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The combinatorics of associated Hermite polynomials

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ABSTRACT. We develop a combinatorial model of the associated Hermite polynomials and their moments, and prove their orthogonality with a sign-reversing involution. We find combinatorial interpretations of the moments as complete matchings, connected complete matchings, oscillating tableaux, and rooted maps and show weight-preserving bijections between these objects. Several identities, linearization formulas, and a second combinatorial model are also derived.

RÉSUMÉ. Nous développons un modèle combinatoire pour les polynômes de Hermite associés et leurs moments, et prouvons leur orthogonalité à l'aide d'une involution changeant le signe. Nous obtenons des interprétations combinatoires des moments en termes de couplages parfaits, de couplages parfaits connexes, de tableaux oscillants et de cartes enracinées, et nous décrivons des bijections préservant le poids entre ces familles d'objets. Plusieurs identités, des formules de linéarisation et un second modèle combinatoire en découlent.

The *associated Hermite polynomials* are a sequence of orthogonal polynomials considered by Askey and Wimp in [**AW84**], who analytically derived a number of results about these polynomials. They are also treated in [**Ism05**, §5.6]. In section 1 we develop a combinatorial interpretation of these polynomials, their moments, and describe an involution that proves the orthogonality and L^2 norms of the polynomials with respect to those moments. Then in section 2 we shall describe several identities and interesting linearization formulas involving associated Hermite polynomials. We finish with weight-preserving bijections between a number of classes of combinatorial objects whose generating functions all yield the moments of the associated Hermites and a second combinatorial model for the polynomials.

We will assume that the reader is familiar with Viennot's general combinatorial theory of orthogonal polynomials [**Vie83, Vie85**] and with the combinatorics of Hermite polynomials; see [**AGV82, dSCV85, LY89**] and also [**Vie83**, §II.6].

In this abstract, $(a)_n$ denotes the rising factorial $a(a+1)\cdots(a+n-1)$, and $a^{\underline{n}}$ denotes the falling factorial $a(a-1)\cdots(a-k+1)$. Figures in this abstract will often have black and green elements (usually edges in graphs). The PDF version will have the colors, of course, but in the print version, the green will appear gray. The reader should understand that green elements of figures appear gray if reading a printed version of this abstract.

1. Definition and orthogonality

The associated Hermite polynomials may be defined by shifting the recurrence relation for the usual Hermite polynomials, which is

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x),$$

to

$$H_{n+1}(x;c) = xH_n(x;c) - (n+c)H_{n-1}(x;c), \tag{1.1}$$

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with $H_0(x) = H_0(x;c) = 1$ and polynomials with negative indices equal to zero. Askey and Wimp use a different normalization than we do; one obtains our normalization from plugging $x/\sqrt{2}$ into their associated Hermites and dividing by $\left(\sqrt{2}\right)^n$.

The usual Hermite polynomial $H_{n+1}(x)$ is the generating function for incomplete matchings of $[n+1]$, in which fixed points have weight x and edges have weight -1; that combinatorial interpretation can be derived from the recurrence relation as follows: the vertex $n+1$ may be fixed with weight x, times the weight of all matchings on $[n]$; or we may connect vertex $n+1$ to any of the n vertices to its left, give the edge weight -1 and multiply by all matchings on the $n-1$ remaining vertices.

For the associated Hermites, we'll build the matchings recursively as described above and think of the parameter c as meaning that one special choice for the edge from $n+1$ may have weight -1 or $-c$. Two natural choices are to make the special choice be the leftmost available vertex, or the rightmost available vertex. Choosing the rightmost available vertex happens to make the orthogonality involution easy to prove, and yields the following result:

THEOREM 1.1. *The nth associated Hermite polynomial is the sum over certain weighted matchings:*

$$H_n(x;c) = \sum_{\text{matchings } M \text{ of } [n]} wt(M), \tag{1.2}$$

in which fixed points have weight x, edges that nest no fixed points or edges and have no left crossings may have weight -1 or $-c$, and all other edges have weight -1.

PROOF. We build the matching from right to left, and if at some point we add an edge and do not choose the rightmost available vertex, then that edge will nest a vertex, and when we come to that vertex, we will either leave it fixed (resulting in a fixed point underneath that edge), connect to another vertex underneath the edge (resulting in an edge nested by the original edge), or connect to a vertex to the left of the edge, resulting in a left crossing for the original edge. Any of these possibilities indicate that the rightmost vertex was not chosen, so edges for which none of those happen are "eligible" for weight $-c$. □

An example of such a weighted matching is shown in Figure 1.

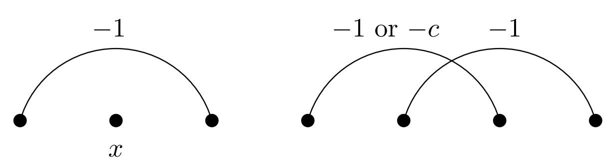


FIGURE 1. A matching on 8 points using the rightmost-choice weighting. This weighting is used throughout this abstract.

We want a linear functional $\mathcal{L}_c$ with respect to which the associated Hermite polynomials are orthogonal. This linear functional is determined by its moments $\mathcal{L}_c(x^n)$, which according to Viennot's general theory, are a sum over weighted Dyck paths in which a northeast edge has weight 1 and a southeast edge leaving from height j has weight $j + c$. There are no Dyck paths of odd length, so the odd moments are zero.

Building matchings as in [**Vie83**, §II.6], we have two combinatorial interpretations for the moments:

THEOREM 1.2. *The nth moment $\mu_n(c)$ of the associated Hermite polynomials is the generating function for complete matchings of $[n]$ weighted by either (1) edges which are not nested by any other edge may have weight 1 or c, and all other edges have weight 1; or (2) edges with no right crossings may have weight 1 or c and all other edges have weight 1.*

The two weightings correspond to giving weight c to the leftmost and rightmost choice, respectively, in the matchings. These interpretations also explain why the odd moments are zero, since there are no complete matchings on an odd number of vertices. For the proof of orthogonality, we shall use the rightmost weighting; later we shall use the leftmost weighting. Figures 2 and 3 show a matching using the two weightings.

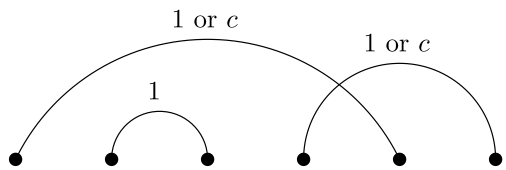


FIGURE 2. A complete matching on 6 points under the leftmost-choice weighting for the moments.

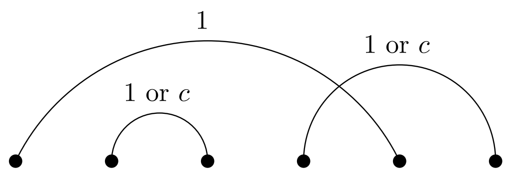


FIGURE 3. The same complete matching under the rightmost-choice weighting. This is used in the orthogonality proof.

1.1. Proof of orthogonality.

We wish to prove the following theorem in a combinatorial manner:

THEOREM 1.3. *The associated Hermite polynomials $H_n(x;c)$ are orthogonal with respect to the linear functional $\mathcal{L}_c$ with the above moments. They satisfy*

$$\mathcal{L}_c(H_n(x;c)H_m(x;c)) = \begin{cases} 0 & n \neq m, \\ (c+1)_n & n = m. \end{cases} \tag{1.3}$$

PROOF. The proof proceeds very similarly to the proof of orthogonality for usual Hermite polynomials. The product $H_n(x;c)H_m(x;c)$ is the generating function for pairs of matchings with, say, black edges, using the rightmost weighting. Applying $\mathcal{L}_c$ has the effect of putting a complete matching with the rightmost weighting with green edges on the fixed points. We will use the phrase *paired matching* to refer to such an object, with black homogeneous edges and arbitrary green edges, weighted as above. This is not standard terminology; it is only for our convenience.

Using Theorems 1.1 and 1.2, the left side of (1.3) is the generating function for paired matchings, where black edges may have weight -1 or $-c$ if they nest no edges, have no green crossings and no left black crossing; otherwise black edges have weight -1. Green edges may have weight 1 or c if they have no right green crossing and weight 1 otherwise. See Figure 4 for an example of such an object for $n = 5$ and $m = 3$.

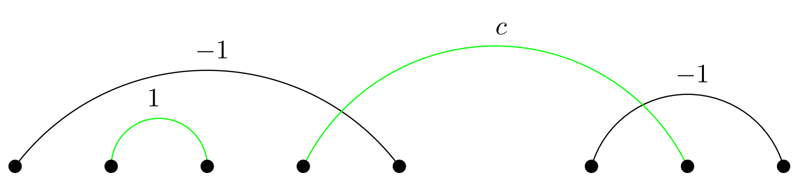


FIGURE 4. A paired matching for $n = 5$ and $m = 3$.

We need an involution that shows the generating function for paired matchings equals zero when $n \neq m$, and equals $(c+1)_n$ otherwise. Assume that $n \geq m$ and put $[n]$ to the left of $[m]$. The involution is the very similar to that used in the combinatorial proof of orthogonality for usual Hermite polynomials:

Find the leftmost homogeneous edge in $[n]$ that nests no other edges and change its color.

For example, in Figure 4, one would change the color of the leftmost green edge that connects vertices 2 and 3 to black. This operation is evidently an involution and will certainly change the sign; to complete the proof, one verifies that the weight of no other edge is affected by this change, and that if we change the color of an edge weighted $\pm c$, the new edge is eligible for a weight of $\mp c$.

If $n > m$, there must be a homogeneous edge in $[n]$; in that case, the above involution has no fixed points, and we have proved that $H_n(x;c)$ is orthogonal to $H_m(x;c)$.

Now we shall prove that the L^2 norm of the associated Hermites is $(c+1)_n$ by interpreting the paired matchings as something whose generating function is known to be $(c+1)_n$: permutations weighted by left-to-right maxima. See [**dMV94, FS84**] for proofs of this fact in the context of Laguerre polynomials. ("Left-to-right maxima" is "éléments saillants inférieurs gauches" in French.) This bijection naturally generalizes the usual combinatorial proof that the L^2 norm of the Hermite polynomials is $n!$.

First, apply the above involution to paired matchings with $n = m$; that involution will cancel all matchings with a homogeneous edge. To set up the bijection, begin with a matching on $[n] \sqcup [n]$ with no homogeneous edges. Number the edges starting from the right as shown in Figure 5. Think of the right side as the domain, and the left side as the range. A simple induction argument demonstrates that edges that get weight 1 or c (here we shall think of that as having weight $1+c$) correspond exactly to digits in the permutation that are left-to-right maxima.

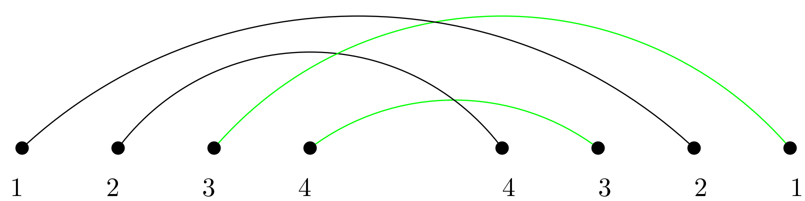


FIGURE 5. The permutation 3142 as a matching. The right side is the domain, the left the range. The digits 3 and 4 are left-to-right maxima in the permutation, and indeed the green edges connecting to 3 and 4 on the left have weight $1+c$ under the associated Hermite moment weighting.

This bijection from the fixed points of the involution to permutations preserves weight, hence the L^2 norms of the associated Hermite polynomials are $(c+1)_n$. This completes the proof of Theorem 1.3. $\square$

We also note that by [**FS84**, Lemma 2.1], the L^2 norm can also be interpreted as the generating function for permutations with cycles weighted by $c+1$.

2. Identities

Our first result is a combinatorial proof of the following identity, found in Askey and Wimp [**AW84**, equation (4.18)]:

THEOREM 2.1. *The associated Hermites may be written as a sum of usual Hermite polynomials:*

$$H_n(x;c) = \sum_{k\geq 0} (-1)^k (c)_k \binom{n-k}{k} H_{n-2k}(x). \tag{2.1}$$

We will need two lemmas to prove Theorem 2.1. We omit their proofs; both can be done with induction.

LEMMA 2.2. $(-1)^k (c)_k$ *is the generating function for complete matchings on $2k$ vertices, with the associated Hermite polynomial weighting, such that all edges of weight -1 have a left crossing by an edge of weight $-c$. Furthermore, in such matchings there are exactly k "slots" available underneath the edges weighted $-c$ where one could place the left endpoint of a new edge of weight -1, and only one "slot" available for the left endpoint of a new edge of weight $-c$.*

Figure 6 shows an example of such a configuration.

LEMMA 2.3. *For such a configuration on $2k$ vertices as described in Lemma 2.2, there are $k+1$ places where the left endpoint of one or more green edges of weight 1 could be placed without affecting the weight of the configuration.*

PROOF OF THEOREM 2.1. Since $H_n(x;c)$ is an even or odd polynomial if n is even or odd, respectively, we can certainly write

$$H_n(x;c) = \sum_{k\geq 0} a_{nk} H_{n-2k}(x) \tag{2.2}$$

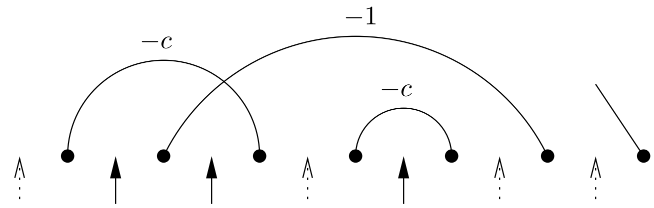


FIGURE 6. A matching on 6 vertices of the type described by Lemma 2.2. If the new edge on the right is to have weight -1 and satisfy the conditions, it must connect to a new vertex in one of the three available slots, indicated by the solid arrows.

for some coefficients a_{nk}. We show that those coefficients equal $(-1)^k(c)_k\binom{n-k}{k}$. Fix k between 0 and $n/2$, multiply both sides by $H_{n-2k}(x)$, and apply the usual Hermite linear functional $\mathcal{L}_0$. On the right side, we use orthogonality and equation (2.2) becomes

$$\mathcal{L}_0(H_n(x;c)H_{n-2k}(x)) = a_{nk}(n-2k)!.$$

Thinking of the left side as paired matchings on $[n]$ and $[n-2k]$ with black edges of weight -1 and $-c$ as appropriate, and green edges all of weight 1, we may apply the following involution: find the leftmost homogeneous edge of weight ± 1 in $[n]$ or $[n-2k]$ and flip its color, *unless* that edge has a left crossing with an edge of weight $-c$. Swapping the colors on such edges does not preserve the weight of the paired matching.

Lemma 2.2 tells us exactly what configurations of edges will remain in $[n]$ after applying the involution; Lemma 2.3 tells us that such configurations may be viewed as consisting of k "chunks" of vertices. Placing the green edges into those chunks is equivalent to forming a weak composition of k into $n-2k+1$ parts; there are $\binom{n-k}{k}$ such compositions, and having chosen where the $n-2k$ edges in $[n]$ start, we can choose their endpoints in $[n-2k]$ in $(n-2k)!$ ways. Together we have

$$(-1)^k(c)_k\binom{n-k}{k}(n-2k)! = \mathcal{L}_0(H_n(x;c)H_{n-2k}(x)) = a_{nk}(n-2k)!$$

which proves the identity of Theorem 2.1. □

2.1. Linearization formulas. In [**Mar94**, theorem 3.1], Markett shows that the linearization coefficients in

$$H_N(x;c)H_M(x;c) = \sum_{j=0}^{\min(N,M)} f(N,M,j)\; H_{N+M-2j}(x;c) \tag{2.3}$$

are

$$f(N,M,j) = (N+M-2j+c+1)_j\; {}_3F_2\left(\begin{matrix} j-N & j-M & -j \\ & j-N-M-c & 1\end{matrix}\right) \tag{2.4}$$

where the ${}_3F_2$ notation indicates a hypergeometric function evaluated at $x=1$. We can prove

THEOREM 2.4. *The linearization coefficients $f(N,M,j)$ of equation (2.4) are polynomials in c with nonnegative integer coefficients.*

PROOF. Omitted; it is mostly simple algebra. □

Note that when $c=0$, the ${}_3F_2$ of (2.4) sums by the Pfaff-Saalschütz identity to

$$\frac{(N+1-j)_j(M+1-j)_j}{j!}, \tag{2.5}$$

and we recover the linearization coefficients for usual Hermite polynomials; the expression above, after multiplying by $(N+M-2j)!$, counts inhomogeneous matchings on $[n]\sqcup[m]\sqcup[n+m-2j]$, as shown by de Sainte-Catherine and Viennot in [**dSCV85**].

A combinatorial interpretation of the coefficients (2.4) is quite desirable but none is known. The best starting points seem to be [**And75, Nan58, AB84**]; the first two papers concern the usual Pfaff-Saalschütz identity, the third features a combinatorial proof of the q-Pfaff-Saalschütz identity.

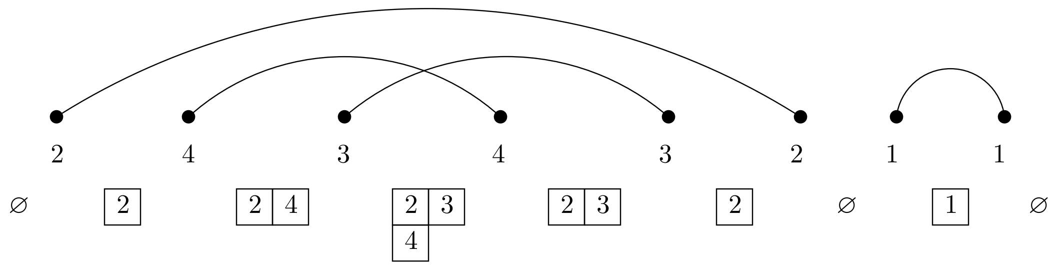


FIGURE 7. A complete matching and the corresponding oscillating tableau. The numbers in the Ferrers shapes are not, strictly speaking, part of the oscillating tableau; they are only used in the bijection from the matching to the tableau.

We also have the following mixed linearization formula.

THEOREM 2.5. *If $n \geq m - 1$, then*

$$H_n(x; c)H_m(x) = \sum_k \binom{n+c}{k}\binom{m}{k} k! H_{n+m-2k}(x; c), \tag{2.6}$$

where the sum runs from 0 *to* $\min(m, \lfloor (n+m)/2 \rfloor)$.

PROOF. Omitted. One uses induction and the recurrence relations for the induction step. □

3. Associated Hermite moments and oscillating tableaux

In this section we will describe a statistic on *oscillating tableaux*, also known as up-down tableaux, and a bijection from these tableaux to complete matchings which is weight-preserving when using the weight for associated Hermite moments. Oscillating tableaux were described by Sundaram [**Sun90**]; see section 5 of [**CDD⁺07**] for discussion of their origins and the bijection to complete matchings. Briefly, an oscillating tableau is a path in the Hasse diagram of the Young lattice in which at each point one either moves up to a partition that covers the current partition, or moves down to a partition covered by the current partition. For our purposes, the path will always begin and end with the empty shape. The *length* of an oscillating tableau is the number of edges in the path. Figure 8 has an example of an oscillating tableau of length 8.

In this section, we use Theorem 1.2's "leftmost-available" weighting of complete matchings, in which edges that are not nested by other edges may have weight 1 or c, and all other edges have weight 1.

Roughly speaking, the bijection from complete matchings to oscillating tableaux works by RSK-inserting numbers when edges start, and deleting them when edges end. More precisely, given a complete matching, number the edges from right to left as in Figure 7. (Equivalently, write the matching as a double occurrence word; see section 4.) We will map this matching to a sequence of Ferrers shapes. Begin with the empty Ferrers shape and read the matching left to right. When edge j starts, RSK-insert a j into the tableau; when edge j ends, delete the box containing j. When done, erase the numbers in the Ferrers shapes. Figure 7 has an example.

There is a possible point of confusion here. A tableau in this context is a path in the Hasse diagram of the Young lattice—a sequence of Ferrers shapes. A standard Young tableau is a path that continually moves up, and therefore it is simple to record the path with a single Ferrers shape filled with numbers that strictly increase in rows and columns. In Figure 7, the Ferrers shapes are written as Young tableaux, which is only for our convenience. The actual image of the complete matching is the same sequence without the numbers in the shapes. The reason for this is that RSK is a bijection, and one can unbump numbers.

Figure 8 describes the inverse map from tableaux to matchings. We read the sequence of Ferrers shapes from right to left. Because of how we number the edges, the first box must have a 1 in it. In general, when the shape gets larger, we put the next-largest number into the new box, because we've started a new edge. The third shape from the right is $\begin{smallmatrix}1&2\\3\end{smallmatrix}$, and the shape to its left must be $\begin{smallmatrix}1\\3\end{smallmatrix}$, because unbumping the 2 is the only way to produce the second shape. This oscillating tableau corresponds to the matching 43412321, using the vertex-numbering scheme described above.

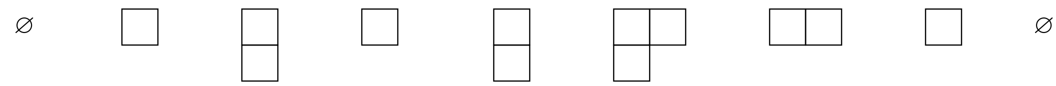


FIGURE 8. An oscillating tableau that corresponds to the complete matching (13)(26)(48)(57).

Let us weight oscillating tableaux with the following statistic: numbers that appear in the first column have weight 1 or c, and all other numbers have weight 1. That statistic is exactly what we need to prove the following theorem.

THEOREM 3.1. *There is a weight-preserving bijection between oscillating tableaux of length $2n$ weighted with the above statistic and complete matchings weighted with the leftmost-available associated Hermite weighting.*

We will use several preliminary results to prove this theorem.

LEMMA 3.2. *In an oscillating tableau, when a number is added to a shape, the corresponding edge is nested by all edges whose corresponding number in the shape is smaller, and has a left crossing from all edges whose corresponding number in the shape is bigger. Edges whose corresponding numbers never appear together in a shape neither nest nor cross one another.*

For example, when we move from $^{2\ 4}$ to $\substack{2\ 3\\ 4}$ in Figure 7, edge 3 is nested by edge 2 and has a left crossing from edge 4. The proof of this is left to the reader; it follows from the way the edges are numbered and in what order we add numbers to the tableau.

The above lemma implies the following facts:

PROPOSITION 3.1. *In an oscillating tableau, edges that get nested by other edges are exactly those whose number appears in the 2nd, 3rd, etc, column of a shape. Edges that have a right crossing are exactly those whose number appears in the 2nd, 3rd, etc row of a shape.*

PROOF OF THEOREM 3.1. The bijection between complete matchings and oscillating tableaux clearly preserves weight: edges that do not get nested by another edge must appear in the first column only. Note also that we could have used the rightmost-available weighting from Theorem 1.2; in that case, we would have needed to make our statistic "entries that appear in the first row and stay there get weight 1 or c". □

4. Associated Hermite moments, rooted maps, and connected matchings

In addition to the weight-preserving bijection between associated Hermite moments and oscillating tableaux of section 3, there is a weight-preserving bijection between associated Hermite moments and rooted maps. See [**Tut73, JV00**] for introductions to maps. A map may be thought of as a graph along with an embedding into a surface. A rooted map is a map in which one edge has been oriented. There is an axiomatic construction of maps that makes it natural to think of the edges in a map as pairs of half-edges or edge ends and we will speak of edge ends in this section.

This connection was motivated by setting $c = 1$; if we have a polynomial that we suspect is the generating function for some objects, setting c to 1 gives us a count of how many objects there are, which facilitates searching. Doing so yields

$$1, 1, 2, 10, 74, 706, 8162, 110410, 1708394, \ldots$$

which is sequence A000698 in [**Slo**]. This sequence likely first appeared in [**Tou52**]; it counts connected matchings (see below).

In Table 1 of [**AB00**, page 10], Arquès and Béraud count rooted maps by number of edges and vertices; that table also describes associated Hermite moments: the entry in the nth row and mth column is the number of rooted maps with n edges and m vertices, and is also the coefficient of c^{m-1} in $\mu_{2n}(c)$. We will weight each vertex in such a map by c *except* the vertex at the head of the root edge, and use the bijection between rooted maps in orientable surfaces and connected matchings found in the work of Ossona de Mendez and Rosenstiehl [**OdMR05, OdMR99**]. A *connected matching* on $2n$ vertices is one in which all vertices except 1 and $2n$ are nested by an edge. Equivalently, one can write a matching as a *double occurrence word* in the letters $1, 2, \ldots, n$ where each letter appears exactly twice; then a matching is connected if the corresponding double occurrence word cannot be written as the concatenation of two double occurrence words.

A double occurrence word is the same thing as the vertex-numbering scheme used in section 3. We shall weight connected matchings by giving weight c to all nonnested edges *except* the edge containing vertex 1. Then we have

THEOREM 4.1. *The function given in* [**OdMR05**] *and* [**OdMR99**] *is a weight-preserving bijection from rooted maps in orientable surfaces with* k *vertices and* n *edges to connected matchings on* $2n+2$ *vertices of weight* c^{k-1}.

PROOF. The idea of the bijection is this: number the edges in the rooted map, add a new loop at the vertex adjacent to the root, then build a double occurrence word by visiting each vertex and adding the edge numbers adjacent to the vertex to the word.

The bijection is weight-preserving because when deciding the next vertex to visit, the algorithm chooses the vertex in the rooted map corresponding to the leftmost unattached vertex in the partially-constructed matching. As we add edge ends to the list, we will add a new edge to the matching that contains that leftmost unattached vertex. No edge can then nest the newly created edge, so every visit to a new vertex in the rooted map results in exactly one nonnested edge in the matching. □

Figure 9 shows an example of the bijection. We will color green the vertices of weight c in the rooted map and the edges of weight c in the connected matching. We start at the head of edge α and read counterclockwise around vertex a; our initial double occurrence word is

$$\alpha\ 1\ 2\ 3\ \alpha.$$

We have visited both ends of α, so we move to the unvisited end of edge 1, go around vertex b and add 4 4 5 2 5 1 to the word, which is now

$$\alpha\ 1\ 2\ 3\ \alpha\ 4\ 4\ 5\ 2\ 5\ 1.$$

Now move to the unvisited end of edge 3 and do the same thing; we just append 3 to the word. We end up with

$$\alpha\ 1\ 2\ 3\ \alpha\ 4\ 4\ 5\ 2\ 5\ 1\ 3,$$

which is double-occurence word for the connected matching $(1,5)(2,11)(3,9)(4,12)(6,7)(8,10)$ where the edges incident to vertices 11 and 12 have weight c because edges 1 and 3 in the rooted map were the edges along which we first visited vertices b and c.

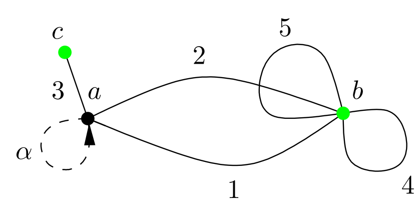
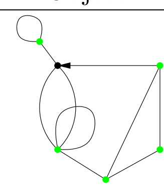
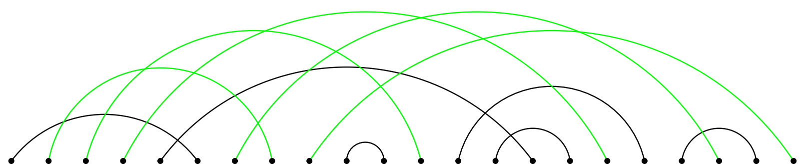
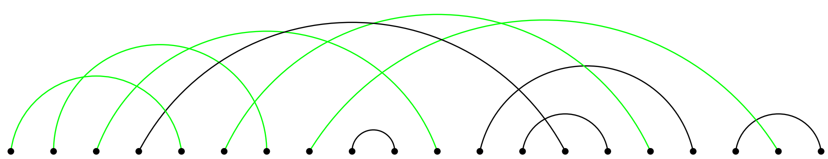
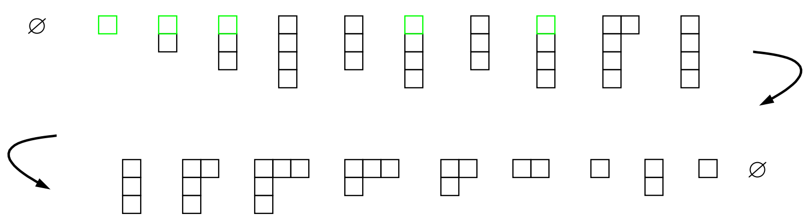


FIGURE 9. A rooted map to which we'll apply the bijection to connected matchings. Green vertices have weight c. We have already added the extra edge, labeled α; the original root was the end of edge 1 incident with vertex a.

Now we need another weight-preserving bijection, this time from weighted connected matchings to one of our original definitions for the moments of associated Hermite polynomials. We will demonstrate such a bijection to the moments weighted with the leftmost-available weighting of Theorem 1.2, in which nonnested edges are eligible for weight c. Call the edge containing vertex 1 the "fake edge".

The bijection works as follows: If the fake edge has no crossings, remove it; the remaining matching on $2n$ vertices, of weight 1, is the result of the bijection. Otherwise, swap the tails of the fake edge and that edge crossing the fake edge which has the leftmost endpoint. That crossing edge must have weight c; give the new edge, which is now nested by the fake edge, weight c also. Continue this tail-swapping process with the fake edge until the fake edge has no crossings, then remove it. The reader may wish to work out an example, and show that the connected matching $(1,5)(2,4)(3,8)(6,7)$ corresponds to the matching $(1,3)(2,4)(5,6)$.

Table 1. A rooted map, a connected matching, a complete matching, and an oscillating tableau, all of weight c^5, that correspond to each other using the weight-preserving bijections of this abstract. In the tableau, we have only colored the first box that corresponds to a number which gets weight c.

Object	What gets weight c
	Vertices not adjacent to head of root edge.
 $(1,6)(2,8)(3,12)(4,17)(5,15)(7,20)(9,22)(10,11)(13,18)(14,16)(19,21)$	Non-nested edges except edge containing vertex 1.
 $(1,5)(2,7)(3,11)(4,14)(6,16)(8,19)(9,10)(12,17)(13,15)(18,20)$	Non-nested edges may have weight 1 or c.
	Numbers that appear in first column may have weight 1 or c.

This map is a bijection because it can be reversed: given such a weighted matching on $2n$ vertices, add a new edge that nests the entire matchings, and swap tails with the green edges (those of weight c) from right to left. Observe that the green edges in the connected matching—which are nonnested—will end up nonnested after the tail-swapping bijection, and vice versa, so this bijection is weight-preserving.

Theorem 4.1 established that the generating functions for rooted maps and connected matchings are the same; that theorem, together with the bijection between connected matchings and arbitrary complete matchings, provides a proof of the following theorem.

Theorem 4.2. *The generating functions for rooted maps with n edges, connected matchings on $2n+2$ vertices, and complete matchings on $2n$ vertices all equal the moment μ_{2n} of the associated Hermite polynomials.*

In the last two sections, we've shown bijections between the moments of the associated Hermites, connected matchings, rooted maps and oscillating tableaux. We summarize these correspondences by going all the way from a rooted map, to a connected matching, to a regular complete matching, to an oscillating tableau in Table 1.

4.1. A second model for associated Hermite polynomials. The above discussion of connected matchings meshes nicely with a second combinatorial model of the associated Hermites, which is motivated

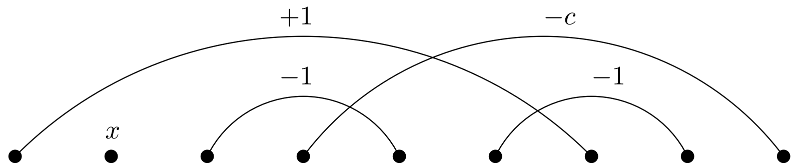


FIGURE 10. A matching on 9 vertices that contributes cx to $H_7(x;c)$ using the combinatorial interpretation of Theorem 4.3. Note the "fake edge" of weight $+1$.

by identity (2.1):

$$H_n(x;c) = \sum_{k\geq 0} (-1)^k (c)_k \binom{n-k}{k} H_{n-2k}(x).$$

The key features of this second model are very similar to those of the connected matching model for the moments: there are no choices for the weights of parts of the matching, and the resulting matching is connected.

THEOREM 4.3. *The associated Hermite polynomial $H_n(x;c)$ is the generating function for certain connected incomplete matchings on $n+2$ vertices with the following weights:*

- *The edge containing vertex 1 has weight 1. Call this edge the "fake edge".*
- *Fixed points have weight x.*
- *Non-nested edges (except the fake edge) have weight $-c$.*
- *Nested edges have weight -1.*

In such matchings, fixed points must be nested by the fake edge. All edges other than the fake edge must either cross or be nested by the fake edge.

An example of such a matching for $H_7(x;c)$ is shown in Figure 10. It is clear that the requirement for nesting and crossing the fake edge yields a connected matching. Note that the connected matching moments of section 4 also have a fake edge.

PROOF. Consider the kth term in the sum (2.1):

$$(-1)^k (c)_k \binom{n-k}{k} H_{n-2k}(x).$$

Begin with the fake edge and put k vertices to the right of it. Put the remaining $n-k$ vertices underneath the fake edge and choose k of them to connect with the edges that will come from the k vertices on the right of the fake edge; that accounts for the binomial coefficient. On the remaining $n-2k$ vertices underneath the fake edge, we put a regular Hermite-style matching; all the edges will have weight -1 since they are nested by the fake edge.

The last thing to do is account for the k edges that come from the right of the fake edge and show that they contribute $(-1)^k(c)_k$. According to Lemma 4.4, the generating function for such a configuration with the edges of weight $+1$ and $+c$ is $(c+1)_{k-1}$, but in our subset, the leftmost edge also gets weight c, so the correct factor is $(c+1)_{k-1} \cdot c = (c)_k$. Also, we must correct for the signs: our edges have weight -1 and $-c$, so we multiply by $(-1)^k$. □

Another proof goes by verifying that the generating function described in the theorem satisfies the three-term recurrence for the associated Hermites (1.1).

The following lemma was used in the proof of Theorem 4.3. It may be proved by induction, similar to Lemma 2.2.

LEMMA 4.4. *The generating function for complete matchings on $2n$ vertices in which all edges go from the "left n" vertices to the "right n" vertices , with all nonnested edges having weight c except the edge containing the leftmost vertex, is $(c+1)_{n-1}$.*

There is a weight-preserving bijection between such matchings and permutations π of $[n]$ weighted by $c^{\mathrm{LRM}(\pi)-1}$ where $\mathrm{LRM}(\pi)$ is the number of left-to-right-maxima of the permutation.

At this point, we have a combinatorial interpretation for both the associated Hermite polynomials (Theorem 4.3) and their moments (Theorem 4.2) in terms of connected matchings with a fake edge; the natural thing to do is combine these to get another proof of orthogonality. This will be quite difficult because it is not at all obvious how to combine a pair of matchings for the polynomials and a matching for the moments to get a paired matching; one would have two fake edges from the polynomials and would need to somehow incorporate the fake edge from the moments into that configuration.

5. Unanswered questions and future directions

We have taken the basic combinatorial model in section 1 for associated Hermite polynomials and their moments and gone in two directions: to oscillating tableaux, and to rooted maps. The appeal of oscillating tableaux is in the recent flurry of work on k-crossings and k-nestings in matchings and set partitions; see [**CDD+07, Kra06, dM06, Kla05, KZ06, Jel07**]. The moments of Charlier polynomials are generating functions for set partitions and it seems likely that some of this work could be used to treat the associated Charlier polynomials.

One disadvantage of the original combinatorial model and oscillating tableaux is that certain parts of the corresponding combinatorial objects may have weight 1 or c. This makes proofs of linearization formulas and other identities more difficult because our proofs are generally "geometric" in nature, in the sense that they depend on the pictorial representation of the objects; we use words like "leftmost" and make other spatial references. In such proofs it is desirable to have a 1-1 correspondence between terms that contribute to the generating function and geometric objects. This makes the sign-reversing involutions we need to find much easier.

Sending c to $c-1$ seems like a solution to this problem, but it would really just shuffle it around: as it stands, the moment weighting for complete matchings on $2n$ points has "1 or c edges", but the weighting for connected complete matchings on $2n+2$ points only has edges of weight c. Shifting c to $c-1$ would fix the problem for matchings on $2n$ vertices, but would give our connected matchings edges of weight c or -1, which is undesirable for the moments.

Observe that in the connected matchings, the rooted maps, and in the second combinatorial model for the associate Hermite polynomials of Theorem 4.3, each model has some sort of "fake edge". Combining the models for the moments and polynomials which both involve connected matchings would be fruitful and eliminate the problems with multiple choices for the weights discussed above, but this has not yet shown promise. A major problem is that each incomplete matching for the polynomial is weighted by x to the number of fixed points—say there are $2k$ fixed points—but the corresponding matchings are matchings on $2k+2$ vertices. It is not clear how to combine these two objects in a geometric or graph-theoretical way that allows a natural and easy proof of orthogonality.

Using rooted maps holds promise, though: de Mendez and Rosenstiehl have generalized the bijection between connected matchings and rooted maps to a bijection between permutations and hypermaps [**OdMR04, OdMR99**]. This suggests an intriguing connection to Laguerre polynomials since hypermaps are built out of permutations in the same way that maps are built out of complete matchings. The paper of Askey and Wimp [**AW84**] which inspired this work devotes much more attention to the associated Laguerres than to Hermites—about two-thirds of the article. It is natural, then, to work out a corresponding combinatorial treatment of those polynomials, especially given the connections between rooted maps and hypermaps. There is also the work of Ismail et al [**ILV88**] who work with the associated Laguerres as birth and death processes—there has been work on birth and death processes and lattice paths [**FG00**] which suggests another avenue for a combinatorial theory of those polynomials.

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