Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Tabloids and Weighted Sums of Characters of Certain Modules of the Symmetric Groups

Yasuhide Numata

Abstract. We consider certain modules of the symmetric groups whose basis elements are called tabloids. As modules of the symmetric groups, some of these are isomorphic to Springer modules. We give a combinatorial description for weighted sums of their characters; we introduce combinatorial objects called $(\rho, \boldsymbol{l})$-tableaux, and rewrite weighted sums of characters as the numbers of these combinatorial objects. We also consider the meaning of these combinatorial objects; we construct a correspondence between $(\rho, \boldsymbol{l})$-tableaux and tabloids whose images are eigenvectors of the action of an element of cycle type ρ in quotient modules.

Résumé. Nous considérons certains modules des groupes symétriques dont les éléments de base s'appelés les tabloïds. Comme modules des groupes symétriques, quelques-uns de ceux-ci sont isomorphes aux modules de Springer. Nous donnons une desctiption combinatoire pour somme pondérés de leir caractères ; nous introduisons des objets combinatoires appelés $(\rho, \boldsymbol{l})$-tabloïds, et récrivon des somme pondérés des caractéres comme les nombres de ces objets combinatoires. Nous considérons la signification de ces objets combinatoires ; nous construisons une correspondance entre $(\rho, \boldsymbol{l})$-tabloïds et taloïds dont les images sont des vecteurs propres de l'action d'un élément de type ρ du cycle dans les modules du quotient.

1. Introduction

Let W be a finite group. In some $\mathbb{Z}$-graded W-modules $R = \bigoplus_d R^d$, we have a phenomenon called "coincidence of dimensions" ([**5, 6, 7, 8**]), i.e., some integers l satisfy the equations

$$\dim \bigoplus_{i \in \mathbb{Z}} R^{il+k} = \dim \bigoplus_{i \in \mathbb{Z}} R^{il+k'}$$

for all k and k'. Induced modules give a proof of the phenomenon. More precisely, let a subgroup $H(l)$ of W and $H(l)$-modules $z(k;l)$ satisfy

$$\bigoplus_{i \in \mathbb{Z}} R^{il+k} \simeq \operatorname{Ind}_{H(l)}^{W} z(k;l), \qquad \dim z(k;l) = \dim z(k';l)$$

for all k and k', where $\operatorname{Ind}_{H(l)}^{W} z(k;l)$ denotes the induced module. Since

$$\dim \bigoplus_{i \in \mathbb{Z}} R^{il+k} = \dim \operatorname{Ind}_{H(l)}^{W} z(k;l) = |W/H(l)| \cdot \dim z(k;l),$$

we can prove the phenomenon by the datum $(H(l), \{ z(k;l) \})$.

We consider the case where W is the m-th symmetric group S_m and R are the S_m-modules R_μ called Springer modules. The Springer modules R_μ are graded algebras parametrized by partitions $\mu \vdash m$. As S_m-modules, R_μ are isomorphic to cohomology rings of the variety of the flags fixed by a unipotent matrix with Jordan blocks of type μ. (See [**2, 9, 10**]. See also [**1, 11**] for algebraic construction.) In [**6**], Morita and Nakajima showed coincidences of dimensions for the Springer modules R_μ. We recall the case where μ is an l-partition, where an l-partition means a partition whose multiplicities are divisible by l. Let $R_\mu(k;l)$ denote

2000 *Mathematics Subject Classification.* Primary 05E10; Secondary 20C30;
Key words and phrases. Tabloids; Springer modules; Green polynomials.

the submodule $\bigoplus_{i\in\mathbb{Z}} R_\mu^{il+k}$ of the Springer module R_μ. In this case, we have $\dim R_\mu(k;l) = \dim R_\mu(k';l)$ for all k, i.e., R_μ has a coincidence of dimensions. Let $H_\mu(l)$ be the semi-direct product $S_\mu \rtimes C_{\mu,l}$ of the Young subgroup S_μ and an l-th cyclic group $C_{\mu,l} = \langle a_{\mu,l}\rangle$. (See [**7**] for the definition of $a_{\mu,l}$) For $k \in \mathbb{Z}$, let $Z_\mu(k;l) : H_\mu(l) \to \mathbb{C}^\times$ denote one-dimensional representations of $H_\mu(l)$ that maps $a_{\mu,l}$ to ζ_l^k and $\sigma \in S_m$ to 1, where ζ_l denotes a primitive l-th root of unity. Then, for $(H_\mu(l), \{ Z_\mu(k;l) \})$, $R_\mu(k;l) \simeq \operatorname{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k;l)$ for all k. To prove it, Morita and Nakajima [**7**] described the values of the Green polynomials at roots of unity, and showed that the characters of the submodules $R_\mu(k;l)$ coincide with those of the induced modules $\operatorname{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k;l)$. These special values of the Green polynomials are nonnegative integers. (See also [**3, 4, 7**].)

Our first motivation for this paper is to describe these nonnegative values of the Green polynomials as numbers of some combinatorial objects. Our second motivation is to give a meaning of the combinatorial objects in terms of modules $\operatorname{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k;l)$ in Morita-Nakajima [**7**]. For these purposes, we introduce some S_m-modules, which are realizations of $\operatorname{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k;l)$ for special parameters, and give a combinatorial description for weighted sums of their characters.

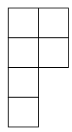
In Section 2, we introduce S_m-modules $M^{\boldsymbol{\mu}}$ and their quotient modules $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ for some n-tuples $\boldsymbol{\mu}$ of Young diagrams. When $n = 1$, this module $M^{(\mu)}(k;(l))$ is a realization of $\operatorname{Ind}_{H_\mu(l)}^{S_m} Z_\mu(k;l)$ in [**7**]. We also introduce combinatorial objects called marked $(\rho,\boldsymbol{l})$-tableaux to describe weighted sums of characters of $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$. When $n = 1$, the number of marked $(\rho,(l))$-tableaux coincides with the right hand side of the explicit formula (3.1) of Green polynomials in [**7**]. Our main result is the description of a weighted sum

$$\sum_{k\in\mathbb{Z}/l\mathbb{Z}} \zeta_l^{jk} \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k;\boldsymbol{l})\right)(\sigma)$$

of characters of $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ as the number of marked $(\rho,\boldsymbol{\gamma})$-tableaux on $\boldsymbol{\mu}$ for the primitive l-th root ζ_l of unity and $\sigma \in S_m$ of cycle type ρ in Section 3. We prove the main result in Section 4 by constructing bijections.

2. Notation and Definition

We identify a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ of m with its Young diagram $\{ (i,j) \in \mathbb{N}^2 \mid 1 \leq j \leq \mu_i \}$ with m boxes. If μ is a Young diagram with m boxes, we write $\mu \vdash m$ and identify a Young diagram μ with the array of m boxes having left-justified rows with the i-th row containing μ_i boxes; for example,

$(2,2,1,1) =$  $\vdash 5.$

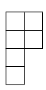
For an integer l, a Young diagram μ is called an *l-partition* if multiplicities $m_i = |\{ k \mid \mu_k = i \}|$ of i are divisible by l for all i. For example, $(2,2,1,1)$ is a 2-partition.

Let μ be a Young diagram with m boxes. We call a map T a *numbering on μ with* $\{1,\ldots,n\}$ if T is an injection $\mu \ni (i,j) \mapsto T_{i,j} \in \{ 1,\ldots,n \}$. We identify a map $T : \mu \to \mathbb{N}$ with a diagram putting $T_{i,j}$ in each box in the (i,j) position; for example,

2	3
6	1
5	
7	

.

For $\mu \vdash m$, t_μ denotes the numbering which maps $(t_\mu)_{i,j} = j + \sum_{k=0}^{i-1} \mu_k$; i.e., the numbering obtained by putting numbers from 1 to m on the boxes of μ from left to right in each row, starting in the top row and moving to the bottom row. For example,

t  $=$

1	2
3	4
5	
6	

.

Two numberings T and T' on $\mu \vdash m$ are said to be *row-equivalent* if their corresponding rows consist of the same numbers. We call a row-equivalence class $\{T\}$ a *tabloid*.

Let T be a numbering on a Young diagram $\mu \vdash m$ with $\{1, \dots, n\}$. Then $\sigma \in S_n$ acts on T from the left by $(\sigma T)_{i,j} = \sigma(T_{i,j})$. For example,

$$(1,2,3,4)\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

This left action induces a left action on tabloids by $\sigma\{T\} = \{\sigma T\}$.

For a numbering T on $\mu \vdash m$ with $\{1, \dots, n\}$, we define S_T to be the subgroup

$$S_{\{\, T_{1,1}, T_{1,2}, \dots, T_{1,\mu_1} \,\}} \times S_{\{\, T_{2,1}, T_{2,2}, \dots, T_{2,\mu_2} \,\}} \times \cdots$$

of the n-th symmetric group S_n, where $S_{\{\, i_1, \dots, i_k \,\}}$ denotes the symmetric group of the letters $\{\, i_1, \dots, i_k \,\}$. It is obvious that S_T and the Young subgroup S_μ are isomorphic as groups for a numbering T on $\mu \vdash m$. It is also clear that $\sigma\{T\} = \{T\}$ for $\sigma \in S_T$.

For a numbering T on an l-partition $\mu \vdash m$, we define $a_{T,l}$ to be the product

$$\prod_{(li+1,j)\in\mu} (T_{li+1,j}, T_{li+2,j}, \dots, T_{li+l,j})$$

of m/l cyclic permutations of length l. For example, $a_{t_{(2,2,1,1)},2} = (13)(24)(56)$. We write $a_{\mu,l}$ for $a_{t_\mu,l}$.

Let μ be an l-partition of m and $\langle a_{\mu,l} \rangle$ the cyclic group of order l generated by $a_{\mu,l}$. For each numbering T on μ with $\{1, \dots, n\}$, there exists $\tau_T \in S_n$ such that $T = \tau_T t_\mu$. Since the map $\tau_T|_{\{\, 1,\dots,m \,\}}$ restricting τ_T to $\{\, 1, \dots, m \,\}$ is unique, $\sigma \in \langle a_{\mu,l} \rangle$ acts on T from right as $T\sigma = \tau_T \sigma t_\mu$. For each numbering T on an l-partition μ, the $(\overline{r} + lq)$-th row of $Ta_{\mu,l}$ is the $(\overline{r+1} + lq)$-th row of T, where $\overline{r}$ and $\overline{r+1}$ are in $\mathbb{Z}/l\mathbb{Z} = \{\, 1, \dots, l \,\}$. This right action also induces a right action on tabloids by $\{T\}\sigma = \{T\sigma\}$.

In this paper, we consider n-tuples of Young diagrams. Throughout this paper, let $\boldsymbol{m} = (m_1, m_2, \dots, m_n)$ and $\boldsymbol{l} = (l_1, l_2, \dots, l_n)$ be n-tuples of positive integers, m the sum $\sum_h m_h$, l the least common multiple of $\{\, l_i \,\}$, and ζ_k the primitive k-th root of unity. We call an n-tuple $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n)})$ of Young diagrams an *$\boldsymbol{l}$-partition of $\boldsymbol{m}$* if $\mu^{(h)}$ is an l_h-partition of m_h for each h. We identify an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ with the disjoint union $\coprod_h \mu^{(h)} = \{\, (i,j;h) \,|\, (i,j) \in \mu^{(h)} \,\}$ of Young diagrams $\mu^{(h)}$. We call an n-tuple $\boldsymbol{T} = (T^{(1)}, \dots, T^{(n)})$ of numberings $T^{(h)}$ on $\mu^{(h)}$ *a numbering on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$* if the map $\boldsymbol{T} : \boldsymbol{\mu} \ni (i,j;h) \mapsto T^{(h)}_{i,j} \in \{\, 1, \dots, m \,\}$ is bijective. For an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$, $\boldsymbol{t_\mu}$ denotes the n-tuple of the numberings $t^{(h)}_{\boldsymbol{\mu}}$ which maps $(i,j;h)$ to $(t^{(h)}_{\boldsymbol{\mu}})_{i,j} = (t_{\mu^{(h)}})_{i,j} + \sum_{k=1}^{h-1} m_k$; i.e., $\boldsymbol{t_\mu}$ is the numbering on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ obtained by putting numbers from 1 to m on boxes of $\boldsymbol{\mu}$ from left to right in each row, starting in the top row and moving to the bottom in each Young diagram, starting from $\mu^{(1)}$ to $\mu^{(n)}$. For example,

$$\boldsymbol{t}_{\left(\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)} = \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\, , \begin{array}{|c|c|}\hline 7 & 8 \\ \hline 9 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right).$$

Let $\boldsymbol{T}$ be a numbering on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$. We define $S_{\boldsymbol{T}}$ to be the subgroup $S_{\boldsymbol{T}} = S_{T^{(1)}} \times S_{T^{(2)}} \times \cdots \times S_{T^{(n)}}$ of S_m. The subgroup $S_{\boldsymbol{T}}$ and the Young subgroup $S_{\overline{\boldsymbol{\mu}}}$ are isomorphic as groups, where $\overline{\boldsymbol{\mu}}$ is the partition obtained from $(\mu^{(1)}_1, \mu^{(2)}_1, \dots, \mu^{(n)}_1, \mu^{(1)}_2, \mu^{(2)}_2, \dots, \mu^{(n)}_2, \dots)$ by sorting in descending order. We write $S_{\boldsymbol{\mu}}$ for $S_{\boldsymbol{t_\mu}}$. We define $a_{\boldsymbol{T},\boldsymbol{l}}$ to be $a_{T^{(1)},l_1} \cdot a_{T^{(2)},l_2} \cdots a_{T^{(n)},l_n}$. We write $a_{\boldsymbol{\mu},\boldsymbol{l}}$ for $a_{\boldsymbol{t_\mu},\boldsymbol{l}}$

Two numberings $\boldsymbol{T}$ and $\boldsymbol{S}$ on an $\boldsymbol{l}$-partition of $\boldsymbol{m}$ are said to be *row-equivalent* if $T^{(h)}$ and $S^{(h)}$ are row-equivalent for each h. The set of numberings whose components are arranged in ascending order in each row is a complete set of representatives for row-equivalence classes. A row-equivalence class of a numbering $\boldsymbol{T}$ on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ is an n-tuple $(\{T^{(1)}\}, \{T^{(2)}\}, \dots, \{T^{(n)}\})$ of tabloids $\{T^{(h)}\}$ on $\mu^{(h)}$. We also call a row-equivalence class $(\{T^{(h)}\})$ of numbering $\boldsymbol{T}$ on an $\boldsymbol{l}$-partition a *tabloid on an $\boldsymbol{l}$-partition*. We also write $\{\boldsymbol{T}\}$ for $(\{T^{(h)}\})$. The set of all tabloids on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ is denoted by $\mathbb{T}_{\boldsymbol{\mu}}$. We define $M^{\boldsymbol{\mu}}$ to be the $\mathbb{C}$-vector space $\mathbb{C}\mathbb{T}_{\boldsymbol{\mu}}$ whose basis is the set $\mathbb{T}_{\boldsymbol{\mu}}$ of tabloids on $\boldsymbol{\mu}$.

Let $\boldsymbol{T}$ be a numbering on an $\boldsymbol{l}$-partition of $\boldsymbol{m}$. Then $\sigma \in S_m$ acts on $\boldsymbol{T}$ from the left by $\sigma(T^{(h)}) = (\sigma T^{(h)})$. This left action induces a left action on tabloids by $\sigma\{\boldsymbol{T}\} = \{\sigma\boldsymbol{T}\}$. For the partition $\overline{\boldsymbol{\mu}} \vdash m$, $M^{\boldsymbol{\mu}}$ and $\mathrm{Ind}_{S_{\overline{\boldsymbol{\mu}}}}^{S_m} 1$ are isomorphic as left S_m-modules, where 1 denotes the trivial module of the Young subgroup $S_{\overline{\boldsymbol{\mu}}}$.

Let $\boldsymbol{T}$ be a numbering on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$. Since there uniquely exists $\tau_{\boldsymbol{T}} \in S_m$ such that $\boldsymbol{T} = \tau_{\boldsymbol{T}} \boldsymbol{t}_{\boldsymbol{\mu}}$, $\sigma \in \langle a_{\boldsymbol{\mu},\boldsymbol{l}} \rangle$ acts on $\boldsymbol{T}$ from right as $\boldsymbol{T}\sigma = \tau_{\boldsymbol{T}} \sigma \boldsymbol{t}_{\boldsymbol{\mu}}$. This right action also induces a right action on tabloids by $\{\boldsymbol{T}\}\sigma = \{\boldsymbol{T}\sigma\}$.

Next we introduce S_m-modules $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$, one of main objects in this paper. We need some definitions to introduce $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

DEFINITION 2.1. Let $\mathbb{T}^{\boldsymbol{l}}_{\boldsymbol{\mu}}$ be the subset $\left\{ a^i_{\boldsymbol{\mu},\boldsymbol{l}} \{\boldsymbol{t}_{\boldsymbol{\mu}}\} \,\middle|\, i \in \mathbb{Z}/l\mathbb{Z} \right\}$ of tabloids for an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$. We define $Z_{\boldsymbol{\mu}}(\boldsymbol{l})$ to be the $\mathbb{C}$-vector space $\mathbb{C}\mathbb{T}^{\boldsymbol{l}}_{\boldsymbol{\mu}}$ whose basis is $\mathbb{T}^{\boldsymbol{l}}_{\boldsymbol{\mu}}$. This l-dimensional vector space is a left module of the semi-direct product $S_{\boldsymbol{\mu}} \rtimes \langle a_{\boldsymbol{\mu},\boldsymbol{l}} \rangle$ and a right module of the cyclic group $\langle a_{\boldsymbol{\mu},\boldsymbol{l}} \rangle$ of order l.

For $k \in \mathbb{Z}/l\mathbb{Z}$, let $I_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ denote the submodule of $Z_{\boldsymbol{\mu}}(\boldsymbol{l})$ generated by

$$\left\{ a^i_{\boldsymbol{\mu},\boldsymbol{l}} \{\boldsymbol{t}_{\boldsymbol{\mu}}\} - \zeta_l^{ki} \{\boldsymbol{t}_{\boldsymbol{\mu}}\} \,\middle|\, i \in \mathbb{Z}/l\mathbb{Z} \right\}.$$

We define $Z_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ to be the quotient module

$$Z_{\boldsymbol{\mu}}(\boldsymbol{l})/I_{\boldsymbol{\mu}}(k;\boldsymbol{l}).$$

For each k, $Z_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is a one-dimensional left module of the semi-direct product $S_{\boldsymbol{\mu}} \rtimes \langle a_{\boldsymbol{\mu},\boldsymbol{l}} \rangle$. This left $S_{\boldsymbol{\mu}} \rtimes \langle a_{\boldsymbol{\mu},\boldsymbol{l}} \rangle$-module $Z_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is generated by $\{\boldsymbol{t}_{\boldsymbol{\mu}}\}$, and $a_{\boldsymbol{\mu},\boldsymbol{l}}$ acts on $\{\boldsymbol{t}_{\boldsymbol{\mu}}\}$ by

$$a_{\boldsymbol{\mu},\boldsymbol{l}} \{\boldsymbol{t}_{\boldsymbol{\mu}}\} = \zeta_l^k \{\boldsymbol{t}_{\boldsymbol{\mu}}\}$$

in $Z_{\boldsymbol{\mu}}(\boldsymbol{l})/I_{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

Let $\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ be $\mathbb{C}[S_m] I_{\boldsymbol{\mu}}(k;\boldsymbol{l})$. Finally, we define an S_m-module $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ to be

$$M^{\boldsymbol{\mu}}/\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l}).$$

By definition, the S_n-module $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is a realization of the induced module $\operatorname{Ind}^{S_m}_{S_{\boldsymbol{\mu}} \rtimes \langle a_{\boldsymbol{\mu},\boldsymbol{l}} \rangle} Z_{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

REMARK 2.2. For an l-partition μ of m, our module $M^{(\mu)}(k;(l))$ gives a realization of the S_m-module $\operatorname{Ind}^{S_m}_{H_\mu(l)} Z_\mu(k;l)$ in Morita-Nakajima [**7**]. For n-tuple $\{ l_h \}$ of integers, $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is a realization of the induced module

$$\operatorname{Ind}^{S_{m_1+\cdots+m_n}}_{S_{m_1} \times \cdots \times S_{m_n}} M^{\mu^{(1)}}(k;l_1) \otimes \cdots \otimes M^{\mu^{(n)}}(k;l_n),$$

where $M^\mu(k;l)$ denotes $M^{(\mu)}(k;(l))$.

REMARK 2.3. Since $\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l}) = \mathbb{C}[S_m] I_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is generated by

$$\left\{ \tau a^i_{\boldsymbol{\mu},\boldsymbol{l}} \{\boldsymbol{t}_{\boldsymbol{\mu}}\} - \zeta_l^{ik} \tau \{\boldsymbol{t}_{\boldsymbol{\mu}}\} \,\middle|\, i \in \mathbb{Z}/l\mathbb{Z}, \tau \in S_m \right\},$$

$\widetilde{I}_{\boldsymbol{\mu}}(k;\boldsymbol{l})$ is also generated by

$$\left\{ \{\boldsymbol{T}\} a^i_{\boldsymbol{\mu},\boldsymbol{l}} - \zeta_l^{ik} \{\boldsymbol{T}\} \,\middle|\, \{\boldsymbol{T}\} \in \mathbb{T}_{\boldsymbol{\mu}}, i \in \mathbb{Z}/l\mathbb{Z} \right\}.$$

Hence $a_{\boldsymbol{\mu},\boldsymbol{l}}$ acts on tabloids $\{\boldsymbol{T}\}$ by

$$\{\boldsymbol{T}\} a_{\boldsymbol{\mu},\boldsymbol{l}} = \zeta_l^k \{\boldsymbol{T}\}$$

in $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

We introduce the following combinatorial objects to describe the characters of $M^{\boldsymbol{\mu}}(k;\boldsymbol{l})$.

DEFINITION 2.4. For a Young diagram $\rho \vdash m$, we call a map $Y : \boldsymbol{\mu} \to \mathbb{N}$ a $(\rho,\boldsymbol{l})$-*tableau* on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ if the following are satisfied:

- $\left| Y^{-1}(\{ k \}) \right| = \rho_k$ for all k,
- for each k, there exist $h \in \mathbb{N}$ and $(i',j') \in \mathbb{N}^2$ such that ρ_k is divisible by l_h and
$$Y^{-1}(\{ k \}) = \left\{ (i+i', j+j'; h) \,\middle|\, (i,j) \in \left(\left(\frac{\rho_k}{l_h} \right)^{l_h} \right) \vdash \rho_k \right\},$$
- for each $(i,j;h)$, $(i,k;h) \in \boldsymbol{\mu}$, $Y(i,j;h) \le Y(i,k;h)$ if $j \le k$.

Example 2.5. For example,

$$\left(\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 3 & 4 \\ \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2 & 2 & 5 \\ \hline \end{array}\right)$$

is a $((4,2,2,2,1),(2,1))$-tableau on $((2,2,2,2),(3))$.

Definition 2.6. We call a pair (Y,c) a *marked* $(\rho,\boldsymbol{l})$*-tableau* on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ if the following are satisfied:

- Y is a $(\rho,\boldsymbol{l})$-tableau on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$,
- c is a map from $\{\, i \,|\, \rho_i \neq 0 \,\}$ to $\coprod_h \mathbb{Z}/l_h\mathbb{Z}$,
- $c(i)$ is in $\mathbb{Z}/l_h\mathbb{Z}$ if $Y^{-1}(\{\, i \,\}) \subset \mu^{(h)}$.

For a marked $(\rho,\boldsymbol{l})$-tableau (Y,c), the inverse image $Y^{-1}(\{\, i \,\})$ has l_h rows and $c(i)$ is in $\mathbb{Z}/l_h\mathbb{Z} = \{\, 1,\dots,l_h \,\}$ if $Y^{-1}(\{\, i \,\})$ is in $\mu^{(h)}$. We identify (Y,c) with the diagram obtained from the diagram of Y by putting $*$ in the left-most box of the $c(i)$-th row of the inverse image $Y^{-1}(\{\, i \,\})$, where we identify $\mathbb{Z}/l_h\mathbb{Z}$ with the set $\{\, 1,\dots,l_h \,\}$ of complete representatives.

Example 2.7. Let

$$Y = \left(\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 3 & 4 \\ \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2 & 2 & 5 \\ \hline \end{array}\right)$$

and let c be the map such that $c(1)=2$, $c(3)=1$, $c(4)=2 \in \mathbb{Z}/2\mathbb{Z}$ and $c(2)=c(5)=1 \in \mathbb{Z}/1\mathbb{Z}$, then (Y,c) is a marked $(2,1)$-tableau. We write

$$\left(\begin{array}{|c|c|}\hline 3^* & 4 \\ \hline 3 & 4^* \\ \hline 1 & 1 \\ \hline 1^* & 1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|c|}\hline 2^* & 2 & 5^* \\ \hline \end{array}\right)$$

for (Y,c).

Remark 2.8. It follows from a direct calculation that the number of marked $(\rho,(l))$-tableaux on an (l)-partition (μ) equals the right hand side of the equation (3.1) in Morita-Nakajima [**7**].

Definition 2.9. Let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition of $\boldsymbol{m}$ and $\boldsymbol{\gamma} = (\gamma_h)$ an n-tuple of integers such that l_h is divisible by γ_h. For a Young diagram $\rho \vdash m$, we call a map $Y : \boldsymbol{\mu} \to \mathbb{N}$ a $(\rho,\boldsymbol{\gamma},\boldsymbol{l})$*-tableau* on $\boldsymbol{\mu}$ if the following are satisfied:

- $\left|Y^{-1}(\{\, k \,\})\right| = \rho_k$ for all k,
- for each k, there exist h and $(i',j') \in \mathbb{N}^2$ such that ρ_k is divisible by γ_h and
$$Y^{-1}(\{\, k \,\}) = \left\{ \left(\frac{il_h}{\gamma_h} + i', j + j'; h\right) \,\middle|\, (i,j) \in \left(\left(\frac{\rho_k}{\gamma_h}\right)^{\gamma_h}\right) \vdash \rho_k \right\},$$
- for each $(i,j;h)$, $(i,k;h) \in \boldsymbol{\mu}$, $Y(i,j;h) \leq Y(i,k;h)$ if $j \leq k$.

Example 2.10. For example,

$$\left(\begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 4 \\ \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 4 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}\ ,\ \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 5 & 5 \\ \hline 6 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\right)$$

is an $((8,6,2,2,2,1),(2,1),(4,1))$-tableau on $((4,4,4,4),(5))$.

A $(\rho,\boldsymbol{l},\boldsymbol{l})$-tableau on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ is a $(\rho,\boldsymbol{l})$-tableau on $\boldsymbol{\mu}$.

For an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and an n-tuple $\boldsymbol{\gamma} = (\gamma_h)$ such that l_h is divisible by γ_h for each h, it follows by definition that

$$|\{\, Y \,|\, \text{a } (\rho,\boldsymbol{\gamma},\boldsymbol{l})\text{-tableau on } \boldsymbol{\mu} \,\}| = |\{\, Y \,|\, \text{a } (\rho,\boldsymbol{\gamma})\text{-tableau on } \boldsymbol{\mu} \,\}| \,.$$

DEFINITION 2.11. We call a pair (Y, c) a *marked* $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$*-tableau* on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ if the following are satisfied:

- Y is a $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-tableau on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$,
- c is a map from $\{ i \mid \rho_i \neq 0 \}$ to $\coprod_h \mathbb{Z}/\gamma_h\mathbb{Z}$,
- $c(i)$ is in $\mathbb{Z}/\gamma_h\mathbb{Z}$ if $Y^{-1}(\{ i \}) \subset \mu^{(h)}$.

Similarly to the case of marked $(\rho, \boldsymbol{l})$-tableaux, we identify a marked $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-tableau (Y, c) with the diagram obtained from the diagram of Y by putting $*$ in the left-most box of the $c(i)$-th row of the inverse image $Y^{-1}(\{ i \})$.

EXAMPLE 2.12. For example,

$$\left(\begin{array}{|c|c|} \hline 3^* & 4 \\ \hline 1 & 1 \\ \hline 3 & 4^* \\ \hline 1^* & 1 \\ \hline \end{array} \, , \; \begin{array}{|c|c|c|} \hline 2^* & 2 & 5^* \\ \hline \end{array} \right)$$

is a marked $((4, 2, 2, 2, 1), (2, 1), (4, 1))$-tableau.

For an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and an n-tuple $\boldsymbol{\gamma} = (\gamma_h)$ such that l_h is divisible by γ_h for each h, it follows by definition that

$$|\{ (Y, c) \mid \text{a marked } (\rho, \boldsymbol{\gamma}, \boldsymbol{l})\text{-tableau on } \boldsymbol{\mu} \}| = |\{ (Y, c) \mid \text{a marked } (\rho, \boldsymbol{\gamma})\text{-tableau on } \boldsymbol{\mu} \}| . \tag{2.1}$$

3. Main Results

The following are the main results of this paper.

THEOREM 3.1. *Let j be an integer. Let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition and $\boldsymbol{\gamma} = (\gamma_h)$ an n-tuple of integers such that each γ_h is the order of $\zeta_{l_h}^j$. For $\sigma \in S_m$ of cycle type ρ,*

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{jk} \operatorname{Char}(M^{\boldsymbol{\mu}}(k; \boldsymbol{l}))(\sigma) = |\{ (Y, c) \mid \textit{a marked } (\rho, \boldsymbol{\gamma})\textit{-tableau on } \boldsymbol{\mu} \}| .$$

THEOREM 3.2. *Let j be an integer. Let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition and $\boldsymbol{\gamma} = (\gamma_h)$ an n-tuple of integers such that each γ_h is the order of $a_{\mu^{(h)}, l_h}^j$. Tabloids $\boldsymbol{T}$ on $\boldsymbol{\mu}$ satisfying $\sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j}$ are parameterized by marked $(\rho_\sigma, \boldsymbol{\gamma}, \boldsymbol{l})$-tableaux on $\boldsymbol{\mu}$, where ρ_σ is the cycle type of σ.*

Applying Theorems 3.1 and 3.2 as $j = 1$, we obtain Propositions 3.1 and 3.2 below.

PROPOSITION 3.1. *For $\sigma \in S_m$ of cycle type ρ and an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$,*

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{k} \operatorname{Char}(M^{\boldsymbol{\mu}}(k; \boldsymbol{l}))(\sigma) = |\{ (Y, c) \mid \textit{a marked } (\rho, \boldsymbol{l})\textit{-tableau on } \boldsymbol{\mu} \}| .$$

PROPOSITION 3.2. *Let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition. Tabloids $\{\boldsymbol{T}\}$ on $\boldsymbol{\mu}$ satisfying $\sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a_{\boldsymbol{\mu},\boldsymbol{l}}^{-1}$ are parameterized by $\boldsymbol{l}$-fillings on $(\rho_\sigma, \boldsymbol{l})$-tableaux on $\boldsymbol{\mu}$, where ρ_σ is the cycle type of σ.*

The following proposition directly follows from Theorem 3.1.

PROPOSITION 3.3. *For an integer j, let $\boldsymbol{\mu}$ be an $\boldsymbol{l}$-partition, $\boldsymbol{\gamma}$ an n-tuple of integers such that γ_h is the order of $\zeta_{l_h}^j$. For $\sigma \in S_m$ of cycle type ρ,*

$$\begin{aligned} \sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{jk} \operatorname{Char}(M^{\boldsymbol{\mu}}(k; \boldsymbol{l}))(\sigma) &= \sum_{k \in \mathbb{Z}/\gamma\mathbb{Z}} \zeta_\gamma^{k} \operatorname{Char}(M^{\boldsymbol{\mu}}(k; \boldsymbol{\gamma}))(\sigma) \\ &= |\{ (Y, c) \mid \textit{a marked } (\rho, \boldsymbol{\gamma})\textit{-tableau on } \boldsymbol{\mu} \}| . \end{aligned}$$

EXAMPLE 3.3. Let $\boldsymbol{\mu} = ((2,2),(4))$ and $\boldsymbol{l} = (2,1)$. First we consider the case where $j = 1$. In this case, all marked $((4,2,2),\boldsymbol{l})$-tableaux on $\boldsymbol{\mu}$ are the following:

$$\left(\begin{array}{|c|c|}\hline 1^* & 1 \\ \hline 1 & 1 \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 2^* & 2 & 3^* & 3 \\ \hline\end{array}\right), \qquad \left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1^* & 1 \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 2^* & 2 & 3^* & 3 \\ \hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|}\hline 2^* & 3^* \\ \hline 2 & 3 \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^* & 1 & 1 & 1 \\ \hline\end{array}\right), \qquad \left(\begin{array}{|c|c|}\hline 2^* & 3 \\ \hline 2 & 3^* \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^* & 1 & 1 & 1 \\ \hline\end{array}\right),$$

$$\left(\begin{array}{|c|c|}\hline 2 & 3^* \\ \hline 2^* & 3 \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^* & 1 & 1 & 1 \\ \hline\end{array}\right), \qquad \left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 2^* & 3^* \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^* & 1 & 1 & 1 \\ \hline\end{array}\right).$$

It follows from Proposition 3.1 that

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^k \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k;\boldsymbol{l})\right)((1234)(56)(78)) = 6.$$

Next consider the case where $j = 2$. Since $\zeta_{l_1} = \zeta_2 = -1$ and $\zeta_{l_2} = \zeta_1 = 1$, we have $\gamma_1 = \left|\left\langle \zeta_{l_1}^2 \right\rangle\right| = 1$ and $\gamma_2 = \left|\left\langle \zeta_{l_2}^2 \right\rangle\right| = 1$. All marked $((4,2,2),(1,1))$-tableaux on $\boldsymbol{\mu}$ are the following:

$$\left(\begin{array}{|c|c|}\hline 2^* & 2 \\ \hline 3^* & 3 \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^* & 1 & 1 & 1 \\ \hline\end{array}\right), \qquad \left(\begin{array}{|c|c|}\hline 3^* & 3 \\ \hline 2^* & 2 \\ \hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^* & 1 & 1 & 1 \\ \hline\end{array}\right).$$

It follows from Theorem 3.1 that

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{2k} \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k;\boldsymbol{l})\right)((1234)(56)(78)) = 2.$$

4. Outline of Proof

In this section, we give an outline of a proof of Theorem 3.1 and Theorem 3.2. First we show Lemma 4.1, that means the equivalence between Theorems 3.1 and 3.2. Next, in Definition 4.3, we define a correspondence φ which provides an explicit parametrization of Theorem 3.2. Then we prove Theorem 3.2 first for the special element σ_ρ of the cycle type ρ, which is Lemma 4.7. We prepare Lemma 4.5 and Lemma 4.6 to prove Lemma 4.7. Finally, in Theorem 4.8, we generalize Lemma 4.7 for general elements of the cycle type ρ. Theorem 4.8 is a realization of Theorem 3.2.

Lemma 4.1 follows from direct calculations of traces.

LEMMA 4.1. *For an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and $\sigma \in S_m$,*

$$\sum_{k \in \mathbb{Z}/l\mathbb{Z}} \zeta_l^{kj} \operatorname{Char}\left(M^{\boldsymbol{\mu}}(k;\boldsymbol{l})\right)(\sigma) = \left|\left\{\, \{\boldsymbol{T}\} \in \mathbb{T}_{\boldsymbol{\mu}} \,\middle|\, \sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\} a_{\boldsymbol{\mu},\boldsymbol{l}}^{-j} \,\right\}\right|.$$

We construct a bijection between marked $(\rho_\sigma, \boldsymbol{\gamma}, \boldsymbol{l})$-tableaux on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and tabloids $\{\boldsymbol{T}\}$ on $\boldsymbol{\mu}$ satisfying $\sigma\{\boldsymbol{T}\} = \{\boldsymbol{T}\} a_{\boldsymbol{\mu},\boldsymbol{l}}^{-1}$ to prove Theorem 3.2.

DEFINITION 4.2. For a Young diagram $\rho \vdash m$, we define $n_{\rho,i}$, $N_{\rho,i}$, $\sigma_{\rho,i}$ and σ_ρ by the following:

$$\begin{aligned} n_{\rho,i} &= 1 + \sum_{j=1}^{i-1} \rho_j, \\ N_{\rho,i} &= \{\, n_{\rho,i}, n_{\rho,i}+1, \ldots, n_{\rho,i}+\rho_i-1 \,\} \subset \{\, 1, \ldots, m \,\}, \\ \sigma_{\rho,i} &= (n_{\rho,i}, n_{\rho,i}+1, \ldots, n_{\rho,i}+\rho_i-1) \in S_m, \\ \sigma_\rho &= \sigma_{\rho,1}\sigma_{\rho,2}\sigma_{\rho,3}\cdots \in S_m. \end{aligned}$$

For a Young diagram $\rho \vdash m$, by definition, $\bigcup_i N_{\rho,i} = \{\, 1, \ldots, m \,\}$, $|N_{\rho,i}| = \rho_i$ and the cycle type of σ_ρ is ρ.

DEFINITION 4.3. Let γ_h be the order of $a^j_{\mu^{(h)},l_h}$. For a marked $(\rho, \boldsymbol{\gamma}, \boldsymbol{l})$-tableau (Y, c) on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$, $\{\varphi_j(Y,c)\}$ denotes the tabloid obtained from the following:

- Put the number $n_{\rho,i}$ on a box in the $c(i)$-th row of the inverse image $Y^{-1}(\{\, i \,\})$ for each i.
- Put the number $\sigma_\rho n$ on a box in the $(\overline{c-j}+ql_h)$-th row of $\mu^{(h)}$ if the number n is in the $(\overline{c}+ql_h)$-th row of $\mu^{(h)}$, where $\overline{c}$, $\overline{c-j} \in \mathbb{Z}/l_h\mathbb{Z} = \{\, 1, \ldots, l_h \,\}$ and $q \in \mathbb{Z}$.

We define $\varphi_j(Y,c)$ to be the numbering sorted in ascending order in each row of $\{\varphi_j(Y,c)\}$.

EXAMPLE 4.4. For a marked $((4,4,1),(2,1))$-tableaux $\left(\begin{array}{|c|c|}\hline 2^* & 2\\\hline 2 & 2\\\hline 1 & 1\\\hline 1^* & 1\\\hline\end{array}, \begin{array}{|c|}\hline 3^*\\\hline\end{array}\right)$,

$$\varphi_1\left(\begin{array}{|c|c|}\hline 2^* & 2\\\hline 2 & 2\\\hline 1 & 1\\\hline 1^* & 1\\\hline\end{array}, \begin{array}{|c|}\hline 3^*\\\hline\end{array}\right) = \left(\begin{array}{|c|c|}\hline 5 & 7\\\hline 6 & 8\\\hline 2 & 4\\\hline 1 & 3\\\hline\end{array}, \begin{array}{|c|}\hline 9\\\hline\end{array}\right).$$

For a marked $((4,4,1),(2,1),(4,1))$-tableau $\left(\begin{array}{|c|c|}\hline 2^* & 2\\\hline 1 & 1\\\hline 2 & 2\\\hline 1^* & 1\\\hline\end{array}, \begin{array}{|c|}\hline 3^*\\\hline\end{array}\right)$,

$$\varphi_2\left(\begin{array}{|c|c|}\hline 2^* & 2\\\hline 1 & 1\\\hline 2 & 2\\\hline 1^* & 1\\\hline\end{array}, \begin{array}{|c|}\hline 3^*\\\hline\end{array}\right) = \left(\begin{array}{|c|c|}\hline 5 & 7\\\hline 2 & 4\\\hline 6 & 8\\\hline 1 & 3\\\hline\end{array}, \begin{array}{|c|}\hline 9\\\hline\end{array}\right).$$

Now we show that this correspondence φ_j provides a realization of Theorem 3.2.
To show Lemma 4.7, we prepare Lemmas 4.5 and 4.6.

LEMMA 4.5. *For a marked $(\rho,\boldsymbol{\gamma},\boldsymbol{l})$-tableau (Y,c) on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$, the tabloid $\{\varphi_j(Y,c)\}$ satisfies*

$$\sigma_\rho\{\varphi_j(Y,c)\} = \{\varphi_j(Y,c)\}a^{-j}_{\boldsymbol{\mu},\boldsymbol{l}},$$

where φ_j is the one defined in Definition 4.3 and γ_h is the order of $a^{-j}_{\mu^{(h)},l_h}$.

LEMMA 4.6. *Let a tabloid $\{\boldsymbol{T}\}$ on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ satisfy $\sigma_\rho\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a^{-j}_{\boldsymbol{\mu},\boldsymbol{l}}$. If $\boldsymbol{T}^{-1}(n_{\rho,k})$ is a box in the $(\overline{r}+l_hq)$-th row of $\mu^{(h)}$, then $n\in N_{\rho,k}$ is in the $(\overline{r-(n-n_{\rho,k})j}+l_hq)$-th row of $\mu^{(h)}$, where $\overline{r}$ and $\overline{r-(n-n_{\rho,k})j}\in\mathbb{Z}/l_h\mathbb{Z}=\{\,1,\dots,l_h\,\}$ and $q\in\mathbb{Z}$.*

LEMMA 4.7. *If γ_h is the order of $a^{j}_{\mu^{(h)},l_h}$, our correspondence φ_j provides a bijection between marked $(\rho,\boldsymbol{\gamma},\boldsymbol{l})$-tableaux on an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ and tabloids $\{\boldsymbol{T}\}$ on $\boldsymbol{\mu}$ satisfying $\sigma_\rho\{\boldsymbol{T}\} = \{\boldsymbol{T}\}a^{-j}_{\boldsymbol{\mu},\boldsymbol{l}}$.*

Finally we consider not only σ_ρ, but also general elements σ whose cycle type is ρ. We explicitly give parameterizations of Theorem 3.2 in the following theorem, which follows from Lemma 4.7.

THEOREM 4.8. *Let the cycle type of $\sigma\in S_m$ be ρ and let $\tau\in S_m$ satisfy $\tau\sigma_\rho\tau^{-1}=\sigma$.*
Then the set $\left\{\,\{\boldsymbol{T}\}\in\mathbb{T}_{\boldsymbol{\mu}}\,\middle|\,\sigma\{\boldsymbol{T}\}=\{\boldsymbol{T}\}a^{-j}_{\boldsymbol{\mu},\boldsymbol{l}}\,\right\}$ equals

$$\{\,\{\tau\varphi_j(Y,c)\}\mid(Y,c)\text{ is a marked }(\rho,\boldsymbol{\gamma},\boldsymbol{l})\text{-tableau on }\boldsymbol{\mu}\,\}$$

for an $\boldsymbol{l}$-partition $\boldsymbol{\mu}$ of $\boldsymbol{m}$ and $\boldsymbol{\gamma}=(\gamma_h)$ such that γ_h is the order of $a^{j}_{\mu^{(h)},l_h}$ for each h.

EXAMPLE 4.9. Let $\boldsymbol{\mu}$, $\boldsymbol{l}$ and ρ be the same as the ones in Example 3.3, i.e., $\boldsymbol{\mu}=((2,2),4)$, $\boldsymbol{l}=(2,1)$ and $\rho=(4,2,2)$. First we consider the case where $j=1$. In this case,

$$\left(\begin{array}{|c|c|}\hline 1^* & 1\\\hline 1 & 1\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 2^* & 2 & 3^* & 3\\\hline\end{array}\right)$$

is a $(\rho,\boldsymbol{l})$-tableau on $\boldsymbol{\mu}$. We have

$$\varphi_1\left(\begin{array}{|c|c|}\hline 1^* & 1\\\hline 1 & 1\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 2^* & 2 & 3^* & 3\\\hline\end{array}\right) = \left(\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 5 & 6 & 7 & 8\\\hline\end{array}\right).$$

Since $\sigma_\rho=(1,2,3,4)(5,6)(7,8)$ acts as

$$(1234)(56)(78)\left(\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & 4\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 5 & 6 & 7 & 8\\\hline\end{array}\right) = \left(\begin{array}{|c|c|}\hline 2 & 4\\\hline 3 & 1\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 6 & 5 & 8 & 7\\\hline\end{array}\right)$$

and

$$\left\{\left(\begin{array}{|c|c|}\hline 2&4\\\hline 3&1\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 6&5&8&7\\\hline\end{array}\right)\right\} = \left\{\left(\begin{array}{|c|c|}\hline 2&4\\\hline 1&3\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 5&6&7&8\\\hline\end{array}\right)\right\}$$
$$= \left\{\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 5&6&7&8\\\hline\end{array}\right)\right\} a_{\boldsymbol{\mu},\boldsymbol{l}}^{-1},$$

it follows that

$$\sigma_\rho\left\{\varphi_1\left(\begin{array}{|c|c|}\hline 1^*&1\\\hline 1&1\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 2^*&2&3^*&3\\\hline\end{array}\right)\right\} = \left\{\varphi_1\left(\begin{array}{|c|c|}\hline 1^*&1\\\hline 1&1\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 2^*&2&3^*&3\\\hline\end{array}\right)\right\} a_{\boldsymbol{\mu},\boldsymbol{l}}^{-1}.$$

Next we consider the case where $j = 2$. In this case,

$$\left(\begin{array}{|c|c|}\hline 2^*&2\\\hline 3^*&3\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^*&1&1&1\\\hline\end{array}\right)$$

is a $(\rho, (1, 1), \boldsymbol{l})$-tableau on $\boldsymbol{\mu}$. We have

$$\varphi_2\left(\begin{array}{|c|c|}\hline 2^*&2\\\hline 3^*&3\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^*&1&1&1\\\hline\end{array}\right) = \left(\begin{array}{|c|c|}\hline 5&6\\\hline 7&8\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}\right).$$

Since σ_ρ acts as

$$(1234)(56)(78)\left\{\left(\begin{array}{|c|c|}\hline 5&6\\\hline 7&8\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}\right)\right\} = \left\{\left(\begin{array}{|c|c|}\hline 6&5\\\hline 8&7\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 2&3&4&1\\\hline\end{array}\right)\right\}$$
$$= \left\{\left(\begin{array}{|c|c|}\hline 5&6\\\hline 7&8\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}\right)\right\}$$

and $a_{\boldsymbol{\mu},\boldsymbol{l}}^{-2} = \varepsilon \in S_8$, it follows that

$$\sigma_\rho\left\{\varphi_1\left(\begin{array}{|c|c|}\hline 2^*&2\\\hline 3^*&3\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^*&1&1&1\\\hline\end{array}\right)\right\} = \left\{\varphi_1\left(\begin{array}{|c|c|}\hline 2^*&2\\\hline 3^*&3\\\hline\end{array}\,,\ \begin{array}{|c|c|c|c|}\hline 1^*&1&1&1\\\hline\end{array}\right)\right\} a_{\boldsymbol{\mu},\boldsymbol{l}}^{-2}.$$

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Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: nu@math.sci.hokudai.ac.jp