On double Hurwitz numbers in genus 0

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ABSTRACT. We study double Hurwitz numbers in genus zero counting the number of covers $\mathbb{C}\mathbf{P}^1 \to \mathbb{C}\mathbf{P}^1$ with two branching points with a given branching behavior. By the recent result due to Goulden, Jackson and Vakil, these numbers are piecewise polynomials in the multiplicities of the preimages of the branching points. We describe the partition of the parameter space into polynomiality domains, called chambers, and provide an expression for the difference of two such polynomials for two neighboring chambers. Besides, we provide an explicit formula for the polynomial in a certain chamber called totally negative, which enables us to calculate double Hurwitz numbers in any given chamber as the polynomial for the totally negative chamber plus the sum of the differences between the neighboring polynomials along a path connecting the totally negative chamber with the given one.

1. Introduction and results

Let $\mu = (\mu_1, \ldots, \mu_m)$ be a partition of an integer d. The *Hurwitz number* h_μ^g is the number of genus g branched covers of $\mathbb{C}\mathbf{P}^1$ with branching corresponding to μ over a fixed point (usually identified with ∞) and an appropriate number of fixed simple branching points. Recall that each cover is counted with the weight c^{-1}, where c is the number of automorphisms of the cover.

Hurwitz numbers possess a rich structure explored by many authors in different fields, such as algebraic geometry, representation theory, integrable systems, combinatorics, and mathematical physics. Being very far from trying to describe all these achievements, we mention only the so-called ELSV-formula (see [**1, 2, 6**]) that relates Hurwitz numbers to intersection theory on the moduli space of curves. This connection has led to many consequences, including several new proofs of Witten's conjecture.

Let $\nu = (\nu_1, \ldots, \nu_n)$ be another partition of the same integer d. The *double Hurwitz number* $h_{\mu;\nu}^g$ is the number of genus g branched covers of $\mathbb{C}\mathbf{P}^1$ with branchings corresponding to μ and ν over two fixed points (in what follows we identify them with ∞ and 0, respectively) and an appropriate number of fixed simple branching points. We denote the latter number $r_{\mu;\nu}^g$; by the Riemann–Hurwitz theorem, $r_{\mu;\nu}^g = m+n+2g-2$. To simplify the exposition, we assume that the points mapped to ∞ and 0 are labelled, so the double Hurwitz numbers under this convention are $|\operatorname{Aut}\mu||\operatorname{Aut}\nu|$ larger than they would be under the usual convention; here $\operatorname{Aut}\mu$ and $\operatorname{Aut}\nu$ stand for the groups of symmetries of the sets μ and ν, respectively. It follows from the Riemann existence theorem that double Hurwitz number can be redefined in purely combinatorial terms as $|\operatorname{Aut}\mu||\operatorname{Aut}\nu|/d!$ times the number of transitive ordered factorizations of the unity in the symmetric group S_d. The latter quantity is the number of $(r+2)$-tuples $(\sigma_1, \tau_1, \tau_2, \ldots, \tau_r, \sigma_2)$ such that τ_i are transpositions, $r = r_{\mu;\nu}^g$, the cycle type of σ_1 equals μ, the cycle type of σ_2 equals ν, $\sigma_1^{-1}\tau_1 \cdots \tau_r \sigma_2 = \operatorname{id}$, and the group generated by $\sigma_1, \tau_1, \ldots, \tau_r$ acts transitively on $[1, d]$.

Most of the known results concerning double Hurwitz numbers treat only the so-called one-part (or polynomial) case, when $m = 1$ and n is arbitrary. One-part double Hurwitz numbers in genus zero where studied in [**13**] (see also [**12**] for an earlier version of the same result). It is proved there that

$$(1.1) \qquad h_{(d);\nu}^0 = (n-1)!d^{n-2};$$

recall that $n - 1 = r^0_{(d);\nu}$. In fact, a combinatorial result much more general than (1.1) was obtained already in [4] using Lagrange inversion; in [13] the same formula was reproved by methods of algebraic geometry, and in this way its algebro-geometric meaning was clarified.

Much less is known for double Hurwitz numbers with arbitrary μ and ν. In fact, there are only two general results. First, it is proved in [9] that the exponent of the generating function for the double Hurwitz numbers is a τ-function for the Toda hierarchy of Ueno and Takasaki. Second, Theorem 2.1 in [5] states that for fixed g, m, and n, double Hurwitz numbers are piecewise polynomial in variables $\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n$, and that the highest degree of this piecewise polynomial is constant and equal to $m + n + 4g - 3$. Moreover, it is proved in [5] that for genus zero case this piecewise polynomial is homogeneous.

In a different direction, Theorem 4.1 of the same paper treats $h^g_{\mu;\nu}$ as a function of g for fixed partitions μ and ν. In principle, any such function can be obtained by recursive computation; the only case done explicitly is $m = n = 2$ and $\mu_1 > \nu_1 > \nu_2 > \mu_2$. Similar results are also obtained in [8] for all double Hurwitz numbers of degree at most 5.

Still another approach suggested in [5] provides exact formulas for double Hurwitz numbers in genus zero when $m = 2$ or $m = 3$. In particular,

$$(1.2) \qquad h^0_{\mu_1,\mu_2;\nu} = \frac{n! |\operatorname{Aut} \nu|}{d} \sum \frac{l(\rho)! \prod_{j \geq 1} \rho_j}{|\operatorname{Aut} \rho||\operatorname{Aut} \sigma||\operatorname{Aut} \tau|} (\mu_1 - |\sigma|)(\mu_2 - |\tau|) \mu_1^{l(\sigma)-1} \mu_2^{l(\tau)-1},$$

where the summation is over partitions $\rho = (\rho_1, \ldots, \rho_{l(\rho)})$, $\sigma = (\sigma_1, \ldots, \sigma_{l(\sigma)})$, $\tau = (\tau_1, \ldots, \tau_{l(\tau)})$ with $\rho \cup \sigma \cup \tau = \nu$ and $|\sigma| = \sum_{i=1}^{l(\sigma)} \sigma_i < \mu_1$, $|\tau| = \sum_{i=1}^{l(\tau)} \tau_i < \mu_2$, see Corollary 5.11 in [5].

In this note we study the homogeneous piecewise polynomial of degree $m + n - 3$ defining double Hurwitz numbers $h^0_{\mu;\nu}$. To formulate the results we need to introduce some notation. For any sequence $a = (a_1, \ldots, a_k)$ and any subset $K \subseteq [1, k]$ we denote by $a(K)$ the subsequence of a consisting of terms a_i, $i \in K$, and by a_K the sum of all elements in $a(K)$. We consider $x_1 = \mu_1, \ldots, x_m = \mu_m, y_1 = \nu_1, \ldots, y_n = \nu_n$ as coordinates of a point in $\mathbb{R}^{m+n}$. The *parameter space* is the cone in $\mathbb{R}^{m+n}$ given by the inequalities $x_1 \geq \cdots \geq x_m \geq 0$, $y_1 \geq \cdots \geq y_n \geq 0$ and the equality $x_{[1,m]} = y_{[1,n]}$. A *resonance* is a hyperplane $x_I = y_J$, where $I \subset [1, m]$ and $J \subset [1, n]$ are proper subsets. The connected components of the complement to the union of all resonances are called *chambers*.

THEOREM 1.1. *Let (μ, ν) vary within a closure of a chamber, then double Hurwitz numbers $h^0_{\mu;\nu}$ are given by a homogeneous polynomial of degree $m + n - 3$.*

Consider a chamber C, and let P_C be the corresponding homogeneous polynomial. The most convenient way to identify C is to pick up a reference point $(\alpha, \beta) \in C$; in this case we write $P_{\alpha;\beta}$ instead of P_C. Observe that the only role of the reference point (α, β) is to indicate the choice of the chamber, and that its coordinates are not necessary integers. It is clear that $P_{\mu;\nu}(\mu, \nu) = h^0_{\mu;\nu}$; however, $P_C(\mu', \nu')$ may differ from $h^0_{\mu';\nu'}$ if $(\mu', \nu') \notin C$.

The total number of resonances is equal to $2(2^{m-1} - 1)(2^{n-1} - 1)$, since each resonance $x_I = y_J$ can be also written as $x_{\bar{I}} = y_{\bar{J}}$, where bar stands for the complement. In what follows we always assume that $1 \notin I$. Therefore, each chamber is defined by a sequence of $w = 2(2^{m-1} - 1)(2^{n-1} - 1)$ signs of the expressions $x_I - y_J$; observe that the total number of chambers is less than 2^w, since certain combinations of signs are impossible. We say that two chambers are *neighboring* along the resonance $x_I = y_J$ if the corresponding sign sequences differ only in the position corresponding to this resonance.

Let C and C' be two chambers neighboring along the resonance $x_I = y_J$; without loss of generality we assume that $x_I - y_J > 0$ in C.

THEOREM 1.2. *Let (μ, ν) be an arbitrary point in C. Then*

$$P_C - P_{C'} = \binom{m + n - 2}{|I| + |J| - 1} (x_I - y_J) P_{\mu(\bar{I}), \mu_I - \nu_J; \nu(\bar{J})} P_{\mu(I); \nu(J), \mu_I - \nu_J}.$$

REMARK 1. Here and in what follows we omit the arguments of polynomials whenever this does not lead to a confusion. Usually, the arguments are formed from the components of x and y according to the same rules as the coordinates of the reference point are formed from the parts of μ and ν. For example, $P_{\mu(I); \nu(J), \mu_I - \nu_J}$ has $|I| + |J| + 1$ arguments. The first $|I|$ of them are x_i, $i \in I$, then follow y_j, $j \in J$, and the last argument is $x_I - y_J$.

Consider the *totally negative* chamber, that is, the one for which all the signs in the corresponding sequence are negative. The following result was conjectured in [**5**] (see Conjecture 5.10).

THEOREM 1.3. *The polynomial corresponding to the totally negative cell is given by* $(m+n-2)! x_1^{n-1}(x_1 + \cdots + x_m)^{m-2}$.

Theorems 1.1–1.3 give rise to recurrence relations expressing double Hurwitz numbers of degree d via double Hurwitz numbers of lesser degrees. In general, these relations are rather cumbersome, however, for $m = 2$ one gets a very simple explicit formula, which is easier than (1.2). The details of the corresponding computations are presented in Section 2. Section 3 contains a sketch of the proof of the main theorems. The proof is divided into a sequence of lemmas. The proofs of the lemmas involve certain techniques from algebraic geometry and are omitted in this version; they will be published elsewhere.

2. Computations

2.1. General recurrence. To find a recurrence relation for the double Hurwitz number $h^0_{\mu;\nu}$ via Theorems 1.1–1.3, one has to pick a path connecting the totally negative chamber with the chamber containing the point (μ, ν) in the parameter space. If (μ, ν) lies on one or more resonances, one can choose any of the adjacent chambers in an arbitrary way. By multiplying all coordinates by a sufficiently big integer $t > d^{m+n}$ and perturbing slightly the resulting point within the same chamber, one can ensure that the obtained point $(t\mu, \nu')$ is in general position, that is, $\nu'_I = \nu'_J$ if and only if $I = J$. Pick the point $(td - m(m-1)/2, m-1, m-2, \ldots, 1, \nu')$ in the totally negative chamber and connect it with $(t\mu, \nu')$ by the following path consisting of $m - 1$ segments. The first segment is of the form

$$(td - m(m-1)/2 - s, m - 1 + s, m - 2, \ldots, 1, \nu'), \qquad s = 0, 1, \ldots, t\mu_2 - m + 1,$$

the second segment is of the form

$$(td - t\mu_2 - (m-1)(m-2)/2 - s, t\mu_2, m - 2 + s, m - 3, \ldots, 1, \nu'), \qquad s = 0, 1, \ldots, t\mu_3 - m + 2,$$

and so on. It is easy to see that each point on this path belongs to at most one resonance. To formulate the recurrence relation, pick arbitrary numbers $\varepsilon_3, \varepsilon_4, \ldots, \varepsilon_m$ and δ satisfying inequalities $0 < \delta < \varepsilon_m < \cdots < \varepsilon_3 < 1/m$ and denote $\varepsilon = (\varepsilon_3, \ldots, \varepsilon_m)$; the exact values of ε_i and δ do not have any meaning, since these numbers will be only used to indicate the corresponding chamber. Clearly, a resonance $x_I = y_J$ is intersected by the above path, and hence contributes to $h^0_{\mu;\nu}$, if and only if $\mu_I > \nu_J$. If this is the case, we consider $I = \{i_1, \ldots, i_{|I|}\}$ and define

$$k = \min\{j \in [1, |I|] : \mu_{i_1} + \cdots + \mu_{i_j} > \nu_J\}.$$

Any $i \in [1, m]$, $i \neq 1, i_k$, can be related to one of the four subsets:

$$I_1 = \{i \in I : i < i_k\}, \qquad \bar{I}_1 = \{i \notin I : 1 < i < i_k\},$$
$$I_2 = \{i \in I : i > i_k\}, \qquad \bar{I}_2 = \{i \notin I : i > i_k\}.$$

We thus get the following result.

THEOREM 2.1. *Double Hurwitz numbers are given by*

$$h^0_{\mu;\nu} = (m + n - 2)! d^{m-2} \mu_1^{n-1} + \sum_{\mu_I > \nu_J} \binom{m + n - 2}{|I| + |J| - 1}(\mu_I - \nu_J)$$

$$\times P_{\xi_1, \mu(\bar{I}_1), \varepsilon(\bar{I}_2), \delta; \nu(\bar{J})}(\mu_1, \mu(\bar{I}_1 \cup \bar{I}_2), \mu_I - \nu_J, \nu(\bar{J})) P_{\mu(I_1), \xi_2, \varepsilon(I_2); \nu(J), \delta}(\mu(I_1), \mu_{i_k}, \mu(I_2), \nu(J), \mu_I - \nu_J),$$

where $\xi_1 = \nu_{\bar{J}} - \mu_{\bar{I}_1} - \varepsilon_{\bar{I}_2} - \delta$, $\xi_2 = \nu_J - \mu_{I_1} - \varepsilon_{I_2} + \delta$.

2.2. Two-part double Hurwitz numbers. The general expression in Theorem 2.1 looks very cumbersome. However, in the case of two-part double Hurwitz numbers, when $m = 2$, it can be written in a very simple way. Indeed, in this case all the resonances are of the form $x_2 = y_J$, therefore $I = \{2\}$, $i_k = 2$, $I_1 = \bar{I}_1 = I_2 = \bar{I}_2 = \varnothing$. Therefore Theorem 2.1 yields

$$h^0_{\mu_1,\mu_2;\nu} = n!\mu_1^{n-1} + \sum_{\mu_2 > \nu_J} \binom{n}{|J|}(\mu_2 - \nu_J)P_{\nu_{\bar{J}}-\delta,\delta;\nu(\bar{J})}(\mu_1, \mu_2 - \nu_J, \nu(\bar{J}))P_{\nu_J+\delta;\nu(J),\delta}(\mu_2, \nu(J), \mu_2 - \nu_J).$$

The second polynomial in the right hand side corresponds to one-part double Hurwitz numbers of total degree μ_2 with $|J| + 1$ zeros, hence, by (1.1),

$$P_{\nu_J+\delta;\nu(J),\delta}(\mu_2, \nu(J), \mu_2 - \nu_J) = |J|!\mu_2^{|J|-1}.$$

The first polynomial in the right hand side corresponds to the totally negative chamber for two-part double Hurwitz numbers of total degree $d - \nu_J$ with $n - |J|$ zeros, hence, by Theorem 1.3,

$$P_{\nu_{\bar{J}}-\delta,\delta;\nu(\bar{J})}(\mu_1, \mu_2 - \nu_J, \nu(\bar{J})) = (n - |J|)!\mu_1^{n-|J|-1}.$$

Observe that the first summand in the right hand side of the above formula can be also included in the regular part of the sum for $J = \varnothing$. In what follows we indicate this by writing $\sum^{\varnothing}$ instead of $\sum$.

We thus obtain the following explicit formula for the two-part double Hurwitz numbers in genus zero, which is simpler than (1.2).

COROLLARY 2.2. *The two-part double Hurwitz numbers are given by*

$$h^0_{\mu_1,\mu_2;\nu} = n!\sum_{\mu_2 > \nu_J}^{\varnothing} (\mu_2 - \nu_J)\mu_1^{n-|J|-1}\mu_2^{|J|-1}.$$

2.3. Three-part double Hurwitz numbers. Consider now the case of three-part double Hurwitz numbers, when $m = 3$. Then the first segment of the path intersects resonances of the form $x_2 = y_J$ and $x_2 + x_3 = y_J$, while the second segment of the path intersects resonances of the form $x_3 = y_J$ and $x_2 + x_3 = y_J$. Therefore, we have the following four types of intersection.

Type 1. $I = \{2\}$, $i_k = 2$, $I_1 = \bar{I}_1 = I_2 = \varnothing$, $\bar{I}_2 = \{3\}$.

By Theorem 2.1, the contribution of such an intersection equals

$$\binom{n+1}{|J|}(\mu_2 - \nu_J)P_{\nu_{\bar{J}}-\varepsilon_3-\delta,\varepsilon_3,\delta;\nu(\bar{J})}(\mu_1, \mu_3, \mu_2 - \nu_J, \nu(\bar{J}))P_{\nu_J+\delta;\nu(J),\delta}(\mu_2, \nu(J), \mu_2 - \nu_J).$$

The second polynomial in the above expression is the same as in the case of two-part double Hurwitz numbers, and its value is equal to $|J|!\mu_2^{|J|-1}$. The first polynomial corresponds to the totally negative chamber for three-part double Hurwitz numbers of total degree $d - \nu_J$ with $n - |J|$ zeros, hence, by Theorem 1.3,

$$P_{\nu_{\bar{J}}-\varepsilon_3-\delta,\varepsilon_3,\delta;\nu(\bar{J})}(\mu_1, \mu_3, \mu_2 - \nu_J, \nu(\bar{J})) = (n - |J| + 1)!\mu_1^{n-|J|-1}(d - \nu_J).$$

Therefore, the total contribution of all intersections of type 1 equals

$$(n+1)! \sum_{\mu_2 > \nu_J} (\mu_2 - \nu_J)\mu_1^{n-|J|-1}\mu_2^{|J|-1}(d - \nu_J).$$

Type 2. $I = \{2,3\}$, $i_k = 2$, $I_1 = \bar{I}_1 = \bar{I}_2 = \varnothing$, $I_2 = \{3\}$.

By Theorem 2.1, the contribution of such an intersection equals

$$\binom{n+1}{|J|+1}(\mu_2 + \mu_3 - \nu_J)P_{\nu_{\bar{J}}-\delta;\nu(\bar{J})}(\mu_1, \mu_2 + \mu_3 - \nu_J, \nu(\bar{J}))P_{\nu_J-\varepsilon_3+\delta,\varepsilon_3;\nu(J),\delta}(\mu_2, \mu_3, \nu(J), \mu_2 + \mu_3 - \nu_J).$$

The first polynomial in the above expression we have already encountered in the case of two-part double Hurwitz numbers, and its value is equal to $(n - |J|)!\mu_1^{n-|J|-1}$. The second polynomial corresponds to the chamber neighboring with the totally negative chamber for two-part double Hurwitz numbers of total degree $\mu_2 + \mu_3$ with $|J| + 1$ zeros, hence, by Corollary 2.2,

$$P_{\nu_J-\varepsilon_3+\delta,\varepsilon_3;\nu(J),\delta}(\mu_2, \mu_3, \nu(J), \mu_2 + \mu_3 - \nu_J) = (|J| + 1)!\left(\mu_2^{|J|} + (\mu_3 - (\mu_2 + \mu_3 - \nu_J))\mu_2^{|J|-1}\right)$$

$$= (|J| + 1)!\mu_2^{|J|-1}\nu_J.$$

Therefore, the total contribution of all intersections of type 2 equals

$$(n+1)! \sum_{\mu_2 > \nu_J} (\mu_2 + \mu_3 - \nu_J) \mu_1^{n-|J|-1} \mu_2^{|J|-1} \nu_J.$$

Type 3. $I = \{3\}$, $i_k = 3$, $I_1 = I_2 = \bar{I}_2 = \varnothing$, $\bar{I}_1 = \{2\}$.
By Theorem 2.1, the contribution of such an intersection equals

$$\binom{n+1}{|J|} (\mu_3 - \nu_J) P_{\nu_{\bar{J}} - \mu_2 - \delta, \mu_2, \delta; \nu(\bar{J})} (\mu_1, \mu_2, \mu_3 - \nu_J, \nu(\bar{J})) P_{\nu_J + \delta; \nu(J), \delta} (\mu_3, \nu(J), \mu_3 - \nu_J).$$

The second polynomial in the above expression is again the same as in the case of two-part double Hurwitz numbers, and its value is equal to $|J|! \mu_3^{|J|-1}$. The first polynomial corresponds to three-part double Hurwitz numbers of total degree $d - \nu_J$ with $n - |J|$ zeros, computed in some chamber intersected by the first segment of our path. To compute these numbers we have to take into account only intersections of types 1 and 2. By the above reasoning, we get

$$P_{\nu_{\bar{J}} - \mu_2 - \delta, \mu_2, \delta; \nu(\bar{J})} (\mu_1, \mu_2, \mu_3 - \nu_J, \nu(\bar{J})) = (n - |J| + 1)! \sum_{\substack{K \subseteq \bar{J} \\ \mu_2 > \nu_K}}^{\varnothing} (\mu_2 - \nu_K)(d - \nu_J - \nu_K) \mu_1^{n-|J|-|K|-1} \mu_2^{|K|-1}$$

$$+ \sum_{\substack{K \subseteq \bar{J} \\ \mu_2 > \nu_K}}^{\varnothing} (\mu_2 + (\mu_3 - \nu_J) - \nu_K) \mu_1^{n-|J|-|K|-1} \mu_2^{|K|-1} \nu_K.$$

Therefore, the total contribution of all intersections of type 3 equals

$$(n+1)! \sum_{\mu_3 > \nu_J} (\mu_3 - \nu_J) \mu_3^{|J|-1} \sum_{\substack{K \subseteq \bar{J} \\ \mu_2 > \nu_K}}^{\varnothing} \mu_1^{n-|J|-|K|-1} \mu_2^{|K|-1} (\mu_2(d - \nu_J) - \nu_K(\mu_1 + \mu_2)).$$

Type 4. $I = \{2, 3\}$, $i_k = 3$, $\bar{I}_1 = I_2 = \bar{I}_2 = \varnothing$, $I_1 = \{2\}$.
By Theorem 2.1, the contribution of such an intersection equals

$$\binom{n+1}{|J|+1} (\mu_2 + \mu_3 - \nu_J) P_{\nu_{\bar{J}} - \delta, \delta; \nu(\bar{J})} (\mu_1, \mu_2 + \mu_3 - \nu_J, \nu(\bar{J})) P_{\mu_2, \nu_J - \mu_2 + \delta; \nu(J), \delta} (\mu_2, \mu_3, \nu(J), \mu_2 + \mu_3 - \nu_J).$$

The first polynomial in the above expression we have already encountered in the case of two-part double Hurwitz numbers, and its value is equal to $(n - |J|)! \mu_1^{n-|J|-1}$. The second polynomial corresponds to arbitrary two-part double Hurwitz numbers of total degree $\mu_2 + \mu_3$ with $|J| + 1$ zeros. By Corollary 2.2, such numbers are given by

$$P_{\mu_2, \nu_J - \mu_2 + \delta; \nu(J), \delta} (\mu_2, \mu_3, \nu(J), \mu_2 + \mu_3 - \nu_J) = (|J| + 1)! \sum_{\substack{K \subseteq J \\ \nu_J - \mu_2 > \nu_K}}^{\varnothing} (\mu_3 - \nu_K) \mu_2^{|J|-|K|} \mu_3^{|K|-1}$$

$$+ (|J| + 1)! \sum_{\substack{K \subseteq J \\ \nu_J - \mu_2 > \nu_K}}^{\varnothing} (\mu_3 - \nu_K)(\mu_3 - \nu_K - (\mu_2 + \mu_3 - \nu_J)) \mu_2^{|J|-|K|-1} \mu_3^{|K|-1};$$

the first sum in the right hand side corresponds to resonances not involving the last y-coordinate, and the second one to those involving this coordinate. Therefore, the total contribution of all intersections of type 4 equals

$$(n+1)! \sum_{\mu_2 < \nu_J < \mu_2 + \mu_3} (\mu_2 + \mu_3 - \nu_J) \mu_1^{n-|J|-1} \sum_{\substack{K \subseteq J \\ \nu_J - \mu_2 > \nu_K}}^{\varnothing} (\mu_3 - \nu_K) \mu_2^{|J|-|K|-1} \mu_3^{|K|-1} (\nu_J \mu_3 - \nu_K(\mu_2 + \mu_3)).$$

Collecting all the summands and taking into account the contribution of the totally negative chamber, we get the following result.

COROLLARY 2.3. *The three-part double Hurwitz numbers are given by*

$$\frac{h^0_{\mu_1,\mu_2,\mu_3;\nu}}{(n+1)!} = \sum_{\mu_2>\nu_J}^{\varnothing} (\mu_2 - \nu_J)\mu_1^{n-|J|-1}\mu_2^{|J|-1}A_J + \sum_{\mu_3>\nu_J} (\mu_3 - \nu_J)\mu_3^{|J|-1} \sum_{\substack{K\subseteq\bar{J} \\ \mu_2>\nu_K}}^{\varnothing} \mu_1^{n-|J|-|K|-1}\mu_2^{|K|-1}B_{JK}$$

$$+ \sum_{\mu_2<\nu_J<\mu_2+\mu_3} (\mu_2 + \mu_3 - \nu_J)\mu_1^{n-|J|-1} \sum_{\substack{K\subseteq J \\ \nu_J-\mu_2>\nu_K}}^{\varnothing} (\mu_3 - \nu_K)\mu_2^{|J|-|K|-1}\mu_3^{|K|-1}C_{JK},$$

where $A_J = d\mu_2 - \nu_J(\mu_1 + \mu_2)$, $B_{JK} = (d - \nu_J)\mu_2 - \nu_K(\mu_1 + \mu_2)$, and $C_{JK} = \nu_J\mu_3 - \nu_K(\mu_2 + \mu_3)$.

3. A sketch of the proof

3.1. Integral representation for double Hurwitz numbers. Define the *Hurwitz space* $\mathcal{H}^0_{\mu;\nu}$ as the space of degree d meromorphic functions on genus 0 curves having $m + n - 2$ simple critical values $z_1, \ldots, z_{m+n-2}$ and monodromies given by μ and ν over two other points x and y; the functions are considered modulo $SL(2, \mathbb{C})$-action in the image. We assume that all preimages of the points $x, y, z_1, \ldots, z_{m+n-2}$ are labelled. The Lyashko–Looijenga map ℓ that takes a function $f \in \mathcal{H}^0_{\mu;\nu}$ to the points $x, y, z_1, \ldots, z_{m+n-2}$ can be viewed as an unramified covering of degree $h^0_{\mu;\nu}(d-2)!^{m+n-2}$ over the moduli space $\mathcal{M}_{0,m+n}$. It is well known that ℓ extends continuously to the mapping $\ell : \overline{\mathcal{H}}^0_{\mu;\nu} \to \overline{\mathcal{M}}_{0,m+n}$, where $\overline{\mathcal{H}}^0_{\mu;\nu}$ is the compactification of $\mathcal{H}^0_{\mu;\nu}$ by admissible covers and $\overline{\mathcal{M}}_{0,m+n}$ is the Deligne–Mumford compactification of the moduli space of genus 0 curves with $m + n$ marked points. On the other hand, let st $: \mathcal{H}^0_{\mu;\nu} \to \mathcal{M}_{0,m+n+(m+n-2)(d-1)}$ be the mapping that takes a function f to the set of all preimages of the points $x, y, z_1, \ldots, z_{m+n-2}$ and let $\pi : \mathcal{M}_{0,m+n+(m+n-2)(d-1)} \to \mathcal{M}_{0,m+n}$ be the projection that forgets the preimages of the points $z_1, \ldots, z_{m+n-2}$. Both these mappings extend continuously to the mappings between the compactified spaces st $: \overline{\mathcal{H}}^0_{\mu;\nu} \to \overline{\mathcal{M}}_{0,m+n+(m+n-2)(d-1)}$ and $\pi : \overline{\mathcal{M}}_{0,m+n+(m+n-2)(d-1)} \to \overline{\mathcal{M}}_{0,m+n}$. Denote by $x_1, \ldots, x_m$ the preimages of x having multiplicities $\mu_1, \ldots, \mu_m$, and by $y_1, \ldots, y_n$ the preimages of y having multiplicities $\nu_1, \ldots, \nu_n$. Finally, denote by D the divisor on $\overline{\mathcal{M}}_{0,m+n+(m+n-2)(d-1)}$ whose generic point is a two-component curve such that x_1 lies on one component and $x_2, \ldots, x_m, y_1, \ldots, y_n$ lie on the other component.

LEMMA 3.1. *One has*

$$h^0_{\mu;\nu} = (m+n-2)!\mu_1^{m+n-3} + (d-2)!^{2-m-n} \sum_{u+v=m+n-4} \mu_1 \int_{\text{st}(\overline{\mathcal{H}}^0_{\mu;\nu})} D\,(\mu_1\pi^*\psi(x_1))^u\,(\text{st}_*\,\ell^*\psi(x))^v,$$

where $\psi(p)$ is the first Chern class of the cotangent bundle at the point p.

3.2. Encoding irreducible components of $\text{st}(\overline{\mathcal{H}}^0_{\mu;\nu}) \cap D$. In view of Lemma 3.1, we will be interested in the description of the irreducible components of $\text{st}(\overline{\mathcal{H}}^0_{\mu;\nu}) \cap D$. Let f be a function whose image belongs to this intersection. Points x and y on the target curve of f belong to different components. Moreover, the number of components is exactly two, for the dimensionality reasons. Therefore, the source curve of f has a number of double points, which are all mapped to the unique double point on the target curve. The components of the source curve are of two types: those covering the component of the target curve containing x, and those covering the component of the target curve containing y. Each component of the first type contains one or more preimages of x, and each component of the second type contains one or more preimages of y. Finally, the component of the source curve containing x_1 does not contain any other preimages of x.

The irreducible components of $\text{st}(\overline{\mathcal{H}}^0_{\mu;\nu}) \cap D$ can be encoded by geometric trees in the following way. Consider two arbitrary partitions $[1, m] = \sqcup_{i=1}^k I_i$, $[1, n] = \sqcup_{j=1}^l J_j$, such that all parts I_i and J_j are nonempty and $I_1 = \{1\}$. Let T be a tree viewed as a bipartite graph with the vertices $I_1, \ldots, I_k$ in one part and $J_1, \ldots, J_l$ in the other part and let γ_e be a weight assigned to an edge e of T in such a way that the sum of γ_e over all edges incident to an arbitrary vertex I_i equals μ_{I_i}, and the sum of γ_e over all edges incident to an arbitrary vertex J_j equals ν_{J_j}. Evidently, γ_e are defined by the above condition in a unique way, and each γ_e is an integer. We say that T is a *geometric tree* if the following three conditions are satisfied:
  (i) at least one among $I_1, \ldots, I_k$ is not a singleton;
  (ii) all γ_e are positive;
  (iii) I_1 is a leaf of T.

The first condition follows from the fact that the target curve of f is stable. A similar condition for $J_1, \ldots, J_l$ follows from (iii) and hence is omitted. The second condition follows from the fact that edges of T correspond to double points of the source curve and weights are the multiplicities at these double points. The third condition is the definition of D. The set of all geometric trees is denoted by $\mathcal{T}_{\mu;\nu}$, the irreducible component of $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu}) \cap D$ encoded by a geometric tree $T \in \mathcal{T}_{\mu;\nu}$ is denoted by D_T.

EXAMPLE 1. Let $\mu = (4, 2, 1)$, $\nu = (5, 2)$. This data defines three geometric trees, presented on Fig. 1a. Three examples of non-geometric trees defined by the same data are given on Fig. 1b. The vertices are labelled by the corresponding subsets of μ and ν, and the edges are labelled by the weights γ_e.
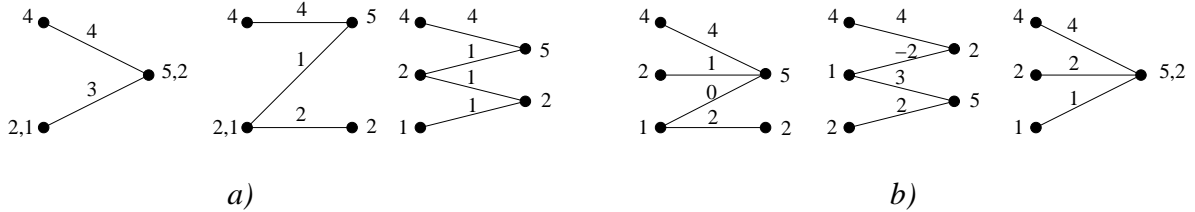

FIGURE 1. a) Geometric trees; b) Several non-geometric trees

LEMMA 3.2. *The set of geometric trees is an invariant of a chamber.*

The integral featuring in Lemma 3.1 can be rewritten as

$$(3.1) \qquad \int_{\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu})} D \left(\mu_1 \pi^* \psi(x_1)\right)^u \left(\mathrm{st}_* \ell^* \psi(x)\right)^v = \sum_{T \in \mathcal{T}_{\mu;\nu}} \delta_T \int_{D_T} \left(\mu_1 \pi^* \psi(x_1)\right)^u \left(\mathrm{st}_* \ell^* \psi(x)\right)^v,$$

where δ_T is the multiplicity arising from the non-transversal intersection of $\mathrm{st}(\overline{\mathcal{H}}^0_{\mu;\nu})$ and D. This multiplicity can be calculated as follows.

LEMMA 3.3. *The multiplicity δ_T is given by*

$$\delta_T = \frac{1}{\mu_1} \prod_{e \in T} \gamma_e.$$

3.3. Essential geometric trees. We say that a geometric tree $T \in \mathcal{T}_{\mu;\nu}$ is *essential* if all vertices J_j except for the one connected to I_1 are singletons. The set of all essential geometric trees for a given pair (μ, ν) is denoted $\mathcal{E}_{\mu;\nu}$. As follows immediately from Lemma 3.2, $\mathcal{E}_{\mu;\nu}$ is an invariant of a chamber. The importance of this notion is revealed in the following statement. Denote by $\mathcal{I}(T, u, v)$ the integral in the right hand side of (3.1).

LEMMA 3.4. *For any inessential tree $T \in \mathcal{T}_{\mu;\nu} \setminus \mathcal{E}_{\mu;\nu}$ one has $\mathcal{I}(T, u, v) = 0$.*

The value of the integral $\mathcal{I}(T, u, v) = 0$ for essential trees is calculated in the following proposition.

LEMMA 3.5. *For any essential tree $T \in \mathcal{E}_{\mu;\nu}$ one has*

$$\mathcal{I}(T, u, v) = \begin{cases} (d-2)!^r r! \mu_1^u \prod_{i=1}^k \frac{h^0_{\mu(I_i);\gamma(I_i)}}{r_i!} \prod_{j \in \bar{J}_1} \nu_j^{-1} & \text{if } u = |J_1| + \deg J_1 - 3 \\ 0 & \text{otherwise,} \end{cases}$$

where $r = m + n - 2$ and $r_i = |I_i| + \deg I_i - 2$.

Let us take a more precise look at essential geometric trees. The structure of such trees is very simple, as shown on Fig. 2. We denote μ-vertices by circles and ν-vertices by squares. Singletons are white and non-singletons are black. An essential geometric tree has a unique black square vertex denoted J_1. All the other black vertices are circles; we denote them $\widehat{I}_i$, $i = 1, \ldots, \widehat{k}$. Note that the collection of all $\widehat{I}_i$'s forms a proper subset of the initial collection of all I_i's. For any black circle vertex $\widehat{I}_i$ we denote by $\widehat{J}_i$ the (possibly empty) union of all white squares incident to it. Note that in general $\widehat{J}_i$ does not coincide with any of the initial J_j.
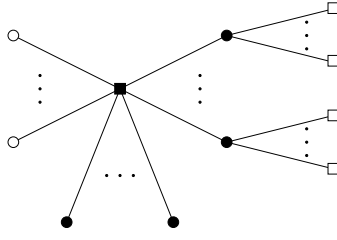
Figure 2. Essential geometric trees

We can now represent double Huritz numbers as a sum over the set of essential geometric trees of products of double Hurwitz numbers of a smaller size. For unification purposes, we extend the set of essential geometric trees by adding the tree with m white circle vertices corresponding to μ_i, and no black circle or white square vertices. The extended set is denoted $\mathcal{E}^*_{\mu;\nu}$.

Theorem 3.6. *Double Hurwitz numbers are given by*

$$(3.2) \qquad h^0_{\mu;\nu} = (m+n-2)! \sum_{T \in \mathcal{E}^*_{\mu;\nu}} \mu_1^{u_T} \prod_{i=1}^{\widehat{k}} (\mu_{\widehat{I}_i} - \nu_{\widehat{J}_i}) \frac{h^0_{\mu(\widehat{I}_i);\nu^*(\widehat{J}_i)}}{(|\widehat{I}_i| + |\widehat{J}_i| - 1)!},$$

where $u_T = |J_1| + \deg J_1 - 3$ and $\nu^*(\widehat{J}_i)$ is obtained from $\nu(\widehat{J}_i)$ by insertion of $\mu_{\widehat{I}_i} - \nu_{\widehat{J}_i}$ at the proper place.

3.4. Proofs of Theorems 1.1–1.3. Let us start from the following observation.

Proposition 3.7. *Resonances for the double Hurwitz numbers $h^0_{\mu(\widehat{I}_i);\nu^*(\widehat{J}_i)}$ correspond bijectively to resonances $x_I = y_J$ for $h^0_{\mu;\nu}$ with $I \subset \widehat{I}_i$, $J \subset \widehat{J}_i$.*

The proof of Theorem 1.1 follows immediately from Lemma 3.2, Theorem 3.6 and Proposition 3.7 by induction over $m + n$. The base of induction is formed by the cases when either m or n equals 1, and there are no resonances.

To prove Theorem 1.2, observe that by Proposition 3.7, (3.2) can be rewritten as

$$P_{\mu;\nu} = (m+n-2)! \sum_{T \in \mathcal{E}^*_{\mu;\nu}} x_1^{u_T} \prod_{i=1}^{\widehat{k}} (x_{\widehat{I}_i} - y_{\widehat{J}_i}) \frac{P_{\mu(\widehat{I}_i);\nu^*(\widehat{J}_i)}}{(|\widehat{I}_i| + |\widehat{J}_i| - 1)!}.$$

We denote by $Q^T_{\mu;\nu}$ the contribution of a tree T to the right hand side of (3.6).

Assume that $(\mu,\nu) \in C$ and $(\mu',\nu') \in C'$. Clearly, $\mathcal{E}^*_{\mu';\nu'} \subset \mathcal{E}^*_{\mu;\nu}$. Therefore, the expression for $P_{\mu;\nu}$ may change for two reasons:

1) birth of new essential geometric trees, and
2) changes in the expressions for $Q^T_{\mu;\nu}$.

Let $\mathcal{E}_{res}$ be the set of essential geometric trees $T \in \mathcal{E}_{\mu';\nu'}$ such that $Q^T_{\mu;\nu} \neq Q^T_{\mu';\nu'}$, and let $\mathcal{E}_{new} = \mathcal{E}_{\mu;\nu} \setminus \mathcal{E}_{\mu';\nu'}$.

Lemma 3.8. *There exists a bijection between $\mathcal{E}_{res} \cup \mathcal{E}_{new}$ and $\mathcal{E}^*_{\mu(\bar{I}),\mu_I - \nu_J;\nu(\bar{J})}$.*

The bijection Φ is presented on Figure 3 below. The upper part of the figure corresponds to the trees in $\mathcal{E}_{new}$, the lower part corresponds to the trees in $\mathcal{E}_{res}$. The asterisque stands for the additional index corresponding to $\mu_I - \nu_J$ in $\mathcal{E}^*_{\mu(\bar{I}),\mu_I - \nu_J;\nu(\bar{J})}$.

It remains to prove Theorem 1.3. We start from expression (3.2). Note that essential geometric trees for the totally negative chamber have a very simple structure: they do not have white square vertices. Therefore, $\mu_{\widehat{I}_i} - \nu_{\widehat{J}_i} = \mu_{\widehat{I}_i}$ and $h^0_{\mu(\widehat{I}_i);\nu^*(\widehat{J}_i)} = (|\widehat{I}_i| - 1)! \mu_{\widehat{I}_i}^{|\widehat{I}_i|-2}$. Finally, $|J_1| = n$, and hence the double Hurwitz numbers for the totally negative chamber are given by

$$(m+n-2)! \mu_1^{n-1} \sum_{k=1}^{m-1} \mu_1^{k-1} \sum_{I_1,\ldots,I_k} \prod_{i=1}^{k} \mu_{I_i}^{|I_i|-1},$$
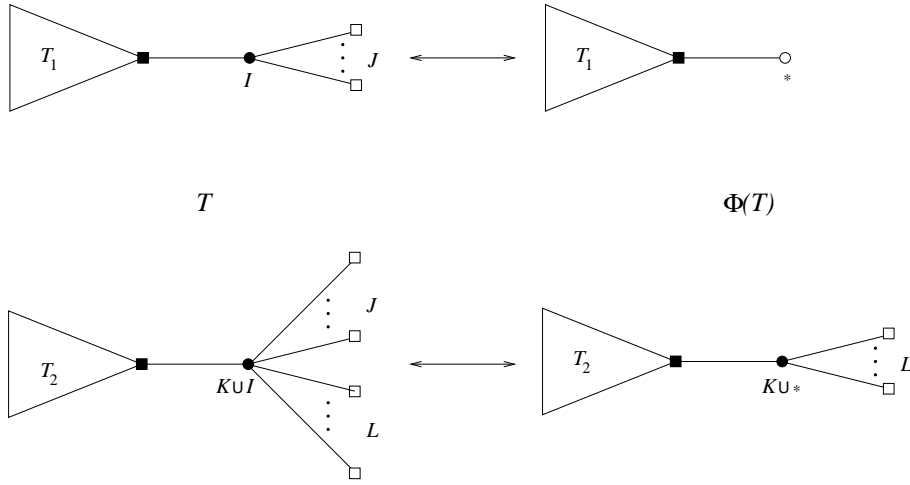
FIGURE 3. The definition of the bijection Φ

where the inner sum is taken over all unordered partitions of $[2, m]$ into k nonempty parts $I_1, \ldots, I_k$. It is easy to see that

$$(3.3) \qquad \sum_{k=1}^{m-1} x_1^k \sum_{I_1, \ldots, I_k} \prod_{i=1}^{k} x_{I_i}^{|I_i|-1} = x_1(x_1 + \cdots + x_m)^{m-2},$$

since both parts of the above formula enumerate trees on $[1, n]$ rooted at 1 classified according to the degrees of the vertices (see e.g. Cayley's tree volume formula in [10]). Therefore, the double Hurwitz numbers in question equal $(m + n - 2)! \mu_1^{n-1} (\mu_1 + \cdots + \mu_m)^{m-2}$, as required.

REMARK 2. Identity (3.3) is a Hurwitz type multinomial identity, see [10] and references therein. Identities of this kind were discovered by Hurwitz in [7] and, apparently, used by him in his studies of Hurwitz numbers (see [11] for a conjectural reconstruction of the original Hurwitz derivation for h_μ^0). It is interesting to note that such identities arose again recently in connection with Gromov–Witten invariants in [3].

References

[1] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein. On Hurwitz numbers and Hodge integrals. *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999) 1175–1180.
[2] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein. Hurwitz numbers and intersections on moduli spaces of curves. *Invent. Math.* **146** (2001) 297–327.
[3] E. Getzler, A. Okounkov, and R. Pandharipande. Multipoint series of Gromov–Witten invariants of $\mathbb{C}\mathbf{P}^1$. *Lett. Math. Phys.* **62** (2002) 159–170.
[4] I. Goulden and D. Jackson. The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group. *Europ. J. Comb.* **13** (1992) 357–365.
[5] I. Goulden, D. Jackson, and R. Vakil. Towards the geometry of double Hurwitz numbers. *Adv. Math.* **198** (2005) 43–92.
[6] T. Graber and R. Vakil. Hodge integrals and Hurwitz numbers via virtual localization. *Compositio Math.* **135** (2003) 25–36.
[7] A. Hurwitz. Über Abel's Verallgemeinerung der binomischen Formel. *Acta Math.* **26** (1902) 199–203.
[8] S. Kuleshov. Ramified coverings of S^2 with two degenerate branching points. Preprint MPIM2001-62.
[9] A. Okounkov. Toda equations for Hurwitz numbers. *Math. Res. Lett.* **7** (2000) 447–453.
[10] J. Pitman. Forest volume decompositions and Abel–Cayley–Hurwitz multinomial expansions. *J. Combin. Theory Ser. A* **98** (2002) 175–191.
[11] V. Strehl. Minimal transitive products of transpositions: the reconstruction of a proof of A. Hurwitz. *Sem. Loth. Comb.* **37** (1996) B37c.
[12] D. Zvonkine. Multiplicities of the Lyashko–Looijenga map on its strata. *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997) 1349–1353.
[13] D. Zvonkin and S. Lando. On multiplicities of the Lyashko–Looijenga mapping on strata of the discriminant. *Funct. Anal. Appl.* **33** (1999) 178–188.

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