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Pattern-avoiding fillings of rectangular shapes

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Abstract. A *constrained table of shape* $r \times s$ is an empty table with r rows and s columns, together with two sequences of nonnegative integers: the *row constraints* $(x_1, \ldots, x_r)$, and the *column constraints* $(y_1, \ldots, y_s)$. A *filling* of the table is an assignment of nonnegative integers into the cells of the table such that the entries of the i-th row sum up to x_i and the entries of the j-th column sum up to y_j. For a given $0-1$-matrix P, we say that the filling *avoids* P, if it contains no submatrix of the same size as P with strictly positive entries in all the positions that correspond to 1-entries of P.

We prove that for a permutation matrix P of order at most three, the number of P-avoiding fillings of a constrained table only depends on the order of P and the (unordered) multiset $\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s\}$. Thus, for example, a 2×10 table with both row constraints equal to 5 and all column constrains equal to 1 has the same number of P-avoiding fillings as the 6×6 table with row constraints $(1, 1, 5, 1, 1, 1)$ and column constraints $(1, 1, 1, 1, 5, 1)$.

Résumé. Un *tableau restreint de forme* $r \times s$, c'est un tableau vide avec r lignes et s colonnes, accompagné de deux suites de nombres entiers non négatifs, que l'on appelle *les restrictions horizontales* $(x_1, \ldots, x_r)$ et *les restrictions verticales* $(y_1, \ldots, y_s)$. Un *remplissage* de ce tableau, c'est un placement des nombres entiers non négatifs dans les cellules du tableau, tel que la somme de cellules qui forment la i-ème ligne (ou i-ème colonne) est égale à x_i (ou y_i). Soit P une $0-1$-matrice. On dit que le remplissage M *évite* P, si M ne contient aucune sous-matrice M' de la même forme que P, telle que les cellules de P avec un 1 correspondent aux cellules de M' avec un nombre positif.

On prouve que si P est une matrice de permutation d'ordre trois ou moins, le nombre de remplissages d'un tableau restreint ne dépend que de l'ordre de P et de la collection $\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s\}$ (qui préserve les multiplicités, mais pas l'ordre de ses éléments). Ainsi, par exemple, le tableau de forme 2×10 avec les restrictions horizontales $(5, 5)$ et les restrictions verticales toutes égales à 1 admet le même nombre de remplissages évitant P que le tableau de forme 6×6 avec les restrictions $(1, 1, 5, 1, 1, 1)$ et $(1, 1, 1, 1, 5, 1)$.

1. Introduction

In this paper, we consider the concept of pattern-avoidance in rectangular tables with prescribed row- and column-sums. This may be seen as a generalization of the popular concept of pattern-avoidance of permutations.

Let us start with basic definitions. A *constrained table of shape* $r \times s$ is an empty table with r rows and s columns, together with two sequences of nonnegative integers: the *row constraints* $(x_1, \ldots, x_r)$, and the *column constraints* $(y_1, \ldots, y_s)$, satisfying

$$\sum_{i=1}^{r} x_i = \sum_{j=1}^{s} y_j.$$

A *filling* of the constrained table is a nonnegative integer matrix $M = (m_{ij})$ with r rows and s columns, such that the sum of the entries in the i-th row is equal to x_i, and the sum of the entries in the j-th column

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is equal to y_j, formally:

$$\forall i \in 1, \dots, r\colon \sum_{j=1}^{s} m_{ij} = x_i$$
$$\forall j \in 1, \dots, s\colon \sum_{i=1}^{r} m_{ij} = y_j.$$

Occasionally, we will refer to 0–1-fillings, i.e. the fillings which only take values 0 and 1.

For two sequences $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_s)$ of nonnegative integers, we let $T[x \times y]$ denote the constrained table with row-constraints x and column-constraints y, and we let $f(x \times y)$ denote the total number of fillings of $T[x \times y]$. The unordered multiset $\{x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_s\}$ will be called *the scoreline* of the table $T[x \times y]$.

We will adopt a convention that the rows of a table or a matrix are numbered from top to bottom and the columns are numbered from left to right.

Let S_n denote the symmetric group of order n, i.e., the group of all the permutations of the set $[n] = \{1, 2, \dots, n\}$. For $\pi \in S_n$ we let M_π denote the permutation matrix of π, i.e., the $n \times n$ matrix whose entry in row i and column j is equal to 1 if $j = \pi(i)$ and equal to 0 otherwise. For a sequence $x = (x_1, \dots, x_r)$ and a permutation $\pi \in S_r$ we write $\pi(x)$ for the sequence $(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(r)})$.

We now formalize our intuitive concept of pattern avoidance of fillings. A *pattern* is a matrix $P = (p_{ij})$ with all entries equal to 0 or 1. Let $M = (m_{ij})$ be a filling of a constrained table $T[x \times y]$ of shape $r \times s$, and let P be a pattern with m rows and n columns. We say that M *contains* P, if M has a (not necessarily contiguous) submatrix M' of the same shape as P such that the following implication holds: if $p_{ij} = 1$ for some $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, then the entry in the i-th row and j-th column of M' is positive. If M does not contain P, we say that M *avoids* P, or M is P-avoiding. We let $f(x \times y; P)$ denote the number of P-avoiding fillings of $T[x \times y]$.

Note that if T is a table of shape $r \times r$ with all the row- and column constraints equal to 1, then the fillings of T are exactly the permutation matrices of order r. Furthermore, if P is itself a permutation matrix of a permutation $\pi \in S_n$, then the P-avoiding fillings of T are precisely the permutation matrices corresponding to the permutations that do not contain π as a subpermutation. Thus, the concept of pattern-avoidance in rectangular fillings is a generalization of pattern-avoidance in permutations.

Notice that for any permutation π of appropriate order, we have the identity $f(x \times y) = f(\pi(x) \times y)$. This is because every filling M of $T[x \times y]$ can be transformed into a filling of $T[\pi(x) \times y]$ by permuting the rows of M according to the permutation π. Of course, this simple bijection in general does not preserve pattern avoidance. However, if P is a permutation matrix of order at most three, not only do we have the identity $f(x \times y; P) = f(\pi(x) \times \rho(y); P)$ for any π and ρ, but in fact, we can prove a stronger identity, stated in the following theorem:

THEOREM 1.1 (Main result). *Let $T[x \times y]$ be a constrained table, let P be a permutation matrix of order at most three. Then $f(x \times y; P)$ is uniquely determined by the scoreline of $T[x \times y]$ and the order of P.*

For example, consider the two tables $T = T[(2, 2) \times (1, 1, 1, 1)]$ and $T' = T[(2, 1, 1) \times (2, 1, 1)]$. Both these tables have the same scoreline $\{2, 2, 1, 1, 1, 1\}$, and thus, they must have the same number of P-avoiding fillings for any permutation matrix P of order at most three. Indeed, if P has order two, then both tables admit exactly one P-avoiding filling, and if P has order three, then all the six fillings of T are P-avoiding, and T' also has six P-avoiding fillings (as well as one filling containing P). This example also shows that Theorem 1.1 cannot be extended to permutation patterns of order greater than three.

The proof of our main result is organized as follows: in the next section, we collect some simple observations that deal with the cases when P has order at most two. We also state previous results on fillings of Ferrers shapes, and show how these results imply that $f(x \times y; P) = f(x \times y; Q)$ for any two permutation matrices P, Q of order three. These arguments imply that it suffices to prove Theorem 1.1 for a single fixed P of order three.

In Section 3, we use the RSK algorithm together with basic results on Young tableaux to prove that if P is a diagonal matrix of any order k, then $f(x \times y; P) = f(\pi(y) \times \rho(y); P)$, where π and ρ are arbitrary permutations of appropriate order.

In Section 4, we study an operation called *corner flip*, defined as follows: let $T[x \times y]$ be a constrained table of shape $r \times s$. Assume that for some $t \le r$ and $u \le s$ we have

$$\sum_{i=1}^{t} x_i = \sum_{j=1}^{u} y_j.$$

A corner flip is an operation that transforms the table $T[x \times y]$ into a table $T[x' \times y']$ of shape $(t+s-u) \times (u+r-t)$, where $x' = (x_1, x_2, \ldots, x_t, y_{u+1}, y_{u+2}, \ldots, y_s)$ and $y' = (y_1, y_2, \ldots, y_u, x_{t+1}, x_{t+2}, \ldots, x_r)$. We will show that corner flips preserve the number of P-avoiding fillings, where P is the permutation matrix of the permutation (321). It is easy to see that any two tables of the same scoreline can be transformed to each other by a sequence of row permutations, column permutations and corner flips. Combining these facts, we obtain the proof of Theorem 1.1.

In the last section, we present some remarks on the connection between the fillings of rectangular shapes, the pattern-avoidance in permutations, and other related concepts.

2. Basic observations and previous results

Let us first deal with the values of $f(x \times y; P)$ when P is a permutation matrix of order at most two. In the trivial case when P has order one, we see that $f(x \times y; P) = 0$ unless all the components of x and y are zero, in which case $f(x \times y; P) = 1$. Let us now turn to the slightly less trivial case of permutation matrices of order two:

LEMMA 2.1. *For every permutation matrix P of order two and any constrained table $T = T[x \times y]$, we have $f(x \times y; P) = 1$.*

PROOF. It suffices to prove the lemma for $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the other case is analogous. Let $x = (x_1, \ldots, x_r)$ and $y = (y_1, \ldots, y_s)$. We proceed by induction on $r + s$. If $r = 1$ or $s = 1$, the claim is clear.

Otherwise, let $k = \min\{x_r, y_s\}$. Observe that if $M = (m_{ij})$ is a P-avoiding filling of $T[x \times y]$, then $m_{rs} = k$, otherwise both the last row and the last column of M would contain a positive entry other than m_{rs}, and these two entries would form the forbidden pattern P. Assume now that $k = x_r$ (the case $k = y_s$ is symmetric). For any P-avoiding filling M of T, the last row of M is equal to $(0, \ldots, 0, k)$. Furthermore, the remaining rows of M form a P-avoiding filling of $T' = T[(x_1, \ldots, x_{r-1}) \times (y_1, \ldots, y_{s-1}, y_s - k)]$. By the induction hypothesis, there is exactly one P-avoiding filling of T', and adding a row $(0, \ldots, 0, k)$ to the bottom of this filling produces a P-avoiding filling of T. $\square$

It remains to prove Theorem 1.1 for permutation patterns of order three. As a first step, we will show that $f(x \times y; P) = f(x \times y; Q)$ for any two patterns P, Q of order three. This result follows directly from a more general theorem by de Mier [**4**], which deals with fillings of arbitrary shapes. To state the result, we first need some more definitions and notation.

Let I_d be the identity matrix of order d, and let J_d be its mirror image, i.e., J_d is the permutation matrix of the permutation $(d, d-1, \ldots, 1)$. For a $k \times l$ matrix X and an $m \times n$ matrix Y we write $\left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$ for the $(k+m) \times (l+n)$ matrix whose first k rows and l columns form the submatrix X, the last m rows and n columns form Y and all the other entries are zero. The matrix $\left(\begin{smallmatrix} 0 & X \\ Y & 0 \end{smallmatrix}\right)$ is defined similarly.

A *partition of* n is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k)$ whose sum is n. A partition λ may be conveniently represented by a *Ferrers shape*, which is a left-justified array whose i-th row has λ_i cells. A *constrained* Ferrers shape is a Ferrers shape whose every row and every column has a prescribed sum. We write $T_\lambda[x \times y]$ for the Ferrers shape determined by the partition λ, with row constraints x and column constraints y. The fillings of constrained Ferrers shapes are a straightforward generalization of the fillings of rectangular tables. For a given (rectangular) pattern P, we say that a filling M of $T_\lambda[x \times y]$ contains P, if M has a rectangular submatrix M' fully contained inside the shape λ such that M' contains P in the sense defined in the introduction. If the shape λ is a rectangle, then this notion of containment coincides with our previous definition. We will write $f_\lambda(x \times y)$ for the number of fillings of $T_\lambda[x \times y]$, and $f_\lambda(x \times y; P)$ for the number of these fillings that avoid P.

Constrained fillings of general shapes have been considered by Backelin, West, and Xin [**1**] and Stankova and West [**5**]; the two papers studied the case when all the constraints are equal to 1. Some of their results were subsequently extended by de Mier [**4**] to fillings of arbitrary shapes with arbitrary constraints. From her paper, we shall use the following theorem, which is a generalization of a result by Backelin, West and Xin [**1**]:

Theorem 2.2 (de Mier [**4**]). *For every positive integer n and every two integers i, j with $0 \le i, j \le n$ and every constrained Ferrers shape $T_\lambda[x \times y]$, the number of fillings of $T_\lambda[x \times y]$ that avoid $\begin{pmatrix} J_j & 0 \\ 0 & I_{n-j} \end{pmatrix}$ is equal to the number of fillings of $T_\lambda[x \times y]$ that avoid $\begin{pmatrix} J_k & 0 \\ 0 & I_{n-k} \end{pmatrix}$.*

Corollary 2.3. *For any constrained rectangular table $T[x \times y]$ and any two permutation matrices P, Q of order three, we have $f(x \times y; P) = f(x \times y; Q)$.*

Proof. For P, Q chosen among I_3, J_3 and $\begin{pmatrix} J_2 & 0 \\ 0 & I_1 \end{pmatrix}$, the claim is a special case of Theorem 1. For the other cases, we can easily establish the required identity by exploiting the symmetries of the rectangle; take, e.g., $P = \begin{pmatrix} 0 & I_1 \\ I_2 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} J_2 & 0 \\ 0 & I_1 \end{pmatrix}$: let us write $x = (x_1, \dots, x_r)$ and let us define $\overline{x} = (x_r, x_{r-1}, \dots, x_1)$. Clearly,

$$f(x \times y; P) = f(\overline{x} \times y; Q) = f(\overline{x} \times y; I_3) = f(x \times y; J_3) = f(x \times y; Q).$$

The remaining cases are settled similarly. □

Let us stress that Corollary 2.3 does not generalize to fillings of arbitrary Ferrers shapes. In fact, apart from the three patterns I_3, J_3 and $\begin{pmatrix} J_2 & 0 \\ 0 & I_1 \end{pmatrix}$, no other two permutations of order three are equinumerous for general fillings of general shapes. (If we restrict ourselves to fillings of arbitrary shapes with unit constraints only, we obtain one other equinumerous pair, namely $\begin{pmatrix} 0 & I_1 \\ I_2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & I_2 \\ I_1 & 0 \end{pmatrix}$, see [**5**]. More on this in Section 5.)

3. The RSK algorithm

In this section, we will prove the following result:

Proposition 3.1. *Let $T[x \times y]$ be a constrained table of shape $r \times s$. For every $\pi \in S_r$ and $\rho \in S_s$, and for every positive integer n, we have $f(x \times y; J_n) = f(\pi(x) \times \rho(y); J_n)$.*

Of course, in the statement of the proposition, we could have used any other matrix from Theorem 2.2. Our choice of J_n is purely a matter of convenience.

Proposition 3.1 is an easy consequence of known results on Young tableaux and the Robinson–Schensted–Knuth (or RSK) algorithm. We will state the necessary results without proof; a useful presentation of several variants of the RSK algorithm and their relation to pattern-avoidance in fillings can be found in [**3**]. The proof of the basic properties of Young tableaux and the RSK correspondence can be found in textbooks of combinatorics, such as [**2**] or [**6**].

We first state the necessary definitions:

A *Young tableau*, or more verbosely, a *column-strict semi-standard Young tableau*, is a filling of a Ferrers shape such that the elements of every row form a weakly increasing sequence and the elements of every column form a strictly increasing sequence.

The *content* of a Young tableau P is a sequence $\mu = (\mu_1, \dots, \mu_k)$ where μ_i is the number of cells of P that contain the number i. The number of Young tableaux of shape λ and content μ is known as the Kostka number, denoted $K_{\lambda,\mu}$.

The proof of the following standard fact can be found e.g. in [**2**]:

Fact 3.2. *Let λ be a partition of n and let $\mu = (\mu_1, \dots, \mu_k)$ be a sequence of nonnegative numbers whose sum is n, let π be a permutation of order k. For $\mu' = \pi(\mu) = (\mu_{\pi(1)}, \dots, \mu_{\pi(k)})$, we have the identity $K_{\lambda\mu} = K_{\lambda\mu'}$.*

Let g_π be a bijection that transforms a Young tableau P of content μ to a Young tableau $g_\pi(P)$ of the same shape and of content $\pi(\mu)$.

We now summarize the needed properties of the RSK algorithm:

Fact 3.3. *The RSK algorithm constructs a bijection between fillings of $T[x \times y]$ and ordered pairs of Young tableaux (P, Q) such that P and Q have the same shape, P has content x and Q has content y. Furthermore, the filling avoids J_n if and only if P and Q have less than n rows.*

These facts immediately imply Proposition 3.1:

Proof of Proposition 3.1. Let x, y, π, ρ be as in Proposition 3.1. The J_n-avoiding fillings of $T[x \times y]$ are mapped by the RSK algorithm to pairs of Young tableaux (P, Q) of the same shape λ of at most $n-1$ rows, where the content of P is x and the content of Q is y. This pair may be transformed into a pair of tableaux $(g_\pi(P), g_\rho(Q))$ of shape λ and content $\pi(x)$ and $\rho(y)$. By the RSK algorithm, such pairs correspond to J_n-avoiding fillings of $T[\pi(x) \times \rho(y)]$. □

We remark that the bijection established above does not, in general, preserve the multiset of the entries used in the corresponding fillings. In particular, it does not send 0–1-filling onto 0–1-fillings. This cannot be avoided, because, for example, $T[(2,1,1) \times (2,1,1)]$ has no 0–1 J_2-avoiding filling, while $T[(1,2,1) \times (2,1,1)]$ has one such filling.

4. The corner flip

Let us fix $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_s)$ such that $\sum_{i=1}^r x_i = \sum_{j=1}^s y_j$. Let us also fix $t \le r$ and $u \le s$ such that $\sum_{i=1}^t x_i = \sum_{j=1}^u y_j$. Recall that a corner flip is an operation that transforms a table $T[x \times y]$ into a table $T[x' \times y']$, where $x' = (x_1, x_2, \dots, x_t, y_{u+1}, y_{u+2}, \dots, y_s)$ and $y' = (y_1, y_2, \dots, y_u, x_{t+1}, x_{t+2}, \dots, x_r)$. We prove the following proposition:

Proposition 4.1. *With the notation as above, $f(x \times y; J_3) = f(x' \times y'; J_3)$.*

We introduce the following terminology: let M be a matrix with at least t rows and at least u columns. The *north-west corner* of M, denoted by M_{NW}, is the submatrix of M formed by the intersection of the first t rows with the first u columns. Similarly, M_{NE} denotes the *north-east corner* of M, which is the intersection of the first t rows of M with the columns of index greater than u. The south-east and south-west corners of M are defined analogously. Thus, a matrix M of shape $r \times s$ can be expressed as $M = \left(\begin{smallmatrix} M_{\mathrm{NW}} & M_{\mathrm{NE}} \\ M_{\mathrm{SW}} & M_{\mathrm{SE}} \end{smallmatrix}\right)$. Notice that if M is a filling of $T[x \times y]$, then the sum of the entries of M_{NE} is equal to the sum of the entries of M_{SW}. The rows of M with indices $1, \dots, t$ are called *the northern rows*, the rows with indices greater than t are *the southern rows*, and similarly for the eastern and western columns.

Let (X, Y) be a pair of matrices. We say that a matrix M *completes X and Y inside* $T[x \times y]$ if M is a J_3-avoiding filling of $T[x \times y]$ with $M_{\mathrm{NW}} = X$ and $M_{\mathrm{SE}} = Y$. The following two lemmas immediately imply Proposition 4.1.

Lemma 4.2. *For any pair of matrices (X, Y), there is at most one M that completes (X, Y) inside $T[x \times y]$.*

Lemma 4.3. *A pair of matrices (X, Y) can be completed inside $T[x \times y]$ if and only if the pair (X, Y^T) can be completed inside $T[x' \times y']$, where Y^T denotes the transpose of Y.*

By these lemmas, there is a bijection ϕ that maps a J_3-avoiding filling M of $T[x \times y]$ to the J_3-avoiding filling $\phi(M) = M'$ of $T[x' \times y']$ uniquely determined by the condition $M'_{\mathrm{NW}} = M_{\mathrm{NW}}$ and $M'_{\mathrm{SE}} = M^{\mathrm{T}}_{\mathrm{SE}}$. The existence of such a bijection implies Proposition 4.1. It remains to prove the two lemmas.

Proof of Lemma 4.2. It is enough to prove that if M is a J_3-avoiding filling of $T[x \times y]$ then both M_{SW} and M_{NE} avoid J_2. By Lemma 2.1, a J_2-avoiding matrix is uniquely determined by its row sums and column sums; in particular, M_{NE} and M_{SW} are determined by x, y and the two matrices M_{SE} and M_{NW}.

Assume that M is a J_3-avoiding filling of $T[x \times y]$ and M_{SW} contains J_2. Since the sum of entries of M_{SW} is equal to the sum of the entries of M_{NE}, we know that M_{NE} contains at least one positive entry. This positive entry and the occurrence of J_2 inside M_{SW} form the forbidden pattern J_3, which is a contradiction, showing that M_{SW} avoids J_2. By the same argument, we obtain that M_{NE} avoids J_2 as well. □

Before we present the proof of Lemma 4.3, we state and prove a lemma that characterizes the pairs (X, Y) that can be completed inside $T[x \times y]$. We will say that a pair of matrices is *plausible* for $T[x \times y]$, if X and Y both avoid J_3, X has shape $t \times u$, Y has shape $(r-t) \times (s-u)$, and the row sums and column sums of the matrix $\left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$ do not exceed the corresponding constraints x and y.

Lemma 4.4. *Let (X, Y) be a pair of matrices, let $M_0 = \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$. Let $\overline{x}_i$ be the sum of the i-th row of M_0 and $\overline{y}_j$ the sum of its j-th column. We say that the i-th row (or j-th column) is* saturated *if $x_i = \overline{x}_i$ (or $y_j = \overline{y}_j$). The pair (X, Y) can be completed inside $T[x \times y]$ if and only if the following conditions are satisfied:*

(a) (X,Y) *is plausible.*
(b) $\sum_{i=1}^{t}(x_i-\bar{x}_i)=\sum_{j=u+1}^{s}(y_j-\bar{y}_j)$ *(which is equivalent to* $\sum_{j=1}^{u}(y_j-\bar{y}_j)=\sum_{i=t+1}^{r}(x_i-\bar{x}_i)$*.)*
(c) *Let i_N be the largest index of a northern row of M_0 such that for every $i<i_N$, the i-th row is saturated (in other words, i_N is the northernmost unsaturated row, or $i_N=t$ if all northern rows are saturated). Similarly, let j_W be the largest index of a western column such that for every $j<j_W$, the j-th column is saturated. The submatrix of M_0 induced by the rows $\{i_N+1,\dots,t\}$ and columns $\{j_W+1,\dots,u\}$ has all entries equal to 0.*
(d) *With i_N and j_W as above, the submatrix of M_0 induced by the rows $\{1,\dots,i_N\}$ and columns $\{j_W+1,\dots,u\}$ avoids J_2. The submatrix induced by the rows $\{i_N+1,\dots,t\}$ and columns $\{1,\dots,j_W\}$ avoids J_2 as well.*
(e) *Let i_S be the smallest row-index of a southern row such that for every $i>i_S$, the i-th row is saturated. Similarly, let j_E be the smallest column index of an eastern column such that for every $j>j_E$, the j-th column is saturated. The submatrix of M_0 induced by the rows $\{t+1,\dots,i_S-1\}$ and columns $\{u+1,\dots,j_E-1\}$ has all entries equal to 0.*
(f) *With i_S and j_E as above, the submatrix of M_0 induced by the rows $\{t+1,\dots,i_S-1\}$ and columns $\{j_E,\dots,s\}$ avoids J_2. The submatrix induced by the rows $\{i_S,\dots,r\}$ and columns $\{u+1,\dots,j_E-1\}$ avoids J_2 as well.*

Proof. We first show that the conditions are necessary. This is obvious in the case of (a) and (b). Assume that M completes X and Y in $T[x\times y]$. Assume, for contradiction, that condition (c) does not hold. Then M has a positive entry $m_{ij}>0$ with $i_N<i\le t$ and $j_W<j\le u$. Since i_N is smaller than t, it is unsaturated, otherwise we would get a contradiction with i_N's maximality. Thus, M has at least one positive entry in row i_N and an eastern column. Similarly, M has a positive entry in column j_W and a southern row. These three positive entries form the forbidden pattern J_3.

Assume now, that condition (d) fails. If the submatrix induced by the rows $1,\dots,i_N$ and columns $\{j_W+1,\dots,u\}$ contains J_2, it means that $j_W<u$ and j_W is unsaturated. Hence, M contains a positive entry in column j_W and a southern row, creating the forbidden J_3. By an analogous argument, there is no J_2 in the submatrix formed by rows $\{i_N+1,\dots,t\}$ and columns $\{1,\dots,j_W\}$.

The arguments for the necessity of (e) and (f) are symmetric copies of the arguments given for the necessity of (c) and (d), respectively.

It remains to show that the conditions (a) to (f) are sufficient. Assume that X and Y satisfy these conditions. Fix J_2-avoiding matrices M_{NE} and M_{SW} in such a way that $M=\left(\begin{smallmatrix} X & M_{\mathrm{NE}}\\ M_{\mathrm{SW}} & Y\end{smallmatrix}\right)$ is a filling of $T[x\times y]$ (we do not know yet that M avoids J_3). By condition (b) and by Lemma 2.1, we know that such M_{NE} and M_{SW} exist and are uniquely determined. By the proof of Lemma 4.2, we know that M is the only candidate for a completion of (X,Y) inside $T[x\times y]$. It remains to show that M avoids J_3. For contradiction, assume that M contains J_3. Fix three positive entries in M forming J_3. Assume that these entries appear in rows $i_1<i_2<i_3$ and columns $j_1>j_2>j_3$. At most one of these three entries is in M_{SW} and at most one is in M_{NE}, because these two corners avoid J_2 by construction. It follows that the copy of J_3 intersects either X or Y. It cannot intersect both X and Y due to their mutual position. Assume that the copy of J_3 intersects X (the other case is symmetric). Then clearly $i_2\le t$ and $j_2\le u$, so the middle entry of the copy of J_3 appears inside X. However, it is not possible to have all the three entries inside X (because (X,Y) is plausible and thus X avoids J_3), so we may assume, losing no generality, that row i_1 and column j_1 intersect in M_{NE}. It follows that i_1 is not saturated, which means that $i_N\le i_1<i_2$. If row i_3 and column j_3 intersect inside M_{SW}, we similarly obtain $j_W\le j_3<j_2$ contradicting condition (c). On the other hand, if row i_3 and column j_3 intersect inside X, then we have a contradiction with condition (c) or (d). □

With the characterization of the matrix pairs (X,Y) that can be completed inside $T[x\times y]$, the proof of Lemma 4.3 is easy:

Proof of Lemma 4.3. It suffices to check that a pair (X,Y) satisfies the conditions of Lemma 4.4 with respect to $T[x\times y]$ if and only if the pair (X,Y^{T}) satisfies these conditions with respect to $T[x'\times y']$. This is obvious for conditions (a) and (b). For the remaining four conditions, we may observe that a saturated northern row or western column of $\left(\begin{smallmatrix} X & 0\\ 0 & Y\end{smallmatrix}\right)$ remains saturated in $\left(\begin{smallmatrix} X & 0\\ 0 & Y^{\mathrm{T}}\end{smallmatrix}\right)$. Similarly, a saturated southern row of index $t+i$ in $\left(\begin{smallmatrix} X & 0\\ 0 & Y\end{smallmatrix}\right)$ corresponds to a saturated eastern column of index $u+i$ in $\left(\begin{smallmatrix} X & 0\\ 0 & Y^{\mathrm{T}}\end{smallmatrix}\right)$ and vice versa. Combining this with the observation that transposition preserves copies of J_2, we see that the last four conditions of Lemma 4.4 are unaffected by the transition from $\left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} X & 0 \\ 0 & Y^{\mathrm{T}} \end{smallmatrix}\right)$ and from $T[x \times y]$ to $T[x' \times y']$. $\square$

Let us now recall and prove the main result:

THEOREM 4.5 (Theorem 1.1, again). *Let $T[x \times y]$ be a constrained table, let P be a permutation matrix of order at most three. Then $f(x \times y; P)$ is uniquely determined by the scoreline of $T[x \times y]$ and the order of P.*

PROOF. We have observed earlier that the result is easy for matrices of order at most two (see Lemma 2.1 and the preceding discussion). Thanks to Corollary 2.3, we only need to prove the theorem for a single permutation matrix P of order three. Our matrix of choice is the J_3. By Propositions 3.1 and 4.1, all we have to do is to realize that for any two tables $T = T[x \times y]$ and $T' = T[x' \times y']$ with the same scoreline, we may transform T into T' by a sequence of permutations and corner flips, which is indeed easily seen. We omit the details. $\square$

5. General Remarks

We conclude the paper with some remarks and examples that put fillings of rectangular shapes into a broader context of pattern-avoidance in fillings.

Let us only consider patterns that are permutation matrices, and let us make no distinction between a permutation and its matrix.

Recall that two permutations π, σ are *Wilf-equivalent* if for every n the number of permutations of order n that avoid π is equal to the number of those that avoid σ; in our notation, this may be written as $f(1^n \times 1^n; \pi) = f(1^n \times 1^n; \sigma)$, where 1^n is the sequence of n ones. Let $\pi \overset{1}{\sim} \sigma$ denote the fact that π and σ are Wilf-equivalent.

In [**1**], the authors introduce the concept of *shape-Wilf equivalence*, which can be defined as follows: π and σ are shape-Wilf equivalent (denoted by $\pi \overset{1,s}{\sim} \sigma$) if for each Ferrers shape λ with n rows and n columns we have $f_\lambda(1^n \times 1^n; \pi) = f_\lambda(1^n \times 1^n; \sigma)$.

Allowing arbitrary constraints, we write $\pi \sim \sigma$ if for every constrained table $T[x \times y]$ we have $f(x \times y; \pi) = f(x \times y; \sigma)$, and we write $\pi \overset{s}{\sim} \sigma$, if the identity holds even for arbitrary shapes, i.e., $f_\lambda(x \times y; \pi) = f_\lambda(x \times y; \sigma)$ for any constrained Ferrers shape $T_\lambda[x \times y]$.

In general, $\overset{s}{\sim}$ is different from $\overset{1,s}{\sim}$; for example, for $\lambda = (4,4,4,3)$ we have

$$18 = f_\lambda((1,1,2,1) \times (2,1,1,1); \left(\begin{smallmatrix} 0 & I_2 \\ I_1 & 0 \end{smallmatrix}\right)) \neq f_\lambda((1,1,2,1) \times (2,1,1,1); \left(\begin{smallmatrix} 0 & I_1 \\ I_2 & 0 \end{smallmatrix}\right)) = 17,$$

while Stankova and West [**5**] have shown that $\left(\begin{smallmatrix} 0 & I_2 \\ I_1 & 0 \end{smallmatrix}\right) \overset{1,s}{\sim} \left(\begin{smallmatrix} 0 & I_1 \\ I_2 & 0 \end{smallmatrix}\right)$. Thus, $\overset{s}{\sim}$ is a proper refinement of $\overset{1,s}{\sim}$.

As we have seen, all the permutations of order three are $\sim$-equivalent, which shows that $\sim$ is different from $\overset{1,s}{\sim}$ and $\overset{s}{\sim}$. To see that $\sim$ is also different from $\overset{1}{\sim}$, consider the following two patterns (we omit the 0-entries for clarity):

$$P = \begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \overline{P} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ & & 1 & \end{pmatrix}.$$

Clearly $P \overset{1}{\sim} \overline{P}$, since $\overline{P}$ is a mirror image of P; on the other hand, for $x = (1,1,1,2,1)$ we have

$$165 = f\left(x \times x; P\right) \neq f\left(x \times x; \overline{P}\right) = 166,$$

which shows that P and $\overline{P}$ are not $\sim$-equivalent. This example can also be interpreted as $f(x \times x; P) \neq f(\overline{x} \times x; P)$, where $\overline{x}$ is the sequence x written backwards. This shows that Proposition 3.1 does not generalize to all forbidden patterns.

The following proposition appears in [**1**] (for $\overset{1,s}{\sim}$), and [**4**] (for $\overset{s}{\sim}$).

PROPOSITION 5.1. *Let A, A' and B be three patterns. If $A \overset{1,s}{\sim} A'$, then $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \overset{1,s}{\sim} \left(\begin{smallmatrix} A' & 0 \\ 0 & B \end{smallmatrix}\right)$, and if $A \overset{s}{\sim} A'$ then $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \overset{s}{\sim} \left(\begin{smallmatrix} A' & 0 \\ 0 & B \end{smallmatrix}\right)$.*

In general, it is not true that $A \sim A'$ implies $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} A' & 0 \\ 0 & B \end{smallmatrix}\right)$ (compare the example P above with I_4); on the other hand, Proposition 5.1 easily implies that if $A \overset{s}{\sim} A'$ and $B \overset{s}{\sim} B'$ then $\begin{pmatrix} A & 0 \\ 0 & \overrightarrow{B} \end{pmatrix} \sim \begin{pmatrix} A' & 0 \\ 0 & \overrightarrow{B}' \end{pmatrix}$, where $\overrightarrow{B}$ and $\overrightarrow{B}'$ are the matrices obtained from B and B' by the rotation of 180 degrees (similar arguments can be made for other symmetries of the square). This allows us to obtain relations such as $I_4 \sim \left(\begin{smallmatrix} J_2 & 0 \\ 0 & J_2 \end{smallmatrix}\right)$, which are not directly covered by previously mentioned theorems.

Despite these observations, a complete understanding of the $\sim$-equivalence, as well as of the other related concepts, seems currently out of reach.

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