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On the Young-Fibonacci insertion algorithm

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Abstract. This work is concerned with some properties of the Young-Fibonacci insertion algorithm and its relation with Fomin's growth diagrams. It also investigates a relation between the combinatorics of Young-Fibonacci tableaux and the study of Okada's algebra associated to the Young-Fibonacci lattice. The original algorithm was introduced by Roby and we redefine it in such a way that both the insertion and recording tableaux of any permutation are *conveniently* interpreted as chains in the Young-Fibonacci lattice. A property of Killpatrick's evacuation is given a simpler proof, but this evacuation is no longer needed in making Roby's and Fomin's constructions coincide. We provide the set of Young-Fibonacci tableaux of size n with a structure of graded poset, induced by the weak order on permutations of the symmetric group, and realized by transitive closure of elementary transformations on tableaux. We show that this poset gives a combinatorial interpretation of the coefficients in the transition matrix from the analogue of complete symmetric functions to analogue of the Schur functions in Okada's algebra. We end with a quite similar observation for four posets on Young-tableaux studied by Taskin.

Résumé. Ce travail s'interresse à quelques propriétés de l'algorithme d'insertion de Young-Fibonacci, plus particulièrement nous montrons une relation simple entre l'approche Schensted et l'approche Fomin de cette correspondance. Nous nous interressons aussi à l'apport de la combinatoire des tableaux de Young-Fibonacci dans l'étude de l'algèbre d'Okada associée au graphe de Young-Fibonacci. Nous redéfissons l'algorithme initial dû à Roby, de manière à ce que les deux tableaux associés à une permutation quelconque aient une interprêtation combinatoire *convenable* en terme de chemin dans le graphe de Young-Fibonacci. Nous munissons l'emsemble des tableaux de Young-Fibonacci de taille n d'un d'ordre partiel gradué induit par la correspondance de Young-Fibonacci et l'ordre faible sur le groupe symétrique. Nous montrons que ce poset donne une interprêtation combinatoire des nombres de Kostka dans l'algèbre d'Okada associée au graphe de Young-Fibonacci. Nous montrons un résultat analogue pour les nombres de Kostka usuels, en relation avec quatre posets sur les tableaux de Young étudiés par Taskin.

Contents

1. Introduction

The Young lattice ($\mathbb{YL}$) is defined on the set of partitions of positive integers, with covering relations given by the natural inclusion order. The differential poset nature of this graph was generalized by Fomin who introduced graph duality [**13**]. With this extension he introduced [**15**] a generalization of the classical

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Robinson-Schensted-Knuth [**1, 2**] algorithm, giving a general scheme for establishing bijective correspondences between couples of saturated chains in dual graded graphs, both starting at a vertex of rank 0 and having a common end point of rank n, on the one hand, and permutations of the symmetric group $\mathfrak{S}_n$ on the other hand. This approach naturally leads to the Robinson-Schensted insertion algorithm.

Roby [**17**] gave an insertion algorithm analogous to the Schensted correspondence, which maps a permutation σ onto a couple made of a Young-Fibonacci tableau $P(\sigma)$ and a path tableau $Q(\sigma)$. Roby's path tableau $Q(\sigma)$ is canonically interpreted as a saturated chain in the Fibonacci lattice $Z(1)$ introduced by Stanley [**11**] and also by Fomin [**14**]. Roby also showed that Fomin's approach is partially equivalent to his construction.

Indeed in Roby's construction, only the saturated chain $\widehat{Q}$ obtained from Fomin's growth diagram has an interpretation as a representation of the path tableau $Q(\sigma)$, while there seems to be no way to translate the Young-Fibonacci tableau $P(\sigma)$ into its equivalent chain $\widehat{P}$. Contrarily to the approach of Killpatrick [**8**] who has used an evacuation to relate the two constructions of Roby and Fomin, we show that with a suitable mechanism for converting a saturated chain in the Young-Fibonacci lattice into a Young-Fibonacci tableau, Roby's construction naturally coincides with Fomin's one.

The paper is organized as follows. In Section 1.1 we recall the definition of the Young-Fibonacci lattice, then in Section 2 we define a mechanism for converting a saturated chain in this lattice into a standard Young-Fibonacci tableau. In the same section, we also introduce a modification in Roby's algorithm, in such a way that both the insertion and recording tableaux of any permutation will have an interpretation in terms of saturated chain in the Young-Fibonacci lattice. In Section 3.1 we relate Roby's algorithm with Fomin's construction using growth diagrams and we compare it to Killpatrick's work. In Section 4, we define an analogue of Kostka numbers for Young-Fibonacci tableaux, and we point out one of their relation with usual Fibonacci numbers. In Section 5 we define and we study some properties of a poset on Young-Fibonacci tableaux. This poset turns out to be a model for the interpretation as well as the computation of another analogue of Kostka numbers, introduced by Okada [**16**] in an analogue of the algebra of symmetric functions, associated to the Young-Fibonacci lattice. We prove this result is Section 6, and in the last section of the paper we prove a similar result relating usual Kostka numbers with four posets on Young-tableaux studied by Taskin [**9**].

1.1. The Young-Fibonacci lattice.

A *Fibonacci diagram* or *snakeshape* of size n is a column by column graphical representation of a composition of an integer n, with parts equal to 1 or 2. The number of such compositions is the n^{th} Fibonacci number. A partial order is defined on the set of all snakeshapes, in such a way to obtain an analogue of the Young lattice of partitions of integers ($\mathbb{YL}$). This lattice is called the Young-Fibonacci lattice ($\mathbb{YFL}$) and it was introduced by Stanley [**11**] and also by Fomin [**14**]. As we will see in the sequel, there is a considerable similarity between the two lattices, as well as the combinatorics of tableaux their induce. The covering relations in $\mathbb{YFL}$ are given below, for any snakeshape u.

(1) u is covered by the snakeshape obtained by attaching a single box just in front ;
(2) u is covered by the snakeshape obtained by adding a single box on top of its first single-boxed column, reading u from left to right.
(3) if u starts with a series of two-boxed columns, then it is covered by all snakeshapes obtained by inserting a single-boxed column just after any of those columns.

The rank $|u|$ of a snakeshape u is the sum of digits of the corresponding Fibonacci word. Its length will be denoted $\ell(u)$. Let u and v be two snakeshapes such that v covers u in $\mathbb{YFL}$, the cell added to u to obtain v is an *inner corner* of v, it is also called an *outer corner* of u.

REMARK 1.1. Young-Fibonacci tableaux ($\mathbb{YFT}$) will naturally appear as numberings of snakeshapes, satisfying certain conditions described in the sequel, the same way as Young tableaux are numberings of partitions of integers with prescribed numbering conditions. The numbering conditions of Young-Fibonacci tableaux are deduced from the description of the Young-Fibonacci insertion algorithm (Section 2.2).

Below is a pictorial representation of a finite realization of $\mathbb{YFL}$, from rank 0 up to rank $n = 4$, with black cells representing inner corners.

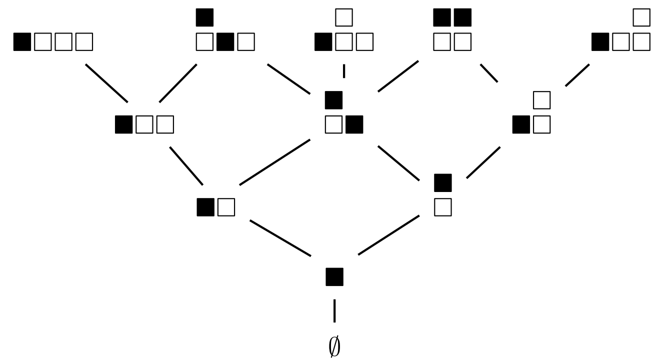


Figure 1. The Young-Fibonacci lattice.

Now let us look at the problem of converting a saturated chain in $\mathbb{YFL}$ into a standard $\mathbb{YFT}$.

2. Young-Fibonacci tableaux and Young-Fibonacci insertion algorithm

In $\mathbb{YL}$, any saturated chain starting at the empty partition can be canonically converted into a standard Young tableau, and this representation is *convenient* in many ways. It consists in labeling the boxes as their occur in the chain. As already observed by Roby [**17**], one question which presents itself is to do the same in $\mathbb{YFL}$ for any saturated chain starting at the empty snakeshape $\emptyset$. *The need of such a conversion mechanism will appear in section 3.1 in the interpretation of two saturated chains in a growth diagram.*

One may also use the canonical labeling to convert a saturated chain of $\mathbb{YFL}$ into a tableau, but Roby had already pointed out that one major problem with this canonical labeling is that *except for the trivial rule that each element in the top row must be greater than the one below it, no other obvious rules govern what numberings are allowed for a given shape.* We suggest that one first defines simple rules governing what numberings are allowed for a given shape, so that it be easy to decide if a numbering of a snakeshape is a legitimate Young-Fibonacci tableau or not. The convention we use is described in the next section.

2.1. Converting a chain in $\mathbb{YFL}$ into a standard Young-Fibonacci tableau.

Since we do not use the same conventions as Roby [**17**] and Fomin [**13**], let us give the following definition of Young-Fibonacci tableaux.

Definition 2.1. A numbering of a snakeshape with distinct nonnegative integers is a standard Young-Fibonacci tableau ($\mathbb{SYFT}$) under the following conditions.

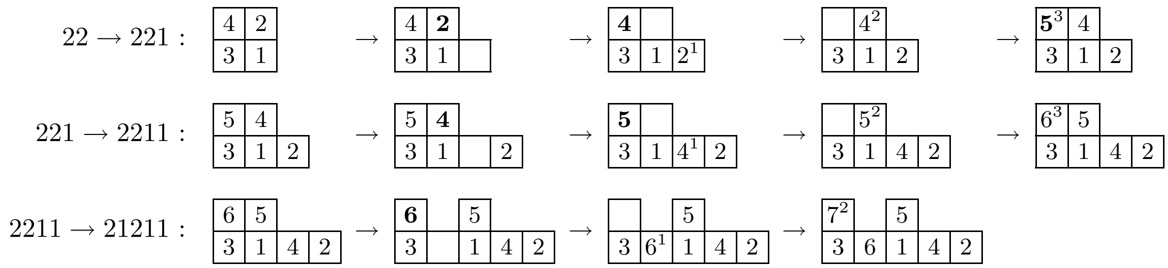
(1) entries are strictly increasing in columns ;
(2) any entry on top in any column has no entry greater than itself on its right.

To convert a chain in $\mathbb{YFL}$ into a standard $\mathbb{YFT}$, *one will follow the canonical approach as far as the new box added to the chain lies in the first column.* Example with the chain $\widehat{Q} = (\emptyset, 1, 2, 12, 22, 221, 2211, 21211)$; the sub-chain $(\emptyset, 1, 2, 12, 22)$ is converted as follows.

$$\emptyset \to \boxed{\mathbf{1}} \to \begin{array}{|c|}\hline \mathbf{2}\\\hline 1\\\hline\end{array} \to \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{2}\\\hline \multicolumn{1}{|c|}{\mathbf{3}} & \multicolumn{1}{c|}{1}\\\hline\end{array} \to \begin{array}{|c|c|}\hline \mathbf{4} & 2\\\hline 3 & 1\\\hline\end{array}$$

Now moving from the shape 22 to the shape 221 in $\mathbb{YFL}$, one inserts a box just after a two-boxed column of the previous shape. *In such a situation, one will move the entry on top in that column into the newly created box, and then shift the other entries of the top row to the right. Finally, if n is the largest entry in the partial tableau obtained, then label the box on top in the first column with $(n+1)$.*

The conversion started above keeps on as follows, x^k means that writing or moving the label x is the k^{th} action performed during the current step.



It easily follows from the description above that this mechanism produces only legitimate $\mathbb{YFT}$ (Definition 2.1), and that the conversion is reversible. Now another question which presents itself is how to count standard $\mathbb{YFT}$ of a given shape $u \neq \emptyset$, we denote this number by $\mathcal{F}_u$. Let us first recall the formula counting linear extensions of a binary tree poset $\mathbb{P}$.

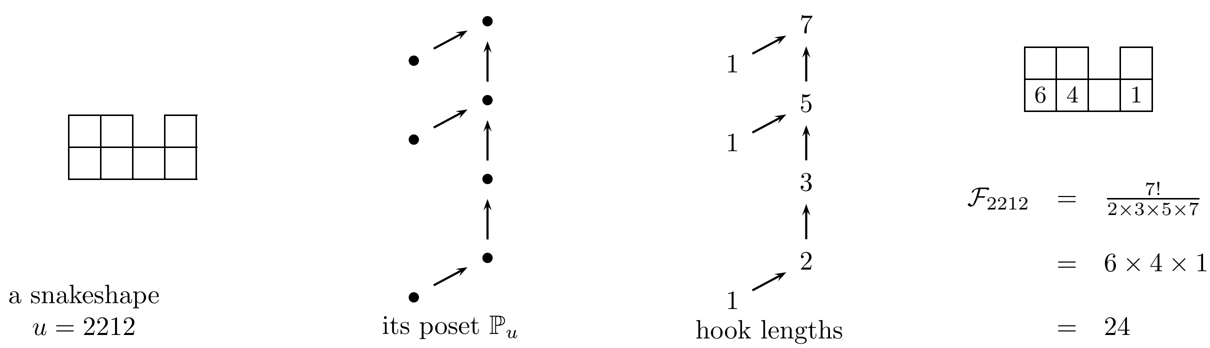
$$|Ext(\mathbb{P})| \;=\; \frac{n!}{d_1 d_2 \cdots d_n} \tag{2.1}$$

where for the i^{th} node v_i, d_i is the number of nodes $v \leq_{\mathbb{P}} v_i$. This formula is due to Knuth [**3**], and since any snakeshape u can be canonically assimilated to a poset $\mathbb{P}_u$, then we have the following.

PROPOSITION 2.2. *Standard Young-Fibonacci tableaux of a given shape are counted by the hook-length formula for binary trees.*

To apply the formula to a snakeshape u, count it cells from right to left and from bottom to top, labeling the first box and each box appearing in the bottom row of any two-boxed column. The number of standard $\mathbb{YFT}$ of the given shape is the product of all the labels obtained.

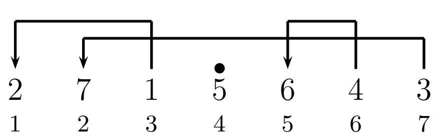
EXAMPLE 2.3. Let us consider $u = 2212$.



2.2. Redefining the Young-Fibonacci Insertion Algorithm.
We refer the reader to [**17, 8**] for a description of the original algorithm ; below is the one we consider.

DEFINITION 2.4. The Young-Fibonacci insertion algorithm maps a permutation σ onto a *couple of standard* $\mathbb{YFT}$ built as follows. The insertion tableau $P(\sigma)$ is built by reading σ from right to left, matching any of the letters encountered (and not yet matched) with the maximal one (not yet matched) on its left if any, provided that the latter be greater than the first. The recording tableau $Q(\sigma)$ records the positions of the letters, in the reverse order of the one in which they are matched.

EXAMPLE 2.5. For $\sigma = 2715643$, we have the following.



$$P(\sigma) = \begin{array}{|c|c|c|c|}\hline 7 & 6 & & 2 \\ \hline 3 & 4 & 5 & 1 \\ \hline \end{array} \quad \text{and} \quad Q(\sigma) = \begin{array}{|c|c|c|c|}\hline 7 & 6 & & 3 \\ \hline 2 & 5 & 4 & 1 \\ \hline \end{array}$$

REMARK 2.6. That both $P(\sigma)$ and $Q(\sigma)$ are standard Young-Fibonacci tableaux (Definition 2.1) is clear from the description of the algorithm. This is not the case in the original algorithm where $P(\sigma)$ and $Q(\sigma)$

are not of the same type. Indeed, with the original insertion algorithm, the insertion tableau is the same as the tableau $P(\sigma)$ above, but the recording tableau $\mathbf{Q}(\sigma)$ which follows does not satisfy Definition 2.1.

$$\mathbf{Q}_{\mathbf{Roby}}(\sigma) = \begin{array}{|c|c|c|c|} \cline{1-2}\cline{4-4} 3 & 7 & \multicolumn{1}{c|}{} & 4 \\ \hline 2 & 6 & 5 & 1 \\ \hline \end{array}$$

The definition of $Q(\sigma)$ we adopt is inspired from the hypoplactic [**4**] and sylvester [**5**] insertion algorithms, where $Q(\sigma)$ also records the positions in σ of the labels of $P(\sigma)$. With this definition, some essential properties of the Young-Fibonacci correspondence have a much easier combinatorial proof, which is not always the case in [**17**]. For example, let us recall the involution property.

THEOREM 2.7. [**17**] *For any permutation σ, $P(\sigma^{-1}) = Q(\sigma)$.*

PROOF. Consider the geometric construction by Killpatrick [**8**], and recall that $P(\sigma)$ corresponds to reading vertical coordinates of the rightmost and uppermost $\mathbf{x}$ in that order, for any broken line. As for $Q(\sigma)$, we have defined it in such a way that it corresponds to reading horizontal coordinates of the uppermost and rightmost $\mathbf{x}$ in that order. The construction for σ^{-1} is obtained by transposing the one for σ. □

Another fundamental property of Roby's algorithm which is easily proved using Definition 2.4 follows.

THEOREM 2.8. [**17**] *Let σ be an involution of the symmetric group, then the cycle decomposition of σ is the column reading of its insertion tableau $P(\sigma)$.*

We give two other canonical words associated with a tableau t ; so if we let $\mathbb{YFC}(t)$ denotes the equivalence class made of permutations having t as insertion tableau, then $\mathbb{YFC}(t)$ has at least three canonical elements. The first canonical element is its canonical involution, that is the only involution the cycles of which coincide with the columns of t, as stated in Theorem 2.8. The two other canonical elements are the maximal (resp. minimal) element for the lexicographical order. We will make use of these elements in Section 5.

LEMMA 2.9. *Let t be a Young-Fibonacci tableau, w_1 the left-to-right reading of its top row and w_2 the <u>right-to-left</u> reading of its bottom row, then $w_1.w_2$ (where . denotes the usual concatenation of words) is the maximal element (for the lexicographical order) of $\mathbb{YFC}(t)$, it is denoted w^t_{max}.*

LEMMA 2.10. *The word consisting of the right-to-left and up-down column reading of t is the minimal element (for the lexicographical order) of $\mathbb{YFC}(t)$, it is denoted w^t_{min}.*

PROOF. Clear from the description of the Young-Fibonacci insertion algorithm. □

An example is given with the tableau t below ; its canonical involution is $(13)(26)(48)(5)(7) = 36185274$, the maximal canonical element is 86315274, and the minimal one is 31562784.

$$t \ = \begin{array}{|c|c|c|c|c|} \cline{1-1}\cline{3-3}\cline{5-5} 8 & \multicolumn{1}{c|}{} & 6 & \multicolumn{1}{c|}{} & 3 \\ \hline 4 & 7 & 2 & 5 & 1 \\ \hline \end{array}$$

We will see (Theorem 5.12) that $\mathbb{YFC}(t)$ is the set of linear extensions of a given poset, and additionally, $\mathbb{YFC}(t)$ is an interval of the weak order on the symmetric group (Theorem 5.9).

3. Young-Fibonacci insertion and growth in differential posets

In this section we show that with the modification we have introduced in Roby's original insertion algorithm, together with the conversion mechanism discussed in section 2.1, the Young-Fibonacci insertion algorithm naturally coincides with Fomin's approach using growth diagrams. So we claim that Killpatrick's evacuation [**8**] is no longer needed in making the two constructions coincide. We give a simplification of Killpatrick's theorem relating Roby's original algorithm to Fomin's one through an evacuation process, and we will later need this evacuation in the proof of Theorem 6.2 giving a combinatorial interpretation of Okada's analogue of Kostka numbers.

Let us recall that Fomin's construction with growth diagrams consists in using some *local rules* in filling a diagram giving rise to a pair of saturated chains in $\mathbb{YL}$. For any permutation σ, the growth diagram $d(\sigma)$ is build the following way. First draw the permutation matrix of σ ; next fill the left and lower boundary of $d(\sigma)$ with the empty snakeshape $\emptyset$. The rest of the construction is iterative ; $d(\sigma)$ is filled from the lower left corner to the upper right corner, following the diagonal. At each step and for any configuration as pictured below, z is obtained by applying the local rules to the vertices t, x, y and the permutation matrix element α. We refer the reader to [**15**] for more details on this construction.

$$\begin{tikzcd} x \arrow[r, "b_2"] & \boxed{z} \\ t \arrow[u, "a_1"] \arrow[r, "a_2"'] \arrow[ur, phantom, "\alpha"] & y \arrow[u, "b_1"'] \end{tikzcd}$$

FIGURE 2. A square in a growth diagram.

Algorithm 1 : local rules for $\mathbb{YFL}$

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if $x \neq y$ and $y \neq t$ **then**
  $z := t$, with a two-boxed column added in front
else
  if $x = y = t$ and $\alpha = 1$ **then**
    $z := t$, with a single-boxed column added in front
  else
    z is defined in such a way that the edge b_i is degenerated whenever a_i is degenerated
  end if
end if

3.1. Equivalence between Roby's and Fomin's constructions.

Let us build Fomin's growth diagram for the permutation $\sigma = 2715643$.

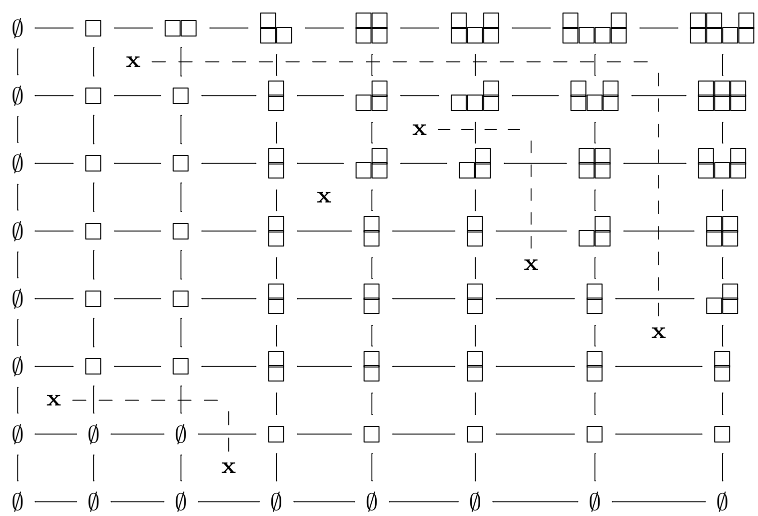
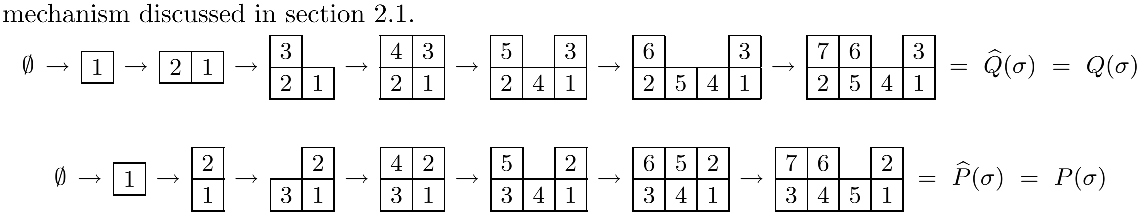


FIGURE 3. Example of growth diagram for the Young-Fibonacci insertion.

We get the paths $\widehat{Q} = (\emptyset, 1, 2, 12, 22, 221, 2211, 21211)$ and $\widehat{P} = (\emptyset, 1, 11, 21, 211, 1211, 2211, 21211)$ on the upper and right boundary respectively. Now let us convert them into Young-Fibonacci tableaux, using the mechanism discussed in section 2.1.



So as we can see on this example, the two constructions naturally coincide.

Remark 3.1. Let us mention that because Roby used the canonical labeling to convert a chain into a tableau, there seemed to be no way to convert the chain $\widehat{P}$ into its equivalent tableau $P(\sigma)$. Killpatrick's algorithm was then an approach to relate $\widehat{P}$ with $P(\sigma)$. Our own approach consists in the introduction of a modification of the original algorithm, and a new labeling process.

Theorem 3.2. *Let $(\widehat{P}(\sigma), \widehat{Q}(\sigma))$ be the pair of Young-Fibonacci tableaux obtained from the permutation σ by using Fomin's growth diagram and let $(P(\sigma), Q(\sigma))$ be the Young-Fibonacci insertion and recording tableaux using Roby's insertion modified (Definition 2.4), then $\widehat{P}(\sigma) = P(\sigma)$ and $\widehat{Q}(\sigma) = Q(\sigma)$.*

Proof. The equality $\widehat{P}(\sigma) = P(\sigma)$ follows from that any snakeshape $\widehat{P}_k$ appearing in $\widehat{P}$ is the shape of the tableau $P(\sigma_{/[1..k]})$ where $\sigma_{/[1..k]}$ is the restriction of σ to the interval $[1..k]$. Indeed, the path $\widehat{P}$ is obtained applying to $P(\sigma)$ the reverse process of the one described in section 2.1. In so doing, the cell added to $\widehat{P}_k$ to get $\widehat{P}_{k+1}$ lies in the first column when either $\sigma_{/[1..k+1]}$ ends with the letter $k+1$ or $\sigma_{/[1..k+1]}$ does not end with the letter $k+1$ but $\sigma_{/[1..k]}$ ends with the letter k. A quite similar reasoning is used to prove the equality $\widehat{Q}(\sigma) = Q(\sigma)$. □

3.2. Another viewpoint of Killpatrick's evacuation for Young-Fibonacci tableaux.
For a tableau t, this operation is defined only for top entries of the columns of t. Let a_0 be such an entry, the tableau resulting from the evacuation of a_0 is denoted $ev(t, a_0)$ and is built as follows.

(1) if a_0 is a single-boxed column, then just delete this column and, if this is necessary, shift one component of the remaining tableau to connect it with the other one (e.g of line 3 in the table below) ;
(2) otherwise, the box containing a_0 is emptied and one compares the entry a_1 that was just below a_0 with the entry a_2 on top of the column just to the right if any. If $a_2 < a_1$ then this terminates the evacuation process (e.g of line 4 in the table below). Otherwise, move a_2 on top of a_1, creating a new empty box in the tableau. If the new empty box is a single-boxed column, then this terminates the evacuation process (e.g of line 7, step 4, in the table below), otherwise, iteratively repeat the process with the entries just below and to the right of this new empty box.

Let t be a tableau of size n and shape u. If one successively evacuates the entries $n, (n-1), \cdots, 1$ from t, labeling the boxes of u according to the positions of the empty cells at the end of the evacuation of entries, one gets a *path tableau* denoted $ev(t)$. Recall that a path tableau is the canonical labeling of a saturated chain.

Remark 3.3. $ev(t)$ is the same tableau as the one described by Killpatrick [**8**], with Young-Fibonacci tableaux defined as in Definition 2.1.

Lemma 3.4. *Let w be a word with no letter repeated, let a_0 be one of its letters appearing as a top element in a column of $P(w)$, and let w_0 be the word obtained from w by deleting the only occurrence of a_0, then $ev(P(w), a_0) = P(w_0)$.*

Proof. Easily from the description of the evacuation and the description of the Young-Fibonacci insertion algorithm (Definition 2.4). □

We give a simpler proof of the following theorem by Killpatrick, relating $ev(P(\sigma))$ with $\widehat{P}$. Indeed, using the canonical labeling, Roby has converted the path $\widehat{P}$ into a path tableau $\widehat{P}(\sigma)$ and

Theorem 3.5. [**8**] *$ev(P(\sigma)) = \widehat{P}(\sigma)$.*

Proof. Follows from Lemma 3.4 and the remark that any snakeshape $\widehat{P}_k$ appearing in $\widehat{P}$ is the shape of the tableau $P(\sigma_{/[1..k]})$ where $\sigma_{/[1..k]}$ is the restriction of σ to the interval $[1..k]$. □

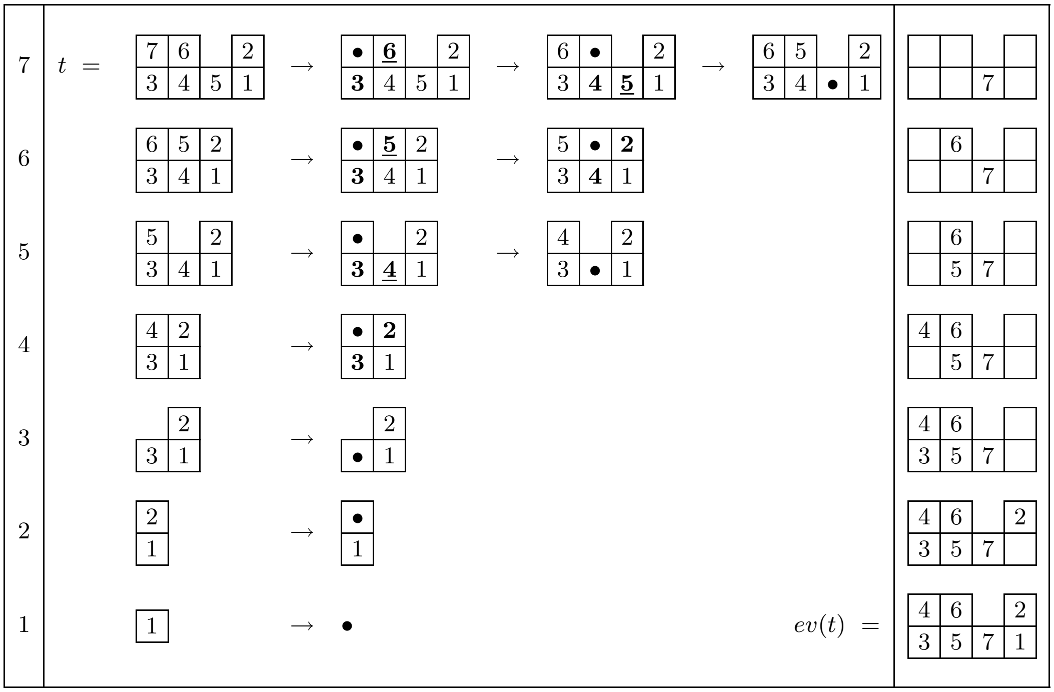


TABLE 1. Evacuation on Young-Fibonacci tableaux.

4. Fibonacci numbers and a statistic on Young-Fibonacci tableaux

In this section we point out a property of Young-Fibonacci numbers defined as an analogue of Kostka numbers. Recall that the usual Kostka numbers $K_{\lambda,\mu}$ are defined for two partitions λ and μ of the same integer n and they appear when expressing Schur functions s_λ in terms of the monomial symmetric functions m_μ, and in the expression of the complete symmetric functions h_μ in terms of Schur functions s_λ.

$$s_\lambda = \sum_\mu K_{\lambda,\mu}\, m_\mu \qquad ; \qquad h_\mu = \sum_\mu K_{\lambda,\mu}\, s_\lambda \tag{4.1}$$

We will not focus on the algebraic interpretation of the $K_{\lambda,\mu}$ but rather on their combinatorial interpretation in terms of tableaux. Indeed, $K_{\lambda,\mu}$ counts the number of distinct semi-standard Young-tableaux of shape λ and valuation μ, that is to say with μ_i entries i for $i = 1\,..\,\ell(\mu)$. It is then natural to introduce the same definition with Young-Fibonacci tableaux.

DEFINITION 4.1. A semi-standard Young-Fibonacci tableau is a numbering of a snakeshape with non-negative integers, not necessarily distinct, preserving the conditions stated in Definition 2.1.

DEFINITION 4.2. Let u and v be two snakeshapes of size n, the Young-Fibonacci number associated to u and v, denoted $\mathcal{N}_{u,v}$ is the number of distinct semi-standard Young-Fibonacci tableaux of shape u and valuation v, that is to say with v_i entries i for $i = 1\,..\,\ell(v)$.

For example, for $u = 221$ and $v = 1211$, there are 4 distinct semi-standard Young-Fibonacci tableaux of shape u and valuation v. So $\mathcal{N}_{221,1211} = 4$.

$$\begin{array}{|c|c|c|}\cline{1-2} 4&3&\multicolumn{1}{c}{}\\ \hline 1&2&2\\ \hline\end{array}\qquad \begin{array}{|c|c|c|}\cline{1-2} 4&3&\multicolumn{1}{c}{}\\ \hline 2&1&2\\ \hline\end{array}\qquad \begin{array}{|c|c|c|}\cline{1-2} 4&3&\multicolumn{1}{c}{}\\ \hline 2&2&1\\ \hline\end{array}\qquad \begin{array}{|c|c|c|}\cline{1-2} 4&2&\multicolumn{1}{c}{}\\ \hline 3&1&2\\ \hline\end{array}$$

PROPOSITION 4.3. *Young-Fibonacci numbers are generated by the recurrence formulas below, where both u and v are snakeshapes.*

$$\left\{\begin{array}{lcllcl} \mathcal{N}_{\emptyset,\emptyset} & = & 1 & ; \ \mathcal{N}_{2,2} & = & 0 \\ \mathcal{N}_{1u,v1} & = & \mathcal{N}_{u,v} & ; \ \mathcal{N}_{1u,v2} & = & \mathcal{N}_{u,v1} \\ \mathcal{N}_{2u,v1} & = & \sum_{w\in v^{1-}} \mathcal{N}_{u,w} & ; \ \mathcal{N}_{2u,v2} & = & \sum_{w\in v^{1-}} \mathcal{N}_{u,w1} \end{array}\right. \tag{4.2}$$

where v^{1^-} denotes the <u>multiset</u> of snakeshapes obtained from v either by deleting a single occurrence of 1, or by decreasing a single entry not equal to 1, for example $2112^{1^-} = [1112, 212, 212, 2111]$.

Proof. Easily from the definition of Young-Fibonacci tableaux and Young-Fibonacci numbers. $\square$

	v =	222	2211	2121	2112	21111	1221	1212	12111	1122	11211	11121	11112	111111
u =	222	2	3	4	5	6	4	5	6	5	7	8	12	**15**
	2211	4	5	5	7	9	5	7	9	7	9	9	12	**15**
	2121	2	3	4	4	5	4	4	5	4	6	8	8	**10**
	2112	1	1	1	1	1	2	2	3	3	4	4	4	**5**
	21111	2	3	3	3	4	3	3	4	3	4	4	4	**5**
	1221	2	2	3	4	4	3	4	4	4	4	6	8	**8**
	1212	1	1	1	1	1	1	2	2	3	3	3	4	**4**
	12111	2	2	2	3	3	2	3	3	3	3	3	4	**4**
	1122	1	1	1	1	1	1	1	1	2	2	3	3	**3**
	11211	1	1	2	2	2	2	2	2	2	2	3	3	**3**
	11121	1	1	1	1	1	1	1	1	2	2	2	2	**2**
	11112	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	$\boxed{0}$	1	1	1	1	1	1	1	**1**
	111111	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

Table 2. Matrix of Young-Fibonacci numbers for $n = 6$.

Theorem 4.4. *Let $n \geq 2$ be a positive integer, then the number of couples (u, v) of snakeshapes of size n such that $\mathcal{N}_{u,v} = 0$ is the $(n-2)^{th}$ Fibonacci number.*

Proof. The proof is done by induction on n. Indeed using equation (4.2) it is easy to see that $\mathcal{N}_{u,v} \neq 0$ whenever $u \neq 1^{n-2}2$. So the problem is equivalent to counting the number of snakeshapes v such that $\mathcal{N}_{1^{n-2}2,v} = 0$. But $\mathcal{N}_{1^{n-2}2,v} = 0$ if and only if there exists a snakeshape w such that $v = 2w$. Then the problem is finally equivalent to counting the snakeshapes of size $(n-2)$, and hence the result. $\square$

5. A weak order on Young-Fibonacci tableaux

In what follows, we introduce a partial and *graded* order denoted $\preceq$ on the set $\mathbb{YFT}_n$ of Young-Fibonacci tableaux of size n. We will see (Theorem 5.8) that this partial ordering on $\mathbb{YFT}_n$ is such that the map from the *weak order* on the symmetric group $\mathfrak{S}_n$ which sends each permutation σ onto its Young-Fibonacci insertion tableau $P(\sigma)$ is order-preserving. More particularly, standard Young-Fibonacci classes on $\mathfrak{S}_n$ are intervals of the weak order on $\mathfrak{S}_n$. Recall that the weak order on permutations of $\mathfrak{S}_n$ is the transitive closure of the relation $\sigma \leq_p \tau$ if $\tau = \sigma\delta_i$ for some i, where δ_i is the adjacent transposition $(i\ i{+}1)$. An inversion of a permutation σ is a couple $(j, i), 1 \leq i < j \leq n$ such that $\sigma^{-1}(i) > \sigma^{-1}(j)$, that is to say j appears on the left of i in σ. <u>*Note that this is not the definition commonly used*</u>. The set of inversions of a permutation σ will be denoted $inv(\sigma)$, and the number of inversions denoted $\#inv(\sigma)$. We will be making use of an analogous notion of non-inversion of a permutation σ which is a couple $(i, j), 1 \leq i < j \leq n$ such that $\sigma^{-1}(i) < \sigma^{-1}(j)$, that is to say i appears on the left of j in σ. The set of non-inversions of a permutation σ will be denoted $ord(\sigma)$.

Definition 5.1. To introduce $\preceq$, we define the operation of *shifting an entry in a tableau t* as follows.

(1) the bottom entry **a** of any column of t may move and bump up the entry **c** on its left if **c** is a single-boxed column of t. In the example below, the letter 1 is the one being shifted.

$$\begin{array}{cccc} 5 & & & \\ 2 & 4 & 3 & 1 \end{array} \xrightarrow{\text{shift the entry 1}} \begin{array}{ccc} 5 & & 3 \\ 2 & 4 & 1 \end{array}$$

(2) In the case **a** was the bottom entry in a two-boxed column, the top entry **b** will just fall down. In the two examples below, the letter 2 (resp. 3) is the one being shifted.

$$\begin{array}{cccc} & 4 & & \\ 5 & 2 & 3 & 1 \end{array} \xrightarrow{\text{shift 2}} \begin{array}{cccc} 5 & & & \\ 2 & 4 & 3 & 1 \end{array} \quad \text{and} \quad \begin{array}{ccc} & 4 & 2 \\ 5 & 3 & 1 \end{array} \xrightarrow{\text{shift 3}} \begin{array}{ccc} 5 & & 2 \\ 3 & 4 & 1 \end{array}$$

(3) In the case the column just to the left of **a** is two-boxed, with bottom entry **c** and **a** < **c**, then **a** may replace **c** which on its turn is shifted to the right in such a way that if **c** < **b** then **c** will just replace **a** ; otherwise **c** is placed as a new single-boxed column between **a** and **b**, and **b** just falls down. In the two examples below, the letter 1 (resp. 2) is shifted.

$$\begin{array}{|c|c|c|} \hline 5 & 4 & \\ \hline 2 & 1 & 3 \\ \hline \end{array} \xrightarrow{\text{shift } 1} \begin{array}{|c|c|c|} \hline 5 & 4 & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline 5 & 3 & \\ \hline 4 & 2 & 1 \\ \hline \end{array} \xrightarrow{\text{shift } 2} \begin{array}{|c|c|c|c|} \hline 5 & & & \\ \hline 2 & 4 & 3 & 1 \\ \hline \end{array}$$

REMARK 5.2. It easily follows from the definition that shifting an entry in a tableau always produces a legitimate tableau of the same size. In an analogous way, given a tableau t, one defines the reverse operation of finding all the tableaux t' such that shifting an entry in t' gives back t. For example, one will check that

$$\begin{array}{ccc} 5 & & 2 \\ 3 & 4 & 1 \end{array} \text{ is obtained from } \begin{array}{ccc} & 4 & 2 \\ 5 & 3 & 1 \end{array}, \quad \begin{array}{ccc} 5 & & 2 \\ 4 & 3 & 1 \end{array} \text{ and } \begin{array}{cccc} 5 & & & \\ 3 & 4 & 2 & 1 \end{array} \text{ by shifting 3 or 1.}$$

Finally it is clear that this operation is antisymmetric, that is to say if t' is obtained from t by shifting a given entry, then t cannot be obtained from t' by shifting an entry.

The latter observation is enforced by the following lemma which also defines the graduation of the poset $(\mathbb{YFT}_n, \preceq)$ we will soon introduce.

LEMMA 5.3. *Let t_2 be a tableau obtained by shifting an entry in a tableau t_1, and let σ_1 (resp. σ_2) be the minimal permutation canonically associated to t_1 (resp. t_2) as stated in Lemma 2.10, then the inversions sets of σ_1 and σ_2 are related by the relation $\#inv(\sigma_2) = \#inv(\sigma_1) + 1$.*

PROOF. The proof takes into account all the situations one can encounter in shifting an entry in t_1.

(1) $t_1 = \mathcal{T}_2 \begin{smallmatrix} & \star \\ c & a \end{smallmatrix} \mathcal{T}_1$ and $t_2 = \mathcal{T}_2 \begin{smallmatrix} c & \\ a & \star \end{smallmatrix} \mathcal{T}_1$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are partial $\mathbb{YFT}$ having minimal canonical words w_1 and w_2 (see Lemma 2.10 for the definition), and $\star$ means any entry preserving the conditions of Definition 2.1, and possibly no entry. The minimal permutations associated to t_1 and t_2 are $\sigma_1 = w_1 \star a c w_2$ and $\sigma_2 = w_1 \star c a w_2$ respectively, and clearly σ_2 has one more inversion than σ_1.

(2) $t_1 = \mathcal{T}_2 \begin{smallmatrix} d & \star \\ c & a \end{smallmatrix} \mathcal{T}_1$ and $t_2 = \mathcal{T}_2 \begin{smallmatrix} d & & \\ a & c & \star \end{smallmatrix} \mathcal{T}_1$, with $a < c < d$; one has $\sigma_1 = w_1 \star a d c w_2$ and $\sigma_2 = w_1 \star c d a w_2$. The inversion (dc) appears in σ_1 but not in σ_2, whereas the inversions (da) and (ca) appear in σ_2 but not in σ_1 ; so σ_2 has one more inversion.

(3) $t_1 = \mathcal{T}_2 \begin{smallmatrix} d & b \\ c & a \end{smallmatrix} \mathcal{T}_1$ and $t_2 = \mathcal{T}_2 \begin{smallmatrix} d & b \\ a & c \end{smallmatrix} \mathcal{T}_1$, with $a < c < b < d$; one has $\sigma_1 = w_1 b a d c w_2$ and $\sigma_2 = w_1 b c d a w_2$. The inversion (dc) appears in σ_1 but not in σ_2, whereas the inversions (da) and (ca) appear in σ_2 but not in σ_1 ; so σ_2 has one more inversion. □

We are now in position to provide $\mathbb{YFT}_n$ with a structure of poset.

DEFINITION 5.4 (weak order on $\mathbb{YFT}_n$). Let t and t' be two tableaux of size n, then t is said smaller than t' and we write $t \preceq t'$ if one can find a sequence $t_0 = t, t_1, \cdots, t_k = t'$ of tableaux of size n such that t_{i+1} be obtained from t_i by shifting an entry, for i from 0 to $k-1$.

PROPOSITION 5.5. $(\mathbb{YFT}_n, \preceq)$ *is a graded poset, the rank of a Young-Fibonacci tableau being the number of inversions of its minimal canonical permutation.*

PROOF. Follows from Lemma 5.3. □

REMARK 5.6. Note that this remarkable property of graduation of the poset of standard Young-Fibonacci tableaux of size n does not apply to the similar poset $\mathbb{YT}_n$ of standard Young tableaux of size n. The reader interested may refer to [**9**] where Taskin studied many nice properties of four partial orders on $\mathbb{YT}_n$.

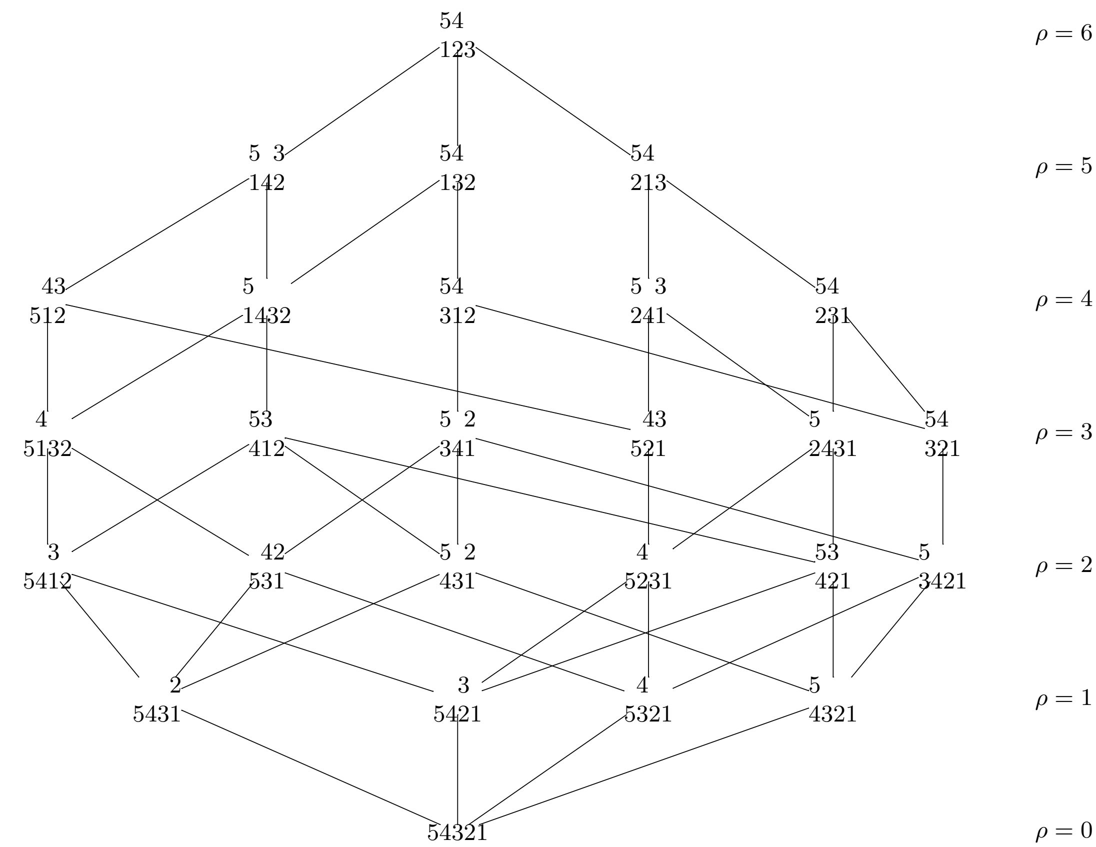


FIGURE 4. The graded weak order on Young-Fibonacci tableaux of size 5.

REMARK 5.7. As one will easily check it on the figure above, $(\mathbb{YFT}_n, \preceq)$ is not a lattice for $n = 5$ for example. Indeed let $a = \begin{array}{l}3\\5421\end{array}$ and $b = \begin{array}{l}5\\3421\end{array}$, then a and b do not have a least upper bound.

THEOREM 5.8. *Let t_1 and t_2 be two tableaux, then $t_1 \preceq t_2$ if and only if one can find two permutations τ_1 and τ_2 such that $P(\tau_1) = t_1$, $P(\tau_2) = t_2$ and $\tau_1 \leq_p \tau_2$.*

PROOF. It is enough to prove this statement for the case t_2 is obtained by shifting an entry in t_1, and the proof is carried out as a parallel process of the proof of Lemma 5.3. So go back to the latter proof and

(1) take $\tau_i = \sigma_i$;
(2) take $\tau_1 = w_1 \star dacw_2$ and $\tau_2 = w_1 \star dcaw_2$;
(3) take $\tau_1 = w_1 bdacw_2$ and $\tau_2 = w_1 bdcaw_2$.

This shows that one can find two permutations τ_1 and τ_2 such that $P(\tau_1) = t_1$, $P(\tau_2) = t_2$ and $\tau_2 = \tau_1 \delta_i$ for some i, whenever $t_1 \preceq t_2$. Reciprocally let τ_1 and τ_2 be two permutations such that $P(\tau_1) = t_1$ and $P(\tau_2) = t_2$ and $\tau_2 = \tau_1 \delta_i$ for some i. Then t_2 is obtained from t_1 by shifting the entry i in t_1. □

We now look at the structure of the Young-Fibonacci classes ; below are two pictures of the poset $(\mathbb{YFT}_4, \preceq)$. On the picture on the left, vertices are Young-Fibonacci classes corresponding to Young-Fibonacci tableaux in the picture on the right. Recall that the rank of a class is the number of inversions of its minimal element in the lexicographical order. The unique involution of any class is enclosed in a rectangle. A double edge means that there are two couples (τ_1, τ_2) and (τ'_1, τ'_2) satisfying the conditions of Theorem 5.8.

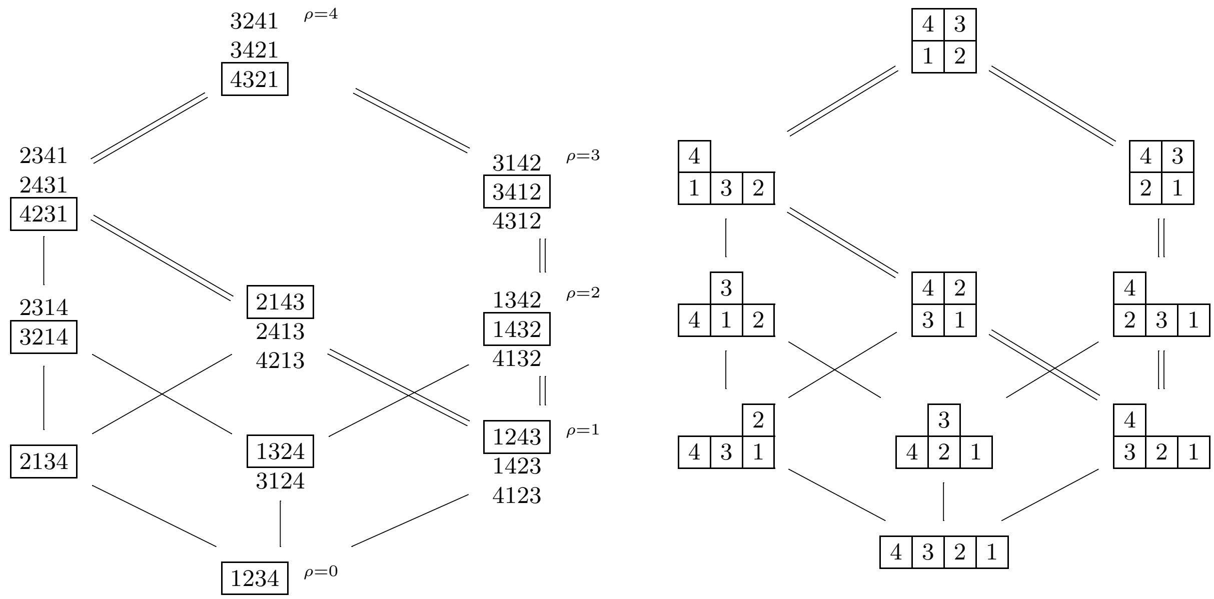


FIGURE 5. The graded weak order on Young-Fibonacci classes of size 4.

It is easy to check that each class appearing as a vertex of the poset $(\mathbb{YFT}_4, \preceq)$ is an interval of the weak order $(\mathfrak{S}_4, \leq_p)$, and this is a general observation.

THEOREM 5.9. *Let t be a standard Young-Fibonacci tableau of size n, then $\mathbb{YFC}(t)$ is an interval of the weak order $(\mathfrak{S}_n, \leq_p)$, more over $\mathbb{YFC}(t) = [w^t_{min}, w^t_{max}]$.*

To prove this statement, we will first relate $\mathbb{YFC}(t)$ with linear extensions of a poset canonically associated to t, and then we will prove that the set of linear extensions of this poset is an interval of the weak order.

DEFINITION 5.10. Let t be a standard Young-Fibonacci tableau of size n, its canonical poset $\mathbb{P}_t$ is the poset defined on the set $\{1, 2, ..., n\}$ with the covering relations below.

(1) the right-to-left reading of the bottom row of t forms a chain in the poset ;
(2) each entry on top in a two-boxed column of t is covered by the corresponding entry on bottom row.

NOTE 5.11. A permutation σ is a *toset* (totally ordered set) with covering relations defined by $\sigma(i) \leq_\sigma \sigma(j)$ whenever $i < j$, that is to say $x \leq_\sigma y$ if x appears to the left of y in σ. Let $\mathbb{P}$ be a poset and σ a permutation, σ is said to be a linear extension of $\mathbb{P}$ if its relations preserve the relations in $\mathbb{P}$, that is to say if $x \leq_{\mathbb{P}} y$ then $x \leq_\sigma y$. The set of linear extensions of a poset $\mathbb{P}$ will be denoted $Ext(\mathbb{P})$.

THEOREM 5.12. *Let t be a standard Young-Fibonacci tableau, then $\mathbb{YFC}(t) = Ext(\mathbb{P}_t)$.*

PROOF. That any permutation σ having t as insertion tableau is a linear extension of $\mathbb{P}_t$ is clear from Definitions 2.4 and 5.10. Conversely, if σ is a linear extension of $\mathbb{P}_t$, then t is naturally built reading σ from right to left following the description given in Definition 2.4. At each new step the first letter one reads is the maximal one (for $\leq_{\mathbb{P}_t}$) not yet read in the chain described in rule (1) of Definition 5.10. $\square$

THEOREM 5.13. *Let t be a standard $\mathbb{YFT}$ of size n, then $Ext(\mathbb{P}_t)$ is the interval $[w^t_{min}, w^t_{max}]$ in $(\mathfrak{S}_n, \leq_p)$.*

To prove this statement we make use of the following well known lemma.

LEMMA 5.14. *Let σ and τ be two permutations of $\mathfrak{S}_n$, then the three properties below are equivalent.*

(1) $\sigma \leq_p \tau$;
(2) $ord(\tau) \subseteq ord(\sigma)$;
(3) $inv(\sigma) \subseteq inv(\tau)$.

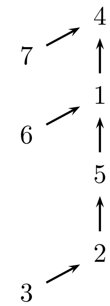
PROOF. (of Theorem 5.13) It easily follows from the definition that $\mathbb{P}_t$ can be partitioned into an antichain $A = (y_1, y_2, \cdots, y_\ell)$ and a chain $C = (x_1 <_{\mathbb{P}_t} x_2 <_{\mathbb{P}_t} \cdots <_{\mathbb{P}_t} x_k)$ such that for $i = 1..\ell$ there exists $j(i) \leq k$ such that $y_i <_{\mathbb{P}_t} x_{j(i)}$, and additionally for $i_1 < i_2$ one has $y_{i_1} < y_{i_2}$ and $x_{j(i_1)} <_{\mathbb{P}_t} x_{j(i_2)}$. For illustrations, we use the following example.

$A = (3, 6, 7)$

$C = (2 <_{\mathbb{P}_t} 5 <_{\mathbb{P}_t} 1 <_{\mathbb{P}_t} 4)$

7	6		3
4	1	5	2

a tableau t
of shape $u = 2212$



its canonical poset $\mathbb{P}_t$

The set I is made of the inversions below.
$(3, 2), (6, 1), (7, 4)$
$(3, 1), (6, 4)$
$(2, 1), (5, 1), (5, 4)$.

The set O is made of the ordered pairs below.
$(2, 5), (2, 4), (1, 4)$
$(3, 5), (3, 4)$.

For $\sigma \in Ext(\mathbb{P})$, $inv(\sigma)$ includes at least the set

$$I \;=\; \left\{ \begin{array}{l} (y_i, x_{j(i)}), i = 1..\ell \\ (y_i, x_r) \;/\; x_{j(i)} > x_r \text{ and } x_{j(i)} <_{\mathbb{P}_t} x_r \\ (x_i, x_j) \;/\; x_i > x_j \text{ and } x_i <_{\mathbb{P}_t} x_j \end{array} \right\}$$

which is nothing but $inv(w^t_{min})$; so by [Lemma 5.14 - (3)], $w^t_{min} \leq_p \sigma$. Moreover, $ord(\sigma)$ includes at least the set

$$O \;=\; \{\; (y_i, x_r) \;/\; x_{j(i)} <_{\mathbb{P}_t} x_r \;\} \;\cup\; \{\; (x_i, x_j) \;/\; x_i < x_j \text{ and } x_i <_{\mathbb{P}_t} x_j \;\}$$

which is nothing but $ord(w^t_{max})$; so by [Lemma 5.14 - (2)], $\sigma \leq_p w^t_{max}$ and hence $\sigma \in [w^t_{min}, w^t_{max}]$. Conversely, for $\sigma \in [w^t_{min}, w^t_{max}]$, applying Lemma 5.14 to w^t_{min}, σ and w^t_{max} it appears that σ has the inversions $y_i \leq_\sigma x_{j(i)}$ for $i = 1..\ell$, and the relations $x_1 <_\sigma x_2 <_\sigma \cdots <_\sigma x_k$. So $P(\sigma) = t$ and hence $\sigma \in Ext(\mathbb{P}_t)$. □

Proof. (of Theorem 5.9) Follows from Theorem 5.12 and Theorem 5.13. □

Definition 5.15. Let u be a snakeshape of size n, the row canonical tableau rT_u is the one such that
(1) top cells of rT_u are labeled with entries n, $n-1$, $\cdots$ from left to right ;
(2) bottom cells in two-boxed columns are labeled with entries 1, 2, $\cdots$ from left to right.

The column canonical tableau cT_u is built by labeling the cells of u from right to left and bottom to top.

Lemma 5.16. *Let u be a snakeshape of size n, then cT_u (resp. rT_u) is the unique tableau of shape u having minimal rank ρ^u_{min} (resp. maximal rank ρ^u_{max}) in the poset $(\mathbb{YFT}_n, \preceq)$. For any snakeshape u, ρ^u_{min} is the number of double-boxed columns of u and ρ^u_{max} is obtained as follows. Label each bottom cell with the number of double-boxed columns on its left and do the same but add 1 for each top cell of double-boxed columns of u. ρ^u_{max} is the sum of labels obtained.*

Proof. (of Lemma 5.16) Easily from the definitions. □

We will now relate $(\mathbb{YFT}_n, \preceq)$ to a transition matrix in Okada's algebra associated to $\mathbb{YFL}$.

6. A connection with Okada's algebra associated to the Young-Fibonacci lattice

A Young-Fibonacci analogue of the ring of symmetric functions [**6**] was given and studied by S. Okada [**16**], with a Young-Fibonacci analogue of Kostka numbers, appearing when expressing the analogue of a complete symmetric function $\mathbf{h}_v$ in terms of the analogue of Schur functions $\mathbf{s}_u$.

$$\mathbf{h}_v \;=\; \sum_u \mathbf{K}_{u,\,v}\, \mathbf{s}_u \tag{6.1}$$

Young-Fibonacci analogue of Kostka numbers are generated by the recurrence formulas below [**16**], where $\mathbf{K}_{a,\,b}$ is defined for two snakeshapes of the same weight and $\rhd$ denotes the covering relation in $\mathbb{YFL}$.

$$\left\{ \begin{array}{lcll} \mathbf{K}_{1u,\,1v} & = & \mathbf{K}_{u,\,v} & (r_1) \\ \mathbf{K}_{2u,\,2v} & = & \mathbf{K}_{u,\,v} & (r_2) \\ \mathbf{K}_{1u,\,2v} & = & 0 & (r_3) \\ \mathbf{K}_{2u,\,1v} & = & \sum_{w \rhd u} \mathbf{K}_{w,\,v} & (r_4) \end{array} \right. \tag{6.2}$$

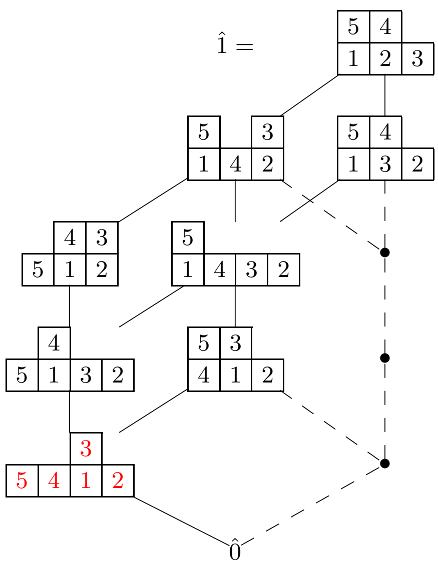
As it is stated below, the hook-length formula for binary trees illustrated in Example 2.3 is an alternative formula for computing $\mathbf{K}_{u,\,1^n} = \mathcal{F}_u$ which is the dimension of a representation in Okada's algebra.

PROPOSITION 6.1. *Let u be a snakeshape of size n, then $\mathcal{F}_u$ is the dimension of the module V_u corresponding to u in the n^{th} homogenous component of Okada's algebra associated to $\mathbb{YFL}$.*

PROOF. $dim(V_u)$ is the number of saturated chains from $\emptyset$ to u in $\mathbb{YFL}$, hence the result. □

Here is a more general statement giving a combinatorial interpretation of $\mathbf{K}_{u,\,v}$ using $(\mathbb{YFT}_n, \preceq)$.

THEOREM 6.2. *Let u and v be two snakeshapes of size n, and let $\hat{1}$ be the maximal tableau in $(\mathbb{YFT}_n, \preceq)$, then $\mathbf{K}_{u,\,v}$ is the number of tableaux t of shape u in the interval $[rT_v, \hat{1}]$.*



EXAMPLE 6.3. In the matrix below, the number $\mathbf{K}_{u,\,1121}$ counts the number of standard Young-Fibonacci tableaux of shape u in the interval $[rT_{1121}, \hat{1}]$.

	221	212	2111	122	1211	1121	1112	1^5
221	1	1	2	1	2	3	4	8
212	.	1	1	1	1	1	3	4
2111	.	.	1	.	1	1	1	4
122	.	.	.	1	1	1	2	3
1211	.	.	.	.	1	1	1	3
1121	.	.	.	.	.	1	1	2
1112	.	.	.	.	.	.	1	1
1^5	.	.	.	.	.	.	.	1

Iterating this for each snakeshape v of size n, one builds the transition matrix for expressing the analogue of complete symmetric function $\mathbf{h}_v$ in terms of the analogue of Schur functions $\mathbf{s}_u$.

FIGURE 6. $(\mathbb{YFT}_5, \preceq)$ and Okada's analogue of Kostka matrix for $n = 5$.

PROOF. (of Theorem 6.2) A proof consists in showing that for any couple (a, b) of snakeshapes appearing in the left hand side of equation (6.2), there is a one-to-one correspondence between tableaux satisfying the conditions of the theorem for (a, b) and those satisfying the conditions of the theorem for the couples of snakeshapes in the corresponding right hand side of the relation. For (r_1), given a tableau t of shape u such that $rT_v \preceq t$, t is mapped onto the tableau t' of shape $1u$ obtained from t by attaching a cell labeled $n+1$ to its left, and $rT_{1v} \preceq t'$. For (r_2), one attaches a two-boxed column to the left of t, with 1 as bottom entry and $n+2$ as top entry, in addition one standardizes t by increasing all its entries. Then t' is of shape $2u$ and $rT_{2v} \preceq t'$. For (r_3) it easily follows from the definition of the operation of shifting an entry in a tableau that there is no tableaux t_1 and t_2 of shape $1u$ and $2v$ respectively, such that $t_2 \preceq t_1$. For (r_4), let t be a tableau of shape $2u$ such that $rT_{1v} \preceq t$, then t is mapped onto the tableau $t' = ev(t, n)$, that is the tableau obtained from t by evacuating its maximal letter (the evacuation process originally due to Killpatrick [**8**] is described in Section 3.2). Indeed, let w be the shape of t', then $w \rhd u$ and $rT_v \preceq t'$. □

7. Kostka numbers, the Littlewood Richardson rule, and four posets on Young tableaux

The poset $(\mathbb{YFT}_n, \preceq)$ of Young-Fibonacci tableaux we defined in Section 5 is an analogue of one among four partial orders on the set $\mathbb{YT}_n$ of standard Young tableaux of size n [**9**]. The weak order $(\mathbb{YT}_n, \preceq_{weak})$ is defined as in Theorem 5.8 with $P(\sigma)$ denoting the Schensted insertion tableau of σ. Let λ and μ be two partitions of lengths $\ell(\lambda)$ and $\ell(\mu)$, λ is said greater than μ in the dominance order and one writes $\lambda \geq_{dom} \mu$ if for each $1 \leq i \leq min(\ell(\lambda), \ell(\mu))$, the inequality $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ holds. Let t be a standard Young tableau of size n, and $1 \leq i \leq j \leq n$. We denote $\lambda(t_{/i,j})$ the shape of the tableau obtained from t by first restricting t to the segment $[i, j]$, then lowering all entries by $i-1$, and finally sliding the skew tableau obtained into normal shape by jeu-de-taquin. The chain order $\preceq_{chain}$ on standard Young tableaux is defined as follows.

Definition 7.1. [**9**] Let t and t' be two standard Young tableaux of size n, then $t \preceq_{chain} t'$ if and only if for each $1 \leq i \leq j \leq n$, $\lambda(t_{/i,j}) \geq_{dom} \lambda(t'_{/i,j})$.

The reader interested may refer to [**9**] for the definition of the two other orders, as well as for the properties of those posets. The four posets happen to coincide up to rank $n = 5$.

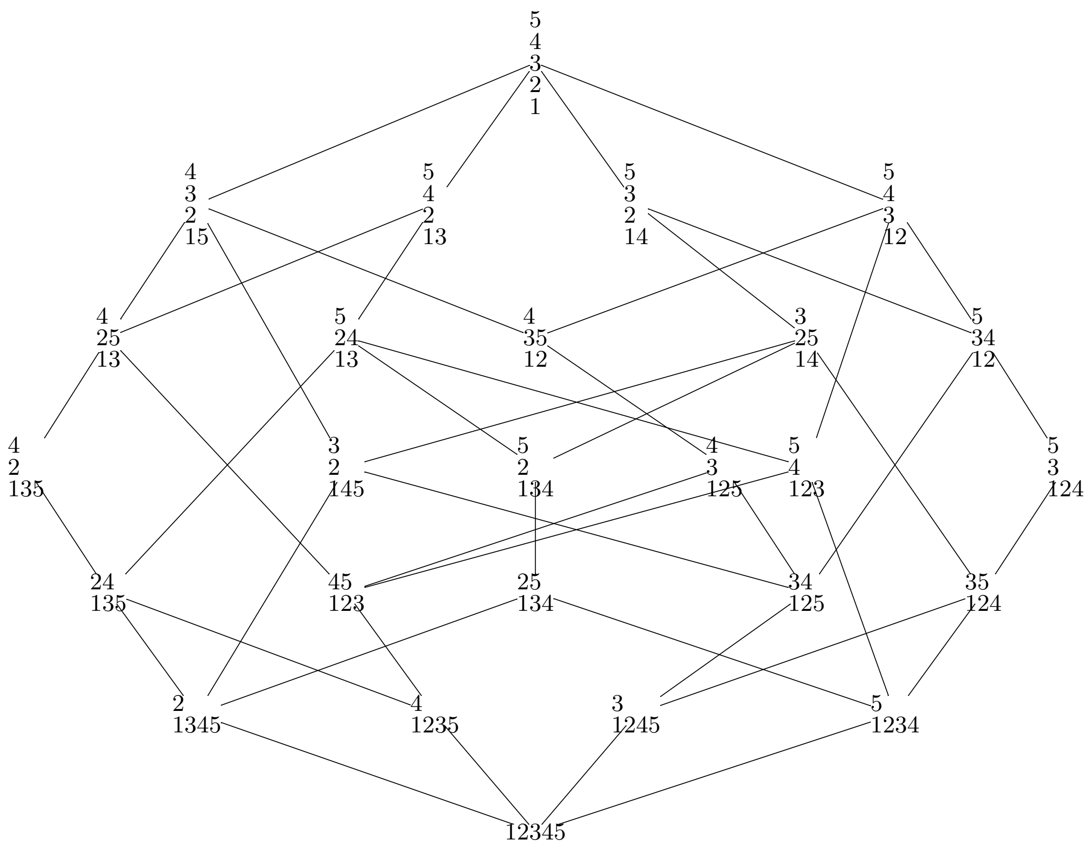
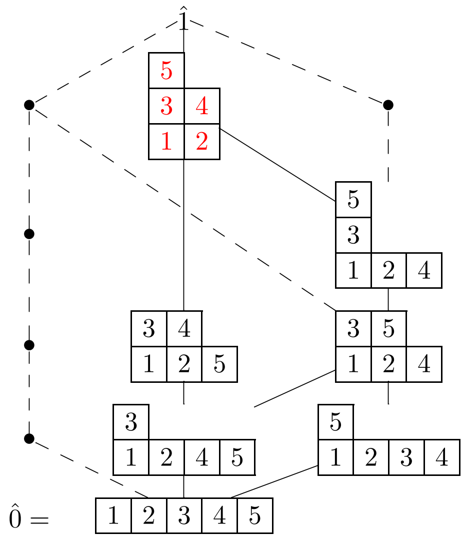


Figure 7. Partial order on Young tableaux of size 5.

Below is a Young tableaux analogue of Theorem 6.2.

Theorem 7.2. *Let λ, μ be two partitions of size n, let rT_μ be the row canonical standard Young tableau of shape μ, that is to say rT_μ has shape μ and is increasingly filled from let to right and bottom to top. And let $\hat{0}$ be the minimal tableau in the poset of standard Young tableaux of size n. Then $K_{\lambda\,\mu}$ is the number of standard Young tableaux of shape λ in the interval $[\hat{0}, rT_\mu]$, for any one of the posets studied in* [**9**].



EXAMPLE 7.3. In the matrix below, the number $K_{\lambda\,221}$ counts the number of standard Young tableaux of shape λ in the interval $[\hat{0}, rT_{221}]$.

	$\mu =$	5	41	32	311	221	2111	11111
$\lambda =$	5	1	1	1	1	1	1	1
	41	.	1	1	2	2	3	4
	32	.	.	1	1	2	3	5
	311	.	.	.	1	1	3	6
	221	.	.	.	.	1	2	5
	2111	.	.	.	.	.	1	4
	11111	.	.	.	.	.	.	1

Iterating this for each partition μ of size n, one builds the transition matrix for expressing the complete symmetric function h_μ in terms of the Schur functions s_λ.

FIGURE 8. Poset of Young tableaux and Kostka matrix for $n = 5$.

PROOF. (of Theorem 7.2) For a given partition μ, let $nscrt(\mu)$ be the row canonical semi-standard Young tableau of shape μ, that is the tableau filled with 1's on its first line, 2's on its second line and so on. Let $nsclt(n)$ be the semi-standard Young tableau of shape n and having μ_i entries i for $i = 1..\ell(\mu)$. Consider the extension of Definition 7.1 to the set $Tab(\mu)$ of semi-standard Young tableaux having μ_i entries i for $i = 1..\ell(\mu)$. Then for each $t \in Tab(\mu)$, one has $nsclt(n) \preceq_{chain} t \preceq_{chain} nscrt(\mu)$. There is a canonical bijection mapping $(Tab(\mu), \preceq_{chain})$ onto $([\hat{0}, rT_\mu], \preceq_{chain})$ and this map is order preserving. So Theorem 7.2 holds for the partial order $\preceq_{chain}$. From ([**9**], Theorem 1.1) and the remark that $[\hat{0}, rT_\mu] = rT_{\mu_1} * rT_{\mu_2} * \cdots * rT_{\mu_{\ell(\mu)}}$, it follows that the set of tableaux in $[\hat{0}, rT_\mu]$ does not depend on the choice of the partial order. □

Concluding remarks and perspectives

There are quite many similarities between the Robinson-Schensted algorithm and the Young-Fibonacci insertion algorithm. As well as between the combinatorics of Young tableaux and the combinatorics of Young-Fibonacci tableaux. One of the questions we have not explored in this paper is the one of the existence of an algebra of Young-Fibonacci tableaux, which would be an analogue of the Poirier-Reutenauer Hopf algebra of Young tableaux [**12**]. Such an algebra would certainly help in giving a combinatorial description (in terms of tableaux) of the product of Schur functions in Okada's algebra associated to the Young-Fibonacci lattice. We are currently looking for a suitable definition of this algebra.

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