ABSTRACT YOUNG PAIRS FOR COXETER GROUPS OF TYPE D (EXTENDED ABSTRACT)

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Abstract. The notion of an Abstract Young (briefly: AY) representation is a natural generalization of the classical Young orthogonal form. The AY representations of the symmetric group are characterized in [U2]. In this paper we present several types of minimal AY representations of D_n, associated with standard D-Young tableaux which are a natural generalization of usual standard Young tableaux. We give an explicit combinatorial view (the representation space is spanned by certain standard tableaux while the action is a generalized Young orthogonal form) of representations which are induced into D_n from minimal AY representations of one of the natural embeddings of S_n into D_n. Then we show that these induced representations are isomorphic to the direct sum of two or three minimal AY representations of D_n also associated with standard D-Young tableaux. It is done by constructing a continuous path between representation matrices where one end of the path is the mentioned direct sum; while the other end is the classical form of induced an representation.

1. Introduction

One of the important problems in combinatorial representation theory is to find a unified combinatorial construction of Weyl groups representations. A very important breakthrough in this area was the introduction of Kazhdan-Lustig cell representations [KL]. Other results were achieved by Vershik [V], Vershik and Okounkov [OV], Cherednik [Ch] and Ram [Ra].

A unified axiomatic approach, in steps of the above works of Vershik and Ram, to the representation theory of Coxeter groups and their Hecke algebras was presented in [U1]. This was carried out by a natural assumption on the representation matrices, avoiding a priori use of external concepts (such as Young tableaux).

Let (W, S) be a Coxeter system, and let $\mathcal{K}$ be a finite subset of W. Let $\mathbb{F}$ be a suitable field of characteristic zero and let ρ be a representation of W on the vector space $V_{\mathcal{K}} := span_{\mathbb{F}}\{C_w \mid w \in \mathcal{K}\}$, with basis vectors indexed by elements of $\mathcal{K}$. Adin, Brenti and Roichman in [U1] and [U2] study the sets $\mathcal{K}$ and representations ρ which satisfy the following axiom:

(A) *For any generator $s \in S$ and any element $w \in \mathcal{K}$ there exist scalars $a_s(w), b_s(w) \in \mathbb{F}$ such that*

$$\rho_s(C_w) = a_s(w)C_w + b_s(w)C_{ws}.$$

If $w \in \mathcal{K}$ but $ws \notin \mathcal{K}$ we assume $b_s(w) = 0$.

A pair $(\rho, \mathcal{K})$ satisfying Axiom (A) is called an *abstract Young (AY) pair;* ρ is an *AY representation,* and $\mathcal{K}$ is an *AY cell.* If $\mathcal{K} \neq \varnothing$ and has no proper subset

$\varnothing \subset \mathcal{K}' \subset \mathcal{K}$ such that $V_{\mathcal{K}'}$ is ρ-invariant, then $(\rho, \mathcal{K})$ is called a *minimal AY pair.* (This is much weaker than assuming ρ to be irreducible.)

In [U1] it was shown that an AY representation of a simply laced Coxeter group is determined by a linear functional on the root space. Thus it may be obtained by restriction of Ram's calibrated representations of affine Hecke algebras (see [Ra]) to the corresponding Weyl groups. In [U2] it is shown that, furthermore, the values of the linear functional on the "boundary" of the AY cell determine the representation.

In [U2] this result is used to characterize AY cells in the symmetric group. This characterization is then applied to show that every irreducible representation of S_n may be realized as a minimal abstract Young representation. AY representations of Weyl groups of type B are not determined by a linear functional. However, it is shown in [U2] that irreducible representations of B_n, similarly to irreducible representations of S_n, may be realized as minimal AY representations.

In this work we present several types of minimal AY representation of D_n which arise from D-Young tableaux introduced in Section 3.1. These D-Young tableaux are a special case of Ram's negative rotationally symmetric tableaux [Ra]. It is shown in [U2, Theorem 3.9] that the representation of a Coxeter group W which is induced from a minimal AY representation of its parabolic subgroup P is a minimal AY representation of W. In section 5.1 of this work we give an explicit combinatorial view (the representation space is spanned by certain standard tableaux while the action is a generalized Young orthogonal form) of representations which are induced into D_n from minimal AY representations of one of the natural embeddings of S_n into D_n. In section 5.2 we show that these induced representations are isomorphic to the direct sum of two or three minimal AY representations of D_n also associated with standard D-Young tableaux. It is done by constructing a continuous path between representation matrices where one end of the path is the mentioned direct sum; another end is the classical form of induced representation (see [U2, Remark 3.10]). The similar results may be obtained for the group B_n instead of D_n, as it is discussed in Section 2.6 of [Cr1] or in Section 6 of [Cr2]. The complete version of this work with proofs and examples can be found in [Cr1] or in [Cr2].

2. Preliminaries and notations

For the necessary background on Coxeter groups see [H]; on symmetric group representations see [J], [JK], [Sa].

2.1. The Coxeter groups of type B.

The Coxeter group of type B, B_n, is the group of all signed permutations. Let S_{2n} be the group of all permutations of the numbers $\pm 1, \pm 2, ..., \pm n$. Then

$$B_n = \{\pi \in S_{2n} \ : \ \pi(-i) = -\pi(i) \text{ for } i = 1, 2, ..., n\}$$

The group B_n is generated by the Coxeter generators $\{s_0, s_1, .., s_{n-1}\}$ defined by:

$$s_0 = (-1, 1)$$

and

$$s_i = (i, i+1)(-i, -i-1), \qquad 1 \leq i \leq n-1.$$

2.2. **The Coxeter groups of type D.** The Coxeter group of type D, D_n, can be defined as the normal subgroup of B_n consisting of all signed permutations π satisfying: $|\{i \in [n] \mid \pi(i) < 0\}|$ is even. We embed D_n in S_{2n} in the natural way. The group D_n is generated by the Coxeter generators $\{s_0, s_1, .., s_{n-1}\}$ defined by:

$$s_0 = (1, -2)(2, -1)$$

and

$$s_i = (i, i+1)(-i, -i-1), \qquad 1 \le i \le n-1.$$

Notice that s_0 of B_n differs from s_0 of D_n while other Coxeter generators are the same. An element of D_n is called a *reflection* if it is conjugate to a Coxeter generator. The reflections of D_n are $(i,j) = (i,j)(-i,-j) \in S_{2n}$. We usually write just (i,j) instead of $(i,j)(-i,-j)$.

Let $V = \mathbb{R}^n$ be the root space of D_n with $\{e_1, \dots, e_n\}$ as its standard basis. The simple roots are $-e_1 - e_2, e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n$. Denote

$$e_{-i} = -e_i \quad \text{for} \quad 1 \le i \le n$$

The positive root $\alpha_{ij} \in \mathbb{R}^n$ corresponding to the reflection $(i,j) \in D_n$ is

$$\alpha_{ij} = e_i - e_j \ \text{ for } \quad i, j \in [\pm n] \ , \ i \ne -j \ , \ i < j$$

Notice that $\alpha_{ij} = \alpha_{-j,-i} = -\alpha_{ji}$. We have the following identity:

$$\alpha_{ir} + \alpha_{rj} = e_i - e_r + e_r - e_j = e_i - e_j = \alpha_{ij}$$

Definition 2.1. For $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ denote $v_{-i} = -v_i$ and define the derived vector

$$\begin{aligned}\Delta v &= (v_2 - v_{-1}, v_2 - v_1, v_3 - v_2, \dots, v_n - v_{n-1}) \\ &= (v_1 + v_2, v_2 - v_1, v_3 - v_2, \dots, v_n - v_{n-1}) \in \mathbb{R}^n \ .\end{aligned}$$

2.3. **Abstract Young Cells and Representations.** This section surveys results from [U1] which will be used in this paper. Recall the definition of AY cells and representations from the introduction.

Observation 2.2. [U1, Observation 3.3] Every nonempty AY cell is a left translate of an AY cell containing the identity element of W.

Proposition 2.3. [U1, Corollary 4.4] *Every minimal AY cell is convex (in the right Cayley graph $X(W,S)$ or, equivalently, under right weak Bruhat order). In particular, $b_s(w) \ne 0$ whenever $s \in S$ and $w, ws \in \mathcal{K}$.*

Important examples of AY cells are (standard) descent classes.

Surprisingly, Axiom (A) leads to very concrete matrices, whose entries are essentially inverse linear.

In [U1] it is shown that, under mild conditions, Axiom (A) is equivalent to the following more specific version. Here T is the set of all reflections in W.

(B) *For any reflection $t \in T$ there exist scalars $\dot{a}_t, \dot{b}_t, \ddot{a}_t, \ddot{b}_t \in \mathbb{F}$ such that, for all $s \in S$ and $w \in \mathcal{K}$:*

$$\rho_s(C_w) = \begin{cases} \dot{a}_{wsw^{-1}} C_w + \dot{b}_{wsw^{-1}} C_{ws}, & if \ell(w) < \ell(ws); \\ \ddot{a}_{wsw^{-1}} C_w + \ddot{b}_{wsw^{-1}} C_{ws}, & if \ell(w) > \ell(ws). \end{cases}$$

If $w \in \mathcal{K}$ and $ws \notin \mathcal{K}$ we assume that $\dot{b}_{wsw^{-1}} = 0$ (if $\ell(w) < \ell(ws)$) or $\ddot{b}_{wsw^{-1}} = 0$ (if $\ell(w) > \ell(ws)$).

Theorem 2.4. [U1, Theorem 5.2] *Let $(\rho, \mathcal{K})$ be a minimal AY pair for the Iwahori-Hecke algebra of (W, S). If $a_s(w) = a_{s'}(w') \Longrightarrow b_s(w) = b_{s'}(w')$ $(\forall s, s' \in S, w, w' \in \mathcal{K})$, then ρ satisfies Axiom (B).*

Theorem 2.5. [U1, Theorem 11.1] *The coefficients $\dot{a}_t$ $(t \in T)$ determine all the character values of ρ.*

The assumption regarding the coefficients $b_s(w)$ in Theorem 2.4 is merely a normalization condition. Theorem 2.4 shows that the coefficients $a_s(w)$ and $b_s(w)$ in Axiom (A) depend only on the reflection $wsw^{-1} \in T$ and on the relation between w and ws in the right weak Bruhat order. It turns out that for simply laced Coxeter groups the coefficients $\dot{a}_t$ are given by a linear functional.

Definition 2.6. For a convex subset $\mathcal{K} \subseteq W$ define:

$$\begin{aligned} T_{\mathcal{K}} &:= \{wsw^{-1} \mid s \in S,\, w \in \mathcal{K},\, ws \in \mathcal{K}\}, \\ T_{\partial\mathcal{K}} &:= \{wsw^{-1} \mid s \in S,\, w \in \mathcal{K},\, ws \notin \mathcal{K}\}. \end{aligned}$$

Definition 2.7. ($\mathcal{K}$-genericity)
Let $\mathcal{K}$ be a convex subset of W containing the identity element. A vector f in the root space V is $\mathcal{K}$-generic if:

(i) For all $t \in T_{\mathcal{K}}$,
$$\langle f, \alpha_t \rangle \notin \{0, 1, -1\}.$$
(ii) For all $t \in T_{\partial\mathcal{K}}$,
$$\langle f, \alpha_t \rangle \in \{1, -1\}.$$
(iii) If $w \in \mathcal{K}$, $s, t \in S$, $m(s,t) = 3$ and $ws, wt \notin \mathcal{K}$, then
$$\langle f, \alpha_{wsw^{-1}} \rangle = \langle f, \alpha_{wtw^{-1}} \rangle \ (= \pm 1).$$

By Observation 2.2, we may assume that $id \in \mathcal{K}$.

Theorem 2.8. [U1, Theorem 7.4] *Let (W, S) be an irreducible simply laced Coxeter system, and let $\mathcal{K}$ be a convex subset of W containing the identity element. If $f \in V^*$ is $\mathcal{K}$-generic, then*

$$\dot{a}_t := \frac{1}{\langle f, \alpha_t \rangle} \qquad (\forall t \in T_{\mathcal{K}} \cup T_{\partial\mathcal{K}}),$$

together with $\ddot{a}_t$, $\dot{b}_t$ and $\ddot{b}_t$ satisfying

$$\begin{aligned} \dot{a}_t + \ddot{a}_t &= 0 \\ \dot{b}_t \cdot \ddot{b}_t &= (1 - \dot{a}_t)(1 - \ddot{a}_t) \end{aligned}$$

define a representation ρ such that $(\rho, \mathcal{K})$ is a minimal AY pair satisfying Axiom (B).

Remark 2.9. Various normalizations for $\dot{b}_t$ and $\ddot{b}_t$ are possible: symmetric ($\ddot{b}_t = \dot{b}_t$), seminormal ($\ddot{b}_t = 1$), row stochastic ($\dot{a}_t + \dot{b}_t = \ddot{a}_t + \ddot{b}_t = 1$), etc. [U1, Subsection 5.2]. By Theorem 2.5, all these normalizations give isomorphic representations.

The following theorem is complementary.

Theorem 2.10. [U1, Theorem 7.5] *Let (W, S) be an irreducible simply laced Coxeter system and let $\mathcal{K}$ be a subset of W containing the identity element. If $(\rho, \mathcal{K})$ is a minimal AY pair satisfying Axiom (B) and $\dot{a}_t \neq 0$ $(\forall t \in T_{\mathcal{K}})$, then there exists a $\mathcal{K}$-generic $f \in V$ such that*

$$\dot{a}_t = \frac{1}{\langle f, \alpha_t \rangle} \qquad (\forall\ t \in T_{\mathcal{K}} \cup T_{\partial\mathcal{K}}).$$

2.4. **Boundary Conditions.** In this subsection it is shown that the action of the group W on the boundary of a minimal AY cell determines the representation up to isomorphism.

Definition 2.11. Let $f \in V$ be an arbitrary vector on the root space V of W.

(1) Define
$$\mathcal{K}^f := \{w \in W \mid \forall t \in A_f, \ell(tw) > \ell(w)\}$$
where
$$A_f = \{t \in T \mid \langle f, \alpha_t \rangle \in \{\pm 1\}\}$$
and ℓ is the Coxeter length.

(2) If f is $\mathcal{K}^f$-generic (as in Definition 2.7), then the corresponding AY representation of W (as in Theorem 2.8), with the symmetric normalization $\ddot{b}_t = \dot{b}_t$ $(\forall t \in T_{\mathcal{K}^f})$, will be denoted ρ^f.

3. D-Young Tableaux and Minimal Cells in D_n

In this section we show that standard Young tableaux of skew shape lead to minimal AY cells.

3.1. **Cells and Skew Shapes of D_n.** In this subsection we study minimal AY cells $\mathcal{K} \subseteq D_n$. By Observation 2.2, every minimal AY cell is a translate of a minimal AY cell containing the identity element; thus we may assume that $id \in \mathcal{K}$.

We identify the root space of D_n with $V = \mathbb{R}^n$.

For a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ recall the notation from Definition 2.1

$$\Delta\, v = (v_1 + v_2, v_2 - v_1, \ldots, v_n - v_{n-1}) \in \mathbb{R}^n.$$

For a (skew) tableau T denote

$$c_k := j - i$$

where k is the entry in row i and column j of T. Call $cont(T) := (c_1, \ldots, c_n)$ the *content vector* of T, and call $\Delta\, cont(T)$ the *derived content vector* of T. Below we will sometimes denote for brevity $cont(T)$ as $c(T)$.

Definition 3.1. Let λ be a diagram of a skew shape. Define a D-Young tableau of shape λ to be a filling of λ by the $2n$ numbers $\pm 1, \pm 2, \ldots, \pm n$ in such a way that $c_{-i} = -c_i$ for $1 \leq i \leq n$. A D-Young tableau is called *standard* if the numbers are increasing in rows and in columns. If $c_i = 0$, then we allow the numbers $\pm i$ to occupy the same box.

Remark 3.2. Our standard D-Young tableau doesn't change if we multiply all its entries by -1 and rotate it by $180°$. Indeed standard D-Young tableaux (except of the case when $\pm i$ occupy the same box) are "negative rotationally symmetric standard tableaux" considered by Ram in [Ra] just with additional requirement that the all assigned boxes form a skew shape.

Remark 3.3. The need of considering the negative entries and the negative contents, as well as the need to allow two numbers in the same box is discussed in subsection 5.2.5 of [Cr2].

Here are three examples:

$$
\begin{array}{ccccc}
 & -9 & -8 & -7 & -6 & -5 \\
 & -4 & -3 & -2 & -1 & \\
 & 1 & 2 & 3 & 4 & \\
5 & 6 & 7 & 8 & 9 &
\end{array}
\qquad
\begin{array}{ccc}
 & -4 & 1 \\
-3 & \pm 2 & 3 \\
-1 & 4 &
\end{array}
\qquad
\begin{array}{cccccc}
 & & -3 & -2 & 1 & 4 \\
-4 & -1 & 2 & 3 & &
\end{array}
\; .
$$

Recall the notations $\mathcal{K}^f$ and ρ^f from Definition 2.11. Also recall that for each $v \in \mathbb{R}^n$, $\langle v, \alpha_{ij} \rangle = v_i - v_j$.

The following theorem generalizes the part of sufficiency of Theorem 4.1 from [U2]. (The work on the generalization of the part of necessity of Theorem 4.1 from [U2] is in progress and seems to be completed soon.)

Theorem 3.4. *Let T be a standard D-Young tableau with $2n$ boxes, let $c = (c_1, ..., c_n) \in \mathbb{Z}^n$ be its content vector. Consider the sets $A_c = \{t \in T \mid \langle c, \alpha_t \rangle \in \{1, -1\}\}$ and $\mathcal{K}^c = \{w \in D_n \mid \ell(tw) > \ell(w), \quad \forall t \in A_c\}$. Then c is K^c-generic and therefore gives rise to a minimal AY pair $(\rho^c, \mathcal{K}^c)$.*

See [Cr1, Theorem 2.3.4] for the proof of Theorem 3.4.

3.2. **DAY Cells: definition and structure.**

Definition 3.5. Let T be a standard D-Young tableau with $2n$ boxes. It follows from Theorem 3.4, that T gives rise to a minimal AY pair $(\mathcal{K}^c, \rho^c)$ where $c = cont(T)$. We call such a cell K^c a DAY cell and such a representation ρ^c a DAY representation.

The following theorem describes a DAY cell by a certain set of standard D-Young tableaux. Its statement and proof are similar to the statement and the proof of Theorem 5.5 from [U2]. The proof of Theorem 3.6 may be found in [Cr1, Theorem 2.3.12].

Theorem 3.6. *Let T be a standard D-Young tableau and let $c = cont(T)$. Then for any $\pi \in D_n$,*

$$\pi \in K^c \iff \text{The tableau } T^{\pi^{-1}} \text{ is a standard } D\text{-Young tableau.}$$

4. Standard D-Young tableaux: construction and enumeration.

We deal here only with tableaux which have at most two boxes on the zero content diagonal. The reason for it is that tableaux without boxes having zero content lead to representations induced to D_n from S_n, while tableaux with one or two boxes of zero content give rise to minimal AY representations which are subrepresentations of these induced representations.

4.1. **Tableaux without zero-content boxes.**

Definition 4.1. Let $m \in \mathbb{N}$. Denote by $T[\lambda, m, +]$ the set consisting of all tableaux satisfying the following conditions:

- The smallest nonnegative content is equal to m.
- The boxes having non negative content form the shape λ.
- the number of negative entries in the boxes with positive contents is even.

The set $T[\lambda, m, -]$ is defined similarly to $T[\lambda, m, +]$, except for the third condition which reads now:

- the number of negative entries in the boxes with positive contents is odd.

Further, when we consider both $T[\lambda, m, +]$ and $T[\lambda, m, -]$ we sometimes write $T[\lambda, m, \pm]$ for brevity.

Obviously from the definition, tableaux from sets $T[\lambda, m, \pm]$ have no zero-content boxes.

Recall that if the number of negative entries in the boxes of T having positive content is even, then for each $\pi \in D_n$ the number of negative entries in the boxes of T^π having positive content is even as well.

Proposition 4.2. *Let λ be a shape (straight or skew) with n boxes and $m \in \mathbb{N}$. Then*

$$\#T[\lambda, m, +] = \#T[\lambda, m, -] = 2^{n-1} f^\lambda$$

See [Cr1, Proposition 2.4.2] for a proof.

4.2. **Tableaux having one or two boxes with zero content.**

Now consider tableaux with one or two boxes on the diagonal of zero content. Such a tableau can be divided (sometimes not uniquely) into two sub tableaux: one contains n boxes with nonnegative contents and the other is a reflection of the first one since $c_{-i} = -c_i$ contains the boxes with non positive contents.

4.2.1. *Tableaux having one box with zero content.* First consider the case in which the tableau has one box with zero content. The condition $c_{-i} = -c_i$ implies that this unique zero-content box must be occupied by two numbers $\pm i$.

Definition 4.3. We define $T[\lambda, \boxtimes]$ to be the set of all standard D-Young tableaux which have one box having zero content (always occupied by $\pm i$ for some i) s.t. the shape of the boxes with nonnegative contents is λ.

For example, the set $T[(2,1), \boxtimes]$ is

$$\begin{array}{ccc} & -3 & -2 \\ & \pm 1 & \\ 2 & 3 & \end{array} \; ; \quad \begin{array}{ccc} & -3 & 1 \\ & \pm 2 & \\ -1 & 3 & \end{array} \; ; \quad \begin{array}{ccc} & -3 & -1 \\ & \pm 2 & \\ 1 & 3 & \end{array} \; ; \quad \begin{array}{ccc} & -3 & 2 \\ & \pm 1 & \\ -2 & 3 & \end{array} \; ; \quad \begin{array}{ccc} & -2 & 3 \\ & \pm 1 & \\ -3 & 2 & \end{array}$$

while the set $T[(4), \boxtimes]$ consists of only one tableau $\begin{array}{|c|c|c|c|c|c|c|} \hline -4 & -3 & -2 & \pm 1 & 2 & 3 & 4 \\ \hline \end{array}$.

4.2.2. *Tableaux having two boxes with zero content.* Now suppose we have a tableau with two different boxes on the zero content diagonal.

Definition 4.4. Denote by $T[\lambda, \cdot|\cdot, +]$ ($T[\lambda, \div, +]$) the set of standard D-Young tableaux which satisfy the following conditions:

- Each tableau has exactly two boxes on the zero content diagonal.

- Each tableau can be divided by the **vertical** (**horizontal**) straight line into two parts (with n boxes in each part)– one consists of the boxes with non-negative contents and the other consists of non positive contents – in such a way that λ is the shape (straight or skew) of the boxes which have non-negative contents after this separation and the number of negative entries in the boxes of λ is even.

When the number of negative entries in the boxes of λ is odd we get $T[\lambda, \cdot|\cdot, -]$ $(T[\lambda, \div, -])$.

Consider, for example, the following tableau:

$$\begin{array}{cccc} & \boxed{-3} & \boxed{1} & \boxed{2} \\ \boxed{-2} & \boxed{-1} & \boxed{3} & \end{array} . \quad \text{Here} \quad \lambda = \begin{array}{cc} \boxed{} & \boxed{} \\ \boxed{} & \end{array} = (2,1)$$

and the number of negative entries in λ is equal to zero, thus even.

The diagonal of zero contents passes through the numbers 3 and -3 and this tableau can be divided as required above by a vertical line:

$$\begin{array}{cccc} & \boxed{-3} & \boxed{1} & \boxed{2} \\ \boxed{-2} & \boxed{-1} & \boxed{3} & \end{array} \longrightarrow \begin{array}{cc} & \boxed{-3} \\ \boxed{-2} & \boxed{-1} \end{array} \qquad \begin{array}{cc} \boxed{1} & \boxed{2} \\ \boxed{3} & \end{array} .$$

4.2.3. *Constructing D-Young tableaux from a shape with n boxes.* We continue to study sets $T[\lambda, \cdot|\cdot, \pm]$, $T[\lambda, \div, \pm]$, $T[\lambda, \boxtimes]$ and $T[\lambda, m, \pm]$ in more detail.

Definition 4.5. Denote $T[\lambda, 0, +] := T[\lambda, \cdot|\cdot, +] \bigcup T[\lambda, \div, +] \bigcup T[\lambda, \boxtimes]$ and $T[\lambda, 0, -] := T[\lambda, \cdot|\cdot, -] \bigcup T[\lambda, \div, -] \bigcup T[\lambda, \boxtimes]$.

Theorem 4.6. *Let λ be a skew or straight shape with n boxes. Then for any $m \in \mathbb{N}$ there exists a natural bijection:*

$$T[\lambda, 0, +] = T[\lambda, \boxtimes] \bigcup T[\lambda, \div, +] \bigcup T[\lambda, \cdot|\cdot, +] \quad \longleftrightarrow \quad T[\lambda, m, +]$$

and $T[\lambda, 0, -] = T[\lambda, \boxtimes] \bigcup T[\lambda, \div, -] \bigcup T[\lambda, \cdot|\cdot, -] \quad \longleftrightarrow \quad T[\lambda, m, -]$.

The Proof of Theorem 4.6 can be found in [Cr1] or [Cr2].

5. Several types of DAY representations.

Let T be a standard D-Young tableau with content vector

$$c(T) = (c_1(T), c_2(T) ..., c_n(T)) .$$

As discussed in the previous sections, $c(T)$ is a generic vector (see Theorem 3.4) and thus gives rise to a minimal AY cell in D_n (see Theorem 2.8). Denote

$$\begin{aligned} f(T) &= (f_0(T), f_1(T), ..., f_{n-1}(T)) \\ &= \Delta c(T) = (c_1 + c_2, c_2 - c_1, c_3 - c_2, ..., c_n - c_{n-1}) . \end{aligned}$$

The corresponding minimal AY representation ρ, which we called a *DAY* representation (see Definition 3.5) acts on the space spanned by all the standard D-Young tableaux obtained from T by the natural action of D_n. The action of the representation matrices of the generators is defined by

$$\rho_{s_i}(T) = \frac{1}{f_i(T)} \, T + \sqrt{1 - \frac{1}{f_i(T)^2}} \, T^{s_i} \quad \text{for } i = 0, 1, 2, ..., n-1 . \qquad (*)$$

Notice (see Theorem 3.6) that T^{s_i} is **not** standard if and only if

$$\sqrt{1-\frac{1}{f_i(T)^2}}\left(=\frac{\sqrt{f_i(T)^2-1}}{|f_i(T)|}\right)=0 \Longleftrightarrow f_i=\pm 1\ .$$

Definition 5.1. For $m \in \{0\}\bigcup\mathbb{N}$ we denote by $\rho^{\lambda,m,+}$, $\rho^{\lambda,m,-}$, $\rho^{\lambda,\cdot|\cdot,+}$, $\rho^{\lambda,\cdot|\cdot,-}$, $\rho^{\lambda,\div,+}$, $\rho^{\lambda,\div,-}$, $\rho^{\lambda,\boxtimes}$ the DAY representations defined above by $(*)$ on the spaces spanned by sets $T[\lambda,m,+]$, $T[\lambda,m,-]$, $T[\lambda,\cdot|\cdot,+]$, $T[\lambda,\cdot|\cdot,-]$, $T[\lambda,\div,+]$, $T[\lambda,\div,-]$, $T[\lambda,\boxtimes]$, respectively.

5.1. **The Representations $\rho^{\lambda,m,\pm}$ are induced from S_n to D_n.** There are two embeddings of S_n in D_n which are relevant for us. Denote them by

$$S_n^1=\langle s_1,s_2,\ldots,s_{n-1}\rangle$$

and

$$S_n^0=\langle s_0,s_2,\ldots,s_{n-1}\rangle \quad .$$

These two embeddings are conjugate in D_n if and only if n is odd.

Let λ be a straight or skew shape with n boxes and let S^λ denote the representation of S_n associated with λ via classical Young orthogonal form. (If λ is a straight shape, then S^λ is an irreducible Specht module, otherwise, S^λ is a skew Specht module.)

Theorem 5.2. *For any natural m*

(1) $\rho^{\lambda,m,+}\cong S^\lambda\uparrow_{S_n^1}^{D_n}$

(2) $\rho^{\lambda,m,-}\cong S^\lambda\uparrow_{S_n^0}^{D_n}$

5.2. **Decomposition of induced representation into minimal AY representations.** The main result of this work is the following decomposition rule:

Theorem 5.3. *Let λ be a straight or skew shape with n boxes and $m\in\mathbb{N}$. Then*

(1) $\rho^{\lambda,\boxtimes}\oplus\rho^{\lambda,\cdot|\cdot,+}\oplus\rho^{\lambda,\div,+}\cong\rho^{\lambda,m,+}$

(2) $\rho^{\lambda,\boxtimes}\oplus\rho^{\lambda,\cdot|\cdot,-}\oplus\rho^{\lambda,\div,-}\cong\rho^{\lambda,m,-}$

Note that, when the set $T[\lambda,\cdot|\cdot,\pm]$ or $T[\lambda,\div,\pm]$ is empty, the representation $\rho^{\lambda,\cdot|\cdot,\pm}$ or $\rho^{\lambda,\div,\pm}$, respectively, is the zero module.

Notice also that in Theorem 5.3 it is enough to deal with $m=1$ because it follows from Theorem 5.2 that the representations $\rho^{\lambda,m_1,+}$ and $\rho^{\lambda,m_2,+}$ for any $m_1,m_2\in\mathbb{N}$ are isomorphic.

5.2.1. *Sketch of the proof of Theorem 5.3.* The complete proof can be found in [Cr1] or [Cr2]. We cite the following lemma from [Cr1]

Lemma 5.4. *There exist matrix functions*

$$g_i\ :\ \mathbb{C}\to\mathcal{M}_d(\mathbb{C})$$

for $i=0,1,2,...,n-1$ and $d=2^{n-1}f^\lambda$ such that

$$g_i(m)=\rho_{s_i}^{\lambda,m,+} \qquad\qquad m=0,1,2,...$$

The entries of $g_i(x)$ are analytic single valued functions on $\{x\in\mathbb{C}\ :\ Re\, x>-1\}$ and are continuous from the right at $x=-1$ as functions of real variable. The same holds for $\rho^{\lambda,m,-}$.

The group D_n has $(n^2+n)/2$ Coxeter relations: $(s_i s_j)^{m_{ij}} = 1$, $0 \leqslant i \leqslant j \leqslant n-1$, where $m_{ii} = 1$ and $m_{ij} = 2$ or $m_{ij} = 3$ for $i < j$. According to these relations, we introduce $(n^2+n)/2$ matrix functions of x for $0 \leqslant i \leqslant j \leqslant n-1$:

$$A_{ij}(x) = (g_i(x+1)g_j(x+1))^{m_{ij}} .$$

For $x = -1, 0, 1, 2, 3, ...$, the matrices $g_i(x+1)$ (for $i = 0, 1, 2, ..., n-1$) are generator matrices of the representations $\rho^{\lambda,0,+}$, $\rho^{\lambda,1,+}$, $\rho^{\lambda,2,+}$,..., and therefore the generator matrices $\rho_{s_i}^{\lambda,0,+}$, $\rho_{s_i}^{\lambda,1,+}$, $\rho_{s_i}^{\lambda,2,+}$, ... must satisfy the defining relations of the group which implies

$$A_{ij}(x) = I \qquad \text{for } 0 \leqslant i \leqslant j \leqslant n-1 \text{ and } x = -1, 0, 1, 2, 3, ...$$

where I is the identity matrix. But the entries of $A_{ij}(x)$ are polynomials in several expressions of the form $\pm\frac{1}{c_i+c_j+2x+2}$, $\sqrt{1-\frac{1}{(c_i+c_j+2x+2)^2}}$, where c_i, c_j are constants. By Argument (1) in the proof of Theorem 5.3, (can be found in [Cr2]) which involves the Carlson's theorem [B] about zeros of an analytic function, we conclude that

$$A_{ij}(x) = I \qquad \text{for } 0 \leqslant i \leqslant j \leqslant n-1 \text{ and for any real } x \geqslant -1 \ ,$$

not only for integers $x = -1, 0, 1, 2, 3, ...$ as before. This means that the matrices $g_i(x+1)$ are generator matrices of representations not only for integers $x = -1, 0, 1, 2, 3, ...$ but for any real $x \geq -1$. Denote this representation by $\rho^{(x+1)}$. Take some fixed $w \in D_n$ and denote by $\chi_w(x)$ the character of the representation $\rho^{(x+1)}$ evaluated at w. The character is a polynomial in the entries of the representation matrices, so in our case $\chi_w(x)$ is a polynomial in several expressions of the form $\pm\frac{1}{c_i+c_j+2x+2}$, $\sqrt{1-\frac{1}{(c_i+c_j+2x+2)^2}}$ and, therefore, $\chi_w(x)$ is a continuous function of $x \geq -1$. By discreteness of the character values and continuity of $\chi_w(x)$, we have $\chi_w(x) = const$ which implies that all the representations $\rho^{(x+1)}$ for $x \geqslant -1$ are isomorphic. In particular, substituting $x = -1, 0, 1, 2, 3, ...$ we get

$$\rho^{\lambda,\boxtimes} \oplus \rho^{\lambda,\div,+} \oplus \rho^{\lambda,\cdot|\cdot,+} \cong \rho^{\lambda,m,+} \ ,$$

and the first statement of Theorem 5.3 is proved. The proof of the second statement of Theorem 5.3 is similar to the one listed above.

Remark 5.5. In this proof we did not use Theorem 5.2. Moreover, we proved that all the representations $\rho^{(x)}$ are isomorphic and denoting

$$\rho_{s_i}^{(\infty)} = \lim_{x\to+\infty} \rho_{s_i}^{(x)} \qquad \text{for } i = 0, 1, 2, ..., n-1$$

we obtain the representation $\rho^{(\infty)}$ which is exactly the classical form of induced representation as given, for example, in [U1], proof of Theorem 9.3, page 32. This observation itself can be used to give another proof of Theorem 5.2 because, as we have seen above, $\chi_w(x) = const$ and has the same value when $x \to \infty$ which implies that the representations $\rho^{(x)}$ (and in particular $\rho^{\lambda,m,+}$ for $m \in \mathbb{N}$) are isomorphic to the representation $\rho^{(\infty)}$ obtained when x tends to infinity.

5.2.2. *Example.* Now we give an example to illustrate the above proof of Theorem 5.3. This example is given with complete detail in [Cr1].

Claim 5.6. *For $m \in \mathbb{N}$*

$$\rho^{(3),\boxtimes} \oplus \rho^{(3),\div,+} \rho^{(3),\cdot|\cdot,+} \cong \rho^{(3),m,+}$$

In this case the set $T[(3),\cdot|\cdot,+]$ is empty. Hence, Claim 5.6 is equivalent to

$$\rho^{(3),\boxtimes}\oplus\rho^{(3),\div,+}\cong\rho^{(3),1,+}\quad .$$

We can "formulate" this claim graphically:

$$\begin{array}{|c|c|c|c|c|}\hline & & 0 & & \\ \hline\end{array}\oplus\begin{array}{|c|c|c|c|}\cline{2-4}\multicolumn{1}{c|}{} & 0 & & \\ \hline & & 0 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array}\cong\begin{array}{|c|c|c|c|c|c|}\cline{4-6}\multicolumn{3}{c|}{} & & & \\ \hline & & & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array}$$

Here zero shows the cells with zero content and each diagram describes the corresponding representation. The representation space of $\rho^{(3),\boxtimes}$ is $T[(3),\boxtimes]=\{\begin{array}{|c|c|c|c|c|}\hline -3 & -2 & \pm1 & 2 & 3\\ \hline\end{array}\}$, the representation space of $\rho^{(3),\div,+}$ is

$$T[(3),\div,+]=\left\{\begin{array}{|c|c|c|c|}\cline{2-4}\multicolumn{1}{c|}{} & -3 & 1 & 2 \\ \hline -2 & -1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array};\begin{array}{|c|c|c|c|}\cline{2-4}\multicolumn{1}{c|}{} & -2 & 1 & 3 \\ \hline -3 & -1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array};\begin{array}{|c|c|c|c|}\cline{2-4}\multicolumn{1}{c|}{} & -1 & 2 & 3 \\ \hline -3 & -2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array}\right\}$$

and the representation space of $\rho^{(3),1,+}$ is $T[(3),1,+]$ which contains four tableaux

$$\begin{array}{|c|c|c|c|c|c|}\cline{4-6}\multicolumn{3}{c|}{} & 1 & 2 & 3 \\ \hline -3 & -2 & -1 & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array}\quad,\quad\begin{array}{|c|c|c|c|c|c|}\cline{4-6}\multicolumn{3}{c|}{} & -3 & 1 & 2 \\ \hline -2 & -1 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array}\quad,$$

$$\begin{array}{|c|c|c|c|c|c|}\cline{4-6}\multicolumn{3}{c|}{} & -2 & 1 & 3 \\ \hline -3 & -1 & 2 & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array}\quad\text{and}\quad\begin{array}{|c|c|c|c|c|c|}\cline{4-6}\multicolumn{3}{c|}{} & -1 & 2 & 3 \\ \hline -3 & -2 & 1 & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array}\quad .$$

Remark 5.7. Using the character table of D_4 one can see that the representation $\rho^{(2^2),\cdot|\cdot,-}$ whose representation space is spanned by

$$\begin{array}{|c|c|c|c|}\hline -4 & -3 & -2 & 1\\ \hline -1 & 2 & 3 & 4\\ \hline\end{array}\;,\;\begin{array}{|c|c|c|c|}\hline -4 & -3 & -1 & 2\\ \hline -2 & 1 & 3 & 4\\ \hline\end{array}\;,\;\begin{array}{|c|c|c|c|}\hline -4 & -2 & -1 & 3\\ \hline -3 & 1 & 2 & 4\\ \hline\end{array}$$

and whose generator matrices are

$$\rho^{(2^2),\cdot|\cdot,-}_{s_0}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\quad,\quad\rho^{(2^2),\cdot|\cdot,-}_{s_1}=\begin{pmatrix}\frac{1}{3}&\frac{\sqrt{8}}{3}&0\\\frac{\sqrt{8}}{3}&-\frac{1}{3}&0\\0&0&1\end{pmatrix}$$

$$\rho^{(2^2),\cdot|\cdot,-}_{s_2}=\begin{pmatrix}1&0&0\\0&-\frac{1}{2}&\frac{\sqrt{3}}{2}\\0&\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}\quad,\quad\rho^{(2^2),\cdot|\cdot,-}_{s_3}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

is isomorphic to the irreducible three dimensional split representation $\{(2),(2)\}^-$ while the representation $\rho^{(2^2),\cdot|\cdot,+}$ whose representation space is spanned by

$$\begin{array}{|c|c|c|c|}\hline -4 & -3 & 1 & 2\\ \hline -2 & -1 & 3 & 4\\ \hline\end{array}\;,\;\begin{array}{|c|c|c|c|}\hline -4 & -2 & 1 & 3\\ \hline -3 & -1 & 2 & 4\\ \hline\end{array}\;,\;\begin{array}{|c|c|c|c|}\hline -4 & -3 & -2 & -1\\ \hline 1 & 2 & 3 & 4\\ \hline\end{array}$$

is isomorphic to another irreducible threedimensional split representation $\{(2),(2)\}^+$. Of course, $\rho^{(2^2),\cdot|\cdot,+}\not\cong\rho^{(2^2),\cdot|\cdot,-}$ because $\{(2),(2)\}^+\not\cong\{(2),(2)\}^-$.

The following conjecture is a generalization of the above two examples from Remark 5.7.

Conjecture 5.8. Let $n\in\mathbb{N}$ be even. Then

$$\rho^{\left(\left(\frac{n}{2}\right)^2\right),\cdot|\cdot,-}\cong\left\{\left(\frac{n}{2}\right),\left(\frac{n}{2}\right)\right\}^-\quad,\quad\rho^{\left(\left(\frac{n}{2}\right)^2\right),\cdot|\cdot,+}\cong\left\{\left(\frac{n}{2}\right),\left(\frac{n}{2}\right)\right\}^+\quad .$$

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