible.

If $k \in K \setminus K'$, then $K \cup N(k) = V(G)$. The neighborhood of k is $V(G) \setminus \{k\}$ since $K \setminus \{k\} \subseteq N(k)$, and k is an isolated point in $\text{Ind}(G)$. Thus all elements of $\mathcal{H}_2$ are maximal faces of $\text{Ind}(G)$. If $\sigma \in \mathcal{H}_1 = \mathcal{G}$, then σ is a maximal face of $\text{Ind}(G \setminus K)$. For each vertex $k \in K$ there is a vertex of σ adjacent to it by assumption, so no vertex of K can be added to σ. Hence σ is also a maximal face of $\text{Ind}(G)$. Therefore, all elements of $\mathcal{H}_1$ are maximal faces of $\text{Ind}(G)$. If $k \in K'$ and $\sigma \in \mathcal{G}_k$, then σ is a maximal face of $\text{Ind}(G \setminus (K \cup N(k))) = \text{lk}_{\text{Ind}(G)}(k)$, so $\{k\} \cup \sigma$ is a maximal face of $\text{Ind}(G)$. All elements of $\mathcal{H}_3$ are therefore maximal faces.

Let Δ_0 be as before, that is $\text{Ind}(G \setminus K) \setminus \mathcal{G}$. For $k \in K'$ let $\Delta'_k = k * (\text{Ind}(G \setminus (K \cup N(k))) \setminus \mathcal{G}_k)$. We will use Lemma 3.4 with Δ_0 and Δ'_k for $k \in K$, which are all contractible. First we show that $\text{Ind}(G) \setminus \mathcal{H} = \Delta_0 \cup (\cup_{k \in K'} \Delta'_k)$. Clearly, $\text{Ind}(G) \setminus \mathcal{H} \supseteq \Delta_0 \cup (\cup_{k \in K'} \Delta'_k)$. If $\sigma \in \text{Ind}(G) \setminus \mathcal{H}$ and no vertex of σ is in K', then $\sigma \in \Delta_0$. If $\sigma \in \text{Ind}(G) \setminus \mathcal{H}$ and $k \in \sigma$, where $k \in K'$, then $\sigma \in \Delta'_k$. If $k_1, k_2 \in K'$ are different, then $\Delta'_{k_1} \cap \Delta'_{k_2} = k_1 * (\text{Ind}(G \setminus (K \cup N(k_1))) \setminus \mathcal{G}_{k_1}) \cap k_2 * (\text{Ind}(G \setminus (K \cup N(k_2))) \setminus \mathcal{G}_{k_2}) = (\text{Ind}(G \setminus (K \cup N(k_1))) \setminus \mathcal{G}_{k_1}) \cap (\text{Ind}(G \setminus (K \cup N(k_2))) \setminus \mathcal{G}_{k_2}) \subseteq \Delta_0$.

By Lemma 3.4

$$\begin{aligned}
\mathrm{Ind}(G)\setminus\mathcal{H} &\simeq \bigvee_{k\in K'} \mathrm{susp}\,(\Delta_0\cap\Delta'_k)\\
&= \bigvee_{k\in K'} \mathrm{susp}\,((\mathrm{Ind}(G\setminus K)\setminus\mathcal{G})\cap(k*(\mathrm{Ind}(G\setminus(K\cup N(k)))\setminus\mathcal{G}_k)))\\
&= \bigvee_{k\in K'} \mathrm{susp}\,((\mathrm{Ind}(G\setminus K)\setminus\mathcal{G})\cap(\mathrm{Ind}(G\setminus(K\cup N(k)))\setminus\mathcal{G}_k))\\
&= \bigvee_{k\in K'} \mathrm{susp}\,(\mathrm{Ind}(G\setminus K)\cap(\mathrm{Ind}(G\setminus(K\cup N(k)))\setminus\mathcal{G}_k))\\
&= \bigvee_{k\in K'} \mathrm{susp}\,(\mathrm{Ind}(G\setminus(K\cup N(k)))\setminus\mathcal{G}_k)\\
&\simeq \bigvee_{k\in K'} \mathrm{susp}\,(\text{point})\\
&\simeq \bigvee_{k\in K'} \text{point}\\
&\simeq \text{point}.
\end{aligned}$$

The equalities need clarification. The first one is by definition. The second one follows from the fact that $k\notin \mathrm{Ind}(G\setminus K)\setminus\mathcal{G}$. Pick a generating face $\sigma\in\mathcal{G}$. By assumption, there is a $v\in\sigma$ for every $k\in K'$, such that v and k are adjacent, that is $v\in N(k)$. Thus $\sigma\notin \mathrm{Ind}(G\setminus(K\cup N(k)))\setminus\mathcal{G}_k$, which gives the next equality. The final one follows from $\mathrm{Ind}(G\setminus K)\supseteq \mathrm{Ind}(G\setminus(K\cup N(k)))\supseteq \mathrm{Ind}(G\setminus(K\cup N(k)))\setminus\mathcal{G}_k$. □

A relative of $\mathcal{L}_n^k$ is its cycle version $\mathcal{C}_n^k$. It is the independence complex of a graph with vertex set $\{1,2,\dots n\}$, and two vertices $i<j$ are adjacent if $j-i<k$ or $(n+i)-j<k$. The case $k=2$ was computed in [**13**], and used by Babson and Kozlov [**1**] in the proof of the Lovász conjecture. The $\mathbb{Z}_2$–homotopy types of $\mathcal{L}_n^2$ and $\mathcal{C}_n^2$ were studied by Živaljević [**16**].

Example 3.8. Using the generating faces of $\mathcal{L}_n^3$ listed in Example 3.6, and Theorem 3.7, with $K=\{1,2\}$, we get these generating faces for $\mathcal{C}_n^3$:

n	generating faces
8	$\{1,5\},\{1,6\},\{2,6\},\{2,7\},\{4,8\}$
9	$\{1,5\},\{1,8\},\{2,6\},\{2,9\},\{4,8\},\{4,9\},\{5,9\}$
13	$\{1,5,9\},\{1,5,10\},\{1,6,10\},\{1,6,11\},\{2,6,10\},\{2,6,11\},$
	$\{2,7,11\},\{2,7,12\},\{4,8,12\},\{4,8,13\},\{4,9,13\},\{5,9,13\}$

Thus $\mathcal{C}_8^3$ is a wedge of five S^1, $\mathcal{C}_9^3$ is a wedge of six S^1, and $\mathcal{C}_{13}^3$ is a wedge of twelve S^2.

This theorem is molded after Theorem 1.1 in [**5**].

Theorem 3.9. *If G is a graph with n vertices and maximal degree d, then* $\mathrm{Ind}(G)$ *is* $\lfloor (n-1)/2d-1\rfloor$*–connected.*

Proof. The proof is by induction on n. If $1\le n\le 2d$ then $\lfloor (n-1)/2d-1\rfloor=-1$ and $\mathrm{Ind}(G)$ is (-1)–connected since it is nonempty.

Recall [**2**, Theorem 10.6(ii)]: If Δ is a simplicial complex and $\{\Delta_i\}_{i\in I}$ is a family of subcomplexes such that $\Delta=\cup_{i\in I}\Delta_i$, and every nonempty intersection $\Delta_{i_1}\cap\Delta_{i_2}\cap\cdots\cap\Delta_{i_t}$ is $(k-t+1)$–connected, then Δ is k–connected if and only if the nerve $\mathcal{N}(\Delta_i)$ is k–connected.

If $n>2d$ define $\Delta_v=\mathrm{Ind}(G\setminus N(v))$ for each $v\in V(G)$. Clearly $\Delta=\cup_{v\in V(G)}\Delta_v$. The complex Δ_v is a cone with apex v and thus $\lfloor (n-1)/2d-1\rfloor$–connected. Let T be a subset of $V(G)$ with $t\ge 2$ elements. There are at most d vertices in a neighborhood and

$$\bigcap_{v\in T}\Delta_v=\bigcap_{v\in T}\mathrm{Ind}(G\setminus N(v))=\mathrm{Ind}\left(G\setminus\bigcup_{v\in T}N(v)\right),$$

so $G\setminus\cup_{v\in T}N(v)$ has at least $n-td$ vertices and $\mathrm{Ind}(G\setminus\cup_{v\in T}N(v))$ is $\lfloor (n-td-1)/2d-1\rfloor$–connected by induction. For $t\ge 2$

$$\left\lfloor\frac{n-1}{2d}-1\right\rfloor-t+1=\left\lfloor\frac{n-td-1}{2d}-\frac{t}{2}\right\rfloor\le\left\lfloor\frac{n-td-1}{2d}-1\right\rfloor,$$

thus $\cap_{v\in T}\Delta_v$ is $(\lfloor (n-1)/2d-1\rfloor-t+1)$–connected as required. We need to show that the nerve is $\lfloor (n-1)/2d-1\rfloor$–connected, and it will follow from that the intersection of $\lfloor (n-1)/2d-1\rfloor+2$ arbitrary Δ_v is nonempty. Indeed, if T is a subset of $V(G)$ with $\lfloor (n-1)/2d-1\rfloor+2$ elements, then $G\setminus\cup_{v\in T}N(v)$ has at least $n-d(\lfloor (n-1)/2d-1\rfloor+2)$ vertices, and

$$n-d\left(\left\lfloor\frac{n-1}{2d}-1\right\rfloor+2\right)\ge n-d\left(\frac{n-1}{2d}-1+2\right)=\frac{n-2d}{2}+\frac{1}{2}>\frac{1}{2},$$

so $\cap_{v\in T}\Delta_v = \mathrm{Ind}(G \setminus \cup_{v\in T} N(v))$ is nonempty. The conditions of [**2**, Theorem 10.6(ii)] are checked and thus $\mathrm{Ind}(G)$ is $\lfloor (n-1)/2d-1 \rfloor$–connected. □

The independence complex of m disjoint complete bipartite graphs $K_{d,d}$ can be collapsed onto the independence complex of m disjoint edges using Lemma 3.2. That complex is homotopy equivalent with an $m-1$ dimensional sphere. The m disjoint $K_{d,d}$ have $2md$ vertices and maximal degree d, thus by Theorem 3.9 the independence complex is $(m-2)$–connected, which is optimal.

4. Anti-Rips complexes

A natural interpretation of $\mathcal{L}_n^k$ is as the complex on $\{1, 2, \ldots n\} \subset \mathbb{R}$, with two different points p and q in the same simplex if, and only if, $|p-q| > k-1$. Most independence complexes in literature can be placed in a metric space, which give rise to this definition.

DEFINITION 4.1. Let P be a subset of a metric space with distance function d, and $r \geq 0$. The *anti-Rips complex* $\mathrm{AR}_r(P)$ have vertex set P, and two different points p and q of P, can only be in the same simplex if $d(p,q) > r$.

Equivalently, $\mathrm{AR}_r(P) = \mathrm{Ind}(G)$, where G is the graph with vertex set P, and two different points p and q are adjacent if $d(p,q) \leq r$. Notice that moving r from 0 to ∞ creates a family of complexes which is ordered by inclusion, and its limits are the simplex on P, and P as disjoint points.

Why name it anti-Rips complexes? Substituting $d(p,q) > r$ with $d(p,q) \leq r$ defines Rips complexes. According to Hausmann [**11**] Lefschetz called them Vietoris complexes, but the notation changed with Rips' reintroduction of them in the study of hyperbolic groups. A contemporary application of Rips complexes is in the approximation of homotopy type of point-cloud data, see for example Carlsson and da Silva [**7**]. Corollary 3.1 can now be generalized.

PROPOSITION 4.1. *If P is a finite subset of $\mathbb{R}$ and $m = \min(P)$ then*

$$\mathrm{AR}_r(P) \simeq \bigvee_{\substack{p\in P \\ m<p\leq m+r}} \mathrm{susp}(\mathrm{AR}_r(\{q \in P \mid q > p + r\}))$$

PROOF. Let $u = m$ in Theorem 3.5. Use that the neighborhood of a point p in the graph corresponding to $\mathrm{AR}_r(P)$, is the set of points q in P, such that $d(p,q) > r$. □

If $P \subset \mathbb{Z}^2$ have n vertices, then $\mathrm{AR}_1(P)$ is $\lfloor (n-9)/8 \rfloor$–connected by Theorem 3.9 since there are at most 4 points within distance one from a point in $\mathbb{Z}^2$. Using the geometry of the plane, $\lfloor (n-9)/8 \rfloor$ can be improved to $\lfloor (n-9)/6 \rfloor$.

PROPOSITION 4.2. *If $P \subset \mathbb{Z}^2$ have n vertices, then $\mathrm{AR}_1(P)$ is $\lfloor (n-9)/6 \rfloor$–connected.*

PROOF. The proof is by induction on n. If $1 \leq n \leq 8$ then $\mathrm{AR}_1(P)$ is $\lfloor (n-9)/6 \rfloor$–connected since it is (-1)–connected, and $\lfloor (n-9)/6 \rfloor \leq -1$. Now assume that $n > 8$. Pick a $u \in P$ such that the sum of its x and y coordinates is maximal among the points in P. It is no restriction to assume that $u = (0,0)$ since the proposition is translation invariant. Define $N : P \to 2^P$ by $N(p) = \{q \in P \mid d(p,q) \leq 1, p \neq q\}$. Let $v = (0,-1)$ and $w = (-1,0)$. Depending on $N(u) \subseteq \{v,w\}$ we have four cases.

If $N(u) = \emptyset$ then $\mathrm{AR}_1(P)$ is a cone and in particular $\lfloor (n-9)/6 \rfloor$–connected.

If $N(u) = \{v\}$ then $\mathrm{AR}_1(P) \simeq \mathrm{susp}(\mathrm{AR}_1(P \setminus (N(u) \cup N(v))))$ by Theorem 3.5. The complex $\mathrm{AR}_1(P \setminus (N(u) \cup N(v)))$ is $\lfloor ((n-5)-9)/6 \rfloor$–connected by induction since $N(u) \cup N(v) \subseteq \{u, v, (1,-1), (0,-2), (-1,-1)\}$. The suspension increases the connectivity by one, and $\lfloor ((n-5)-9)/6 \rfloor + 1 \geq \lfloor (n-9)/6 \rfloor$, thus $\mathrm{AR}_1(P)$ is $\lfloor (n-9)/6 \rfloor$–connected. The case $N(u) = \{w\}$ is analogous.

The final case is $N(u) = \{v,w\}$. By induction $\mathrm{AR}_1(P \setminus (N(u) \cup N(v) \cup N(w)))$ is $(\lfloor (n-9)/6 \rfloor - 2)$–connected since $N(u) \cup N(v) \cup N(w) \subseteq \{u, v, w, (-1,1), (-2,0), (-1,-1), (1,-1), (0,-2)\}$ and $\lfloor ((n-8)-9)/6 \rfloor \geq \lfloor (n-9)/6 \rfloor - 2$. The complex $\mathrm{AR}_1(P \setminus (N(u) \cup N(v)))$ is $(\lfloor (n-9)/6 \rfloor - 1)$–connected by induction since $N(u) \cup N(v) \subseteq \{u, v, w, (-1,-1), (1,-1), (0,-2)\}$. Analogously $\mathrm{AR}_1(P \setminus (N(u) \cup N(w)))$ is $(\lfloor (n-9)/6 \rfloor - 1)$–connected. By Corollary 3.2, $\mathrm{AR}_1(P)$ is $\lfloor (n-9)/6 \rfloor$–connected. □

5. Open questions

We conclude with some open questions.

QUESTION 5.1. One approach to bound the connectivity of a simplicial complex is to chop it up in pieces for which the connectivity can be calculated easily, and then use the Nerve Lemma (cf. [**5, 9**] and Theorem 3.9). Suitable subcomplexes for complexes of directed trees are those with the same roots and Theorem 2.10 shows their connectedness. However their intersections are in general cumbersome. Can this class of subcomplexes be adapted to prove a nontrivial bound for the connectivity?

QUESTION 5.2. Theorem 3.7 puts conditions on the generating faces which in general are hard to verify, and maybe even not possible to achieve by choosing the generating faces correctly. In practice, when all generating faces are "far from a certain vertex", that vertex can often be collapsed away, or discrete Morse theory [**10**] can be used. Can this be formalized to a method for removing vertices not in generating faces?

QUESTION 5.3. Kozlov [**14**] found explicit homotopy equivalences between $\mathcal{L}_{2n}^2$ and complexes described by Shapiro and Welker [**15**]. Is it possible to generalize to $\mathcal{L}_n^k$, or even to anti-Rips complexes on $\mathbb{R}$?

QUESTION 5.4. The homotopy type of $\mathcal{C}_n^k$ in general is still unsolved. A larger class to investigate is the anti-Rips complex of a finite subset of a circle.

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