The Cyclic Sieving Phenomenon for Faces of Generalized Cluster Complexes

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ABSTRACT. The notion of cyclic sieving phenomenon was introduced by Reiner, Stanton, and White as a generalization of Stembridge's $q = -1$ phenomenon. The generalized cluster complexes associated to root systems were given by Fomin and Reading as a generalization of the cluster complexes found by Fomin and Zelevinsky. In this paper, the faces of various dimensions of the generalized cluster complexes in type A_n, B_n, D_n, and $I_2(a)$ are shown to exhibit the cyclic sieving phenomenon under a cyclic group action. For the cluster complexes of exceptional type E_6, E_7, E_8, F_4, H_3, and H_4, a verification for such a phenomenon on their maximal faces is given.

1. Introduction

In [**8**], Reiner, Stanton, and White introduced the notion of cyclic sieving phenomenon as a generalization of Stembridge's $q = -1$ phenomenon for generating functions of a set of combinatorial structures with a cyclic group action. Namely, a triple $(X, X(q), C)$ consisting of a finite set X, a polynomial $X(q) \in \mathbb{Z}[q]$ with the property that $X(1) = |X|$, and a cyclic group C that acts on X is said to exhibit the *cyclic sieving phenomenon* if for every $c \in C$,

$$[X(q)]_{q=\omega} = |\{x \in X : c(x) = x\}|, \tag{1}$$

where the complex number ω is a root of unity of the same multiplicative order as c. Equivalently, if $X(q)$ is expanded as $X(q) \equiv a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \pmod{q^n - 1}$, where n is the order of C, then a_k counts the number of orbits on X under C, the stabilizer-order of which divides k. In particular, a_0 counts the total number of orbits, a_1 counts the number of free orbits, and $a_2 - a_1$ is the number of orbits that have a stabilizer of order 2. This paper is motivated by the following concrete example. Here we use the notation $\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{[n]!_q}{[i]!_q [n-i]!_q}$, where $[n]!_q = [1]_q [2]_q \cdots [n]_q$ and $[i]_q = 1 + q + \cdots + q^{i-1}$.

THEOREM 1.1. ([**8**, Theorem 7.1]) *Let X be the set of dissections of a regular $(n + 2)$-gon using k noncrossing diagonals $(0 \le k \le n - 1)$. Let*

$$X(q) := \frac{1}{[k+1]_q} \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{2}$$

Let the cyclic group C of order $n + 2$ act on X by cyclic rotation of the polygon. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

Note that $X(1) = |X|$ is the well-known Kirkman-Cayley number. In [**4**], Fomin and Zelevinsky introduced a simplicial complex $\Delta(\Phi)$, called *cluster complex*, associated to a root system Φ, which can be realized by a combinatorial structure constructed in terms of polygon-dissections. In fact, Theorem 1.1 proves the cyclic sieving phenomenon for the k-faces of the cluster complex $\Delta(\Phi)$ in type A_{n-1}, under a cyclic group

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generated by a deformation Γ (defined in Section 2) of a Coxeter element of Φ. This connection will be explained in the next section.

From [4, Theorem 1.9], the number of *facets* (i.e., maximal faces) of $\Delta(\Phi)$ can be expressed uniformly as

$$(3) \qquad \mathrm{Cat}(\Phi) := \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1},$$

known as the *generalized Catalan numbers* $\mathrm{Cat}(\Phi)$, where h is the Coxeter number and $e_1, \ldots, e_n$ are the exponents of Φ. In particular, $\mathrm{Cat}(A_{n-1}) = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. The cyclic group generated by Γ is of order $h + 2$. Along with the q-analogue of (3) defined by

$$(4) \qquad \mathrm{Cat}(\Phi, q) := \prod_{i=1}^{n} \frac{[h + e_i + 1]_q}{[e_i + 1]_q},$$

one of our main results is to prove the following theorem, case by case under the classification of Φ.

THEOREM 1.2. *Let X be the set of facets of the cluster complex $\Delta(\Phi)$. Let $X(q) := Cat(\Phi, q)$ be defined in (4). Let the cyclic group C of order $h + 2$ generated by Γ act on X. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

Moreover, in [3], Fomin and Reading defined the *generalized cluster complexes* $\Delta^s(\Phi)$ associated to a root system Φ and a positive integer s, which specializes at $s = 1$ to $\Delta(\Phi)$. The purpose of this paper is to study the cyclic sieving phenomenon for the faces of $\Delta^s(\Phi)$, along with a q-analogue $X(q)$ of face numbers, under a cyclic group action. Making use of Fomin and Reading's results, we prove the cyclic sieving phenomenon by a combinatorial approach for $\Delta^s(\Phi)$ in type A_n, B_n, D_n, and $I_2(a)$. For Φ of exceptional type E_6, E_7, E_8, F_4, H_3, and H_4, although a systematic method is not available, we verify such a phenomenon by computer for the facets of $\Delta^s(\Phi)$ when $s = 1$.

This paper is organized as follows. We review backgrounds of cluster complexes and generalized cluster complexes in Section 2. As the main results of this paper, the cyclic sieving phenomenon for the generalized cluster complexes in type A_n, B_n, D_n, and $I_2(a)$ are given in Sections 3, 4, 5, and 6, respectively. The cases of exceptional type are given in Section 7.

2. Backgrounds

In this section, we review basic facts of cluster complexes from [4] and interpret Theorem 1.1 in terms of cluster complexes (type A_{n-1}). Then we review the definition of generalized cluster complexes from [3] and introduce the main purpose of this paper. Most of this section follows the materials in [3, Sections 2 and 3]

2.1. Cluster complexes. Let Φ be an irreducible root system of rank n. Let $\Phi_{>0}$ denote the set of positive roots in Φ, and let $\Pi = \{\alpha_i : i \in I\}$ denote the set of simple roots in Φ, where $I = \{1, \ldots, n\}$. Accordingly, $-\Pi = \{-\alpha_i : i \in I\}$ is the set of negative simple roots. The set $S = \{s_i : i \in I\}$ of reflections corresponding to simple roots α_i generates a finite reflection group W that naturally acts on Φ. The pair (W, S) is a Coxeter system.

Let $I = I_+ \cup I_-$ be a partition of I such that each of sets I_+ and I_- is totally disconnected in the Coxeter diagram. Let $\Phi_{\geq -1} = \Phi_{>0} \cup (-\Pi)$. Define the involutions $\tau_\pm : \Phi_{\geq -1} \to \Phi_{\geq -1}$ by

$$\tau_\epsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i, \text{ for } i \in I_{-\epsilon}, \\ \left(\prod_{i \in I_\epsilon} s_i\right)(\alpha) & \text{otherwise,} \end{cases}$$

for $\epsilon \in \{+, -\}$. The product $\Gamma = \tau_- \tau_+$ generates a cyclic group $\langle \Gamma \rangle$ that acts on $\Phi_{\geq -1}$. For example, let Φ be the root system of type A_2, with $I = \{1, 2\}$. The set $\Phi_{\geq -1} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2\}$ of roots is shown in Figure 1. Setting $I_+ = \{1\}$ and $I_- = \{2\}$, one can check that Γ acts on $\Phi_{\geq -1}$ by $\alpha_1 \to -\alpha_1 \to \alpha_1 + \alpha_2 \to -\alpha_2 \to \alpha_2 \to \alpha_1$.

From [4, Section 3.1], a relation of *compatibility* on $\Phi_{\geq -1}$ is defined by means of Γ such that (i) $\alpha, \beta \in \Phi_{\geq -1}$ are compatible if and only if $\Gamma(\alpha)$ and $\Gamma(\beta)$ are compatible; (ii) $-\alpha_i \in -\Pi$ and $\beta \in \Phi_{>0}$ are compatible if and only if the simple root expansion of β does not involve α_i. Following [4, p. 983], the *cluster complex* $\Delta(\Phi)$ is defined to be the simplicial complex whose faces are subsets of roots in $\Phi_{\geq -1}$, which are pairwise compatible. For Φ in type A_n, B_n, and D_n, the cluster complex $\Delta(\Phi)$ can be realized by dissections of
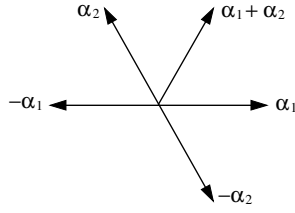
FIGURE 1. The set $\Phi_{\geq -1}$ in type A_2.

a regular polygon such that $\langle \Gamma \rangle$ corresponds to a group action on the dissections by rotation of the given polygon.

Specifically, consider the root system Φ of type A_n. Then $\Phi_{\geq -1}$ consists of positive roots of the form $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, for $1 \leq i \leq j \leq n$, and negative simple roots $-\alpha_i$, for $1 \leq i \leq n$. Let P be a regular polygon with $n + 3$ vertices, labeled by $\{1, 2, \ldots, n + 3\}$ counterclockwise. The roots in $\Phi_{\geq -1}$ are identified with the diagonals of P as follows. For $1 \leq i \leq \frac{n+1}{2}$, the root $-\alpha_{2i-1} \in -\Pi$ is identified with the diagonal connecting vertices i and $n + 3 - i$. For $1 \leq i \leq \frac{n}{2}$, the root $-\alpha_{2i} \in -\Pi$ is identified with the diagonal connecting vertices $i + 1$ and $n + 3 - i$. These diagonals form a 'snake' of negative simple roots. Figure 2 shows this snake in type A_5. The positive roots of $\Phi_{>0}$ are identified with the remaining diagonals as follows. Each α_{ij} is identified with the unique diagonal that intersects the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ and no other diagonals in the snake. For example, Figure 3 is a realization of the set $\Phi_{\geq -1}$ in type A_2.

Under this bijection, every pair of compatible roots is carried to a pair of noncrossing diagonals. Hence each k-face (i.e., k-element simplex) of the cluster complex $\Delta(A_n)$ corresponds to a dissection of P using k noncrossing diagonals. Moreover, the map Γ corresponds to a clockwise rotation of P that carries point 2 to point 1, etc. Therefore, Theorem 1.1 can be interpreted in terms of $\Delta(A_{n-1})$, i.e., let X be the set of k-faces of $\Delta(A_{n-1})$, let $X(q)$ be the polynomial defined by (2). Then $(X, X(q), \langle \Gamma \rangle)$ exhibits the cyclic sieving phenomenon.
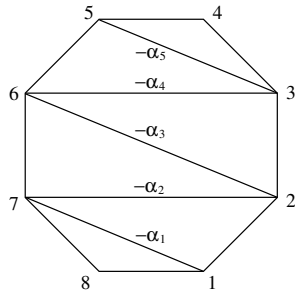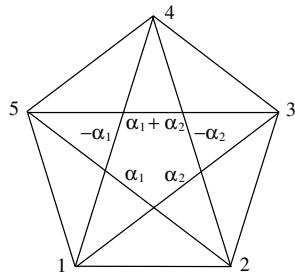

FIGURE 2. The snake in type A_5.

FIGURE 3. A representation for the set $\Phi_{\geq -1}$ of roots in type A_2.

2.2. Generalized cluster complexes. Let s be a positive integer. For each $\alpha \in \Phi_{>0}$, let $\alpha^1, \ldots, \alpha^s$ denote the s 'colored' copies of α. Define

$$\Phi^s_{\geq -1} = \{\alpha^k : \alpha \in \Phi_{>0}, 1 \leq k \leq s\} \cup \{(-\alpha_i)^1 : i \in I\},$$

i.e., $\Phi^s_{\geq -1}$ consists of s copies of the positive roots and one copy of the negative simple roots. The relation of compatibility on $\Phi^s_{\geq -1}$ can be defined by an s-analogue of Γ. For $\alpha^k \in \Phi^s_{\geq -1}$, define

$$\Gamma_s(\alpha^k) = \begin{cases} \alpha^{k+1} & \text{if } \alpha \in \Phi_{>0} \text{ and } k < s, \\ (\Gamma(\alpha))^1 & \text{otherwise.} \end{cases}$$

By [**3**, Theorem 3.4], this relation is determined by the following two conditions (i) α^k and β^l are compatible if and only if $\Gamma_s(\alpha^k)$ and $\Gamma_s(\beta^l)$ are compatible; (ii) $(-\alpha_i)^1$ and β^l are compatible if and only if the simple root expansion of β^l does not involve α_i. The *generalized cluster complex* $\Delta^s(\Phi)$ associated to a root system Φ is defined to be the simplicial complex whose faces are subsets of roots in $\Phi^s_{\geq -1}$, which are pairwise compatible.

A feasible q-analogue $X(q)$ of face numbers plays an essential role in the cyclic sieving phenomenon. Fomin and Reading derived a formula for the face numbers of various dimensions of $\Delta^s(\Phi)$ in terms of Coxeter numbers and exponents (see [**3**, Theorem 8.5]). Let $f_k(\Phi, s)$ denote the *k-face number* of $\Delta^s(\Phi)$. For Φ of type A_n, B_n, D_n, and $I_2(a)$, the k-face numbers can be expressed explicitly as follows.

THEOREM 2.1. *For Φ of type A_n, B_n, D_n, and $I_2(a)$, the face numbers of the generalized cluster complex $\Delta^s(\Phi)$ are given by*

(i) $f_k(A_n, s) = \dfrac{1}{k+1}\dbinom{s(n+1)+k+1}{k}\dbinom{n}{k}$,

(ii) $f_k(B_n, s) = \dbinom{sn+k}{k}\dbinom{n}{k}$,

(iii) $f_k(D_n, s) = \dbinom{s(n-1)+k}{k}\dbinom{n}{k} + \dbinom{s(n-1)+k-1}{k}\dbinom{n-2}{k-2}$,

(iv) $f_1(I_2(a), s) = sa + 2$, *and* $f_2(I_2(a), s) = \dfrac{(sa+2)(s+1)}{2}$.

Case (i) of Theorem 2.1 is due to J. H. Przytycki and A. S. Sikora [**7**] and case (ii) is due to E. Tzanaki [**11**]. We remark that Fomin and Reading's formula is given as a polynomial involving s, which is in the form of the unique factorization of that polynomial into irreducibles. However, it is impractical to derive feasible q-analogues $X(q)$ of face numbers from that formula except for $k = n$. The q-analogues $X(q)$ that serve our purpose are derived case by case from Theorem 2.1. (See Theorems 3.1, 4.1, 5.1, and 6.1).

For the special case $k = n$, from [**3**, Theorem 8.4], the number of facets of $\Delta^s(\Phi)$ can be expressed uniformly as

$$(5) \qquad\qquad \mathrm{Cat}^{(s)}(\Phi) := \prod_{i=1}^{n} \frac{sh + e_i + 1}{e_i + 1},$$

In this paper, we aim to prove the cyclic sieving phenomenon for the generalized cluster complexes $\Delta^s(\Phi)$ in the framework that X is the set of k-faces of $\Delta^s(\Phi)$, C is the cyclic group of order $sh + 2$ generated by Γ_s, and $X(q)$ is a q-analogue of the k-face numbers. For Φ of type A_n, B_n, and D_n, our results rely on Fomin and Reading's realization constructed in terms of polygon-dissections [**3**, Section 5]. Under this realization, the cyclic group C corresponds to rotation of the given polygon. In type $I_2(a)$, we make use of the graph-representation of $\Delta^s(I_2(a))$ given in [**3**, Example 4.4]. For Φ of exceptional type E_6, E_7, E_8, F_4, H_3, and H_4, a complete verification of such a phenomenon is given only for the facets of $\Delta^s(\Phi)$ and only when $s = 1$.

When $k = n$, our polynomials $X(q)$ for the facets of $\Delta^s(\Phi)$ agree with the generalized q-Catalan numbers defined by

$$(6) \qquad\qquad \mathrm{Cat}^{(s)}(\Phi, q) := \prod_{i=1}^{n} \frac{[sh + e_i + 1]_q}{[e_i + 1]_q}.$$

As a result, we prove the following conjecture mentioned by Reiner-Stanton-White, for $\Delta^s(\Phi)$ in types A_n, B_n, D_n, and $I_2(a)$.

CONJECTURE 2.2. ([**9**]) *For a positive integer s, let X be the set of facets of the generalized cluster complex $\Delta^s(\Phi)$. Let $X(q) = Cat^{(s)}(\Phi, q)$ be defined in (6). Let the cyclic group C of order $sh + 2$ generated by Γ_s act on X. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

For $k < n$, our polynomials $X(q)$ for the k-faces of $\Delta^s(\Phi)$ seem to be artificially tailored to serve the purpose of cyclic sieving phenomenon (especially (9) in type D_n). We are interested in finding the genuine q-analogue $X(q)$ of face numbers, which involves the Coxeter number and exponents of Φ in the spirit of (6). Such a polynomial $X(q)$ should not only lead to a unified result for the k-faces of $\Delta^s(\Phi)$ as the one in Conjecture 2.2 but also be consistent with other combinatorial structures (e.g. noncrossing partitions) in connection with Coxeter groups. We leave it as an open problem.

Open Problem. What is the genuine q-analogue $X(q)$ of the k-face numbers of $\Delta^s(\Phi)$?

3. The cyclic sieving phenomenon for $\Delta^s(A_{n-1})$

Let P be a regular polygon with $sn + 2$ vertices labeled by $\{1, 2, \ldots, sn + 2\}$ counterclockwise. Consider the set of dissections of P into $(sj + 2)$-gons $(1 \leq j \leq n - 1)$ by noncrossing diagonals. Such dissections are called *s-divisible*. For convenience, a diagonal in an s-divisible dissection is called *s-divisible*. Consider a root system Φ of type A_{n-1}. Following [**3**, Section 5.1], the roots of $\Phi^s_{\geq -1}$ can be identified with the s-divisible diagonals of P as follows. For $1 \leq i \leq \frac{n}{2}$, the root $-\alpha_{2i-1}$ is identified with the diagonal connecting points $s(i - 1) + 1$ and $s(n - i) + 2$. For $1 \leq i \leq \frac{n-1}{2}$, the root $-\alpha_{2i}$ is identified with the diagonal connecting points $si + 1$ and $s(n - i) + 2$. These $n - 1$ diagonals form an s-snake of negative simple roots. For each positive root $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$ $(1 \leq i \leq j \leq n - 1)$, there are exactly s diagonals, which are s-divisible, intersecting the diagonals $-\alpha_i, \ldots, -\alpha_j$ and no other diagonals in the s-snake. This collection of diagonals is of the form $D, \Gamma_s^1(D), \ldots, \Gamma_s^{s-1}(D)$, for some diagonal D. For $1 \leq k \leq s$, we identify α_{ij}^k with $\Gamma_s^{k-1}(D)$. Figure 4 shows the s-snake for $s = 3$ and $n = 4$, along with the diagonals identified with the colored roots α_{23}^1, α_{23}^2, and α_{23}^3. Under this bijection, the k-faces of $\Delta^s(A_{n-1})$ correspond to the s-divisible dissections of P using k noncrossing diagonals, and Γ_s corresponds to clockwise rotation of P carrying point 2 to point 1, etc.
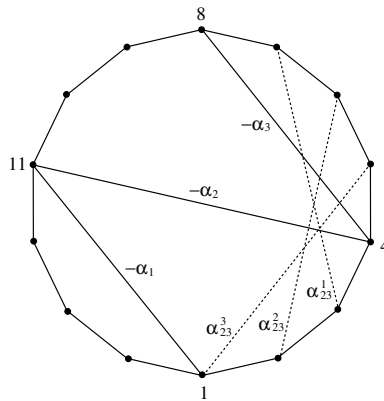

FIGURE 4. The s-snake in type A_3.

A feasible polynomial for $X(q)$ is the natural q-analogue of Theorem 2.1(i). Define

$$(7) \qquad G(s, n, k; q) = \frac{1}{[k + 1]_q} \begin{bmatrix} sn + k + 1 \\ k \end{bmatrix}_q \begin{bmatrix} n - 1 \\ k \end{bmatrix}_q,$$

for $0 \leq k \leq n - 1$. Note that $G(s, n, k; 1) = f_k(A_{n-1}, s)$. As a generalization of Theorem 1.1, we prove that the k-faces of $\Delta^s(A_{n-1})$ exhibit the cyclic sieving phenomenon under the group action $\langle \Gamma_s \rangle$.

THEOREM 3.1. *For positive integers s and n, let X be the set of s-divisible dissections of an $(sn+2)$-gon using k noncrossing diagonals. Let the cyclic group C of order $sn + 2$ act on X by cyclic rotation of the polygon. Let $X(q) := G(s, n, k; q)$. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

For example, take $s = 2$, $n = 3$, and $k = 2$. Then $X(q) \equiv 2 + q + 2q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7$ (mod $q^8 - 1$). As shown in Figure 5, there are 12 2-divisible dissections of an octagon using 2 noncrossing diagonals. These dissections are partitioned into two orbits under a group action by cyclic rotation, one of which is free and the other has a stabilizer of order 2.
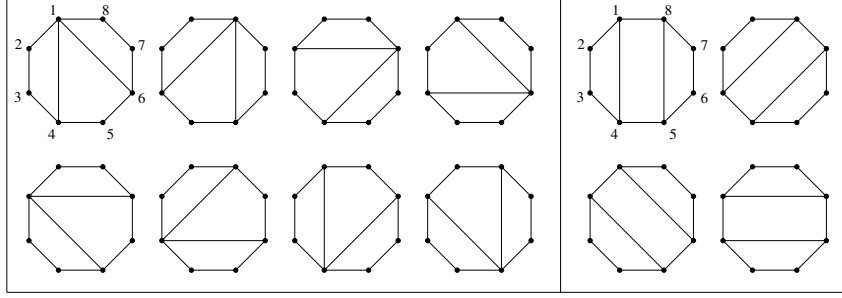

FIGURE 5. The 2-divisible dissections of an octagon using 2 noncrossing diagonals.

By verifying condition (1) mentioned in the introduction, Theorem 3.1 is proved by Proposition 3.2 and Corollary 3.5.

PROPOSITION 3.2. *For $d \geq 2$ a divisor of $sn + 2$, let ω be a primitive d-th root of unity. Then*

$$[G(s, n, k; q)]_{q=\omega} = \begin{cases} \binom{\frac{sn+k+1}{2}}{\frac{k+1}{2}}\binom{\frac{n-2}{2}}{\frac{k-1}{2}} & \text{if } d = 2, \ k \text{ odd, and } n \text{ even} \\ \binom{\frac{sn+2+k}{d} - 1}{\frac{k}{d}}\binom{\lfloor \frac{n-1}{d} \rfloor}{\frac{k}{d}} & \text{if } d \geq 2 \text{ and } d | k, \\ 0 & \text{otherwise.} \end{cases}$$

Let X be the set of s-divisible dissections of an $(sn + 2)$-gon P using k noncrossing diagonals. Let C be the cyclic group of order $sn + 2$ acting on X by cyclic rotation of P. For $d \geq 2$ a divisor of $sn + 2$, let $U(s, n, k, d) \subseteq X$ denote the set of dissections that are invariant under d-fold rotation (i.e., a subgroup C_d of order d of C generated by a $\frac{2\pi}{d}$-rotation of P). In the following, we shall enumerate $U(s, n, k, d)$ to complete the proof of Theorem 3.1.

For any centrally symmetric dissection, we observe that there is either a diameter or an $(sj + 2)$-gon in the center. Hence if $U(s, n, k, d)$ is nonempty, then either $d = 2$ and k is odd, or $d \geq 2$ and d divides k. These two cases are treated in Propositions 3.3 and 3.4, respectively.

PROPOSITION 3.3. *For positive integers s and n, with sn even and k odd, we have*

$$|U(s, n, k, 2)| = \begin{cases} \binom{\frac{sn+k+1}{2}}{\frac{k+1}{2}}\binom{\frac{n-2}{2}}{\frac{k-1}{2}} & \text{if } n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

For the latter case, the result relies on a bijection (Proposition 3.4), which is inspired by a work of Tzanaki [11]. In fact, for the special case $d = 2$ and n even, the result has been obtained by Tzanaki in [11, Corollary 3.2] by a bijection similar to the one given by Przytycki and Sikora in [7, Theorem 1]. We extend this method to enumerate d-fold rotationally symmetric dissections for all $d \geq 2$.

PROPOSITION 3.4. *For $d \geq 2$ a common divisor of $sn + 2$ and k, there is a bijection between the set $U(s, n, k, d)$ and the cartesian product of the set of sequences $\{1 \leq a_1 \leq a_2 \leq \cdots \leq a_{\frac{k}{d}} \leq \frac{sn+2}{d}\}$ and the set of sequences $(\epsilon_1, \ldots, \epsilon_m) \in \{0, 1\}^m$ with exactly $\frac{k}{d}$ entries equal to 1, where $m = \lfloor \frac{n-1}{d} \rfloor$.*

COROLLARY 3.5. *For $d \geq 2$ a common divisor of $sn + 2$ and k, we have*

$$|U(s,n,k,d)| = \binom{\frac{sn+2+k}{d} - 1}{\frac{k}{d}} \binom{\lfloor \frac{n-1}{d} \rfloor}{\frac{k}{d}}.$$

Since the results of Proposition 3.3 and Corollary 3.5 agree with that of Proposition 3.2, the proof of Theorem 3.1 is completed.

4. The cyclic sieving phenomenon for $\Delta^s(B_n)$

Following [**3**, Section 5.2], the generalized cluster complex $\Delta^s(\Phi)$ for type B_n can be realized as follows (see also [**10, 11**]). Let P be a regular polygon with $2sn+2$ vertices. The vertices are labeled by $\{1, 2, \ldots, sn+1, \overline{1}, \overline{2}, \ldots, \overline{sn+1}\}$ counterclockwise. A *B-diagonal* of P is either (i) a *diameter*, i.e., a diagonal that connects a pair of antipodal points $i, \overline{i}$, for some $1 \leq i \leq sn+1$, or (ii) a pair of s-divisible diagonals ij, $\overline{ij}$ for two distinct $i, j \in \{1, 2, \ldots, sn+1, \overline{1}, \overline{2}, \ldots, \overline{sn+1}\}$, nonconsecutive around the boundary of the polygon. (It is understood that if $a = \overline{i}$, then $\overline{a} = i$). Note that a B-diagonal dissects P into a pair of $(sm+2)$-gons and a centrally symmetric $(2s(n-m)+2)$-gon $(1 \leq m \leq n)$. The vertices of $\Delta^s(B_n)$ correspond to the B-diagonals of P, and the faces of $\Delta^s(B_n)$ correspond to s-divisible dissections of P using B-diagonals. The maximal faces correspond to centrally symmetric dissections of P into $(s+2)$-gons. For $s = 1$, this complex is the dual complex of the n-dimensional *cyclohedron*, or *Bott-Taubes polytop* (see [**2**, Lecture 3]). Under this bijection, the map Γ_s corresponds to clockwise rotation of P carrying point 2 to point 1, etc. Taking Gaussian coefficients with base q^2 in Theorem 2.1(ii), we define

$$(8) \qquad H(s,n,k;q) = \begin{bmatrix} sn+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2},$$

for $0 \leq k \leq n$. Note that $H(s,n,k;1) = f_k(B_n, s)$. We prove that the faces of $\Delta^s(B_n)$ exhibit the cyclic sieving phenomenon under the group action $\langle \Gamma_s \rangle$.
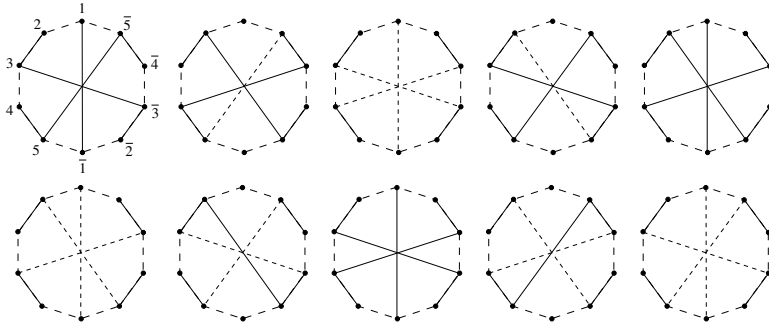
THEOREM 4.1. *For positive integers s and n, let X be the set of s-divisible dissections of a $(2sn+2)$-gon using k noncrossing B-diagonals. Let the cyclic group C of order $2sn+2$ act on X by cyclic rotation of the polygon. Let $X(q) := H(s,n,k;q)$. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

5. The cyclic sieving phenomenon for $\Delta^s(D_n)$

Following [**3**, Section 5.3], the generalized cluster complex $\Delta^s(\Phi)$ for type D_n can be realized as follows. Let P be a regular polygon with $2s(n-1)+2$ vertices. The vertices are labeled by $\{1, 2, \ldots, s(n-1)+1, \overline{1}, \overline{2}, \ldots, \overline{s(n-1)+1}\}$ counterclockwise. There are two copies of each diameter, one colored red and the other colored blue. A *D-diagonal* of P is either (i) a red or a blue diameter, or (ii) a non-diameter B-diagonal. For $1 \leq i \leq s(n-1)+1$, let L_i denote the diameter connecting $i, \overline{i}$, and let $\kappa(L_i)$ denote the color of L_i. The map Γ_s acts on $\Delta^s(D_n)$ by rotating P clockwise, carrying point 2 to point 1, and switching the colors of certain diameters. Specifically, Γ_s carries the ordered pair $(L_j, \kappa(L_j))$ to $(L_{j-1}, \kappa(L_{j-1}))$, where $\kappa(L_{j-1}) \neq \kappa(L_j)$ if $j = 1$ or $j \equiv 2 \pmod{s}$, and $\kappa(L_{j-1}) = \kappa(L_j)$ otherwise. A relation of compatibility among D-diagonals is defined as follows. Two diameters with the same endpoints and different colors are compatible. Two diameters with distinct endpoints are compatible if and only if applying Γ_s repeatedly until either of them is carried to L_1 results in diameters of the same color. In all the other cases, two D-diagonals are compatible if they are noncrossing in the sense of type-B dissections. The faces of $\Delta^s(D_n)$ correspond to dissections of P using compatible D-diagonals. For convenience, the set $\{(sj+2, sj+1) : 0 \leq j \leq n-1\}$ of edges of P are called *color-switchers*, where $s(n-1)+2 = \overline{1}$. Figure 6 shows the orbit of a maximal face of $\Delta^2(D_3)$ under the action of Γ_2, along with the color-switchers and their opposite edges, drawn as broken edges, indicating the locations at which diameters change colors.

We define the polynomial $F(s,n,k;q)$ by

$$(9) \qquad \begin{aligned} F(s,n,k;q) := & \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q^2} \cdot q^n \\ & + \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} \cdot q^n. \end{aligned}$$

FIGURE 6. The orbit of a maximal face of $\Delta^2(D_3)$.

Note that $F(s, n, k; 1) = f_k(D_n, s)$. We shall prove that the k-faces of $\Delta^s(D_n)$, along with $F(s, n, k; q)$, exhibit the cyclic sieving phenomenon.

THEOREM 5.1. *For positive integers s and n, let X be the set of s-divisible dissections of a $(2s(n-1)+2)$-gon using k compatible D-diagonals. Let C be the cyclic group of order $2s(n-1) + 2$ generated by Γ_s that acts on X. Let $X(q) := F(s, n, k; q)$. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

6. The cyclic sieving phenomenon for $\Delta^s(I_2(a))$

For a root system Φ of type $I_2(a)$, the complex $\Delta^s(I_2(a))$ is an $(s+1)$-regular graph on $sa + 2$ vertices, whose edges are facets. As shown in [**3**, Example 4.4], this graph can be constructed in the plane on a circle of $sa + 2$ points labeled from 0 to $sa + 1$ clockwise. For a odd, the edge set has $(am + 2)$-fold rotational symmetry and connects each vertex v to the $s + 1$ vertices $v + \frac{s(a-1)}{2} + j \pmod{sa + 2}$, for $j = 1, \ldots, s + 1$. Figure 7(a) shows this graph for $s = 2$ and $a = 5$. In this case, the map Γ_s corresponds to a counterclockwise rotation of the graph by $\frac{2\pi}{sa+2}$. For a even, fixing an odd integer i, the edge set has $\left(\frac{sa+2}{2}\right)$-fold rotational symmetry and connects 0 to the vertices $i, i+2, \ldots, i+2s$. Figure 7(b) is $\Delta^2(I_2(4))$ drawn in this style with $i = 1$. In this case, the map Γ_s acts by a counterclockwise rotation of the graph by $\frac{4\pi}{sa+2}$. Along with the q-analogue $X(q)$ of Theorem 2.1(iv), we prove the cyclic sieving phenomenon for the facets of $\Delta^s(\Phi)$.
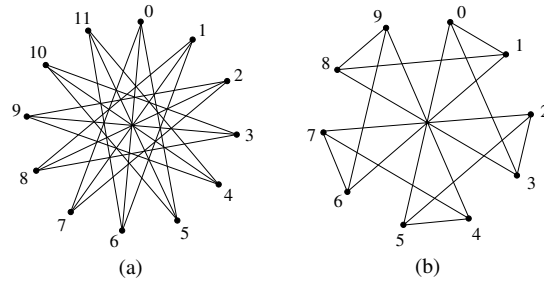
FIGURE 7. Representation of $\Delta^2(I_2(5))$ and $\Delta^2(I_2(4))$.

THEOREM 6.1. *Let X be the edge set of the graph $\Delta^s(I_2(a))$. Define*

$$X(q) := \frac{[sa + 2]_q}{[2]_q} \cdot \frac{[sa + a]_q}{[a]_q}.$$

Let $C = \mathbb{Z}_{sa+2}$ be the cyclic group that acts on X by cyclic rotation of the graph. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

7. The cases of exceptional types

In this section, we consider the cluster complex $\Delta(\Phi)$ of exceptional type E_6, E_7, E_8, F_4, H_3, and H_4.

When $k = n$, the polynomial $X(q) = \text{Cat}(\Phi, q)$ defined in (4) is a feasible q-analogue of the number of facets of $\Delta(\Phi)$. We verify Theorem 1.2 affirmatively for $\Delta(\Phi)$ of exceptional type. To see this, with the Coxeter numbers and exponents of Φ listed in Figure **??**, the polynomials $X(q)$ are expanded as follows.

  (i) In type E_6, $X(q) \equiv 67 + 52q + 67q^2 + 52q^3 + \cdots + 67q^{12} + 52q^{13} \pmod{q^{14} - 1}$.
  (ii) In type E_7, $X(q) \equiv 416 + 416q^2 + 416q^4 + \cdots + 416q^{18} \pmod{q^{20} - 1}$.
  (iii) In type E_8, $X(q) \equiv 1574 + 1562q^2 + 1572q^4 + 1562q^6 + \cdots + 1574q^{24} + 1562q^{26} + 1572q^{28} + 1562q^{30}$
      $\pmod{q^{32} - 1}$.
  (iv) In type F_4, $X(q) \equiv 15 + 15q^2 + 15q^4 + \cdots + 15q^{12} \pmod{q^{14} - 1}$.
  (v) In type H_3, $X(q) \equiv 6 + 5q^2 + 5q^4 + 6q^6 + 5q^8 + 5q^{10} \pmod{q^{12} - 1}$.
  (vi) In type H_4, $X(q) \equiv 18 + 17q^2 + 18q^4 + 17q^6 + \cdots + 18q^{28} + 17q^{30} \pmod{q^{32} - 1}$.

As searched by a computer, the orbit-structures for the k-faces of $\Delta(\Phi)$ under the cyclic group C generated by Γ are shown in Figure 8. We write $a_1(b_1)$, $a_2(b_2), \ldots$, $a_t(b_t)$ for the orbit-structure of the k-faces that are partitioned into b_i orbits of size a_i, for $1 \le i \le t$, in which case the number of k-faces is equal to $a_1 b_1 + \cdots + a_t b_t$.

k	E_6	E_7	E_8	F_4	H_3	H_4
1	14(2), 7(2)	10(7)	16(8)	7(4)	6(3)	16(4)
2	14(26), 7(5)	10(94), 5(1)	16(149), 8(3)	7(19)	6(8)	16(21), 8(1)
3	14(104), 7(13)	10(518)	16(1121)	7(30)	6(5), 2(1)	16(35)
4	14(195), 7(18)	10(1410), 5(1)	16(4211), 8(13), 4(2)	7(15)		16(17), 8(1)
5	14(171), 7(15)	10(2020), 2(1)	16(8778)			
6	14(52), 7(15)	10(1456)	16(10230), 8(22)			
7		10(416)	16(6270)			
8			16(1562), 8(10), 4(2)			

FIGURE 8. The orbit-structures of the k-faces of $\Delta(\Phi)$ of exceptional types.

We observe that the orbit-structures for the facets of $\Delta(\Phi)$ shown in Figure 8 agree with the expansions (i)-(vi) of $X(q) \pmod{q^{h+2} - 1}$. Hence the cyclic sieving phenomenon holds. Consequently, together with the special case $s = 1$ and $k = n$ in Theorems 3.1, 4.1, 5.1, and 6.1, Theorem 1.2 is proved.

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