PERMUTATION TABLEAUX AND THE ASYMMETRIC EXCLUSION PROCESS

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Abstract. The partially asymmetric exclusion process (PASEP) is an important model from statistical mechanics which describes a system of interacting particles hopping left and right on a one-dimensional lattice of N sites. It is partially asymmetric in the sense that the probability of hopping left is q times the probability of hopping right. Additionally, particles may enter from the left with probability α and exit to the right with probability β.

It has been observed that the (unique) stationary distribution of the PASEP has remarkable connections to combinatorics. In this paper we will show that in fact the (normalized) probability of being in a particular state of the PASEP can be viewed as a certain weight generating function for *permutation tableaux* of a fixed shape. Our first proof is algebraic and uses the *matrix ansatz* of Derrida *et al.* Our second proof involves defining a Markov chain – which we call the PT chain – on the set of permutation tableaux which *projects* to the PASEP in a very strong sense, thus revealing a hidden structure behind the PASEP. Via the bijection from permutation tableaux to permutations, the PT chain can also be viewed as a Markov chain on the symmetric group. Another nice feature of the PT chain is that it possesses a certain symmetry which extends the *particle-hole symmetry* of the PASEP.

Résumé. Le processus d'exclusion partiellement asymétrique (PASEP) est un modèle important de mécanique statistique qui décrit un système de particules en interaction sautant à droite et à gauche sur un réseau à une dimension comportant N emplacements. Ce modèle est partiellement asymétrique en ce sens que la probabilité de sauter à gauche est q fois la probabilité de sauter à droite. De plus, les particules peuvent entrer par la gauche avec probabilité α et sortir par la droite avec probabilité β.

Il a été observé que l'unique distribution stationnaire du PASEP a un rapport remarquable avec la combinatoire. Dans cet article nous démontrons qu'en fait la probabilité normalisée d'être dans un état particulier du PASEP peut être vue comme une fonction génératrice pour les tableaux de permutations d'une forme fixée. Notre première preuve est algébrique et utilise le *matrix ansatz* de Derrida *et al.* Notre deuxième preuve repose sur la construction d'une chaîne de Markov – que nous appelons la chaîne PT – sur l'ensemble des tableaux de permutations qui se projette sur le PASEP dans un sens très fort, ainsi révélant une structure cachée derrière le PASEP. En utilisant la bijection entre tableaux de permutations et permutations, la chaîne PT peut aussi être vue comme une chaîne de Markov sur le groupe symétrique. Une autre caractéristique remarquable de la chaîne PT est qu'elle possède une certaine symétrie qui étend la symétrie particule-trou du PASEP.

1. Introduction

The partially asymmetric exclusion process (PASEP) is an important model from statistical mechanics which is quite simple but surprisingly rich: it exhibits boundary-induced phase transitions,

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spontaneous symmetry breaking, and phase separation. The PASEP is regarded as a primitive model for biopolymerization [10], traffic flow [12], and formation of shocks [7]; it also appears in a kind of sequence alignment problem in computation biology [1].

In brief, the PASEP describes a system of particles hopping left and right on a one-dimensional lattice of N sites. Particles may enter the system from the left with a rate αdt and may exit the system from the right at a rate βdt. The probability of hopping left is q times the probability of hopping right.

It has been observed that the (unique) stationary distribution of the PASEP has remarkable connections to combinatorics. Derrida *et al* [5, 6] proved a connection to Catalan and Narayana numbers in the case where $q = 0$ (TASEP) and $\alpha = \beta = 1$; Duchi and Schaeffer [8] gave a combinatorial explanation of this result by constructing a new Markov chain on "complete configurations" (enumerated by Catalan numbers) that projects to the TASEP, for general α and β. Subsequently Corteel [2] proved a connection of the PASEP to the q-Eulerian numbers of [14], thereby generalizing the result of Derrida *et al* to include the case where again $\alpha = \beta = 1$ but q is general.

In this paper we demonstrate a stronger result in the case of general α, β and q, showing that in fact the (normalized) probability of being in a particular state of the PASEP can be viewed as a certain *weight generating function* for *permutation tableaux* of a fixed shape – this is a Laurent polynomial in α, β, and q. Our first proof is algebraic and uses the *matrix ansatz* of Derrida *et al* [6].

Our second proof involves constructing a Markov chain on permutation tableaux (which we call the PT chain) which *projects* to the PASEP: that is, after projection via a certain surjective map between the spaces of states, a walk on the state diagram of the PT chain is indistinguishable from a walk on the state diagram of the PASEP. The steady state distribution of the PT chain has the nice property that the (normalized) probability of being in a particular state (i.e. a permutation tableau) is the *weight* of that permutation tableau – this is a Laurent *monomial* in α, β, and q. Note that our construction generalizes the work of Duchi and Schaeffer [8], whose work can be viewed as the $q = 0$ case of ours.

The structure of this paper is as follows. In Section 2 we define the PASEP. In Section 3 we define permutation tableaux, certain $0-1$ tableaux which are naturally in bijection with permutations. Section 4 outlines a proof of our main result which uses the matrix ansatz. Section 5 defines the PT chain; it is clear from the definition that the PT chain projects to the PASEP. Section 6 describes an involution on the state-diagram of the PT chain which extends the particle-hole symmetry. Finally, Section 7 briefly mentions the connection of permutation tableaux with permutations and what this means for the PT chain and its involution.

It would be interesting to explore whether the PT chain has any physical significance, and whether this larger chain may shed some insight on the PASEP itself.

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2. The PASEP

In the physics literature, the PASEP is usually defined on a continuous time-scale. However, this formulation is equivalent to the following Markov chain.

Definition 2.1. *Let* B_N *be the set of all* 2^N *words in the language* $\{\circ, \bullet\}^*$. *The PASEP is the Markov chain on* B_N *with transition probabilities:*

- *If* $X = A \bullet \circ B$ *and* $Y = A \circ \bullet B$ *then* $P_{X,Y} = \frac{1}{N+1}$ *(particle hops right) and* $P_{Y,X} = \frac{q}{N+1}$ *(particle hops left).*
- *If* $X = \circ B$ *and* $Y = \bullet B$ *then* $P_{X,Y} = \frac{\alpha}{N+1}$ *(particle enters from left).*
- *If* $X = B\bullet$ *and* $Y = B\circ$ *then* $P_{X,Y} = \frac{\beta}{N+1}$ *(particle exits to the right).*
- *Otherwise* $P_{X,Y} = 0$ *for* $Y \neq X$ *and* $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$*.*

See Figure 1 for an illustration of the state diagram for $N = 2$.

Remark 2.2. *Note that we will sometimes denote a state of the PASEP as a word in* $\{0,1\}^N$ *and sometimes as a word in* $\{\circ, \bullet\}^N$*. In the latter notation, the symbol* $\circ$ *denotes the absence of a particle, which one can also think of as a white particle.*

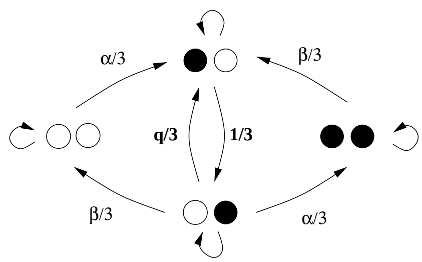


FIGURE 1. The state diagram of the PASEP for $N = 2$

In the long time limit, the system reaches a steady state where all the probabilities $P_N(\tau_1, \tau_2, \ldots, \tau_N)$ of finding the system in configurations $(\tau_1, \tau_2, \ldots, \tau_N)$ are stationary, i.e. satisfy

$$\frac{d}{dt} P_N(\tau_1, \ldots, \tau_N) = 0.$$

Moreover, the stationary distribution is unique [6], as shown by Derrida *et al.*

The question is now to solve for the probabilities $P_N(\tau_1, \ldots, \tau_N)$. For convenience, we define unnormalized weights $g_N(\tau_1, \ldots, \tau_N)$, which are equal to the $P_N(\tau_1, \ldots, \tau_N)$ up to a constant:

$$P_N(\tau_1, \ldots, \tau_N) = g_N(\tau_1, \ldots, \tau_N)/Z_N,$$

where Z_N is the *partition function* $\sum_\tau g_N(\tau_1, \ldots, \tau_N)$. The sum defining Z_N is over all possible configurations $\tau \in \{0,1\}^N$.

Remark 2.3. *There is an obvious particle-hole symmetry* [6] *in the PASEP: since (black) particles enter at the left with probability* α *and exit to the right with probability* β*, it is equivalent to saying that holes (or white particles) are injected at the right with probability* β *and are removed at the left end with probability* α*.*

Let us define $\overline{(\tau_1, \ldots, \tau_N)} = (1 - \tau_N, 1 - \tau_{N-1}, \ldots, 1 - \tau_1)$*. Clearly this operation is an involution on states of the PASEP. Because of the particle-hole symmetry, one always has that* $g_N^{q,\alpha,\beta}(\tau) = g_N^{q,\beta,\alpha}(\overline{\tau})$*.*

The notation for denoting a state of the PASEP as a word in $\{\circ, \bullet\}^N$ *is particularly suggestive of the particle-hole symmetry in the PASEP.*

3. Connection with permutation tableaux

Recall that a *partition* $\lambda = (\lambda_1, \dots, \lambda_K)$ is a weakly decreasing sequence of nonnegative integers. For a partition λ, where $\sum \lambda_i = m$, the *Young diagram* Y_λ of shape λ is a left-justified diagram of m boxes, with λ_i boxes in the ith row. We define the *half-perimeter* of λ or Y_λ to be the sum of the number of rows and the number of columns. The *length* of a row or column of a Young diagram is the number of boxes in that row or column. Note that we will allow a row to have length 0.

We will often identify a Young diagram Y_λ of half-perimeter t with the lattice path $p(\lambda)$ of length t which takes unit steps south and west, beginning at the north-east corner of Y_λ and ending at the south-west corner. Note that such a lattice path always begins with a step south. See Figure 2 for the path corresponding to the Young diagram of shape $(2, 1, 0)$.

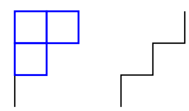


Figure 2. Young diagram and path for $\lambda = (2, 1, 0)$

If $\tau \in \{0, 1\}^N$, we define a Young diagram $\lambda(\tau)$ of half-perimeter $N + 1$ as follows. First we define a path $p = (p_1, \dots, p_{N+1}) \in \{S, W\}^{N+1}$ such that $p_1 = S$, and $p_{i+1} = S$ if and only if $\tau_i = 1$. We then define $\lambda(\tau)$ to be the partition associated to this path p. This map is clearly a bijection between the set of Young diagrams of half-perimeter $N + 1$ and the set of N-tuples in $\{0, 1\}^N$, and we denote the inverse map similarly: given a Young diagram λ of half-perimeter $N + 1$, we define $\tau(\lambda)$ to be the corresponding N-tuple.

As in [13], we define a *permutation tableau* $\mathcal{T}$ to be a partition λ together with a filling of the boxes of Y_λ with 0's and 1's such that the following properties hold:

(1) Each column of the rectangle contains at least one 1.
(2) There is no 0 which has a 1 above it in the same column *and* a 1 to its left in the same row.

We call such a filling a *valid* filling of Y_λ.

Remark 3.1. *Permutation tableaux are closely connected to total positivity for the Grassmannian* [11, 14]. *More precisely, if we forget the requirement (1) above we recover the definition of a* Ⅎ-*diagram, an object which represents a cell in the totally nonnegative part of the Grassmannian.*

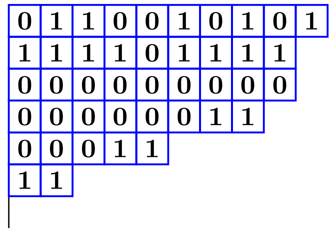
Remark 3.2. *Sometimes we will depict permutation tableaux slightly differently, replacing the* 1*'s with black dots and omitting the* 0*'s entirely, as in Figure 4.*

We will now define a few statistics on permutation tableaux. We define the *rank* $\mathrm{rk}(\mathcal{T})$ of a permutation tableau $\mathcal{T}$ with m columns to be the total number of 1's in the filling minus m. (We subtract m since there must be at least m 1's in a valid filling of a tableau with m columns.)

We define $f(\mathcal{T})$ to be the number of 1's in the first row of $\mathcal{T}$.

We say that a zero in a permutation tableau is *restricted* if there is a one above it in the same column. And we say that a row is *unrestricted* if it does not contain a restricted entry. Define $u(\mathcal{T})$ to be the number of unrestricted rows of $\mathcal{T}$ minus 1. (We subtract 1 since the top row of a tableau is always unrestricted.)

Figure 3 gives an example of a permutation tableau $\mathcal{T}$ with rank $19 - 10 = 9$ and half-perimeter 17, such that $u(\mathcal{T}) = 3$ and $f(\mathcal{T}) = 5$.



$\lambda = (10, 9, 9, 8, 5, 2, 0)$

FIGURE 3. A permutation tableau

We define the *weight* of a tableau $\mathcal{T}$ to be the monomial $\mathrm{wt}(\mathcal{T}) := q^{\mathrm{rk}(\mathcal{T})}\alpha^{-f(\mathcal{T})}\beta^{-u(\mathcal{T})}$, and we define $F_\lambda(q)$ to be the (Laurent) polynomial $\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, where the sum ranges over all permutation tableaux $\mathcal{T}$ of shape λ.

Our main result is the following.

Theorem 3.3. *Fix $\tau = (\tau_1, \dots, \tau_N) \in \{0,1\}^N$, and let $\lambda := \lambda(\tau)$. (Note that* half-perim$(\lambda) = N+1$.*) The probability of finding the PASEP in configuration $(\tau_1, \dots, \tau_N)$ in the steady state is*

$$\frac{F_\lambda(q)}{Z_N}.$$

Here, $F_\lambda(q)$ is the weight-generating function for permutation tableaux of shape λ. Moreover, the partition function Z_N for the PASEP is equal to the weight-generating function for all permutation tableaux of half-perimeter $N+1$.

4. FIRST PROOF – THE MATRIX ANSATZ APPROACH

The "matrix ansatz" has been used by Derrida et al [6] to obtain exact expressions for all the $P_n(\tau_1, \dots, \tau_n)$. More precisely, they show the following.

Theorem 4.1. [6] *Suppose that D and E are matrices, V is a column vector, and W is a row vector, such that the following conditions hold:*

$$DE - qED = D + E$$
$$DV = \frac{1}{\beta}V$$
$$WE = \frac{1}{\alpha}W$$

Then

$$f_n(\tau_1, \dots, \tau_n) = W(\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E))V.$$

Note that $\prod_{i=1}^{n}(\tau_i D + (1-\tau_i)E)$ is simply a product of n matrices D or E with matrix D at position i if site i is empty ($\tau_i = 0$).

Remark 4.2. *It follows from Theorem 4.1 that the partition function Z_n is equal to $W(D+E)^n V$.*

We will now describe a solution (D_1, E_1, V_1, W_1) to the matrix ansatz. Our solution has an interpretation in terms of permutation tableaux.

Let D_1 be the (infinite) upper triangular matrix (d_{ij}) such that $d_{i,i+1} = \beta^{-1}$ and $d_{i,j} = 0$ for $j \neq i+1$. Let E_1 be the (infinite) lower triangular matrix (e_{ij}) such that for $j \leq i$, $e_{ij} = \beta^{i-j}(\alpha^{-1}q^{j-1}\binom{i-1}{j-1} + \sum_{r=0}^{j-2}\binom{i-j+r}{r}q^r)$. Otherwise, $e_{ij} = 0$.

Observe that when $\alpha = \beta = 1$, we have $e_{ij} = \frac{[i]^{(i-j)}}{(i-j)!}$. Here, $[i]^{(k)}$ represents the kth derivative of $[i]$ with respect to q, and $[i]$ is the q-analog of the number i, namely $1 + q + \cdots + q^{i-1}$. And then E_1 becomes the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & [2] & 0 & 0 & 0 & \dots \\ 1 & [3]' & [3] & 0 & 0 & \dots \\ 1 & \frac{[4]''}{2} & [4]' & [4] & 0 & \dots \\ 1 & \frac{[5]'''}{6} & \frac{[5]''}{2} & [5]' & [5] & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let W_1 be the (row) vector $(1, 0, 0, \dots)$ and V_1 be the (column) vector $(1, 1, 1, \dots)$. It is now easy to check that the required relations hold.

Lemma 4.3. *With the definitions of D_1, E_1, V_1, W_1 above, the following relations hold: $D_1E_1 - qE_1D_1 = D_1 + E_1$, $DV_1 = \frac{1}{\beta}V_1$, and $W_1E = \frac{1}{\alpha}W_1$.*

It turns out that the matrix product $W_1(\prod_{i=1}^n(\tau_iD_1 + (1-\tau_i)E_1))V_1$ from Theorem 4.1 enumerates permutation tableaux of shape $\lambda(\tau)$ according to weight. This leads to the connection between steady state probabilities of the PASEP and weight-generating functions for permutation tableaux.

5. The PT chain

The goal of this section is to define a Markov chain on permutation tableaux which projects to the PASEP; we will call this chain the *PT chain*.

Definition 5.1. *We define a projection operator* pr *which projects a state of the PT, i.e. a permutation tableau, to a state of the PASEP. If $\mathcal{T}$ is a permutation tableau of shape λ and half-perimeter $N+1$, then we define $\text{pr}(\mathcal{T}) := \tau(\lambda)$. This is a state of the PASEP with N sites.*

Before defining the PT chain, we show an example: the state diagram of the chain for $N = 3$. In Figure 4, the $24 = 4!$ states of the PT chain are arranged into $8 = 2^3$ groups according to partition shape (four of size 1, two of size 3, two of size 7). All elements of a fixed group of tableaux project to the same state of the PASEP, depicted just above that group. We have not included the transition probabilities in Figure 4, but they are easy to calculate: if there is a transition $\mathcal{S} \to T$ in the PT chain, then $\text{prob}_{PT}(\mathcal{S} \to T) = \text{prob}_{PASEP}(\text{pr}(\mathcal{S}) \to \text{pr}(T))$. Finally, observe that there is a reflective left-right symmetry in the figure.

Remark 5.2. *The PT projects to the PASEP in the following sense:*

- *The operator* pr *is a surjective map from the set of permutation tableaux of half-perimeter $N+1$ to the states of the PASEP with N sites.*
- *If $\mathcal{S}$ and $\mathcal{T}$ are states of the PT such that $\text{prob}(\mathcal{S} \to \mathcal{T}) > 0$, then $\text{prob}_{PT}(\mathcal{S} \to \mathcal{T}) = \text{prob}_{PASEP}(\text{pr}(\mathcal{S}) \to \text{pr}(\mathcal{T}))$.*

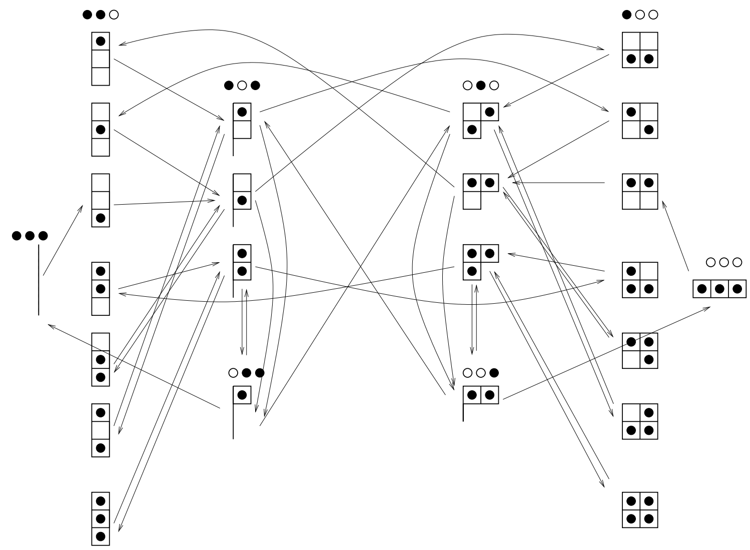


FIGURE 4. The state diagram of the PT chain for $N = 3$

- *If A and B are states of the PASEP, then for each state $\mathcal{S}$ of the PT such that* $\mathrm{pr}(\mathcal{S}) = A$*, there is a unique state $\mathcal{T}$ of the PT such that* $\mathrm{pr}(\mathcal{T}) = B$ *and* $\mathrm{prob}_{PT}(\mathcal{S} \to \mathcal{T}) > 0$*; moreover,* $\mathrm{prob}_{PT}(\mathcal{S} \to \mathcal{T}) = \mathrm{prob}_{PASEP}(A \to B)$.

The above statements will be clear from the definition of the PT chain.

Given these statements, it is clear that after applying the projection operator, a walk on the state diagram of the PT chain is indistinguishable from a walk on the state diagram of the PASEP.

We now define all possible transitions in the PT chain, together with the transition probabilities. There are four kinds of transitions, which correspond to the four kinds of transitions in the PASEP. In what follows, we will assume that $\mathcal{S}$ is a permutation tableau of half-perimeter $N + 1$, whose shape is $\lambda = (\lambda_1, \ldots, \lambda_m, \ldots, \lambda_t)$ where $\lambda_1 \geq \cdots \geq \lambda_m > 0$, and $\lambda_r = 0$ for $r > m$.

For convenience, we introduce a few more definitions for permutation tableaux. A 1 is *topmost* if it has only 0's above it. A 1 is *superfluous* if it is not topmost. And a 1 is *necessary* if it is the unique 1 in its column. Note that $\mathrm{rk}(\mathcal{T})$ is equal to the number of superfluous 1's in $\mathcal{T}$.

5.1. **Particle enters from the left.** If the rightmost column of $\mathcal{S}$ has length 1, then there is a transition in PT from $\mathcal{S}$ that corresponds to a particle entering from the left in the PASEP.

We now define a new permutation tableau $\mathcal{T}$ as follows: delete the rightmost column of $\mathcal{S}$ and add a new all-zero row of length $\lambda_1 - 1$ to $\mathcal{S}$, inserting it as far south as possible (subject to the constraint that the lengths of the rows of a permutation tableau must weakly decrease). See Figure 5.

We define $\mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\alpha}{N+1}$. It is easy to see that $\mathrm{wt}(\mathcal{T}) = \alpha \cdot \mathrm{wt}(\mathcal{S})$, and therefore $\mathrm{wt}(\mathcal{S}) \cdot \mathrm{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\mathrm{wt}(\mathcal{T})}{N+1}$.

5.2. **Particle hops right.** If some row $\lambda_j > \lambda_{j+1}$ in $\mathcal{S}$ (resp. if $\lambda_t > 0$), then there is a transition in PT from $\mathcal{S}$ that corresponds to the $(j-1)$st black particle (resp. $(t-1)$st black particle) in $\mathrm{pr}(\mathcal{S})$ hopping to the right in the PASEP.

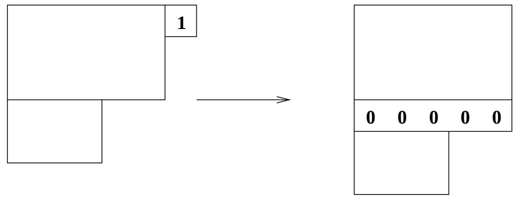


Figure 5

We now define a new permutation tableau $\mathcal{T}$ as follows, based on the rightmost entry of the jth row of $\mathcal{S}$.

5.2.1. *Case 1.* Suppose that the rightmost entry of the jth row is a 0. Then we define a new tableau $\mathcal{T}$ by deleting the jth row of $\mathcal{S}$ and adding a new row of $\lambda_j - 1$ 0's, inserting it as far south as possible. See Figure 6.

We define $\text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{1}{N+1}$. If $\lambda_j > 1$ then it is easy to see that $\text{wt}(\mathcal{T}) = \text{wt}(\mathcal{S})$, and therefore $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\text{wt}(\mathcal{T})}{N+1}$.

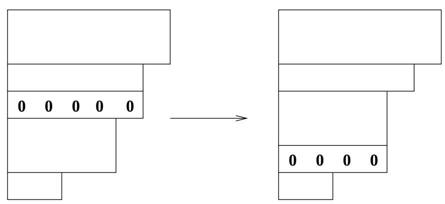


Figure 6

In the special case that $\lambda_j = 1$, then $\text{wt}(\mathcal{T}) = \beta^{-1}\,\text{wt}(\mathcal{S})$, and therefore $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\beta\,\text{wt}(\mathcal{T})}{N+1}$. Note that if this special occurs then $\text{pr}(\mathcal{T})$ ends with a black particle. And given such a $\mathcal{T}$, there is only one such $\mathcal{S}$ with such a transition $\mathcal{S} \to \mathcal{T}$.

5.2.2. *Case 2.* Suppose that the rightmost entry of the jth row of $\mathcal{S}$ is a superfluous 1. Then we define a new tableau $\mathcal{T}$ by deleting that 1 from $\mathcal{S}$. See Figure 7.

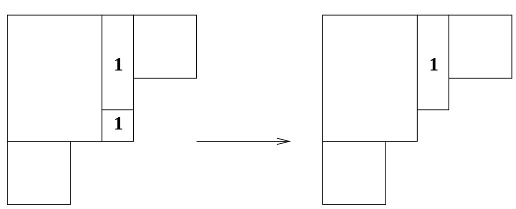


Figure 7

We define $\text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{1}{N+1}$. It is easy to see that $\text{wt}(\mathcal{T}) = q^{-1}\,\text{wt}(\mathcal{S})$, and therefore $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{q\,\text{wt}(\mathcal{T})}{N+1}$.

5.2.3. *Case 3.* Suppose that the rightmost entry of the jth row of $\mathcal{S}$ is a necessary 1. Then we define a new tableau $\mathcal{T}$ by deleting the column containing the necessary 1 and adding a new column whose length is 1 less. That new column consists entirely of 0's except for a necessary 1 at the bottom, and it is inserted as far east a possible. See Figure 8.

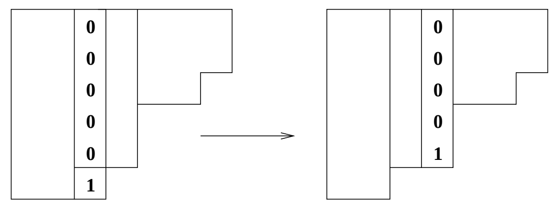


FIGURE 8

We define $\text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{1}{N+1}$. If the column in $\mathcal{S}$ containing the necessary 1 has length at least 2 then $\text{wt}(\mathcal{T}) = \text{wt}(\mathcal{S})$. Thus $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\text{wt}(\mathcal{T})}{N+1}$.

In the special case that the column in $\mathcal{S}$ containing the necessary 1 has length exactly 2 then $\text{wt}(\mathcal{T}) = \alpha^{-1}\,\text{wt}(\mathcal{S})$. Thus $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\alpha\,\text{wt}(\mathcal{T})}{N+1}$.

5.3. **Particle exits to the right.** If $\mathcal{S}$ contains a row of length 0 then there is a transition in PT from $\mathcal{S}$ that corresponds to a particle in $\text{pr}(\mathcal{S})$ exiting the PASEP to the right. We define a new tableau $\mathcal{T}$ by deleting the tth row of $\mathcal{S}$ (which has length 0) and adding a new column of length $t-1$ which consists of $t-2$ 0's followed by a 1 (read top-to-bottom), inserting this column into the tableau as far to the right as possible. See Figure 9.

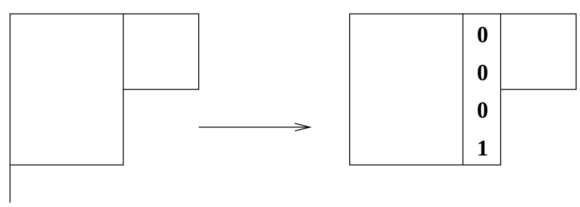


FIGURE 9

We define $\text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\beta}{N+1}$. It is easy to see that $\text{wt}(\mathcal{T}) = \beta\,\text{wt}(\mathcal{S})$. Thus $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\text{wt}(\mathcal{T})}{N+1}$.

5.4. **Particle hops left.** If some row $\lambda_j > \lambda_{j+1}$ in $\mathcal{S}$ then there is a transition in PT from $\mathcal{S}$ that corresponds to the jth black particle in $\text{pr}(\mathcal{S})$ hopping to the left in the PASEP. We define a new tableau $\mathcal{T}$ by increasing the length of the $(j+1)$st row by 1 and filling the extra square with a 1. See Figure 10.

We define $\text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{q}{N+1}$. It is easy to see that $\text{wt}(\mathcal{T}) = q\,\text{wt}(\mathcal{S})$. Thus $\text{wt}(\mathcal{S}) \cdot \text{prob}(\mathcal{S} \to \mathcal{T}) = \frac{\text{wt}(\mathcal{T})}{N+1}$.

Theorem 5.3. *Consider the PT chain on permutation tableaux of half-perimeter $N+1$ and fix a permutation tableau $\mathcal{T}$ (of half-perimeter $N+1$). Then the steady state probability of finding the PT chain in state $\mathcal{T}$ is $\frac{\text{wt}(\mathcal{T})}{\sum_{\mathcal{S}} \text{wt}(\mathcal{S})}$. Here, the sum is over all permutation tableaux of half-perimeter $N+1$.*

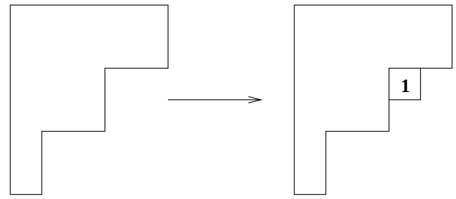


FIGURE 10

Since the PT chain projects to the PASEP in the sense of Remark 5.2, it is clear that Theorem 5.3 implies Theorem 3.3.

6. The involution

There is an involution I on permutation tableaux which generalizes the particle-hole symmetry of the PASEP, and reveals a symmetry in the PT chain. This is depicted as a reflective symmetry from left to right in Figure 4. As before, we will use both $\overline{\mathcal{T}}$ and $I(\mathcal{T})$ to denote the image of a tableau $\mathcal{T}$ under the involution.

More concretely, we have the following result.

Theorem 6.1. *There exists an involution I on the set of permutation tableaux of half-perimeter $N+1$, which has the following properties:*

(1) $\mathrm{pr}(\mathcal{T}) = \overline{\mathrm{pr}(\overline{\mathcal{T}})}$. *In other words,* $\mathrm{pr}(\mathcal{T})$ *and* $\mathrm{pr}(\overline{\mathcal{T}})$ *are related via the particle-hole symmetry.*
(2) $\mathrm{rk}(\mathcal{T}) = \mathrm{rk}(\overline{\mathcal{T}})$.
(3) $u(\mathcal{T}) = f(\overline{\mathcal{T}})$.
(4) $f(\mathcal{T}) = u(\overline{\mathcal{T}})$.

Recall that $\mathrm{wt}^{q,\alpha,\beta}(\mathcal{T}) = q^{rk(\mathcal{T})}\alpha^{-f(\mathcal{T})}\beta^{-u(\mathcal{T})}$. Theorem 6.1 and Theorem 5.3 then immediately imply the following result.

Corollary 6.2. *Conider the PT chain on permutation tableaux of half-perimeter $N+1$ and fix a permutation tableaux $\mathcal{T}$ (of half-perimeter $N+1$). Then* $\mathrm{wt}(\mathcal{T}, q, \alpha, \beta) = \mathrm{wt}(\overline{\mathcal{T}}, q, \beta, \alpha)$. *Moreover, the steady state probability of finding the PT chain in state $\mathcal{T}$ is equal to the steady state probability of finding the PT chain in state $\overline{\mathcal{T}}$.*

This can be seen as an extension of the particle-hole symmetry that was mentioned in Remark 2.3. Indeed our Theorem 3.1 states that the probability to be in state τ is the (normalized) weight-generating function $\frac{F_{\lambda(\tau)}(q)}{Z_N}$ for all permutation tableaux of shape $\lambda(\tau)$. Corollary 6.2 immediately implies that $F_{\lambda(\tau)}(q) = F_{\lambda(\overline{\tau})}(q)$.

Remark 6.3. *Philippe Duchon has kindly informed us that he independently discovered such an involution* [9].

Additionally, the following result reveals a symmetry in the state diagram of the PT chain.

Theorem 6.4. *There is a transition in the PT chain from $\mathcal{T}$ to $\mathcal{U}$ if and only if there is a transition from $\overline{\mathcal{T}}$ to $\overline{\mathcal{U}}$. Furthermore, the transition probabilities are related as follows:*

(1) $\mathrm{prob}(\mathcal{T} \to \mathcal{U}) = \frac{\alpha}{N+1}$ *if and only if* $\mathrm{prob}(\overline{\mathcal{T}} \to \overline{\mathcal{U}}) = \frac{\beta}{N+1}$.
(2) $\mathrm{prob}(\mathcal{T} \to \mathcal{U}) = \frac{\beta}{N+1}$ *if and only if* $\mathrm{prob}(\overline{\mathcal{T}} \to \overline{\mathcal{U}}) = \frac{\alpha}{N+1}$.

(3) $\mathrm{prob}(\mathcal{T} \to \mathcal{U}) = \frac{1}{N+1}$ *if and only if* $\mathrm{prob}(\overline{\mathcal{T}} \to \overline{\mathcal{U}}) = \frac{1}{N+1}$.
(4) $\mathrm{prob}(\mathcal{T} \to \mathcal{U}) = \frac{q}{N+1}$ *if and only if* $\mathrm{prob}(\overline{\mathcal{T}} \to \overline{\mathcal{U}}) = \frac{q}{N+1}$.

6.1. **Notation.** Recall the definitions of restricted 0's and rows, and of topmost 1's from Section 3. For each restricted row, the *rightmost restricted zero* is the restricted zero that has no restricted zeros to its right.

Let $\mathcal{T}$ be a permutation tableau with K rows and $N+1-K$ columns and shape $\lambda = (\lambda_1, \ldots, \lambda_K)$. Numbering the rows from top to bottom and the columns from left to right, let $\mathcal{T}(i,j)$ denote the filling of the cell (i,j) of $\mathcal{T}$. The *conjugate* of $\mathcal{T}$, which we shall denote by $\mathcal{T}'$, is the tableau of shape λ' such that $\mathcal{T}'(i,j) = \mathcal{T}(j,i)$ for all i,j. Here $\lambda' = (\lambda'_1, \ldots, \lambda'_{N+1-K})$ is the conjugate partition, i.e. the partition formed by the *columns* of λ.

If $a \in \{0,1\}$, let a^c denote $1-a$.

1	0	0	0	1	1	1
1	0	0	1	1		
0	0	0				
0	1	1				

FIGURE 11. The permutation tableau $\mathcal{T}$

6.2. **The map.** Let $\overline{\mathcal{T}}$ be the tableau of shape $(K-1, \lambda'_1-1, \lambda'_2-1, \ldots, \lambda'_{N+1-K}-1)$ whose entries are as follows.

(1) $\overline{\mathcal{T}}(1,j) = 1$ if row $j+1$ of $\mathcal{T}$ is unrestricted and 0 otherwise for $1 \le j \le K-1$.
(2) $\overline{\mathcal{T}}(i,j) = \mathcal{T}(j+1,i-1)^c$ if cell $(j+1,i-1)$ of $\mathcal{T}$ contains a topmost one or a rightmost restricted zero, and $\mathcal{T}(j+1,i-1)$ otherwise.

Proposition 6.5. *I is an involution.*

7. The PT chain as a Markov chain on permutations

There is a bijection Φ between the set of permutation tableaux of half-perimeter n and the permutations in S_n [13], which translates various statistics on permutation tableaux into statistics on permutations. This bijection allows us to interpret the PT chain as a Markov chain on permutations [4]. Additionally, we may describe the involution I from Section 6 in terms of permutations.

Theorem 7.1. *Let $\mathcal{T}$ be a permutation tableau of half-perimeter $N+1$. Denote $\Phi(\mathcal{T})$ by $\pi = (\pi(1), \ldots, \pi(N+1))$ and $\Phi(\overline{\mathcal{T}})$ by $\overline{\pi} = (\overline{\pi}(1), \ldots, \overline{\pi}(N+1))$. Then*

$$\begin{aligned} \overline{\pi}(1) &= N+2-\pi(1) \\ \overline{\pi}(i) &= N+2-\pi(N+3-i), \quad 2 \le i \le N+1. \end{aligned}$$

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