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Clusters, noncrossing partitions and the Coxeter plane

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Abstract. The present research is a contribution to W-Catalan combinatorics, which concerns generalizations of objects counted by the Catalan number to an arbitrary finite Coxeter group W. Triangulations of a polygon are a special case of the combinatorial clusters appearing in the theory of cluster algebras of finite type. Planar diagrams analogous to triangulations encode the combinatorics of clusters in the classical types, but no planar models (or other combinatorial models) are known for clusters associated to exceptional finite Coxeter groups. The usual noncrossing partitions can be seen as encoding the combinatorics of a certain nonstandard presentation of the symmetric group, and noncrossing partitions for any finite Coxeter group can be defined by an analogous construction. However, planar diagrams for these generalized noncrossing partitions are unknown outside of the classical types. In this extended abstract, we report on work in progress to construct planar diagrams for clusters and noncrossing partitions of arbitrary finite type.

Contents

1. Introduction

The "classical" noncrossing partitions are set partitions Π of $[n] := \{1, 2, \ldots n\}$ with the noncrossing property: namely, when the elements of $[n]$ are placed in cyclic order on a circle, the convex hulls of the parts of Π are disjoint. Thus for example, every partition of [4] is noncrossing except 13|24, which is "crossing" because the line segment defined by $\{1, 3\}$ intersects the line segment defined by $\{2, 4\}$. The noncrossing partitions were first studied by Kreweras [**11**] in 1972 and since then their rich combinatorial and enumerative structure has been widely studied. (See for example the expository articles [**2**, Chapter 4] and [**13**].) Noncrossing partitions are counted by the Catalan number.

Recently, through the merger of lines of research in algebraic combinatorics [**1, 12**] and geometric group theory [**3, 4, 6**] it became apparent that the classical noncrossing partitions are a special case (the case $W = S_n$) of a construction valid for any finite Coxeter group W. The W-noncrossing partitions are counted by the W-Catalan number $\mathbf{Cat}(W)$, a number given by an elegant formula in terms of fundamental numerical invariants of W. In the case is of type A (i.e. $W = A_{n-1} = S_n$) the combinatorics of the W-noncrossing

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partitions are encoded by planar diagrams (the classical noncrossing partitions). When W is one of the other classical finite Coxeter groups (B_n or D_n) there also exist planar diagrams for W-noncrossing partitions.

Another set of objects (several hundred years "more classical") counted by the Catalan number has also recently been generalized to the context of finite Coxeter groups: triangulations of a convex polygon. This generalization springs from the theory of cluster algebras. Fomin and Zelevinsky [**9**] showed that a cluster algebra satisfies a certain finiteness condition if and only if the underlying combinatorics of the cluster algebra is governed by a finite Coxeter group W (or more precisely, a finite crystallographic root system associated to W). The combinatorial structure of the cluster algebra is a simplicial complex whose vertex set is a subset (the "almost positive roots") of the root system. The faces of the complex are sets of almost positive roots that are pairwise "compatible" in a certain sense. The complex is dual to a simple polytope called the generalized associahedron.

When W is the symmetric group, the almost positive roots are in bijection with the diagonals of a convex polygon. Two diagonals are compatible if they do not cross. Thus a maximal set of compatible roots (or "cluster") corresponds to a maximal set of noncrossing diagonals of a polygon. The latter is equivalent to a triangulation of the polygon. When W is a Coxeter group of type B_n or D_n, there are planar models (only slightly more complicated) which encode compatibility. Fomin and Zelevinsky [**9**] posed the problem of finding combinatorial models for compatibility for the remaining types.

Outside of the classical types (A, B and D), planar models for noncrossing partitions or for clusters have not previously been constructed. The presence of planar models in some types but not in others is the starting point of the research (in progress) described here. The aim is to provide answers to the following questions:

QUESTION 1. Do planar models exist for noncrossing partitions in finite Coxeter groups of non-classical type?

QUESTION 2. Do planar models exist for clusters of almost positive roots in finite Coxeter groups of non-classical type?

QUESTION 3. If the answer to Question 1 and/or Question 2 is "yes," why **planar** models?

QUESTION 4. Is there a uniform (i.e. not type-by-type) construction of such planar models?

QUESTION 5. For W of type A, B or D, the combinatorics of noncrossing partitions and of clusters is simple enough that planar models could be found using *ad hoc* constructions. On the other hand, planar models (or other combinatorial models) for the non-classical Coxeter groups have been elusive and perhaps don't exist. What difference between Coxeter groups of types A, B and D and Coxeter groups of other types explains this disparity?

To answer Question 1 in the affirmative for a particular finite Coxeter group, one would give a procedure for producing a planar diagram for each "partition" of type W and a combinatorial criterion for determining which partitions are crossing. In Section 2 we describe a uniform construction that produces a planar diagram for each partition for any finite irreducible W. Uniform criteria for detecting crossing partitions are not immediately clear: to handle even a relatively simple case like F_4, we currently have to resort to criteria which are both unwieldy and fail decisively in other groups.

To answer Question 2 in the affirmative would mean creating a planar model for all almost positive roots (analogous to the collection of all diagonals of a convex polygon) and a combinatorial rule describing when two roots are compatible (analogous to deciding whether two diagonals cross). In Section 4 we describe a uniform construction of such diagrams for any finite irreducible W. In this case, in contrast to the case of noncrossing partitions, the problem of finding a criterion seems likely to have a meaningful solution. For example, a fairly simple criterion describes compatibility in F_4, and initial investigations suggest that analogous criteria may work in the cases where the diagrams are more complex.

The definitions of clusters and noncrossing partitions involve the choice of a *Coxeter element* for W. This is an element of W which can be expressed as the product of some permutation of the set S of simple generators of W. The order of a Coxeter element is the *Coxeter number* h of W and the *exponents* of W are certain integers that can be read off from the eigenvalues of a Coxeter element. (These are well-defined because any two Coxeter elements are conjugate in W.) The combinatorics and geometry of Coxeter elements plays a critical (if sometimes hidden) role in the theory of finite Coxeter groups and indeed in most areas of mathematics where the underlying combinatorial datum is a finite Coxeter group or root system.

The properties of Coxeter elements are closely tied to a certain (2-dimensional) plane which we call the *Coxeter plane*. This plane was apparently first considered in generality by Coxeter [**8**]. A careful analysis of the Coxeter plane by Steinberg [**14**] provided the first uniform proofs of the key properties of Coxeter elements The central importance of the Coxeter element (and thus of the Coxeter plane) to the theory of finite Coxeter groups provides an answer to Question 3: Planar diagrams for noncrossing partitions (and less directly, the planar diagrams for clusters) "live" in the Coxeter plane.

One can not rule out the possibility that the assumptions of Question 5 are flawed. Perhaps there exist simple combinatorial models for general finite Coxeter groups. However, the present research suggests a plausible answer to Question 5 by putting the combinatorial models for types A, B and D into a common framework and then showing that the analogous constructions for most other types are considerably more complicated. To further explain the difference between classical and non-classical types, we consider the details of the combinatorial models in the classical types. As mentioned above, the noncrossing partitions for $W = S_n$ are the classical "noncrossing partitions of a cycle." As constructed in [**1**] and [**12**], the noncrossing partitions for W of type B_n are certain partitions of the set $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$ and the noncrossing partitions of type D_n are certain other partitions of $\pm[n]$.

The reader familiar with the classical Coxeter groups will recognize the sets $[n]$ and $\pm[n]$ as groundsets of the standard permutation representations of S_n, B_n and D_n. Namely S_n is the group of all permutations of $[n]$, while B_n is the group of permutations of $\pm[n]$ which are symmetric with respect to reversal of signs and D_n is the subgroup of B_n consisting of elements satisfying a certain evenness condition. Thus to answer Question 5 one presumably needs to explain in a Coxeter-theoretic sense what the sets $[n]$, $\pm[n]$ and $\pm[n]$ respectively have to to do with S_n, B_n and D_n. The explanation, while well known, is often obscured in purely combinatorial treatments of the classical Coxeter groups. The sets $[n]$, $\pm[n]$ and $\pm[n]$ respectively encode the smallest nontrivial orbits of the Coxeter groups S_n, B_n and D_n. More specifically, the usual reflection representation of S_n acts by permuting the standard basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}_n$, while B_n and D_n each permute the vectors $\pm e_1, \ldots, \pm e_n$. In each case, the associated permutation representation is faithful, because the permuted set of vectors spans the whole space. It appears that the reason that the classical Coxeter groups have simple planar models for clusters and noncrossing partitions is the same as the reason that the classical Coxeter groups have simple combinatorial models as permutation groups, namely because the classical groups have small nontrivial orbits.

For the purposes of planar diagrams, we will see that "small" should be read as "on the order of the Coxeter number h." This is because the essential construction is to take a W-orbit and project it orthogonally to the Coxeter plane P. Since the Coxeter element c is in particular an orthogonal transformation and fixes P, it commutes with projection to P, so that the projected orbit decomposes as the union of c-orbits in P. When the size of the W-orbit is on the order of h, the projected W-orbit consists of a single c-orbit (forming a regular polygon in P) and perhaps several points at the origin. In addition to the classical Coxeter groups, the groups of type H_3 and $I_2(m)$ also have small orbits and the constructions proposed here work just as simply in these types as in the classical types.

For W of type A, B or I_2, the smallest orbit is of size h, so the projection onto P consists of the vertices of an h-gon. The smallest orbit in type D_n or H_3 is of size $h + 2$, and the projection onto P consists of an h-gon and two points at the origin. Figure 1 shows, for each exceptional Coxeter group W, the projection of a smallest orbit of W onto P. In the cases where points in the projected orbit project to the origin, the multiplicity of points at the origin is indicated in the figure. No other points in the projected orbits have multiplicity.

Even if the current research leads to affirmative answers to Questions 1 and/or 2, the affirmative answers might in essence amount to a negative result about planar models for clusters and/or noncrossing partitions, for the following reason: In the cases (e.g. H_4 and E_8) where the smallest orbit of W is quite large relative to h, the planar diagrams produced will necessarily be extremely complicated. Since these complicated models are produced by the same method that gives simple models in the classical types, one is led to conclude that simple models probably don't exist for groups without small orbits. However, an affirmative answer to Question 4 could make even the complicated models useful. For example, one might hope to use a uniform model of noncrossing partitions to give a uniform proof (different from that in [**7**]) that the noncrossing partitions form a lattice under refinement order. (In the classical types, the lattice property is trivial once planar models have been established.) Similarly, a uniform combinatorial model of compatibility could

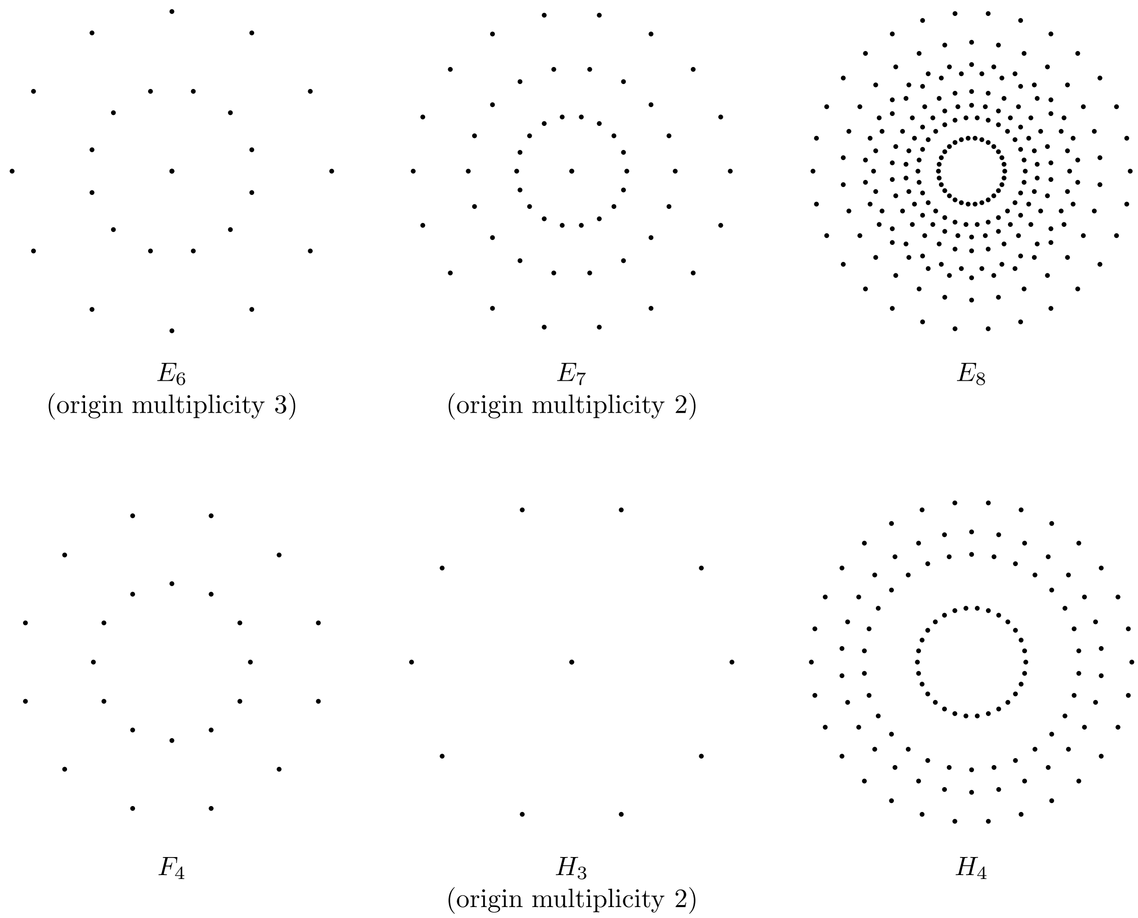

FIGURE 1. The smallest orbit of W, projected orthogonally to the Coxeter plane.

conceivably lead to uniform proofs of some of the fundamental facts about clusters (e.g. [**9**, Theorem 1.8]) which currently have only type-by-type proofs.

2. Diagrams for partitions

In this section, we discuss the analog of "partitions" associated to W and describe a procedure for constructing plane diagrams for partitions. Given a finite Coxeter group W with simple generators S, the *reflections* T of W are the elements of W which act as ortigonal reflections in the usual geometric representation of W. The reflections are exactly the conjugates of elements of S. For any $J \subseteq S$, the subgroup generated by J is called a *standard parabolic subgroup*. More generally, a *parabolic subgroup* of W is any subgroup conjugate to a standard parabolic subgroup. Any parabolic subgroup is in particular a *reflection subgroup* (a subgroup generated by reflections in W).

When $W = S_n$, every reflection subgroup is parabolic, but in other finite Coxeter groups there are non-parabolic reflection subgroups. The reflection subgroups of S_n are thus subgroups W' generated by transpositions. Such a subgroup W' decomposes the set $[n]$ into W'-orbits. This decomposition induces a bijection from parabolic subgroups of S_n to set partitions of $[n]$. The inverse map takes a set partition Π of $[n]$ and forms the group W' generated by transpositions $(i\ j)$ such that i and j are in the same part of Π.

For an arbitrary finite Coxeter group W, we take the parabolic subgroups as an analog of partitions. To give the parabolic subgroups of W a definite realization as set partitions, consider any non-trivial W-orbit o. Each parabolic subgroup W' decomposes o into W'-orbits. We will call this decomposition the W'-partition of o. Given a partition of o that is known to the be a W'-partition for some W', one can read off W' from the partition of o. For $W = S_n$ with the usual reflection representation, taking o to be the set $e_1, \ldots, e_n$, the W'-partitions of o correspond to the partitions of $[n]$.

We next explain the notion of crossing or noncrossing parabolic subgroups of W, quoting results from [**3, 6, 7**]. Any element $w \in W$ can be written as a word in the alphabet T of reflections of W. Since words in the alphabet S are so commonly used in the study of Coxeter groups, to avoid confusion we will refer a word in the alphabet T as a *T-word.* A *reduced* T-word for $w \in W$ is a T-word for w which has minimal length among all T-words for w. Fix a Coxeter element c and consider the set of words $t_1 \cdots t_i$ where $t_1 \cdots t_n$ varies over all reduced T-words for c and i varies from 0 to n. For each such word, the group $\langle t_1, \ldots, t_i \rangle$ is a parabolic subgroup of W. Two different such words give rise to the same parabolic subgroup if and only if the products of the two words are the same element of W. The elements of W arising as products $t_1 \cdots t_i$ in this manner are called *noncrossing partitions.* The parabolic subgroups arising in the manner are called *noncrossing parabolic subgroups*[1] (or simply *noncrossing parabolics*). All other parabolic subgroups are *crossing.*

If W is reducible as a direct product of nontrivial Coxeter groups, the noncrossing parabolics of W are exactly those parabolics whose intersection with each factor is a noncrossing parabolic in the factor. Thus it is harmless (and in fact essential in what follows) to restrict to the case where W is irreducible. Furthermore, since all Coxeter elements are conjugate, the exact choice of Coxeter element c in the above construction is not of critical importance. Changing Coxeter elements merely conjugates the whole set of noncrossing parabolics (and the set of noncrossing partitions) by some element of W. It will be useful to fix a particularly convenient Coxeter element. Let $S_+ \cup S_-$ be a bipartition of the Coxeter diagram for W. (This is possible because the diagram for W is a tree.) Define involutions

$$c_+ := \prod_{s \in S_+} s \quad \text{and} \quad c_- := \prod_{s \in S_-} s$$

so that $c := c_- c_+$ is a Coxeter element. Since the elements of S_+ commute pairwise (there are no edges between them in the diagram) the product c_+ is well-defined, and similarly c_- is well-defined.

We now describe a procedure which produces a plane diagram for each parabolic. This diagram is constructed in the *Coxeter plane* P. Details on P can be found in [**8**], in [**14**], in Sections 3.16–3.20 of [**10**] or in Section V.6.2 of [**5**]. The important properties of P are as follows: P is fixed as a set by the action of $\langle c_+, c_- \rangle$, and $\langle c_+, c_- \rangle$ acts on P as a dihedral reflection group of order $2h$. In P there are two lines L_+ and L_- with the following properties: A reflecting hyperplane H for a reflection in W intersects P in L_ε if and only if H is the reflecting hyperplane for some simple reflection $s \in S_\varepsilon$. The other reflecting hyperplanes intersect P in lines which are the images of $L_\pm$ under $\langle c_+, c_- \rangle$. In particular, every reflecting hyperplane is in the c-orbit of H_s for some $s \in S$. Equivalently, every reflection in W is in the orbit (under conjugation by c) of some simple reflection s.

Fix a W-orbit o. For each parabolic subgroup W', construct the W'-partition of o. A planar diagram for W' is obtained by projecting the W'-partition of o orthogonally to P. The diagram is a coloring of the projection of o to P. In some cases a point of P is allowed to have multiple colors: when the projection of o to P is not one-to-one, repeated points in the projected orbit are considered as multiple copies of the same point, and may have different colors. This problem can be minimized by taking o as small as possible, in which case the only point of the projected orbit having multiplicity is the origin. The multiplicity is at most 3. (See Figure 1.)

3. Diagrams for noncrossing partitions

Recall that the definition of planar diagrams requires the choice of a W-orbit o. In this section we consistently choose o to minimize $|o|$ over all nontrivial W-orbits and speak of "the" diagram of a parabolic W', meaning the diagram obtained for this smallest choice of o. The *parts* of the diagram are the projections to P of the parts of the W'-partition of o, with the understanding, as in the last paragraph of Section 2 that some points of P may belong to multiple parts. We continue to assume that W is irreducible.

In this section, we discuss criteria for deciding, based on the diagram, whether a parabolic subgroup is crossing or noncrossing. Given that a uniform criterion is still lacking, we necessarily discuss the criteria in a type-by-type manner. Furthermore, as the general theory cannot be applied without a uniform criterion,

[1] Arguably, it makes more sense to call the parabolic subgroups "noncrossing partitions" rather than the group elements, because in the motivating S_n case, the noncrossing partitions are parabolic subgroups, not group elements. Unfortunately, the use of the term "noncrossing partitions" to denote the group elements appears to have become generally accepted. We use the term "noncrossing parabolics" to avoid confusion.

we necessarily approach the problem in an empirical manner. We treat the diagrams as the primary object and treat the crossing/noncrossing definition as a "black box" which merely declares each diagram to be crossing or noncrossing, and attempt to explain the black box in terms of the combinatorics of the diagram. (We do, however take advantage of one item of general theory: the dihedral symmetry of $\langle c_+, c_- \rangle$ acting on parabolic subgroups by conjugation is known to preserve crossing or noncrossing status. Thus we may as well consider the diagrams up to this dihedral symmetry.) We discuss the classical cases and the groups of type $I_2(m)$ and H_3, which work out well because these groups have small orbits. We then discuss the difficulty of giving a reasonable criterion in the other exceptional types.

3.1. The classical types. In types A and B, one sees empirically that a parabolic subgroup is noncrossing if and only if the following criterion holds:

CRITERION 1. The convex hulls of any two parts are disjoint.

One can prove that this criterion correctly described the noncrossing property in types A and B by comparing this construction of planar diagrams with classical noncrossing partitions and with Reiner's construction [**12**] of noncrossing partitions of type B. In type D, one observes that a parabolic subgroup is noncrossing if and only if

CRITERION 2. The relative interiors of the convex hulls of any two parts are disjoint.

The validity of this criterion in type D can be proved by comparing this construction with Athanasiadis' and Reiner's construction [**1**]. The formally weaker Criterion 1 is equally valid in types A and B. We omit further details about the classical types.

3.2. The dihedral types. For a dihedral Coxeter group W of type $I_2(m)$, the parabolic subgroups are $\{1, t\}$ for each reflection t as well as the trivial subgroup $\{1\}$ and the entire group W. In this case, $h = m$ and the plane P is the plane on which W acts (so that projection to P is the identity map). The orbit o can be taken to be the vertices of a regular m-gon. For any finite Coxeter group W, the trivial subgroup, the subgroups $\{1, t\}$ for reflections T and the whole group W are noncrossing parabolics. Thus in type $I_2(m)$ we must declare every planar diagram to be noncrossing. This can be accomplished with either Criterion 1 or Criterion 2. The validity of either criterion is trivial for $W' = \{1\}$ and $W' = W$. In the remaining cases, either criterion is valid because any parabolic subgroup $\{1, t\}$ decomposes o into a collection of non-overlapping parallel line segments.

3.3. Type H_3. The Coxeter group W of type H_3 is the symmetry group of the icosahedron and of the dodecahedron. The smallest nontrivial orbit of W consists of the 12 vertices of the isocahedron. The Coxeter plane P is the plane that is a perpendicular bisector of the line segment connecting a pair of opposite vertices. This pair of opposite vertices projects to the origin in P and no other point in the projected orbit has multiplicity. The Coxeter element c acts by reflecting through P and rotating P by 1/10 of a turn. In particular, $h = 10$. The existence of an orbit of size not much larger than h makes the diagrams of parabolic subgroups fairly simple. In particular, the noncrossing property is described by Criterion 2. Figure 2 shows one representative of each c-orbit of noncrossing parabolics, omitting the parabolics $\{1\}$ and W. Call the points at the origin ± 0. If a diagram has a part shown in red (or the lighter of the two grays) then the red part contains -0 but not $+0$. The c-orbit of each noncrossing parabolic shown is of cardinality 5. To see that the cardinality is 5 and not 10 in some cases, it is necessary to recall that the action of c on the diagrams not only rotates by 1/10 of a turn, but also switches the two points at the origin. Figure 3 shows one representative of each c-orbit of crossing parabolics. In this figure, instead of drawing the convex hulls of the parts, we have drawn line segments indicating that two points are related by a reflection in the parabolic. The W'-partition of o is thus the closure of the relation shown by the line segments. The parts are also indicated by colors of vertices. The first three diagrams have both a red and a yellow vertex at the origin. The last diagram has two singleton parts at the origin (shown in black).

3.4. Type F_4. The Coxeter group W of type F_4 is the smallest example of a Coxeter group having no orbit of cardinality approximately h. A smallest orbit o of F_4 has 24 elements, while h is 12. The projection of o to P consists of two regular 12-gons centered at the origin as shown in Figure 1. The presence of two "rings" in the planar diagrams for parabolics causes problems for Criteria 1 and 2 which appear already in the case where the parabolic is generated by a single reflection. Recall from the discussion of $I_2(m)$ that any

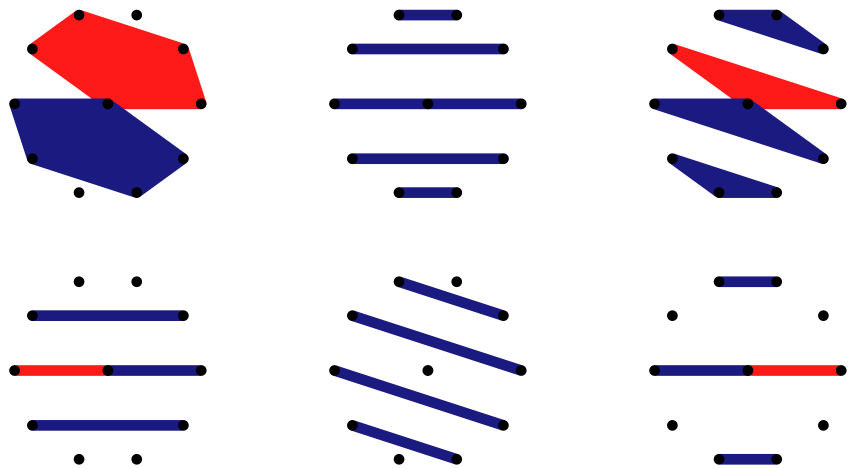

FIGURE 2. Noncrossing parabolic subgroups of H_3.

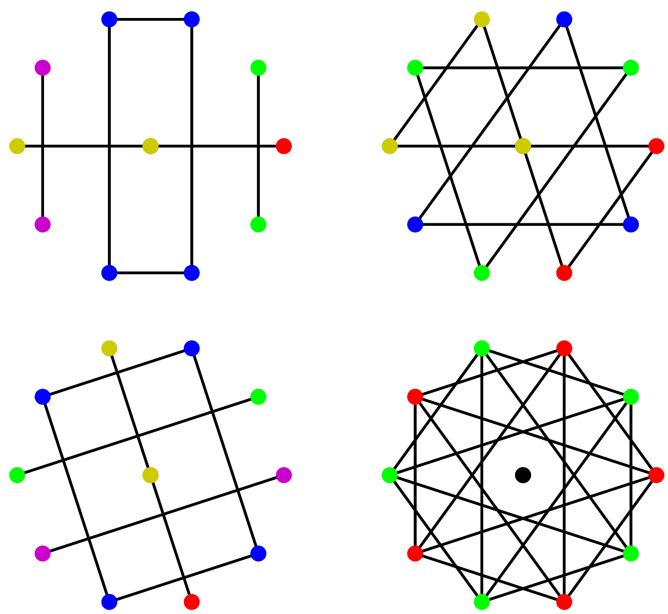

FIGURE 3. Crossing parabolic subgroups of H_3.

such parabolic is noncrossing. However, Figure 4 shows the diagram of such a parabolic which fails Criteria 1 and 2. The convex hulls of the parts are the connected pieces of the diagram, except as indicated by color.

The existence of reflections whose diagrams have parts with overlapping convex hulls means that most noncrossing parabolics will also have crossing diagrams. Thus one tries to find a criterion of the form "convex hulls of parts can't overlap very much." To illustrate the difficulty of defining such a criterion, observe for example that the criterion must declare Figure 5.a to be crossing while declaring Figure 5.b to be noncrossing. In this figure, as in Figure 3, line segments indicate that two points are related by a reflection in the parabolic and the parts are also indicated by colors of vertices.

A rather unsatisfying criterion can be devised to detect crossing parabolics in F_4. From there, one would hope that, as more complicated groups are considered, the sequence of criteria would "converge" to a criterion which is uniformly valid. Indeed, such convergence is essentially guaranteed by the fact that there are only finitely many groups remaining and only finitely many diagrams for each group. However, the prospects for convergence to a meaningful criterion are uncertain. To underscore the uncertainty, we point out that the unwieldy criterion for F_4 fails dramatically for small parabolics in H_4.

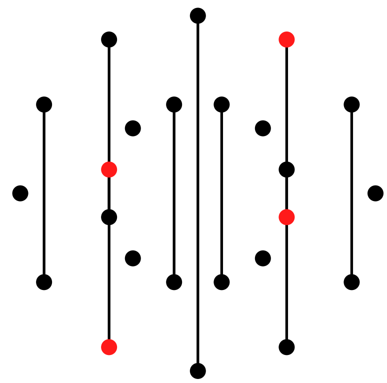

FIGURE 4. A (noncrossing) parabolic of rank 1 in F_4.

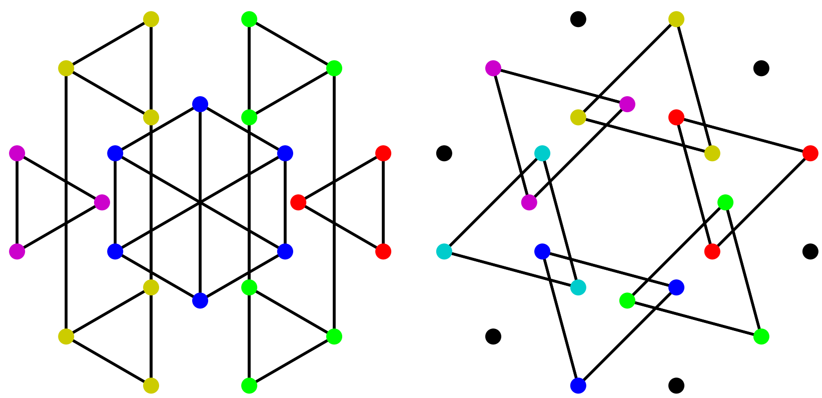

FIGURE 5. a: A crossing parabolic in F_4. b: A noncrossing parabolic in F_4.

4. Diagrams for clusters

Because of space limitations, we omit a detailed description of the construction of diagrams of clusters, in favor of a brief sketch which can be made precise. The main idea for the construction springs from the similarity, in the classical types, between diagrams for noncrossing partitions and diagrams for clusters. For example, in the case $W = S_n$, diagrams for noncrossing partitions involve an n-gon, while diagrams for clusters are triangulations of an $(n+2)$-gon. The "n" in this example is the Coxeter number h for S_n. Similarly, diagrams for noncrossing partitions in types B and D involve an h-gon, while diagrams for clusters live on an $(h+2)$-gon.

In the noncrossing case, it is "parts" of parabolic subgroups which may not "cross," while in the case of clusters, pairs of roots must be "compatible." For any W, the compatibility relation has a dihedral symmetry with a rotation of order $h+2$, while the noncrossing condition on parabolics has a dihedral symmetry with a reflection of order h.

With these ideas in mind, we sketch the construction in general terms. Start with the projection of the smallest orbit onto P. For each reflection t, a set of line segments (the diagram of the parabolic $\{1, t\}$) represents t. The h-gons in the diagram are then expanded into $(h+2)$-gons. This allows some additional "reflections" to fit into the picture, consistent with the fact that the set of almost positive roots is slightly bigger than the set of reflections. The question becomes how to translate the correspondence between positive roots (i.e. reflections) and collections of segments connecting points of the h-gons to a correspondence between almost positive roots and collections of segments connecting points of the $(h+2)$-gons. The properties of the Coxeter plane P, and in particular the description of c-orbits of reflections, provide the key.

In the classical types, we obtain diagrams with compatibility criteria equivalent to the constructions given by Fomin and Zelevinsky [**9**]. We now comment on a few of the remaining types.

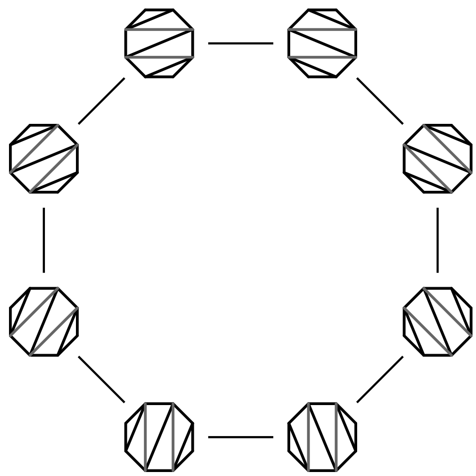

FIGURE 6. The G_2 associahedron.

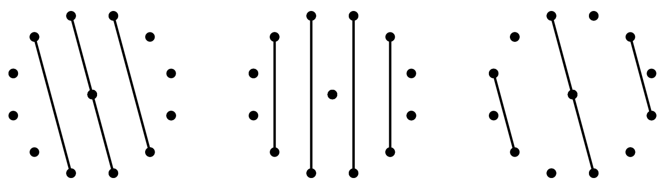

FIGURE 7. Individual roots in H_3.

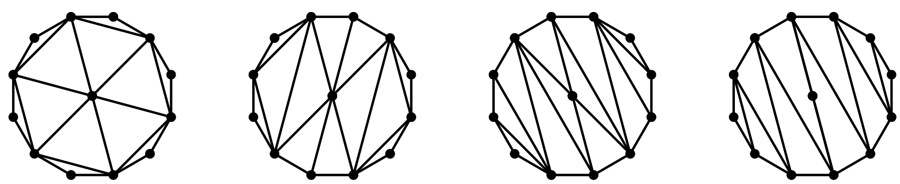

FIGURE 8. Clusters of type H_3.

4.1. The dihedral types. In this case, $h = m$ and W has an orbit of size m. The projected orbit is an m-gon, which is deformed to an $(m+2)$-gon for the purpose of cluster diagrams. There are $m+2$ almost positive roots, and each corresponds to a maximal set of parallel diagonals of the $(m+2)$-gon. Here the criterion for compatibility is that the diagonals involved may not cross. The clusters are shown in Figure 6 for the case $m = 2$. (This is the crystallographic type G_2.) The edges shown connecting clusters are the edges of the corresponding associahedron.

4.2. Type H_3. In this case, $h = 10$ and W has an orbit of size 12. The projected orbit is a 10-gon plus two points at the origin. This is deformed to a 12-gon plus two points at the origin. Again, each almost positive root is represented by a collection of parallel line-segments. Some of the segments may connect a vertex of the 12-gon to one of the points at the origin. However, no segments pass through the origin and for each almost positive root, at most one segment connects to each point at the origin. A representative set of almost positive roots is shown in Figure 7

The criterion for compatibility is that segments may not cross. However, segments representing different roots may coincide. A representative of each orbit of clusters is shown in Figure 8. In this figure, the edges of the polygon are also drawn.

4.3. Other types. Work is still in progress on the criteria for compatibility in the other types of finite Coxeter groups. The situation for clusters is much more promising than for noncrossing partitions: In the case of noncrossing partition, great difficulties arose as soon as (in F_4) the diagram had more than one "ring," and the difficulties appear to be compounded as additional rings are required. However, in the case of clusters, the interaction between the two rings in F_4 is not hard to deal with, and initial investigations of other types suggests that similar criteria will work in the presence of 3 or more rings. In seems likely that eventually a meaningful uniform statement can be made of the criterion for compatibility in planar diagrams for all types.

5. Acknowledgments

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References

[1] C. Athanasiadis and V. Reiner, *Noncrossing partitions for the group D_n.* SIAM J. Discrete Math **18** (2004), no. 2, 397–417.
[2] D. Armstrong, *Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups.* math.CO/0611106.
[3] D. Bessis, *The dual braid monoid.* Ann. Sci. École Norm. Sup. (4) **36** (2003) no. 5, 647–683.
[4] P. Biane, *Some properties of crossings and partitions.* Discrete Math. **175** (1997) no. 1-3, 41–53.
[5] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6.* Springer-Verlag, Berlin, 2002.
[6] T. Brady and C. Watt, *$K(\pi, 1)$'s for Artin groups of finite type.* Geom. Dedicata **94** (2002), 225–250.
[7] T. Brady and C. Watt, *Lattices in finite real reflection groups.* Trans. Amer. Math. Soc., to appear.
[8] H. Coxeter, *Regular Polytopes*, Methuen & Co., Ltd., London, 1948.
[9] S. Fomin and A. Zelevinsky, *Y-systems and generalized associahedra.* Ann. of Math. **158** (2003), 977–1018.
[10] J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, **29**, Cambridge Univ. Press, 1990.
[11] G. Kreweras, *Sur les partitions non croisées d'un cycle.* Discrete Math. **1** (1972) no. 4, 333–350.
[12] V. Reiner, *Non-crossing partitions for classical reflection groups.* Discrete Math. **177** (1997) no. 1-3, 195–222.
[13] R. Simion, *Noncrossing partitions.* Discrete Math. **217** (2000) no. 1-3, 367–409.
[14] R. Steinberg, *Finite reflection groups.* Trans. Amer. Math. Soc. **91** (1959) 493–504.

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