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Dual graded graphs for Kac-Moody algebras
Extended Abstract

Thomas Lam and Mark Shimozono

Abstract. Motivated by affine Schubert calculus, we construct a family of dual graded graphs (Γ_s, Γ_w) for an arbitrary Kac-Moody algebra $\mathfrak{g}$. The graded graphs have the Weyl group W of $\mathfrak{g}$ as vertex set and are labeled versions of the strong and weak orders of W respectively. Using a construction of Lusztig for quivers with an admissible automorphism, we define folded insertion for a Kac-Moody algebra and obtain Sagan-Worley shifted insertion from Robinson-Schensted insertion as a special case. Drawing on work of Proctor and Stembridge, we analyze the induced subgraphs of (Γ_s, Γ_w) which are distributive posets.

1. Introduction

The Robinson-Schensted algorithm is possibly the most important algorithm in algebraic combinatorics. It exhibits a bijection between permutations and pairs of standard Young tableaux of the same shape. Stanley investigated the class of differential posets [**19**] (also studied in [**2**]) and Fomin studied the more general notion of a dual graded graph [**3**] to formalize local conditions which would be sufficient, in more general situations, to imply the existence of a Robinson-Schensted style algorithm, sometimes just called an insertion algorithm.

In this article, we construct a family of dual graded graphs (Γ_s, Γ_w) associated to each Kac-Moody algebra $\mathfrak{g}$. These graded graphs have as vertex set the Weyl group W of $\mathfrak{g}$. The pair (Γ_s, Γ_w) depends on additional parameters, but in every case Γ_s and Γ_w are respectively obtained as edge-labelings of the Hasse diagrams of the strong (Bruhat) order and weak order on W. These labelings were strongly motivated by the geometry of Kac-Moody flag manifolds G/P where G is a Kac-Moody group and P is a parabolic subgroup. In general the edge labels of the strong graph Γ_s are nonnegative integer linear combinations of certain Schubert structure constants for $H^*(G/B)$ known as Chevalley coefficients.

In the case of the affine Grassmannian Gr (which takes the form G/P where G is an affine Kac-Moody group and P is a certain maximal parabolic), the dual graded graph structure arises from the pair of dual graded Hopf algebras given by $H_*(\mathrm{Gr})$ and $H^*(\mathrm{Gr})$. The down operator is defined by the action of the Schubert homology class ξ_{s_0} indexed by the zero-th simple reflection s_0 in the affine Weyl group, on the Schubert basis of $H^*(\mathrm{Gr})$. The up operator is defined by multiplication by the cohomology Schubert class ξ^{s_i} for a fixed simple reflection $s_i \in W$. It is a general phenomenon that pairs of dual graded combinatorial Hopf algebras yield dual graded graphs; we shall pursue this in a separate publication [**12**].

For each pair of Kac-Moody dual graded graphs (Γ_s, Γ_w) we define standard weak and strong tableaux. Via Fomin's theory we obtain an enumerative identity that gives the number of colored permutations as a sum over pairs of tableaux of the same shape, one strong and one weak. For $\mathfrak{g}$ of affine type $A_{n-1}^{(1)}$ we recover the dual graded graphs that were implicitly studied in our joint work with Lapointe and Morse [**10**]. The

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weak and strong tableaux in [**10**] are nontrivial semistandard generalizations of the corresponding objects here. In [**10**], an affine insertion algorithm was explicitly constructed for semistandard weak and strong tableaux, and a number of applications were given to the geometry of the affine Grassmannian Gr and to the theory of symmetric functions. In the limit $n \to \infty$ of the $A_{n-1}^{(1)}$ case, our construction reproduces Young's lattice, which is the self-dual graded graph that gives rise to the Robinson-Schensted algorithm.

Using a construction of Lusztig for quivers with an admissible automorphism, given any symmetrizable Kac-Moody algebra $\mathfrak{g}(A)$, there is a symmetric Kac-Moody algebra $\mathfrak{g}(B)$ and an embedding $\mathfrak{g}(A) \to \mathfrak{g}(B)$. Our combinatorics is compatible with these embeddings: any pair of our dual graded graphs (Γ_s^A, Γ_w^A) for $\mathfrak{g}(A)$, can be realized using a similar pair (Γ_s^A, Γ_w^B) for $\mathfrak{g}(B)$. We define folded insertion, which realizes a Schensted bijection for (Γ_s^A, Γ_w^A), in terms of a Schensted algorithm for the related pair (Γ_s^B, Γ_w^B). Taking A and B to be of affine types $C_n^{(1)}$ and $A_{2n-1}^{(1)}$ respectively, in the $n \to \infty$ limit we obtain Sagan-Worley shifted insertion [**17, 24**] from Robinson-Schensted insertion as a special case of folded insertion.

Our second aim is to investigate induced subgraphs of the graphs (Γ_s, Γ_w) which are distributive lattices, when considered as posets. These are precisely the conditions under which one may describe our strong and weak tableaux by "filling cells with numbers" as in a usual standard Young tableau. Here we draw on work of Proctor [**15**] and Stembridge [**21**] classifying the parabolic quotients of Coxeter groups which have weak or strong orders that are distributive lattices. We sharpen these results slightly to show that in these cases, the distributivity is compatible (see Section 3.2) with the edge-labeling of our graphs (Γ_s, Γ_w). These distributive parabolic quotients have also appeared recently in the geometric work of Thomas and Yong [**23**], who show that the *jeu-de-taquin* can be used to calculate structure constants of the cohomology of (co)minuscule flag varieties. We do not recover this result, but observe that their notion of standard tableau fits nicely into our general framework: they are given by either our strong or weak tableaux with edge labels forgotten.

Among the topics that are omitted here but are present in the full article [**11**], are: Lusztig's embedding and folded insertion, a new family of Schensted algorithms coming from affine type C that includes Sagan-Worley insertion, analogues of the mixed and left-right insertion of Haiman [**6**], a simplified description of the standard case of the affine Robinson-Schensted-Knuth correspondence of [**10**] in terms of cores, and a new Chevalley formula for the homology of all affine Grassmannians.

2. Dual graded graphs for Kac-Moody algebras

2.1. Fomin's dual graded graphs. We recall Fomin's notion of dual graded graphs [**3**]. A graded graph is a directed graph $\Gamma = (V, E, h, m)$ with vertex set V and set of directed edges $E \subset V^2$, together with a grading function $h : V \to \mathbb{Z}_{\geq 0}$, such that every directed edge $(v, w) \in E$ satisfies $h(w) = h(v) + 1$ and has a label $m(v, w) \in \mathbb{Z}_{\geq 0}$. Sometimes we will use poset-theoretic language when discussing a graded graph Γ by treating Γ as the Hasse diagram (graph of cover relations) of a poset P_Γ.

Γ is locally-finite if, for every $v \in V$, there are finitely many $w \in V$ such that $(v, w) \in E$ and finitely many $u \in V$ such that $(u, v) \in E$; we shall assume this condition without further mention. For a graded graph $\Gamma = (V, E, h, m)$ define the $\mathbb{Z}$-linear operators $D, U : \mathbb{Z}V \to \mathbb{Z}V$ on the free abelian group $\mathbb{Z}V$ of formal $\mathbb{Z}$-linear combinations of vertices, by

$$U_\Gamma(v) = \sum_{(v,w)\in E} m(v, w) w \qquad D_\Gamma(w) = \sum_{(v,w)\in E} m(v, w) v$$

A pair of graded graphs (Γ, Γ') is dual if they have the same vertex sets and grading function (but possibly different edge sets and edge labels) such that

$$D_{\Gamma'} U_\Gamma - U_\Gamma D_{\Gamma'} = rI \tag{2.1}$$

for some fixed $r \in \mathbb{Z}_{>0}$. We call r the *differential coefficient*. When $\Gamma = \Gamma'$ and all the edges have multiplicity one, we obtain the r-differential posets of [**19**]. The prototypical example of dual graded graphs is (Γ, Γ) where Γ is Young's lattice of partitions, with edges (μ, λ) if the Ferrers diagram of λ is obtained by adding a cell to that of μ, and every edge label is 1.

Our first aim is to exhibit a new family of dual graded graphs, which depend on a Kac-Moody algebra $\mathfrak{g}$, an element K which lies in the center of $\mathfrak{g}$, and a weight $\Lambda \in \mathfrak{h}^*$ where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra.

2.2. The labeled Kac-Moody weak and strong orders. Let $\mathfrak{g} = \mathfrak{g}(A)$ denote the Kac-Moody algebra over $\mathbb{C}$ associated to the generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ where I is the Dynkin node set and $a_{ij} = \langle \alpha_j\,,\,\alpha_i^\vee\rangle$ [**8**], where $\langle\cdot\,,\,\cdot\rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ is the natural pairing between the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and its dual $\mathfrak{h}^*$. Let $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, and $\Lambda_i \in \mathfrak{h}^*$ be the simple roots, simple coroots, and fundamental weights, for $i \in I$. We assume that the simple roots are linearly independent and the dimension of $\mathfrak{h}$ is chosen to be minimal. Let W be the Weyl group of $\mathfrak{g}$ with generators s_i for $i \in I$, and let Δ_{re}^+ be the set of positive real roots, with reflection $s_\gamma \in W$ and coroot $\gamma^\vee$ for $\gamma \in \Delta_{re}^+$. The *(left) weak order* $\prec$ of W is generated by the cover relations $w \prec s_j w$ whenever $\ell(w) < \ell(s_j w)$ and $j \in I$, $w \in W$. The *strong order* (or Bruhat order) $<$ of W is generated by the cover relations $w \lessdot ws_\gamma$ whenever $\ell(ws_\gamma) = \ell(w) + 1$ and $\gamma \in \Delta_{re}^+$, $w \in W$. For any $v \in W$ define its left descent set by $\mathrm{Des}(v) = \{j \in I \mid s_j v < v\}$.

Given Λ in the set P^+ of dominant integral weights, let $\Gamma_s(\Lambda)$ be the graded graph whose vertex set is W and whose edges are given by $(v, w) \in W^2$ where $v \lessdot w$. Letting $\gamma \in \Delta_{re}^+$ be such that $w = vs_\gamma$, the edge (v, w) is labeled by $m_\Lambda(v, w) = \langle \Lambda\,,\,\gamma^\vee\rangle$.

Let Z^+ denote the set of elements $K \in \mathfrak{h}$ that lie in the center of $\mathfrak{g}$, which have the form $K = \sum_{j\in I} k_j \alpha_j^\vee$ for $k_j \in \mathbb{Z}_{\geq 0}$. The condition that K is central implies that the vector $(k_j)_{j\in I}$ defines a linear dependence amongst the rows of A.

Given $K \in Z^+$, let $\Gamma_w(K)$ be the graded graph whose vertex set is W, and whose edges are given by weak covering relations $(v, w) \in W^2$ with $v \prec w = s_j v$ for some $j \in I$. The edge (v, w) is labeled by $n_K(v, w) = k_j = \langle \Lambda_j\,,\,K\rangle$.

Theorem 2.1. *Let $(\Lambda, K) \in P^+ \times Z^+$. Then $(\Gamma_s(\Lambda), \Gamma_w(K))$ is a pair of dual graded graphs with differential coefficient $r = \langle \Lambda\,,\,K\rangle$.*

Proof. Let $U = U_{\Gamma_s(\Lambda)}$ and $D = D_{\Gamma_w(K)}$. First consider the coefficient of $u \neq v$ in $(DU - UD)v$. It is given by

$$\sum_{\substack{(j,\gamma)\in I\times\Delta_{re}^+\\ v\lessdot vs_\gamma\\ u=s_jvs_\gamma<vs_\gamma}} k_j\langle\Lambda\,,\,\gamma^\vee\rangle - \sum_{\substack{(j,\gamma)\in I\times\Delta_{re}^+\\ s_jv<v\\ s_jv\lessdot s_jvs_\gamma=u}} k_j\langle\Lambda\,,\,\gamma^\vee\rangle.$$

Using two versions of [**7**, Lemma 5.11] we see that the two indexing sets for the sums are equal. Thus the coefficient of $u \neq v$ in $(DU - UD)v$ is zero.

Now let us calculate the coefficient of v in $(DU - UD)v$. Since $s_j v$ is either covered by v or covers v in the left weak Bruhat order, the coefficient of v in $(DU - UD)v$ is

$$\begin{aligned}
&\sum_{j\in I\setminus\mathrm{Des}(v)} k_j\langle\Lambda\,,\,v^{-1}\alpha_j^\vee\rangle - \sum_{j\in\mathrm{Des}(v)} k_j\langle\Lambda\,,\,(s_jv)^{-1}\alpha_j^\vee\rangle\\
&= \sum_{j\in I} k_j\langle\Lambda\,,\,v^{-1}\alpha_j^\vee\rangle = \sum_{j\in I} k_j\langle v\Lambda\,,\,\alpha_j^\vee\rangle\\
&= \langle v\Lambda\,,\,K\rangle = \langle\Lambda\,,\,v^{-1}(K)\rangle = \langle\Lambda\,,\,K\rangle.
\end{aligned}$$

We have used the W-invariance of $\langle\cdot\,,\,\cdot\rangle$ and K, plus the fact that if $v < s_j v$ then $s_j v = vs_\gamma$ where $\gamma = v^{-1}\alpha_j \in \Delta_{re}^+$. □

Remark 2.2. If $\Lambda = \Lambda_j$ the graph $\Gamma_s(\Lambda_j)$ is a labeling of the strong Bruhat order on W by certain Chevalley coefficients. Namely, the edge $(v, vs_\alpha = w)$ is labeled with the coefficient of ξ^w in $\xi^{s_j}\xi^v$, where $\xi^w \in H^*(G/B)$ is the Schubert cohomology class for the affine flag manifold G/B, with G the Kac-Moody group with associated Lie algebra $\mathfrak{g}$ and B a Borel subgroup [**9**].

2.3. Paths and tableaux. Suppose Γ is a graded graph and $v, w \in \Gamma$ are two vertices. A *tableau* T of shape v/w is a directed path

$$T = (v = v_0 \xrightarrow{m_1} v_1 \xrightarrow{m_2} \cdots \xrightarrow{m_k} v_k = w)$$

such that each directed edge $v_{i-1} \to v_i$ has been marked with an integer $1 \leq m_i \leq m(v_{i-1}, v_i)$. Alternatively, one thinks of m_i as indicating which of the $m(v_{i-1}, v_i)$ edges from v_{i-1} to v_i the path transverses. In the case that Γ is Young's lattice, these tableaux are the usual standard Young tableaux.

If Γ has a unique minimum element $\hat{0}$, we say T has shape v if it has shape $v/\hat{0}$. This is the case for our graphs $\Gamma_s(\Lambda)$ and $\Gamma_w(K)$. We call the tableaux in $\Gamma_s(\Lambda)$ and $\Gamma_w(K)$ *(standard) Λ-strong tableaux* and *(standard) K-weak tableaux* respectively.

Given a pair of dual graded graphs (Γ, Γ') one may deduce a number of combinatorial identities [**3**] concerning the number of tableaux. In the case that $\Gamma = \Gamma'$ is Young's lattice, one obtains the well-known identity $n! = \sum_\lambda f_\lambda^2$ where λ ranges over the partitions of n and f_λ is the number of standard Young tableaux of shape λ. So Λ-strong and K-weak tableaux are both analogues of standard Young tableaux.

Applying Fomin's general theory [**3**] we obtain the following result.

COROLLARY 2.3. *Suppose $\mathfrak{g}$ is a Kac-Moody algebra and $(\Lambda, K) \in P^+ \times Z^+$. Then for each $n \in \mathbb{Z}_{\geq 0}$ we have*

$$\sum_{\substack{w \in W \\ \ell(w)=n}} f^w_{\text{strong}} f^w_{\text{weak}} = r^n\, n! \tag{2.2}$$

where $r = \langle \Lambda\,,\, K\rangle$, f^w_{strong} is the number of Λ-strong tableaux of shape w and f^w_{weak} is the number of K-weak tableaux of shape w.

2.4. From commutation relations to combinatorial algorithms. Dual graded graphs are closely related to insertion algorithms. If a local bijection is chosen which exhibits the equation (2.1) at each vertex v, then there is an induced bijection which proves the combinatorial identity (2.2) [**4**]. Two particular choices of local bijections for (Γ, Γ) give the row and column insertion Robinson-Schensted bijections when Γ is Young's lattice [**18**].

In the case of the dual graded graphs $(\Gamma_s(\Lambda), \Gamma_w(K))$ there is a natural bijection which proves that $w \neq v$ does not occur in $(DU - UD)v$, for $v, w \in W$. This natural bijection is essentially obtained from [**7**, Lemma 5.11], just as in the proof of Theorem 2.1. The marked (down-up) path $v \xrightarrow{m} s_j v \xrightarrow{n} s_j v s_\alpha = w$ is bijected to the (up-down) path $v \xrightarrow{n} v s_\alpha \xrightarrow{m} s_j v s_\alpha = w$. Here $1 \leq m \leq \langle \Lambda_j\,,\, K\rangle$ and $1 \leq n \leq \langle \Lambda\,,\, \alpha^\vee\rangle$ indicates the markings on the edges.

Currently we are not aware of a natural bijection which exhibits the coefficient of v in $(DU - UD)v$ as $\langle \Lambda\,,\, K\rangle$. We shall give a special case where such a bijection has been constructed explicitly.

2.5. The affine case. If the generalized Cartan matrix A is of finite type, then $Z^+ = 0$ and all edge labels in $\Gamma_w(K)$ are zero. If A is of affine type, it is known that Z^+ is the set of nonnegative integer multiples of the canonical central element $K = K_{\text{can}} = \sum_{j \in I} a_j^\vee \alpha_j^\vee$. The vector $(a_j^\vee)_{j \in I}$ is the unique linear dependence of the rows of A given by positive, relatively prime integers [**8**]. In this case, we lose little generality by only considering K_{can}, so we define $\Gamma_w := \Gamma_w(K_{\text{can}})$. We also define $\Gamma_s^{(i)} = \Gamma_s(\Lambda_i)$ for convenience.

REMARK 2.4. If $\mathfrak{g}$ is an untwisted affine algebra, the edge multiplicities of the weak graph Γ_w are related to the multiplication in the *homology* $H_*(\mathrm{Gr})$ of the affine Grassmannian $\mathrm{Gr} = \mathrm{Gr}_G$ of the simple Lie group G whose Lie algebra is the canonical simple Lie subalgebra of the affine algebra $\mathfrak{g}$. The multiplication in $H_*(\mathrm{Gr})$ comes from the identification $H_*(\mathrm{Gr}_G) \cong H_*(\Omega K)$ [**5**] [**16**] where ΩK is the topological group given by continuous maps from the circle S^1 into the maximal compact form K of G that send a basepoint of the circle to $1 \in K$. The group structure on ΩK induces a multiplication on $H_*(\Omega K)$; see [**1**] [**14**].

2.6. Affine type A and LLMS insertion. For this subsection we assume that $\mathfrak{g}$ is the affine Lie algebra of type $A_{n-1}^{(1)}$. In this case the combinatorics of the pair of dual graded graphs $(\Gamma_s^{(i)}, \Gamma_w)$ was studied carefully in [**10**] and was one of the main motivations of the current work. One may interpret the affine insertion algorithm of [**10**] (which we shall call LLMS insertion) as furnishing an explicit bijection proving the duality of $(\Gamma_s^{(i)}, \Gamma_w)$. Furthermore the LLMS insertion involves nontrivial extensions of the notion of tableaux to semistandard weak and strong tableaux, and proves Pieri rules in the homology $H_*(\mathrm{Gr})$ and cohomology $H^*(\mathrm{Gr})$ of the affine Grassmannian of $SL(n, \mathbb{C})$ [**10**].

For $\mathfrak{g} = A_{n-1}^{(1)}$ we have $a_j^\vee = 1$ for all $j \in I$, so the weak graph Γ_w is multiplicity free. By the symmetry of the affine Dynkin diagram, we may assume that $\Lambda = \Lambda_0$ and for brevity we write Γ_s for $\Gamma_s^{(0)}$. To describe Γ_s explicitly, we recall that the affine symmetric group $W = \tilde{S}_n$ has the concrete description as the group of bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying (a) $f(i + n) = f(i)$ for each $i \in \mathbb{Z}$; and (b) $\sum_{i=1}^n f(i) = 1 + 2 + \cdots + n$, with function composition being group multiplication. The reflections t_{ij} in $\tilde{S}_n$ are indexed by a pair of integers

(i,j) satisfying $i<j$ and $i \neq j \mod n$. Suppose $v \lessdot vt_{ij} = w$ is a cover in $\tilde{S}_n$. Then the edge (v,w) in Γ_s has multiplicity equal to $\#\{k \in \mathbb{Z} \mid v(i) \leq k < v(j) \text{ and } k = 0 \mod n\}$.

Let $\tilde{S}_n^0 \subset \tilde{S}_n$ denote the set of minimal length representatives in the cosets of $\tilde{S}_n/S_n$. These are the *affine Grassmannian* elements. The restrictions of Γ_s and Γ_w to $\tilde{S}_n^0 \subset W$ yield a pair (Γ_s^0, Γ_w^0) of dual graded graphs. We will now explain LLMS insertion for $\tilde{S}_n^0$ and standard strong and weak tableaux, in the language of cores. An n-ribbon is a connected skew shape λ/μ containing no 2×2 square. An n-core is a partition λ such that no n-ribbon can be removed to obtain another partition. We let $\mathcal{C}_n$ denote the set of n-cores.

Proposition 2.1 ([**10, 13**]). *There exists a unique bijection $c : \tilde{S}_n^0 \to \mathcal{C}_n$ such that if $w = s_k v$ with $k \in I = \{0,1,\ldots,n-1\}$ and $v,w \in \tilde{S}_n^0$ with $\ell(w) = \ell(v)+1$, then $c(w)$ is obtained from $c(v)$ by adjoining every addable corner cell (i,j) such that $j-i = k \mod n$. Moreover, for $v,w \in \tilde{S}_n^0$, we have $v < w$ if and only if $c(v) \subset c(w)$, and if $v \lessdot w$ then $c(w)/c(v)$ is a disjoint union of translates of some ribbon R, and the number of components of $c(w)/c(v)$ is equal to the multiplicity $m(v,w)$ in Γ_s.*

We say that $\mu \in \mathcal{C}_n$ covers $\lambda \in \mathcal{C}_n$ if $c^{-1}(\mu) \gtrdot c^{-1}(\lambda)$. Thus a standard strong tableau in Γ_s is a sequence $\lambda = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^l = \mu$ such that λ^i covers λ^{i-1} and one of the components of λ^i/λ^{i-1} has been marked. In the case that $v \lessdot s_j v = w$, the ribbon R of Proposition 2.1 is always a single box. In this case we say that $c(v) \subset c(w)$ is a weak cover.

As we remarked in Section 2.4, there is a natural bijection which shows that $DU - UD$ is diagonal for (Γ_s, Γ_w). To show that $(DU-UD)v = v$ for $v \in \tilde{S}_n^0$, the following map was used in [**10**]. We say a square (i,j) has *residue* k if $j - i = k \mod n$. Let $\lambda = c(v)$ be the n-core corresponding to v. If $\lambda \subset \mu$ is a weak cover then by Proposition 2.1, μ/λ consists of all the outer corners of λ which have a fixed residue. As μ varies over all the weak covers of λ we obtain all the outer corners in this way. Thus choosing a weak cover μ of λ and choosing a component of μ/λ is equivalent to choosing an outer corner (or addable corner) of λ. Similarly, components of cores ν weak covered by λ correspond to inner corners (or removable corners) of λ. These components are exactly the (two-step) paths involved in calculating the coefficient of v in $(DU-UD)v$. The injection establishing $(DU-UD)v = v$ is the one used in the row insertion Schensted algorithm. It sends an inner corner of λ to the outer corner of λ in the next row. The unique outer corner on the first row of λ is distinguished and not in the image of this injection.

3. Distributive parabolic quotients

3.1. Classification. Let W be a finite irreducible Weyl group with simple generators $\{s_i \mid i \in I\}$, set of reflections T, and length function $\ell : W \to \mathbb{Z}$. For $J \subset I$ we let W_J denote the parabolic subgroup generated by $\{s_i \mid i \in J\}$ and W^J denote the set of minimal length coset representatives of W/W_J. We have the following characterization of W^J:

$$W^J = \{w \in W \mid w < ws_i \text{ for any } i \in J\}.$$

The weak and strong orders of W restrict to W^J. Proctor [**15**] classified the cases when W^J is a distributive lattice under the weak order. Stembridge [**21**] showed that these are the same cases that W^J is a distributive lattice under the strong order. In all such cases, the weak and strong orders agree on W^J and W_J is a maximal parabolic subgroup of W, that is $J = I \setminus \{i\}$ for some $i \in I$. We call such W^J *distributive parabolic quotients*.

Theorem 3.1 ([**15**]). *The distributive parabolic quotients are:*

(1) $W \simeq A_n$; $J = I \setminus \{i\}$ *for any* $i \in I$.
(2) $W \simeq B_n$; $W_J \simeq B_{n-1}$ *or* $W_J \simeq A_{n-1}$.
(3) $W \simeq D_n$; $W_J \simeq D_{n-1}$ *or* $W_J \simeq A_{n-1}$.
(4) $W \simeq G_2$; $J = I \setminus \{i\}$ *for any* $i \in I$.

In [**21**], it is shown that these cases are also exactly the parabolic quotients W^J of Weyl groups such that every element $w \in W^J$ is fully commutative, that is, every two reduced decompositions of w can be obtained from each other using just the relations of the form $s_i s_j = s_j s_i$ for $i,j \in I$.

3.2. Distributive labeled posets. We need a slightly more precise form of the results of Proctor and Stembridge. If Q is a finite poset we let $J(Q)$ denote the poset of (lower) order ideals of Q. The poset $J(Q)$ is a distributive lattice and the fundamental theorem of finite distributive posets [**20**] says that the correspondence $Q \mapsto J(Q)$ is a bijection between finite posets and finite distributive lattices. Suppose P is a finite poset and $\omega : \{x \lessdot y\} \to A$ is a labeling of the edges of the Hasse diagram of P with elements of some set A. We call (P, ω) an edge-labeled poset. We say that (P, ω) is a *distributively labeled lattice* if

(1) $P = J(Q)$ is a distributive lattice; and
(2) there is a vertex (element) labeling $\pi : Q \to A$ such that
$$\omega(I \setminus \{q\} \lessdot I) = \pi(q)$$
for any $I \in J(Q)$ and q maximal in I.

If W is a Weyl group, we may label the edges of the Hasse diagram of the weak order $(W, \prec)$ with simple reflections: the cover $w \prec s_i w$ is labeled with s_i. We denote the resulting edge-labeled poset by W_{weak}. Similarly define W_{strong} to be the strong order where $w \lessdot wt$ is labeled with $t \in T$. These labeled posets restrict to give labeled posets W^J_{weak} and W^J_{strong}. Note that each cover relation in W^J under either order is itself a cover relation in W. Thus W^J_{weak} and W^J_{strong} are induced subgraphs of W_{weak} and W_{strong}.

THEOREM 3.2. *Suppose W^J is a distributive parabolic quotient. Then W^J_{weak} and W^J_{strong} are distributively labeled posets.*

The statement concerning W^J_{weak} is immediate from Stembridge's work [**21**, Theorem 2.2]. We give a self-contained proof of Theorem 3.2 which recovers Stembridge's result that the weak and strong orders agree on such W^J.

3.3. Cominuscule parabolic quotients. Let Φ be an irreducible finite root system and W be its Weyl group. We let $\Delta = \{\alpha_i \mid i \in I\}$ denote a system of simple roots and $\Phi = \Phi^+ \cup \Phi^-$ denote the decomposition of the roots into the disjoint subsets of positive and negative roots. Let $\theta = \sum_{i \in I} a_i \alpha_i$ denote the highest root of Φ. We say that $i \in I$ is *cominuscule* if $a_i = 1$. Apart from the case $\Phi = G_2$, each distributive parabolic quotient corresponds to some W^J where $J = I \setminus \{i\}$ for a cominuscule node $i \in I$. This can be checked case-by-case using Theorem 3.1, or alternatively compared with the dual (minuscule) statement [**21**, Theorem 6.1]. There are two cases of distributive parabolic quotients whose removed node i is not cominuscule, namely, B_n/A_{n-1} and C_n/C_{n-1}; these are precisely the minuscule-but-not-cominuscule cases in [**23**]. However in these cases one may use the isomorphic quotients given by their duals C_n/A_{n-1} and B_n/B_{n-1}, which are cominuscule.

For now we suppose that a cominuscule node $i \in I$ has been fixed and let $J = I \setminus \{i\}$. If α and β are two roots, we say $\alpha \geq \beta$ if $\alpha - \beta$ is a sum of positive roots. Recall that θ is the unique maximal root under this order. Let $\Phi^{(i)}$ denote the poset of positive roots which lie above α_i. Clearly $\theta \in \Phi^{(i)}$. The inversion set of $w \in W$ is defined by
$$\mathrm{Inv}(w) = \{\alpha \in \Phi^+ \mid w\alpha < 0\}.$$

LEMMA 3.3. *Suppose $\alpha, \beta \in \Phi^{(i)}$. Write $s_\alpha \beta = \beta + k\alpha$ where $k = -\langle \alpha^\vee , \beta \rangle \in \mathbb{Z}$. Then $k \in \{0, -1, -2\}$ and*

(1) *If α and β are incomparable then $s_\alpha \beta = \beta$.*
(2) *If $\alpha > \beta$ then $s_\alpha \beta$ is equal to one of the following: (i) β; (ii) $-\gamma$ where $\gamma \in \Phi^+ \setminus \Phi^{(i)}$; or (iii) $-\gamma$ where $\gamma > \alpha$.*

PROOF. To obtain the bounds on k we observe that for all roots $\gamma \in \Phi$, $-\theta \leq \gamma \leq \theta$, so that the coefficient of α_i in γ, lies between the corresponding coefficients in $-\theta$ and θ, which are -1 and 1 by the assumption that i is cominuscule.

Suppose that α and β are incomparable. Then $\beta - \alpha$ is neither positive nor negative and hence not a root. Since the roots in Φ occur in strings, we must have $k = 0$.

If $\alpha > \beta$, the three cases correspond to $k = 0$, $k = -1$, and $k = -2$. □

For our results on distributive parabolic quotients, we require the following result, which is a slight strengthening of [**23**, Prop. 2.1, Lemma 2.2]. We include a self-contained proof, part of which is the same as the proof of [**23**, Prop. 2.1].

PROPOSITION 3.1. *The map $w \longmapsto \mathrm{Inv}(w)$ defines an isomorphism of posets $\mathrm{Inv}|_{W^J} : (W^J, \leq) \to J(\Phi^{(i)})$. Moreover, if $u \lessdot w$ for $u, w \in W^J$, then writing $w = us_\alpha$ for $\alpha \in \Phi^+$, we have $\alpha \in \Phi^{(i)}$ and $\mathrm{Inv}(w) = \mathrm{Inv}(u) \cup \{\alpha\}$.*

Proof. Let $w \in W^J$. First we show that $\mathrm{Inv}(w) \subset \Phi^{(i)}$. Suppose that $\gamma \in \mathrm{Inv}(w) \setminus \Phi^{(i)}$. If $\gamma = \alpha_k$ where $k \neq i$ this means $ws_k < w$ which contradicts the assumption that $w \in W^J$. Otherwise $\gamma = \delta + \rho$ where $\delta, \rho \in \Phi^+ \setminus \Phi^{(i)}$. Since $w\gamma < 0$ we have $w\delta < 0$ or $w\rho < 0$ so the same argument applies. Repeating we obtain a contradiction.

Now we show that $\mathrm{Inv}(w) \in J(\Phi^{(i)})$. Suppose $\alpha \in \mathrm{Inv}(w)$ and $\beta < \alpha$. Then $\gamma = \alpha - \beta \in \Phi^+ \setminus \Phi^{(i)}$ since the coefficient of α_i in γ is zero. Since $\mathrm{Inv}(w) \subset \Phi^{(i)}$, we have $\gamma \notin \mathrm{Inv}(w)$, that is, $w\alpha - w\beta = w\gamma > 0$. Since $w\alpha < 0$ this shows that $w\beta < 0$ as desired. Thus $\mathrm{Inv}|_{W^J}$ is well-defined.

Next we show that $\mathrm{Inv}|_{W^J}$ sends covers to covers. Let $u \lessdot w$ with $u, w \in W^J$ and $\alpha \in \Phi^+$ such that $w = us_\alpha$. Then $0 > w\alpha = -u\alpha$ so $\alpha \in \mathrm{Inv}(w) \setminus \mathrm{Inv}(u)$. For all $\beta \in \mathrm{Inv}(u)$, since $\mathrm{Inv}(u) \in J(\Phi^{(i)})$, $\alpha > \beta$ or $\alpha > \beta$. Either way we have $w\beta = us_\alpha\beta < 0$, since by Lemma 3.3, $s_\alpha\beta$ is either equal to β or $-\gamma$ for $\gamma \in \Phi^+ \setminus \mathrm{Inv}(u)$. That is, $\mathrm{Inv}(u) \subset \mathrm{Inv}(w)$. Since $|\mathrm{Inv}(w)| = |\mathrm{Inv}(u)| + 1$ it follows that $\mathrm{Inv}(w) = \mathrm{Inv}(u) \cup \{\alpha\}$, so that $\mathrm{Inv}(u) \subset \mathrm{Inv}(w)$ is a covering relation in $J(\Phi^{(i)})$.

Next we show that every covering relation in $J(\Phi^{(i)})$ is the image of a covering relation in W^J, and in particular, that $\mathrm{Inv}|_{W^J}$ is onto. An arbitrary covering relation in $J(\Phi^{(i)})$ is given by $S \setminus \{\alpha\} \subset S$ where $S \in J(\Phi^{(i)})$ and α is maximal in S.

By induction there is a $u \in W^J$ such that $\mathrm{Inv}(u) = S \setminus \{\alpha\}$. Let $w = us_\alpha$. It suffices to show that

$$\mathrm{Inv}(w) = S \quad\text{and}\quad w \in W^J.$$

The second claim follows from the first since none of the α_k for $k \neq i$ lie in $\mathrm{Inv}(w)$. For the first claim, since $\alpha \in \Phi^{(i)} \setminus \mathrm{Inv}(u)$, we may argue as before to show that $S = \mathrm{Inv}(u) \cup \{\alpha\} \subset \mathrm{Inv}(w)$.

For the opposite inclusion, suppose $\beta \in \Phi^+ \setminus S$. We must show that $w\beta > 0$. Write $s_\alpha\beta = \beta + k\alpha$ for $k \in \mathbb{Z}$. If $k = 0$ then we are done as before. If $k > 0$ then $s_\alpha\beta > \alpha$, so that $s_\alpha\beta \in \Phi^+ \setminus S$ since S is an order ideal. But then $s_\alpha\beta \notin \mathrm{Inv}(u)$ so $w\beta > 0$. So we may assume that $k < 0$.

Suppose first that $\beta \in \Phi^{(i)}$. We may assume that α and β are comparable by Lemma 3.3. Since S is an order ideal we have $\beta > \alpha$. If $k = -1$ then $s_\alpha\beta = \beta - \alpha \in \Phi^+ \setminus \Phi^{(i)}$ since the coefficient of α_i is 1 in both α and β. In particular $s_\alpha\beta \notin \mathrm{Inv}(u)$ so $w\beta > 0$. Otherwise $k = -2$. Then $s_\alpha\beta = \beta - 2\alpha < 0$. We have $0 < \beta - \alpha < \alpha$ and $-s_\alpha\beta = 2\alpha - \beta = \alpha - (\beta - \alpha) < \alpha$. Since S is an order ideal it follows that $-s_\alpha\beta \in \mathrm{Inv}(u)$ and $w\beta = us_\alpha\beta > 0$ as desired.

Otherwise $\beta \in \Phi^+ \setminus \Phi^{(i)}$. Since i is cominuscule we have $k \in \{-1, 0, 1\}$. We assume $k = -1$ as the other cases were already done. Then $s_\alpha\beta = \beta - \alpha < 0$ since its coefficient of α_i is -1. Moreover $\alpha - \beta \in \Phi^{(i)}$. Since $\alpha > \alpha - \beta$ and S is an order ideal, it follows that $\alpha - \beta \in \mathrm{Inv}(u)$. Therefore $w\beta = us_\alpha\beta > 0$ as desired.

We have shown that every cover in $J(\Phi^{(i)})$ is the image under $\mathrm{Inv}|_{W^J}$ of a cover in $(W^J, \leq)$.

The bijectivity of $\mathrm{Inv}|_{W^J}$ follows by induction and the explicit description of the image of a cover under $\mathrm{Inv}|_{W^J}$. □

Proof of Theorem 3.2. For the case $W = G_2$, both labeled posets W^J_{weak} and W^J_{strong} are chains, so the result follows immediately. Thus we may assume that W^J is a cominuscule parabolic quotient.

For W^J_{strong} the result follows from Proposition 3.1. We label the vertices of $\Phi^{(i)}$ by reflections, defining $\pi : \Phi^{(i)} \to T$ by $\pi(\alpha) = s_\alpha$. Each cover $w \lessdot ws_\alpha$ in W^J_{strong} corresponds to adding $\alpha \in \Phi^{(i)}$ to $\mathrm{Inv}(w)$. Thus the edge label of $w \lessdot ws_\alpha$ agrees with the vertex label $\pi(\alpha) = s_\alpha$.

For the weak order W^J_{weak} let us consider two covers $w \lessdot ws_\alpha = s_\beta w$ and $v \lessdot vs_\alpha = s_{\beta'}v$ which have the same label s_α in W^J_{strong}. We claim that $s_\beta = s_{\beta'} = s_k$ for some $k \in I$. The elements w and v differ by right multiplication by some s_γ's where $\gamma \in \Phi^{(i)}$ is incomparable with α; this is accomplished by passing between w or v to the element $u \in W^J$ such that $\mathrm{Inv}(u) = \mathrm{Inv}(w) \cap \mathrm{Inv}(v)$. By Lemma 3.3 these s_γ's commute with s_α, and so $w\alpha = v\alpha$. This gives us a map $f : \Phi^{(i)} \to \Phi^+$ defined by $f(\alpha) = \beta = w\alpha$, which does not depend on $w \in W^J$ as long as $w \lessdot ws_\alpha$.

To show that $f(\alpha)$ is simple for each $\alpha \in \Phi^{(i)}$, consider a reduced word $ws_\alpha = s_{k_1}s_{k_2}\cdots s_{k_l}$. We know that $w^{(r)} = s_{k_r}\cdots s_{k_l} \in W^J$ and that $\mathrm{Inv}(w^{(r)})$ differs from $\mathrm{Inv}(w^{(r+1)})$ by some root in $\Phi^{(i)}$ since $w^{(r+1)} \lessdot w^{(r)}$. For some value $r = r^*$, this root is α and by the well-definedness just proved $f(\alpha) = \alpha_{k_{r^*}}$, since $w^{(r^*)} = w^{(r^*+1)}s_\alpha$. This shows that the strong order and weak order on W^J coincide, and that W^J_{weak} is isomorphic to the poset of order ideals of $\Phi^{(i)}$ where $\Phi^{(i)}$ is labeled with $\pi(\alpha) = f(\alpha)$. □

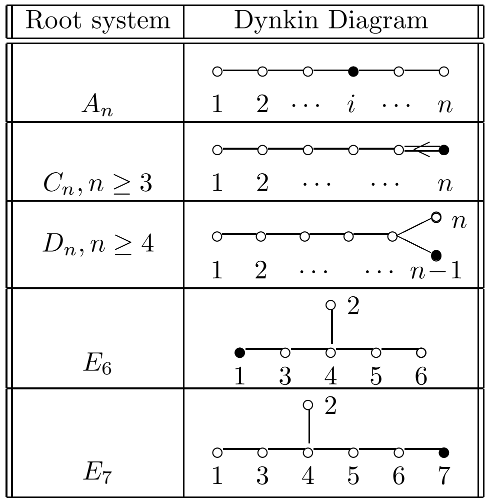

FIGURE 1. Some cominuscule parabolic quotients

4. Distributive subgraphs of Kac-Moody graded graphs

In this section we apply Theorem 3.2 to the dual graded graphs constructed in Section 2.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody algebra associated to the generalized Cartan matrix A and let W be its Weyl group. Let $W^{\text{fin}} \subset W$ be a finite parabolic subgroup corresponding to some index set $I' \subset I$. Now suppose that W^{fin} has a distributive parabolic quotient as in Theorem 3.1 corresponding to $J \subset I'$. We let $W^J \subset W^{\text{fin}}$ denote the distributive parabolic quotient.

Now let $(\Lambda, K) \in P^+ \times Z^+$ and let $(\Gamma_s(\Lambda), \Gamma_w(K))$ be the pair of dual graded graphs constructed in Section 2. By restricting to the subset of vertices $W^J \subset W^{\text{fin}} \subset W$ we obtain induced subgraphs $\Gamma_s^J(\Lambda)$ and $\Gamma_w^J(K)$. These graded graphs will no longer be dual, since they are finite, but they still have rich combinatorics.

THEOREM 4.1. *The induced graded subgraphs $\Gamma_s^J(\Lambda)$ and $\Gamma_w^J(K)$ are (naturally) distributively labeled posets.*

PROOF. By definition the graded graphs $\Gamma_s^J(\Lambda)$ and $\Gamma_w^J(K)$ have the same edges as W^J_{strong} and W^J_{weak}. To obtain $\Gamma_s^J(\Lambda)$ from W^J_{strong} we replace each edge $(w \lessdot ws_\alpha)$ labeled with the reflection s_α with the edge (w, ws_α) and integer label $\langle \Lambda \,,\, \alpha^\vee \rangle$. The distributivity of the labeling follows from Theorem 3.2. To obtain $\Gamma_w^J(K)$ from W^J_{weak} we replace each edge $(w \lessdot s_j w)$ labeled with the simple reflection s_j with the edge $(w, s_j w)$ and integer label $\langle \Lambda_j \,,\, K \rangle$. □

Thus $\Gamma_s^J(\Lambda)$ and $\Gamma_w^J(K)$ can be thought of as the poset of order ideals in some integer labeled poset P^J and Q^J. The Λ-strong and K-weak tableaux can be thought of as linear extensions of P^J and Q^J with additional markings.

In the rest of the paper, we give examples of the posets P^J and Q^J and relate them to classically understood tableaux. In each case we let $\mathfrak{g}$ be of untwisted affine type, $I' = I \setminus \{0\}$ and $J = I \setminus \{i\}$ for a fixed node $i \in I'$ to be specified. We use the canonical central element for K and Λ_i for the dominant weight. In this case P^J and Q^J are both labelings of the poset $\Phi^{(i)} \subset \Phi^+$ for the simple Lie algebra $\mathfrak{g}'$ whose Dynkin diagram is the subdiagram of that of $\mathfrak{g}$ given by removing the 0 node. These examples, with the exception of G_2, can be viewed as providing some additional data for the posets $\Phi^{(i)}$, whose unlabeled versions were given explicitly in [**23**]. As in [**23**] we shall rotate the labeled Hasse diagrams clockwise by 45 degrees so that the minimal element is in the southwest corner. In the following, we let V^J_{strong} and V^J_{weak} denote the vertex-labeled posets such that $W^J_{\text{strong}} = J(V^J_{\text{strong}})$ and $W^J_{\text{weak}} = J(V^J_{\text{weak}})$.

4.1. Type $A_n^{(1)}$.

Let $i \in I'$ be arbitrary. The poset $\Phi^{(i)}$ consists of elements $\alpha_{p,q} = \alpha_p + \cdots + \alpha_q$ for $1 \leq p \leq i \leq q \leq n$. The weak labeling of $\Phi^{(i)}$ is given by $\alpha_{p,q} \mapsto s_{p+q-i}$. For example, for $n = 7$ and $i = 3$

and abbreviating $\alpha_{p,q}$ by pq and s_j by j, the labelings of $\Phi^{(i)}$ by positive roots and simple reflections are given by:

$$V^J_{\text{strong}} = \begin{array}{|c|c|c|c|c|} \hline 13&14&15&16&17 \\ \hline 23&24&25&26&27 \\ \hline 33&34&35&36&37 \\ \hline \end{array} \qquad V^J_{\text{weak}} = \begin{array}{|c|c|c|c|c|} \hline 1&2&3&4&5 \\ \hline 2&3&4&5&6 \\ \hline 3&4&5&6&7 \\ \hline \end{array}$$

All labelings in P^J and Q^J are given by the constant 1. The resulting strong and weak tableaux are usual standard tableaux.

4.2. Type $C_n^{(1)}$. Let $i = n$. Let $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = 2e_n$ where e_i is the i-th standard basis element of the weight lattice $\mathbb{Z}^n$. Then $\Phi^{(n)}$ consists of the roots $\alpha_{i,j} = e_i + e_j$ for $1 \le i \le j \le n$. We have $a_i^\vee = 1$ for all i. For $n = 4$ we have

$$V^J_{\text{strong}} = \begin{array}{cccc} 14&13&12&11 \\ 24&23&22& \\ 34&33&& \\ 44&&& \end{array} \quad V^J_{\text{weak}} = \begin{array}{cccc} 1&2&3&4 \\ 2&3&4& \\ 3&4&& \\ 4&&& \end{array} \quad P^J = \begin{array}{cccc} 2&2&2&1 \\ 2&2&1& \\ 2&1&& \\ 1&&& \end{array} \quad Q^J = \begin{array}{cccc} 1&1&1&1 \\ 1&1&1& \\ 1&1&& \\ 1&&& \end{array}$$

The strong tableaux are shifted standard tableaux with two kinds of markings on offdiagonal entries; these are the standard recording tableaux for shifted insertion [**17**]. The weak tableaux are standard shifted tableaux.

4.3. Type $D_n^{(1)}$. Let $i = n$. Letting $\alpha_i = e_i - e_{i+1}$ for $1 \le i \le n-1$ and $\alpha_n = e_{n-1} + e_n$, the roots of $\Phi^{(n)}$ are given by $\alpha_{p,q} = e_p + e_q$ for $1 \le p < q \le n$. We have $a_j^\vee = 1$ for $j \in \{0, 1, n-1, n\}$ and $a_j^\vee = 2$ otherwise. For $n = 5$ we give the labelings of $\Phi^{(n)}$ below. Note the 1 in the upper left corner of Q^J.

$$V^J_{\text{strong}} = \begin{array}{cccc} 15&14&13&12 \\ 25&24&23& \\ 35&34&& \\ 45&&& \end{array} \quad V^J_{\text{weak}} = \begin{array}{cccc} 1&2&3&4 \\ 2&3&5& \\ 3&4&& \\ 5&&& \end{array} \quad P^J = \begin{array}{cccc} 1&1&1&1 \\ 1&1&1& \\ 1&1&& \\ 1&&& \end{array} \quad Q^J = \begin{array}{cccc} 1&2&2&1 \\ 2&2&1& \\ 2&1&& \\ 1&&& \end{array}$$

4.4. Type E. The computations in this section were made using Stembridge's Coxeter/Weyl package [**22**]. In both of the following cases, P^J has all labels 1.

For $E_6^{(1)}$ and $i = 1$ with the Dynkin labeling in Figure 1,

$$V^J_{\text{weak}} = \begin{array}{cccccccc} &&&1&3&4&5&6 \\ &&&3&4&2&& \\ &&2&4&5&&& \\ 1&3&4&5&6&&& \end{array} \qquad Q^J = \begin{array}{cccccccc} &&&1&2&3&2&1 \\ &&&2&3&2&& \\ &&2&3&2&1&& \\ 1&2&3&2&1&&& \end{array}.$$

For $E_7^{(1)}$ and $i = 7$ with the Dynkin labeling in Figure 1,

$$V^J_{\text{weak}} = \begin{array}{ccccccccc} &&&&&&&&7 \\ &&&&&&&&6 \\ &&&&&&&&5 \\ &&&&&&&2&4 \\ &&&&7&6&5&4&3 \\ &&&&6&5&4&3&1 \\ &&&&5&4&2&& \\ &&&2&4&3&&& \\ 7&6&5&4&3&1&&& \end{array} \qquad Q^J = \begin{array}{ccccccccc} &&&&&&&&1 \\ &&&&&&&&2 \\ &&&&&&&&3 \\ &&&&&&&2&4 \\ &&&&1&2&3&4&3 \\ &&&&2&3&4&3&2 \\ &&&&3&4&2&& \\ &&&2&4&3&&& \\ 1&2&3&4&3&2&&& \end{array}$$

4.5. Type $G_2^{(1)}$. This case does *not* correspond to a cominuscule root. Pick $i = 1$ and let α_1, α_2 be the two simple roots with α_1 the short root, so that the highest root is $3\alpha_1 + 2\alpha_2$. Then $a_1^\vee = 1$ and $a_2^\vee = 2$. Abbreviating the reflection $s_{p\alpha_1+q\alpha_2}$ by pq, we have

$$V^J_{\text{strong}} = \begin{array}{|c|c|c|c|c|} \hline 1&31&21&32&11 \\ \hline \end{array} \quad P^J = \begin{array}{|c|c|c|c|c|} \hline 1&3&2&3&1 \\ \hline \end{array}$$

and

$$V^J_{\text{weak}} = \begin{array}{|c|c|c|c|c|} \hline 1&2&1&2&1 \\ \hline \end{array} \quad Q^J = \begin{array}{|c|c|c|c|c|} \hline 1&2&1&2&1 \\ \hline \end{array}$$

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E-mail address: `tfylam@math.harvard.edu`

E-mail address: `mshimo@vt.edu`