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The Alternating Sign Matrix Polytope

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ABSTRACT. The Birkhoff (permutation) polytope, B_n, consists of the $n \times n$ nonnegative doubly stochastic matrices, has dimension $(n-1)^2$, and has n^2 facets. A new analogue, the alternating sign matrix polytope, ASM_n, is introduced and characterized. Its vertices are the $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ $n \times n$ alternating sign matrices. It has dimension $(n-1)^2$, has $4[(n-2)^2+1]$ facets, and has a simple inequality description. Its face lattice and projection to the permutohedron are also described.

RÉSUMÉ. Le polytope B_n de permutation (aussi dit de Birkhoff) consiste en les matrices double stochastiques $n \times n$ non négatives. Il est de dimension $(n-1)^2$, et a n^2 facettes. Un nouvel analogue, le polytope des matrices à signe alternant, ASM_n, est présenté et caractérisé. Ses sommets sont les $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ matrices $n \times n$ à signe alternant. Il est de dimension $(n-1)^2$, a $4[(n-2)^2+1]$ facettes, et est décrit par une inégalité simple. Le treillis de ses faces et sa projection sur le permutoèdre sont également décrits.

1. Background and Summary

The Birkhoff (permutation) polytope B_n is defined as the convex hull of n-by-n permutation matrices. Its dimension is $(n-1)^2$, it has $n!$ vertices, and has n^2 facets (each facet is made up of all doubly stochastic matrices with a 0 in a specified entry) [**9**]. Many analogous polytopes have been studied which are subsets of B_n. In contrast, the alternating sign matrix polytope ASM_n is formed by taking the convex hull of n-by-n alternating sign matrices, which is a set of matrices containing the permutations. Thus B_n is contained in ASM_n.

DEFINITION 1.1. Alternating sign matrices (ASMs) are square matrices with the following properties [**7**]:

- entries $\in \{0, 1, -1\}$
- the entries in each row and column sum to 1
- nonzero entries in each row and column alternate in sign

Permutation matrices, then, are the alternating sign matrices whose entries are nonnegative. The connection between these two sets of matrices, though, is much deeper. There exists a partial ordering on alternating sign matrices that is a distributive lattice. This lattice contains as a subposet the Bruhat order on the symmetric group, and in fact, it is the smallest lattice that does so (i.e. it is the MacNeille completion of the Bruhat order) [**5**]. Given this close relationship between permutations and ASMs it is natural to hope for something relating their polytopes.

The dimension of ASM_n is $(n-1)^2$ because the last entry in each row and column must be precisely what is needed to make that row or column sum equal 1. ASM_n has $4[(n-2)^2+1]$ facets and its vertices are the alternating sign matrices (proofs in section 3), whose count is given by [**4**]:

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

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2. The Inequality Description of ASM_n

B_n can be described not only as the convex hull of the permutation matrices, but equivalently as the set of all doubly stochastic matrices (matrices with row and column sums equaling 1) whose entries are nonnegative [**3**].

The inequality description of ASM_n is similar to that of B_n. It consists of the subset of doubly stochastic matrices (now allowed to have negative entries) whose partial sums in each row and column are between 0 and 1. The proof uses the idea of the proof of the inequality description of B_n found in [**8**]. For the statement and proof of a similar theorem, see [**1**].

THEOREM 2.1. *The convex hull of n-by-n alternating sign matrices consists of all n-by-n real matrices* $X = \{x_{ij}\}$ *such that:*

$$0 \leq \sum_{i=1}^{i'} x_{ij} \leq 1 \;\forall\; 1 \leq i' \leq n \tag{2.1}$$

$$0 \leq \sum_{j=1}^{j'} x_{ij} \leq 1 \;\forall\; 1 \leq j' \leq n \tag{2.2}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \;\forall\; 1 \leq j \leq n \tag{2.3}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \;\forall\; 1 \leq i \leq n \tag{2.4}$$

PROOF. Call the subset of $\mathbb{R}^{n^2}$ given by the above inequalities $P(n)$. It is easy to check that the convex hull of the alternating sign matrices is contained in the set $P(n)$. It remains to show that any $X \in P(n)$ can be written as a convex combination of alternating sign matrices.

Let $X \in P(n)$. Form two matrices, R and C, where the entries of R are the row partial sums of X and the entries of C are the column partial sums of X. So R and C are matrices with entries r_{ij} and c_{ij} such that $0 \leq r_{ij}, c_{ij} \leq 1$. Now form a recording matrix Y with entries $\{r\}$, $\{c\}$, $\{r, c\}$, or $\emptyset$ as follows:

$$\begin{cases} r \in y_{ij} & \text{if } r_{ij} \notin \{0,1\} \\ c \in y_{ij} & \text{if } c_{ij} \notin \{0,1\} \end{cases}$$

If Y is empty then X is an ASM, since the partial row and column sums of an ASM are always 0 or 1 and ASMs are the only matrices in $P(n)$ with this property. Thus the proof will proceed by induction on the number of r's plus the number of c's in Y.

We need the following lemma.

LEMMA 2.2. *There exists a circuit in Y with the following properties (see figure 1):*

- *Every row leader must contain an r.*
- *Every column leader must contain a c*
- *All entries in a horizontal line of the circuit following the row leader except the last must contain an r.*
- *All entries in a vertical line of the circuit following the column leader except the last must contain a c.*

Pick such a circuit in Y and label the corners alternately $(+)$ and $(-)$. Form a new matrix X' by adding a fixed constant k to the entries of X labelled $(+)$ and subtracting k from the entries labelled $(-)$. Note that subtracting k from a column leader $x_{i_0 j}$ and adding k to the corresponding column tail $x_{i_1 j}$ affects all the partial sums c_{ij}, $i_0 \leq i \leq (i_1 - 1)$, but no others, and similarly for rows. Thus the value of k depends on the row and column partial sums for every entry in the circuit, not only the corners. More specifically:

$$k = \min\left(\min_{(-)\; rows} r_{ij}, \min_{(+)\; rows} 1 - r_{ij}, \min_{(-)\; columns} c_{ij}, \min_{(+)\; columns} 1 - c_{ij}\right)$$

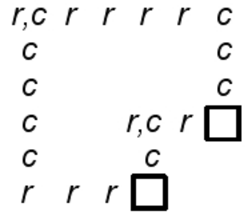

Figure 1. Circuit in Y matrix

where, for example, "$(-)$ *rows*" indicates that the minimum is over all rows whose row leader is labelled $(-)$. By this definition of k, there will be at least one additional partial sum equaling 0 or 1, thus there will be at least one less r or c in Y'. So by induction X' is a convex combination of ASMs. Now form X'' and Y'' by switching the sign on each of the corners of the circuit and subtracting/adding another constant k_0 as defined above. X'' is also a convex combination of ASMs. Then there exists λ with $0 < \lambda < 1$ depending on k and k_0 such that $X = \lambda X' + (1-\lambda)X''$. Thus the proof is complete by induction. □

It remains, then, to prove the lemma to show that such a circuit exists:

Proof. Suppose Y is nonempty (that is, Y is not an ASM). Pick any nonempty entry in Y for the starting point. We will show that one can trace around the matrix in a path to form a circuit of adjacent entries in Y as follows (the first item describes the general pattern the path follows and thus does not require proof, while the others do require proof):

- Generally, if the path is at an r, move to the right along r's until it reaches a c; if the path is on a c, move down along c's until it reaches an r (and if the path is on an r, c it can follow the pattern for either r or c).
- If the path reaches a blank spot while going down or to the right in Y, it must move up or to the left in order to continue, since blank spots can only occur at the end of a row and column in the circuit. We need to prove that this is always possible (i.e. in this situation there is always a c above or an r to the left of such a blank spot).

  Proof. Suppose the path reaches a blank spot y_{ij} while going down. Then $y_{(i-1)j}$ contains a c.

  Case 1: Suppose $c_{ij} = 0$. Then since $c_{(i-1)j} > 0$ it must be that $x_{ij} < 0$, and also $x_{ij} > -1$. But then the reason that an r does not appear in y_{ij} must be because $r_{ij} = 0$, so $r_{i(j-1)} > 0$ and also $r_{i(j-1)} < 1$. So there exists an r to the left of the blank spot in Y.

  Case 2: Suppose $c_{ij} = 1$. Then since $0 < c_{(i-1)j} < 1$ it must be that $x_{ij} > 0$, and also $x_{ij} < 1$. But then the reason that an r does not appear in y_{ij} must be because $r_{ij} = 1$, thus $r_{i(j-1)} < 1$ and also $r_{i(j-1)} > 0$. So there exists an r to the left of the blank spot in Y.

  Analogously, if the path reaches a blank spot in Y while going to the right along r's, there exists a c in the entry just above the blank one. □

- If the path reaches a blank while moving in the opposite direction (up along c's or left along r's), there is always a way to back up one space and turn in a different direction instead (at right angles to the previous direction).

  Proof. Suppose while going up along c's the path reaches a blank spot y_{ij}. If $y_{(i+1)j}$ is an rc then start going to the right along r's. If not, there are again two cases.

  Case 1: Suppose $c_{ij} = 0$. Then since $0 < c_{(i+1)j} < 1$ it must be that $0 < x_{(i+1)j} < 1$. This means, since there is no r in $y_{(i+1)j}$ that $r_{(i+1)j} = 1$. Thus $0 < r_{(i+1)(j-1)} < 1$ so the path can start going to the left along r's.

  Case 2: Suppose $c_{ij} = 1$. Then $-1 < x_{(i+1)j} < 0$ and $r_{(i+1)j} = 0$. So then $0 < r_{(i+1)(j-1)} < 1$, so the path can again start going to the left along r's.

  Similarly, if the path reaches a blank spot while going to the left along r's, it is always possible to back up one space and start going up along c's. □

- The first and last rows and columns cause no difficulties.

PROOF. Since the row and column sums of X equal 1, there are no r entries in the last column of Y and there are no c entries in the last row of Y. Thus the path will never reach a point in the last row where it is forced to turn down, and the path will never get to a point in the last column where it is forced to go right. Now we need to show that the path never reaches a point in the first row where it must go up, and that the path never reaches a point in the first column where it is forced to go left.

Since all entries in the first row of X must be nonnegative, the row partial sum for the first row is increasing to the right. Thus in Y the r's of the first row must be one after another, corresponding to all the entries in X that are between the first nonzero entry and the last (including the first and not including the last). The c's of the first row correspond to the nonzero entries in that row in X. So the first nonempty entry of the first row of Y must be an rc, the last must be a c, and in between each entry contains an r. So if the path is at a c in the first row it can start going either to the right or left along r's, and when it gets to the end of the r's it can go down along c's.

The proof that you never get to a point in the first column where you are forced to go left follows similarly. ☐

- The process terminates when the path reaches a nonempty entry of Y that was already in the circuit.

PROOF. So far we have shown that the path can proceed from entry to entry in Y producing a path satisfying the conditions of the lemma.. So if, for example, the path reaches a previously hit entry y_{ij} from the right, then y_{ij} must contain an r. If y_{ij} had previously been in the path in the interior of a vertical line, then y_{ij} must also contain a c. So we can forget about the entries in the circuit from the beginning until the first time y_{ij} was hit by the circuit, and just take the part from y_{ij} back to itself. By the finiteness of the matrix, the process will always terminate. ☐

Thus the desired circuit exists. ☐

3. Properties of ASM_n

Now that we can describe ASM_n in terms of inequalities, let us examine some of the properties of ASM_n, namely, its facets, its vertices, and its projection to the permutohedron.

To make the proofs of the next two theorems more transparent, we introduce *simple flow grids* which will be used more extensively in section 4. Define for each ASM A a directed graph $g(A)$ with n^2 vertices arranged in a square grid where each vertex represents the entry in the corresponding position of A. For vertices v and w directly north, south, east, or west of each other in $g(A)$ let there be an edge from v to w if the partial sum from the border of the matrix to the entry corresponding to v in the direction pointing toward w equals 1. By the definition of alternating sign matrices, there will be exactly one edge between each pair of bordering vertices. Call $g(A)$ the simple flow grid corresponding to A (see Figure 2).

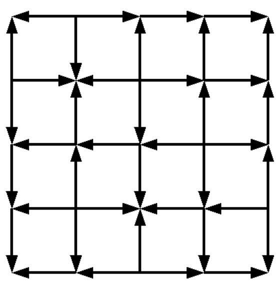

FIGURE 2. The simple flow grid corresponding to $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

Vertices of $g(A)$ corresponding to 1's are sources and vertices corresponding to -1's are the sinks. The directions of the rest of the edges in the grid are determined by the placement of the 1's, in that there is a series of arrows emanating from the 1's and continuing until they reach a sink or leave the grid.

We can define simple flow grids without reference to ASMs as follows.

Definition 3.1. A *simple flow grid* is a directed graph on an $n \times n$ array of vertices with exactly one directed edge between neighboring vertices in the array, and in which each of the vertices not on the border is either a source, a sink, or is such that their vertical edges both point in the same direction (north or south) and their horizonal edges both point in the same direction (east or west). The border vertices must also satisfy these conditions if one imagines at each border vertex, arrows pointing out of the grid.

Simple flow grids are in one-to-one correspondence with ASMs. Simple flow grids are, in fact, almost the same as the six-vertex model of square ice in statistical physics (see the discussion in [**4**]), the only difference being that the horizontal arrows point in the opposite direction.

Theorem 3.2. *ASM_n has $4[(n-2)^2+1]$ facets.*

Proof. By counting the defining inequalities, one sees that there could be at most $4n^2$ facets. It is left to determine how many of these inequalities are linearly dependent on the others. Note that the statement that the row and column partial sums must be ≤ 1 can be written in another way, that is that the row and column partial sums from the opposite direction must be ≥ 0. So the $4n^2$ defining inequalities for $X \in ASM_n$ can be written:

$$\sum_{i=1}^{i'} x_{ij} \geq 0 \qquad \sum_{j=1}^{j'} x_{ij} \geq 0$$

$$\sum_{i=i'}^{n} x_{ij} \geq 0 \qquad \sum_{j=j'}^{n} x_{ij} \geq 0$$

Thus these inequalities place four constraints on each entry x_{ij} of X, namely that the partial sum up to x_{ij} along row i or column j in either direction is ≥ 0. For a general entry in the matrix all four constraints are needed, but for entries near the border some of the inequalities depend on others.

First, since the full row and column sums always equal 1, the inequalities such as $\sum_{i=1}^{n} x_{ij} \geq 0$ are unneccessary. For entries x_{ij} with $i \in \{1, n\}$ or $j \in \{1, n\}$ only one inequality is needed, $x_{ij} \geq 0$, since if all the border entries are ≥ 0 then the partial sums of border entries in any direction will also be ≥ 0. Thus there is one facet for each border entry for a total of $4(n-1)$ facets.

Now any border entry x_{ij} of $X \in ASM_n$ must be at most 1 since the sum of the first row, for example, is a sum of nonnegative entries which equals 1. Therefore inequalities such as $\sum_{i=2}^{n} x_{ij} \geq 0$ are unnecessary since this is implied from the fact that $x_{1j} \leq 1$.

We can count the number of inequalities remaining using simple flow grids. Recall that an arrow in a simple flow grid $g(A)$ represents a location in an ASM A where the partial sum equals 1, thus an arrow missing from $g(A)$ represents a location in A where the partial sum equals 0. Thus if we remove the border vertices from $g(A)$ along with all arrows that begin from or terminate at the border vertices, the arrows pointing in the opposite directions to each of the arrows in the $(n-2)\times(n-2)$ array remaining each represent a facet on which A lies. There are $2(n-2)(n-3)$ such arrows in any ASM and two directions in which any of these arrows can point, thus there are $4(n-2)(n-3)$ total facets obtained from inequalities involving interior entries of the matrix.

This gives us a final count of $4(n-1) + 4(n-2)(n-3) = 4[(n-2)^2+1]$ inequalities. Thus ASM_n has at most $4[(n-2)^2+1]$ facets, each determined by making one of the inequalities an equality. They are facets (not just faces) since each inequality determines exactly one more entry of the matrix, decreasing the dimension by one.

Now given any two facets F_1 and F_2, it is easy to exhibit a pair of ASMs $\{X_1, X_2\}$ such that X_1 lies on F_1 and not on F_2. Simply include the arrow corresponding to F_2 but not the arrow corresponding to F_1 in $g(X_1)$, then do the opposite for X_2. Thus each of the $4[(n-2)^2+1]$ inequalities gives rise to a unique facet. □

COROLLARY 3.3. *The number of facets of ASM_n on which an ASM A lies is given by*

$$\begin{cases} 2(n-1)(n-2)+2, & \text{if } A \text{ has two corner 1's} \\ 2(n-1)(n-2)+1, & \text{if } A \text{ has one corner 1} \\ 2(n-1)(n-2), & \text{otherwise} \end{cases}$$

PROOF. Each 0 around the border of A represents one facet. Thus the number of facets corresponding to border zeros of A equals $4(n-1) - (\#$ 1's around the border of $A)$. Then there are $2(n-2)(n-3)$ facets represented by arrows pointing in the opposite directions to the arrows in the $(n-2)\times(n-2)$ interior array of $g(A)$. The sum of these numbers gives the above count. □

THEOREM 3.4. *The vertices of ASM_n are the alternating sign matrices.*

PROOF. Fix an ASM A. In order to show that A is a vertex of ASM_n, we simply need to find a hyperplane with A on one side and all the other ASMs on the other side. Then since ASM_n is the convex hull of $n \times n$ ASMs, A would necessarily be a vertex.

Consider the simple flow grid corresponding to A. In any simple flow grid there are, by definition, $2n(n-1)$ directed edges, where for each entry of A there is an arrow whenever the partial sum in that direction up to that point equals 1. Since the number of directed edges in a simple flow grid is fixed, A is the only ASM with all of those partial sums equaling 1. Thus the hyperplane where the sum of those partial sums equals $2n(n-1) - \frac{1}{2}$ (and the row and column sums still equal 1) will have A on one side and all the other ASMs on the other. Thus the ASMs are the vertices of ASM_n. □

Another interesting property of the ASM polytope is its relationship to the permutohedron. The permutohedron P_z corresponding to a vector $z = (z_1, z_2, \ldots, z_n)$ is the convex hull of the permutations of the coordinates of z. That is,

$$P_z = \text{conv}\{(z_{\omega(1)}, z_{\omega(2)}, \ldots, z_{\omega(n)}) \mid \omega \in S_n\}$$

It is well known that P_z is the image of B_n under the projection $\phi_z : \mathbb{R}^{n^2} \to \mathbb{R}^n$ defined by $\phi_z(X) = zX$ [**2**]. When the same projection map is applied to ASM_n, the same permutohedron is the result whenever z is a strictly monotone vector.

THEOREM 3.5. *Let $z = (z_1, z_2, \ldots, z_n)$ be a strictly increasing (or decreasing) vector and X an $n\times n$ ASM. Then $\phi_z(X) = zX$ is in the convex hull of the permutations of $\{z_1, z_2, z_3, \ldots, z_n\}$ so that $\phi_z(ASM_n) = P_z$. That is, matrix multiplication by a strictly monotone vector z projects ASM_n onto P_z.*

PROOF. For this proof we need the concept of majorization. Let x and y be vectors of length n. Then $x \preceq y$ (that is x is *majorized* by y) if

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, & \text{for } 1 \le k \le n-1 \\ \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \end{cases} \tag{3.1}$$

where the vector $(x_{[1]}, x_{[2]}, \ldots, x_{[n]})$ is obtained from x by rearranging its components so that they are in decreasing order, and similarly for y [**6**]. Then there is a proposition of Rado which states that for vectors x and y of length n, $x \preceq y$ if and only if x lies in the convex hull of the $n!$ permutations of the entries of y [**6**].

For ease of notation we pick z to be a strictly decreasing n-vector (so that $z_i = z_{[i]}$) and $X = \{x_{ij}\}$ an ASM of order n.

Let $zX = y = (y_1, y_2, \ldots, y_n)$. The proof will be completed by showing $y \preceq z$.

To verify the second condition of (3.1) note that since X is an ASM, each y_i has the form

$$y_i = z_{i_1} - z_{i_2} + z_{i_3} - \cdots + z_{i_m}$$

where $i_1 < i_2 < \ldots < i_m$. Then since the rows of X must each sum to 1, $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i$.

To verify the first condition of (3.1) we show given any $J \subseteq [n], |J| = k$, that $\sum_{j\in J} y_j \le \sum_{j=1}^{k} z_j$. Consider the vector v made up of the sum by row of the entries of the columns $\{c_j, j \in J\}$. That is, $v = (v_1, v_2, \ldots, v_n)$ where $v_i = \sum_{j\in J} x_{ij}$. We will need to verify the following two inequalities:

$$\sum_{i=1}^{m} v_i \le m \text{ for all } m \in [n] \tag{3.2}$$

$$\sum_{i=1}^{n} v_i = k \tag{3.3}$$

To prove (3.2) note that $\sum_{i=1}^{m} v_i$ equals the number of columns in $\{c_j \mid j \in J\}$ whose partial sum from the top of the matrix until row m equals 1. I claim (and will show below) that there are only m such columns in the entire matrix with this property, so there are at most m such columns in $\{c_j \mid j \in J\}$.

The partial sum in column j from the top of the matrix to row m equals 1 exactly when there exists $i_0 \leq m$ such that $x_{i_0 j} = 1$ and $x_{ij} = 0$ for all $i_0 + 1 \leq i \leq m$. Now the top row of any ASM has exactly one 1 and the rest of the entries are 0. In the second row there may be a -1 directly under the 1 from the top row, but no -1's anywhere else, so row 2 contains either a single 1 and the rest 0's, or two 1's and one -1. In either case there are only two columns whose partial sum in row 2 equals 1. Suppose row $i-1$ has $i-1$ columns whose partial sum equals 1. Row i then has $\ell \leq i$ occurrences of 1 and $\ell - 1$ occurrences of -1. But each of the -1's must be in one of the columns whose partial sum in row $i-1$ equals 1. Thus there are $\ell - 1$ columns whose partial sum in row $i-1$ equals 1 and in row i equals 0, and ℓ columns whose partial sum in row $i-1$ equals 0 but in row i equals 1. Thus row i has $(i-1) - (\ell - 1) + \ell = i$ columns whose partial sum equals 1. Thus by induction, $\sum_{i=1}^{m} v_i \leq m$ for all $m \in [n]$.

To prove (3.3) observe,

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} \sum_{j \in J} x_{ij} = \sum_{j \in J} \sum_{i=1}^{n} x_{ij} = k$$

since the columns of X each sum to 1.

Therefore

$$v \cdot z = v_1 z_1 + v_2 z_2 + \ldots + v_n z_n \leq 1 \cdot z_1 + 1 \cdot z_2 + \ldots + 1 \cdot z_k.$$

since z is decreasing.

So finally,

$$\sum_{j \in J} y_j = v \cdot z \leq \sum_{j=1}^{k} z_j.$$

Thus $y \preceq z$ and so zX is contained in the convex hull of the permutations of z. Therefore $\phi_z(ASM_n) = \phi_z(B_n) = P_z$. □

4. Face Lattice

Another nice result concerning the Birkhoff polytope is the structure of its face lattice [**2**]. Associate to each permutation matrix X a bipartite graph with vertices $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ where there is an edge connecting x_i and y_j if and only if there is a 1 in the (i, j) position of X. Such a graph will be a perfect matching. A graph G is called elementary if every edge is a member of some perfect matching of G.

PROPOSITION 4.1. *The face lattice of the Birkhoff polytope is isomorphic to the lattice of elementary subgraphs of $K_{n,n}$ ordered by inclusion* [**2**].

A similar statement can be made about the ASM polytope using simple flow grids (see section 3) in the place of perfect matchings and *elementary flow grids* in place of elementary graphs.

DEFINITION 4.1. An *elementary flow grid* G is a directed graph on an $n \times n$ array of vertices such that the edge set of G is the union of the edge sets of simple flow grids.

The proof of Theorem 3.2 shows that for entries around the border of a matrix $X \in ASM_n$, not all four directions of partial sums yield facets. Therefore there are some directed edges near the border of a simple flow grid whose presence or absence from the grid is determined by the placement of other edges. Thus their absence from the grid does not determine a facet.

Now for any face F of ASM_n define the grid corresponding to the face, $g(F)$, to be the union over all the vertices of F of the simple flow grids corresponding to the vertices. That is,

$$g(F) = \bigcup_{vertices\ A \in F} g(A)$$

Thus $g(F)$ is an elementary flow grid since its edge set is the union of the edge sets of simple flow grids.

Now we wish to define the converse, that is, given an elementary flow grid G we would like to know the corresponding face $f(G)$ of ASM_n. Define $f(G)$ to be the convex hull of the vertices of ASM_n whose corresponding flow grids are contained in the grid G. So let

$$f(G) = \text{conv}\{\text{vertices } A \in ASM_n \mid g(A) \subseteq G\}$$

The directed edges that are not in G (with the exception of some border edges as discussed earlier) represent facets that contain $f(G)$. Let the collection of these directed edges be called $\{e_1, e_2, \ldots, e_k\}$ and their corresponding facets $\{F_1, F_2, \ldots, F_k\}$. Let $I = \bigcap_{j=1}^{k} F_j$ be the intersection of these facets. Thus I is a face of ASM_n and $f(G) \subseteq I$.

We wish to show that $f(G)$ equals I. So suppose $f(G) \subsetneq I$. Then since I is a face of ASM_n and $f(G)$ is defined as the convex hull of vertices of ASM_n there exists an additional vertex $B \in I$ of ASM_n such that $B \notin f(G)$. But $g(B)$ must be missing the edges $e_1, e_2, \ldots, e_k$ since $B \in I$, thus all the directed edges of $g(B)$ must be in G. Therefore $g(B) \subseteq G$ so that $B \in f(G)$ which is a contradiction. So $f(G) = I$. Thus $f(G)$ is a face of ASM_n since it is the intersection of faces of ASM_n.

It can easily be seen that $f(g(F)) = F$ and $g(f(G)) = G$. Also if F_1 and F_2 are faces of ASM_n then $F_1 \subseteq F_2$ if and only if $g(F_1) \subseteq g(F_2)$.

Thus elementary flow grids are in bijection with the faces of ASM_n (if we also regard the empty grid as an elementary flow grid). Elementary flow grids can be made into a lattice by inclusion, where the join is the union of the edge sets and the meet is the largest elementary flow grid made up of the edges from the intersection of the edges sets.

This discussion yields the following theorem:

THEOREM 4.2. *The face lattice of ASM_n is isomorphic to the lattice of all $n \times n$ elementary flow grids ordered by inclusion.*

The dimension of any face of ASM_n can be determined by looking at $g(F)$ as in the following corollary. The characterization of edges of ASM_n is analagous to the result for the Birkhoff polytope which states that the graphs representing edges of B_n are the elementary subgraphs of $K_{n,n}$ which have exactly one cycle [**2**].

Given an elementary flow grid G, define a *doubly directed region* as a collection of cells in G completely bounded by double directed edges but containing no double directed edges in the interior (see figure 3). Let $\alpha(G)$ denote the number of doubly directed regions in G.

COROLLARY 4.3. *The m–dimensional faces of ASM_n are represented by the elementary flow grids in which the number of doubly directed regions equals m. In particular, the edges of ASM_n are represented by elementary flow grids containing exactly one cycle (which is traversable in both directions).*

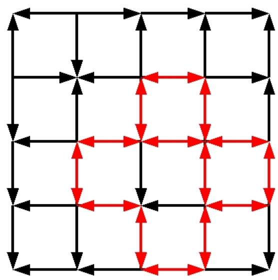

FIGURE 3. An elementary flow grid containing 3 doubly directed regions, corresponding to a 3-dimensional face of ASM_5

PROOF. We proceed by induction on the dimension of the face of ASM_n. The simple flow grid corresponding to any ASM X has no double edges, thus $\alpha(g(X)) = 0$. Now suppose for every m–dimensional face of ASM_n, the number of doubly directed regions of the elementary flow grid corresponding to the face equals m. Let F be an $(m+1)$–dimensional face of ASM_n and F' an m–dimensional subface of F. Let A and A' be ASMs such that $A \subset F - F'$ and $A' \subset F'$. Then by induction $\alpha(g(F')) = m$.

Now $g(F)$ is the elementary flow grid whose edge set is the union of the edge sets of $g(F')$ and $g(A)$. Since every vertex in a simple flow grid must have even indegree and even outdegree, in order to obtain $g(A)$

from $g(A')$ by reversing some directed edges, the number of directed edges reversed at each vertex must be even. Thus taking the union of the directed edges of $g(A')$ with the directed edges of $g(A)$ forms one or more circuits of double directed edges, where at least one of the double directed edges is not in $g(F')$. Therefore $g(F)$ has at least one more doubly directed region than $g(F')$, so $\alpha(g(F)) \geq m+1$. Then since $g(ASM_n)$ is the elementary flow grid with all possible directed edges,

$$\alpha(g(ASM_n)) = (n-1)^2 = \dim(ASM_n).$$

Therefore moving up the face lattice one rank increases the number of doubly directed regions by exactly one, so $\alpha(g(F)) = m+1$. □

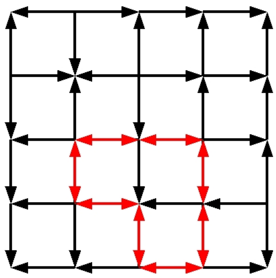

FIGURE 4. The elementary flow grid representing the edge between $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ in ASM_5

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