Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Combinatorial realisation of Hall-Littlewood polynomials at $t = -1$

R.C. King and A.M. Hamel

Abstract. A generalisation is given of Hall-Littlewood polynomials, $P_\lambda(\mathbf{x};t)$, to the case of an arbitrary simple Lie algebra $\mathbf{g}$. For $t=0$ and $t=1$ these polynomials are just the appropriate Weyl group symmetric monomial function and Weyl's character of an irreducible representation, respectively. Here the case $t=-1$ is discussed in some detail. First a factorisation result is established for all $\mathbf{g}$, and then a combinatorial realisation of $P_\lambda(\mathbf{x};-1)$ is offered for both $gl(n)$ and $sp(2n)$ in terms of certain Q-functions that are shown to be Weyl group symmetric. The Q-functions are defined in terms of primed shifted tableaux. In addition a lattice path interpretation is provided in the form of determinantal expansions.

Résumé. Nous donnons un généralisation des polynômes Hall-Littlewood, $P_\lambda(\mathbf{x};t)$, à une algèbre Lie simple arbitraire $\mathbf{g}$. À $t=0$ et $t=1$ ces polynômes ne sont que la fonction symmétrique monomiale du groupe Weyl et le caractère Weyl d'une représentation irréductible. Nous discutons ici en détail le cas $t=-1$. Nous établissons d'abord un résultat de factorisation pour tout choix de $\mathbf{g}$, et ensuite nous dérivons une réalisation combinatoire de $P_\lambda(\mathbf{x};-1)$ pour $gl(n)$ et $sp(2n)$ en termes de certaines Q-fonctions qui sont symmétriques dans le groupe Weyl. Les Q-fonctions sont définies en termes des tableaux primés et décalés. De plus, nous donnons une interprétation de chemins de treillis sous forme d'expansion de déterminant.

1. Introduction

Let $\mathbf{g}$ be a simple Lie algebra of rank r, with Cartan subalgebra $\mathbf{h}$ and dual $\mathbf{h}^*$. Let the corresponding Weyl group be W and let the sets of positive roots and simple roots be denoted by Δ^+ and Π, respectively. For each $w \in W$, let $\ell(w)$ be the minimal length of w when expressed as a word in the generators, w_i with $i = 1, 2, \ldots, r$, of the Weyl group W.

An arbitrary weight vector, $\mu \in \mathbf{h}^*$, is said to be integral, dominant integral or strongly dominant integral if $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}$, $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}_{>0}$ for all $\alpha \in \Pi$. In such cases we write $\mu \in P$, P^+ or P^{++}, respectively. The Weyl vector $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$ is such that $\langle \rho, \alpha \rangle = 1$ for all $\alpha \in \Pi$. It follows that for each $\mu \in P^{++}$ we can write $\mu = \lambda + \rho$ with $\lambda \in P^+$.

Let $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be a sequence of orthonormal vectors spanning $\mathbf{h}^*$ and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with $x_k = e^{\epsilon_k}$ for $k = 1, 2, \ldots, n$. Then for each $\mu \in P^+$ we have $e^\mu = \mathbf{x}^\mu = (x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n})$. In this situation the action of the Weyl group is defined by $w(e^\mu) = e^{w(\mu)} = \mathbf{x}^{w(\mu)}$. The corresponding monomial Weyl-symmetric function is defined by:

$$m_\mu(\mathbf{x}) = \sum_{w \in W^\mu = W/W_\mu} \mathbf{x}^{w(\mu)}\,, \tag{1.1}$$

where $W_\mu = \{w \in W \,|\, w(\mu) = \mu\}$, and W^μ is a set of minimal length left coset representatives of W with respect to W_μ.

Macdonald [**6, 7**] introduced a two-parameter family of orthogonal polynomials, $P_\mu(\mathbf{x};q,t)$ with $\mu \in P^+$, uniquely characterised by the conditions:

$$\begin{array}{ll} \text{(i)} & P_\mu(\mathbf{x};q,t) = m_\mu(\mathbf{x}) + \sum_{\nu<\mu} a_{\nu\mu}(q,t)\, m_\nu(\mathbf{x}) \quad \text{with} \quad a_{\nu\mu} \in \mathbb{Q}(q,t) \\ \text{(ii)} & \langle P_\mu(\mathbf{x};q,t)\,,\, P_\nu(\mathbf{x};q,t)\rangle = 0 \quad \text{if} \quad \nu \neq \mu\,, \end{array} \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 17B10; Secondary 05E10.
Key words and phrases. algebraic combinatorics, symmetric functions, ECO systems, enumeration, partitions.

for a suitably defined partial order relation $<$ and symmetric scalar product $\langle \cdot\,,\,\cdot\rangle$ [**6, 7**].

Here we confine our attention to the case $q \to 0$, for which the Macdonald polynomials $P_\mu(\mathbf{x};0,t)$ coincide with the Hall-Littlewood polynomials $P_\mu(\mathbf{x};t)$. For these cases there exists the explicit formula [**5, 6, 7**]:

$$P_\mu(\mathbf{x};t) = \frac{1}{W_\mu(t)} \sum_{w\in W} w\left(\mathbf{x}^\mu \prod_{\alpha\in\Delta^+}\left(\frac{1-t\,\mathbf{x}^{-\alpha}}{1-\mathbf{x}^{-\alpha}}\right)\right), \tag{1.3}$$

with

$$W_\mu(t) = \sum_{w\in W_\mu} t^{\ell(w)}\,. \tag{1.4}$$

The special case $t=1$ gives the monomial Weyl-symmetric functions

$$P_\mu(\mathbf{x};1) = \frac{1}{|W_\mu|}\sum_{w\in W} w(\mathbf{x}^\mu) = \sum_{w\in W^\mu = W/W_\mu} \mathbf{x}^{w(\mu)} = m_\mu(\mathbf{x})\,. \tag{1.5}$$

For $t=0$ we obtain

$$P_\mu(\mathbf{x};0) = \sum_{w\in W} w\left(\mathbf{x}^\mu \prod_{\alpha\in\Delta^+}\left(\frac{1}{1-\mathbf{x}^{-\alpha}}\right)\right) = \sum_{w\in W} w\left(\frac{\mathbf{x}^{\mu+\rho}}{a_\rho(\mathbf{x})}\right), \tag{1.6}$$

where $a_\rho(\mathbf{x})$ is Weyl's denominator function

$$a_\rho(\mathbf{x}) = \mathbf{x}^\rho \prod_{\alpha\in\Delta^+}\left(1-\mathbf{x}^{-\alpha}\right) = \prod_{\alpha\in\Delta^+}\left(\mathbf{x}^{\alpha/2}-\mathbf{x}^{-\alpha/2}\right). \tag{1.7}$$

However, $w(a_\rho(\mathbf{x})) = (-1)^{\ell(w)}\, a_\rho(\mathbf{x})$, so that

$$P_\mu(\mathbf{x};0)) = \frac{1}{a_\rho(\mathbf{x})}\sum_{w\in W}(-1)^{\ell(w)}\mathbf{x}^{w(\mu+\rho)} = \frac{a_{\mu+\rho}(\mathbf{x})}{a_\rho(\mathbf{x})}\,. \tag{1.8}$$

This serves to define $P_\mu(\mathbf{x};0)$ for all weights μ, but if μ is an integral dominant weight, that is $\mu\in P^+$, then this is nothing other than Weyl's formula for the character of the irreducible representation V^μ of $\mathbf{g}$ having highest weight μ. That is to say, for each $\mu\in P^+$, we have

$$\mathrm{ch}\,V^\mu(\mathbf{x}) = P_\mu(\mathbf{x};0)\,. \tag{1.9}$$

The special case $\mu=0$, for which $\mathrm{ch}\,V^0=1$, gives Weyl's denominator identity:

$$a_\rho(\mathbf{x}) = \mathbf{x}^\rho\prod_{\alpha\in\Delta^+}\left(1-\mathbf{x}^{-\alpha}\right) = \sum_{w\in W}(-1)^{\ell(w)}\mathbf{x}^{w(\rho)}\,. \tag{1.10}$$

It follows from the second form of $a_\rho(\mathbf{x})$ given here that $a_{2\rho}(\mathbf{x}) = a_\rho(\mathbf{x^2})$, where $\mathbf{x^2}$ is obtained from $\mathbf{x}$ by squaring every component. This implies in turn that

$$\mathrm{ch}\,V^\rho(\mathbf{x}) = P_\rho(\mathbf{x};0) = \frac{a_{2\rho}(\mathbf{x})}{a_\rho(\mathbf{x})} = \frac{a_\rho(\mathbf{x^2})}{a_\rho(\mathbf{x})} = \frac{\mathbf{x}^{2\rho}}{\mathbf{x}^\rho}\prod_{\alpha\in\Delta^+}\left(\frac{1-\mathbf{x}^{-2\alpha}}{1-\mathbf{x}^{-\alpha}}\right) = \mathbf{x}^\rho\prod_{\alpha\in\Delta^+}\left(1+\mathbf{x}^{-\alpha}\right). \tag{1.11}$$

The final expression can be thought of as a rather simple deformation of Weyl's denominator function (1.7).

The interesting case $t=-1$ of $P_\mu(\mathbf{x};t)$ defies, in general, such a simple analysis, if for no other reason than the singular nature of $1/W_\mu(-1)$ preventing in some cases the direct use of the formula (1.3). However, if μ is strongly integral dominant, that is $\mu\in P^{++}$ then $w(\mu)=\mu$ if and only if $w=1$, the identity element of W. In such a case $W_\mu(-1)=1$. This allows us to see from (1.3) that for all $\mu\in P^{++}$

$$\begin{aligned} P_\mu(\mathbf{x};-1) &= \sum_{w\in W} w\left(\mathbf{x}^\mu\prod_{\alpha\in\Delta^+}\left(\frac{1+\mathbf{x}^{-\alpha}}{1-\mathbf{x}^{-\alpha}}\right)\right) = \sum_{w\in W} w\left(\mathbf{x}^\mu\frac{\mathbf{x}^{2\rho}}{(\mathbf{x}^\rho)^2}\prod_{\alpha\in\Delta^+}\left(\frac{(1-\mathbf{x}^{-2\alpha})}{(1-\mathbf{x}^{-\alpha})^2}\right)\right) \\ &= \frac{a_\rho(\mathbf{x^2})}{(a_\rho(\mathbf{x}))^2}\sum_{w\in W}(-1)^{\ell(w)}\mathbf{x}^{w(\mu)} = \frac{1}{a_\rho(\mathbf{x})}\sum_{w\in W}(-1)^{\ell(w)}\mathbf{x}^{w(2\rho)}\;\frac{1}{a_\rho(\mathbf{x})}\sum_{w\in W}(-1)^{\ell(w)}\mathbf{x}^{w(\mu)}\,, \end{aligned} \tag{1.12}$$

where use has been made of (1.10) with $\mathbf{x}$ replaced by $\mathbf{x^2}$. If we now set $\mu=\lambda+\rho$, with $\lambda\in P^+$ by virtue of our assumption that $\mu\in P^{++}$, it then follows from (1.8) and (1.9) that

$$P_{\lambda+\rho}(\mathbf{x};-1) = P_\rho(\mathbf{x};0)\;P_\lambda(\mathbf{x};0) = \mathrm{ch}\,V^\rho(\mathbf{x})\;\mathrm{ch}\,V^\lambda(\mathbf{x})\,. \tag{1.13}$$

This identity is true for all $\lambda \in P^+$.

In the case $\mathbf{g} = sl(n)$ or $gl(n)$ the above definitions are such that $P_\lambda(\mathbf{x};1) = m_\lambda(\mathbf{x})$ and $P_\lambda(\mathbf{x};0) = s_\lambda(\mathbf{x}) = \mathrm{ch}\, V^\lambda(\mathbf{x})$ for all partitions λ, where $m_\lambda(\mathbf{x})$ and $s_\lambda(\mathbf{x})$ are the classical monomial and Schur symmetric functions, respectively [**5**]. Moreover, for all partitions λ whose parts are distinct, $P_\lambda(\mathbf{x};-1)$ is nothing other than $P_\lambda(\mathbf{x})$, that is Schur's P-function [**5, 11**]. Both $s_\lambda(\mathbf{x})$ and $P_\lambda(\mathbf{x})$ have a combinatorial realisation in terms of semistandard and primed shifted semistandard Young tableaux, $T \in \mathcal{T}^\lambda(n)$ and $PST \in \mathcal{PST}^\lambda(n)$, respectively. In addition, the identity (1.13) in the $gl(n)$ case [**10, 13, 9, 5**] admits a proof by way of a bijection [**3**] between each $PST \in \mathcal{PST}^\lambda(n)$ and a pair (PD, T), consisting of a triangular primed tableau, $PD \in \mathcal{PD}^\rho(n)$, and a semistandard Young tableau, $T \in \mathcal{T}^\lambda(n)$. This is described in the next Section.

Much of this can be extended to the case $\mathbf{g} = sp(2n)$, as described in Section 3. In particular, for $sp(2n)$ there exists a combinatorial realisation both of $P_\lambda(\mathbf{x};0) = sp_\lambda(\mathbf{x}) = \mathrm{ch}\, V^\lambda$ for all partitions λ and of $P_\lambda(\mathbf{x};-1)$ for all partitions λ having n distinct non-vanishing parts. It is conjectured that this realisation extends to the case of all partitions λ having no more than n distinct parts. These realisations are by way of symplectic tableaux $T \in \mathcal{T}^\lambda(n,\overline{n})$ [**4**] and primed symplectic shifted tableaux, $PST \in \mathcal{PST}^\lambda(n,\overline{n})$ [**2, 3**], respectively. Moreover these realisations lead rather naturally to a bijective, tableaux based proof of (1.13) for $sp(2n)$ [**3**].

The combinatorial realisation of $P_\lambda(\mathbf{x};-1)$ by way of primed shifted tableaux does not immediately make clear the symmetry of these polynomials under the action of the Weyl group. This is discussed in Section 4, followed in Section 5 by a lattice path interpretation of $P_\lambda(\mathbf{x};-1)$ leading to determinantal expansions.

2. The $gl(n)$ case

For $\mathbf{g} = sl(n)$, which has rank $n-1$, it is convenient to work in an n-dimensional space spanned by Euclidean basis vectors ϵ_i for $i = 1,2,\ldots,n$ and project down to the subspace in which $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0$, wherever necessary. With respect to this basis $\Delta^+ = \{\epsilon_i - \epsilon_j \,|\, 1 \le i < j \le n\}$ and, taking the projection into account, $\rho = (n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n) = (n, n-1, \ldots, 1)$. Setting $x_k = e^{\epsilon_k}$ for $i = 1,2,\ldots,n$ we have $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with $x_1 x_2 \cdots x_n = 1$. For $\mathbf{g} = gl(n)$ we can use the same positive roots, the same ρ and the same $\mathbf{x}$ and just drop the constraint $x_1 x_2 \cdots x_n = 1$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of length $\ell(\lambda) \le n$ with $n - \ell(\lambda)$ trailing zeros. Then for both $gl(n)$ and $sl(n)$ there exits a finite-dimensional polynomial irreducible representation V^λ having highest weight λ. The partition λ also serves to define a Young diagram consisting of $\ell(\lambda)$ rows of boxes of lengths λ_i, for $i = 1,2,\ldots,\ell(\lambda)$, left adjusted to a vertical line.

Each semistandard tableaux $T \in \mathcal{T}^\lambda(n)$ is a filling of the boxes of F^λ with entries from the ordered set $\{1 < 2 < \cdots < n\}$ weakly increasing across rows and strictly increasing down columns. The weight, $\mathrm{wgt}(T)$, of such a tableau is given by $\kappa(T) = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ where κ_k is the number of entries k in T. For example, for $n = 5$ and $\lambda = (8,8,5,4,0)$ we have:

$$F^\lambda = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ \\ \cline{1-5} \ & \ & \ & \ \\ \cline{1-4}\end{array} \qquad T = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&2&3&3 \\ \hline 2&2&2&3&3&3&4&4 \\ \hline 3&3&4&4&5 \\ \cline{1-5} 5&5&5&5 \\ \cline{1-4}\end{array} \qquad \mathrm{wgt}(T) = (4,5,7,4,5)\,. \tag{2.1}$$

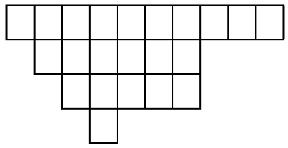
The corresponding Schur function then takes the form

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathcal{T}^\lambda(n)} \mathbf{x}^{\mathrm{wgt}(T)}\,, \tag{2.2}$$

where, quite generally for any $\mathbf{x}$ and κ, we have $\mathbf{x}^\kappa = x_1^{\kappa_1} x_2^{\kappa_2} \cdots x_n^{\kappa_n}$.

Now we turn to Schur's P and Q-functions, as described by Stembridge [**11**]. Let λ be a partition, all of whose parts are distinct. Such a partition defines a shifted Young diagram SF^λ in which the row lengths are as before the parts of λ, but the rows are left adjusted to a diagonal line.

Each primed semistandard shifted tableaux $PST \in \mathcal{QST}^\lambda(n)$ is a filling of the boxes of SF^λ with entries taken from the ordered set $\{1' < 1 < 2' < 2 < \cdots < n' < n\}$ weakly increasing across rows and down columns, with each column containing at most one k and each row at most one k' for all $k = 1,2,\ldots,n$. The weight, $\mathrm{wgt}(PST)$, of such a tableau is given by $\kappa(T) = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ where κ_k is the total number of entries k and k' in PST. For example, for $n=5$ and $\lambda=(10,6,5,1,0)$ we have

(2.3) $SF^\lambda =$  $\quad PST = \begin{array}{cccccccccc} \mathbf{1'} & 1 & 2' & 2 & 2 & 2 & 4' & 4 & 4 & \mathbf{5} \\ & \mathbf{2} & 2 & \mathbf{3'} & 3 & 3 & \mathbf{4'} & & & \\ & & \mathbf{4'} & 4 & 4 & \mathbf{5} & 5 & & & \\ & & & \mathbf{5} & & & & & & \end{array} \quad \mathrm{wgt}(PST) = (2,6,3,7,4)$

In general each ribbon strip $\mathrm{str}_k(PST)$, consisting of the set of all boxes having entries k or k', may be composed of more than one connected component. The only entries on such a strip for which there is any degree of freedom in specifying whether a given entry is k or k' are those entries that lie at the lower left-hand end of a connected component of the ribbon strip. These have been indicated in boldface type.

For each partition λ of length $\ell(\lambda) \le n$ whose parts are all distinct, the corresponding Schur Q-function takes the form:

$$Q_\lambda(\mathbf{x}) = \sum_{PST \in \mathcal{QST}^\lambda(n)} \mathbf{x}^{\mathrm{wgt}(PST)} . \tag{2.4}$$

Since every entry on the main diagonal may be either unprimed or primed, $Q_\lambda(\mathbf{x})$ is divisible by $2^{\ell(\lambda)}$ and Schur's P-function is defined to be

$$P_\lambda(\mathbf{x}) = 2^{-\ell(\lambda)}\ Q_\lambda(\mathbf{x}) = \sum_{PST \in \mathcal{PST}^\lambda(n)} \mathbf{x}^{\mathrm{wgt}(PST)} , \tag{2.5}$$

where $\mathcal{PST}^\lambda(n)$ is the subset of $\mathcal{QST}^\lambda(n)$ consisting of all those primed shifted tableaux PST with no primes on the main diagonal.

With this definition, we have [**11, 5**] the following combinatorial realisation of the $t=-1$ specialisation of $gl(n)$ or $sl(n)$ Hall-Littlewood polynomials:

Proposition 2.1. Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and let λ be a partition of length $\ell(\lambda)\le n$ all of whose parts are distinct, then the Hall-Littlewood polynomial $P_\lambda(\mathbf{x};t)$ of $gl(n)$ is such that

$$P_\lambda(\mathbf{x};-1) = P_\lambda(\mathbf{x}) . \tag{2.6}$$

Turning to (1.13), in the case of $gl(n)$ this factorisation was established by Macdonald [**5**], see Ex2, p259. A combinatorial proof requires the notion of primed triangular tableaux $PD \in \mathcal{QD}^\rho(n)$. Each such PD is a tableau of shifted shape SF^ρ, where $\rho=(n,n-1,\ldots,1)$, with entries taken from the set $\{1',1,2',2,\ldots,n',n\}$, in which each unprimed entry k lies in the kth row and each primed entry k' lies in the kth column. The weight of such a primed tableau is given as usual by $\mathrm{wgt}(PD)=\kappa=(\kappa_1,\kappa_2\ldots,\kappa_n)$ with κ_k equal to the total number of entries k or k'. For example:

$$PD = \begin{array}{cccccc} 1 & 2' & 1 & 4' & 5' & 6' \\ & 2 & 3' & 2 & 5' & 2 \\ & & 3' & 4' & 3 & 3 \\ & & & 4 & 5' & 6' \\ & & & & 5' & 5 \\ & & & & & 6 \end{array} \qquad \mathrm{wgt}(PD) = (2,4,4,3,5,3) \tag{2.7}$$

To enumerate all possible $PD \in \mathcal{QD}^\rho(n)$ it is merely necessary to note that at position (i,j) in the ith row and jth column the entry must be either i or j'. Since each entry k or k' contributes x_k to $\mathbf{x}^{\mathrm{wgt}(PD)}$, it follows that

$$\sum_{PD \in \mathcal{QD}^\rho(n)} \mathbf{x}^{\mathrm{wgt}(PD)} = \prod_{1\le i\le j\le n} (x_i + x_j) . \tag{2.8}$$

Since each entry on the main diagonal may be unprimed or primed, this expression may be divided by $2^{\ell(\rho)} = 2^n$ to give

$$\sum_{PD \in \mathcal{PD}^\rho(n)} \mathbf{x}^{\mathrm{wgt}(PD)} = \prod_{1\le i\le n} x_i \prod_{1\le i<j\le n} (x_i + x_j) . \tag{2.9}$$

where $\mathcal{PD}^\rho(n)$ is the subset of $\mathcal{QD}^\rho(n)$ consisting only of those primed triangular tableaux PD having no primed entries on the main diagonal.

However, with $\rho = (n, n-1, \ldots, 1)$ and positive roots $\alpha = \epsilon_i - \epsilon_j$ for $1 \le i < j \le n$, it follows from (1.11) and the above that

$$s_\rho(\mathbf{x}) = \mathbf{x}^\rho \prod_{1\le i<j\le n} (1 - x_i^{-1} x_j) = \prod_{1\le i\le n} x_i \prod_{1\le i<j\le n} (x_i + x_j) = \sum_{PD\in\mathcal{PD}^\rho(n)} \mathbf{x}^{\mathrm{wgt}(PD)} . \tag{2.10}$$

As mentioned in the Introduction, there exists a bijection [**3**] between $PST \in \mathcal{PST}^{\lambda+\rho}(n)$ and pairs (PD, T) with $PD \in \mathcal{PD}(n)$ and $T \in \mathcal{T}^\lambda(n)$. This is illustrated by the map:

$$PST = \begin{array}{ccccccccc} 1&1&1&2'&2&2&3&3&5 \\ &2&2&3'&3&4'&5'&5&6' \\ &&3&3&4'&4&5'&6& \\ &&&4&5'&5&5&& \\ &&&&5&6'&6&& \\ &&&&&6&&& \end{array} \quad \longrightarrow \quad (PD, T) = \begin{array}{cccccc} 1&2'&1&4'&5'&6' \\ &2&3'&2&5'&2 \\ &&3&4'&3&3 \\ &&&4&5'&6' \\ &&&&5&5 \\ &&&&&6 \end{array} \; , \; \begin{array}{ccc} 1&2&3 \\ 3&5&5 \\ 4&6& \\ 5&& \\ 6&& \end{array} \tag{2.11}$$

To effect this map one moves all primed entries k' north west into their own column, that is the kth column, taking k' in turn equal to $1', 2', \ldots, n'$ and dealing with each k' in turn from top to bottom. This is done by a sequence of jeu de taquin moves in the form of transpositions of k' with their north or west unprimed neighbours, i or j, respectively:

$$\begin{array}{cc} & j \\ i & k' \end{array} \longrightarrow \begin{array}{cc} & k' \\ i & j \end{array} \quad \text{if } i \le j\,; \qquad \begin{array}{cc} & j \\ i & k' \end{array} \longrightarrow \begin{array}{cc} & j \\ k' & i \end{array} \quad \text{if } i > j\,. \tag{2.12}$$

The chosen move is the unique one that ensures that the unprimed entries remain weakly increasing across rows and strongly increasing down columns. Of course k' must move west if it is already in the 1st row but not yet in the kth column. When it reaches the kth column it may still move north by means of a transposition of the type:

$$\begin{array}{c} i \\ k' \end{array} \longrightarrow \begin{array}{c} k' \\ i \end{array} \tag{2.13}$$

unless i is in the ith row, in which case the k' moves no further. It has been shown [**3**] that each k' always reaches a position off the main diagonal in the kth column and that in the first n columns all unprimed entries lie in their own row. Moreover, by reversing the individual jeu de taquin steps it has been shown that this map is bijective. Finally, it should be noted that the map is weight preserving since all the steps involve interchanging the positions of entries, without any alteration of their value.

By virtue of the combinatorial realisation of $P_{\lambda+\rho}(\mathbf{x})$ with $\ell(\lambda+\rho) = n$ that is implied by (2.6), and those of $s_\lambda(\mathbf{x})$ and $s_\rho(\mathbf{x})$ given in (2.2) and (2.10), respectively, it follows from the existence of the above bijection that for all partitions λ:

$$P_{\lambda+\rho}(\mathbf{x}; -1) = P_{\lambda+\rho}(\mathbf{x}) = s_\rho(\mathbf{x})\; s_\lambda(\mathbf{x})\,. \tag{2.14}$$

3. The $sp(2n)$ case

In the case $\mathbf{g} = sp(2n)$, which has rank n, we work in the n-dimensional space spanned by Euclidean basis vectors ϵ_i with $i = 1, 2, \ldots, n$. The set of positive roots is given by $\Delta^+ = \{\epsilon_i - \epsilon_j \,|\, 1 \le i < j \le n\} \cup \{2\epsilon_i \,|\, 1 \le i \le n\}$ and $\rho = n\epsilon_1 + (n-1)\epsilon_2 + \cdots + \epsilon_n = (n, n-1, \ldots, 1)$. We set $x_k = e^{\epsilon_k}$ and $\overline{x}_k = x_k^{-1} = e^{-\epsilon_k}$ for $k = 1, 2, \ldots, n$.

Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with no more than n non-vanishing parts specifies a finite-dimensional irreducible representation V^λ of $sp(2n)$. Its character is given by $\mathrm{ch}\, V^\lambda(\mathbf{x}) = sp_\lambda(\mathbf{x})$, where $sp_\lambda(\mathbf{x})$ may be referred to as a symplectic Schur function. This may be given a combinatorial definition as follows [**4**]. Each symplectic tableau $T \in \mathcal{T}^\lambda(n, \overline{n})$ is filling of the boxes of F^λ with entries from the ordered set $\{\overline{1} < 1 < \overline{2} < 2 < \cdots < \overline{n} < n\}$ weakly increasing across rows and strictly increasing down columns, with no entry k or $\overline{k}$

lying below the kth row. The weight of such a tableau is given by $\mathrm{wgt}(T) = \kappa(T) = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ with κ_k equal to the total number of entries k minus the total number of entries $\overline{k}$, for $k = 1, 2, \ldots, n$. For example for $n = 5$ and $\lambda = (4, 3, 3, 0, 0)$ we have

$$F^\lambda = \begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & \overline{2} & 4 \\ \hline 3 & \overline{4} & \overline{4} & \\ \cline{1-3} \overline{4} & 4 & 4 & \\ \cline{1-3} \end{array} \qquad T = \begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & \overline{2} & 4 \\ \hline 3 & \overline{4} & \overline{4} & \\ \cline{1-3} \overline{4} & 4 & 4 & \\ \cline{1-3} \end{array} \qquad \mathrm{wgt}(T) = (0, -1, 1, 0, 0)\,. \tag{3.1}$$

In terms of these symplectic tableaux and their weights, symplectic Schur functions take the form:

$$sp_\lambda(\mathbf{x}) = \sum_{T \in \mathcal{T}^\lambda(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(T)}\,. \tag{3.2}$$

Proceeding as in the $gl(n)$ case, we now turn to what we call symplectic Schur Q-functions defined in terms of primed shifted symplectic tableaux QST [**3**]. Let λ be partition, all of whose parts are distinct, and let SF^λ be the corresponding shifted Young diagram. Then each primed shifted symplectic tableau $QST \in \mathcal{QST}^\lambda(n,\overline{n})$ is a filling of the boxes of SF^λ with entries from the ordered set $\{\overline{1}' < \overline{1} < 1 < 1 < \overline{2}' < \overline{2} < 2' < 2 < \cdots < \overline{n}' < \overline{n} < n' < n\}$ weakly increasing across rows and down columns, with each column containing at most one k and at most one $\overline{k}$, each row at most one k' and at most one $\overline{k}'$, and the main diagonal at most one entry from the subset $\{\overline{k}', \overline{k}, k', k\}$ for each $k = 1, 2, \ldots, n$. The weight $\mathrm{wgt}(QST)$ of such a tableau is given by $\kappa = (\kappa_1, \kappa_2 \ldots, \kappa_n)$ where κ_k is the total number of entries k or k' minus the total number of entries $\overline{k}$ or $\overline{k}'$. For example, for $n = 5$ and $\lambda = (10, 6, 5, 1, 0)$ we have

(3.3)

$$SF^\lambda = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ & \ & \ & \ & \ \\ \hline \multicolumn{1}{c|}{} & \ & \ & \ & \ & \ & \ & \multicolumn{3}{c}{} \\ \cline{2-7} \multicolumn{2}{c|}{} & \ & \ & \ & \ & \ & \multicolumn{3}{c}{} \\ \cline{3-7} \multicolumn{3}{c|}{} & \ & \multicolumn{6}{c}{} \\ \cline{4-4} \end{array} \qquad QST = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline \overline{\mathbf{1}}' & 1 & \overline{\mathbf{2}}' & \overline{2} & \overline{2} & \mathbf{2} & 4' & 4 & 4 & \mathbf{5} \\ \hline \multicolumn{1}{c|}{} & \mathbf{2} & 2 & \mathbf{3}' & \overline{\mathbf{4}} & \overline{4} & \mathbf{4}' & \multicolumn{3}{c}{} \\ \cline{2-7} \multicolumn{2}{c|}{} & \mathbf{4}' & 4 & 4 & 4 & \overline{\mathbf{5}} & \multicolumn{3}{c}{} \\ \cline{3-7} \multicolumn{3}{c|}{} & \overline{\mathbf{5}} & \multicolumn{6}{c}{} \\ \cline{4-4} \end{array} \qquad \mathrm{wgt}(QST) = (0, 0, 1, 6, -1)$$

Once again boldface type has been used to indicate those entries for which there is a degree of freedom as to whether or not they are primed.

If we now let

$$Q_\lambda(\mathbf{x}, \mathbf{y}) = \sum_{QST \in \mathcal{QST}^\lambda(n,\overline{n})} \prod_{k=1}^{n} x_k^{(\#k \in QST)+(\#k' \in QST)}\ y_k^{(\#\overline{k} \in QST)+(\#\overline{k}' \in QST)}\,, \tag{3.4}$$

then our symplectic Schur Q-functions are defined by:

$$Q^{sp}_\lambda(\mathbf{x}) = Q_\lambda(\mathbf{x}, , \overline{\mathbf{x}}) = \sum_{QST \in \mathcal{QST}^\lambda(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(QST)}\,, \tag{3.5}$$

where $\mathrm{wgt}(QST) = (\#k \in QST) + (\#k' \in QST) - (\#\overline{k} \in QST) - (\#\overline{k}' \in QST)$. Once again taking into account the fact that each element on the main diagonal may be either unprimed or primed, we define symplectic Schur P-functions by

$$P^{sp}_\lambda(\mathbf{x}) = 2^{-\ell(\lambda)} Q_\lambda(\mathbf{x}, \overline{\mathbf{x}}) = \sum_{PST \in \mathcal{PST}^\lambda(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(PST)}\,, \tag{3.6}$$

where $\mathcal{PST}^\lambda(n,\overline{n})$ is the subset of $\mathcal{QST}^\lambda(n,\overline{n})$ consisting of those primed shifted tableaux PST having no primes on the main diagonal.

With this definition, we offer the following conjecture regarding the $t = -1$ specialisation of $sp(2n)$ Hall-Littlewood polynomials :

CONJECTURE 3.1. Let $\mathbf{x} = (x_1, x_2, \ldots, n)$ and let λ be a partition of length $\ell(\lambda) \leq n$ all of whose parts are distinct, then the Hall-Littlewood polynomial, $P_\lambda(\mathbf{x}; t)$, of $sp(2n)$ is such that:

$$P_\lambda(\mathbf{x}; -1) = P^{sp}_\lambda(\mathbf{x})\,. \tag{3.7}$$

It is far from obvious that the function $P_\lambda^{sp}(\mathbf{x})$ does, as claimed, coincides with the $t=-1$ specialisation of the $sp(2n)$ Hall-Littlewood polynomial $P_\lambda(\mathbf{x};t)$, or even that it is symmetric with respect to the Weyl group of $sp(2n)$, which includes both all the permutations of the components of $\mathbf{x}$ and any combination of the inversions $x_k \mapsto x_k^{-1}$ with $k \in \{1,2,\ldots,n\}$. However, in the case that λ is a partition with n non-vanishing distinct parts these properties can be established, thanks to the existence of the factorisation property (1.13).

To see how this comes about it is necessary to introduce, as our final set of tableaux, the set $\mathcal{QD}^\rho(n,\overline{n})$. Each $PD \in \mathcal{QD}^\rho(n,\overline{n})$ is a tableau of shifted shape SF^ρ where $\rho=(n,n-1,\ldots,1)$, with entries taken from the set $\{\overline{1}',\overline{1},1',1,\overline{2}',\overline{2},2',2,\cdots,\overline{n}',\overline{n},n',n\}$, in which each entry k or $\overline{k}$ lies in the kth row and each entry k' or $\overline{k}'$ lies in the kth column. The weight of such a primed tableau is given by $\mathrm{wgt}(PD)=\kappa=(\kappa_1,\kappa_2\ldots,\kappa_n)$, where κ_k equals the total number of entries k or k' minus the total number of entries $\overline{k}$ or $\overline{k}'$. Typically for $n=5$ we have

$$PD = \begin{array}{ccccc} \overline{1} & 1 & \overline{3}' & 4' & \overline{1} \\ & 2 & 2 & 2 & \overline{2} \\ & & \overline{3}' & 3 & \overline{3} \\ & & & 4' & 4 \\ & & & & \overline{5}' \end{array} \qquad \mathrm{wgt}(PD)=(-1,2,-2,-1,-1)\,. \tag{3.8}$$

It follows from our definition of $\mathcal{QD}^\rho(n,\overline{n})$ that

$$\sum_{PD\in\mathcal{QD}^\rho(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(PD)} = \prod_{1\le i\le j\le n} (x_i+\overline{x}_i+x_j+\overline{x}_j)\,. \tag{3.9}$$

As usual this expression is divisible by $2^{\ell(\rho)}=2^n$ and we have

$$\sum_{PD\in\mathcal{PD}^\rho(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(PD)} = \prod_{1\le i\le n} (x_i+\overline{x}_i) \prod_{1\le i<j\le n} (x_i+\overline{x}_i+x_j+\overline{x}_j)\,, \tag{3.10}$$

where $\mathcal{PD}^\rho(n,\overline{n})$ is the subset of $\mathcal{QD}^\rho(n,\overline{n})$ consisting of those PD that have no primes on their main diagonal.

However, with $\rho=(n,n-1,\ldots,1)$ and positive roots $\epsilon_i\pm\epsilon_j$ for $1\le i<j\le n$ and $2\epsilon_i$ for $1\le i\le n$, it follows from (1.11) that

$$\begin{aligned} sp_\rho(\mathbf{x}) &= \mathbf{x}^\rho \prod_{1\le i\le n}(1+x_i^{-2}) \prod_{1\le i<j\le n}(1+x_i^{-1}x_j)(1+x_i^{-1}x_j^{-1}) \\ &= \prod_{1\le i\le n}(x_i+x_i^{-1}) \prod_{1\le i<j\le n}(x_i+x_i^{-1}+x_j+x_j^{-1}) = \sum_{PD\in\mathcal{PD}^\rho(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(PD)}\,, \end{aligned} \tag{3.11}$$

where in the last step use has been made of the fact that $\overline{x}_k=x_k^{-1}$ for all $k=1,2,\ldots,n$.

This time the required bijection [**3**] is one between $PST \in \mathcal{QST}^{\lambda+\rho}(n,\overline{n})$ and pairs (PD,T) with $PD\in\mathcal{QD}^\rho(n,\overline{n})$ and $T\in\mathcal{T}^\lambda(n,\overline{n})$. This is illustrated in the case $n=5$ by the map:

$$QST = \begin{array}{ccccccccc} \overline{1} & 1' & \overline{2}' & 2' & \overline{3}' & 3 & \overline{4} & \overline{4} & 5 \\ & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\ & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\ & & & 4 & 4 & & & & \\ & & & & 5' & & & & \end{array} \longrightarrow (PD,T) = \begin{array}{ccccc} 1' & 1 & \overline{3}' & 4' & \overline{1} \\ & \overline{2}' & \overline{2} & 4' & 5' \\ & & \overline{3} & \overline{4}' & \overline{3} \\ & & & 4 & 4 \\ & & & & 5' \end{array} \;,\; \begin{array}{cccc} \overline{1} & \overline{4} & \overline{4} & 5 \\ 3 & 4 & 4 & \\ \overline{4} & 5 & 5 & \end{array} \tag{3.12}$$

The map is effected by moving all primed entries k' and $\overline{k}'$ into their own column by means of a sequence of *jeu de taquin* moves, dealing first with all $\overline{k}'$s and the all k's. The allowed moves include all those described in the $gl(n)$. These are applied first to all $\overline{k}'$s, and in doing so no obstacles are encountered. However, in moving the k's one may encounter a final move in which the destination site in the ith row and kth column is already occupied by a $\overline{k}'$. In this case we use the following transposition:

$$\begin{array}{cc} \overline{k}' & k' \end{array} \longrightarrow \begin{array}{cc} i & \overline{i} \end{array} \tag{3.13}$$

Similarly, to avoid a pair i and $\overline{i}$ in the kth column, which would be non-standard in that they cannot both be in the ith row, one replaces such a pair by a $\overline{k}'$ and k' pair as below:

$$\begin{array}{|c|}\hline \overline{i} \\ \hline i \\ \hline \end{array} \longrightarrow \begin{array}{|c|}\hline k' \\ \hline \overline{k}' \\ \hline \end{array} \tag{3.14}$$

It is this last transformation that forces one to allow primed entries on the main diagonal, and thus establish the bijection at the level of $\mathcal{QST}^{\lambda+\rho}(n,\overline{n})$, rather than $\mathcal{PST}^{\lambda+\rho}(n,\overline{n})$. Having done this it has been shown that each primed entry k' or $\overline{k}'$ reaches the kth column, and that in the first n columns each unprimed entry lies in its own row. Just like the previous moves, the two new moves described above are weight preserving since the contribution to the weight of any pair of entries $k\,\overline{k}$ or $k'\,\overline{k}'$ is zero. It follows that the map is a weight preserving bijection. This allows us to see that

$$\sum_{QST\in\mathcal{QST}^{\lambda+\rho}(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(QST)} = \sum_{PD\in\mathcal{PD}^{\rho}(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(PD)} \sum_{T\in\mathcal{T}^{\lambda}(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(T)}\,. \tag{3.15}$$

Taking into account the fact that $\ell(\lambda+\rho)=\ell(\rho)=n$ this leads to the factorisation:

$$P^{sp}_{\lambda+\rho}(\mathbf{x}) = sp_\rho(\mathbf{x})\ sp_\lambda(\mathbf{x})\,. \tag{3.16}$$

Thanks to the general factorisation identity (1.13), this suffices to prove our Conjecture 3.1 in any case for which λ has length $\ell(\lambda)=n$.

4. Symmetry properties

4.1. The Weyl symmetry of Schur Q-functions. Before embarking on a direct proof of the fact that $Q_\lambda(\mathbf{x})$, as defined combinatorially by (2.4), is a symmetric function of the components of $\mathbf{x}$, it is worth both extending the definition to the case of skew Q-functions $Q_{\lambda/\mu}(\mathbf{x})$ and introducing supersymmetric skew Schur functions $s_{\tau/\sigma}(\mathbf{x}/\mathbf{y})$.

Dealing first with the latter, if τ and σ are partitions such that $F^{\tau/\sigma}=F^\tau\backslash F^\sigma$ is a skew Young diagram, then for any $\mathbf{x}=(x_1,\ldots,x_m)$ and $\mathbf{y}=(y_1,\ldots,y_n)$, we have

$$s_{\tau/\sigma}(\mathbf{x}/\mathbf{y}) = \sum_{T\in\mathcal{T}^{\tau/\sigma}(m/n)} \prod_{i=1}^m x_i^{\#i\in T} \prod_{j=1}^n y_j^{\#j'\in T}\,, \tag{4.1}$$

where $\mathcal{T}^{\tau/\sigma}(m/n)$ is the set of all skew supertableaux, T, of shape $F^{\tau/\sigma}$ with entries taken from the set $\{1,\ldots,m,1',\ldots,n'\}$, subject to the ordering $1<2<\cdots<m<1'<2'<\cdots<n'$, with unprimed entries increasing weakly across rows and strictly down columns, and primed entries increasing strictly across rows and weakly down columns. This function is supersymmetric in the sense that it is symmetric under any permutation of $(x_1,\ldots,x_m)$, symmetric under any permutation of $(y_1,\ldots,y_n)$ and is independent of z if we set $x_i=z$ and $y_j=-z$ for any $i\in\{1,\ldots,m\}$ and $j\in\{1,\ldots,n\}$. It is a remarkable, and very useful fact that $s_{\tau/\sigma}(\mathbf{x}/\mathbf{y})$ is independent of the order relation on its entries [**5**], see Ex23, p90. In particular, in the case $m=n$ it is convenient to adopt the ordering $1'<1<2'<2<\cdots<n'<n$.

Similarly, if λ and μ are partitions whose parts are distinct, such that $SF^{\lambda/\mu}=SF^\lambda\backslash SF^\mu$ is a skew shifted Young diagram, then for any $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ we have

$$Q_{\lambda/\mu}(\mathbf{x}) = \sum_{PST\in\mathcal{QST}^{\lambda/\mu}(n)} \prod_{k=1}^n x_k^{(\#k\in PST)+(\#k'\in PST)}\,, \tag{4.2}$$

where $\mathcal{QST}^{\lambda/\mu}(n)$ is the set of all primed semistandard shifted skew tableaux, PST, of shape $SF^{\lambda/\mu}$, with entries taken from the set $\{1',1,2',2,\ldots,n',n\}$ subject to the ordering $1'<1<2'<2<\cdots<n'<n$, weakly increasing across rows and down columns, with no two k's in the same row, and no two ks in the same column.

It should then be noted that each $PST\in\mathcal{QST}^\lambda(n)$, with λ a partition whose parts are all distinct, has a structure consisting of a sequence of ribbon strips $\mathrm{str}_k(PST)$ with $k=1,2,\ldots,n$. This structure is specified by means of a sequence $\mathrm{Seq}(\lambda)$ of the form $\lambda^{(0)}\subseteq\lambda^{(1)}\subseteq\cdots\subseteq\lambda^{(n)}$, with $\lambda^{(0)}=0$ and $\lambda^{(n)}=\lambda$, where each $\lambda^{(k)}$ is a partition all of whose parts are distinct, and $\lambda^{(k)}/\lambda^{(k-1)}$ is a skew partition in the shape of the ribbon strip $\mathrm{str}(PST)$ for $k = 1, 2, \ldots, n$. With this notation,

$$Q_\lambda(\mathbf{x}) = \sum_{\mathrm{Seq}(\lambda)} \prod_{k=1}^{n} Q_{\lambda^{(k)}/\lambda^{(k-1)}}(x_k)\,. \tag{4.3}$$

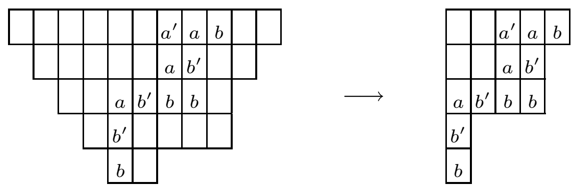
Moreover,

$$Q_\lambda(\mathbf{x}) = \sum_{\mathrm{Seq}(\lambda)} Q_{\lambda^{(k-1)}}(x_1,\ldots,x_{k-1})\; Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_k,x_{k+1})\; Q_{\lambda^{(n)}/\lambda^{(k+1)}}(x_{k+2},\ldots,x_n)\,. \tag{4.4}$$

It follows that the contribution to $Q_\lambda(\mathbf{x})$ arising from the entries $k', k, (k+1)', (k+1)$ in all those $PST \in \mathcal{QST}^\lambda(n)$ whose ribbon strip structure is specified by $\mathrm{Seq}(\lambda)$ is given by $Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_k, x_{k+1})$.

In what follows, we let $a = k$ and $b = k+1$. Then there are two types of case to consider. Firstly, we have those cases for which $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ is a skew Young diagram $F^{\tau/\sigma}$ specified by some pair of partitions τ and σ. In the case $\lambda = (11, 9, 7, 6, 2)$, an example of this type is provided by:

(4.5)



where $\lambda^{(b)} = (9, 7, 6, 2, 1)$ and $\lambda^{(a-1)} = (6, 5, 2, 1)$ lead to $\tau = (5, 4, 4, 1, 1)$ and $\sigma = (2, 2)$.

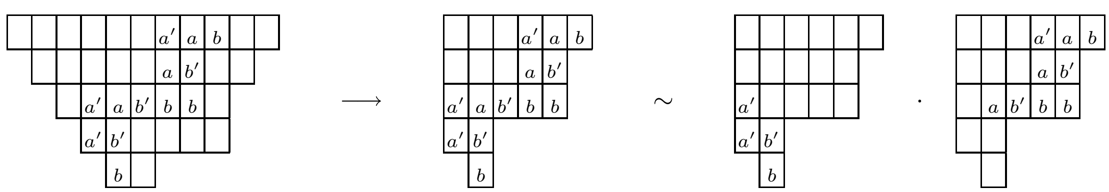
In such a case, it follows from the definitions given above that

$$Q_{\lambda^{(b)}/\lambda^{(a-1)}}(x_a, x_b) = s_{\tau/\sigma}(x_a, x_b/x_a, x_b)\,. \tag{4.6}$$

Since $s_{\tau/\sigma}(x_a, x_b/y_a, y_b)$ is symmetric under the interchange of x_a and x_b, and of y_a and y_b, it follows that $Q_{\lambda^{(b)}/\lambda^{(a-1)}}(x_a, x_b)$ is symmetric under the interchange of x_a and x_b.

Secondly, we have those cases for which $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ is not a skew Young diagram $F^{\tau/\sigma}$. Such a case arises if and only if $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ contains two boxes on the main diagonal of SF^λ. In the case $\lambda = (11, 9, 7, 6, 2)$, this is illustrated by:

(4.7)



To show that we still have symmetry in this case it should be noted that $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ involves a connected component with pairs of boxes on a sequence of one or more, say d, consecutive diagonals, together with a single box on the next diagonal. This single box together with all the remaining boxes of $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ necessarily constitute a skew Young diagram $F^{\tau/\sigma}$ for some pair of partitions τ and σ. In the above example, $d = 2$, $\tau = (5, 4, 4)$ and $\sigma = (2, 2)$. We refer to the single box as the initial box of $F^{\tau/\sigma}$.

Each $PST \in \mathcal{QST}^\lambda(n)$ whose ribbon strip structure is specified by $\mathrm{Seq}(\lambda)$ is such that the pair of boxes on the main diagonal of $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ may be filled with entries (a', b'), (a', b), (a, b') or (a, b), the pairs of boxes on the next $d-1$ diagonals are then filled with either (a', b') or (a, b), depending on their position relative to the preceding pair, and the entry, say z, in the initial box of $F^{\tau/\sigma}$ can in every case be either a or b', but neither a' nor b. However, in the supertableaux $T \in \mathcal{T}^{\tau/\sigma}(2/2)$ contributing to $s_{\tau/\sigma}(x_a, x_b/x_a, x_b)$ the entry in the initial box of $F^{\tau/\sigma}$ can be a', a, b' or b. Fortunately, with the ordering $a' < a < b' < b$, if any supertableaux T with entry z in its initial box belongs to $\mathcal{T}^{\tau/\sigma}(2/2)$, then so does T', where T' is obtained from T by either adding or subtracting a prime to z. Thus all contributing supertableaux arise in pairs T and T'. However, the entries $z = a'$ and $z = a$ both contribute a factor x_a to $Q_{\lambda^{(b)}/\lambda^{(a-1)}}(x_a, x_b)$, while the entries $z = b'$ and $z = b$ both contribute a factor x_b. It follows that

$$Q_{\lambda^{(b)}/\lambda^{(a-1)}}(x_a, x_b) = 4\,(x_a\,x_b)^d\;\frac{1}{2}\;s_{\tau/\sigma}(x_a, x_b/x_a, x_b) = 2\,(x_a\,x_b)^d\;s_{\tau/\sigma}(x_a, x_b/x_a, x_b)\,. \tag{4.8}$$

This shows, once again, that $Q_{\lambda^{(b)}/\lambda^{(a-1)}}(x_a, x_b)$ is symmetric under the interchange of x_a and x_b.

It follows that all contributions $Q_{\lambda^{(b)}/\lambda^{(a-1)}}(x_a, x_b)$ to $Q_\lambda(\mathbf{x})$ are symmetric under the interchange of $x_a = x_k$ and $x_b = x_{k+1}$. Applying this in the cases $k = 1, 2, \ldots, n-1$ is sufficient to prove our required result that $Q_\lambda(\mathbf{x})$, as defined by (2.4), is symmetric under all permutations of the components of $\mathbf{x}$.

4.2. The Weyl symmetry of symplectic Q-functions. Now, the aim is to show that $Q^{sp}_\lambda(\mathbf{x})$, as defined combinatorially by (3.5), is not only a symmetric function of the components of $\mathbf{x}$ but also symmetric with respect to any combination of the inversions $x_k \mapsto x_k^{-1}$ with $k \in \{1, 2, \ldots, n\}$. Once again, it is convenient to extend the definition to the case of skew symplectic Q-functions as follows. If λ and μ are partitions whose parts are distinct, such that $SF^{\lambda/\mu} = SF^\lambda \backslash SF^\mu$ is a skew shifted Young diagram, then for any $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ we let

$$Q_{\lambda/\mu}(\mathbf{x}, \mathbf{y}) = \sum_{QST \in \mathcal{QST}^{\lambda/\mu}(n,\overline{n})} \prod_{k=1}^{n} x_k^{(\#k \in QST)+(\#k' \in QST)} \prod_{k=1}^{n} y_k^{(\#\overline{k} \in QST)+(\#\overline{k}' \in QST)}, \tag{4.9}$$

where $\mathcal{QST}^{\lambda/\mu}(n,\overline{n})$ is the set of all primed semistandard shifted symplectic skew tableaux, QST, of shape $SF^{\lambda/\mu}$, with entries taken from the set $\{\overline{1}', \overline{1}, 1', 1, \overline{2}', \overline{2}, 2', 2, \ldots, \overline{n}', \overline{n}, n', n\}$ subject to the ordering $\overline{1}' < \overline{1} < 1' < 1 < \overline{2}' < \overline{2} < 2' < 2 < \cdots < \overline{n}' < \overline{n} < n' < n$, weakly increasing across rows and down columns, with no two k's or $\overline{k}'$s in the same row, and no two ks or $\overline{k}$s in the same column. This allows us to define

$$Q^{sp}_{\lambda/\mu}(\mathbf{x}) = Q_{\lambda/\mu}(\mathbf{x}, \overline{\mathbf{x}}) = \sum_{QST \in \mathcal{QST}^{\lambda/\mu}(n,\overline{n})} x_k^{\mathrm{wgt}(QST)}, \tag{4.10}$$

where $\mathrm{wgt}(QST) = (\#k \in QST) + (\#k' \in QST) - (\#\overline{k} \in QST) - (\#\overline{k}' \in QST)$.

It should then be noted that each $QST \in \mathcal{QST}^\lambda(n,\overline{n})$, with λ a partition whose parts are all distinct, has a structure consisting of a sequence of ribbon strips $\mathrm{str}_{\overline{k}}(QST)$ and $\mathrm{str}_k(QST)$ with $k = 1, 2, \ldots, n$. This structure is specified by means of a sequence $\mathrm{Seq}(\lambda)$ of the form $\lambda^{(0)} \subseteq \lambda^{(\overline{1})} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(\overline{n})} \subseteq \lambda^{(n)}$, with $\lambda^{(0)} = 0$ and $\lambda^{(n)} = \lambda$, where each $\lambda^{(a)}$ is a partition all of whose parts are distinct, and $\lambda^{(a)}/\lambda^{(a-1)}$ is a skew partition in the shape of the ribbon strip $\mathrm{str}_a(PST)$ for $a = \overline{1}, 1, \overline{2}, 2, \ldots, \overline{n}, n$. Here $a-1$ is to be interpreted as the symbol immediately to the left of a in the sequence $(0, \overline{1}, 1, \overline{2}, 2, \ldots, \overline{n}, n)$, that is to say $a - 1 = k - 1$ if $a = \overline{k}$ and $a - 1 = \overline{k}$ if $a = k$. With this notation,

$$Q^{sp}_\lambda(\mathbf{x}) = \sum_{\mathrm{Seq}(\lambda)} \prod_{a=\overline{1}}^{n} Q_{\lambda^{(a)}/\lambda^{(a-1)}}(x_a). \tag{4.11}$$

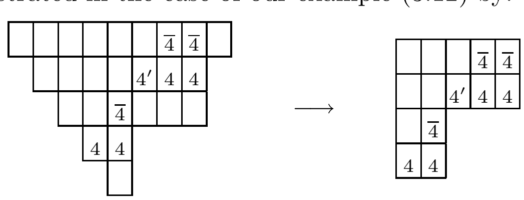
Moreover, with similar definitions of $a+1$ and $a+2$, one and two steps, respectively, to the right of a in the sequence $(0, \overline{1}, 1, \overline{2}, 2, \ldots, \overline{n}, n)$, we have

$$Q^{sp}_\lambda(\mathbf{x}) = \sum_{\mathrm{Seq}(\lambda)} Q_{\lambda^{(a-1)}}(x_1, \ldots, x_{a-1})\; Q_{\lambda^{(a+1)}/\lambda^{(a-1)}}(x_a, x_{a+1})\; Q_{\lambda^{(n)}/\lambda^{(a+1)}}(x_{a+2}, \ldots, x_n). \tag{4.12}$$

It follows that the contribution to $Q_\lambda(\mathbf{x})$ arising from the entries $a', a, (a+1)', (a+1)$ in all those $QST \in \mathcal{QST}^\lambda(n,\overline{n})$ whose ribbon strip structure is specified by $\mathrm{Seq}(\lambda)$ is given by $Q_{\lambda^{(a+1)}/\lambda^{(a-1)}}(x_a, x_{a+1})$.

In the case $a = \overline{k}$, so that $b = a+1 = k$ and $a - 1 = k - 1$, then at most one of $\{a', a, b', b\}$ may appear on the main diagonal. It follows that $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ is always a skew Young diagram $F^{\tau/\sigma}$ for some pair of partitions τ and σ. This is illustrated in the case of our example (3.12) by:

(4.13)



where it can be seen that $\tau = (5, 5, 2, 2)$ and $\sigma = (3, 2, 1)$. It then follows that in each such case

$$Q_{\lambda^{(k)}/\lambda^{(k-1)}}(x_{\overline{k}}, x_k) = s_{\tau/\sigma}(x_{\overline{k}}, x_k/x_{\overline{k}}, x_k). \tag{4.14}$$

Since $s_{\tau/\sigma}(x_a, x_b/y_a, y_b)$ is symmetric under the interchange of x_a and x_b, and of y_a and y_b, it follows that $Q_{\lambda^{(k)}/\lambda^{(k-1)}}(x_{\overline{k}}, x_k)$ is symmetric under the interchange of $x_{\overline{k}}$ and x_k. This result is valid for all $k = 1, 2, \ldots, n$

Thanks to the identification $x_{\overline{k}} = x_k^{-1}$ for $k = 1, 2, \ldots, n$, this implies that $Q^{sp}_\lambda(\mathbf{x})$ is symmetric with respect to any combination of the inversions $x_k \mapsto x_k^{-1}$ with $k \in \{1, 2, \ldots, n\}$.

On other hand, if $a = k$, so that $b = a + 1 = \overline{k+1}$ and $a - 1 = \overline{k}$, the situation is the same as that of $a = k$ and $b = k + 1$ for $gl(n)$. Two types of case occur, namely those for which $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ is a skew Young diagram $F^{\tau/\sigma}$ for a pair of partitions τ and σ, and those for which $SF^{\lambda^{(b)}/\lambda^{(a-1)}}$ contains a connected sequence of pairs of boxes on d consecutive diagonals, starting with the main diagonal, linked to at least one box, the initial box, of a skew Young diagram $F^{\tau/\sigma}$ for a pair of partitions τ and σ. The analysis goes through exactly as in the $gl(n)$ case, giving either

$$Q_{\lambda^{(\overline{k+1})}/\lambda^{\overline{k}}}(x_k, x_{\overline{k+1}}) = s_{\tau/\sigma}(x_k, x_{\overline{k+1}}/x_k, x_{\overline{k+1}}) \tag{4.15}$$

or

$$Q_{\lambda^{(\overline{k+1})}/\lambda^{\overline{k}}}(x_k, x_{\overline{k+1}}) = 2\,(x_k, x_{\overline{k}})^d\; s_{\tau/\sigma}(x_k, x_{\overline{k+1}}/x_k, x_{\overline{k+1}})\,. \tag{4.16}$$

In both cases $Q_{\lambda^{(\overline{k+1})}/\lambda^{\overline{k}}}(x_k, x_{\overline{k+1}})$ is symmetric under the interchange of x_k and $x_{\overline{k+1}}$. This result is valid for all $k = 1, 2, \ldots, n-1$.

Now, returning to the symplectic Schur Q-function $Q^{sp}_\lambda(\mathbf{x})$ itself, its dependence on x_k and x_{k+1} can be isolated by means of the following decomposition

$$Q^{sp}_\lambda(\mathbf{x}) = \sum_{\mathrm{Seq}(\lambda)} Q_{\lambda^{(k-1)}}(x_{\overline{1}}, \ldots, x_{k-1})\; Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_{\overline{k}}, x_k, x_{\overline{k+1}}, x_{k+1})\; Q_{\lambda^{(n)}/\lambda^{(k+1)}}(x_{\overline{k+2}}, \ldots, x_n)\,, \tag{4.17}$$

where
(4.18)

$$\begin{aligned}
Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_{\overline{k}}, x_k, x_{\overline{k+1}}, x_{k+1}) &= Q_{\lambda^{(k)}/\lambda^{(k-1)}}(x_{\overline{k}}, x_k)\; Q_{\lambda^{\overline{k+1}}/\lambda^{(k)}}(x_{\overline{k+1}})\; Q_{\lambda^{(k+1)}/\lambda^{(\overline{k-1})}}(x_{k+1})\\
&= Q_{\lambda^{(\overline{k})}/\lambda^{(k-1)}}(x_{\overline{k}})\; Q_{\lambda^{(\overline{k+1})}/\lambda^{(\overline{k})})}(x_k, x_{\overline{k+1}})\; Q_{\lambda^{(k+1)}/\lambda^{(\overline{k-1})}}(x_{k+1})\\
&= Q_{\lambda^{(\overline{k})}/\lambda^{(k-1)}}(x_{\overline{k}})\; Q_{\lambda^{(k)}/\lambda^{(\overline{k})}}(x_k)\; Q_{\lambda^{(k+1)}/\lambda^{(k)}}(x_{\overline{k+1}}, x_{k+1}).
\end{aligned}$$

The fact that the first, second and third expressions are symmetric with respect to the transpositions $x_{\overline{k}} \leftrightarrow x_k$, $x_k \leftrightarrow x_{\overline{k+1}}$ and $x_{\overline{k+1}} \leftrightarrow x_{k+1}$, respectively, ensures that $Q_{\lambda^{(k+1)}/\lambda^{(k-1)}}(x_{\overline{k}}, x_k, x_{\overline{k+1}}, x_{k+1})$ is symmetric with respect to all permutations of its arguments, including the permutation $(x_{\overline{k}}, x_k, x_{\overline{k+1}}, x_{k+1}) \mapsto (x_{\overline{k+1}}, x_{k+1}, x_{\overline{k}}, x_k)$. This implies in turn that $Q^{sp}_\lambda(\mathbf{x})$ is invariant under the transposition $x_k \leftrightarrow x_{k+1}$. Since this is true for all $k = 1, 2, \ldots, n-1$, it follows that $Q^{sp}_\lambda(\mathbf{x})$ is symmetric with respect to all permutations of the components of $\mathbf{x}$. Combining this with the invariance of $Q^{sp}_\lambda(\mathbf{x})$ under inversions, it follows that $Q^{sp}_\lambda(\mathbf{x})$ is Weyl group invariant, as required.

5. Lattice path approach

It is well-known that tableaux translate nicely to lattice paths. In the case of Schur Q-function, Stembridge [**12**] has provided a lattice path interpretation and used this to derive a pfaffian result for $Q_\lambda(\mathbf{x})$. Hamel [**1**] extended this approach to obtain pfaffians for more general decompositions of tableaux. Hamel also included a determinantal expression for $Q_\lambda(\mathbf{x})$ due to Okada [**8**] and showed an extension to it too. Here we derive a determinantal expression for $Q^{sp}_\lambda(\mathbf{x})$ and prove it using lattice paths.

The lattice path grid is defined as follows. Label the y-axis with $0, \frac{1}{2}, \overline{1'}, \overline{1}, \overline{1\frac{1}{2}}, 1', 1, 1\frac{1}{2}, \overline{2'}, \overline{2}, \overline{2\frac{1}{2}}, 2', 2, \ldots$. Define lattice paths with three types of permissible steps: up-vertical steps that increase the y-coordinate by 1; horizontal steps at unprimed levels that increase the x-coordinate by 1; and up-diagonal steps from unprimed levels to primed levels that increase the x-coordinate by 1 and increase the y-coordinate by 1.

For each tableau with m parts we can construct an m-tuple of noninteresecting lattice paths. There will be one path for each row in the tableau. Construct a path as follows: if a box in the row contains an i or $\overline{i}$ and is at coordinates (a, b) in the tableau, put a horizontal step from $(a-b, i)$ to $(a-b+1, i)$ (resp. $(a-b, \overline{i})$ to $(a-b+1, \overline{i})$); if a box in the row contains an i' or $\overline{i'}$ and is at coordinates (a, b) in the tableau, put an up-diagonal step from $(a-b, \overline{i})$ to $(a-b+1, i')$ (resp. $(a-b, i-1)$ to $(a-b+1, \overline{i'})$). If, however, the box is on the main diagonal of the tableau then there is a minor modification. In that case a box containing i' or $\overline{i'}$ causes an up-diagonal step from $(0, \overline{i\frac{1}{2}})$ to $(1, i')$ (resp. $(0, (i-1)\frac{1}{2})$ to $1, \overline{i'})$). These steps are then connected with vertical steps.

Okada [**8**] proved a plane partition version of the following result using lattice paths. Hamel [**1**] reinterpreted it in the form described above. In the theorem, $a_1, a_2, \ldots, a_l$ is the sequence of elements on the main diagonal of the tableau, and $|a_j| = k$ if $a_j = k$ or k'.

Theorem 5.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a shifted shape partition. Then*

$$Q_\lambda(\mathbf{x}) = \sum_{a_1 < a_2 < \ldots < a_l} \det \left[x_{|a_j|} Q_{\lambda_i - 1}(x_{|a_j|}, x_{|a_j|+1}, \ldots) \right].$$

Here we generalize this theorem to the symplectic case $Q^{sp}_\lambda(\mathbf{x})$. We now take $|a_j| = k$ if $a_j = k, k', \overline{k}$ or $\overline{k'}$.

Theorem 5.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a shifted shape partition. Then*

$$\begin{aligned} Q^{sp}_\lambda(\mathbf{x}) \quad = \quad & \sum_{a_1 < a_2 \ldots < a_l} \Bigg(\delta_{a_j, \overline{\mathbf{k}}} \ \det \left[x^{-1}_{|a_j|} Q_{\lambda_i - 1}(x^{-1}_{|a_j|}, x_{|a_j|}, x_{|a_j|+1}, \ldots) \right] \\ & + \delta_{a_j, \mathbf{k}} \ \det \left[x_{|a_j|} Q_{\lambda_i - 1}(x_{|a_j|}, x^{-1}_{|a_j|+1}, x_{|a_j|+1}, \ldots) \right] \Bigg), \end{aligned}$$

where $\delta_{a_j, \overline{\mathbf{k}}}$ is 1 if a_j is barred and 0 otherwise, and similarly $\delta_{a_j, \mathbf{k}}$ is 1 if a_j is unbarred and 0 otherwise.

The Q generating function in the determinant reflects the fact that with the first element removed from the row, the row is now of size $\lambda_i - 1$ and the second element must have a weight of $|a_j|$ or greater.

Proof: Let $a_1, a_2, \ldots, a_l$ be the set of elements in the boxes on the main diagonal of the shifted tableau. Using the lattice path set-up as described above with the additional constraint that the first step in a path must be a step to $(1, a_j)$.

A Lindström-Gessel-Viennot argument will supply the proof. In particular, Theorem 1.2 of Stembridge [**12**] applies. Summing over all permissible sequences a gives the full generality of the result. This completes the proof.

References

[1] A.M. Hamel, Pfaffians and determinants for Schur Q-functions, *J. Comb. Theory A* **75** (1996), 328–340.

[2] A.M. Hamel and R.C. King, Symplectic shifted tableaux and deformations of Weyl's denominator formula for $sp(2n)$, *J. Algebraic Comb.*, **16** (2002), 269–300.

[3] A.M. Hamel and R.C. King, Bijective proofs of shifted tableau and alternating sign matrix identities, *J. Algebraic Comb. to appear.*

[4] R.C. King, Weight multiplicities for the classical groups, in *Lecture Notes in Physics*, Vol 50, Berlin: Springer, 1976, pp497–499.

[5] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd. Ed. Oxford: Oxford Univerity Press, 1995.

[6] I.G. Macdonald, *Symmetric Functions and Orthogonal Polynomials*, University Lecture Series, Vol. 12, Providence, R.I.: American Mathematical Society, 1998.

[7] I.G. Macdonald, Orthogonal polynomials associated with root systems, *Sém. Lotharingien Comb.* **45** (2000) Article B45a.

[8] S. Okada, Partially strict shifted plane partitions, *J. Comb. Theory A* **53** (1990), 143–156.

[9] P. Pragacz, Algebro-geometric applications of Schur S- and Q-polynomials, in *Lecture Notes in Maths.*, Vol 1478, Berlin: Springer, 1991, pp130–191.

[10] R.P. Stanley, Problem 4 in Problem Session, Combinatorics and Algebra, *Contemporary Maths.*, Vol 34, Providence, R.I., American Mathematical Society, 1994.

[11] J.R. Stembridge, Shifted tableaux and the projective representations of symmetric groups, *Adv. Math.* **74** (1989), 87–134.

[12] J.R. Stembridge, Nonintersecting paths, pfaffians, and plane partitions, *Advances in Math.* **83** (1990), 63-131.

[13] T. Tokuyama, A generating function of strict Gelfand patterns and some formulas on characters of general linear groups, *J. Math. Soc. Japan* **40** (1988), 671-685.

School of Mathematics, University of Southampton, Southamptpon, SO17 1BJ, England

E-mail address: R.C.King@soton.ac.uk

Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, Ontario N2L 3C5, Canada

E-mail address: ahamel@wlu.ca