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An additive theorem related to Latin transversals

Zhi-Wei Sun

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
`http://math.nju.edu.cn/~zwsun`

Abstract

Let G be any additive abelian group with cyclic torsion subgroup, and let A, B and C be finite subsets of G with cardinality $n > 0$. We show that there is a numbering $\{a_i\}_{i=1}^n$ of the elements of A, a numbering $\{b_i\}_{i=1}^n$ of the elements of B and a numbering $\{c_i\}_{i=1}^n$ of the elements of C, such that all the sums $a_i + b_i + c_i$ $(1 \leqslant i \leqslant n)$ are distinct. Consequently, each subcube of the Latin cube formed by the Cayley addition table of $\mathbb{Z}/N\mathbb{Z}$ contains a Latin transversal. This additive theorem is an essential result which can be further extended via restricted sumsets in a field. The whole paper is available from `arXiv:math.CO/0610981` or the author's homepage.

In 1999 Snevily [Sn99] raised the following original conjecture in additive combinatorics.

Snevily's Conjecture. *Let G be an additive abelian group with $|G|$ odd. Let A and B be subsets of G with cardinality $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then there is a numbering $\{a_i\}_{i=1}^n$ of the elements of A and a numbering $\{b_i\}_{i=1}^n$ of the elements of B such that the sums $a_1 + b_1, \ldots, a_n + b_n$ are distinct.*

In our opinion, Snevily's conjecture belongs to the central part of combinatorial number theory due to its simplicity and beauty.

When $|G|$ is an odd prime, this conjecture was proved by Alon [A00] via the polynomial method (cf. Alon [A99], and Tao and Vu [TV06, pp. 329-345]). In 2001 Dasgupta, Károlyi, Serra and Szegedy [DKSS] confirmed Snevily's conjecture for any cyclic group of odd order. In 2003 Sun [Su03] obtained some further extensions of the Dasgupta-Károlyi-Serra-Szegedy result via restricted sums in a field.

In Snevily's conjecture the abelian group is required to have odd order. For a general abelian group G with cyclic torsion subgroup, what additive properties can we impose on several subsets of G with cardinality n? In this direction we establish the following new theorem of additive nature.

Theorem 1.1. *Let G be any additive abelian group with cyclic torsion subgroup, and let $A_1, \ldots, A_m$ be arbitrary subsets of G with cardinality $n \in \mathbb{Z}^+$, where m is odd. Then the elements of A_i $(1 \leqslant i \leqslant m)$ can be listed in a suitable order $a_{i1}, \ldots, a_{in}$, so that all the sums $\sum_{i=1}^m a_{ij}$ $(1 \leqslant j \leqslant n)$ are distinct. In other words, for a certain subset A_{m+1} of G*

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with $|A_{m+1}| = n$, there is a matrix $(a_{ij})_{1\leqslant i\leqslant m+1,\,1\leqslant j\leqslant n}$ such that $\{a_{i1},\ldots,a_{in}\} = A_i$ for all $i = 1,\ldots,m+1$ and the column sum $\sum_{i=1}^{m+1} a_{ij}$ vanishes for every $j = 1,\ldots,n$.

Remark 1.1. Theorem 1.1 in the case $m = 3$ is essential; the result for $m = 5, 7, \ldots$ can be obtained by repeated use of the case $m = 3$. The group G in Theorem 1.1 cannot be replaced by an arbitrary abelian group, there are counter-examples even for the Klein quaternion group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Recall that a line of an $n \times n$ matrix is a row or column of the matrix. We define a line of an $n \times n \times n$ cube in a similar way. A *Latin cube* over a set S of cardinality n is an $n \times n \times n$ cube whose entries come from the set S and no line of which contains a repeated element. A *transversal* of an $n \times n \times n$ cube is a collection of n cells no two of which lie in the same line. A *Latin transversal* of a cube is a transversal whose cells contain no repeated element.

Corollary 1.1. *Let N be any positive integer. For the $N \times N \times N$ Latin cube over $\mathbb{Z}/N\mathbb{Z}$ formed by the Cayley addition table, each $n \times n \times n$ subcube with $n \leqslant N$ contains a Latin transversal.*

In 1967 H. J. Ryser [R] conjectured that every Latin square of odd order has a Latin transversal. Another conjecture of Brualdi (cf. [D], [DK, p. 103] and [EHNS]) states that every Latin square of order n has a partial Latin transversal of size $n - 1$. These and Corollary 1.1 suggest that our following conjecture might be reasonable.

Conjecture 1.1. *Every $n \times n \times n$ Latin cube contains a Latin transversal.*

Note that Conjecture 1.1 does not imply Theorem 1.1 since an $n \times n \times n$ subcube of a Latin cube might have more than n distinct entries.

In Theorem 1.1 the condition $2 \nmid m$ is indispensable. Let G be an additive cyclic group of even order n. Then G has a unique element g of order 2 and hence $a \neq -a$ for all $a \in G \setminus \{0, g\}$. Thus $\sum_{a\in G} a = 0 + g = g$. For each $i = 1,\ldots,m$ let $a_{i1},\ldots,a_{in}$ be a list of the n elements of G. If those $\sum_{i=1}^{m} a_{ij}$ with $1 \leqslant j \leqslant n$ are distinct, then

$$\sum_{a\in G} a = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} = m\sum_{a\in G} a,$$

hence $(m-1)g = (m-1)\sum_{a\in G} a = 0$ and therefore m is odd.

Combining Theorem 1.1 with [Su03, Theorem 1.1(ii)], we obtain the following consequence.

Corollary 1.2. *Let G be any additive abelian group with cyclic torsion subgroup, and let $A_1,\ldots,A_m$ be subsets of G with cardinality $n \in \mathbb{Z}^+$, where m is even. Suppose that all the elements of A_m have odd order. Then the elements of A_i $(1 \leqslant i \leqslant m)$ can be listed in a suitable order $a_{i1},\ldots,a_{in}$, so that all the sums $\sum_{i=1}^{m} a_{ij}$ $(1 \leqslant j \leqslant n)$ are distinct.*

As an essential result, Theorem 1.1 might have various potential applications in additive number theory and combinatorial designs. We can also extend Theorem 1.1 via restricted sumsets in a field.

We can prove Theorem 1.1 in two ways. A direct proof involves the following lemma.

Lemma 1.1. *Let R be a commutative ring with identity, and let $a_{ij} \in R$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$, where $m \in \{3, 5, \ldots\}$. The we have the identity*

$$\begin{aligned}\sum_{\sigma_1,\ldots,\sigma_{m-1}\in S_n} \operatorname{sign}(\sigma_1\cdots\sigma_{m-1}) \prod_{1\leqslant i<j\leqslant n} \left(a_{mj}\prod_{s=1}^{m-1} a_{s\sigma_s(j)} - a_{mi}\prod_{s=1}^{m-1} a_{s\sigma_s(i)}\right) \\ = \prod_{1\leqslant i<j\leqslant n} (a_{1j} - a_{1i})\cdots(a_{mj} - a_{mi}),\end{aligned}$$

where S_n *denotes the symmetric group of all permutation on* $\{1,\ldots,n\}$*, and* $\mathrm{sign}(\sigma)$ *takes* 1 *or* -1 *according as* $\sigma \in S_n$ *is even or odd.*

Another proof of Theorem 1.1 makes use of the following powerful tool.

Combinatorial Nullstellensatz [A99]**.** *Let* $A_1,\ldots,A_n$ *be finite subsets of a field* F *with* $|A_i| > k_i \geqslant 0$ *for* $i = 1,\ldots,n$*. If the total degree of* $f(x_1,\ldots,x_n) \in F[x_1,\ldots,x_n]$ *is* $k_1+\cdots+k_n$ *and the coefficient of the monomial* $x_1^{k_1}\cdots x_n^{k_n}$ *in* $f(x_1,\ldots,x_n)$ *is nonzero, then* $f(a_1,\ldots,a_n) \neq 0$ *for some* $a_1 \in A_1,\ldots,a_n \in A_n$*.*

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