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Dual equivalence graphs, ribbon tableaux and Macdonald polynomials

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Abstract. We make a systematic study of a new combinatorial construction called a *dual equivalence graph*. Motivated by the dual equivalence relation on standard Young tableaux introduced by Haiman, we axiomatize such constructions and prove that the generating functions of these graphs are Schur positive. We construct a graph on k-ribbon tableaux which we conjecture to be a dual equivalence graph, and we prove the conjecture for $k \leq 3$. This implies the Schur positivity of the k-ribbon tableaux generating functions, $\widetilde{G}_{\mu}^{(k)}(x;q)$, introduced by Lascoux, Leclerc and Thibon. From Haglund's monomial expansion for Macdonald polynomials, this has the further consequence of a combinatorial Schur expansion of the transformed Macdonald polynomials $\widetilde{H}_{\mu}(x;q,t)$ when μ is a partition with at most 3 columns.

1. Introduction

The immediate purpose of this paper is to establish a combinatorial formula for the Schur expansion of the k-ribbon tableaux generating functions known as LLT polynomials when $k \leq 3$. As a corollary, this yields a combinatorial formula for the Kostka-Macdonald polynomials for partitions with at most 3 columns. Furthermore, we conjecture that the construction used generalizes to arbitrary k. Our real purpose, however, is not only to obtain the above results, but also to introduce a new combinatorial construction, called a *dual equivalence graph*, by which one can establish the Schur positivity of polynomials expressed in terms of monomials.

In Section 2, we introduce notation for familiar objects in symmetric function theory, for the most part following the notation of [**Mac95**]. Section 3 is devoted to the development of the theory of dual equivalence graphs. We review the original definition of dual equivalence given in [**Hai92**], and in Section 3.1 show how from this we obtain a graph whose vertices are given by standard Young tableaux and whose connected components are indexed by partitions. In Section 3.2, we present an axiomatization of a general dual equivalence graph and state two main theorems which justify this axiomatization and indicate its significance.

With these constructions in place, Section 4 contains the first application of dual equivalence graphs from which we obtain a combinatorial interpretation of the Schur expansion of LLT polynomials. Section 4.1 begins by recalling the original definition given in [**LLT97**], and in Section 4.2 we give an equivalent framework which will facilitate the constructions to follow. In Section 4.3, we present the main theorem of this section, a combinatorial proof of Schur positivity of LLT polynomials when $k \leq 3$, and we give the idea of the proof and an indication of how it may generalize to arbitrary k. Extending this example, in Section 4.4, we use the combinatorial expansion of Macdonald polynomials in terms of LLT polynomials from [**Hag04, HHL05a**] to give a combinatorial description of the (q,t)-Kostka numbers.

2. Preliminaries

We represent an integer partition λ by a decreasing sequence of its (nonzero) parts

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m), \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0.$$

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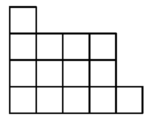
We may also write

$$\lambda = (1^{m_1}, 2^{m_2}, \ldots)$$

where m_i is the number of times i occurs as a part of λ. We denote the size of λ by $|\lambda| = \sum_i \lambda_i$. If $|\lambda| = n$, we say that λ is a *partition of* n. Given λ, we have the *conjugate partition* λ' defined by

$$\lambda'_j = \sum_{i \geq j} m_i.$$

The *Young diagram* of λ is the set of points (i, j) in the $\mathbb{Z} \times \mathbb{Z}$ plane such that $1 \leq i \leq \lambda_j$. We draw the diagram so that each point (i, j) actually gives the *cell* of the grid that is southwest of the point. For example, the Young diagram for $\lambda = (5, 4, 4, 1)$ is



We will write λ for both the partition and its diagram.

For a partition diagram, the *content* of a cell indexes the diagonal on which it occurs, i.e. $c(x) = j - i$ when the cell x lies in position $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

For partitions λ, μ, we write $\mu \subset \lambda$ whenever the diagram for μ is contained within the diagram for λ, equivalently $\mu_i \leq \lambda_i$ for $i \geq 1$. In this case, we define the *skew diagram* $\nu = \lambda/\mu$ to be the set theoretic difference $\lambda - \mu$.

Let $\mathcal{A}_n, \mathcal{A}$ denote the alphabets $\{1, \ldots, n\}, \mathbb{N}$, respectively. A *filling* of a (skew) diagram λ is a map

$$S : \lambda \longrightarrow \mathcal{A}.$$

A *semi-standard Young tableau* is a filling which is weakly increasing along each row of λ and strictly increasing along each column. A semi-standard Young tableau is *standard* if it is a bijection from λ to $\mathcal{A}_n$, where $n = |\lambda|$. For λ a partition of n and μ a composition of n, we define

$$\begin{aligned} \mathrm{SSYT}(\lambda) &= \{\text{semi-standard tableaux } T : \lambda \to \mathcal{A}\} \\ \mathrm{SSYT}(\lambda, \mu) &= \{\text{semi-standard tableaux } T : \lambda \to \mathcal{A} \text{ with entries } 1^{\mu_1}, 2^{\mu_2}, \ldots\} \\ \mathrm{SYT}(\lambda) &= \{\text{standard tableaux } T : \lambda \to \mathcal{A}_n\} = \mathrm{SSYT}(\lambda, (1^n)). \end{aligned}$$

For $T \in \mathrm{SSYT}(\lambda, \mu)$, we say that T has *shape* λ and *weight* μ. Given a semi-standard Young tableau T, the *content reading word* of T is the word obtained by reading the entries of T along diagonals from southwest to northeast, starting with the smallest content.

We have the familiar bases for the ring of symmetric function from [**Mac95**]: the monomial symmetric functions m_λ, the elementary symmetric functions e_λ, the complete homogeneous symmetric functions h_λ, the power sum symmetric functions p_λ and the almighty Schur functions s_λ.

Recall the Hall inner product $\langle -, - \rangle$ on symmetric functions defined by

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} = \langle m_\lambda, h_\mu \rangle,$$

which makes the Schur functions into an orthonormal basis. The *Kostka numbers*, $K_{\lambda,\mu}$, give the change of basis from the complete homogeneous symmetric functions to the Schur functions,

$$\langle h_\mu, s_\lambda \rangle = K_{\lambda,\mu} = \langle s_\lambda, m_\mu \rangle,$$

or equivalently

$$h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda; \quad s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu.$$

Thus the Kostka numbers also give the highest weight multiplicities for GL_n modules. Throughout this paper, we are interested in certain one- and two-parameter generalizations of the Kostka numbers.

3. Dual equivalence graphs

Dual equivalence was first defined by Haiman [**Hai92**]. The relation to which it is "dual" is *jeu-de-taquin* equivalence under the Schensted correspondence. In particular, two permutations are dual equivalent exactly when the inverse permutations are *jeu-de-taquin* equivalent.

DEFINITION 3.1. An *elementary dual equivalence* on three consecutive letters $i-1, i, i+1$ of a permutation is given by switching the outer two letters whenever the middle letter is not i.

$$\cdots\ i\ \cdots\ i \pm 1\ \cdots\ i \mp 1\ \cdots \equiv^* \cdots\ i \mp 1\ \cdots\ i \pm 1\ \cdots\ i\ \cdots$$

Two permutations are *dual equivalent* if one can be transformed into the other by some sequence of elementary dual equivalences. Two standard tableau are *dual equivalent* if their content reading words are.

3.1. The standard dual equivalence graph $\mathcal{G}_\lambda$. We can construct a colored graph whose vertices are standard tableau from this relation in the following way. Whenever two standard tableaux T, U have content reading words which differ by an elementary dual equivalence for $i-1, i, i+1$, connect T and U with an edge colored by i. It is clear that the connected components of the graph so constructed will correspond to the dual equivalence classes of standard tableaux. Let $\mathcal{G}_\lambda$ denote the subgraph of tableaux of shape λ. The following proposition tells us that the $\mathcal{G}_\lambda$ exactly give the connected components of the dual equivalence graph.

PROPOSITION 3.2 ([**Hai92**]). *Two standard tableaux on straight shapes are dual equivalent if and only if they have the same shape.*

For any subset $D \subset \{1, \ldots, n-1\}$, Gessel [**Ges84**] defined the *quasi-symmetric function*

$$Q_{n,D}(x) = \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow j \notin D}} x_{i_1} \cdots x_{i_n}. \tag{3.1}$$

We can use Gessel's quasi-symmetric functions to define a generating function on the vertices of a dual equivalence graph. First we add a signature for each vertex, which may be regarded as an indicator function for a subset of $\{1, \ldots, n-1\}$, setting $i \in D$ if and only if $\sigma_i = -1$.

DEFINITION 3.3. Let T be a standard tableau on $\mathcal{A}_n$ with content reading word w. Define the *descent signature* $\sigma(T) \in \{\pm 1\}^{n-1}$ by

$$\sigma(T)_i \;=\; \begin{cases} +1 & \text{if } i \text{ appears to the left of } i+1 \text{ in } w \\ -1 & \text{if } i+1 \text{ appears to the left of } i \text{ in } w \end{cases} \tag{3.2}$$

The generating function associated to a connected component $\mathcal{G}_\lambda$, $|\lambda| = n$, is given by

$$g_\lambda(x) = \sum_{v \in V(\mathcal{G}_\lambda)} Q_{n,\sigma(v)}(x). \tag{3.3}$$

Recall the following combinatorial formula for (skew) Schur functions

$$s_\lambda(x) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T,$$

where x^T is the monomial $x_1^{\mu_1} x_2^{\mu_2} \cdots$ when T has weight μ. Gessel changed this formula into a sum over standard tableaux, which shows that $g_\lambda(x)$ is nothing new, but just that ubiquitous Schur function $s_\lambda(x)$.

PROPOSITION 3.4 ([**Ges84**]). *The (skew) Schur function $s_\lambda(x)$, $|\lambda| = n$, can be expressed in terms of quasi-symmetric functions by*

$$s_\lambda(x) = \sum_{T \in \mathrm{SYT}(\lambda)} Q_{n,\sigma(T)}(x).$$

3.2. Axiomatization of dual equivalence. Our goal now is to characterize when a given colored graph $\mathcal{G}$ with signed vertices "looks like" a dual equivalence graph. The axiomatization given below comes from analyzing the local properties of the standard dual equivalence graph, and abstracting away the dependence on tableaux.

Let V be a vertex set with signatures given by $\sigma : V \to \{\pm 1\}^{N-1}$. Let E be a collection of colored edges between (distinct) vertices of V on the palette $\{2, 3, \ldots, n-1\}$, with $n \leq N$. Let $E_i \subset E$ denote those edges colored by i.

DEFINITION 3.5. $\mathcal{G} = (V, \sigma, E)$ is a *dual equivalence graph* if the following hold:

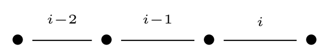
(ax1) For $w \in V$ and $1 < i < n$, $\sigma(w)_{i-1} = -\sigma(w)_i$ if and only if there exists $x \in V$ such that $\{w, x\} \in E_i$. Moreover, x is unique when it exists.

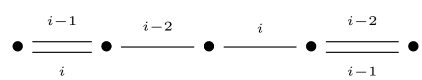
(ax2) Whenever $\{w, x\} \in E_i$,

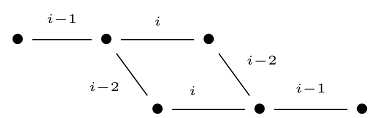
$$\begin{array}{rclll} \sigma(w)_j & = & -\sigma(x)_j & \text{for } j = i-1, i; \\ \sigma(w)_h & = & \sigma(x)_h & \text{for } h < i-2 \text{ and } i+1 < h. \end{array}$$

(ax3) Whenever $\{w, x\} \in E_i$,
 (a) if $\sigma(w)_{i-2} = -\sigma(x)_{i-2}$, then $\{w, x\} \in E_{i-1}$;
 (b) if $\sigma(w)_{i+1} = -\sigma(x)_{i+1}$, then $\{w, x\} \in E_{i+1}$ if $i+1 < n$, and $\sigma(w)_i = -\sigma(w)_{i+1}$ if $i+1 = n$.

(ax4) For $3 < i < n$, every non-trivial connected component of the subgraph $(V, \sigma, E_{i-2} \cup E_{i-1} \cup E_i)$ is one of the following:

(a) 

(b) 

(c) 

(ax5) Whenever $|i - j| \geq 3$, $\{w, x\} \in E_i$ and $\{x, y\} \in E_j$, there exists $v \in V$ such that $\{w, v\} \in E_j$ and $\{v, y\} \in E_i$.

We can quickly verify that each $\mathcal{G}_\lambda$ satisfies the above axioms. Axiom 1 follows from the fact that T admits an elementary dual equivalence for $i{-}1, i, i{+}1$ exactly when i does not occur between $i-1$ and $i+1$ in the content reading word. For axiom 2, simply note that only two letters are interchanged by an elementary dual equivalence, so σ doesn't change much. Axiom 3 comes into play exactly when both $i-1$ and $i+2$ lie between i and $i+1$, so that an elementary dual equivalence applies to $i{-}1, i, i{+}1$ as well as $i, i{+}1, i{+}2$. Axiom 4 is a straightforward check, and axiom 5 is obvious since $i{-}1, i, i{+}1$ and $j{-}1, j, j{+}1$ have no common letters when $|i - j| \geq 3$.

This, however, is only half of the story. Let us say that two signed, colored graphs are *isomorphic* if there exists a graph isomorphism from one to the other which respects the signature of vertices and color of edges. By looking at the structure of graphs satisfying the given axioms, we prove by induction on the colors of the edges that the given definition is correct in the following sense.

THEOREM 3.6. *Every connected component of a dual equivalence graph is isomorphic to $\mathcal{G}_\lambda$ for a unique partition λ.*

For a given signed, colored graph $\mathcal{G}$ for which every vertex is assigned some additional statistic α, we define the generating function $G(x; q)$ by

$$G(x; q) = \sum_{v \in V(\mathcal{G})} q^{\alpha(v)} Q_{n, \sigma(v)}(x). \tag{3.4}$$

We can, of course, include multivariate statistics, but as our immediate purpose is to apply this theory to LLT polynomials, a single parameter suffices for now.

Theorem 3.6 and Proposition 3.4 together give a criterion implying that $G(x;q)$ is symmetric and Schur positive, and establish a combinatorial interpretation of the Schur expansion.

Corollary 3.7. *Let $\mathcal{G}$ be a dual equivalence graph with a vertex statistic α which is constant on connected components of $\mathcal{G}$. Let $C(\lambda)$ denote the set of connected components of $\mathcal{G}$ which are isomorphic to $\mathcal{G}_\lambda$. Then*

$$G(x;q) = \sum_{\lambda} \sum_{\mathcal{C} \in C(\lambda)} q^{\alpha(\mathcal{C})} s_\lambda(x).$$

4. An application: LLT polynomials

In order to demonstrate one of the main uses of dual equivalence graphs, we present the following application to the ribbon tableaux generating functions known as LLT polynomials.

In 1997, Lascoux, Leclerc and Thibon introduced in [**LLT97**] a new family of symmetric functions which are q-analogs of products of Schur functions, denoted $\widetilde{G}^{(k)}_{\mu}(x;q)$. Using Fock space representations of quantum affine Lie algebras constructed by Kashiwara, Miwa and Stern [**KMS95**], Lascoux, Leclerc and Thibon proved that $\widetilde{G}^{(k)}_{\mu}(x;q)$ is a symmetric function [**LLT97**]. Thus we may define the Schur coefficients, $\widetilde{K}^{(k)}_{\lambda,\mu}(q)$ by

$$\widetilde{G}^{(k)}_{\mu}(x;q) = \sum_{\lambda} \widetilde{K}^{(k)}_{\lambda,\mu}(q) s_\lambda(x) .$$

Using Kazhdan-Lusztig theory, Leclerc and Thibon proved $\widetilde{K}^{(k)}_{\lambda,\mu}(q) \in \mathbb{N}[q]$ for straight shapes μ [**LT00**]. This has recently been extended by Grojnowski and Haiman for skew shapes [**GH**].

An incomplete combinatorial proof of $\widetilde{K}^{(2)}_{\lambda,\mu}(q) \in \mathbb{N}[q]$ was given by Carré and Leclerc in [**CL95**]. This proof was later completed by van Leeuwen using the theory of crystals [**vL05**]. The proof relies heavily on special properties of $k = 2$ which fail for $k \geq 3$.

Finding a combinatorial formula for $\widetilde{K}^{(k)}_{\lambda,\mu}(q)$ remains open for $k > 3$, and the proof for $k = 3$ is one of the main results of this paper. To that end, our goal is to establish the existence of a dual equivalence structure on standard ribbon tableaux of a given (skew) shape which preserves cospin, thereby proving that the coefficient of s_λ in $\widetilde{G}^{(k)}_{\mu}$ will q-count the number of connected components of the graph which are isomorphic to $\mathcal{G}_\lambda$.

4.1. k-ribbon tableaux and cospin. Recall that a k-ribbon is a connected skew diagram with k cells which contains no 2×2 block. To each partition λ is associated a k-core, denoted $\lambda_{(k)}$, which is the unique partition obtained from λ by successively removing k-ribbons in such a way that at every stage a partition diagram remains.

By labeling each k-ribbon with a letter of $\mathcal{A}$, we obtain a *k-ribbon filling* of $\lambda/\lambda_{(k)}$. Such a filling gives a *semi-standard k-ribbon tableau* if the ribbons labeled i form a horizontal k-ribbon strip for each i, and the union of the ribbons with labels $< i$ form a skew k-ribbon tiling for all i. A semi-standard k-ribbon tableau is called *standard* if the ribbons are labeled from 1 to n without repetition. Amending prior notation, define

$$\mathrm{SSYT}_k(\lambda) = \{\text{semi-standard } k\text{-ribbon tableaux of shape } \lambda/\lambda_{(k)}\}.$$

The k-quotient of a k-ribbon tableau $T \in \mathrm{SSYT}_k(\lambda)$ is the k-tuple of tableaux $(T^{(0)}, \ldots, T^{(k-1)})$, some of which may be empty, which corresponds to T under the Stanton-White correspondence [**SW85**]. This correspondence gives a bijection between semi-standard k-ribbon tableaux on $\lambda/\lambda_{(k)}$, and k-tuples of semi-standard tableaux on $(\lambda^{(0)}, \ldots, \lambda^{(k-1)})$, the k-quotient of λ. Therefore the generating function for ribbon tableaux reduces to a product of Schur functions,

$$G^{(k)}_{\lambda}(x) \;=\; \sum_{T \in \mathrm{SSYT}_k(\lambda)} x^T \;=\; \prod_{i=0}^{k-1} \sum_{T^{(i)} \in \mathrm{SSYT}(\lambda^{(i)})} x^{T^{(i)}} \;=\; \prod_{i=0}^{k-1} s_{\lambda^{(i)}}(x) . \tag{4.1}$$

For a complete discussion of cores and quotients, see [**JK81**].

Define the *spin* of a ribbon R to be

$$s(R) = \frac{\mathrm{ht}(R) - 1}{2} ,$$

where $\mathrm{ht}(R)$ denotes the height of the ribbon. Extending this to a ribbon tableau T, define $s(T)$ to be the sum of the spins of the ribbons of T.

Given a shape λ, define

$$s_k^*(\lambda) \;=\; \max\{s(T) | T \in \mathrm{SSYT}_k(\lambda)\}\,.$$

The *cospin* $\widetilde{s}(T)$ of a k-ribbon tableau T of shape λ is given by

$$\widetilde{s}(T) \;=\; s_k^*(\lambda) - s(T)\,.$$

Adding in q with the cospin statistic gives the LLT polynomial

$$\widetilde{G}_{\mu}^{(k)}(x;q) \;=\; \sum_{T\in \mathrm{SSYT}_k(\mu)} q^{\widetilde{s}(T)} x^T \;=\; \sum_{T\in \mathrm{SYT}_k(\mu)} q^{\widetilde{s}(T)} Q_{n,\sigma(T)}(x)\,, \tag{4.2}$$

where $\sigma(T)$ may be defined analogous to equation (3.2).

4.2. k-tuples of SSYT **and k-inversions.** In order to establish a dual equivalence graph on ribbon tableaux, we consider the k-quotient under the Stanton-White correspondence. For this to be a viable approach, we need to translate the cospin statistic into a statistic on the quotient, which was done in [**SSW03, HHL+05b**]. We will use the statistics given in [**HHL+05b**].

To each piece of the k-quotient we assign a distinct integer modulo k, say $(s_0, \ldots, s_{k-1})$, with $s_i \equiv i \pmod{k}$. In the k-quotient, we adjust the content of a cell x of $\lambda^{(i)}$ by

$$\widetilde{c}(x) \;=\; kc(x) + s_i\,. \tag{4.3}$$

The result is that the shifted content of the labels in the quotient correspond precisely with the content of the labels of the ribbons in the ribbon tableau.

Now we can define a new statistic on the k-quotient, called the k-inversion number, which will differ from cospin by some constant depending only on the shape. We say that cells x and y form a *k-inversion* if $k > \widetilde{c}(y) - \widetilde{c}(x) > 0$ and the entry of x is larger than the entry of y. The k-inversion number, denoted inv_k, for a k-tuple of tableaux is the number of such k-inversions.

This statistic motivates the following encoding of a k-tuple of semi-standard Young tableaux. Construct a word whose letters are subsets of $\{1, \ldots, n\}$ by defining the jth letter to be the set of entries with shifted content j. Note that $\lambda^{(i)}$ contributes all letters x with $\widetilde{c}(x) \equiv i \pmod{k}$. Therefore the shape of $\lambda^{(i)}$ may be recovered from this word from the descent set of the $i \pmod{k}$ subword. With this motivation, we say x and y form a *k-descent* if $\widetilde{c}(y) - \widetilde{c}(x) = k$ and the entry of x is larger than the entry of y.

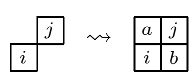
The vertices for which we wish to establish a dual equivalence graph will be words arising as the content readings words of standard k-tuples of Young tableaux. Fixing the shapes of the k-tuples amounts to fixing the k-descent set, and in order for cospin to be a function on connected components, we must ensure that the edges of the graph preserve the k-inversion number.

We define the signature $\sigma : V \longrightarrow \{\pm 1\}^{n-1}$ by

$$\sigma(w)_j \;=\; \begin{cases} +1 & \text{if } \widetilde{c}(j) < \widetilde{c}(j+1) \\ -1 & \text{if } \widetilde{c}(j) > \widetilde{c}(j+1) \end{cases} \tag{4.4}$$

for $j = 1, \ldots, n-1$, where $\widetilde{c}(j)$ is the shifted content of the cell containing the letter j in w.

Note that since we are considering only words coming from standard tableaux, if i and j occur in the same content position in w, say $i < j$,



then cells for a and b must exist, and $i < a, b < j$, so we may conclude that $j - i \geq 3$. This observation shows that equation (4.4) defines σ for all j.

4.3. Constructing a dual equivalence graph. The following involutions are the basic ingredients in constructing edges which preserve k-descents and the k-inversion number. Recall from the previous section that none of $i-1, i, i+1$ may occur with the same content. In line with dual equivalence axiom 1, say that a word w *admits an i-edge* if $\sigma(w)_{i-1} = -\sigma(w)_i$, i.e. w would like to be part of an i-edge. We regard the distance between two entries of w as the difference in their contents, with the obvious extension $\mathrm{dist}(a_1, \ldots, a_l) = \max_{i,j}(\mathrm{dist}(a_i, a_j))$.

Definition 4.1. Define involutions on words admitting an i-edge as follows:

$$\begin{aligned} d_i(\cdots\, i\, \cdots\, i\pm 1\, \cdots\, i\mp 1\, \cdots) &= \cdots\, i\mp 1\, \cdots\, i\pm 1\, \cdots\, i\, \cdots\,, \\ \widetilde{d}_i(\cdots\, i\, \cdots\, i\pm 1\, \cdots\, i\mp 1\, \cdots) &= \cdots\, i\pm 1\, \cdots\, i\mp 1\, \cdots\, i\, \cdots\,, \end{aligned}$$

with all other entries remaining fixed, and define the involution $D_i^{(k)}$ by

$$D_i^{(k)}(w) = \begin{cases} d_i(w) & \mathrm{dist}(i-1,i,i+1) > k \\ \widetilde{d}_i(w) & \mathrm{dist}(i-1,i,i+1) \le k \end{cases}$$

If $\mathrm{dist}(i-1,i,i+1) > k$ in w, then d_i preserves the k-inversion number and k-descent set, otherwise $\widetilde{d}_i$ preserves both these statistics. Therefore $D_i^{(k)}$ preserves both the k-inversion number and the k-descent set for all words.

When $k = 1$, the definition of $D_i^{(1)}$ reduces to the standard elementary dual equivalence since necessarily $\mathrm{dist}(i-1, i, i+1) > 1$. In this case, allowing $D_i^{(1)}$ to define edges in the obvious way recovers the standard dual equivalence graph on tableaux.

When $k = 2$ the situation is not much more complicated. Now the full description of $D_i^{(2)}$ is needed, however, it is still relatively straightforward to check that allowing $D_i^{(2)}$ to define i-edges in the obvious way gives a dual equivalence graph. For a (skew) partition μ, denote this graph by $\mathcal{G}_\mu^{(2)}$.

Theorem 4.2. *$\mathcal{G}_\mu^{(2)}$ is a dual equivalence graph for which the cospin statistic is constant on connected components.*

Corollary 4.3. $\widetilde{K}_{\lambda,\mu}^{(2)}(q) \in \mathbb{N}[q]$.

The proof of Theorem 4.2 is surprisingly short, and offers a much simpler combinatorial proof of Corollary 4.3 than the proof using crystals. Sadly, when $k \ge 3$, $D_i^{(k)}$ will not give the edges of a dual equivalence graph. For instance, if w has the pattern 2431 with $\mathrm{dist}(1,2,3) \le k$, then $\widetilde{d}_2(w)$ contains the pattern 3412, resulting in a necessary double edge for E_2 and E_3 by axiom 3. This implies that E_i may, and sometimes must, change the positions of entries less than $i-1$.

In general, we will construct edges E_i inductively, beginning with $D_i^{(k)}$, thereby ensuring that the k-descent set and k-inversion number will be preserved. The process considers "forced" double edges of the kind discussed above and adjusts the graph slightly to account for these. The algorithm for constructing edges depends heavily on the fact that the subgraph of all lower colored edges is a dual equivalence graph, and relies on many properties of dual equivalence graphs in order to be well-defined. Let $\mathcal{G}_\mu^{(k)}$ denote the graph so constructed whose vertices are given by standard k-ribbon tableaux of (skew) shape μ with σ defined by equation (4.4).

Conjecture 4.4. *$\mathcal{G}_\mu^{(k)}$ is well-defined and is a dual equivalence graph for which the cospin statistic is constant on connected components.*

To prove Conjecture 4.4, we look more closely at the resulting edges E_i defined by the algorithm. Interestingly, the edges break into two cases based on $\mathrm{dist}(i-1,i,i+1)$. When $\mathrm{dist}(i-1,i,i+1) > k$, the edges are simply given by the elementary dual equivalences $D_i^{(k)} = d_i$, and proving that the resulting subgraph is well-defined and satisfies the axioms can be done for arbitrary k. When $\mathrm{dist}(i-1,i,i+1) \le k$, the situation is much more complicated. The crux of the argument comes down to one key lemma which characterizes the connection between $D_i^{(k)}$ and double edges. While this lemma remains a conjecture for $k > 3$, we do have the following result.

Theorem 4.5. *Conjecture 4.4 is true for $k \le 3$.*

Corollary 4.6. *For each partition λ, let $C_\mu^{(k)}(\lambda)$ denote the set of connected components of $\mathcal{G}_\mu^{(k)}$ which are isomorphic to $\mathcal{G}_\lambda$. Then for $k \le 3$,*

$$\widetilde{G}_\mu^{(k)}(x;q) = \sum_\lambda \left(\sum_{\mathcal{C}\in C_\mu^{(k)}(\lambda)} q^{\widetilde{s}(\mathcal{C})} \right) s_\lambda(x).$$

In particular, $\widetilde{G}_\mu^{(k)}(x;q)$ is Schur positive.

4.4. Macdonald polynomials. In 1988, Macdonald [**Mac88**] found a remarkable new basis of symmetric functions in two parameters which specializes to Schur functions, complete homogeneous, elementary and monomial symmetric functions and Hall-Littlewood functions, among others. The transformed Macdonald polynomials $\widetilde{H}_\mu(x;q,t)$ are uniquely characterized by certain orthogonality and triangularity conditions as follows.

PROPOSITION 4.7 ([**Hai99**]). *The $\widetilde{H}_\mu(x;q,t)$ are the unique functions satisfying the following:*

(1) $\widetilde{H}_\mu(x;q,t) \in \mathbb{Q}(q,t)\{s_\lambda[X/(1-q)] | \lambda \geq \mu\}$;
(2) $\widetilde{H}_\mu(x;q,t) \in \mathbb{Q}(q,t)\{s_\lambda[X/(1-t)] | \lambda \geq \mu'\}$;
(3) $\widetilde{H}_\mu[1;q,t] = 1$.

The square brackets in Proposition 4.7 stand for *plethystic substitution.* In short, $s_\lambda[A]$ means s_λ applied as a Λ-ring operator to the expression A, where Λ is the ring of symmetric functions. For a thorough account of plethysm, see [**Hai99**].

Of particular interest are the (q,t)-Kostka polynomials $\widetilde{K}_{\lambda,\mu}(q,t)$ which give the Schur expansion of Macdonald polynomials:

$$\widetilde{H}_\mu(x;q,t) = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q,t) s_\lambda(x). \tag{4.5}$$

The Macdonald positivity conjecture states that $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$. Garsia and Haiman [**GH93**] conjectured that the transformed Macdonald polynomials $\widetilde{H}_\mu(x;q,t)$ could be realized as the bigraded characters of the diagonal action of S_n on two sets of variables. By analyzing the algebraic geometry of the Hilbert scheme of n points in the plane, Haiman [**Hai01**] was able to prove this conjecture and consequently establish $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

Another breakthrough in the study of Macdonald polynomials came with Haglund's combinatorial formula for the monomial expansion of $\widetilde{H}_\mu(x;q,t)$ [**Hag04**]. This formula, which was proven by Haglund, Haiman and Loehr [**HHL05a**], does not give a combinatorial proof of $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$, but does make the problem more accessible. Combining Theorem 2.3, Proposition 3.4 and equation (23) from [**HHL05a**], to give a combinatorial description of $\widetilde{K}_{\lambda,\mu}(q,t)$ it suffices to give a description of the Schur expansion of certain LLT polynomials. Below we recall Haglund's formula and show how the graphs constructed for LLT polynomials also apply to Macdonald polynomials.

For a cell c in the diagram of λ, define the *arm*, $a(c)$, to be the number of cells to the right of c, and the *leg*, $l(c)$, to be the number of cells above c.

Let S be a standard filling of λ, that is $S : \lambda \to \mathcal{A}_n$ where $|\lambda| = n$. A *descent* of S is a cell (i,j) of λ such that $S(i,j) > S(i,j-1)$. Denote by $\mathrm{Des}(S)$ the set of all descents of S. Then define the *major index* of S by

$$\mathrm{maj}(S) \stackrel{\text{def}}{=} |\mathrm{Des}(S)| + \sum_{c \in \mathrm{Des}(S)} l(c). \tag{4.6}$$

An ordered pair of cells $((i,j),(g,h))$ is called *attacking* if $j = h$ and $i < g$, or $j = h+1$ and $i > g$. An *inversion pair* of S is an attacking pair $((i,j),(g,h))$ such that $S(i,j) > S(g,h)$. Denote by $\mathrm{Inv}(S)$ the set of inversion pairs of S. Then define the *inversion number* of S by

$$\mathrm{inv}(S) \stackrel{\text{def}}{=} |\mathrm{Inv}(S)| - \sum_{c \in \mathrm{Des}(S)} a(c). \tag{4.7}$$

Expressed in terms of quasi-symmetric functions, Haglund's formula is

$$\widetilde{H}_\mu(x;q,t) = \sum_{S:\mu \to [n]} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} Q_{n,\sigma(S)}(x). \tag{4.8}$$

For a given filling S, we may define the spaced row-reading word of S, denoted $w(S)$, to be the row-reading word of S augmented with $\emptyset$'s in each cell of $(\mu_1^{\mu_1})/\mu$. The inversion pairs defined above exactly give the k-inversions of $w(S)$, and similarly the descent set of S corresponds precisely with the k-descent set of $w(S)$, as defined in Section 4.2. The arm and leg statistics remain unchanged, thus the k-inversion number and k-descent set for $w(S)$ completely determine $\mathrm{inv}(S)$ and $\mathrm{maj}(S)$ as given in equations 4.6 and 4.7. Therefore the dual equivalence graphs constructed for LLT polynomials hold these statistics constant on connected components, giving rise to the following corollary to Conjecture 4.4.

THEOREM 4.8. *Let μ be a partition with $\mu_1 = k$. Assuming Conjecture 4.4 holds, let $\mathcal{G}^{(k)}_{\mu_D}$ be the dual equivalence graph for standard k-ribbon tableaux whose content reading words have* $\mathrm{Des}_k = D$. *For each partition λ, let $C^{(k)}_{\mu_D}(\lambda)$ denote the set of connected components of $\mathcal{G}^{(k)}_{\mu_D}$ which are isomorphic to $\mathcal{G}_\lambda$. Then*

$$\widetilde{K}_{\lambda,\mu}(q,t) = \sum_{D\subset\mu} \sum_{\mathcal{C}\in C^{(k)}_{\mu_D}(\lambda)} q^{\mathrm{inv}(\mathcal{C})} t^{\mathrm{maj}(\mathcal{C})}.$$

In particular, by Theorem 4.5, we have a combinatorial formula for $\widetilde{K}_{\lambda,\mu}(q,t)$ when μ is a partition with at most 3 columns.

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