Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Generating functions from the point of view of Rota–Baxter algebras

Li Guo

Abstract. We study generating functions in the context of Rota-Baxter algebras. We show that exponential generating functions can be naturally viewed in a special free complete Rota-Baxter algebra. This allows us to use free Rota–Baxter algebras to give a wide class of algebraic structures where generalizations of generating functions can be studied. We illustrate this by several cases and examples.

1. Introduction

The power series (or ordinary) generating function of a number sequence $a_0, a_1, a_2, \cdots$ is the power series

$$\mathbf{A}(z) = \sum_{n\geq 0} a_n z^n.$$

The exponential generating function (EGF) of a number sequence $a_1, a_2, \cdots$ is the exponential power series

$$\mathbf{B}(z) = \sum_{n\geq 1} a_n \frac{z^n}{n!}.$$

For some sequences, the exponential series has a better formula. For example, the Bell numbers $b(n)$ have exponential generating function

$$\sum_{n\geq 0} B(n)\frac{x^n}{n!} = \exp(e^x - 1).$$

The Bernoulli numbers B_n have exponential generating function

$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

Such nice analytic expressions are not available in terms of power series generating functions.

One way to view the power series generating functions and exponential generating functions in the same framework is that they each give a way to encode a number sequence $a_n, n \geq 0$ as the coefficients of a linear combination with respect to a basis of the power series algebra $\mathbb{R}[[z]]$. The basis is $z^n, n \geq 0$ for power series generating functions and is the divided powers $z^n/n!, n \geq 0$ for exponential generating functions. The distinction is the two different bases for the same power series ring. We can also view the difference externally: $v_k = z^k, k \geq 0$ is the standard basis of the algebra

$$\mathcal{A} := \prod_{k\geq 0} \mathbb{R}\, v_k, \quad \text{with componentwise product } v_m v_n = v_{m+n}$$

2000 *Mathematics Subject Classification.* Primary 05A15; secondary 05A18, 16W99.

Key words and phrases. exponential generating functions, shuffle products, free Rota-Baxter algebras, divided powers, Stirling numbers.

The author is supported in part by NSF grant DMS-0505643.

while $w_k = z^k/k!$ is the standard basis of the divided power algebra

$$\mathcal{B} := \prod_{k\geq 0} \mathbb{R}\, w_k, \text{ with componentwise product } w_m w_n = \binom{m+n}{m} w_{m+n}.$$

We can take this point of view further and consider the following general framework for generating functions: A complete filtered $\mathbb{R}$-algebra is a $\mathbb{R}$-algebra A with ideals $A_n, n \geq 0$, such that $A_m A_n \subseteq A_{m+n}$ and A is complete with respect to the metric on A induced by A_n. In other words, the natural map

$$A \to \varprojlim A/A_n$$

is bijective. Let $\mathcal{U} := \{u_j, j \in J\}$, be a basis of A that is compatible with its filtration in the sense that $\mathcal{U} \cap A_k$ is a basis of A_k, $k \geq 0$. Then a **$\mathcal{U}$-generating function** of a family of numbers $a_j \in \mathbb{R}, j \in J$, is the element $\sum_{j\in J} a_j u_j$ in A. In this context, a power series generating function is a $\mathcal{U}$-generating function when $\mathcal{U}$ is taken to be the basis $v_k = \{z^k, k \geq 0\}$ in the complete filtered algebra

$$\mathbb{R}[[z]] = \prod_{k\geq 0} \mathbb{R}\, z^k \cong \prod_{k\geq 0} \mathbb{R}\, v_k = \mathcal{A}$$

and an exponential generating function is a $\mathcal{U}$-generating function when $\mathcal{U}$ is taken to be the basis $\{w_k = z^k/k!, k \geq 0\}$ in the complete filtered algebra

$$\mathbb{R}[[z]] = \prod_{k\geq 0} \mathbb{R}\, \frac{z^k}{k!} \cong \prod_{k\geq 0} \mathbb{R}\, w_k = \mathcal{B}.$$

In both cases, the complete filtration on $\mathbb{R}[[z]]$ is given by the ideals $z^n\mathbb{R}[[z]]$ which is also $u_k\mathcal{A}$ (resp. $v_k\mathcal{B}$).

Of course such a formal definition in such generality is of little use unless

(1) it can be naturally related to the ordinary generating functions or exponential generating functions;
(2) it is useful in the study of number sequences and number families.

We will show that free Rota-Baxter algebras do give a generalization with these conditions. In Section 2 we review the construction of free commutative Rota–Baxter algebras and show that their completions give a large class of complete filtered algebras. In Section 3 we show that such complete free Rota-Baxter algebras give the exponential generating functions in a very special case. We then show how other instances of complete free Rota-Baxter algebras give rise to interesting generating functions of sequences of numbers or multi-indexed families of numbers, such as the Stirling numbers of the second kind and partition numbers.

2. Complete free commutative Rota-Baxter algebras

Let $\lambda \in \mathbb{R}$ be a constant. A **Rota–Baxter algebra** of weight λ is a pair (R, P) where R is a unitary $\mathbf{k}$-algebra and $P : R \to R$ is a linear operator such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \tag{2.1}$$

for any $x, y \in R$. Often $\theta = -\lambda$ is used, especially in the physics literature. Let (R, P) and (R', P') be Rota-Baxter algebras of weight λ. A **Rota-Baxter algebra homomorphism** f from (R, P) to (R', P') is an algebra homomorphism $f : R \to R'$ such that $f \circ P = P' \circ f$.

The study of Rota-Baxter algebras was started by G. Baxter in 1960 and was popularized largely by Rota and his school in 1960s and 70s and again in 1990s. In recently years, there have been several interesting developments of Rota–Baxter algebras in theoretical physics and mathematics, including quantum field theory, Yang–Baxter equations, shuffle products, operads, Hopf algebras, combinatorics and number theory. The most prominent of these is the work of Connes and Kreimer in their Hopf algebraic approach to renormalization theory in perturbative quantum field theory [**8, 9, 13**].

2.1. Free commutative Rota-Baxter algebras. We recall the construction of free commutative Rota-Baxter algebras in terms of mixable shuffles. See [**20, 21, 19**] for details.

Given a commutative algebra C which will often be taken as $\mathbb{R}$, $\lambda \in C$, and a C-algebra A, the free commutative Rota-Baxter C-algebra on A is defined to be a Rota-Baxter C-algebra $(Ш_{C,\lambda}(A), P_A)$ together with a C-algebra homomorphism $j_A : A \to Ш_{C,\lambda}(A)$ with the property that, for any Rota-Baxter C-algebra (R, P) and any C-algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter C-algebra homomorphism $\tilde{f} : (Ш_{C,\lambda}(A), P_A) \to (R, P)$ such that $j_A \circ \tilde{f} = f$ as C-algebra homomorphisms.

One realization of this free commutative Rota-Baxter algebra is given by the mixable shuffle Rota-Baxter algebra. The mixable shuffle Rota-Baxter algebra is a pair $(Ш_{C,\lambda}(A), P_A)$, where $Ш_{C,\lambda}(A)$ is a C-algebra in which

- the C-module structure is given by the direct sum
$$\bigoplus_{n=1}^{\infty} A^{\otimes n}, \quad \text{where } A^{\otimes n} = \underbrace{A \otimes_C \ldots \otimes_C A}_{n-\text{factors}};$$
- the multiplication is given by the augmented mixable shuffle product $\diamond$, recursively defined on $A^{\otimes m} \otimes A^{\otimes n}$ by
$$\begin{aligned} a_0 \diamond (b_0 \otimes b_1 \otimes \ldots \otimes b_n) &= a_0 b_0 \otimes b_1 \otimes \ldots \otimes b_n, \\ (a_0 \otimes a_1 \otimes \ldots \otimes a_m) \diamond b_0 &= a_0 b_0 \otimes a_1 \otimes \ldots \otimes a_m, \ a_i, \, b_j \in A^{\otimes 1} = A, \end{aligned}$$
and

$$\begin{aligned} &(a_0 \otimes a_1 \otimes \ldots \otimes a_m) \diamond (b_0 \otimes b_1 \otimes \ldots \otimes b_n) \\ = \ & (a_0 b_0) \otimes \big((a_1 \otimes \ldots \otimes a_m) \diamond (1 \otimes b_1 \otimes \ldots \otimes b_n)\big) \\ & + (a_0 b_0) \otimes \big((1 \otimes a_1 \otimes \ldots \otimes a_m) \diamond (b_1 \otimes \ldots \otimes b_n)\big) \\ & + \lambda a_0 b_0 \otimes \big((a_1 \otimes \ldots \otimes a_m) \diamond (b_1 \otimes \ldots \otimes b_n)\big), \ a_i, \ b_j \in A \end{aligned} \tag{2.2}$$

with the convention that
$$a \diamond (1 \otimes b) = a \otimes b, \ (1 \otimes a) \diamond b = b \otimes a, \ a \diamond b = ab, \quad \text{for } a, b \in A.$$
The Baxter operator P_A is defined by
$$P_A(a_1 \otimes \ldots \otimes a_m) = 1 \otimes a_1 \otimes \ldots \otimes a_m, \ a_1 \otimes \ldots \otimes a_m \in A^{\otimes m}, \ m \geq 1.$$
Since the mixable shuffle product is compatible with the product on A, we will suppress the notation $\diamond$. We will also express C and λ from $Ш_{C,\lambda}(A)$ when there is not danger of confusion.

Note that assuming P_A is a Rota-Baxter operator and thus satisfies Eq. (2.1), then Eq. (2.2) follows. For example, we have
$$\begin{aligned} &(a_0 \otimes a_1 \otimes a_2)(b_0 \otimes b_1) \\ = \ & a_0 b_0 \otimes ((a_1 \otimes a_2)(1 \otimes b_1) + b_1(1 \otimes a_1 \otimes a_2) + \lambda (a_1 \otimes a_2) b_1) \\ = \ & a_0 b_0 \otimes (a_1 \otimes (a_2(1 \otimes b_1) + (b_1(1 \otimes a_2)) + \lambda a_2 b_1) + b_1 \otimes a_1 \otimes a_2 + a_1 b_1 \otimes a_2) \\ = \ & a_0 b_0 \otimes (a_1 \otimes a_2 \otimes b_1 + a_1 \otimes b_1 \otimes a_2 + \lambda a_1 \otimes a_2 b_1 + b_1 \otimes a_1 \otimes a_2 + \lambda a_1 b_1 \otimes a_2). \end{aligned}$$

THEOREM 2.1. **(**[**20**, Theorem 4.1]**)** *For any C-algebra A, $(Ш(A), P_A)$, together with the natural embedding $j_A : A \to Ш(A)$, is a free Baxter C-algebra on A (of weight λ) in the sense that the triple $(Ш(A), P_A, j_A)$ satisfies the following universal property: For any Baxter C-algebra (R, P) and any C-algebra map $\varphi : A \to R$, there exists a unique Baxter C-algebra homomorphism $\tilde{\varphi} : (Ш(A), P_A) \to (R, P)$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "j_A"] \arrow[dr, "\varphi"'] & Ш(A) \arrow[d, "\tilde{\varphi}"] \\ & R \end{tikzcd}$$

commutes.

Alternatively, $Ш(A)$ can defined to be the tensor product algebra $A \otimes Ш^+(A)$ where the multiplication on $Ш^+(A) = \bigoplus_{k \geq 1} A^{\otimes k}$ is given in the explicit form by the mixable shuffle product [**20**] and in the recursive form by a generalization [**10, 23**] of the **quasi-shuffle algebra** defined by Hoffman [**26**] in the study of multiple zeta values.

Quasi-shuffle is also known as harmonic product [**25**] and coincides with the stuffle product [**2, 4**] in the study of multiple zeta values. Variations of the stuffle product have also appeared in [**6, 14**]. It is shown [**10**] to be the same as the mixable shuffle product [**20, 21**] which is also call overlapping shuffles [**24**] and generalized shuffles [**16**], and can be interpreted in terms of Delannoy paths [**1, 15, 28**].

2.2. Complete free Rota-Baxter algebras. For a given commutative algebra A, it is easy to see that the submodules

$$\mathrm{Fil}^k\ Ш(A) := \bigoplus_{i\geq k} A^{\otimes i},\ i\geq 0$$

of $Ш(A)$ are ideals of $Ш(A)$. They are in fact Rota-Baxter ideals of $Ш(A)$ in the sense that $P_A(\mathrm{Fil}^k\ Ш(A)) \subseteq \mathrm{Fil}^k\ Ш(A)$. Further, $\cap_{k\geq 0}\mathrm{Fil}^k\ Ш(A)=0$. Thus

$$\widehat{Ш}(A) := \varprojlim\ Ш(A)/\mathrm{Fil}^k\ Ш(A) \cong \prod_{k\geq 0} A^{\otimes k}$$

is a complete filtered algebra and contains $Ш(A)$ as a subalgebra. It coincides with the complete free commutative Rota-Baxter algebra defined in [**21**].

3. Generating functions from Rota-Baxter algebras

We first interpret exponential generating functions in terms of free commutative Rota-Baxter algebras. We then consider other cases where free commutative Rota-Baxter algebras give rise to generating functions.

3.1. Connection with exponential power series. Let $A=\mathbb{R}$. Then

$$Ш_\lambda(A) = Ш_\lambda(\mathbb{R}) = \oplus_{k\geq 1}\mathbb{R}^{\otimes k} = \oplus_{k\geq 1}\mathbb{R}\,\mathbf{1}_k, \tag{3.1}$$

where $\mathbf{1}_k=\underbrace{1\otimes\cdots\otimes 1}_{(k+1)-\mathrm{terms}}$. The augmented shuffle product in this special case is

$$\mathbf{1}_m \diamond \mathbf{1}_n = \sum_{i=0}^{m}\lambda^k \binom{m+n-k}{m}\binom{m}{k}\mathbf{1}_{m+n-k}. \tag{3.2}$$

When $\lambda=0$, we have

$$\mathbf{1}_m \diamond \mathbf{1}_n = \binom{m+n}{m}\mathbf{1}_{m+n}.$$

We then have $\widehat{Ш}_\lambda(\mathbb{R}) = \prod_{k\geq 0}\mathbb{R}\mathbf{1}_k$. This is also the cofree differential algebra [**27**] with the differential operator $d(x_n)=x_{n-1}, d(1)=0$.

Denote $x_n=x^n/n!$ (divided powers). Then as an algebra,

$$\mathbb{R}[x]=\oplus_{n\geq 0}\mathbb{R}x_n$$

with multiplication given by $x_m x_n = \binom{m+n}{m} x_{m+n}$. This extends to an isomorphism

$$\mathbb{R}[[x]] \to \widehat{Ш}(\mathbb{R}) = \prod_{k\geq 0}\mathbb{R}\,\mathbf{1}_k, \quad x_k\mapsto \mathbf{1}_k,\ k\geq 0.$$

Through this isomorphism the theory of exponential generating function is translated to a theory of generating functions in $\widehat{Ш}(\mathbb{R})$, and can be generalized to $\widehat{Ш}_\lambda(A)$ for other algebras A and other weight λ. We will next demonstrate how this can be done in several cases. More systematic results will be provided in the full paper.

3.2. λ-exponential generating functions. We first consider free Rota-Baxter algebra $Ш(\mathbb{R})$ on $\mathbb{R}$ of weight 1. It is given by Eq. (3.1) with product given by Eq. (3.2) with $\lambda=1$.

We quote from [**32**], Proposition 5.1.1, the following simple yet fundamental property of exponential generating functions which underlies the prominent role played by these generating functions. See [**32**] for details. For any function $f:\mathbb{N}\to\mathbb{R}$, let $E_f(x)=\sum_{k\geq 0} f(n)x^n/n!$ be the exponential generating function of f.

THEOREM 3.1. *Let $\#Y$ be the cardinality of a finite set Y. Given functions $f,g:\mathbb{N}\to\mathbb{R}$, define a new function $h:\mathbb{N}\to\mathbb{R}$ by the rule*

$$h(\#X) = \sum_{(S,T)} f(\#S)g(\#T),$$

where X is a finite set, and where (S,T) ranges over all weak ordered partitions of X into two blocks, i.e., $S\cap T=\emptyset$ and $S\cup T=X$. Then

$$E_h(x)=E_f(x)E_g(x). \tag{3.3}$$

We prove the following generalization of Theorem 3.1. For $f:\mathbb{N}\to\mathbb{R}$, call $E_f^{(\lambda)}(x):=\sum_{k\geq 0} f(n)\mathbf{1}_k\in \widehat{Ш}_\lambda(\mathbb{R})$ be the **λ-exponential generating function of f**. When $\lambda=0$, this recovers the exponential generating function.

Theorem 3.2. *Given functions $f,g:\mathbb{N}\to\mathbb{R}$, define a new function $h:\mathbb{N}\to\mathbb{R}$ by the rule*

$$h(\#X)=\sum_{(S,T)}\lambda^{\#(S\cap T)}f(\#S)g(\#T),$$

where X is a finite set, and where (S,T) ranges over all ordered subsets of X such that $S\cup T=X$. Then

$$E_h^{(\lambda)}(x)=E_f^{(\lambda)}(x)E_g^{(\lambda)}(x)\in \widehat{Ш}_\lambda(\mathbb{R}). \tag{3.4}$$

Proof. We have

$$\begin{aligned} E_f^{(\lambda)}(x)E_g^{(\lambda)}(x) &= \big(\sum_{m\geq 0} f(m)\mathbf{1}_m\big)\big(\sum_{n\geq 0} g(n)\mathbf{1}_n\big)\\ &= \sum_{m,n} f(m)g(n)\mathbf{1}_m\mathbf{1}_n\\ &= \sum_{m,n} f(m)g(n)\sum_{i=0}^{m}\lambda^i\binom{m+n-k}{m}\binom{m}{k}\mathbf{1}_{m+n-k}\end{aligned}$$

by Eq. (3.2). Note that $\binom{m+n-k}{m}\binom{m}{k}=\binom{m+n-k}{k,m-k,m+n-2k}$. So setting $u=m+n-k$, we have

$$\begin{aligned} E_f^{(\lambda)}(x)E_g^{(\lambda)}(x) &= \sum_{u=0}^{\infty}\Big(\sum_{m=0}^{u}\sum_{k=0}^{m}\lambda^k\binom{u}{k,m-k,u-m}f(m)g(u-m+k)\Big)\mathbf{1}_k\\ &= \sum_{u=0}^{\infty}\Big(\sum_{(u_1,u_2,u_3)\in\mathbb{N}^3,u_1+u_2+u_3=u}\lambda^{u_1}\binom{u}{u_1,u_2,u_3}f(u_1+u_2)g(u_1+u_3)\Big)\mathbf{1}_u.\end{aligned}$$

Now the theorem follows since, for given $(u_1,u_2,u_3)\in\mathbb{N}^3$, $\binom{u}{u_1,u_2,u_3}$ is the number of ways of partitioning a size u set into subsets of size u_1,u_2 and u_3. It is also the number of ways of taking subsets S of size u_1+u_2 and T of size u_1+u_3 of a X of size $u_1+u_2+u_3$ such that $S\cup T=X$ and $\#(S\cap T)=u_1$. □

3.3. Stirling numbers. We next give an example of weight 1 exponential generating functions. Stirling numbers of the first kind, $s(n,k)$, and the second kind, $S(n,k)$, $n,\ k\in\mathbb{N}$, are defined by

$$(x)_n=\sum_{k=0}^{n}s(n,k)x^k,\quad x^n=\sum_{k=0}^{n}S(n,k)(x)_k.$$

Stirling numbers of the second kind have the recursive formula

$$S(n+1,k+1)=S(n,k)+(k+1)S(n,k+1),\ n,\ k\geq 0$$

with $S(0,0)=1$, $S(n,0)=S(0,k)=0$ for $n,\ k\geq 1$. It follows that $S(n,1)=S(n,n)=1, n\geq 1$. The importance of these numbers in combinatorics is well-known [**32**]. For instance, $S(n,k)$ is the number of partitions of n objects into k non-empty cells. They have the generating function

$$\exp[t(e^u-1)]=\sum_{n=0}^{\infty}\sum_{k=0}^{n}S(n,k)t^ku^n/n!. \tag{3.5}$$

THEOREM 3.3. *We have the generating function*

$$e^{(1\otimes 1)u} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S(n,k)k!\mathbf{1}_k u^n/n! \in \widehat{Ш}_1(\mathbb{R})[[u]].$$

It has a much simpler form compared with Eq. (3.5). $k!S(n,k)$ is the number of ways to put n different objects into m different non-empty cells, and is the number of surjective maps from $\{1,\ldots,n\}$ to $\{1,\ldots,k\}$.

4. Free commutative Rota-Baxter algebras of weight 0

Let X be a set. Then the free commutative Rota-Baxter algebra of weight 0 $Ш_0(X) := Ш_0(\mathbb{R}[X])$ is the vector space

$$Ш_0(X) = \oplus_{n\geq 0}\mathbb{R}X^n \quad (\text{ identifying } x_1\otimes\cdots\otimes x_n \leftrightarrow (x_1,\cdots,x_n))$$

with the product defined by Eq. (2.2) with $\lambda = 0$. For example,

$$(a_0\otimes a_1)(b_0\otimes b_1) = (a_0b_0\otimes a_1\otimes b_1) + (a_0b_0\otimes b_1\otimes a_1).$$

In general

$$(a_0\otimes a_1\otimes\cdots\otimes a_m)(b_0\otimes b_1\otimes\cdots\otimes b_n) = a_0b_0\otimes\Big(\sum\big(\text{shuffles of } (a_1\otimes\cdots\otimes a_m) \text{ and } (b_1\otimes\cdots\otimes b_n)\big)\Big).$$

We have the following examples of generating functions in the completion $\widehat{Ш}_0(X)$ of $Ш_0(X)$.

$$\frac{1}{1-1\otimes x} = \sum_{n\geq 0}(1\otimes x)^n = \sum_{n\geq 0} n!(1\otimes x^{\otimes n}).$$

$$\frac{1}{1-1\otimes x\otimes x} = \sum(1\otimes x\otimes x)^n = \sum\frac{(2n)!}{2^n}(1\otimes x^{\otimes(2n)}).$$

5. Free commutative Rota-Baxter algebras of weight 1

Let

$$Ш_1(x) := Ш_1(\mathbb{R}[x]) = \bigoplus_{k=1}^{\infty}\mathbb{R}[x]^{\otimes k} = \bigoplus_{k=1}^{\infty}\mathbb{R}\{x^m \mid m\geq 1\}^k.$$

Recall that the product in $Ш_1(x)$ and hence in $\widehat{Ш}_1(x)$ is given by the mixable shuffle product. For example,

$$(a_0\otimes a_1\otimes a_2)(b_0\otimes b_1) = a_0b_0\otimes\big(a_1\otimes a_2\otimes b_1 + a_1\otimes b_1\otimes a_2 + a_1\otimes a_2b_1 + b_1\otimes a_1\otimes a_2 + a_1b_1\otimes a_2\big).$$

Consider powers $(1\otimes x)^n \in Ш_1(x)$. We have

$$\begin{aligned}
(1\otimes x)^2 &= 2(1\otimes x\otimes x) + 1\otimes x^2,\\
(1\otimes x)^3 &= 6(1\otimes x\otimes x\otimes x) + 3(1\otimes x\otimes x^2)\\
&\quad +3(1\otimes x^2\otimes x) + (1\otimes x^3).\\
(1\otimes x)^4 &= 24(1\otimes x\otimes x\otimes x\otimes x) + 12(1\otimes x\otimes x\otimes x^2) + 12(1\otimes x\otimes x^2\otimes x)\\
&\quad +12(1\otimes x^2\otimes x\otimes x) + 6(1\otimes x^2\otimes x^2) + 4(1\otimes x\otimes x^3)\\
&\quad +4(1\otimes x^3\otimes x) + (1\otimes x^4).
\end{aligned}$$

Note that the terms on the right hand side for $(1\otimes x)^3$ correspond to the ordered partitions $n = 1+1+1 = 1+2 = 2+1 = 3$ and the coefficients are the multi-nomial coefficients $\binom{3}{1,1,1}, \binom{3}{1,2}, \binom{3}{2,1}, \binom{3}{3}$. The same is true for $(1\otimes x)^{\otimes 4}$.

In general, for $I = (i_1,\cdots,i_k)\in\mathbb{N}^k$, denote $x^{\otimes I} = x^{i_1}\otimes\cdots\otimes x^{i_k}$. So $x^{\otimes(2,3)} = x^2\otimes x^3$. Let $\mathfrak{I} = \mathbb{N}_{>0}^k$. For $I\in\mathfrak{I}$, define the norm of I to be $|I| = \sum_{s=1}^k i_s$, length $\ell(I) = k$ and $\binom{n}{I} = \binom{n}{i_1,\cdots,i_k}$.

THEOREM 5.1. *We have the generating function*

$$\frac{1}{1-(1\otimes x)} = \sum_{n=0}^{\infty}(1\otimes x)^n = \sum_{n=0}^{\infty}\sum_{|I|=n}\binom{n}{I}(1\otimes x^{\otimes I}) \in \widehat{\text{Ш}}_1(x).$$

Here we have used the convention that, for $I=\emptyset$, take $|I|=0$, $\binom{|I|}{I}=1, (1\otimes x)^I = 1\otimes x^{\otimes I}=1$.

PROOF. It follows from the following facts [**19**].

(1) $(1\otimes x)^n = \sum_{|I|=n}\binom{n}{I}(1\otimes x^{\otimes I})$.
(2) The sum is over the set of ordered partitions of n.
(3) $\displaystyle\sum_{|I|=n,\ell(I)=k}\binom{n}{I} = k!S(n,k)$.

□

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, RUTGERS UNIVERSITY, NEWARK, NJ 07102
E-mail address: `liguo@newark.rutgers.edu`
URL: `http://newark.rutgers.edu/~liguo`