Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Transversal and cotransversal matroids via their representations.

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Abstract. It is known that the duals of transversal matroids are precisely the strict gammoids. We show that, by representing these two families of matroids geometrically, one obtains a simple proof of their duality.

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This note gives a new proof of the theorem, due to Ingleton and Piff, that the duals of transversal matroids are precisely the strict gammoids. Section 2 defines the relevant objects. Section 3 presents explicit representations of the families of transversal matroids and strict gammoids. Section 4 uses these representations to prove the duality of these two families.

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2.1. Matroids and duality. A *matroid* $M = (E, \mathcal{B})$ is a finite set E, together with a non-empty collection $\mathcal{B}$ of subsets of E, called the *bases* of M, which satisfy the following axiom: If B_1, B_2 are bases and e is in $B_1 - B_2$, there exists f in $B_2 - B_1$ such that $(B_1 - e) \cup f$ is a basis. If $M = (E, \mathcal{B})$ is a matroid, then $\mathcal{B}^* = \{E - B \,|\, B \in \mathcal{B}\}$ is also the collection of bases of a matroid $M^* = (E, \mathcal{B}^*)$, called the *dual* of M.

2.2. Representable matroids. Matroids provide a combinatorial abstraction of linear independence. If V is a set of vectors in a vector space and $\mathcal{B}$ is the collection of maximal linearly independent subsets of V, then $M = (V, \mathcal{B})$ is a matroid. Such a matroid is called *representable*; V is called a *representation* of M.

2.3. Transversal matroids. Let $A_1, \ldots, A_r$ be subsets of $[n] = \{1, \ldots, n\}$. A *transversal* (or *system of distinct representatives*) of $(A_1, \ldots, A_r)$ is a subset $\{e_1, \ldots, e_r\}$ of $[n]$ such that e_i is in A_i for each i. The transversals of $(A_1, \ldots, A_r)$ are the bases of a matroid on $[n]$.

Such a matroid is called a *transversal matroid*, and $(A_1, \ldots, A_r)$ is called a *presentation* of the matroid. This presentation can be encoded in the bipartite graph H with "top" vertex set $T = [n]$, "bottom" vertex set $B = \{\widehat{1}, \ldots, \widehat{r}\}$, and an edge joining j and $\widehat{i}$ whenever j is in A_i. The transversals are the r-sets in T that can be matched to B. We will denote this transversal matroid by $M[H]$.

2.4. Strict gammoids. Let G be a directed graph with vertex set $[n]$, and let $A = \{v_1, \ldots, v_r\}$ be a subset of $[n]$. We say that an r-subset B of $[n]$ *can be linked to* A if there exist r vertex-disjoint directed paths whose initial vertex is in B and whose final vertex is in A. Such a collection of r paths is called a *routing* from B to A. The collection of r-subsets which can be linked to A are the bases of a matroid denoted $L(G, A)$. We can assume that the vertices in A are sinks of G; that is, that there are no edges coming out of them. This is because the removal of those edges does not affect the matroid $L(G, A)$.

The matroids that arise in this way are called *strict gammoids* or *cotransversal matroids*. The purpose of this note is to give a new proof of the following theorem, due to Ingleton and Piff:

Theorem 2.1. [**3**] *Strict gammoids are precisely the duals of transversal matroids.*

2000 *Mathematics Subject Classification.* 05B35; 05C38; 05A99.
Key words and phrases. matroids, transversal, cotransversal, matchings, routings.
Supported by NSF grant DMS-9983797.

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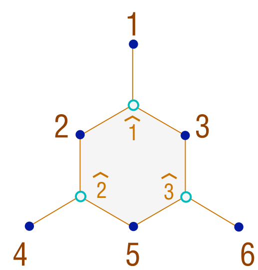
Let $\mathbb{K}$ be the field of fractions of the ring of formal power series in the indeterminates x_{ij} indexed by $1 \leq i \leq r$ and $1 \leq j \leq n$. We now show how transversal matroids and strict gammoids can be represented over $\mathbb{K}$.[1]

3.1. A representation of transversal matroids. Let M be a transversal matroid on the set $[n]$ with presentation $(A_1, \ldots, A_r)$. Let X be the $r \times n$ matrix whose (i, j) entry is $-x_{ij}$ if $j \in A_i$ and is 0 otherwise. The columns of X are a representation of M in the vector space $\mathbb{K}^r$. To see this, consider the columns $j_1, \ldots, j_r$. They are independent when their determinant is non-zero, and this happens as soon as one of the $r!$ summands of the determinant is non-zero. But $\pm X_{\sigma_1 j_1} \cdots X_{\sigma_r j_r}$ (where σ is a permutation of $[r]$) is non-zero if and only if $j_1 \in A_{\sigma_1}, \ldots, j_r \in A_{\sigma_r}$. So the determinant is non-zero if and only if $\{j_1, \ldots, j_r\}$ is a transversal.

We will find it convenient to choose a transversal $j_1 \in A_1, \ldots, j_r \in A_r$ at the outset, and normalize the rows to have the (i, j_i) entry be $-x_{ij_i} = 1$ for $1 \leq i \leq r$.

Example 1. Let $n = 6$ and $A_1 = \{1, 2, 3\}, A_2 = \{2, 4, 5\}, A_3 = \{3, 5, 6\}$. The corresponding bipartite graph H is shown below.



If we choose the transversal $1 \in A_1, 2 \in A_2, 3 \in A_3$, we obtain a representation for the transversal matroid $M[H]$, given by the columns of the following matrix:

$$X = \begin{pmatrix} 1 & -a & -b & 0 & 0 & 0 \\ 0 & 1 & 0 & -c & -d & 0 \\ 0 & 0 & 1 & 0 & -e & -f \end{pmatrix}$$

3.2. A representation of strict gammoids. Let $M = L(G, A)$ be a strict gammoid. Say G has vertex set $[n]$, and assume $A = \{r + 1, \ldots, n\}$. Any edge $i \to j$ of G has $i \leq r$, so we can assign to it weight x_{ij}. Define the weight of a path in G to be the product of the weights on its edges. For each vertex i of G and each sink a in A, let p_{ia} be the sum of the weights of all paths in G which start at vertex i and end at sink a. There may be infinitely many such paths, but the number of paths of a given weight is finite, so p_{ia} is a well-defined element of $\mathbb{K}$. [2]

Let Y be the $(n - r) \times n$ matrix whose (a, i) entry is p_{ia}. The columns of Y are a representation of M. To see this, recall the *Lindström-Gessel-Viennot theorem*, which states that the determinant of the matrix with columns $i_1, \ldots, i_{n-r}$ is equal to the signed sum[3] of the routings from $\{i_1, \ldots, i_{n-r}\}$ to A. It is clear that two routings cannot have the same weight, so this signed sum is non-zero if and only if it is non-empty; that is, if and only if $\{i_1, \ldots, i_{n-r}\}$ can be linked to A.

Example 2. Consider the directed graph G shown below, where all edges point down, and the set $A = \{4, 5, 6\}$.

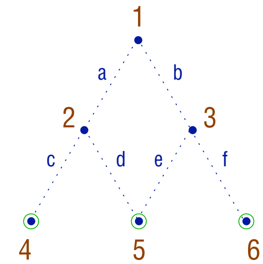
The representation we obtain for the strict gammoid $L(G, A)$ is given by the columns of the following matrix:

$$Y = \begin{pmatrix} ac & c & 0 & 1 & 0 & 0 \\ ad + be & d & e & 0 & 1 & 0 \\ bf & 0 & f & 0 & 0 & 1 \end{pmatrix}$$

[1]It is possible to carry out the same constructions over $\mathbb{R}$, but special care is required to handle the issue of convergence of the infinite sums that will arise.

[2]In fact, p_{ia} is a rational function in the x_{ij}s. For a proof, see [**10**, Theorem 4.7.2].

[3]The sign of a routing is determined by the permutation that matches the starting and ending points of its paths.



3.3. Representations of dual matroids. If a rank-r matroid M is represented by the columns of an $r \times n$ matrix A, we can think of M as being represented by the r-dimensional subspace $V = \text{rowspace}(A)$ in $\mathbb{K}^n$. The reason is that, if we consider any other $r \times n$ matrix A' with $V = \text{rowspace}(A')$, the columns of A' also represent M.

This point of view is very amenable to matroid duality. If M is represented by the r-dimensional subspace V of $\mathbb{K}^n$, then the dual matroid M^* is represented by the $(n-r)$-dimensional orthogonal complement $V^\perp$ of $\mathbb{K}^n$.

Notice that the rowspaces of the matrices X and Y in the examples above are orthogonally complementary. That is, essentially, the punchline of this story.

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4.1. Directed graphs with sinks and bipartite graphs with complete matchings. Given a directed graph G and a subset A of its set of sinks, we construct an undirected graph H as follows. We first split each vertex v not in A into a "top, incoming" vertex v and a "bottom, outgoing" vertex $\widehat{v}$, and draw an edge between them. Then we replace each edge $u \to v$ of G with an edge between the outgoing $\widehat{u}$ and the incoming v.

More concretely, given a directed graph G with vertex set V, and given a set A of sinks of G, we construct a bipartite graph H, together with a fixed bipartition and a fixed complete matching. The top vertex set in the bipartition is V, and the bottom vertex set is a copy $\widehat{V} - \widehat{A}$ of $V - A$. The complete matching is obtained by joining the top u and the bottom $\widehat{u}$ for each u in $V - A$. Then, for $u \neq v$, we join the bottom $\widehat{u}$ and the top v in H if and only if $u \to v$ is an edge of G.

Conversely, if we are given the bipartite graph H, a bipartition of H [4] and a complete matching of H, it is clear how to recover G and A. The resulting G and A will depend on which bipartition and matching are used. Observe that if we start with the directed graph G and sinks A of Example 2, we obtain the bipartite graph H of Example 1.

4.2. Proof of Theorem 2.1: Duality of transversal matroids and strict gammoids. Having laid the necessary groundwork, we are ready to present our proof of Theorem 2.1. We constructed a correspondence between a directed graph G with a specified subset A of its set of sinks, and a bipartite graph H with a specified bipartition and a specified complete matching. Now we show that, in this correspondence, the strict gammoid $L(G, A)$ is dual to the transversal matroid $M[H]$. We have constructed a subspace of $\mathbb{K}^n$ representing each one of them. By the remarks of Section 3.3, it suffices to show that these two subspaces are orthogonally complementary, as observed in Examples 1 and 2.

Our representation of $M[H]$ is given by the columns of the $r \times n$ matrix X whose (i,i) entry is 1, and whose (i,j) entry, for $i \neq j$, is $-x_{ij}$ if $i \to j$ is an edge of G and is 0 otherwise. Think of the x_{ij}s as weights on the edges of G. For a vector $y \in \mathbb{K}^n$, the ith entry of the column vector Xy is $y_i - \sum_{j \in N(i)} x_{ij} y_j$, where the sum is over the set $N(i)$ of vertices j such that $i \to j$ is an edge of G. It follows that y is in the nullspace of X when, for each vertex i of G,

$$y_i = \sum_{j \in N(i)} x_{ij} y_j.$$

[4]which is unique if H is connected

As before, let p_{ia} be the sum of the weights of the paths from i to a in G. Notice that

$$p_{ia} = \sum_{j \in N(i)} x_{ij} p_{ja},$$

since the left hand side enumerates all paths from i to a, and the right hand side enumerates the same paths by grouping them according to the first vertex j that they visit after i. Therefore $(p_{1a}, \ldots, p_{na})$, the ath row of our representation Y of $L(G, A)$, is in the nullspace of X. Since each row of Y is in the nullspace of X, rowspace$(Y) \subseteq$ nullspace(X). But

$$\begin{aligned} \dim(\text{rowspace}(Y)) &= \text{rank}(L(G, A)) = n - r, \text{ and} \\ \dim(\text{nullspace}(X)) &= n - \dim(\text{rowspace}(X)) = n - \text{rank}(M[H]) = n - r, \end{aligned}$$

so in fact these two subspaces are equal. It follows that rowspace(X) and rowspace(Y) are orthogonal complements. This completes our proof of Theorem 2.1. □

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For more information on matroid theory, Oxley's book [**8**] is a wonderful place to start. The representation of transversal matroids shown here is due to Mirsky and Perfect [**7**]. The representation of strict gammoids that we use was constructed by Mason [**6**] and further explained by Lindström [**5**][5]. The theorem that strict gammoids are precisely the cotransversal matroids is due to Ingleton and Piff [**3**]. Our proof of this result appears to be new.

This note is a small side project of [**1**]. While studying the combinatorics of generic flag arrangements and its implications on the Schubert calculus. We became interested in the strict gammoid of Example 2 and its representations, since we proved that it is the matroid of the arrangement of lines determined by intersecting three generic complete flags in $\mathbb{C}^3$. Similarly, the analogous strict gammoid in a triangular array of size n is the matroid of the line arrangement determined by three generic flags in $\mathbb{C}^n$.

I would like to thank Sara Billey for several helpful discussions, and Laci Lovász and Jim Oxley for help with the references. I also thank the referees for suggestions which improved the presentation.

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[5]It is in this context that he discovered what is now commonly known as the Lindström-Gessel-Viennot theorem [**2**]. This theorem was also discovered and used earlier by Karlin and MacGregor [**4**].