Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Dumont permutations of the third kind

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Abstract. We consider the set of permutations all of whose descents are from an even value to an even value. Proving a conjecture of Kitaev and Remmel, we show that these permutations are enumerated by Genocchi numbers, hence equinumerous to Dumont permutations of the first (and second) kind, and thus may be called *Dumont permutations of the third kind*. We also define the related *Dumont permutations of the fourth kind*. We find certain statistics on Dumont permutations of the third kind that generate the Seidel triangle for Genocchi numbers. Finally, we consider and enumerate certain sets of pattern-restricted Dumont permutations of the first and third kinds.

Résumé.

1. Introduction

Dumont [4] showed that the certain sets of permutations are enumerated by Genocchi numbers G_{2n}, which are multiples of Bernoulli numbers B_{2n}, namely $G_{2n} = 2(1-2^{2n})B_{2n}$, so that

$$\sum_{n=1}^{\infty} G_{2n}\frac{x^{2n}}{(2n)!} = x\tan\frac{x}{2}, \qquad \sum_{n=1}^{\infty} (-1)^n G_{2n}\frac{x^{2n}}{(2n)!} = \frac{2x}{e^x+1} - x = -x\tanh\frac{x}{2}.$$

Definition 1.1. A *Dumont permutation of the first kind* (or a *Dumont-I* permutation, for short) is a permutation $\pi \in \mathfrak{S}_{2n}$ in which each even entry begins a descent and each odd entry begins an ascent or ends the string. In other words, for every $i = 1, 2, \ldots, 2n$,

$$\pi(i) \text{ is even} \implies i < 2n \text{ and } \pi(i) > \pi(i+1),$$
$$\pi(i) \text{ is odd} \implies \pi(i) < \pi(i+1) \text{ or } i = 2n.$$

Definition 1.2. A *Dumont permutation of the second kind* (or *Dumont-II* permutation, for short) is a permutation $\pi \in \mathfrak{S}_{2n}$ in which all entries at even positions are deficiencies and all entries at odd positions are fixed points or excedances. In other words, for every $i = 1, 2, \ldots, n$,

$$\pi(2i) < 2i, \qquad \pi(2i-1) \geq 2i-1.$$

Notation 1.3. We denote the set of Dumont permutations of the first (resp. second) kind of length $2n$ by $\mathfrak{D}^1_{2n}$ (resp. $\mathfrak{D}^2_{2n}$). We also let $[m] = \{1, 2, \ldots, m\}$ for a given integer m.

Example 1.4. $\mathfrak{D}^1_2 = \mathfrak{D}^2_2 = \{21\}$, $\mathfrak{D}^1_4 = \{2143, 3421, 4213\}$, $\mathfrak{D}^2_4 = \{2143, 3142, 4132\}$.

Dumont [4] proved that $|\mathfrak{D}^1_{2n}| = |\mathfrak{D}^2_{2n}| = G_{2n}$.

In this paper, we define two more sets of permutations whose cardinalities we also prove to be Genocchi numbers.

2000 *Mathematics Subject Classification.* Primary 05A05; Secondary 05A15.

Key words and phrases. Dumont permutations, permutation statistics, Genocchi numbers, Seidel triangle, descents, patterns, surjective pistols.

*Research supported in part by the NSA Young Investigator Grant H98230-06-1-0037.

DEFINITION 1.5. Let $\mathfrak{D}^3_{2n}$ be the set of permutations $\pi \in \mathfrak{S}_{2n}$ in which all descents occur only from an even value to an even value; in other words, for every $i = 1, 2, \dots, 2n-1$,

$$\pi(i) > \pi(i+1) \implies \pi(i) \text{ and } \pi(i+1) \text{ are even.}$$

Note that this implies that all permutations in $\mathfrak{D}^3_{2n}$ $(n \geq 1)$ must start with 1. Also note that the set of permutations in $\mathfrak{D}^3_{2n}$ are almost centrally symmetric: if $\pi \in \mathfrak{D}^3_{2n}$ and $\rho \in \mathfrak{S}_{2n}$ is given by $\rho(1) = 1$, $\rho(i) = 2n + 2 - \pi(2n + 2 - i)$ for $i \geq 2$, then $\rho \in \mathfrak{D}^3_{2n}$ as well. If we form the *diagram* of π by plotting the points $(i, \pi(i))$ in the xy plane, this means that reflecting the diagram through the point $(n+1, n+1)$ (except for the point $(1, 1)$) yields a permutation in $\mathfrak{D}^3_{2n}$.

EXAMPLE 1.6. $\mathfrak{D}^3_2 = \{12\}$, $\mathfrak{D}^3_4 = \{1234, 1342, 1423\}$.

Kitaev and Remmel [**9, 10**] conjectured that the sets of permutations where all descents occur only from an even value to an even value are also enumerated by Genocchi numbers. Our main result is a combinatorial proof of their conjecture.

THEOREM 1.7 (Main Theorem). *There is a bijection between sets $\mathfrak{D}^3_{2n}$ and $\mathfrak{D}^1_{2n}$, and hence $|\mathfrak{D}^3_{2n}| = |\mathfrak{D}^1_{2n}|$.*

DEFINITION 1.8. Given the result of our Main Theorem, we call permutations in $\mathfrak{D}^3_{2n}$ *Dumont permutations of the third kind* (or *Dumont-III* permutations, for short).

There is a simple natural bijection [**6**] between Dumont-I and Dumont-II permutations. Given a Dumont-I permutation written in a one-line form, insert parentheses to indicate cycles so that cycles start with each successive left-to-right maximum (an entry $\pi(j)$ for which $\pi(j) > \pi(i)$ whenever $i < j$). This yields a Dumont-II permutation.

EXAMPLE 1.9.

$$\mathfrak{D}^1_4 \ni 2143 \mapsto (21)(43) = 2143 \in \mathfrak{D}^2_4,$$
$$\mathfrak{D}^1_4 \ni 3421 \mapsto (3)(421) = 4132 \in \mathfrak{D}^2_4,$$
$$\mathfrak{D}^1_4 \ni 4213 \mapsto (4213) \quad = 3142 \in \mathfrak{D}^2_4.$$

The same bijection can be applied to Dumont-III permutations to obtain the set $\mathfrak{D}^4_{2n}$ that we may call *Dumont permutations of the fourth kind*, or *Dumont-IV* permutations. One can easily show the following.

PROPOSITION 1.10. *The set $\mathfrak{D}^4_{2n}$ consists exactly of all permutations $\pi \in \mathfrak{S}_{2n}$ all of whose deficiencies occur at even positions and have even values, in other words,*

$$\pi(i) < i \implies i \text{ and } \pi(i) \text{ are even.}$$

EXAMPLE 1.11. $\mathfrak{D}^4_2 = \{12\}$, $\mathfrak{D}^4_4 = \{1234, 1342, 1432\}$.

$$\mathfrak{D}^3_4 \ni 1234 \mapsto (1)(2)(3)(4) = 1234 \in \mathfrak{D}^4_4,$$
$$\mathfrak{D}^3_4 \ni 1342 \mapsto (1)(3)(42) \quad = 1432 \in \mathfrak{D}^4_4,$$
$$\mathfrak{D}^3_4 \ni 1423 \mapsto (1)(423) \quad = 1342 \in \mathfrak{D}^4_4.$$

Note that that all permutations in $\mathfrak{D}^4_{2n}$ $(n \geq 1)$ must also start with 1, and that the set of graphs of all $\pi \in \mathfrak{D}^4_{2n}$ with "1" deleted is invariant under reflection across the antidiagonal $x + y = 2n + 2$. In other words, if the permutation $\varrho \in \mathfrak{S}_{2n-1}$ is such that $\pi = (1, \varrho + 1) \in \mathfrak{D}^4_{2n}$, then $\pi' = (1, irc(\varrho) + 1) \in \mathfrak{D}^4_{2n}$, where irc is the map of "inverse of reversal of complement," i.e. the reflection of ϱ across the antidiagonal $x + y = 2n$.

2. A bijection between Dumont-I and Dumont-III permutations

Our strategy for proving our main theorem will be to define a combinatorial structure called a *signature* for each Dumont-I permutation, and, in a different way, a signature for each Dumont-III permutation. We will then count the Dumont-I permutations and Dumont-III permutations corresponding to each possible signature. To help with the counting, we define a *signature function* for each signature. We will see that the number of Dumont-I permutations with a given signature is equal to the product of the values of the associated signature function, and the same is true of Dumont-III permutations. The counting procedure allows us to define a bijection from Dumont-I permutations to Dumont-III permutations, in which signatures are preserved.

2.1. Signatures and signature functions. We will first define signatures in the abstract, and then show how to associate signatures with Dumont-I and Dumont-III permutations.

DEFINITION 2.1. A *signature* of order k and size $2n$ consists of a set of k even numbers from $[2n]$ called the *peaks* and a set of k odd numbers from $[2n]$ called the *pits*. We also insist that peaks include $2n$ and pits include 1. Given a particular signature, let $\mathrm{pk}(x)$ and $\mathrm{pt}(x)$ be the numbers of peaks and pits, respectively, that are at least x. The associated *signature function* has domain $[2n]$ and is given by

$$f(x) = \mathrm{pk}(x) - \mathrm{pt}(x) + (1 \text{ if } x \text{ is odd}).$$

We are not interested in signatures for which $f(x)$ takes on non-positive values, since they will not correspond to any Dumont-I or Dumont-III permutations.

EXAMPLE 2.2. $(\{6, 8\}, \{1, 3\})$ is a signature of size 8 and order 2. Its signature function is given by

x	1	2	3	4	5	6	7	8
$f(x)$	1	1	2	2	3	2	2	1

The product of these 8 values is 48. Note that $f(1) = 1$ and $f(2n) = 1$ for every signature function f of size $2n$.

2.2. Signatures of Dumont-I permutations. The *peaks* of a Dumont-I permutation are the even values that do not end descents, and the *pits* are the odd values that do end descents. (In other words: the peaks are the even non-descent-bottoms, and the pits are the odd descent-bottoms. Or, equivalently: the peaks are the local maxima, possibly including the initial entry, and the peaks are the local minima, possibly including the terminal entry.) The sequence from each peak to the next pit is a *descent block*; it can contain only descents, and can contain only even values except for the pit itself. The gaps before, after, and between descent blocks are called *ascent blocks*; they can contain only odd values, and any or all of them may be empty. Note that each descent block includes exactly one peak and one pit. The ascent blocks do not include peaks or pits, and (including the empty ones) there is one more ascent block than there are descent blocks.

NOTATION 2.3. We denote a descent (resp. ascent) block starting from a and ending at b by $a \searrow b$ (resp. $a \nearrow b$).

The *signature* of a Dumont-I permutation consists of the set of its peaks and the set of its pits, without any indication of their sequence.

EXAMPLE 2.4. 58364217 is a Dumont-I permutation. Its peaks are 6 and 8, its pits are 1 and 3, and we may write its signature as $(\{6, 8\}, \{1, 3\})$. The descent blocks are 83 and 6421. The ascent blocks are 5 and 7, plus an empty ascent block between 3 and 6.

THEOREM 2.5. *Given a signature of size $2n$ and its associated signature function f, the number of Dumont-I permutations with that signature is exactly $\prod_{x=1}^{2n} f(x)$.*

PROOF. To construct a Dumont-I permutation with a given signature, we must first match the pits with the peaks to form descent blocks. We construct the matching beginning with the largest pit, x_1. There are $\mathrm{pk}(x_1)$ peaks to which it might be assigned. That is equal to $f(x_1)$, since there is exactly one pit $\geq x_1$ (x_1 itself) and x_1 is odd.

After picking one of those assignments, let x_2 be the next largest pit. It has $\mathrm{pk}(x_2)$ peaks to which it might be assigned, minus the peak assigned to x_1; the number of choices is therefore $\mathrm{pk}(x_2) - 1$, which is $f(x_2)$.

Continuing to assign the pits in decreasing order, the number of choices we have when dealing with pit x is precisely $\mathrm{pk}(x) - \mathrm{pt}(x) + 1 = f(x)$.

The next step in constructing a Dumont-I permutation is to put our new descent blocks into some sequence. We start with the block continuing the smallest peak; now call that peak y_1. It can become the first descent block of the permutation, or it can follow any block whose pit is less than y_1 — excluding the pit that has been matched with y_1. The number of choices is therefore

$$1 + (\#\text{pits} < y_1) - 1 = (\#\text{pits} < y_1) = k - \mathrm{pt}(x_1) = \mathrm{pk}(y_1) - \mathrm{pt}(y_1) = f(y_1)$$

since every peak is at least y_1.

Letting y_2 be the next smallest peak, we can place its descent block at the start (if that choice remains) or after any pit that is less than y_2 (excluding the pit matched with y_2 and the pit, if any, followed by y_1's block). The number of choices is

$$1 + (\#\text{pits} < y_2) - 2 = (\#\text{pits} < y_2) - 1 = k - pt(y_2) - 1 = \text{pk}(y_2) - \text{pt}(y_2) = f(y_2).$$

Continuing to assign the peaks in increasing order, we see that the number of locations available for any peak y is $f(y)$. Making assignments for each descent block determines the sequence of the blocks.

Now we have all of the peaks and pits in order, and we know the bounds of each descent block and each ascent block.

We now assign each even value (other than peaks) to a descent block, and each odd value (other than pits) to an ascent block. These choices can be made independently, and they complete the construction of the permutation, because the order in which values occur within any block is determined by the Dumont definition. The number of descent blocks available to any even value x is $f(x)$, and the number of ascent blocks available to any odd value x is $f(x)$.

We have completed the construction of the permutation. For each $x \in [2n]$, we made a choice from among $f(x)$ alternatives, so the number of Dumont-I permutations that can be constructed from this signature is exactly the product of the values $f(x)$. □

EXAMPLE 2.6. There are exactly 48 Dumont-I permutations with signature $(\{6, 8\}, \{1, 3\})$. For example, the permutation 58364217 is constructed as follows:

(1) match pit 3 with peak 8 (it could have been matched with 6; $f(3) = 2$)
(2) match pit 1 with peak 6 (only choice; $f(1) = 1$)
(3) put the descent sequence $6 \searrow 1$ immediately after $8 \searrow 3$ (it could have been put at the start of the permutation; $f(6) = 2$)
(4) put the descent sequence $8 \searrow 3$ at the start of the permutation (only choice; $f(8) = 1$)
(5) insert 2 into $6 \searrow 1$ (only choice; $f(2) = 1$)
(6) insert 4 into $6 \searrow 1$ (it could have fit into $8 \searrow 3$; $f(4) = 2$)
(7) insert 5 before $8 \searrow 3$ (it could have gone between $8 \searrow 3$ and $6 \searrow 1$ or after $6 \searrow 1$; $f(5) = 3$)
(8) insert 7 after $6 \searrow 1$ (it could have gone before $8 \searrow 3$; $f(7) = 2$).

It is not hard to assign numbers to the alternatives available at each stage, and so to construct a bijection between Dumont permutations with signature f, and the set of integer functions g on $[2n]$ satisfying $1 \leq g(x) \leq f(x)$ for each $x \in [2n]$. We will define such a bijection in an example at the end of the next subsection.

2.3. Signatures of Dumont-III permutations. We will define peaks, pits, and signatures differently for Dumont-III permutations than for Dumont-I permutations. Some of the definitions may seem peculiar. This is because we are not trying to describe the Dumont-III permutation itself. Instead, we are trying to identify the ghost of a Dumont-I permutation hidden within it.

The *peaks* of a Dumont-III permutation are the even values that do not end descents). The *pits* of a Dumont-III permutation are the even values that do not begin descents. A peak that is also a pit is called a *singleton*.

(In other words: the peaks are the even non-descent-bottoms, and the pits are the even non-descent-tops. The non-singleton peaks are local maxima and the non-singleton pits are local minima, but the singletons are need not be maxima or minima.)

The sequence from each peak to the following pit (or to itself if it is a singleton) is called a *descent block*. Note that a singleton is called a descent block even though it contains zero descents. The (possibly empty) sequences before, after, and between the descent blocks are called *ascent blocks*. Descent blocks contain only even values; ascent blocks contain only odd values, if any.

The *signature* of a Dumont-III permutation consists of

(1) a list of its peaks, and
(2) a list of the values of the form $x - 1$ for each pit x.

The second list contains odd numbers that we call *pit list entries*.

Example 2.7. 15846237 is a Dumont-III permutation. Its peaks are 6 and 8. Its pits are 2 and 4, so the pit list entries are 1 and 3, and its signature is $(\{6,8\},\{1,3\})$. The descent blocks are 84 and 62. The ascent blocks are 15 and 37, plus an empty ascent block between 4 and 6.

Theorem 2.8. *Given a signature with size $2n$ and its associated signature function f, the number of Dumont-III permutations with that signature is exactly $\prod_{x=1}^{2n} f(x)$.*

Proof. We can construct a Dumont-III permutation as follows.

From the signature, determine the pits. In order from the largest pit to the smallest, assign the pits to peaks to form descent blocks. We only need to make assignments for non-singleton pits since a unique assignment is made by definition in the case of the singletons.

When assigning a pit x, we may choose any peak larger than x, excluding those chosen for pits greater than x. The number of alternatives is therefore

$$\mathrm{pk}(x) - (\mathrm{pt}(x) - 1) = \mathrm{pk}(x) - \mathrm{pt}(x) + 1 = f(x).$$

Note that we are acquiring a factor of $f(x)$ only when x is a non-singleton pit. Singletons do not contribute factors at this stage.

We next assemble the descent blocks in sequence. We proceed in the order from smallest peak to largest. When dealing with a peak x, we can put its descent block at the start of the permutation (if that choice has not been taken) or immediately after any pit that is smaller than x (excluding those already taken by smaller peaks, and the pit matched with x itself). Thus, the number of alternatives is

$$1 + (\#\text{pits} < x) - (\#\text{peaks} < x) - 1 = (\#\text{pits} < x) - (\#\text{peaks} < x) = \mathrm{pk}(x) - \mathrm{pt}(x) = f(x).$$

We have now acquired a factor of $f(x)$ for each peak x, singleton or not.

We now have the full sequence of peaks and pits, including singletons. We can assign the remaining odd numbers and even numbers to ascent blocks and descent blocks exactly as in the case of Dumont-I permutations, hence the theorem follows. □

Example 2.9. There are exactly 48 Dumont-III permutations with signature $(\{6,8\},\{1,3\})$. The permutation 15846237 is constructed as follows:

(1) match pit 4 with peak 8 (it could have been matched with 6; $f(4) = 2$)
(2) match pit 2 with peak 6 (only choice; $f(2) = 1$)
(3) put the descent sequence $6 \searrow 2$ immediately after $8 \searrow 4$ (it could have been put at the start of the permutation; $f(6) = 2$)
(4) put the descent sequence $8 \searrow 4$ at the start of the permutation (only choice; $f(8) = 1$)
(5) insert 1 before $8 \searrow 4$ (only choice; $f(1) = 1$)
(6) insert 3 after $6 \searrow 2$ (it could have fit before $8 \searrow 4$; $f(3) = 2$)
(7) insert 5 before $8 \searrow 4$ (it could have gone between $8 \searrow 4$ and $6 \searrow 2$ or after $6 \searrow 2$; $f(5) = 3$)
(8) insert 7 after $6 \searrow 2$ (it could have gone before $8 \searrow 4$; $f(7) = 2$).

Corollary 2.10. *Given a signature, there are as many Dumont-III permutations with that signature as Dumont-I permutations with that signature.*

Corollary 2.11. *For each n, there are as many Dumont-III permutations of size $2n$ as Dumont-I permutations of size $2n$. In other words, $|\mathfrak{D}^3_{2n}| = |\mathfrak{D}^1_{2n}|$.*

It is now easy to construct a bijection from Dumont-III permutations with a given signature and associated signature function f to the set of integer functions g on $[2n]$ satisfying $1 \le g(x) \le f(x)$ for all $x \in [2n]$.

Example 2.12. Here is an example of the bijections. Suppose that we number the choices at each stage of the Examples 2.6 and 2.9 from left to right. When assigning pits to peaks we number the choices from the smallest peak to the largest, and when sequencing the peaks we number the choices starting with the initial position, and then from the smallest predecessor-peak to the largest.

Then the Dumont-I permutation 58364217 corresponds to the function g with values $(1,1,2,2,1,2,2,1)$. Similarly, the Dumont-III permutation 15846237 corresponds to the function g with values $(1,1,2,2,1,2,2,1)$. So they correspond naturally to each other.

Note that this implies that the bijection preserves the value of the leftmost peak, a fact that we will use later on.

An interesting combinatorial lemma is hidden in the above argument.

LEMMA 2.13. *Given two disjoint sets I and J of k integers each (such as in a signature), suppose that there are N ways to match the elements of I and J so that each element $i \in I$ is greater than its matching element $j \in J$. Then for each such matching, the number of ways to order the pairs so as to form an "alternating sequence" (descent from each $i \in I$, then ascent from each $j \in J$ except final) is also N. Therefore, the total number of ways to form an alternating sequence from the pair of sets (I, J) is N^2, always a square.*

PROOF. [1] Consider a bipartite multigraph G with vertices $\{x_i \mid i \in I\} \cup \{y_j \mid j \in J\}$. Given two matchings, M_1 and M_2 of I and J, connect vertices x_i and y_j once for each time the pair (i, j) belongs to one of the matchings. Then G is a union of disjoint cycles (possibly including 2-cycles). In each of these cycles, write down the vertex indices along the cycle starting with an edge induced by M_1 and ending with the smallest index (necessarily in J), then concatenate the resulting strings in the order of increasing smallest (and last) element. This forms the desired "alternating sequence" from the pairs in one of the matchings using the other matching to order those pairs. □

EXAMPLE 2.14. Let the sets be $I = \{1, 3, 5, 7, 9, 11\}$, $J = \{4, 6, 8, 10, 12, 14\}$, and the matchings be $M_1 = \{(4, 1), (6, 3), (8, 5), (10, 7), (14, 9), (12, 11)\}$ and $M_2 = \{(10, 1), (4, 3), (6, 5), (8, 7), (12, 9), (14, 11)\}$. Then the alternating sequence formed from M_1 using M_2 is $(10, 7, 8, 5, 6, 3, 4, 1, 12, 11, 14, 9)$ and the alternating sequence formed from M_2 using M_1 is $(4, 3, 6, 5, 8, 7, 10, 1, 14, 11, 12, 9)$.

3. Dumont-III permutations and surjective pistols

We start by defining *surjective pistols* and *alternating pistols* also enumerated by Genocchi numbers [**6, 14**].

DEFINITION 3.1. A *surjective pistol* on $[2n]$ is a surjective map $p : [2n] \to 2[n] = \{2, 4, \ldots, 2n\}$ such that $p(i) \geq i$ for each $i \in [2n]$. Denote the set of surjective pistols on $[2n]$ by $\mathcal{SP}_{2n}$.

DEFINITION 3.2. A *alternating pistol* on $[2n]$ is a map $p : [2n] \to [n]$ such that $p(2i) \leq p(2i-1) \leq i$ for each $i \in [n]$ and $p(2i) \leq p(2i+1)$ for each $i \in [n-1]$. If, in fact, $p(2i) < p(2i+1)$ for each $i \in [n-1]$, then p is called a *strict alternating pistol*. Denote the set of (strict) alternating pistols on $[2n]$ by $\mathcal{AP}_{2n}$ (resp. $\mathcal{SAP}_{2n}$).

We will now give a bijection from surjective pistols to $\mathfrak{D}^3$-permutations of the same length.

DEFINITION 3.3. A *subexcedent* function on $[n]$ is a function $\alpha : [n] \to [n]$ such that $\alpha(i) \leq i$ for all $i \in [n]$. Let SE_n be the set of all subexcedent functions on $[n]$.

EXAMPLE 3.4. If p is a surjective pistol on $[2n]$, then the function α on $[2n]$ given by $\alpha(i) = 2n+1-p(x)$ is a subexcedent function. Note that, in this particular case, $f(i)$ is odd for all $i \in [2n]$.

It is well known that there SE_n is equinumerous to $\mathfrak{S}_n$. Here we give two more bijections $IAX : SE_n \to \mathfrak{S}_n$ and $IBOP : SE_n \to \mathfrak{S}_n$ such that the set of subexcedent functions on $[2m]$ that take exactly all odd values in $[2m]$ is mapped onto $\mathfrak{D}^1_{2m}$ and $\mathfrak{D}^3_{2m}$, respectively. IAX and IBOP here stand for "*insert after, with exchange*" and "*insert before opposite parity*," respectively.

If $\alpha \in SE_n$, then construct a permutation $\pi = IAX(\alpha)$ as follows. For each $i \in [n]$, in the order from 1 to n:

(1) If $\alpha(i) = i$, insert i at the end.
(2) If $\alpha(i) < i$, then insert i after $\alpha(i)$, *except* if that would put i at the end, insert i at the beginning instead.

If $\alpha \in SE_n$, then construct a permutation $\pi = IBOP(\alpha)$ as follows. For each $i \in [n]$, in the order from 1 to n:

(1) If $\alpha(i)$ is even, then insert the value i before the value $\alpha(i) - 1$. Thus, if $\alpha(i)$ is even, we insert i before an odd value.

[1]The authors thank David Chudzicki for supplying the proof of Lemma 2.13.

(2) If $\alpha(i)$ is odd and the value $\alpha(i)$ happens to precede an even value, then insert the value i between them. Thus, in this case, if $\alpha(i)$ is odd, insert before an even value.
(3) Suppose that $\alpha(i)$ is odd and is not a value that precedes an even value, and in fact, suppose that $\alpha(i)$ is the k-th smallest of such values. Then insert the value i in the k-th leftmost available insertion point, where the available points are
   - before evens that don't follow odds, and
   - at the end.

   Thus in this case too, if $\alpha(i)$ is odd, insert before an even or at the end.

Note that the "opposite parity" in IBOP is the parity opposite to that of $\alpha(i)$, not i.

THEOREM 3.5.
(1) *IAX and IBOP are indeed bijections from SE_n to $\mathfrak{S}_n$.*
(2) *If $\alpha \in SE_n$, then $IBOP(\alpha)$ contains exactly all odd numbers at most n if and only if $\pi = IBOP(\alpha)$ is a $\mathfrak{D}^3$-permutation. Likewise for IAX and $\mathfrak{D}^1$. Therefore, $\mathfrak{D}^3$-permutations, $\mathfrak{D}^1$-permutations and surjective pistols are equinumerous.*
(3) *$IBOP(\alpha)$ contains only odd numbers, but not necessarily all of them, if and only if $\pi = IBOP(\alpha)$ is a potential $\mathfrak{D}^3$-permutation, meaning that π represents the order of $1, 2, \ldots, n$ in some longer $\mathfrak{D}^3$-permutation. Likewise for IAX and $\mathfrak{D}^1$.*

EXAMPLE 3.6. $IBOP(1133) = 1234$, $IBOP(1131) = 1423$, $IBOP(1113) = 1342$. Note that 111 does not contain all odd integers in $\{1, 2, 3\}$, so $IBOP(111) = 132$ is not a $\mathfrak{D}^3$-permutation, but does occur as a subsequence of a larger $\mathfrak{D}^3$-permutation 1342.

Likewise, $IAX(1133) = 4213$, $IAX(1131) = 2143$, $IAX(1113) = 3421$. Note also that $IAX(111) = 321$ is not a $\mathfrak{D}^1$-permutation, but does occur as a subsequence of a larger $\mathfrak{D}^1$-permutation 3421.

Note that the known bijections from $\mathfrak{D}^1$-permutations to surjective pistols, composed with complementation (as in Example 3.4) and IBOP, yield a different bijection from $\mathfrak{D}^1_{2n}$ to $\mathfrak{D}^3_{2n}$ than the one described in Section 2.

4. Seidel triangle generation by statistics on $\mathfrak{D}^3$- and $\mathfrak{D}^4$-permutations

The Seidel triangle for Genocchi numbers [**7, 8, 13, 14**] is a Pascal triangle-type array of integers $(g_{i,j})_{i,j\geq 1}$ that is a refinement Genocchi numbers. It is defined by the following recursive relation:

$$\begin{cases} g_{2i+1,j} = g_{2i+1,j-1} + g_{2i,j} \text{ for } j = 1, 2, \ldots, i+1, \\ g_{2i,j} = g_{2i,j+1} + g_{2i-1,j} \text{ for } j = i, i-1, \ldots, 1, \end{cases} \tag{4.1}$$

where $g_{1,1} = 1$ and $g_{i,j} = 0$ if $j \leq 0$ or $i \leq 0$ or $i > \lceil j/2 \rceil$. Then

$$g_{2n-1,n-1} = g_{2n-1,n} = g_{2n,n} = G_{2n-2}, \qquad g_{2n-1,1} = g_{2n-2,1} = H_{2n-1},$$

where G_{2n} is the nth Genocchi number, and H_{2n-1} is the nth *median Genocchi number* (or nth *Genocchi number of the second kind*).

Median Genocchi numbers H_{2n-1} count, for example, $\mathfrak{D}^1_{2n}$-permutations that start with n or $n+1$, $\mathfrak{D}^2_{2n}$-permutations with no fixed points, and the *strict alternating pistols* $\mathcal{SAP}_{2n}$.

The first few values of the Seidel triangle are as follows (numbering rows i from bottom to top and columns j from left to right):

i/j	1	2	3	4	5	6	7	8	9	10
5									155	155
4							17	17	155	310
3					3	3	17	34	138	448
2			1	1	3	6	14	48	104	552
1	1	1	1	2	2	8	8	56	56	608

Note that we are summing up in odd columns and summing down in even columns. Also, note that the odd columns $2n-1$ sum to H_{2n-1}, while even columns $2n$ sum to G_{2n}.

In this section, we determine some and conjecture other statistics on $\mathfrak{D}^3$ and $\mathfrak{D}^4$ that generate the Seidel triangle.

DEFINITION 4.1. Let $\mathcal{H}^3_{2n,2k}$ be the set of permutations $\pi \in \mathfrak{D}^3_{2n}$ such that $\pi(2) = 2k$. Let $\mathcal{G}^3_{2n,2k}$ be the set of permutations $\pi \in \mathfrak{D}^3_{2n}$ whose leftmost even letter is $2k$. Similarly, let $\mathcal{G}^1_{2n,2k}$ be the set of permutations $\pi \in \mathfrak{D}^1_{2n}$ whose leftmost even letter is $2k$, and let $\mathcal{H}^1_{2n,2k}$ be the set of permutations $\pi \in \mathfrak{D}^1_{2n}$ such that $\pi(1) = 2k$ and the descending run starting at $2k$ ends at 1.

THEOREM 4.2. $|\mathcal{H}^3_{2n,2k}| = g_{2n-1,n-k+1}$, *in particular,* $H_{2n-1} = |\mathcal{H}^3_{2n,2n}|$ *and* $H_{2n+1} = |\cup_{k=1}^{n} \mathcal{H}^3_{2n,2k}| = |\mathcal{H}^3_{2n,even}|$.

PROOF. Note that the function $h(n,k) = g_{2n-1,n-k+1}$ satisfies the recurrence

$$h(n,k) = h(n,k+1) + \sum_{j=1}^{k-1} h(n-1,j).$$

We will give a combinatorial proof that $|\mathcal{H}^3_{2n,2k}|$ satisfies the same recurrence.

We will write a permutation $\pi \in \mathcal{H}^3_{2n,2k}$ as a concatenation of "1" and four other blocks and interchange two of them to get a permutation $\pi' \in \mathcal{H}^3_{2n,2k+2}$. Let $\pi \in \mathcal{H}^3_{2n,2k}$ and write it as $\pi = 1ABCD$, where A, B, C, D are certain subblocks of π. If $A \neq \emptyset$, let $l(A)$ and $r(A)$ be the leftmost and rightmost letters of the block A, and define $l(B)$, $r(B)$, etc., similarly.

Then $l(A) = 2k$, $l(C) = 2k+2$, and π will map to $\pi' = f(\pi) = 1CBAD \in \mathcal{H}^3_{2n,2k+2}$. The blocks B and D are defined as follows.

All letters of B, if any, are greater than $2k+2$. All letters of D, if any, are odd and greater than $2k$ (so D forms an increasing sequence). Also, $D = \emptyset$ if $r(\pi) < 2k$ or $r(\pi) = \text{even} > 2k$.

If $r(AB) < 2k+2$, then $B = \emptyset$ and $r(C)$ is the rightmost letter of π that can precede $2k$ in a $\mathfrak{D}^3$-permutation (i.e. any $< 2k$ or even $> 2k$). So $l(D)$ is the leftmost odd letter of π greater than $2k$ that does not occur before any even letter.

Now suppose that $r(AB) > 2k+2$, then $B \neq \emptyset$, and of course $r(AB) = r(B)$ is even, since it is the letter preceding $2k+2$. If $r(\pi) < 2k$ or $r(\pi) = \text{even} > 2k$, then $D = \emptyset$. If $r(\pi) = \text{odd} > 2k$, then $D \neq \emptyset$. Consider the maximal suffix of π consisting of odd letters greater than $2k$. Then $l(D)$ is the rightmost letter in that suffix that is less than $r(B)$. Now B is the maximal block of letters greater than $l(D)$ and ending with $r(B)$.

Notice that the crucial step in constructing an inverse map from $1CBAD$ to $1ABCD$ is to find $l(D)$, and that $l(D)$ is the same for π and $f(\pi)$, *except* when $l(D) = 2k+1$ in π, so we have an "almost"-bijection $f : \mathcal{H}^3_{2n,2k} \to \mathcal{H}^3_{2n,2k+2}$. "Almost" here means that it is a bijection on permutations π where $D = \emptyset$ or $l(D) > 2k+1$. If $\pi \in \mathcal{H}^3_{2n,2k}$ has $l(D) = 2k+1$, then there is exactly one other permutation in $\mathcal{H}_{2n,2k}$ with $l(D) > 2k+1$ that has the same image as π.

Now consider $f(\pi) = 1CBAD$ such that

(1) $r(C) = c < 2k$,
(2) if $B \neq \emptyset$, then all letters of B are greater than $2k+2$, and B ends on some even $b = r(B) > 2k+2$,
(3) $l(A) = 2k$ and $r(A) = a \leq 2k$,
(4) $l(D) = 2k+1$ and D is an (increasing) string of odd letters,
(5) if $D = 2k+1 < d < \dots$, then $c < d$, and if $B \neq \emptyset$ and $r(B) = b$, then b¡d.

Now let $C = 2k+2, C'$, $D = 2k+1, D'$, and note that $\rho = 1C'BAD'$ is also "$\mathfrak{D}^3$-legal" (i.e. has descents only from even to even elements). Now subtract 2 from each letter of ρ greater than $2k$ to obtain a $\mathfrak{D}^3$-permutation $\sigma = 1C''B''A''D''$ of length $2n-2$. Now we only have to map it to a permutation τ that starts with an even element less than $2k$.

Note that $l(A'') = 2k$, $r(A'') = a$, $r(C'') = c$, all letters of B'' are greater than $2k$, $r(B'') = b'' = b-2 > 2k$ is even, $l(D'') = d'' = d-2$ is odd, and $b'' < d''$ and $c < 2k < d''$.

Let $C'' = EF$, where E is maximal prefix of C'' such that each of its letters is greater than $2k$. Then $r(E) = \text{even} > 2k$, and $l(F) = \text{even} < 2k$. From the above, it follows that $F \neq \emptyset$. However, either of E and B'' may be empty.

Then given $\sigma = 1EFB''A''D''$, define the map by $\tau = 1FEA''B''D''$. Note that τ is $\mathfrak{D}^3$-legal (see above), so $\tau \in \mathcal{H}^3_{2n,2j}$ for some $j < k$, and given any $\mathfrak{D}^3$-permutation $\tau \in \mathcal{H}^3_{2n,2j}$ with $j < k$, we can find F, E, A'', B'', D'' uniquely (easy to see from the above construction).

This ends the proof. □

EXAMPLE 4.3. Let $2n = 10$ and $2k = 4$, then

(1) $f(1\,4\,5\,6\,7\,9\,10\,2\,3\,8) = 1\,6\,7\,9\,10\,2\,3\,8\,4\,5$ since $r(\pi) = 8$ is even, so $B = D = \emptyset$.
(2) $f(1\,4\,7\,8\,10\,6\,2\,3\,5\,9) = 1\,6\,2\,3\,5\,10\,4\,7\,8\,9$ since $r(AB) = 10 > 6$, so $r(B) = 9$, $l(D) = 9$, $B = 10$, $D = 9$.
(3) $f(1\,4\,7\,10\,8\,6\,2\,3\,5\,9) = 1\,6\,2\,3\,7\,10\,8\,4\,5\,9$ since $r(AB) = 8 > 6$, so $r(B) = 8$, $l(D) = 5$, so $B = 7\,10\,8$, $D = 5\,9$. Since $l(D) = 5 = 4+1$, this is one of the "extra" permutations, so we delete 5 and 6 from the image and subtract 2 from every letter greater than 4 to get $\sigma = 1\,2\,3\,5\,8\,6\,4\,7$. Hence, $A'' = 4$, $B = 5\,8\,6$, $C'' = 2\,3$, $D'' = 7$, so $E = \emptyset$, $F = 2\,3$, and therefore $\tau = 1\,2\,3\,4\,5\,8\,6\,7 \in \mathcal{H}_{8,2}$.

Another, less direct way to prove this theorem is to notice that the $\mathfrak{D}^1_{2n} \to \mathfrak{D}^3_{2n}$ bijection described in Example 2.12 preserves the value of the leftmost peak, i.e. the value of the first even letter. In fact, the following is true.

Lemma 4.4. *The map $\mathfrak{D}^1_{2n} \to \mathfrak{D}^3_{2n}$ from Section 2 maps $\mathcal{H}^1_{2n,2k}$ onto $\mathcal{H}^3_{2n,2k}$ and $\mathcal{G}^1_{2n,2k}$ onto $\mathcal{G}^3_{2n,2k}$.*

Proof. Omitted. □

Then $|\mathcal{H}^1_{2n,2k}| = |\mathcal{H}^3_{2n,2k}|$ and $|\mathcal{G}^1_{2n,2k}| = |\mathcal{G}^3_{2n,2k}|$, so we only need to prove the following theorem.

Theorem 4.5. *$|\mathcal{H}^1_{2n,2k}| = g_{2n-1,n-k+1}$ and $|\mathcal{G}^1_{2n,2k}| = g_{2n,n-k+1}$. In particular, $H_{2n-1} = |\mathcal{H}^1_{2n,2n}|$, $H_{2n+1} = |\cup_{k=1}^n \mathcal{H}^1_{2n,2k}| = |\mathcal{H}^1_{2n,even}|$, $G_{2n-2} = |\mathcal{G}^1_{2n,2}|$, $G_{2n} = |\cup_{k=1}^n \mathcal{G}^1_{2n,2k}| = |\mathcal{G}^1_{2n,even}| = |\mathfrak{D}^1_{2n}|$.*

Proof. Again, we will prove that that the functions $h(n,k) = |\mathcal{H}^1_{2n,2k}|$ and $g(n,k) = |\mathcal{G}^1_{2n,2k}|$ satisfy the same recurrences as $g_{2n-1,n-k+1}$ and $g_{2n,n-k+1}$, respectively:

$$h(n,k) = h(n,k+1) + \sum_{j=1}^{k-1} h(n-1,j), \tag{4.2}$$

$$g(n,k) = g(n,k-1) + \sum_{j=k-1}^{n-1} g(n-1,j). \tag{4.3}$$

We will now describe two bijections, β_1 and β_2, that prove the recurrences (4.2) and (4.3).

Bijection β_1. Given a permutation that starts with $2k, \dots, 2, 1$:

(1) exchange $2k$ and $2k+2$ if $2k+2$ is not preceded or followed by $2k+1$,
(2) if $2k+2$ is followed by $2k+1$, remove $2k+2$ and insert it in front of $2k$ (i.e. at the beginning),
(3) if $2k+2$ is preceded by $2k+1$, then remove $2k$ (at the beginning), replace the block $2k+1, 2k+2$ with $2k$, and subtract 2 from every letter greater than $2k+2$.

The first two cases yield a permutation of the same size that starts with $2k+2, \dots, 2, 1$. The last case yields a permutation of size 2 less than the original permutation, that starts with an even letter less than $2k$.

Bijection β_2. Given a permutation with first even letter $2k+2$:

(1) if $2k+2$ is not preceded or followed by $2k+1$ and not followed by $2k$, then exchange $2k$ and $2k+2$,
(2) if $2k+2$ is followed by $2k$ and not preceded by $2k+1$, then remove $2k+2$ and insert it in front of $2k+1$,
(3) if $2k+2$ is followed by $2k+1$, remove $2k+2, 2k+1$ and subtract 2 from every letter greater than $2k+2$.
(4) if $2k+2$ is preceded by $2k+1$ and followed by $2k$, remove $2k+1, 2k+2$ and subtract 2 from every letter greater than $2k+2$.

The first two cases yield a permutation of the same size that with first even letter $2k$. The last two cases yield a permutation of size 2 less than the original permutation, with the first even letter at least $2k$. □

We conjecture that Theorem 4.5 can be refined further as follows. Consider $\mathfrak{D}^3$-permutations in the set difference $\mathcal{G}^3_{2n,2k} \setminus \mathcal{H}^3_{2n,2k}$, i.e. those whose leftmost even value $2k$ does not occur immediately after 1.

Conjecture 4.6. *Given $m < k \le n$, let $f(n,k,m)$ be the number of permutations in $\mathcal{G}^3_{2n,2k} \setminus \mathcal{H}^3_{2n,2k}$ whose rightmost (alternatively, leftmost after $2k$) even entry less than $2k$ is $2m$. Then $f(n,k,m) = h(n,m) = g_{2n,n-m+1}$.*

Note that $f(n,k,m)$ is apparently independent of k. In fact, trying to prove Conjecture 4.6 by the same method as Theorem 4.2, we conjecture that the overage

$$f(n,k,m) - f(n,k,m+1) = h(n,m) - h(n,m+1) = g_{2n-2,n-m+1} = f(n-1,k-1,m-1)$$

counts permutations where $2k$ is the leftmost even value, $2m$ is the rightmost even value less than $2k$, and $2m-1$ immediately precedes $2m$.

Finally, it seems that we can also obtain the even columns of the Seidel triangle using a different statistic on $\mathfrak{D}^3$.

CONJECTURE 4.7. *The number of $\mathfrak{D}^3$-permutations where $2k$ immediately follows $2n$ is $g_{2n,k}$. Equivalently, the number of $\mathfrak{D}^3$-permutations where $2k$ immediately precedes 2 is $g_{2n,n-k+1}$.*

This last conjecture can be easily restated for $\mathfrak{D}^4$-permutations as follows.

CONJECTURE 4.8. *The number of $\mathfrak{D}^4$-permutations that end on $2k$ is $g_{2n,k}$. Equivalently, the number of $\mathfrak{D}^3$-permutations where 2 is in position $2k$ is $g_{2n,n-k+1}$.*

5. Restricted Dumont-III permutations

Here we will consider some pattern-restricted sets of Dumont-III permutations. Recall that a *pattern* is an order-isomorphism type of a subsequence. Thus, for example, a instance of a pattern 231 in a permutation π is a subsequence (a,b,c) of π such that $c < a < b$ and $\pi^{-1}(a) < \pi^{-1}(b) < \pi^{-1}(c)$. A permutation *avoids* a pattern if it contains no instances of it. A *generalized pattern* [**1**] is a pattern where some consecutive terms of a subsequence must also be consecutive in the whole permutation. For example, an instance of 2-31 in π is an instance of 231 where elements corresponding to "3" and "1" are consecutive in π. We denote the subset of permutations of a set S avoiding a certain pattern τ by $S(\tau)$.

Dumont-I and Dumont-II permutations avoiding various small patterns were considered in [**2, 3, 11**]. We will mention one result in particular:

THEOREM 5.1 (Mansour [**11**]). $|\mathfrak{D}^1_{2n}(231)| = |\mathfrak{D}^1_{2n}(312)| = C_n$*, the n-th Catalan number.*

On $\mathfrak{D}^3$, we also have a similar theorem.

THEOREM 5.2. $|\mathfrak{D}^3_{2n}(231)| = |\mathfrak{D}^3_{2n}(312)| = C_n$.

PROOF. The first equality follows from the fact that $\mathfrak{D}^3$-permutations without the leftmost value 1 are centrally symmetric, and 312 is obtained from 231 by 180°-rotation. Now let us prove that $|\mathfrak{D}^3_{2n}(231)| = C_n$.

Let $\pi \in \mathfrak{D}^3_{2n}(231)$, and let $a = \pi(2)$. (Recall that $\pi(1) = 1$.) Then $\pi = 1a\pi'\pi''$, where $\pi' < a < \pi''$, since all values of π smaller than a must occur in to the left of all values greater than a so as to avoid an instance of pattern 231. Thus, either $\pi' = \emptyset$ and hence $a = 2$, or $\pi'' \neq \emptyset$, and hence a is followed by a descent, so a is even (and $a > 2$). Thus, in either case a must be even, say $a = 2k$. Hence, it is easy to see that $\pi_1 = 1\pi' \in \mathfrak{D}^3_{2k-2}(231)$ and $\pi_2 = \pi'' - 2k \in \mathfrak{D}^3_{2n-2k}(231)$ (i.e. $\pi_2(i) = \pi''(i) - 2k$ for all i). This implies the theorem. □

The recursive argument in the above proof can be easily adapted to give a direct bijection between $\mathfrak{D}^1_{2n}(231)$ and $\mathfrak{D}^3_{2n}(231)$.

This is a nice result, but much more seems to be true.

CONJECTURE 5.3. *The bijection of Section 2 between $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^3_{2n}$ preserves the bistatistic of the numbers of occurrences of patterns 2-31 and 31-2.*

Since 2-31 and 31-2 are obviously equidistributed on $\mathfrak{D}^3$ as reverse complements of each other, this would imply that 2-31 and 31-2 are also equidistributed on $\mathfrak{D}^1$.

Moreover, for $\mathfrak{D}^1$-permutations we conjecture that the distribution of the number of occurrences of patterns 2-31 and 31-2 satisfies the following property.

CONJECTURE 5.4. *Let $d(2n,m,i,j)$ be the number of $\mathfrak{D}^1$-permutations of $[2n]$ starting with m and containing i occurrences of 2-31 and j occurrences of 31-2. Then for each $k \in [n-1]$,*

$$d(2n, n+1-k, i, j) = d(2n, n+1+k, i, j+k),$$

and $d(2n,m,i,j) = 0$ if $i > \binom{2n}{2} - (n+1-m)$ or $j > \binom{2n-1}{2} - \lceil \frac{2n-m}{2} \rceil$.

We intend to prove some of these conjectures in the journal version of the paper.

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