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Descent Patterns in Permutations

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ABSTRACT. We consider the problem of finding generating functions for permutations which contain a given descent set. We expand on recent work of Remmel and Riehl, who developed a systematic way to find generating functions for permutations which contain a given finite descent set S, by showing that in many cases we can find generating functions for permutations σ in the symmetric group S_n whose descent set contains $S \cap \{1, \ldots, n\}$ for certain infinite sets S of natural numbers.

RÉSUMÉ.Nous considérons le probléme de trouver des fonctions génératives pour les permutations qui contiennent un ensemble donné de descente. Nous examinons les travaux récents de Remmel et de Riehl, qui ont développé une maniére systématique de trouver des fonctions génératives pour les permutations qui contiennent une descente finie donnée S réglé, en prouvant que dans plusieurs cas nous pouvons trouver des fonctions génératives pour la permutation σ dedans le groupe symétrique S_n dont l'ensemble de descente contient $S \cap \{1, \ldots, n\}$ pour certains ensembles infinis S des nombres normaux.

1. Introduction

There has been a long line of research [**2**], [**3**], [**1**], [**8**], [**9**], [**12**], [**13**], [**14**], [**17**], [**11**], which shows that a large number of generating functions for permutation statistics can be obtained by applying homomorphisms defined on the ring of symmetric functions Λ to simple symmetric function identities. For example, the n-th elementary symmetric function, e_n and the n-th homogeneous symmetric function, h_n, are defined by the generating functions $E(t) = \sum_{n\geq 0} e_n t^n = \prod_i (1 + x_i t)$ and $H(t) = \sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1-x_i t}$. We let $P(t) = \sum_{n\geq 0} p_n t^n$ where $p_n = \sum_i x_i^n$ is the n-th power symmetric function. For any partition, $\mu = (\mu_1, \ldots, \mu_\ell)$, we let $h_\mu = \prod_{i=1}^{\ell} h_{\mu_i}$, $e_\mu = \prod_{i=1}^{\ell} e_{\mu_i}$, and $p_\mu = \prod_{i=1}^{\ell} p_{\mu_i}$. Now it is well known that

$$H(t) = 1/E(-t) \tag{1.1}$$

and

$$P(t) = \frac{\sum_{n\geq 1}(-1)^{n-1} n e_n t^n}{E(-t)}. \tag{1.2}$$

A surprising large number of results on generating functions for various permutation statistics in the literature and large number of new generating functions can be derived by applying homomorphisms on Λ to simple identities such as (1.1) and (1.2).

Let S_n denote the symmetric group and write $\sigma \in S_n$ in one line notation as $\sigma = \sigma_1 \ldots \sigma_n$. In this section, we shall consider the following statistics on S_n.

$$\begin{array}{ll} Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} & Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\} \\ des(\sigma) = |Des(\sigma)| & rise(\sigma) = |Rise(\sigma)| \\ inv(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j) & coinv(\sigma) = \sum_{i<j} \chi(\sigma_i < \sigma_j) \end{array}$$

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where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Also if $\alpha^1, \ldots, \alpha^k \in S_n$, then we shall write $comdes(\alpha^1, \ldots, \alpha^k) = |\bigcap_{i=1}^k Des(\alpha^i)|$. We should also note that these definitions make sense for any sequence $\sigma = \sigma_1 \cdots \sigma_n$ of natural numbers. We shall also use standard notation for q and p,q analogues. That is, we let $[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, and $\begin{bmatrix} n \\ \lambda_1,\ldots,\lambda_\ell \end{bmatrix}_q = \frac{[n]_q!}{[\lambda_1]_q!\cdots[\lambda_\ell]_q!}$. Similarly we can define p,q-analogues of all these formulas by replacing $[n]_q$ by $[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + p^1q^{n-2} + q^{n-1} = \frac{p^n - q^n}{p-q}$ in the formulas. Then all of the following results can proved by applying a suitable homomorphism to the identity (1.1).

1) $\sum_{n=0}^\infty \frac{u^n}{n!} \sum_{\sigma \in S_n} x^{des(\sigma)} = \frac{1-x}{-x+e^{u(x-1)}}$.

2) (Carlitz 1970) [**4**]
$\sum_{n=0}^\infty \frac{u^n}{(n!)^2} \sum_{(\sigma,\tau) \in S_n \times S_n} x^{comdes(\sigma,\tau)} = \frac{1-x}{-x+J(u(x-1))}$.

3) (Stanley 1979) [**16**]
$\sum_{n=0}^\infty \frac{u^n}{[n]!} \sum_{\sigma \in S_n} x^{des(\sigma)} q^{inv(\sigma)} = \frac{1-x}{-x+e_q(u(x-1))}$.

4) (Stanley 1979) [**16**]
$\sum_{n=0}^\infty \frac{u^n}{[n]!} \sum_{\sigma \in S_n} x^{des(\sigma)} q^{coinv(\sigma)} = \frac{1-x}{-x+E_q(u(x-1))}$.

5) (Fedou and Rawlings 1995) [**7**]
$\sum_{n=0}^\infty \frac{u^n}{[n]_q![n]_p!} \sum_{(\sigma,\tau) \in S_n \times S_n} x^{comdes(\sigma,\tau)} q^{inv(\sigma)} p^{inv(\tau)} = \frac{1-x}{-x+J_{q,p}(u(x-1))}$,
where $J(u) = \sum_{n \geq 0} \frac{u^n}{n!n!}$, $e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} q^{\binom{n}{2}}$, $E_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!}$, $J_{q,p}(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} q^{\binom{n}{2}} p^{\binom{n}{2}}$.

Recently, Remmel and Riehl [**15**] developed a method to extend results like then ones given in (1)-(5) to find similar generating functions involving the sum over all permutations $\sigma \in S_n$ such that $S \subseteq Des(\sigma)$ where S is some fixed finite subset of the positive numbers $\{1, 2, \ldots\}$. For example, their methods gave a systematic way to find the following generating functions for any $S \subset \{1, 2, \ldots\}$.

1_S) $\sum_{n=0}^\infty \frac{u^n}{n!} \sum_{\sigma \in S_n, S \subseteq Des(\sigma)} x^{des(\sigma)} y^{ris(\sigma)}$

2_S) $\sum_{n=0}^\infty \frac{u^n}{(n!)^2} \sum_{(\sigma,\tau) \in S_n \times S_n, S \subseteq Comdes(\sigma,\tau)} x^{comdes(\sigma,\tau)} y^{comris(\sigma,\tau)}$

3_S) $\sum_{n=0}^\infty \frac{u^n}{[n]_q!} \sum_{\sigma \in S_n, S \subseteq Des(\sigma)} x^{des(\sigma)} y^{ris(\sigma)} q^{inv(\sigma)}$.

4_S) $\sum_{n=0}^\infty \frac{u^n}{[n]_{q,p}!} \sum_{\sigma \in S_n, S \subseteq Des(\sigma)} x^{des(\sigma)} y^{ris(\sigma)} q^{inv(\sigma)} p^{coinv(\sigma)}$.

5_S) $\sum_{n=0}^\infty \frac{u^n}{[n]_q![n]_p!} \sum_{(\sigma,\tau) \in S_n \times S_n, S \subseteq Comdes(\sigma,\tau)} x^{comdes(\sigma,\tau)} q^{inv(\sigma)} p^{inv(\tau)}$.

Let $\mathbb{N}$ denote the set of natural numbers and S be an infinite set $S \subseteq \mathbb{N}$. Then one can also ask for generating functions for the set of permutations σ of S_n such that $S \cap \{1, \ldots, n\} \subseteq Des(\sigma)$ or $S \cap \{1, \ldots, n\} = Des(\sigma)$. For example, if O equals the odd numbers, then saying $S \cap \{1, \ldots, n\} = Des(\sigma)$ implies that σ is an up-down permutation or alternating permutations. It is well known that the exponential generating function of the set of alternating permutations of odd length is $tan(x)$ and the exponential generating function of the set of alternating permutations of even length is $sec(x)$. The main goal of this paper is to outline some methods that will allow us to find generating functions like (1)-(5) for permutations σ of S_n such that $S \cap \{1, \ldots, n\} = Des(\sigma)$ where S is of the form $k\mathbf{N} - T$ where T is finite set and $k\mathbb{N} = \{k, 2k, 3k, \ldots\}$.

The outline of this paper is as follows. In section 1, we shall supply the necessary background on symmetric functions and the combinatorics of the entries of the transition matrices between various bases of symmetric functions that we need for our developments. In section 2, we shall review the methods of Remmel and Riehl [**15**] to find generating functions (1_S) - (5_S) where S is finite set. In section 3, we shall explain how to find generating functions for permutations $\sigma \in S_n$ such that $Des(\sigma) = k\mathbb{N} \cap \{1, \ldots, n\}$ based on the ideas of Mendes [**11**]. Then in section 4, we shall describe how we can combine the two methods to find generating functions for permutations $\sigma \in S_n$ such that $Des(\sigma) = (k\mathbb{N} - T) \cap \{1, \ldots, n\}$ where T is any finite subset of $k\mathbb{N}$.

2. Symmetric functions and transition matrices

In this section, we shall present the background on symmetric functions and the combinatorics of the transition matrices between various bases of symmetric functions that will be needed for our methods.

Let Λ_n denote the space of homogeneous symmetric functions of degree n over infinitely many variables $x_1, x_2, \ldots$. We say that $\lambda = (0 \leq \lambda_1 \leq \cdots \leq \lambda_k)$ is a partition of n, written $\lambda \vdash n$ if $\lambda_1 + \ldots + \lambda_k = n = |\lambda|$. We let $\ell(\lambda)$ denote the number of parts of λ. It is well known that $\{h_\lambda : \lambda \vdash n\}$, $\{e_\lambda : \lambda \vdash n\}$, and $\{p_\lambda : \lambda \vdash n\}$ are all bases of Λ_n, see [**10**].

We let F_λ denote the Ferrers diagram of λ. If $\mu = (\mu_1, \ldots, \mu_m)$ is a partition where $m \leq k$ and $\lambda_i \geq \mu_i$ for all $i \leq m$, we let $F_{\lambda/\mu}$ denote the skew shape that results by removing the cells of F_μ from F_λ. For example, Figure 1 pictures the skew diagram $(1,2,3,3)/(1,2)$ on the left. A *column-strict tableau* T of shape λ/μ is any filling of $F_{\lambda/\mu}$ with natural numbers such that entries in each row are weakly increasing from left to right, and entries in each column are strictly increasing from bottom to top. We define the weight of T to be $w(T) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ where α_i is the number of times that i occurs in T. For example, on the right of Figure 1, we have pictured a column strict tableau of shape $(1,2,3,3)/(1,2)$ and weight $x_1^2 x_2 x_3 x_4^2$. Then the skew Schur function $s_{\lambda/\mu} = \sum_T w(T)$ where the sum runs over all column strict tableaux of shape λ/μ. We define a ribbon (or zigzag) shape to be a connected skew shape that contains no 2 x 2 array of

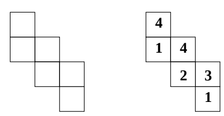

Figure 1. The skew Ferrers diagram and column strict tableau of shape $(1,2,3,3)/(1,2)$.

boxes. Ribbon (or zigzag) Schur functions are the skew Schur functions with a ribbon shape and are indexed by compositions. A composition $\beta = (\beta_1, \ldots, \beta_k)$ of n, denoted $\beta \models n$, is a sequence of positive integers such that $\beta_1 + \beta_2 + \cdots + \beta_k = n$. Given a composition $\beta = (\beta_1, \ldots, \beta_k)$, we let Z_β denote the skew Schur function corresponding to the zigzag shape whose row lengths are $\beta_1, \ldots, \beta_k$ reading from top to bottom. For example Figure 2 shows the zigzag shape corresponding to the composition $(2,3,1,4)$. We let $\lambda(\beta)$ denote the partition that arises from β by arranging its parts in weakly increasing order and $\ell(\beta)$ denote the number of parts of β. For example, if $\beta = (2,3,1,2)$, then $\lambda(\beta) = (1,2,2,3)$. We also define *shape*$(\beta) = \lambda/\nu$ such that $F_\beta = F_{\lambda/\nu}$. A *rim hook* of λ is a connected sequence of cells, h, along the northeast boundary of F_λ

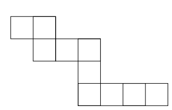

Figure 2. The ribbon shape corresponding to the composition $(2,3,1,4)$, so that $s_{(2,4,4,7)/(1,3,3)} = Z_{(2,3,1,4)}$.

which has a ribbon shape and is such that if we remove h from F_λ, we are left with the Ferrers diagram of another partition. More generally, h is a rim hook of a skew shape λ/μ if h is a rim hook of λ which does not intersect μ. We say that h is a *special rim hook* of λ/μ if h starts in the cell which occupies the north-west corner of λ/μ. We say that h is a *transposed special rim hook* of λ/μ if h ends in the cell which occupies the south-east corner of λ/μ.

A *special rim hook tabloid* (transposed special rim hook tabloid) of shape λ/μ and type ν, T, is a sequence of partitions $T = (\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \lambda^{(k)} = \lambda)$, such that for each $1 \leq i \leq k, \lambda^{(i)}/\lambda^{(i-1)}$ is a special rim hook (transposed special rim hook) of $\lambda^{(i)}$ such that the weakly increasing rearrangement of $(\lambda^{(1)}/\lambda^{(0)}, \cdots, \lambda^{(k)}/\lambda^{(k-1)})$ is equal to ν. We show an example of a special rim hook tabloid and a transposed special rim hook tabloid of shape $(4,5,6,6)/(1,3,3)$ in 3. We define the sign of a special rim hook $h_i = \lambda^{(i)}/\lambda^{(i-1)}$ to be $sgn(h_i) = (-1)^{r(h_i)-1}$, where $r(h_i)$ is the number of rows that h_i occupies. Likewise we define the transposed sign of a transposed special rim hook to be $t\text{-}sgn(h_i) = (-1)^{c(h_i)-1}$, where

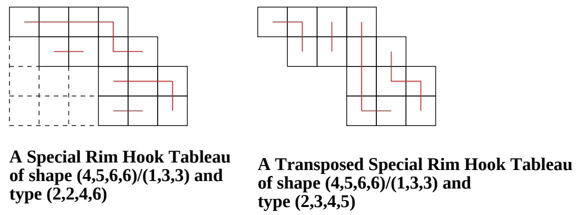

FIGURE 3. An example of a special rim hook tableau and a transposed special rim hook tableau.

$c(h_i)$ is the number of columns that h_i occupies. Let $SRHT(\nu,\lambda/\mu)$ (t-$SRHT(\nu,\lambda/\mu)$) equal the set of special rim hook tabloids (transposed special rim hook tabloids) of type ν and shape λ/μ. If $T \in SRHT(\nu,\lambda/\mu)$, we let $sgn(T) = \prod_{H\in T} sgn(H)$. If $T \in t\text{-}SRHT(\nu,\lambda/\mu)$, then $t\text{-}sgn(T) = \prod_{H\in T} t\text{-}sgn(H)$. For $|\lambda/\mu| = |\nu|$, we let

$$K^{-1}_{\nu,\lambda/\mu} = \sum_{T\in SRHT(\nu,\lambda/\mu)} sgn(T) \quad \text{and} \quad TK^{-1}_{\nu,\lambda/\mu} = \sum_{T\in t\text{-}SRHT(\nu,\lambda/\mu)} sgn(T). \tag{2.1}$$

Then Eğecioğlu and Remmel [**5**] proved that

$$s_{\lambda/\mu} = \sum_{\nu} K^{-1}_{\nu,\lambda/\mu} h_\nu \quad \text{and} \quad s_{\lambda/\mu} = \sum_{\nu} TK^{-1}_{\nu,\lambda/\mu} e_\nu. \tag{2.2}$$

Eğecioğlu and Remmel [**6**] also proved that

$$h_\mu = \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,\mu} e_\lambda, \tag{2.3}$$

where $B_{\lambda,\mu}$ is the number of λ-brick tabloids of shape μ. Here a λ-brick tabloid T of shape μ is filling of F_μ with bricks of sizes corresponding the parts of λ such that (i) no two bricks overlap and (ii) each brick lies within a single row. For example, the $(1,1,2,2)$-brick tabloids of shape $(2,4)$ are pictured in Figure 4. More

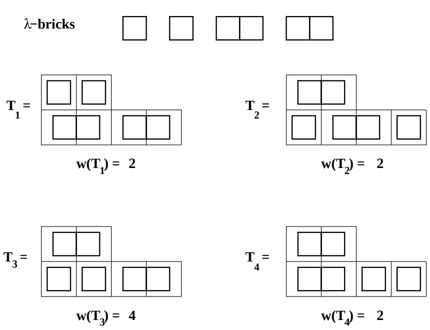

FIGURE 4. $B_{\lambda,\mu}$ and $w(B_{\lambda,\mu})$ for $\lambda = (1,1,2,2)$ and $\mu = (2,4)$.

generally, let $\mathcal{B}_{\lambda,\mu}$ denote the set of λ-brick tabloids of shape $\mu = (\mu_1,\ldots,\mu_k)$. Suppose that R is a ring and we are given any sequence of $\vec{u} = (u_1,u_2,\ldots)$ of elements of R. Then for any brick tabloid $T \in \mathcal{B}_{\lambda,\mu}$ we let $(b_1,\ldots,b_k)$ denote the lengths of the bricks which lie at the right end of the rows of T reading from top to bottom and we set $w_{\vec{u}}(T) = u_{b_1}\cdots u_{b_k}$. We then set $w_{\vec{u}}(B_{\lambda,\mu}) = \sum_{T\in\mathcal{B}_{\lambda,\mu}} w_{\vec{u}}(T)$. For example if $u = (1,2,3,\ldots)$, then $w_{\vec{u}}(T) = w(T)$ is just the product of the lengths of the bricks that lie at the end of the rows of T. We have given $w(T)$ for each of the brick tabloids in Figure 4. This given, we can define a new family of symmetric functions $p^{\vec{u}}_\lambda$ as follows. First we let $p^{\vec{u}}_0 = 1$ and

$$p^{\vec{u}}_n = \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} w_{\vec{u}}(B_{\lambda,\mu}) e_\lambda \tag{2.4}$$

for $n \geq 1$. Finally if $\mu = (\mu_1,\ldots,\mu_k)$ is a partition of n, we set $p^{\vec{u}}_\mu = p^{\vec{u}}_{\mu_1}\cdots p^{\vec{u}}_{\mu_k}$. We note that it follows from results of Eğecioğlu and Remmel [**6**] that if $u = (1,2,3,\ldots)$, then $p^{\vec{u}}_n$ is just the usual power symmetric function p_n. Thus we call $p^{\vec{u}}_n$ a generalized power symmetric function.

Mendes and Remmel [**13, 12**] proved the following:

$$\sum_{n\geq 1} p_n^{\vec{u}} t^n = \frac{\sum_{n\geq 1}(-1)^{n-1}u_n e_n t^n}{E(-t)} \quad \text{and} \quad 1+\sum_{n\geq 1} p_n^{\vec{u}} t^n = \frac{1+\sum_{n\geq 1}(-1)^{n}(e_n - u_n e_n) t^n}{E(-t)} \tag{2.5}$$

3. The methods of Remmel and Riehl

In this section, we shall briefly outline the methods that Remmel and Riehl [**15**] used to find expressions for (1_S)-(5_S) where S is any finite set of positive integers. We will exhibit this in only the simplest case, namely, we will outline how one can compute

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in S_n, S\subseteq Des(\sigma)} x^{des(\sigma)}. \tag{3.1}$$

The method proceeds in three steps. First, for any composition $\alpha=(\alpha_1,\ldots,\alpha_k)$, of n, define $h_\alpha = h_{\alpha_1}\cdots h_{\alpha_k}$ and $Set(\alpha)=\{\alpha_1,\alpha_1+\alpha_2,\ldots,\alpha_1+\cdots+\alpha_{k-1}\}$. Then for $\sigma\in S_n$, we define $Des_\alpha(\sigma)=Des(\sigma)-Set(\alpha)$ and $des_\alpha(\sigma)=|Des_\alpha(\sigma)|$. Define a homomorphism ξ from the ring of symmetric functions Λ to the polynomial ring $\mathbb{Q}[x]$ by setting

$$\xi(e_n)=\frac{(1-x)^{n-1}}{n!}. \tag{3.2}$$

Then the following result is due to Beck and Remmel [**1**].

THEOREM 3.1.

$$n!\xi(h_\alpha)=\sum_{\sigma\in S_n} x^{des_\alpha(\sigma)}. \tag{3.3}$$

Next given a composition of n, $\alpha=(\alpha_1,\ldots,\alpha_k)$, let F_α denote the ribbon shape corresponding to α and Z_α denote the ribbon Schur function corresponding to α. Then Remmel and Riehl proved the following.

THEOREM 3.2.

$$n!\xi(Z_\alpha)=\frac{(x-1)^{k-1}}{x^{k-1}}\sum_{\sigma\in S_n, Set(\alpha)\subseteq Des(\sigma)} x^{des(\sigma)}. \tag{3.4}$$

PROOF. Here we give a sketch of the proof. We can expand the Z_α using using (2.2) as

$$Z_\alpha = S_{\lambda/\nu} = \sum_{\mu\vdash|\lambda/\nu|} h_\mu \sum_{T\in SRHT(\mu,\lambda/\nu)} sgn(T). \tag{3.5}$$

Applying ξ to both sides of (3.5),

$$\begin{aligned} n!\xi(Z_\alpha) &= n!\sum_{\mu\vdash n}\xi(h_\mu)\sum_{T\in SRHT(\mu,\lambda/\nu)} sgn(T) = \sum_{T\in SRHT(shape(\alpha))} sgn(T) n!\xi(h_{\beta(T)}) \\ &= \sum_{T\in SRHT(shape(\alpha))} sgn(T)\sum_{\sigma\in S_n} x^{des_{\beta(T)}(\sigma)}. \end{aligned}$$

where $\beta(T)$ is composition induced by reading the rim hooks in T from top to bottom. Thus we can identify $n!\xi(Z_\alpha)$ with a sum over filled special rim hook tabloids $F(T)$ where each cell of the underlying special rim hook tabloid is filled with a number in such a way that when we read the numbers in cells starting from the top, we get a permutation $\sigma_{F(T)}\in S_n$. In figure 5, we show several examples of such fillings with special rim hook tabloids when $\alpha=(3,1,3,2)$. The special rim hook tabloids T on the left have $\beta(T)=(4,5)$ while the special rim hook tabloids on the right have $\beta(T)=(3,1,5)$. The weight of such a filled special rim hook tabloid $F(T)$ is $sgn(T)x^{des_\alpha(\sigma_{F(T)})}$ so we have put an x on top of each cell that contributes to $des_\alpha(\sigma_{F(T)})$.

Let us examine the fillings in pairs with identical integer fillings and identical special rim hooks except that one has a break in the special rim hooks at the first vertical segment and the other one does not. It is easy to see that the sign of the underlying special rim hook tabloid T is -1 raised to number of vertical segments that are part of rim hooks so that any two such special rim hook tabloids must differ in sign. Then one can easily see that if there is an increase in the permutation over the first vertical step, then there is no difference in weights of the two filled special rim hook tabloids so that that any two such filled special rim

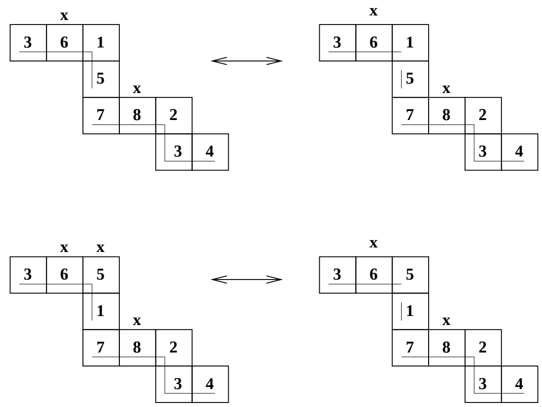

FIGURE 5. Pairing of special rim hook tabloids of shape $F_{(3,1,3,2)}$

hook tabloids contribute 0 to $\sum_{T\in SRHT(shape(\alpha))} sgn(T)\sum_{\sigma\in S_n} x^{des_{\beta(T)}(\sigma)}$. However if there is a decrease in the permutation over the first vertical step, then the first vertical segment will contributes a factor of $-x$ if the special rim hook contains the first vertical segment and contributes a factor of 1 if the special rim hook does not contain the first vertical segment. It this situation, we can replace the pair by $(1-x)$ times the weight of the filled special rim hook tabloid whose first special rim hook does not contain the first vertical segment. Thus we can replace the sum $\sum_{T\in SRHT(shape(\alpha))} sgn(T)\sum_{\sigma\in S_n} x^{des_{\beta(T)}(\sigma)}$ by the sum $(1-x)\sum_{F(T),\sigma_{F(T)}} x^{des_{\beta(T)}(\sigma_{F(T)})}$ where the sum runs over all filled special rim hook tabloids $F(T)$ such that first special rim hook of T is horizontal and the permutation $\sigma_{F(T)}$ has a descent at the first vertical segment. We can further partition such filled special rim hook tabloids according to whether the second vertical segment is part of a special rim hook and use a similar argument to conclude that we can replace the sum $(1-x)\sum_{F(T),\sigma_{F(T)}} x^{des_{\beta(T)}(\sigma_{F(T)})}$ by $(1-x)^2\sum_{F(T*),\sigma_{F(T*)}} x^{des_{\beta(T*)}(\sigma_{F(T*)})}$ where the sum runs over all filled special rim hook tabloids $F(T*)$ such that first 2 special rim hooks of $T*$ are horizontal and the permutation $\sigma_{F(T*)}$ has descents at the first two vertical segments.

Continuing on in this way, we can show that if α has k parts, then $n!\xi(Z_\alpha)$ is equal to the sum $(1-x)^{k-1}\sum_{F(T**),\sigma_{F(T**)}} x^{des_{\beta(T**)}(\sigma_{F(T**)})}$ where the sum runs over all filled special rim hook tabloids $F(T**)$ such that all special rim hooks of $T**$ are horizontal and the permutation $\sigma_{F(T**)}$ has descents at any vertical segment. That is,

$$\begin{aligned} n!\xi(Z_\alpha) &= \sum_{T\in SRHT(shape(\alpha))} sgn(T)\sum_{\sigma\in S_n} x^{des_{\beta(T)}(\sigma)} \\ &= (1-x)^{k-1}\sum_{\sigma\in S_n, Set(\alpha)\subseteq Des(\sigma))} x^{des_\alpha(\sigma)} \\ &= \frac{(1-x)^{k-1}}{x^{k-1}}\sum_{\sigma\in S_n, Set(\alpha)\subseteq Des(\sigma))} x^{des(\sigma)}. \end{aligned}$$

□

The third step of the Remmel-Riehl method is to obtain an appropriate generating function for ribbon Schur functions. To this end, let $\alpha=(\alpha_k,\alpha_{k-1},\ldots,\alpha_1)$ be a composition. For $j=1,\ldots,k-1$, let $\alpha^{(j)}=(\alpha_k,\ldots,\alpha_{j+1})$. In other words, $\alpha^{(j)}$ removes the last j parts of α, with $\alpha^{(0)}=\alpha$ and $\alpha^{(k)}=\emptyset$. Let $(\alpha,n)=(\alpha_k,\alpha_{k-1},\ldots,\alpha_1,n)$. In other words, we add n as the final part of the composition. Let $Z_\emptyset=1$. Let $\overline{Set}_\alpha=\{\alpha_k,\alpha_k+\alpha_{k-1},\ldots,\alpha_k+\cdots+\alpha_1\}$. Finally, we define $E(-t)=\sum_{n\geq 0}(-t)^n e_n$ and $E_{q,p}(w)=\sum_{n\geq 0}\frac{(w)^n q^{\binom{n}{2}}}{[n]_{p,q}!}$.

THEOREM 3.3. *Let α be a composition of length k. Then*

$$\sum_{n\geq 1} Z_{\alpha,n} t^{n+|\alpha|} = \frac{\sum_{j=0}^{k}(-1)^j Z_{\alpha^{(j)}} t^{|\alpha^{(j)}|}}{E(-t)} - ((-1)^k + \sum_{j=1}^{k}(-1)^j \sum_{r=1}^{\alpha_j -1} Z_{(\alpha^{(j)},r)} t^{r+|\alpha^{(j)}|})$$

The proof follows by grouping by the length of the last hook in the special rim hook filling used to expand Z_α in terms of h_μ. For example, if $\alpha = (3,2,1,3)$, then

$$\begin{aligned}\sum_{n\geq 1} Z_{(\alpha,n)} t^{n+|\alpha|} &= \frac{Z_{(3,2,1,3)}t^9 - Z_{(3,2,1)}t^6 + Z_{(3,2)}t^5 - Z_{(3)}t^3 + 1}{E(-t)} - \\ & (1 - (Z_{(3,2,1,2)}t^8 + Z_{(3,2,1,1)}t^7) - (Z_{(3,1)}t^4) + (Z_{(2)}t^2 + Z_{(1)}t)).\end{aligned}$$

We can then combine Theorem 3.3 with Theorem 4.1 to obtain the following.

THEOREM 3.4.

$$\begin{aligned}\sum_{n\geq 1} \frac{t^{n+|\alpha|}}{(n+|\alpha|)!} \sum_{\sigma\in S_n, Set(\alpha)\subseteq Des(\sigma)} x^{des(\sigma)} &= \frac{x^k}{(x-1)^{k-1}}\xi\Bigg((\frac{\sum_{j=0}^{k} Z_{\alpha(j)} t^{|\alpha^{(j)}|}}{x-e^{t(x-1)}}) - \\ & ((-1)^k + \sum_{j=1}^{k}(-1)^j \sum_{r=1}^{\alpha_j-1} Z_{(\alpha^{(j)},r)} t^{r+|\alpha^{(j)}}) \Bigg)\end{aligned}$$

One can then expand the ribbon Schur function Z_β that appear on the right side of (3.6) in terms of the elementary functions and apply ξ to the resulting sum to obtain an explicit generating function. For example, if $\alpha = (3,1,1)$, then one can use this method to show that

$$\sum_{n\geq 1} \frac{t^{5+n}}{(5+n)!} \sum_{\sigma\in S_{5+n}, \{3,4,5\}\subseteq Des(\sigma)} x^{des(\sigma)} = \frac{x^3}{120(1-x)^2(-x+e^{t(x-1)})}P(x,t) + \frac{x^3}{2(1-x)^3}Q(x,t)$$

where $P(x,t) = (6+x-19x^2+11x^3+x^4)t^5 - (15+25x-35x^2-5x^3)t^4 + (20+80x+20x^2)t^3 - 120$ and $Q(x,t) = 2+2t+(1+x)t^2$.

4. Permutations that have descents on at $k\mathbb{N}$

In this section, we shall use ideas of Mendes [**11**] to find the generating functions for permutations σ such that $i \in Des(\sigma)$ if and only if $i \cong 0 \mod k$. We will give an example in the case where $k=3$. Let $S_n^{(3)}$ denote the set of all permutations σ of S_n such that $i \in Des(\sigma)$ iff $i \in k\mathbb{N}$ and let $A_{n,3} = |S_n^{(3)}|$.

First we let $\xi^{(3)}$ denote the ring homomorphism which maps Λ into the rationals $\mathbb{Q}$ defined by

$$\begin{aligned}\xi^{(3)}(e_0) &= 1 \\ \xi^{(3)}(e_k) &= 0 \ if\ k = 3n+1 \ or\ k=3n+2 \\ \xi^{(3)}(e_k) &= \frac{(-1)^{3n}}{3n!}(-1)^n \ if\ k=3n>0.\end{aligned}$$

We will also consider two weighting functions $\vec{u} = (u_1, u_2, \ldots)$ where $u_i = i$ for all i and $\vec{v} = (v_1, v_2, \ldots)$ where $v_i = i(i-1)$ for all i.

Then we claim that we have the following result.

THEOREM 4.1. (1) *$(3n)!\xi^{(3)}(h_{3n}) = A_{3n,3}$ for $n \geq 1$ and $(3k)!\xi^{(3)}(h_k) = 0$ if k is not equivalent to 0 mod 3.*
(2) *$(3n+1)!\xi^{(3)}(p^{\vec{v}}_{3n+3}) = A_{3n+1,3}$ for $n \geq 1$ and $(3k)!\xi^{(3)}(p^{\vec{v}}_k) = 0$ if k is not equivalent to 0 mod 3.*
(3) *$(3n+2)!\xi^{(3)}(p^{\vec{u}}_{3n+3}) = A_{3n+2,3}$ for $n \geq 1$ and $(3k)!\xi^{(3)}(p^{\vec{u}}_k) = 0$ if k is not equivalent to 0 mod 3.*

PROOF. We start by proving part (1). Using (2.3), we have that $\xi^{(3)}(h_k) = \sum_{\mu\vdash k}(-1)^{k-\ell(\mu)}B_{\mu,(k)}\xi^{(3)}(e_\mu)$. Clearly, if k is not equivalent to 0 mod 3, then every partition μ of k must have a part which is not equivalent of 0 mod 3 and hence $\xi^{(3)}(e_\mu) = 0$. Thus if k is not equivalent 0 mod 3, then $\xi^{(3)}(h_k) = 0$. A similar argument will show that if k is not equivalent to 0 mod 3, then $\xi^{(3)}(p^{\vec{v}}_k) = \xi^{(3)}(p^{\vec{u}}_k) = 0$.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, we let $3\lambda = (3\lambda_1, \ldots, 3\lambda_k)$. Clearly the only partitions of $3n$ such that $\xi^{(3)}(e_\mu) \neq 0$ are partitions where $\mu = 3\lambda$ for some partition λ of n. Thus we can use (2.3) to conclude that

$$\begin{aligned}(3n)!\xi^{(3)}(h_{3n}) &= (3n)!\sum_{\lambda \vdash n}(-1)^{3n-\ell(\lambda)} B_{3\lambda,(3n)}\xi^{(3)}(e_{3\lambda}) \\ &= (3n)!\sum_{\lambda \vdash n}(-1)^{3n-\ell(\lambda)} B_{3\lambda,(3n)} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{3\lambda_i}}{(3\lambda_i)!}(-1)^{\lambda_i} \\ &= \sum_{\lambda \vdash n} B_{3\lambda,(3n)} \binom{3n}{3\lambda_1, \ldots, 3\lambda_{\ell(\lambda)}} \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i - 1}. \end{aligned} \tag{4.1}$$

Next we interpret the left-hand side of (4.1) as weighted sum of objects. That is, given brick tabloids $T \in \mathcal{B}_{3\lambda,(3n)}$, we write $T = (b_1, \ldots, b_{\ell(\lambda)})$ if the sizes of the bricks in T, reading from left to right, are $b_1, \ldots, b_{\ell(\lambda)}$. Then clearly, we have $\binom{3n}{3\lambda_1,\ldots,3\lambda_{\ell(\lambda)}} = \binom{3n}{b_1,\ldots,b_{\ell(\lambda)}}$ so that we can think of the binomial coefficient $\binom{3n}{b_1,\ldots,b_{\ell(\lambda)}}$ as choosing sets of size $b_1, \ldots, b_{\ell(\lambda)}$ from $\{1, \ldots, 3n\}$ to place in the bricks of T in such a way that the elements within a brick are in increasing order and each number is used only once. Thus a typical filled brick tabloid F that we would produce in the case where $n = 5$ is pictured in Figure 6. Then the factor $\prod_{i=1}^{\ell(\lambda)}(-1)^{\lambda_i-1}$ allows us to place a -1 in every third square in a brick except for the last cell. We can then define the weight of the filled brick tabloid F to be the product of the -1's on top of the cells in F. Hence the filled brick tabloid in Figure 6 has weight 1. In this way, we can interpret $(3n)!\xi^{(3)}(h_{3n})$ as the sum of the weights $W(F)$ over the set $\mathcal{F}BT_{3n}$ of all filled brick tabloids $F = (B_1, \ldots, B_{\ell(\lambda)})$ of shape $(3n)$ such that size of each brick B_i is a multiple of 3.

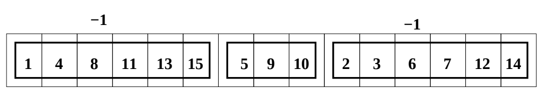

FIGURE 6. A filled brick tabloid

We can define a involution I on $\mathcal{F}BT_{3n}$ as follows. We scan the filled brick tabloid F from left to right until either
(i) we find a cell which has a -1 on top in which case we split the brick into two bricks at that cell and change the -1 to 1 or
(ii) we find two consecutive bricks B_i and B_{i+1} such that the elements in these bricks form an increasing sequence reading from left to right in which case we combine the two bricks into a single brick B and place a -1 on top of the last cell of B_i.
If both (i) and (ii) fail, then we define $I(F) = F$. For example, if F is the filled brick tabloid in Figure 6, then $I(F)$ is the filled brick tabloid pictured in Figure 7.

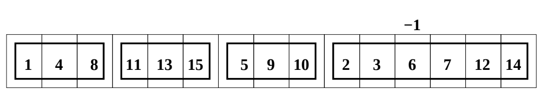

FIGURE 7. $I(F)$ for F in Figure 6.

It is easy to see that I is an involution and proves that $(3n)!\xi^{(3)}(h_{3n}) = \sum_{F \in \mathcal{F}BT_{3n}, I(F)=F} W(F)$. Thus we must examine the fixed points of I. Note that $I(F) = F$, then F must consist entirely of bricks of size 3 since any brick of size greater than 3 must have at least one cell which has a -1 on top. Moreover there must be a decrease between any two bricks. Thus $W(F) = 1$ and the permutation $\sigma(F)$ that we obtain by reading the entries of F from left to right must have $Des(\sigma_F) = \{3, 6, \ldots, 3(n-1)\}$. Hence $(3n)!\xi^{(3)}(h_{3n}) = A_{3n,3}$ as desired.

For part 2, we can use the same reasoning to show that

$$
\begin{aligned}
(3n+1)!\xi^{(3)}(p_{3n+3}^{\vec{v}}) &= (3n+1)! \sum_{\lambda \vdash n+1} (-1)^{3n+3-\ell(\lambda)} w_{\vec{v}}(B_{3\lambda,(3n+3)}) \xi^{(3)}(e_{3\lambda}) \\
&= (3n+1)! \sum_{\lambda \vdash n+1} (-1)^{3n-\ell(\lambda)} w_{\vec{v}}(B_{3\lambda,(3n+3)}) \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{3\lambda_i}}{(3\lambda_i)!}(-1)^{\lambda_i} \\
&= \sum_{\lambda \vdash n+1} w_{\vec{v}}(B_{3\lambda,(3n+3)}) \frac{(3n+1)!}{(3\lambda_1)!\cdots(3\lambda_{\ell(\lambda)})!} \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i - 1}. \qquad (4.2)
\end{aligned}
$$

Once again we interpret the left-hand side of (4.2) as a weighted sum of objects. That is, if $T = (b_1, \ldots, b_{\ell(\lambda)})$, then

$$
w_{\vec{v}}(T) \frac{(3n+1)!}{(3\lambda_1)!\cdots(3\lambda_{\ell(\lambda)})!} = b_{\ell(\lambda)}(b_{\ell(\lambda)}-1) \frac{(3n+1)!}{b_1!\cdots b_{\ell(\lambda)-1}! b_{\ell(\lambda)}!} = \binom{3n+1}{b_1, \ldots, b_{\ell(\lambda)-1}, b_{\ell(\lambda)}-2}.
$$

Thus, in this case, we can think of the binomial coefficient $\binom{3n+1}{b_1,\ldots,b_{\ell(\lambda)-1},b_{\ell(\lambda)}-2}$ as choosing sets of size $b_1, \ldots, b_{\ell(\lambda)-1}, b_{\ell(\lambda)} - 2$ to place in the bricks of T in such a way that the elements within a brick are in increasing order and the last two cells of the last brick are empty. Thus a typical filled brick tabloid F that we would produce in the case where $n = 5$ is pictured in Figure 8.

Then the factor $\prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i - 1}$ allows us to place a -1 in every third square in a brick except for the last cell. We can then define the weight of the filled brick tabloid F to be the product of the -1's on top of the cells in F. Hence the filled brick tabloid in Figure 8 has weight -1. In this way, we can interpret $(3n+1)!\xi^{(3)}(p_{3n+3}^{\vec{v}})$ as the sum of the weights $W(F)$ over the set $\mathcal{F}BT_{3n+3}$ of all filled brick tabloids $F = (B_1, \ldots, B_{\ell(\lambda)})$ of shape $(3n+3)$ such that the last two cells are empty and the size of each brick B_i is a multiple of 3. We

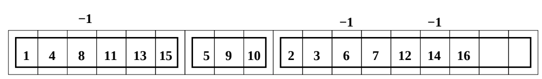

FIGURE 8. A filled brick tabloid with the last 2 cells empty

can then use the exact same involution I in part (1) to prove that $(3n+1)!\xi^{(3)}(p_{3n+3}^{\vec{v}}) = A_{3n+1,3}$.

Finally the proof a part (3) is similar to the proof of part (2) except in this case we will get filled brick tabloids $F = (B_1, \ldots, B_{\ell(\lambda)})$ of shape $(3n+3)$ such that the last cell is empty and the size of each brick B_i is a multiple of 3.

□

One can then apply $\xi^{(3)}$ to (1.1) and (2.5) to prove the following generating functions.

$$
\begin{aligned}
\sum_{n\geq 0} \frac{A_{3n,3}t^{3n}}{(3n)!} &= \frac{1}{1+\sum_{n\geq 1}\frac{(-t)^{3n}}{(3n)!}} \\
\sum_{n\geq 1} \frac{A_{3n-1,3}t^{3n-1}}{(3n-1)!} &= \frac{\sum_{n\geq 1}\frac{(-1)^{n-1}t^{3n-1}}{(3n-1)!}}{1+\sum_{n\geq 1}\frac{(-t)^{3n}}{(3n)!}} \\
\sum_{n\geq 1} \frac{A_{3n-2,3}t^{3n-2}}{(3n-2)!} &= \frac{\sum_{n\geq 1}\frac{(-1)^{n-1}t^{3n-2}}{(3n-2)!}}{1+\sum_{n\geq 1}\frac{(-t)^{3n}}{(3n)!}}.
\end{aligned}
$$

5. Main Results

In this section, we shall outline how we can obtain generating functions for the number of permutations $\sigma \in S_n$ such that $i \in Des(\sigma)$ if and only if $i \in k\mathbb{N} - A$ where A is some finite subset of $k\mathbb{N}$. We shall illustrate our example, in the case where $k = 3$ and $A = \{3, 9\}$. First, we let $S_n^{(3,A)}$ denote the set of all $\sigma \in S_n$ such that $i \in Des(\sigma)$ if and only if $i \in 3\mathbb{N} - A$.

Now suppose that $\alpha = (\alpha_1, \ldots, \alpha_k)$ is composition of n. Then define $h_{3\alpha} = h_{3\alpha_1} \cdots h_{3\alpha_k}$ and $Set(3\alpha) = \{3\alpha_1, 3\alpha_1 + 3\alpha_2, \ldots, 3\alpha_1 + \cdots + 3\alpha_{k-1}\}$. Let $\beta = 1 + 3s$ and $\gamma = 2 + 3t$. First we define $T_n^{(3,Set(3\alpha))}$ by declaring that $\sigma \in S_n$ is in $T_n^{(3,Set(3\alpha))}$ if and only if

$$(3\mathbb{N} - Set(3\alpha)) \cap \{1, \ldots, n-1\} \subseteq Des(\sigma) \subseteq 3\mathbb{N} \cap \{1, \ldots, n-1\}.$$

Let $\vec{u} = (u_1, u_2, \ldots)$ and $\vec{v} = (v_1, v_2, \ldots)$ where $u_i = i$ and $v_i = i(i-1)$ for all i. Then we claim that we have the following.

THEOREM 5.1. *If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is composition of n and $\beta = 1 + 3s$ and $\tau = 2 + 3t$, then*

(1) $(3n)!\xi^{(3)}(h_{3\alpha}) = |T_{3n}^{(3,Set(3\alpha))}|$.
(2) $(3n+3s+1)!\xi^{(3)}(h_{3\alpha} p_{3s+3}^{\vec{v}}) = |T_{3n+3s+1}^{(3,Set(3\alpha))}|$.
(3) $(3n+3t+2)!\xi^{(3)}(h_{3\alpha} p_{3t+3}^{\vec{u}}) = |T_{3n+3t+2}^{(3,Set(3\alpha))}|$.

PROOF. The proof is a variation of the proof of Theorem 4.1. For example, for part (1), suppose that $\alpha = (2, 3, 2)$. Then clearly $h_{3\alpha} = h_\gamma$ where $\gamma = (6, 6, 9)$. Thus

$$\begin{aligned}
(21)!\xi^{(3)}(h_{3\alpha}) &= \sum_{\mu \vdash 7} (-1)^{21-\ell(\mu)} B_{3\mu,\gamma} \xi^{(3)}(e_{3\mu}) \\
&= \sum_{\mu \vdash 7} (-1)^{21-\ell(\mu)} \sum_{T=(b_1,\ldots,b_k) \in \mathcal{B}_{3\mu,\gamma}} \binom{21}{b_1, \ldots, b_k} \prod_{i=1}^{k} (-1)^{3b_i + b_i/3} \\
&= \sum_{\mu \vdash 7} \sum_{T=(b_1,\ldots,b_k) \in \mathcal{B}_{3\mu,\gamma}} \binom{21}{b_1, \ldots, b_k} \prod_{i=1}^{k} (-1)^{(b_i/3)-1}.
\end{aligned}$$

The main difference in this case is we start out with a brick tabloid T of shape γ filled with bricks whose size is a multiple of 3 instead of a brick tabloid T of shape $(3n)$ filled with bricks whose size is a multiple of 3. However, we can still use the binomial coefficient $\binom{21}{b_1,\ldots,b_k}$ to fill the bricks with the numbers $1, \ldots, n$ so that numbers in each brick increase. For example, Figure 9 pictures such two filled brick tabloids. Again we use the factor $\prod_{i=1}^{k} (-1)^{(b_i/3)-1}$ to place a -1 on top of every third cell of a brick except the final cell and we define the weight $W(F)$ of such a filled brick tabloid as the product of -1's on top of the cells in F. In this way, we interpret $(3n)!\xi^{(3)}(h_{3\alpha})$ as the sum of the weights $W(F)$ over the set $\mathcal{FBT}_{3\alpha}$ of all filled brick tabloids $F = (B_1, \ldots, B_k)$ of shape γ where γ is the partition induced by the parts of 3α such that size of each brick is a multiple of 3.

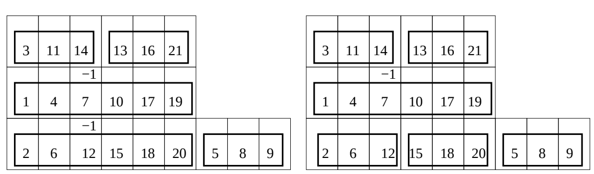

FIGURE 9. A filled brick tabloid of shape γ

We can then use the same involution I described in Theorem 4.1 except this time we read the bricks in rows from bottom to top and within rows from left to right to determine which is the first cell which has a -1 on top or where two bricks can be combined. We then define $I(F)$ by either breaking a brick into two or combining two consecutive bricks that lie in a row just as in Theorem 4.1. For example, for the filled brick tabloid F pictured at the top of Figure 9, $I(F)$ is pictured at the bottom of Figure 9 since the first cell with a -1 on top in our ordering of cells is the cell containing the number 12.

Then as before, I shows that $(3n)!\xi^{(3)}(h_{3\alpha})$ is the sum over the weights of the fixed points of I. Once again, if $I(F) = F$, then F must consist entirely of bricks of size 3 and there must be decreases between any two consecutive bricks that lie in the same row. For example, Figure 10 pictures such a fixed point of I. The only thing to do is interpret such a fixed point as a permutation. In our case, since $3\alpha = (6, 9, 6)$, we will

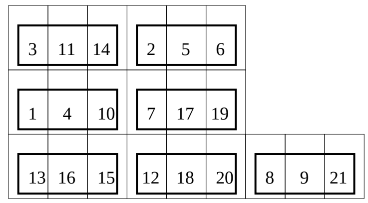

FIGURE 10. A fixed point of I

read the cells in each row from left to right and read the rows so that we read the top row of size 6, followed by the row of size 9, followed by the bottom row of size 6 to obtain the permutation

$$3\ 11\ 14\ 2\ 5\ 6\ 13\ 16\ 15\ 12\ 18\ 20\ 8\ 9\ 21\ 1\ 4\ 10\ 7\ 17\ 19.$$

In general, we make the convention that the rows of the same size in F reading from top to bottom corresponds to the parts of 3α of the same size, reading from left to right. Note that we are guaranteed that there is a descent at each position i which is multiple of 3 unless i corresponds to a cell which at the end of row. In such cases, i is not forced to be the position of descent so that i may or may not be in $Des(\sigma_F)$. In this way, we see that $(3n)!\xi^{(3)}(h_{3\alpha}) = |T_{3n}^{(3,Set(3\alpha))}|$ as desired.

The proof of parts (2) and (3) is similar expect that the row corresponding to $3s+3$ or $3t+3$ will have a weight on the last brick which will make sure that one achieves the appropriate binomial coefficient to fill in all but the last one or two cells of the last brick in that row. Also, when we read the permutation off of a fixed point, we will assume that that elements in that brick are read last.

□

Finally, we shall show how we can use ribbon Schur functions to get our desired generating function in the case where $k=3$ and $A=\{3,9\}$. We start out by finding the image of $Z_{(3,6,3n)}$ under $\xi^{(3)}$. Expanding $Z_{(3,6,3n)}$ in terms of the homogeneous symmetric functions as we did in section 2 and using Theorem 5.1, we can interpret $(9+3n)!\xi^{(3)}(Z_{(3,6,3n)})$ as sum of filled special rim hook tabloids of the form pictured in Figure 11. That is, it is easy to see that all the special rim hooks of shape $F_{(3,6,3n)}$ must all have sizes which are multiples of 3. Thus, if we read the rim hooks from top to bottom, then we will induce a composition of the form 3α for some composition α of $3+n$. For example, in Figure 11, $n=2$ and reading the rim hooks of the filled special rim hook tabloid at the top, induces the composition $3\alpha=(9,6)$. Now $(9+3n)!\xi^{(3)}(h_{(9,6)})$ equals the number of permutations σ of S_{15} such that $\{3,6,12\} \subseteq Des(\sigma) \subseteq \{3,6,9,12\}$ so we have filled the special rim hook with such a permutation. On the other hand, reading the rim hooks of the filled special rim hook tabloid which is in the middle of Figure **??**, induces the composition $(3,6,6)$. Now $(9+3n)!\xi^{(3)}(h_{(3,6,6)})$ equals the number of permutations σ of S_{15} such that $\{6,12\} \subseteq Des(\sigma) \subseteq \{3,6,9,12\}$ so we can use the same permutation to fill it. Now these two filled special rim hook tabloids only differ in the first vertical segment is part of the first filled special rim hook tabloid while the it is not part of the second filled special rim hook tabloid. Thus these two filled special rim hook tabloids will cancel as before.

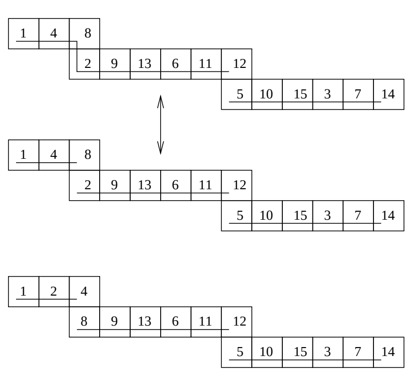

FIGURE 11. A fixed point of I

However, if σ is a permutation which has a rise over the first vertical segment like the filled special rim hook tabloid at the bottom of Figure 11, then that filled special rim hook tabloid cannot have that vertical segment be part of that special rim hook tabloid. Thus our cancellation leaves us only with filled special rim hook tabloids whose first special rim hook is horizontal and whose permutation has a rise over the first vertical segment. Then we perform a similar cancellation for the remaining filled special rim hook tabloids relative to the second vertical segment to conclude that $(9+3n)!\xi^{(3)}(Z_{(3,6,3n)})$ equals the sum of the weights of the filled special rim hook tabloids whose first two special rim hooks are horizontal and whose permutation has a rise over the first two vertical segments. Continuing on this way, we can prove that

$$(9+3n)!\xi^{(3)}(Z_{(3,6,3n)}) = |\{\sigma \in S_{9+3n} : Des(\sigma) = \{6\} \cup \{3k : 12 \leq k \leq n-1\}|.$$

In general, we can prove

THEOREM 5.2.

$$(3\alpha + 3n)!\xi^{(3)}(Z_{(3\alpha,3n)}) = |\{\sigma \in S_{3|\alpha|+3n} : Des(\sigma) = \{3k : k = 1, \ldots, |\alpha| + n - 1\} - Set(3\alpha)\}| \tag{5.1}$$

Then we can apply $\xi^{(3)}$ to both sides of (3.3) to get our desired generating function.

To obtain generating functions for lengths which are not divisible of 3, we have to use a variant of the ribbon Schur functions. That is, we will use the same collection of special rim hook tabloids that would appear in the expansion of $Z_{(3\alpha,3n+3)}$ in terms of homogeneous symmetric functions. However, we must weight each such special rim hook tabloid differently. That is, in the usual expansion of $Z_{(3\alpha,3n+3)}$ each special rim hook tabloid T whose hook lengths are $(a_1, \ldots, a_r)$ reading from top to bottom is weighted with $sgn(T)h_{a_1} \cdots h_{a_r}$. If we are interested in lengths which are equivalent to $1 \mod 3$, then we will weight T with $sgn(T)h_{a_1} \cdots h_{a_{r-1}} p_{a_r}^{\vec{v}}$ so that when we apply $\xi^{(3)}$, we can pick up the proper weight for the last special rim hook which has the effect of allowing us to have the last two cells of the special rim hook empty. Similarly, if we are interested in lengths which are equivalent to $2 \mod 3$, then we will weight T with $sgn(T)h_{a_1} \cdots h_{a_{r-1}} p_{a_r}^{\vec{u}}$ so that when we apply $\xi^{(3)}$, we can pick up the proper weight for the last special rim hook which has the effect of allowing us to have the last cell of that special rim hook empty.

References

[1] D. Beck, J. Remmel, and T. Whitehead, The combinatorics of transition matrices between the bases of the symmetric functions and the B_n analogues, *Discrete Mathematics* **153** (1996), 3–27.
[2] F. Brenti, Permutation enumeration, symmetric functions, and unimodality, *Pacific J. Math.* **157** (1993), 1–28.
[3] F. Brenti, *A class of q-symmetric functions arising from plethysm*, J. Comb. Th. Ser. A, **91** (2000), 137–170.
[4] L. Carlitz, Sequences and inversions, *Duke Math. J.* **37** (1970), 193-198.
[5] Ö. Eğecioğlu and J.B. Remmel, *A Combinatorial interpretation of the Inverse Kostka Matrix*, Linear and Multilinear Algebra , **26**, (1990), pp 59–84.
[6] Ö. Eğecioğlu and J. Remmel, *Brick tabloids and connection matrices between bases of symmetric functions*, Discrete Applied Math. **34** (1991), 107–120.
[7] J-M. Fédou and D. Rawlings, *Statistics on pairs of permutations*, Discrete Math. **143** (1995), 31–45.
[8] T.M. Langley, *Alternative transition matrices for Brenti's q-symmetric functions and a class of q,t-symmetric functions on the hyperoctahedral group*, Proceedings of the 2002 Conference on Formal Power Series and Algebraic Combinatorics, Melbourne Australia.
[9] T.M. Langley and J.B. Remmel, *Enumeration of m-tuples of permuations and a new class of power bases for the space of symmetric functions*, Advances in App. Math. **36** (2006), 30-66.
[10] I. G. MacDonald, "Symmetric Functions and Hall Polynomials," Oxford Univ. Press, London/New York, 1979.
[11] A. Mendes, *Building generating functions brick by brick*, Ph.D. thesis, University of California, San Diego, 2004.
[12] A. Mendes and J.B. Remmel, Permutations and words counted by consecutive patterns, preprint.
[13] A. Mendes and J.B. Remmel, Generating Functions from Symmetric Functions, preprint.
[14] D. Ram, J.B. Remmel, and T. Whitehead, *Combinatorics of the q-basis of symmetric functions*, J. Comb. Th. Series A, **76**(2) (1996), 231–271.
[15] Jeffrey B. Remmel and Amanda Riehl, Generating functions for permutations that contain a given descent set, in preparation.
[16] R. P. Stanley, *Binomial posets, Möbius inversion and permuation enumeration*, J. Comb. Th. (A), **20** (1976), 336-356.
[17] Jennifer D. Wagner, The permutation enumeration of wreath products and cyclic and symmetric groups, Advances in Appl. Math. **30** (2003), 343-368.

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