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Major Indices, Mahonian Identities and Ordered Generating Systems (Extended Abstract)

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Abstract. A classical result of MacMahon shows that the length function and the major index are equidistributed over the symmetric group. A long standing open problem is to extend the notion of *major index* and MacMahon identity to other groups. A partial solution was given in [**3**] and [**5**], where this result was extended to classical Weyl groups. In this paper, it is proved that various permutation groups may be decomposed into a set-wise direct product of cyclic subgroups. This property is then used to extend the notion of major index and MacMahon identity to these groups.

1. Background

1.1. The major index. Let S_n be the symmetric group on n letters. Denote $s_i := (i, i+1)$ the adjacent transposition. S_n is a Coxeter group with respect to the the Coxeter generating set $S = \{s_i \mid 1 \leq i < n\}$. Let $\ell(\pi)$ be the length of $\pi \in S_n$, with respect to S,

$$Des(\pi) := \{1 \leq i < n | \; \ell(\pi s_i) < \ell(\pi)\}$$

the descent set of π, and

$$maj(\pi) := \sum_{i \in Des(\pi)} i$$

the major index of π. It is well known that

$$\ell(\pi) = \#\{i < j \mid pi(i) > \pi(j)\}$$

the number of inversions in π, and that

$$Des(\pi) = \{1 \leq i \leq n-1 | \; \pi(i) > \pi(i+1)\}.$$

The study of permutation statistics dates back to Euler. A remarkable (and nearly 100 years old) theorem of MacMahon says that the major index and the number of inversions of a permutation are equidistributed over S_n.

MacMahon Identity: [**24**]

$$\sum_{\pi \in S_n} q^{\ell(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)}.$$

Various refinements and extensions of this identity appear in the literature. Most of the extensions are in two directions: the study of equidistributed parameters over subsets of S_n, and the study of such statistics on other, related groups, such as the hyperoctahedral group. For these extensions and for the algebraic role of MacMahon identity see e.g. [**15, 19, 20, 8, 16, 1, 2, 5, 22, 25, 4**].

The following problem was first posed by Foata in the early nineties.

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PROBLEM 1.1. *Extend the notion of major index and MacMahon identity to other permutation groups; in particular, to the group of signed permutations.*

Several "major index" statistics have been introduced and studied for signed and colored permutations (see, e.g., [**11, 12, 13, 26, 27, 34**]). However, these major indices are not equidistributed with a natural length function on this permutation group. A generalization of MacMahon's result, which involves "ordered generation" is suggested in this paper.

1.2. Ordered Generating Systems.

DEFINITION 1.2. *The sequence* $(a_1, \cdots, a_n)$ *is called an* Ordered Generating System (OGS) *for a group* G*, if every element* $g \in G$ *has a unique presentation:*

$$g = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n},$$

where $0 \le k_i < |a_i|$*, for every* $1 \le i \le n$*.*

In other words, there exist subgroups $C_1, \ldots, C_n$, such that

$$G = C_1 \cdots C_n \tag{1}$$

$$\forall i \;\; C_i \cap C_1 \cdots \hat{C}_i \cdots C_n = 1 \tag{2}$$

$$\forall i \;\; C_i \text{ is cyclic }. \tag{3}$$

Examples:

1) pq groups have an OGS [**30**].
2) Q_8 - the group of quaternions has **no** OGS [**30**].

The major index of a permutation has an algebraic interpretation in terms of ordered generating systems.

OBSERVATION 1.3. [**3**, Claim 2.1] *For every permutation* $\pi \in S_n$

$$maj(\pi) = \sum_{i=1}^{n-1} k_i,$$

where π *is written in the form:*

$$t_{n-1}{}^{k_{n-1}} \cdots t_1{}^{k_1},$$

and

$$t_i := s_i s_{i-1} \cdots s_1.$$

This observation was applied in [**3**] to solve Problem 1.1 for the hyperoctahedral group, see Propositions 3.1(2) and 3.3 below. In this paper, this approach is extended to other permutation groups.

2. Dihedral Groups

We start with a "baby case". Recall that the dihedral group $I_2(n)$ is the group of reflections of the plane, generated by two reflections s and t along lines which meet at an angle $\frac{\pi}{n}$. Then $(st)^n = 1$. $I_2(n)$ has another generating set $\{r, t\}$, where $r = st$ is a rotation through $\frac{2\pi}{n}$ and t is the above reflection. Every element in $I_2(n)$ has a unique presentation

$$\pi = t^i r^j \qquad 0 \le i \le 1,\; 0 \le j \le n-1.$$

Thus (t, r) is an ordered generating system.

Let

$$\mathrm{fmaj}_I(\pi) := i + j$$

and $\ell(\pi)$ be the length of π with respect to the Coxeter generating set $\{s, t\}$. A mahonian identity holds.

PROPOSITION 2.1.

$$\sum_{\pi \in I_2(n)} q^{\mathrm{fmaj}_I(\pi)} = \sum_{\pi \in I_2(n)} q^{\ell(\pi)} = (1+q)[n]_q,$$

where $[n]_q := \frac{q^n - 1}{q-1}$.

3. Classical Weyl Groups

Recall the presentations of the classical Weyl groups as permutation groups, see e.g. [**7**].

PROPOSITION 3.1. 1. *For $1 \le i \le n$ let α_i denote the permutation $[i, 1, 2, \ldots, i-1, i+1, i+2, \ldots, n]$ in the symmetric group S_n. The sequence $(\alpha_n, \alpha_{n-1}, \ldots, \alpha_2)$ is an ordered generating system for S_n.*

2. *For $1 \le i \le n$ let β_i denotes the signed permutation $[-i, 1, 2, \ldots, i-1, i+1, i+2, \ldots, n]$ in the hyperoctahedral group B_n. The sequence $(\beta_n, \beta_{n-1}, \ldots, \beta_1)$ is an ordered generating system for B_n.*
3. *For $1 \le i < n$ let*

$$\delta_i := [-i, 1, 2, \ldots, i-1, i+1, i+2, \ldots, -n]$$

and

$$\delta_n = [n, 1, 2, \ldots, n].$$

The sequence $(\delta_{n-1}, \ldots, \delta_1, \delta_n)$ is an ordered generating system for the group of even signed permutations D_n.

REMARK 3.2. *The statements (1) and (2) for types A and B are proved in* [**3**].

Proof of Proposition 3.1(3). Let $\hat{B}_i$ be a subgroup of D_n, consisting of all even signed permutations for which $\pi(n) = \pm n$ and $\pi(r) = r$ for all $r > i+1$. In order to prove that $\delta_{n-1} \ldots, \delta_1, \delta_n$ is an OGS for D_n, we will note, first, that $\delta_1, \ldots, \delta_{i-1}$ are elements in $\hat{B}_{i-1}$. For any pair of exponents $0 \le k_1 < k_2 < i$, $(\delta_i^{k_1})^{-1}\delta_i^{k_2}(i) = \delta_i^{k_2-k_1}(i) \ne i$, hence $(\delta_i^{k_1})^{-1}\delta_i^{k_2} \notin \hat{B}_{i-1}$, thus the cosets $\delta_i^{k_1}\hat{B}_{i-1}$ and $\delta_i^{k_2}\hat{B}_{i-1}$ are distinct. This shows that for every $1 < i \le n$, the set $\{\delta_i^k |\ 0 \le k < i\}$ consists of all left coset representatives of $\hat{B}_{i-1}$ in $\hat{B}_i$.

It remains to show that $\{\delta_n^k |\ 0 \le k < n\}$ is a complete set of left coset representatives of $\hat{B}_{n-1}$ in D_n. By a similar argument, for $0 \le k_1 < k_2 < n$, $(\delta_n^{k_1})^{-1}\delta_n^{k_2}(n) \ne \pm n$, thus the corresponding (left as well as right) cosets are distinct. This completes the proof.

□

PROPOSITION 3.3. [**3**] *For $\pi \in B_n$, define*

$$\mathrm{fmaj}_B(\pi) := \sum_{i=1}^{n} k_i,$$

where π is written in the form: ${\beta_n}^{k_n} \cdots {\beta_1}^{k_1}$, and $0 \le i < 2i$ for $1 \le i \le n$. Then fmaj_B plays a similar algebraic role to maj; in particular,

$$\sum_{\pi \in B_n} q^{\ell(\pi)} = \sum_{\pi \in B_n} q^{\mathrm{fmaj}_B(\pi)},$$

where ℓ is the standard length function with respect to the Coxeter generating set of B_n.

DEFINITION 3.4. *For $\pi \in D_n$, define*

$$\mathrm{fmaj}_D(\pi) := \sum_{i=1}^{n} k_i,$$

where π is written in the form:

$${\delta_{n-1}}^{k_{n-1}} \cdots {\delta_1}^{k_1} {\delta_n}^{k_n} \qquad 0 \le k_i \le 2i \text{ for } 1 \le i < n \text{ and } 0 \le k_n < n.$$

The following is a type D analogue of Proposition 3.3.

PROPOSITION 3.5.

$$\sum_{\pi \in D_n} q^{\ell(\pi)} = \sum_{\pi \in D_n} q^{\mathrm{fmaj}_D(\pi)},$$

where ℓ is the standard length function with respect to the Coxeter generating set of D_n.

Recall from [**3**] that fmaj_B has a direct combinatorial interpretation.

Proposition 3.6. [**3**, Theorem 3.1] *For every* $\pi \in B_n$, $\mathrm{fmaj}_B(\pi) = 2 \cdot maj(\pi) + neg(\pi)$ *where maj is taken with respect to the order* $-1 < -2 < \cdots < 1 < 2 < \cdots$ *and neg is the number of negative values in* π.

A direct combinatorial interpretation of fmaj_D is given in the following proposition.

Proposition 3.7. *For every* $\pi = [\pi(1), \ldots, \pi(n)] \in D_n$ *let* $j = |\pi^{-1}(n)|$ *and* $\hat{\pi} \in B_{n-1}$ *be the signed permutation* $[\pi(j+1), \ldots, \pi(n), \pi(1), \ldots, \pi(j-1)]$. *Then*

$$\mathrm{fmaj}_D(\pi) = \mathrm{fmaj}_B(\hat{\pi}) + j.$$

Example. Let $\pi = [-3, 5, -2, 1, 4]$. Then $j = |\pi^{-1}(5)| = 2$ and $\hat{\pi} = [-2, 1, 4, -3]$. Thus, by Propositions 3.7 and 3.6 $\mathrm{fmaj}_B(\hat{\pi}) = 2 \cdot maj(\hat{\pi}) + neg(\hat{\pi}) = 2 \cdot 3 + 2 = 8$ and $\mathrm{fmaj}_D(\pi) = 8 + 2 = 10$.

Note that our fmaj_D is different from the flag major index for D_n, which was introduced by Biagioli and Caselli [**5**].

4. The Alternating Group of B_n

Let $A(B_n)$ be the alternating group of the Coxeter group of type B; namely, the subgroup consisting all elements in B_n of even length.

Let $r_1 := [2, -1, 3, \ldots, n]$ and $r_i := [-1, 2, \ldots, i+1, i, i+2, i+3, \ldots, n]$, $(2 \le i \le n-1)$.
$R := \{r_i \mid 1 \le i \le n-1\}$ is a set of generators of $A(B_n)$ with Coxeter like relations [**9**, Chapter IV Section 1 Exercise 9]. The defining relations are:

$$r_i^2 = 1 \qquad (1 < i < n)$$

$$r_1^4 = 1$$

$$(r_i r_j)^2 = 1 \qquad (|i-j| > 1)$$

$$(r_i r_{i+1})^3 = 1 \qquad (1 \le i < n).$$

Let $\ell(\pi)$ be the length of $\pi \in A(B_n)$ with respect to $R \cup R^{-1}$. Then

Proposition 4.1. [**10**]

$$\sum_{\pi \in A(B_n)} q^{\ell(\pi)} = \prod_{i=1}^{n-1} [2i]_q [n]_q.$$

For $1 \le i \le n-2$ let

$$\gamma_i := r_i r_{i-1} \cdots r_1^2$$

and

$$\gamma_n := r_{n-1} r_{n-2} \cdots r_2 r_1^{-1}.$$

Proposition 4.2. *The sequence* $(\gamma_n, \ldots, \gamma_1)$ *is an ordered generating system for* $A(B_n)$. *Thus every element* $\pi \in A(B_n)$ *has a unique presentation*

$$\pi = \gamma_n^{k_n} \gamma_{n-1}^{k_{n-1}} \cdots \gamma_1^{k_1} \qquad 0 \le k_i \le 2i \text{ for } 1 \le i < n \text{ and } 0 \le k_n < n..$$

Let $\mathrm{fmaj}_A(\pi) := \sum_{i=1}^{n} k_i$. *Then*

$$\sum_{\pi \in A(B_n)} q^{\mathrm{fmaj}_A(\pi)} = \sum_{\pi \in A(B_n)} q^{\ell(\pi)}$$

The proof is obtained using a bijective map from words in $A(B_n)$ to words in the even signed permutation group D_n, which is presented in [**10**].

A major index and a mahonian identity on the alternating group of S_n may be found in [**25**]. It should be noted that while the length function is defined there with respect to an analogous generating set to the above $R \cup R^{-1}$, it seems that there is no simple interpretation of the major index for the alternating group of S_n via an ordered generating system.

5. Finite Coxeter Groups

The following proposition follows from techniques used in [**30**].

PROPOSITION 5.1. *Every finite Coxeter group has an ordered generating system.*

Having an ordered generating system for a Coxeter group W one can define the flag major index of an element $w \in W$ as the sum of the exponents in its unique presentation, as done above. Unfortunately, for most finite Coxeter groups (except of the classical Weyl groups and the dihedral groups), we were not able to find ordered generating systems, for which the resulting flag major index is equidistributed with the length function with repect to the Coxeter generators.

6. Complex Reflection Groups

The colored permutation group $G(r,n)$ is the wreath product of Z_r by S_n, where:

$$G(r,n) := \{((c_1, \cdots, c_n), \sigma) \mid c_i \in [0, r-1],\ \sigma \in S_n\}.$$

The complex reflection group $G(r,p,n)$ is defined in [**31**] as the subgroup of index p of $G(r,n)$ which consists of the elements

$$G(r,p,n) := \{g \in G(r,n) \mid \sum_{i=1}^{n} c_i \equiv 0 (mod) p\}.$$

The following proposition due to Adin and Roichman generalizes the special case of $G(2,1,n) = B_n$.

PROPOSITION 6.1. [**3**] *Let* $t_i := [i+1, 1, 2, \ldots, i-1, i+1, \ldots, n]$ *and* $u_i := ((1,0,\ldots,0), t_i)$. *Then* $(t_{n-1}, \ldots, t_1, t_0)$ *is an ordered generating system of* $G(r,n)$.

This proposition of Adin and Roichman generalizes the special case of $G(2,1,n) = B_n$. The following proposition generalizes the special case of $G(2,2,n) = D_n$.

PROPOSITION 6.2. *Let* $t_i = [i+1, 1, ..., i, i+2, ..., n]$ *and* $u_i = \{(1, 0, \cdots, 0, -1), t_i)$. *Then* $(t_{n-1}, \ldots, t_1, t_0)$ *is an ordered generating system of* $G(r,r,n)$.
Every element $\pi \in G(r,r,n)$ *has a unique presentation of the form:*

$$\pi = u_{n-1}^{j_{n-1}} \cdots u_1^{j_1} u_0^{j_0},$$

where $0 \le j_i \le (i+1)r - 1$, *for* $0 \le i \le n-2$, *and* $0 \le j_{n-1} \le n-1$.

For $\pi \in G(r,r,n)$ define

$$\text{fmaj}(\pi) = \sum_{i=0}^{n-1} j_i.$$

COROLLARY 6.3. $\sum\limits_{w \in G(r,r,n)} q^{\text{fmaj}(w)}$ *is equal to the Hilbert series of the coinvariant algebra of* $G(r,r,n)$.

This result extends the well known special case of the symmetric group. For closely related results and other extensions see e.g. [**32, 2, 6, 4**].

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