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Inequalities for Symmetric Means

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Abstract. We study Muirhead-type generalizations of families of inequalities due to Newton, Maclaurin and others. Each family is defined in terms of a commonly used basis of the ring of symmetric functions in n variables. Inequalities corresponding to elementary symmetric functions and power sum symmetric functions are characterized by the same simple poset which generalizes the majorization order. Some analogous results are also obtained for the Schur, homogeneous, and monomial cases.

Résumé. Nous étudions les inégalités de Muirhead et ses analogues qui géneralise les inégalités de Newton, Maclaurin, et des autres. Chaque famille d'inégalités correspond a une base de l'anneau de polinômes symmétriques avec n variables. Avec un seule poset qui géneralise l'ordre "majorization", nous caracterisons les familles qui d'inégalités qui correspondent aux bases $\{e_\lambda\}$ et $\{p_\lambda\}$. Nous avons des résultats analougues pour les bases $\{h_\lambda\}$, $\{s_\lambda\}$, $\{m_\lambda\}$.

1. Introduction

Commonly used bases for the vector space Λ_n^r of homogeneous of degree r symmetric functions in n variables $\mathbf{x} = (x_1, \ldots, x_n)$ are the *monomial* symmetric functions $\{m_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, *elementary* symmetric functions $\{e_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, (complete) *homogeneous* symmetric functions $\{h_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, *power sum* symmetric functions $\{p_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, and *Schur* functions $\{s_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$. (See [**Sta99**, Ch. 7] for definitions.) When $r = 0$ we adopt the conventions $m_0(\mathbf{x}) = e_0(\mathbf{x}) = h_0(\mathbf{x}) = s_0(\mathbf{x}) = 1$ and $p_0(\mathbf{x}) = n$.

To each element $g_\lambda(\mathbf{x})$ of these bases, we will associate a *term-normalized* symmetric function $G_\lambda(\mathbf{x})$ and a *mean* $\mathfrak{G}_\lambda(\mathbf{x})$ by

$$G_\lambda(\mathbf{x}) = \frac{g_\lambda(\mathbf{x})}{g_\lambda(1^n)}, \qquad \mathfrak{G}_\lambda(\mathbf{x}) = \sqrt[d]{G_\lambda(\mathbf{x})}. \tag{1.1}$$

Note that $\{G_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$ forms a basis of Λ_n^r, and that the functions $\{\mathfrak{G}_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$, while symmetric, are not polynomials in $\mathbf{x}$ and therefore do not belong to the *ring of symmetric functions* Λ_n. In the definition of $\mathfrak{G}_\lambda(\mathbf{x})$, we assume $r > 0$.

The term *symmetric mean* is often used to describe a function $\mathfrak{G}(\mathbf{x}) : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is symmetric in $x_1, \ldots, x_n$ and which for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n_{\geq 0}$, $c \in \mathbb{R}_{\geq 0}$ satisfies

(1) $\min(\mathbf{a}) \leq \mathfrak{G}(\mathbf{a}) \leq \max(\mathbf{a})$,
(2) $\mathbf{a} \leq \mathbf{b}$ (componentwise) implies $\mathfrak{G}(\mathbf{a}) \leq \mathfrak{G}(\mathbf{b})$,
(3) $\lim_{\mathbf{b} \to \mathbf{0}} \mathfrak{G}(\mathbf{a} + \mathbf{b}) = \mathfrak{G}(\mathbf{a})$.
(4) $\mathfrak{G}(c\mathbf{a}) = c\mathfrak{G}(\mathbf{a})$.

(See, e.g., [**Bul03**, p. 62].) For fixed n and two means $\mathfrak{F}$, $\mathfrak{G}$, we will write $\mathfrak{F}(\mathbf{x}) \leq \mathfrak{G}(\mathbf{x})$ or $\mathfrak{G}(\mathbf{x}) - \mathfrak{F}(\mathbf{x}) \geq 0$ if we have $\mathfrak{F}(\mathbf{a}) \leq \mathfrak{G}(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{R}^n_{\geq 0}$. We define the inequality $F(\mathbf{x}) \leq G(\mathbf{x})$ analogously. Note that if the degrees of $F(\mathbf{x})$ and $G(\mathbf{x})$ are equal, then we have $F(\mathbf{x}) \leq G(\mathbf{x})$ if and only if we have $\mathfrak{F}(\mathbf{x}) \leq \mathfrak{G}(\mathbf{x})$.

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The study of inequalities in symmetric means has quite a long history. (See, e.g., [**Bul03**], [**HLP34**].) Perhaps the best known such inequality is that of the *arithmetic* and *geometric* means,

$$\mathfrak{E}_1(\mathbf{x}) \geq \mathfrak{E}_n(\mathbf{x}).$$

Other symmetric function inequalities which may be stated in terms of means are due to

(1) Muirhead [**Mui03**]: For $\lambda, \mu \vdash r$,

$$\begin{aligned} M_\lambda(\mathbf{x}) &\leq M_\mu(\mathbf{x}) \qquad \text{if and only if } \mu \text{ majorizes } \lambda, \text{ equivalently,} \\ \mathfrak{M}_\lambda(\mathbf{x}) &\leq \mathfrak{M}_\mu(\mathbf{x}) \qquad \text{if and only if } \mu \text{ majorizes } \lambda, \end{aligned}$$

(2) Maclaurin [**Mac29**]: For $1 \leq i \leq j \leq n$,

$$\mathfrak{E}_i(\mathbf{x}) \geq \mathfrak{E}_j(\mathbf{x}),$$

(3) Newton [**New**, p. 173]: For $1 \leq k \leq n-1$,

$$\begin{aligned} E_{k,k}(\mathbf{x}) &\geq E_{k+1,k-1}(\mathbf{x}), \text{ equivalently,} \\ \mathfrak{E}_{k,k}(\mathbf{x}) &\geq \mathfrak{E}_{k+1,k-1}(\mathbf{x}), \end{aligned}$$

(4) Schlömilch [**Sch58**]: For $1 \leq i \leq j$,

$$\mathfrak{P}_i(\mathbf{x}) \leq \mathfrak{P}_j(\mathbf{x}),$$

(5) Gantmacher [**Gan59**, p. 203]: For $k \geq 1$,

$$\begin{aligned} p_{k,k}(\mathbf{x}) &\leq p_{k+1,k-1}(\mathbf{x}), \text{ equivalently,} \\ P_{k,k}(\mathbf{x}) &\leq P_{k+1,k-1}(\mathbf{x}), \text{ equivalently,} \\ \mathfrak{P}_{k,k}(\mathbf{x}) &\leq \mathfrak{P}_{k+1,k-1}(\mathbf{x}), \end{aligned}$$

(6) Popoviciu [**Pop34**]: For $1 \leq i \leq j$,

$$\mathfrak{H}_i(\mathbf{x}) \leq \mathfrak{H}_j(\mathbf{x}),$$

(7) Schur [**HLP34**, p. 164]: For $k \geq 1$,

$$\begin{aligned} H_{k,k}(\mathbf{x}) &\leq H_{k+1,k-1}(\mathbf{x}), \text{ equivalently,} \\ \mathfrak{H}_{k,k}(\mathbf{x}) &\leq \mathfrak{H}_{k+1,k-1}(\mathbf{x}). \end{aligned}$$

It is easy to use the inequalities of Newton, Gantmacher, and Schur to derive the inequalities of Maclaurin, Schlömilch, and Popoviciu, respectively. Furthermore, we will show in Section 4 that Muirhead's inequalities imply those of Newton and Gantmacher.

Note that the term-normalized symmetric functions and mean which we have associated to a symmetric function in (1.1) are defined only for a finite number n of variables. Nevertheless, we may essentially eliminate dependence upon n from the inequalities enumerated above by considering them to be inequalities in sequences of functions,

$$\begin{aligned} G &= (G(x_1), G(x_1, x_2), G(x_1, x_2, x_3), \dots), \\ \mathfrak{G} &= (\mathfrak{G}(x_1), \mathfrak{G}(x_1, x_2), \mathfrak{G}(x_1, x_2, x_3), \dots). \end{aligned}$$

More generally, we will define partial orders on such sequences by declaring $F \leq G$ if we have $F(\mathbf{x}) \leq G(\mathbf{x})$ for all $n > 0$, and $\mathfrak{F} \leq \mathfrak{G}$ if we have $\mathfrak{F}(\mathbf{x}) \leq \mathfrak{G}(\mathbf{x})$ for all $n > 0$. The principal goal of this paper is to characterize the infinite partial orders on the sequences $\{\mathfrak{G}_\lambda \mid \lambda \vdash 1, 2, \dots\}$ corresponding to the common bases of the ring of symmetric functions.

Note that Muirhead's inequalities are indexed by pairs of arbitrary partitions of a fixed integer, while other inequalities are indexed by one- or two-part partitions, not necessarily of a fixed integer. Using the common bases of Λ_n^r, we will state and prove inequalities in pairs $(\mathfrak{G}_\lambda, \mathfrak{G}_\mu)$ of means for which λ and μ are arbitrary and do not necessarily partition the same integer. We consider elementary means and power sum means in Sections 2-3, and other means and inequalities in Section 4.

2. Elementary Means

Generalizing Maclaurin's and Newton's inequalities are inequalities of the form $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$, which we characterize in Theorem 2.3 in terms of sequence *majorization.* Let $\alpha = (\alpha_1, \alpha_2, \dots)$ be a weakly decreasing nonnegative sequence. If the components of α are integers (respectively, rationals) which sum to r, we say that α is an *integer partition* (respectively, a *rational partition*) of r and write $\alpha \vdash r$ or $|\alpha| = r$.

Given two sequences α, β, with $|\alpha| = |\beta| < \infty$, we write $\alpha \preceq \beta$ and say that β *majorizes* α if for each index i the corresponding initial partial sums satisfy

$$\alpha_1 + \cdots + \alpha_i \leq \beta_1 + \cdots + \beta_i.$$

(If necessary, append zeros to the end of either sequence so that the partial sums are defined.) For $c \in \mathbb{R}_{\geq 0}$, $d \in \mathbb{N}$, we define operations of scalar multiplication $\alpha \mapsto c\alpha$, and repetition $\alpha \mapsto \alpha^d$ by

$$\begin{gathered} c\alpha = (c\alpha_1, c\alpha_2, \dots), \\ \alpha^d = (\underbrace{\alpha_1, \dots, \alpha_1}_{d}, \underbrace{\alpha_2, \dots, \alpha_2}_{d}, \dots). \end{gathered}$$

Given an integer partition $\lambda = (\lambda_1, \lambda_2, \dots)$, we will define the *transpose partition* $\lambda^\top = (\lambda^\top_1, \lambda^\top_2, \dots)$ by

$$\lambda^\top_j = \max\{i \,|\, \lambda_i \geq j\}.$$

The following facts are either well-known or easy to see.

Observation 2.1. Let λ, μ be integer partitions with $|\lambda| = |\mu|$, let c be a positive real number, and let d be a positive integer.

(1) $(\lambda^\top)^\top = \lambda$.
(2) $(d\lambda^\top)^\top = \lambda^d$.
(3) $(((\lambda^\top)^d)^\top = d\lambda$.
(4) $\lambda \preceq \mu \iff c\lambda \preceq c\mu \iff \lambda^d \preceq \mu^d \iff \lambda^\top \succeq \mu^\top$.

For $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash r$, the term-normalized elementary symmetric function $E_\lambda(\mathbf{x})$ is given by

$$E_\lambda(\mathbf{x}) = \frac{e_\lambda(\mathbf{x})}{\binom{n}{\lambda_1} \cdots \binom{n}{\lambda_\ell}},$$

and the corresponding elementary mean is the rth root of this, $\mathfrak{E}_\lambda(\mathbf{x}) = \sqrt[r]{E_\lambda(\mathbf{x})}$. Since λ^d is a partition of dr, we have the following stability property of elementary means under the repetition operation.

Observation 2.2. For any partition λ and integer $d \geq 1$ we have $\mathfrak{E}_\lambda = \mathfrak{E}_{\lambda^d}$.

The following theorem shows that inequalities of the form $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$ are characterized by the majorization order on rational partitions of 1.

Theorem 2.3. *Given integer partitions λ and μ, we have*

$$\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu \qquad \text{if and only if} \qquad \frac{\lambda^\top}{|\lambda|} \preceq \frac{\mu^\top}{|\mu|}.$$

Proof. First let us consider the case that $|\lambda| = |\mu|$. This case, asserting the equivalence of $E_\lambda \leq E_\mu$ and $\lambda^\top \preceq \mu^\top$, was first proved in [**Cut06**, Thm. 5.7].

($\Rightarrow$) Suppose that $\lambda^\top \not\preceq \mu^\top$. Then for some index i we have

$$\lambda^\top_1 + \cdots + \lambda^\top_i > \mu^\top_1 + \cdots + \mu^\top_i.$$

Choosing a number $n \geq \max(\lambda_1, \mu_1)$ and specializing the symmetric functions $E_\lambda(\mathbf{x}), E_\mu(\mathbf{x})$ at

$$\begin{gathered} x_1 = \cdots = x_i = t, \\ x_{i+1} = \cdots = x_n = 1, \end{gathered}$$

we obtain polynomials in $\mathbb{N}[t]$ of degrees $\lambda^\top_1 + \cdots + \lambda^\top_i$ and $\mu^\top_1 + \cdots + \mu^\top_i$, respectively. Thus we have

$$\lim_{t\to\infty} [E_\lambda(t, \dots, t, 1, \dots, 1) - E_\mu(t, \dots, t, 1, \dots, 1)] = \infty,$$

which implies that $E_\lambda \not\leq E_\mu$.

($\Leftarrow$) Now suppose that $\lambda^\top \preceq \mu^\top$, or equivalently that $\lambda \succeq \mu$, and write $\lambda = (\lambda_1, \ldots, \lambda_\ell)$. If λ covers μ in the majorization order, then there exist indices $1 \leq j < k \leq \ell$ for which we have

$$\mu = (\lambda_1, \ldots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \ldots, \lambda_{k-1}, \lambda_k + 1, \lambda_{k+1}, \ldots, \lambda_\ell).$$

Thus $E_\mu(\mathbf{x}) - E_\lambda(\mathbf{x})$ is equal to

$$\frac{E_\lambda(\mathbf{x})}{E_{\lambda_j}(\mathbf{x})E_{\lambda_k}(\mathbf{x})}(E_{\lambda_j-1}(\mathbf{x})E_{\lambda_k+1}(\mathbf{x}) - E_{\lambda_j}(\mathbf{x})E_{\lambda_k}(\mathbf{x})).$$

Rewriting Newton's inequalities as

$$\frac{E_1(\mathbf{x})}{E_0(\mathbf{x})} \geq \frac{E_2(\mathbf{x})}{E_1(\mathbf{x})} \geq \frac{E_3(\mathbf{x})}{E_2(\mathbf{x})} \geq \cdots,$$

we see that $E_{\lambda_j}(\mathbf{x})E_{\lambda_k}(\mathbf{x}) \leq E_{\lambda_j-1}(\mathbf{x})E_{\lambda_k+1}(\mathbf{x})$, which implies that $E_\lambda \leq E_\mu$. If λ does not cover μ in the majorization order, then there exists a sequence of partitions

$$\mu = \nu^{(0)} \leq \nu^{(1)} \leq \cdots \leq \nu^{(m)} = \lambda,$$

in which each comparison of consecutive partitions is a covering relation, such that we have

$$E_\mu(\mathbf{x}) - E_\lambda(\mathbf{x}) = \sum_{i=0}^{m-1} (E_{\nu^{(i)}}(\mathbf{x}) - E_{\nu^{(i+1)}}(\mathbf{x})) \geq 0,$$

and consequently $E_\lambda \leq E_\mu$.

Now consider the case that $|\lambda|$ and $|\mu|$ are not equal. By Observation 2.2, we have

$$\mathfrak{E}_\lambda(\mathbf{x}) = \mathfrak{E}_{\lambda^{|\mu|}}(\mathbf{x}), \qquad \mathfrak{E}_\mu(\mathbf{x}) = \mathfrak{E}_{\mu^{|\lambda|}}(\mathbf{x}).$$

Since $\lambda^{|\mu|}$ and $\mu^{|\lambda|}$ are both partitions of $|\lambda| \cdot |\mu|$, we have $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$ if and only if $(\lambda^{|\mu|})^\top \preceq (\mu^{|\lambda|})^\top$. By Observation 2.1 this is equivalent to the condition $\frac{\lambda^\top}{|\lambda|} \preceq \frac{\mu^\top}{|\mu|}$. $\square$

From Theorem 2.3, we see that the poset describing inequalities $\mathfrak{E}_\lambda \leq \mathfrak{E}_\mu$ of elementary means is isomorphic to the dual of the majorization order on rational partitions of 1. For all $r \geq 1$, this poset contains the (dual of the) majorization order on partitions of r. Figure 1 shows the restriction of this poset to elementary means $\mathfrak{E}_\lambda$ corresponding to integer partitions of $1, \ldots, 6$. Note that some integer partitions do not appear in this poset because of the equalities implied by Observation 2.2. Partitions of integers dividing 6 have been emphasized to show the embedding in this poset of the dual of the majorization order on partitions of 6.

From Theorem 2.3, we see that this poset is not finitary: between any two partitions, there must be another. In particular, if $|\lambda| = |\mu| = r$ and λ is covered by μ in the majorization order on partitions of r, then λ^2 is not covered by μ^2 in the majorization order of paritItions of $2r$.

Define the semiring $\mathbf{N}$ to be the set of all nonnegative linear combinations of products of symmetric functions of the forms

$$\{E_{i,j}(\mathbf{x}) - E_{i+1,j-1}(\mathbf{x}) \,|\, 1 \leq j \leq i \leq n-1\} \cup \{E_i(\mathbf{x}) \,|\, 1 \leq i \leq n\}.$$

The proof of Theorem 2.3 shows that each difference $E_\lambda - E_\mu$ with μ majorizing λ belongs to $\mathbf{N}$.

3. Power Sum Means

We now generalize Schlömilch's and Gantmacher's inequalities by characterizing inequalities of the form $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$. Note that for $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash r$, the term-normalized power sum symmetric function $P_\lambda(\mathbf{x})$ is given by

$$P_\lambda(\mathbf{x}) = \frac{p_\lambda(\mathbf{x})}{n^\ell},$$

and the corresponding power sum mean is the rth root of this, $\mathfrak{P}_\lambda(\mathbf{x}) = \sqrt[r]{P_\lambda(\mathbf{x})}$. Like the elementary basis $\{e_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$ of Λ_n^r, the power sum basis $\{p_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$ is multiplicative. We therefore have the following equalities.

OBSERVATION 3.1. For any partition λ and integer $d \geq 1$ we have $\mathfrak{P}_\lambda = \mathfrak{P}_{\lambda^d}$.

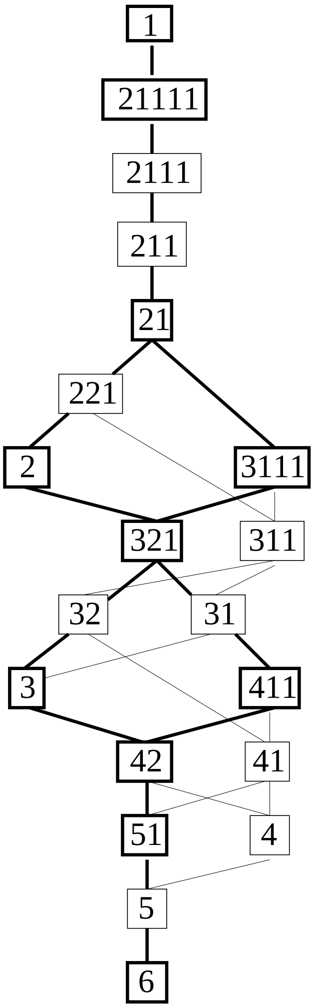

FIGURE 1. Equivalence classes of partitions of $1, \ldots, 6$ ordered by inequalities in elementary means.

To prove a power sum analog of Theorem 2.3, we whall use the following alternative characterization of the majorization order on integer partitions.

LEMMA 3.2. *Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, define a function $\psi_\lambda : \mathbb{N} \to \mathbb{N}$ by*

$$\psi_\lambda(u) = \max_{1 \le k \le \ell} \{\lambda_1 + \cdots + \lambda_k - ku\}.$$

Then for two partitions $\lambda, \mu \vdash r$, we have that $\lambda \preceq \mu$ if and only if $\psi_\lambda(u) \le \psi_\lambda(u)$ for $u = 0, \ldots, r$.

PROOF. Note that $\psi_\lambda(u)$ is equal to the number of boxes in columns $u+1, \ldots, r$ of the Young diagram of λ,

$$\psi_\lambda(u) = r - (\lambda_1^\top + \cdots + \lambda_u^\top).$$

Thus the condition $\psi_\lambda(u) \le \psi_\mu(u)$ for $u = 0, \ldots, r$ is equivalent to the condition $\lambda^\top \succeq \mu^\top$, which in turn is equivalent to $\lambda \preceq \mu$. $\square$

The following result shows that inequalities of the form $\mathfrak{P}_\lambda \le \mathfrak{P}_\mu$, are characterized by the majorization order on of rational partitions of 1. In fact, we have $\mathfrak{P}_\lambda \le \mathfrak{P}_\mu$ if and only if we have $\mathfrak{E}_\lambda \ge \mathfrak{E}_\mu$.

THEOREM 3.3. *Given integer partitions λ and μ, we have*

$$\mathfrak{P}_\lambda \le \mathfrak{P}_\mu \qquad \text{if and only if} \qquad \frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}.$$

PROOF. Let $\lambda = (\lambda_1, \dots, \lambda_m\ell)$, $\mu = (\mu_1, \dots, \mu_m)$. Let us first consider the case that $|\lambda| = |\mu|$. In this case the theorem asserts the equivalence of $P_\lambda \leq P_\mu$ and $\lambda \preceq \mu$.

($\Rightarrow$) Suppose that $\lambda \not\preceq \mu$. Using Lemma 3.2, choose an index j such that $\psi_\lambda(j) > \psi_\mu(j)$, and consider the functions $\phi_\lambda(t)$, $\phi_\mu(t)$ defined by

$$\phi_\lambda(t) = P_\lambda(t, \underbrace{1, \dots, 1}_{t^j}) = \frac{1}{t^{j\ell}} \prod_{i=1}^{\ell} (t^{\lambda_i} + t^j) = \sum_{k=0}^{\ell} \sum_{\{i_1, \dots, i_k\}} t^{\lambda_{i_1} + \cdots + \lambda_{i_k} - kj},$$

$$\phi_\mu(t) = P_\mu(t, \underbrace{1, \dots, 1}_{t^j}) = \frac{1}{t^{jm}} \prod_{i=1}^{m} (t^{\mu_i} + t^j) = \sum_{k=0}^{m} \sum_{\{i_1, \dots, i_k\}} t^{\mu_{i_1} + \cdots + \mu_{i_k} - kj}.$$

These are polynomials in t of degrees $\psi_\lambda(j)$ and $\psi_\mu(j)$, respectively. Thus we have

$$\lim_{t \to \infty} [\phi_\lambda(t) - \phi_\mu(t)] = \infty,$$

which implies $P_\lambda \not\leq P_\mu$.

($\Leftarrow$) Suppose that $\lambda \preceq \mu$. Then we proceed as in the proof of Theorem 2.3 with Gantmacher's inequalities replacing those of Newton and conclude that $P_\lambda \leq P_\mu$.

In the case that λ and μ are not equal, we again proceed as in the proof of Theorem 2.3. Applying Observation 3.1, we see that

$$\mathfrak{P}_\lambda(\mathbf{x}) = \mathfrak{P}_{\lambda^{|\mu|}}(\mathbf{x}), \qquad \mathfrak{P}_\mu(\mathbf{x}) = \mathfrak{P}_{\mu^{|\lambda|}}(\mathbf{x}).$$

Thus we have $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$ if and only if $\lambda^{|\mu|} \preceq \mu^{|\lambda|}$, or equivalently, if and only if $\frac{\lambda^\top}{|\lambda|} \preceq \frac{\mu^\top}{|\mu|}$. □

The similarity of Theorems 2.3 and 3.3 is somewhat curious. Apparently we have that $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$ if and only if $\mathfrak{E}_\lambda \geq \mathfrak{E}_\mu$. Thus the poset of power sum means is dual to the poset of elementary means.

Define the semiring $\mathbf{G}$ to be the set of all nonnegative linear combinations of products of symmetric functions of the forms

$$\{P_{i+1,j-1}(\mathbf{x}) - P_{i,j}(\mathbf{x}) \,|\, 1 \leq j \leq i \leq n-1\} \cup \{P_i(\mathbf{x}) \,|\, 1 \leq i \leq n\}.$$

The proof of Theorem 3.3 shows that each difference $P_\mu(\mathbf{x}) - P_\lambda(\mathbf{x})$ with μ majorizing λ belongs to $\mathbf{G}$.

4. More Inequalities

The authors have obtained some results and have formulated some conjectures concerning characterization of inequalities of the forms $\mathfrak{S}_\lambda \leq \mathfrak{S}_\mu$, $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$, $\mathfrak{H}_\lambda \leq \mathfrak{H}_\mu$, for λ and μ not necessarily partitions of the same integer.

Let us consider the generalization of Muirhead's inequalities which would characterize all inequalities of the form $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$. Note that a formula for the term-normalized monomial symmetric function is given by

$$M_\lambda(\mathbf{x}) = \frac{m_\lambda(\mathbf{x})}{\binom{n}{\alpha_1, \dots, \alpha_r, n-\ell}},$$

where α_j is equal to the number of parts of λ which are equal to j. Unlike the elementary and power sum bases, the monomial basis $\{m_\lambda(\mathbf{x}) \,|\, \lambda \vdash r\}$ of Λ_n^r, is not multiplicative. Nevertheless, rational partitions of 1 seem to characterize monomial means as follows.

CONJECTURE 4.1. Given integer partitions λ and μ, we have

$$\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu \qquad \text{if and only if} \qquad \frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|} \text{ and } \frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}.$$

Equivalently, $\mathfrak{M}_\lambda \leq \mathfrak{M}_\mu$ if and only if $\mathfrak{E}_{\lambda^\top} \leq \mathfrak{E}_{\mu^\top}$ and $\mathfrak{P}_\lambda \leq \mathfrak{P}_\mu$.

This conjecture suggests defining a *double majorization order* on integer partitions by $\lambda \trianglelefteq \mu$ if $\frac{\lambda}{|\lambda|} \preceq \frac{\mu}{|\mu|}$ and $\frac{\lambda^\top}{|\lambda|} \succeq \frac{\mu^\top}{|\mu|}$. It is easy to see that when $|\lambda| = |\mu|$, double majorization becomes ordinary majorization. Thus for each $r \in \mathbb{N}$, the majorization order of partitions of r is a subposet of the double majorization order. Figure 2 shows the restriction of this poset to integer partitions of of $1, \dots, 5$.

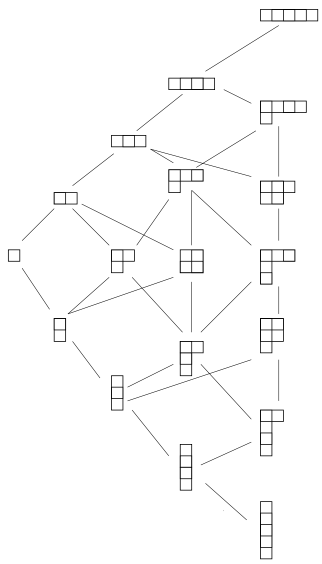

Figure 2. Double majorization of partitions of $1, \ldots, 5$.

It is well-known that Muirhead's inequalities imply many other inequalities. For instance, a straightforward computation shows that Gantmacher's inqualities are essentially special cases of those of Muirhead,

$$P_{k+1,k}(\mathbf{x}) - P_{k,k}(\mathbf{x}) = \frac{n-1}{n}(M_{k+1,k-1}(\mathbf{x}) - M_{k,k}(\mathbf{x})).$$

A lengthier computation shows that one can obtain Newton's inequalities by adding various inequalities of Muirhead.

Theorem 4.2. *We have*

$$E_{k,k}(\mathbf{x}) - E_{k+1,k-1}(\mathbf{x}) = \sum_{j=0}^{k-1} d_j (M_{2^{k-j}1^{i+2j-k}}(\mathbf{x}) - M_{2^{k-j-1}1^{2j+2}}(\mathbf{x})),$$

where

$$d_j = \frac{\binom{k-1}{j}\binom{n-k}{j+1}}{(n-k)\binom{n}{k}}.$$

Proof. Omitted. □

Thus by the comments at the ends of Sections 2 and 3, the inequalities in these sections can be derived from those of Muirhead. It would therefore be interesting to express homogeneous and Schur differences $H_\lambda(\mathbf{x}) - H_\mu(\mathbf{x})$ and $S_\lambda(\mathbf{x}) - S_\mu(\mathbf{x})$ as nonnegative linear combinations of Muirhead differences $M_\lambda(\mathbf{x}) - M_\mu(\mathbf{x})$. The authors have obtained partial results suggesting that this is possible in many cases.

Generalizing all of the inequalities considered so far are those of the form $\mathfrak{F} \leq \mathfrak{G}$, where the means are constructed from homogeneous elements $f(\mathbf{x}) \in \Lambda^k$ and $g(\mathbf{x}) \in \Lambda^\ell$ which need not be commonly used basis elements of these spaces,

$$f(\mathbf{x}) = \sum_{\kappa \vdash k} c_\kappa m_\kappa, \qquad g(\mathbf{x}) = \sum_{\lambda \vdash \ell} d_\lambda m_\lambda.$$

It would be interesting to characterize these inequalities in terms of the coordinate sequences $(c_\kappa)_{\kappa \vdash k}$. By the homogeneity of means, it would be sufficient to restrict one's attention to coordinate sequences which sum to 1.

To ensure that the means are well defined, one would have to require that the symmetric functions $F(\mathbf{x})$ and $G(\mathbf{x})$ evaluate nonnegatively on all nonnegative vectors (of finite support). This condition does *not* imply F and G to be monomial nonnegative, or even to be nonnegative linear combinations of Muirhead differences. This suggests the following question.

QUESTION 4.3. Which symmetric functions $f(\mathbf{x}) \in \Lambda$ ($F(\mathbf{x}) \in \Lambda_n^r$) satisfy $f(\mathbf{a}) \geq 0$ ($F(\mathbf{a}) \geq 0$) for all real nonnegative vectors $\mathbf{a}$ of finite support?

5. Acknowledgments

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