THE HOOK FORMULA

JASON BANDLOW

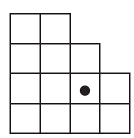
ABSTRACT. The hook-length formula is a well known result expressing the number of standard tableaux of shape λ in terms of the lengths of the hooks in the diagram of λ. Many proofs of this fact have been given, of varying complexity. We present here a new simple proof which will be accessible to anyone familiar with some standard power series expansions. This proof is of interest to combinatorialists for more than its simplicity, as it illustrates a surprising connection between hook lengths and the contents of the inner and outer corners of a Young diagram. Here the content of a cell (i, j) in a Young diagram is defined to be the number $j - i$.

1. INTRODUCTION

For a natural number n, we say λ is a *partition* of n, and write $\lambda \vdash n$, if λ is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of natual numbers satisfying

(1) $\sum_{i=1}^{k} \lambda_i = n$ and
(2) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$

The *Young diagram* of a partition is an array of boxes, or cells, in the plane, left-justified, with λ_i cells in the i^{th} row from the bottom. We label these cells (i, j), with i denoting the row and j the column. For example, in the following Young diagram of $(4, 4, 3, 2)$, the cell $(2, 3)$ is marked:



We will identify a partition with its Young diagram throughout. The *hook length* of a cell $c \in \lambda$ is the number of cells weakly above and strictly to the right of c. We denote this by $h_\lambda(c)$. For example, in the diagram above, $h_\lambda((2, 3)) = 3$.

A *standard tableau* of shape λ is a labelling of the cells of the Young diagram of λ with the numbers 1 to n so that the labels increase moving up the columns and to the right across rows. For example, there are only two standard tableaux of shape $(2, 1)$:

$$\begin{array}{|c|c|} \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array}$$

We denote the number of standard tableaux of shape λ by f_λ. This number has implications beyond combinatorics; Alfred Young showed that it also gives the dimension of the irreducible representation of the symmetric group S_n indexed by λ.

The hook length formula

$$f_\lambda = \frac{n!}{\prod_{s\in\lambda} h_\lambda(s)} \tag{1}$$

was first given by Frame, Robinson and Thrall [FRT54]. Since that time, many different proofs have been given ([GNW79], [Kra95], [NPS97] for just some examples). These proofs are quite useful; [GNW79], for example, provides an "intuitive" reason to believe the formula, while [Kra95] and [NPS97] provide bijective proofs. However, these have the disadvantage of appearing somewhat complicated to those readers unfamiliar with probability theory or the combinatorics of Young diagrams. We offer our proof as a simple approach for the non-specialist. In addition, we hope that more experienced combinatorialists will be as interested as ourselves by the connections which the proof reveals.

2. A proof of the hook formula

Given partitions $\lambda \vdash n$, $\mu \vdash (n-1)$, we say that μ precedes λ (denoted by $\mu \to \lambda$) if the diagram of μ is contained in the diagram of λ. Given a standard tableau of shape λ, it is immediate that removing the cell containing n gives a standard tableau of shape $\mu \to \lambda$. Thus we see that the number of standard tableaux satisfies the recursion

$$f_\lambda = \sum_{\mu\to\lambda} f_\mu$$

Our goal is to show the right side of (1) satisfies the same recursion. That is, we wish to show

$$\frac{n!}{\prod_{s\in\lambda} h_\lambda(s)} = \sum_{\mu\to\lambda} \frac{(n-1)!}{\prod_{s\in\mu} h_\mu(s)}$$

or, more simply,

$$\sum_{\mu\to\lambda} \frac{\prod_{s\in\lambda} h_\lambda(s)}{\prod_{s\in\mu} h_\mu(s)} = n \tag{2}$$

The *outer corners* of a Young diagram for λ are the cells which can be removed to give the diagram of a partition $\mu \to \lambda$. For a fixed λ we label the outer corners from top to bottom as $A_i = (\alpha_i, \beta_i)$, for $1 \le i \le m$. We then set $B_i = (\alpha_{i+1}, \beta_i)$, for $0 \le i \le m$, where we set $\beta_0 = 0 = \alpha_{m+1}$. We also define the cell $A_0 = (\alpha_0, \beta_0) = (0,0)$. Note that the cells A_0, B_0, B_m are outside of the diagram of the partition. An example of this labelling is given in Figure 1. For $1 \le i \le m$, we denote by $\mu^{(i)}$ the partition $\lambda \setminus A_i$.

The *content* of a cell $c = (i,j)$ is defined to be $j-i$, and is denoted by $ct(c)$. For $0 \le i \le m$, we set $x_i = ct(A_i)$ and $y_i = ct(B_i)$. We shall see that, remarkably, the left side of (2) has a nice expression in terms of the x_i and y_i. These numbers also satisy some important relations. We give the first of these here:

$$\begin{aligned} \sum_{i=0}^{m} (x_i - y_i) &= \sum_{i=0}^{m} (\beta_i - \alpha_i) - (\beta_i - \alpha_{i+1}) \\ &= \alpha_0 - \alpha_{m+1} = 0 \end{aligned} \tag{3}$$

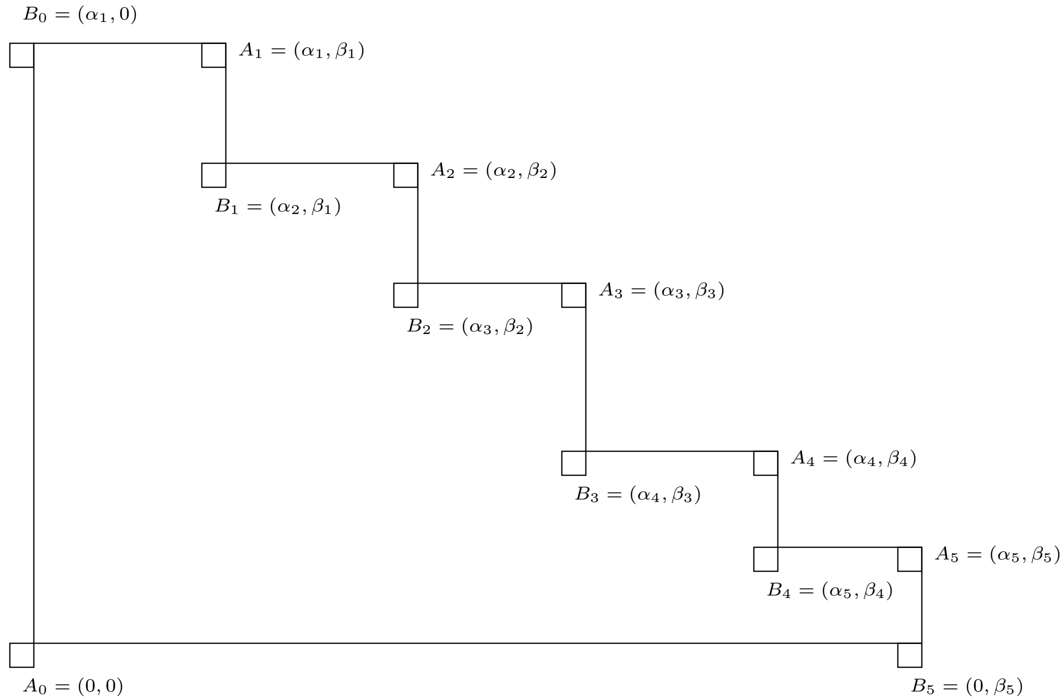


FIGURE 1. Partition with labelled corners

The second relation we will use is perhaps more surprising:

$$
\begin{aligned}
\sum_{i=0}^{m} x_i^2 - y_i^2 &= \sum_{i=0}^{m} (x_i - y_i)(x_i + y_i) \\
&= \sum_{i=0}^{m} \Big((\beta_i - \alpha_i) - (\beta_i - \alpha_{i+1}\Big)\Big((\beta_i - \alpha_i) + (\beta_i - \alpha_{i+1}\Big) \\
&= -\sum_{i=0}^{m}(\alpha_i - \alpha_{i+1})(2\beta_i) + \sum_{i=0}^{m}(\alpha_i - \alpha_{i+1})(\alpha_i + \alpha_{i+1}) \\
&= -2\sum_{i=0}^{m}(\alpha_i - \alpha_{i+1})(\beta_i) + \sum_{i=0}^{m}(\alpha_i^2 - \alpha_{i+1}^2) \\
&= -2n + \alpha_0^2 - \alpha_{m+1}^2 = -2n
\end{aligned} \tag{4}
$$

The penultimate equality comes from considering the diagram of λ as the disjoint union of rectangles of width β_i and height $(\alpha_i - \alpha_{i+1})$.

We now express the left side of (2) in terms of the x_i and y_i. For fixed i, it is clear that every cell not in the same row or column as A_i cancels in equation (2). In fact, other cells will cancel as well. That is, most pairs of cells of the form

$$(\alpha_i, b) \in \lambda \quad \leftrightarrow \quad (\alpha_i, b-1) \in \mu^{(i)}$$

and

$$(a, \beta_i) \in \lambda \quad \leftrightarrow \quad (a-1, \beta_i) \in \mu^{(i)}$$

will cancel. The only places this won't work are cells below or across from corner cells. To be precise, the cells in λ which will not cancel are the cells

$$R_j = (\alpha_i, \beta_j + 1) \qquad \text{for } 0 \le j < i$$

and

$$C_j = (\alpha_{j+1} + 1, \beta_i) \qquad \text{for } i \le j \le k$$

Similarly, the cells in $\mu^{(i)}$ which won't cancel are the cells

$$R'_j = (\alpha_i, \beta_j) \qquad \text{for } 1 \le j < i$$

and

$$C'_j = (\alpha_{j+1}, \beta_i) \qquad \text{for } i \le j \le k-1$$

These cells are illustrated in Figure 2.

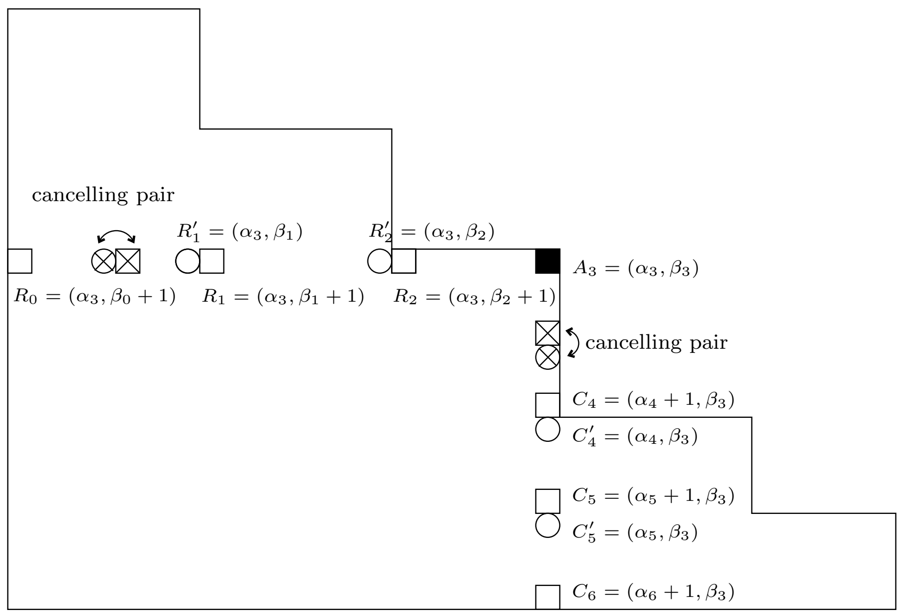


FIGURE 2. Non-cancelling cells. Squares should be viewed as cells in λ, circles as cells in $\lambda \setminus A_3$.

We now write the hook lengths of these cells in terms of the x_i and y_i.

$$\begin{aligned}
h_\lambda(R_j) &= (\alpha_{j+1} - \alpha_i) + (\beta_i - (\beta_j + 1)) + 1 \\
&= (\beta_i - \alpha_i) - (\beta_j - \alpha_{j+1}) = x_i - y_j \\
h_\lambda(C_j) &= ((\alpha_i - 1) - \alpha_{j+1}) + (\beta_j - \beta_i) + 1 \\
&= (\beta_j - \alpha_{j+1}) - (\beta_i - \alpha_i) = y_j - x_i \\
h_\lambda(R'_j) &= (\alpha_j - \alpha_i) + ((\beta_i - 1) - \beta_j) + 1 \\
&= (\beta_i - \alpha_i) - (\beta_j - \alpha_j) = x_i - x_j \\
h_\lambda(C'_j) &= ((\alpha_i - 1) - \alpha_{j+1}) + (\beta_{j+1} - \beta_i) + 1 \\
&= (\beta_{j+1} - \alpha_{j+1}) + (\beta_i - \alpha_i) = x_{j+1} - x_i
\end{aligned}$$

and this reduces the left hand side of (2) to

$$\sum_{i=1}^{m} \frac{\prod_{j=0}^{i-1}(x_i - y_j)\prod_{j=i}^{k}(y_j - x_i)}{\prod_{j=0}^{i-1}(x_i - x_j)\prod_{j=i}^{k-1}(x_{j+1} - x_i)} = (-1)\sum_{i=1}^{m} \frac{\prod_{j=0}^{k}(x_i - y_j)}{\prod_{j=0}^{k\ (i)}(x_i - x_j)} \tag{5}$$

where we use the notation $\prod_{j=0}^{k\ (i)}$ to indicate that the i^{th} term is omitted.

We wish to show this quantity is, in fact, just n. This will come from the partial fraction expansion of the following rational function:

$$\frac{\prod_{j=0}^{m}(1 - ty_j)}{\prod_{j=1}^{m}(1 - tx_j)} = \sum_{s=1}^{m} \frac{P_s}{1 - tx_s} + c_1 t + c_0 \tag{6}$$

It will develop that taking the coefficient of t^2 on both sides of this equation will give us exactly what we need. On the right hand side, we expand the denominators with power series expansions and see that the coefficient of t^2 is

$$\sum_{s=1}^{m} x_s^2 P_s$$

To find the P_s, we use the usual partial fractions trick of multiplying both sides of (6) by $(1 - tx_s)$, and then specializing $t \to \frac{1}{x_s}$. This gives

$$P_s = \frac{1}{x_s^2}\frac{\prod_{j=0}^{m}(x_s - y_j)}{\prod_{j=1}^{m\ (s)}(x_s - x_j)}$$

Comparing this to (5), we see that it remains to show that the coefficient of t^2 in

$$\frac{\prod_{j=0}^{m}(1 - ty_j)}{\prod_{j=1}^{m}(1 - tx_j)} \tag{7}$$

is simply $-n$. We rewrite (7) using the expansion

$$\frac{1}{1-x} = exp\left(\sum_{k\geq 1} \frac{x^k}{k}\right)$$

to obtain

$$\frac{\prod_{j=0}^{m}(1 - ty_j)}{\prod_{j=1}^{m}(1 - tx_j)} = exp\left(\sum_{k\geq 1} \frac{t^k}{k}\left(\sum_{i=1}^{m}(x_i^k) - \sum_{i=0}^{m}(y_i^k)\right)\right)$$

The $k = 1$ term is 0 by (3), so we are left with

$$exp\left(\sum_{k\geq 2} \frac{t^k}{k}\left(\sum_{i=1}^{m}(x_i^k) - \sum_{i=0}^{m}(y_i^k)\right)\right) = 1 + \frac{t^2}{2}\left(\sum_{i=1}^{m}(x_i^2) - \sum_{i=0}^{m}(y_i^2)\right) + \dots$$

The coefficient of t^2 is the quantity we are trying to evaluate, and since $x_0 = 0$, we can rewrite this as

$$\frac{1}{2}\sum_{i=0}^{m}\left(x_i^2 - y_i^2\right) = -n$$

by (4). This completes the proof.

References

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