Extended Abstract

q-EULERIAN POLYNOMIALS: EXCEDANCE NUMBER AND MAJOR INDEX

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ABSTRACT. We derive a new q-analog of a classical formula for the exponential generating function of the Eulerian polynomials. This arose in our work on poset topology and was presented as a conjecture at the second author's FPSAC 2006 lecture. We have since proved our conjecture and a symmetric function generalization of it. Our q-Eulerian polynomials are the enumerators for the joint distribution of the excedance statistic and the major index. There is a vast literature on q-Eulerian polynomials that involves other combinations of Eulerian and Mahonian permutation statistics, but this is the first result to address the combination of excedance number and major index. Our proof involves an intriguing new class of symmetric functions and a bijection of Gessel and Reutenauer, which can be viewed as a necklace analog of Stanley's theory of P-partitions. We also discuss connections with (1) the representation of the symmetric group on the homology of a poset introduced by Björner and Welker, (2) the representation of the symmetric group on the cohomology of the toric variety associated with the Coxeter complex of the symmetric group, studied by Procesi, Stanley, Stembridge, Dolgachev and Lunts, (3) the enumeration of words with no adjacent repeats studied by Carlitz, Scoville and Vaughan and by Dollhopf, Goulden and Greene, and (4) Stanley's chromatic symmetric functions.

1. INTRODUCTION

The subject of permutation statistics originated in the early 20th century work of Major Percy MacMahon [22, Vol. I, pp. 135, 186; Vol. II, p. viii], [23] and has developed into an active and important area of enumerative combinatorics over the last four decades. It deals with the enumeration of permutations according to natural statistics. A permutation statistic is simply a function from the symmetric group $\mathfrak{S}_n$ to the set of nonnegative integers. MacMahon studied four fundamental permutation statistics, the inversion index, the major index, the descent number and the excedance number, which we define below.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For each $\sigma \in \mathfrak{S}_n$, the descent set of σ is defined to be

$$\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\},$$

and the excedance set is defined to be

$$\mathrm{EXC}(\sigma) := \{i \in [n-1] : \sigma(i) > i\}.$$

The descent number and excedance number are defined respectively by

$$\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)| \qquad \text{and} \qquad \mathrm{exc}(\sigma) := |\mathrm{EXC}(\sigma)|.$$

For example, if $\sigma = 32541$, written in one line notation, then

$$\mathrm{DES}(\sigma) = \{1, 3, 4\} \qquad \text{and} \qquad \mathrm{EXC}(\sigma) = \{1, 3\};$$

hence $\mathrm{des}(\sigma) = 3$ and $\mathrm{exc}(\sigma) = 2$. If $i \in \mathrm{DES}(\sigma)$ we say that σ has a descent at i. If $i \in \mathrm{EXC}(\sigma)$ we say that $\sigma(i)$ is an excedance of σ.

MacMahon [22, Vol. I, p. 186] observed that the descent number and excedance number are equidistributed, that is, the number of permutations in $\mathfrak{S}_n$ with j descents equals the number of permutations with j excedances for all j. (There is a well-known combinatorial proof of this fact

due to Foata [11, 14].) These numbers were first studied by Euler and have come to be known as the Eulerian numbers. They are the coefficients of the *Eulerian polynomials*

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}.$$

Any permutation statistic that is equidistributed with des and exc is said to be an *Eulerian statistic.*

The Eulerian numbers and the Eulerian polynomials have been extensively studied in many different contexts in the mathematics and computer science literature. For excellent treatments of this subject, see the classic lecture notes of Foata and Schutzenberger [15] and Section 5.1 of Knuth's classic book series "The Art of Computer Programming" [21]. The exponential generating function formula,

$$(1.1) \qquad\qquad \sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t}$$

where $A_0(t) = 1$, is attributed to Euler in [21].

The major index of a permutation $\sigma \in \mathfrak{S}_n$ is defined by

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i,$$

and the inversion statistic is defined by

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \leq i < j \leq n \ \& \ \sigma(i) > \sigma(j)\}|.$$

MacMahon [23] showed that the major index is equidistributed with the inversion statistic by establishing the first equality in

$$(1.2) \qquad\qquad \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q! = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)},$$

where

$$[n]_q := 1 + q + \cdots + q^{n-1}$$

and

$$[n]_q! := [n]_q [n-1]_q \cdots [1]_q.$$

Rodrigues [28] had earlier obtained the second equality. (An elegant combinatorial proof of the equidistribution of maj and inv was obtained by Foata [12, 14].) Any permutation statistic that is equidistributed with the major index and inversion index is said to be a *Mahonian statistic.*

Note that by setting $q = 1$ in (1.2), one gets the formula $n!$ for the number of permutations. Equation (1.2) is a beautiful "q-analog" of this formula and is the fundamental example of the subject of permutation statistics and q-analogs, in which one seeks to obtain nice q-analogs of enumeration formulas.

One can look for nice q-analogs of the Eulerian polynomials by considering the joint distributions of the Mahonian and Eulerian statistics given above. Consider the four possibilities,

$$A_n^{\mathrm{inv,des}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{maj,des}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{inv,exc}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

$$A_n^{\mathrm{maj,exc}}(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}.$$

There are many interesting results on the first three q-Eulerian polynomials and on multivariate distributions of all sorts of combinations of Eulerian and Mahonian statistics (for a sample see

[1, 2, 5, 7, 13, 16, 17, 18, 19, 20, 26, 27, 30, 31, 32, 39]). These include Stanley's [32] q-analog of (1.1) given by,

$$\sum_{n \geq 0} A_n^{\mathrm{inv,des}}(q,t) \frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1)) - t}$$

where $A_0^{\mathrm{inv,des}}(q,t) = 1$ and

$$\mathrm{Exp}_q(z) := \sum_{n \geq 0} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

Surprisingly, we have found no mention of the fourth q-Eulerian polynomial $A_n^{\mathrm{maj,exc}}(q,t)$ anywhere in the literature. Here we announce the following remarkable q-analog of (1.1).

Theorem 1.1. *The q-exponential generating function for $A_n^{\mathrm{maj,exc}}(q,t)$ is given by*

(1.3)
$$\sum_{n \geq 0} A_n^{\mathrm{maj,exc}}(q,t) \frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq) - tq\exp_q(z)} \ ,$$

where $A_0^{\mathrm{maj,exc}}(q,t) = 1$ and

$$\exp_q(z) := \sum_{n \geq 0} \frac{z^n}{[n]_q!}.$$

An alternative formulation of (1.3) more closely analogous to (1.1) is given by

(1.4)
$$\sum_{n \geq 0} A_n^{\mathrm{maj,exc}}(q,t) \frac{z^n}{[n]_q!} = \frac{1-tq}{\exp_q(ztq)\mathrm{Exp}_q(-z) - tq} \ .$$

In fact, we prove the following more general result, which reduces to Theorem 1.1 when $r = 1$.

Theorem 1.2. *Let*

$$A_n^{\mathrm{maj.exc,fix}}(q,t,r) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)} r^{\mathrm{fix}(\sigma)},$$

where $\mathrm{fix}(\sigma)$ denotes the number of fixed points of $\sigma \in \mathfrak{S}_n$, i.e., the number of $i \in [n]$ such that $\sigma(i) = i$. Then

(1.5)
$$\sum_{n \geq 0} A_n^{\mathrm{maj.exc,fix}}(q,t,r) \frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(rz)}{\exp_q(ztq) - tq\exp_q(z)}.$$

By setting $t = 1$ in (1.5) one obtains a formula of Gessel and Reutenauer [19], and by setting $q = 1$ one obtains a formula that is equivalent to (1.1). By setting $r = 0$, we obtain the new result,

$$\sum_{n \geq 0} \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)} \frac{z^n}{[n]_q!} = \frac{1-tq}{\exp_q(ztq) - tq\exp_q(z)},$$

where $\mathcal{D}_n$ is the set of derangements in $\mathfrak{S}_n$.

Alternative formulations of (1.5) are given by the recurrence relation

$$A_n^{\mathrm{maj.exc,fix}}(q,t,r) = r^n + \sum_{k=0}^{n-2} \begin{bmatrix} n \\ k \end{bmatrix}_q A_k^{\mathrm{maj.exc,fix}}(q,t,r) \, tq[n-k-1]_{tq},$$

and the formula

$$A_n^{\mathrm{maj.exc,fix}}(q,t,r) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (tq)^m \sum_{\substack{k_0 \geq 0 \\ k_1, \ldots, k_m \geq 2 \\ \sum k_i = n}} \begin{bmatrix} n \\ k_0, \ldots, k_m \end{bmatrix}_q r^{k_0} \prod_{i=1}^{m} [k_i - 1]_{tq},$$

where
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \text{ and } \begin{bmatrix} n \\ k_0, \dots, k_m \end{bmatrix}_q = \frac{[n]_q!}{[k_0]_q![k_1]_q! \cdots [k_m]_q!}.$$

In the next section we describe the techniques that were used to prove Theorem 1.2. They involve an interesting class of symmetric functions and a symmetric function identity (Theorem 2.1), which generalizes Theorem 1.2. We prove the symmetric function identity by modifying a bijection of Gessel and Reutenauer [19] and generalizing a bijection of Stembridge [36]. After learning of our work in [29], Foata and Han extended Theorem 1.2, finding the generating function for the joint distribution of maj, exc, fix and des. Their result can also be obtained by specializing our symmetric function identity.

In Section 3 we discuss a connection with two graded representations of the symmetric group, which turn out to be isomorphic. We show that a specialization of the Frobenius characteristic of these representations yields $A^{\mathrm{maj},\mathrm{exc}}(q,t)$. One of the representations is the representation of the symmetric group on the cohomology of the toric variety associated with the Coxeter complex of the symmetric group. This representation was studied by Procesi [25], Stanley [33], Stembridge [36], [37], and Dolgachev and Lunts [9]. The other representation is the representation of the symmetric group on the homology of maximal intervals of a certain intriguing poset introduced by Björner and Welker [4] in their study of connections between poset topology and commutative algebra. In fact, our study of the latter representation is what led us to discover formula (1.1) and its symmetric function generalization, in the first place.

Various authors have studied Mahonian (resp. Eulerian) partners to Eulerian (resp. Mahonian) statistics whose joint distribution is equal to a known Euler-Mahonian distribution. We mention, for example, Foata [13], Foata and Zeilberger [17], Clarke, Steingrímsson and Zeng [7], Haglund [20], Babson and Steingrímsson [1], and Skandera [30]. In Section 4 we define a new Mahonian statistic to serve as a partner for des in the (maj, exc) distribution. We do not have a simple proof of the equidistribution. We have a highly nontrivial proof which uses tools from poset topology and the symmetric function results discussed in Sections 2 and 3.

Details of the proofs discussed in this extended abstract, as well as further consequences and open problems, will appear in a forthcoming paper.

2. Symmetric function generalization

In this section we present a symmetric function generalization of Theorem 1.2.
Let
$$H(z) = H(\mathbf{x}, z) := \sum_{n \geq 0} h_n(\mathbf{x}) z^n,$$

where $h_n(\mathbf{x})$ denotes the complete homogeneous symmetric function in the indeterminates $\mathbf{x} = (x_1, x_2, \dots)$, that is
$$h_n(\mathbf{x}) := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \dots x_{i_n}$$

for $n \geq 1$, and $h_0 = 1$. By setting $x_i := q^{i-1}$, for all i, and $z := z(1-q)$ in $H(\mathbf{x}, z)$, one obtains $\exp_q(z)$, see [34]. It follows that

$$(2.1) \qquad \left. \frac{(1-t)H(\mathbf{x}, zr)}{H(\mathbf{x}, zt) - tH(\mathbf{x}, z)} \right|_{\substack{x_i := q^{i-1} \\ z := z(1-q)}} = \frac{(1-t)\exp_q(zr)}{\exp_q(zt) - t\exp_q(z)}.$$

We will construct for each $n, j, k \geq 0$, a quasisymmetric function $Q_{n,j,k}(\mathbf{x})$ whose generating function $\sum_{n,j,k \geq 0} Q_{n,j,k}(\mathbf{x}) t^j r^k z^n$ specializes to

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} r^{\mathrm{fix}(\sigma)} \frac{z^n}{[n]_q!}$$

when we set $x_i := q^{i-1}$ and $z := z(1-q)$. Thus by taking specializations of both sides of (2.4) below and setting $t := tq$, we obtain (1.5).

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be the barred word obtained from σ by placing a bar above each excedance. For example, if $\sigma = 531462$ then $\bar{\sigma} = \bar{5}\bar{3}14\bar{6}2$. View $\bar{\sigma}$ as a word over the ordered alphabet

$$\{\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n\}.$$

We extend the definition of descent set from permutations to words w of length n over an ordered alphabet by letting

$$\mathrm{DES}(w) := \{i \in [n-1] : w_i > w_{i+1}\},$$

where w_i is the ith letter of w. Now define the excedance-descent set of a permutation $\sigma \in \mathfrak{S}_n$ to be

$$\mathrm{EXD}(\sigma) := \mathrm{DES}(\bar{\sigma}).$$

For example, $\mathrm{EXD}(531462) = \mathrm{DES}(\bar{5}\bar{3}14\bar{6}2) = \{1, 4\}$. The interesting thing about EXD is that for all $\sigma \in \mathfrak{S}_n$,

$$(2.2) \qquad \sum_{i \in \mathrm{EXD}(\sigma)} i = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma).$$

For $S \subseteq [n-1]$ and $n \geq 1$, define the quasisymmetric function

$$F_{S,n}(x_1, x_2, \dots) := \sum_{\substack{i_1 \geq \cdots \geq i_n \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \dots x_{i_n},$$

and let $F_{\emptyset,0} = 1$.

A basic result in Gessel's theory of quasisymmetric functions (see eg., [34]) is that

$$F_{S,n}(1, q, q^2, \dots) = \frac{q^{\sum_{s \in S} s}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Hence it follows from (2.2) that for all $\sigma \in \mathfrak{S}_n$,

$$F_{\mathrm{EXD}(\sigma),n}(1, q, q^2, \dots) = \frac{q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

For any $n, j, k \geq 0$, let

$$Q_{n,j,k} = Q_{n,j,k}(\mathbf{x}) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ \mathrm{fix}(\sigma) = k}} F_{\mathrm{EXD}(\sigma),n}(\mathbf{x}).$$

By taking the specialization of the generating function we get,

$$(2.3) \qquad \sum_{n,j,k \geq 0} Q_{n,j,k}(\mathbf{x})t^j r^k z^n \Bigg|_{\substack{x_i := q^{i-1} \\ z := z(1-q)}} = \sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} r^{\mathrm{fix}(\sigma)} \frac{z^n}{[n]_q!}.$$

It follows from (2.1) and (2.3) that by setting $x_i := q^{i-1}, z := z(1-q)$ and $t := tq$ in the following result we obtain Theorem 1.2.

Theorem 2.1.

$$(2.4) \qquad \sum_{n,j,k \geq 0} Q_{n,j,k} t^j r^k z^n = \frac{(1-t)H(zr)}{H(zt) - tH(z)}.$$

The proof of this theorem requires an alternative characterization of $Q_{n,j,k}$ which is established by adapting a bijection that Gessel and Reutenauer [19] introduced to enumerate permutations with a fixed descent set and a fixed cycle type. Gessel and Reutenauer deal with circular words over the alphabet of positive integers. We consider circular words over the alphabet of barred and unbarred positive integers. For each such circular word and each starting position, one gets a linear word by reading the circular word in a clockwise direction. If one gets a distinct linear word for each starting position, then the circular word is said to be *primitive*. For example $(\bar{1}, 1, 1)$ is primitive while $(\bar{1}, 2, \bar{1}, 2)$ is not. The *absolute value* of a letter is the letter obtained by erasing the bar if there is one. We will say that a primitive circular word is a *necklace* if each letter that is followed (clockwise) by a letter greater in absolute value is barred and each letter that is followed by a letter smaller in absolute value is unbarred. Letters that are followed by letters equal in absolute value have the option of being barred or not. A circular word consisting of one barred letter is not a necklace. For example the following circular words are necklaces:

$$(\bar{1}, 3, 1, \bar{1}, 2, 2), (\bar{1}, 3, \bar{1}, \bar{1}, 2, 2), (\bar{1}, 3, 1, \bar{1}, \bar{2}, 2), (\bar{1}, 3, \bar{1}, \bar{1}, \bar{2}, 2), (3),$$

while $(\bar{1}, \bar{3}, 1, 1, 2, \bar{2})$ and $(\bar{3})$ are not.

An *ornament* is a multiset of necklaces. The type $\lambda(R)$ of an ornament R is the partition whose parts are the sizes of the necklaces in R. The weight $w(R)$ of an ornament R is the product of the weights of the letters of R, where the weight of the letter a is the indeterminate $x_{|a|}$, where $|a|$ denotes the absolute value of a. For example

$$\lambda((\bar{1}, 2, 2), (\bar{1}, \bar{2}, 3, 3, 2)) = (5, 3)$$

and

$$w((\bar{1}, 2, 2), (\bar{1}, \bar{2}, 3, 3, 2)) = x_1^2 x_2^4 x_3^2.$$

For each partition λ and nonnegative integer j, let $\mathfrak{R}_{\lambda,j}$ be the set of ornaments of type λ with j bars.

Theorem 2.2. *For all $\lambda \vdash n$ and $j = 0, 1, \ldots, n-1$, let*

$$Q_{\lambda,j} := \sum_{\sigma} F_{\mathrm{EXD}(\sigma),n}$$

summed over all permutations of cycle type λ with j excedances. Then

$$Q_{\lambda,j} = \sum_{R \in \mathfrak{R}_{\lambda,j}} w(R).$$

This theorem is proved via a bijection between ornaments of type λ with j bars and permutations of cycle type λ with j excedances paired with "compatible" weakly decreasing sequences of *positive* integers. Our bijection is an adaptation of the Gessel-Reutenauer bijection which sends multisets of primitive circular words over an ordered alphabet to permutations paired with compatible weakly decreasing sequences over the *same* alphabet. The Gessel-Reutenauer bijection, which is also described in [8], can be viewed as a necklace analog of the bijection in Stanley's theory of P-partitions [31]. Here we need to order our alphabet by

$$1 < \bar{1} < 2 < \bar{2} < \ldots.$$

Our map is obtained by first applying the Gessel-Reutenauer map to our ornaments and then removing the bars from the barred letters in the compatible weakly decreasing sequence to obtain a weakly decreasing sequence of positive integers.

Theorem 2.2 has several interesting consequences. For one thing, it can be used to prove that the quasisymmetric functions $Q_{\lambda,j}$ and $Q_{n,j,k}$ are actually symmetric. It also has the following useful consequence.

Corollary 2.3. *For all n, j, k,*

$$Q_{n,j,k} = h_k Q_{n-k,j,0}.$$

It follows from Corollary 2.3 that Theorem 2.1 is equivalent to

$$(2.5) \qquad \sum_{n,j \geq 0} Q_{n,j,0} t^j z^n = \frac{1-t}{H(zt) - tH(z)},$$

which in turn, is equivalent to the recurrence relation

$$(2.6) \qquad Q_{n,j,0} = \sum_{\substack{0 \leq m \leq n-2 \\ j+m-n < i < j}} Q_{m,i,0} h_{n-m}.$$

We establish this recurrence relation by introducing another type of configuration, closely related to ornaments.

Define a *banner* B to be a word over the alphabet of barred and unbarred positive integers, where $B(i)$ is barred if $|B(i)| < |B(i+1)|$ and $B(i)$ is unbarred if $|B(i)| > |B(i+1)|$ or $i = length(B)$. All other letters have the option of being barred. The weight of a banner is the product of the weights of its letters.

A *Lyndon word* over an ordered alphabet is a word that is strictly lexicographically smaller than all its circular rearrangements. A *Lyndon factorization* of a word over an ordered alphabet is a factorization into a weakly lexicographically decreasing sequence of Lyndon words. It is a result of Lyndon (see [24, Theorem 5.1.5]) that every word has a unique Lyndon factorization. The Lyndon type of a word is the partition whose parts are the lengths of the words in its Lyndon factorization. For each partition λ and positive integer j, let $\mathfrak{B}_{\lambda,j}$ be the set of banners with j bars whose Lyndon type is λ.

By turning the Lyndon words in the Lyndon factorization of a banner into circular words, we obtain an ornament. This map from banners to ornaments is the bijection whose existence is asserted in the following proposition.

Proposition 2.4. *For any partition λ and nonnegative integer j, there is a weight-preserving bijection from $\mathfrak{B}_{\lambda,j}$ to $\mathfrak{R}_{\lambda,j}$.*

Corollary 2.5. *Let $\mathfrak{B}_{n,j}$ be the set of banners of length n with j bars whose Lyndon type has no parts of size 1. Then*

$$Q_{n,j,0} = \sum_{B \in \mathfrak{B}_{n,j}} w(B)$$

Define a *marked sequence* (α, i) to be a weakly increasing finite sequence α of positive integers together with an integer i such that $1 \leq i \leq \text{length}(\alpha) - 1$. Let $\mathfrak{M}_n$ be the set of marked sequences of length n and let $\mathfrak{B}_n$ be the set of banners of length n whose Lyndon type has no parts of size 1.

Theorem 2.6. *For all $n \geq 2$, there is a bijection*

$$\gamma : \mathfrak{B}_n \to \bigcup_{0 \leq m \leq n-2} \mathfrak{B}_m \times \mathfrak{M}_{n-m},$$

such that if $\gamma(B) = (B', (\alpha, i))$ then

$$w(B) = w(B') w(\alpha)$$

and

$$\text{bar}(B) = \text{bar}(B') + i,$$

where $\text{bar}(B)$ denotes the number of bars of B.

We will not describe the bijection here except to say that, when restricted to banners with distinct letters, it reduces to a bijection from permutations to marked words that Stembridge [36] constructed to study the representation of the symmetric group on the cohomology of the toric variety assoiciated with the type A Coxeter complex. (We discuss this representation in Section 3.) Banners in $\mathfrak{B}_n$ admit a certain kind of decomposition, called a decreasing decomposition in [8]. The decreasing

decomposition plays the role in our bijection that the cycle decomposition of permutations plays in Stembridge's bijection.

Corollary 2.5 and Theorem 2.6 are all that is needed to establish the recurrence relation (2.6), which yields our main result, Theorem 2.1.

Remark. If one applies the standard involution ω to the symmetric function appearing on the right hand side of (2.4), one gets a refinement of a symmetric function that has been studied by Carlitz, Scoville and Vaughan [6] and Dollhopf, Goulden and Greene [10] in connection with the enumeration of words with no adjacent repeats. It was pointed out to us by Richard Stanley that these words and our banners can be viewed as "dual" graph colorings in the sense of [35, Theorem 4.2].

3. Some Representation Theoretic Consequences

The Frobenius characteristic ch is a fundamental homomorphism from the ring of representations of symmetric groups to the ring of symmetric functions. In this section we present two representations whose Frobenius characteristic is $Q_{n,j} := \sum_{k=0}^{n} Q_{n,j,k}$.

The first representation involves the toric variety associated with the Coxeter complex of a Weyl group. Let X_n be the toric variety associated with the Coxeter complex of $\mathfrak{S}_n$. The action of $\mathfrak{S}_n$ on X_n induces a representation of $\mathfrak{S}_n$ on the cohomology $H^{2j}(X_n)$ for each $j = 0, \ldots, n-1$. (Cohomology in odd degree vanishes.) Stanley [33], using a formula of Processi [25], proves that

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} \mathrm{ch} H^{2j}(X_n)\, t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}.$$

Combining this with Theorem 2.1 yields the following conclusion.

Theorem 3.1. *For all $j = 0, 1, \ldots, n-1$,*

$$\mathrm{ch} H^{2j}(X_n) = Q_{n,j}.$$

For the second representation we consider the homology of the order complex of the Rees product of two simple posets. The Rees product is a poset construction recently introduced by Björner and Welker [4] in their study of relations between poset topology and commutative algebra.

Definition 3.2. *Let P and Q be pure (ranked) posets with respective rank functions r_P and r_Q. The Rees product $P * Q$ of P and Q is defined as follows:*

$$P * Q := \{(p, q) \in P \times Q : r_P(p) \geq r_Q(q)\}$$

with order relation given by $(p_1, q_1) \leq (p_2, q_2)$ if the following holds

- $p_1 \leq_P p_2$
- $q_1 \leq_Q q_2$
- $r_P(p_2) - r_P(p_1) \geq r_Q(q_2) - r_Q(q_1)$

Let B_n be the Boolean algebra (ie., the lattice of subsets of $[n]$ ordered by inclusion) and let C_n be the chain $1 < 2 < \cdots < n$. The maximal elements of $(B_n \setminus \{\emptyset\}) * C_n$ are of the form $([n], j)$, where $j = 1, \ldots, n$. Let $I_{n,j}$ be the set of elements of $(B_n \setminus \{\emptyset\}) * C_n$ that are smaller than $([n], j)$ and let $\tilde{H}_i(I_{n,j})$ be the reduced simplicial (complex) homology of the order complex of $I_{n,j}$. It follows from results of Björner and Welker that homology vanishes below the top dimension $n - 2$. The symmetric group $\mathfrak{S}_n$ acts on $I_{n,j}$ in an obvious way and this induces a representation on $\tilde{H}_{n-2}(I_{n,j})$. We prove the following result using techniques from poset topology.

Theorem 3.3.

$$1 + \sum_{n \geq 1} \sum_{j=1}^{n} \mathrm{ch}(\tilde{H}_{n-2}(I_{n,j}) \otimes \mathrm{sgn})\, t^{j-1} z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)},$$

where sgn *denotes the sign representation. Consequently for all* n, j,

$$\mathrm{ch}(\tilde{H}_{n-2}(I_{n,j}) \otimes \mathrm{sgn}) = Q_{n,j-1}$$

and as $\mathfrak{S}_n$-*modules*

$$\tilde{H}_{n-2}(I_{n,j}) \otimes \mathrm{sgn} \cong H^{2j-2}(X_n).$$

We conjecture that for all λ and j, the symmetric function $Q_{\lambda,j}$ is also the Frobenius characteristic of some representation. One consequence of Theorem 2.2 is that $Q_{\lambda,j}$ can be described as a product of plethysms of symmetric functions of the form $Q_{(n),i}$, where (n) denotes a partition with a single part. Hence if the conjecture holds for all $Q_{(n),i}$ then it holds in general. We use ornaments and banners to show that if the conjecture does hold then the restriction to $\mathfrak{S}_{n-1}$ of the representation whose Frobenius characteristic is $Q_{(n),i}$, has Frobenius characteristic $Q_{n-1,i-1}$.

4. A NEW MAHONIAN STATISTIC

In this section we describe a new Mahonian statistic whose joint distribution with des is the same as the joint distribution of maj and exc.

An *admissible inversion* of $\sigma \in \mathfrak{S}_n$ is a pair $(\sigma(i), \sigma(j))$ such that the following conditions hold:

- $i < j$
- $\sigma(i) > \sigma(j)$
- either
  - $\sigma(j) < \sigma(j+1)$ or
  - $\exists k$ such that $i < k < j$ and $\sigma(k) < \sigma(j)$.

Let $\mathrm{ai}(\sigma) := \#$ admissible inversions of σ. Define the statistic

$$\mathrm{aid}(\sigma) := \mathrm{ai}(\sigma) + \mathrm{des}(\sigma).$$

For example, the admissible inversions of 24153 are $(2, 1), (4, 1)$ and $(4, 3)$. So $\mathrm{aid}(24153) = 3 + 2$.

Theorem 4.1. *For all* $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{aid}(\sigma)} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)}.$$

We do not have a direct proof of this simple identity except when t or q is 1. Our proof relies on Theorem 1.1, a q-analog of Theorem 3.3, and techniques from poset topology. We consider the Rees product $(B_n(q) \setminus \{(0)\}) * C_n$, where $B_n(q)$ is the lattice of subspaces of the vector space $\mathbb{F}_q^n$. Let $I_{n,j}(q)$ be the set of elements in $(B_n(q) \setminus \{(0)\}) * C_n$ that are less than the maximal element $(\mathbb{F}_q^n, j)$. We first use a well-known tool from poset topology, called lexicographic shellability [3, 38], to prove that

$$(4.1) \qquad \dim \tilde{H}_{n-2}(I_{n,j}(q)) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{des}(\sigma) = j - 1}} q^{\mathrm{ai}(\sigma)}.$$

We then use other tools from poset topology to prove a theorem analogous to Theorem 3.3, which states that

$$(4.2) \qquad \sum_{n \geq 0} \sum_{j=1}^{n} \dim \tilde{H}_{n-2}(I_{n,j}(q)) t^{j-1} \frac{z^n}{[n]_q!} = \frac{(1-t) \exp_q(z)}{\exp_q(zt) - t \exp_q(z)}.$$

Theorem 4.1 now follows from Theorem 1.1 and equation (4.1).

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