Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Partially directed paths in a symmetric wedge

E. J. Janse van Rensburg, T. Prellberg, and A. Rechnitzer

Abstract. The enumeration of lattice paths in wedges poses unique mathematical challenges. These models are not translationally invariant, and the absence of this symmetry complicates both the derivation of a functional recurrence for the generating function, and its solution. In this paper we consider a model of partially directed walks from the origin in the square lattice confined to a symmetric wedge defined by $Y = \pm pX$, where $p > 0$ is an integer.

We prove that the growth constant for all these models is equal to $1 + \sqrt{2}$, independent of the angle of the wedge. We derive a functional equation for the generating function of the model, and obtain an explicit solution when $p = 1$. From this we find asymptotic formulas for the number of partially directed paths of length n in a wedge when $p = 1$.

The functional equation is solved by a variation of the kernel method, which we call the "iterated kernel method". This method appears to be similar to the obstinate kernel method used by Bousquet-Mélou (see, for example [6, 7, 8]). This method requires us to consider iterated compositions of the roots of the kernel. These compositions turn out to be surprisingly tractable, and we are able to find simple explicit expressions for them. However, in spite of this, the generating function turns out to be similar in form to Jacobi θ-functions, and has a natural boundary on the unit circle.

1. Introduction

The problem of counting random walks on lattices is perhaps one of the oldest problems in enumerative combinatorics, with a history that dates back at least 100 years [1]. The related problem of counting random walks on the slit plane, the quarter plane and other restricted geometries has seen a great deal of recent activity (for example [3, 5, 6, 9, 14, 15, 16, 19, 23, 24]).

Such models of paths and walks frequently appear as simple models of polymers in dilute solution in the physics literature [30]. For example, the steric stabilisation of colloids by polymers results when polymers are confined to the spaces between colloidal particles [22]. This situation has been modelled by studying paths confined to the slab between two planes, see for example [10, 12]. Random walk models of polymers in confined geometries are generally more tractable. These models can generally be solved, at least in principle, by a Bethe ansatz or constant term formulation. This technique has been used to solve for random walks in a half-space and which interacts with the boundary of the space [21]. Such random walk models, however, do not take into account the volume exclusion of monomers in a polymer. A more realistic model is the self-avoiding walk [20]. This model is non-Markovian, and while much is known about it from constructive [17, 18] and conformal invariance techniques (in two dimensions) [11], solving it remains beyond the current techniques in combinatorics.

The introduction of directedness in a self-avoiding walk in a wedge may give models of directed or partially directed walks in a wedge which are both self-avoiding and which may in some cases be solvable. Models of directed and partially directed walks in wedge geometries (see Figure 1) have been studied previously [13, 25, 28]. These models include directed paths in a wedge; a model which is related to Dyck paths. It is interesting that the radius of convergence of the generating function is known in this model, even for wedges with wedge-angles of irrational cotangent [25, 28].

2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 82B41.
Key words and phrases. Enumeration, lattice paths, asymptotics, kernel method.

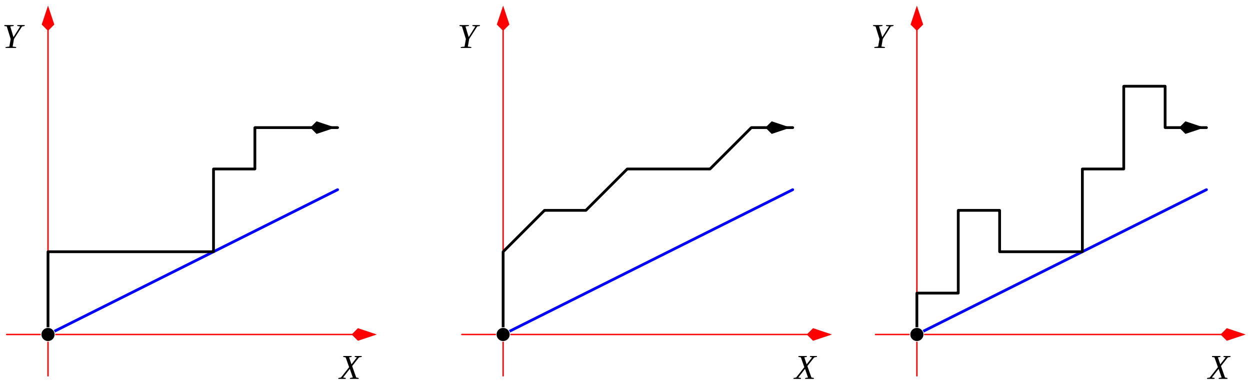

FIGURE 1. Models of directed walks above a line $Y = +pX$. The walks are constrained to take only north and east steps (left); north, east and north-east steps (middle); and north, east and south steps (right).

In this paper we consider models of a partially directed path confined in wedges (see Figure 2). These models are similar to the directed path models in Figure 1, however, they are also substantially more challenging, since the path interacts with the wedge on two sides, rather than on only one side. As a result, it is much harder to find their generating functions and analyse their asymptotics. An extended version of this manuscript has been submitted to JCTA [29].

1.1. Directed and partially directed paths. A directed walk on the square lattice is a path taking unit steps only in the north and east directions. Such objects are necessarily self-avoiding; they cannot revisit the same vertex. Partially directed paths may take unit steps only in the north, south and east directions with the further condition that no vertex is visited twice — *ie* they are self-avoiding. Hence, north steps cannot be followed by south steps and *vice-versa*. The generating function of such walks can be derived using standard techniques:

$$W(t) = \sum_{n \geq 0} c_n t^n = \frac{1+t}{1-2t-t^2}, \tag{1.1}$$

where c_n is the number of walks of length n and t is the length generating variable. An expansion of $W(t)$ in t produces an explicit expression for c_n:

$$c_n = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right). \tag{1.2}$$

The exponential growth constant is the exponential rate at which c_n increases with n. This is given by

$$\mu = \lim_{n\to\infty} c_n^{1/n} = 1+\sqrt{2}. \tag{1.3}$$

This is the most fundamental quantity in this model from a statistical mechanics point of view. The radius of convergence of $W(t)$ is μ^{-1}, and the limiting free energy is the logarithm of the growth constant, $\kappa = \log \mu$, this defines the explicit connection between the combinatorial properties of the model and its thermodynamic properties.

We show some models of directed and partially directed paths in wedge geometries in Figure 1. These models have been considered previously in [13, 25, 27, 26]. In general the growth constant is a (non-trivial) function of the wedge angle. In this paper we consider the variant illustrated in Figure 2 — a model of a partially directed path in a wedge formed by the lines $Y = \pm pX$. More precisely we consider partially directed paths confined to the symmetric p-wedge $\mathcal{V}_p$ defined by

$$\mathcal{V}_p = \left\{(n,m) \in \mathbb{Z}^2 \,\middle|\, \text{where } n \geq 0 \text{ and } -pn \leq m \leq pn\right\}. \tag{1.4}$$

The number, $v_{p,n}$, of partially directed walks in $\mathcal{V}_p$ of length n is the focus of this paper.

In the next section we establish some basic facts about the asymptotic growth of $v_{p,n}$ as $n \to \infty$. In Section 3 we find functional equations satisfied by the corresponding generating functions, which we solve

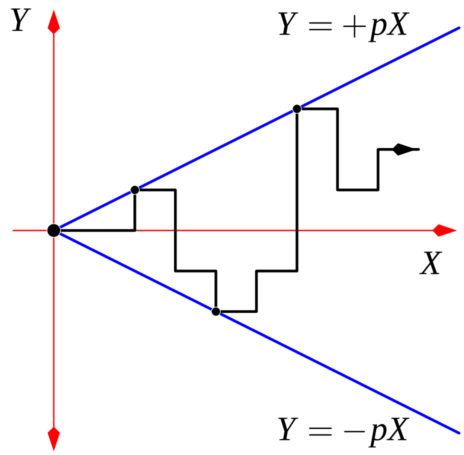

FIGURE 2. The symmetric model of a partially directed path in a p-wedge formed by the lines $Y = \pm pX$.

in Section 4. We then analyse the generating functions to determine $v_{1,n}$ to leading order. We show in particular that

$$v_{1,n} = A_0(1+\sqrt{2})^n + \frac{\sqrt{5}^n}{\sqrt{(n+1)^3}}\left(A_1 + (-1)^n A_2 + O(1/n)\right) \tag{1.5}$$

for the number of paths in a symmetric wedge when $p = 1$ where A_0, A_1 and A_2 are constants.

2. Growth constants in wedges

In this section we prove that the growth constant for partially directed walks is independent of the angle of the wedge and is equal to that of unrestricted partially directed walks. First, let b_n be the number of partially directed walks in the wedge defined by the lines $X = 0$ and $Y = 0$, whose last vertex lies in the line $Y = 0$. The generating function of these paths, $B(t)$, can be found by standard techniques and is:

$$B(t) = \frac{1}{2t^3}\left(1 - t - t^2 - t^3 - \sqrt{(1-t^4)(1-2t-t^2)}\right) \tag{2.1}$$

Singularity analysis then gives the following lemma.

LEMMA 2.1. *The growth constant of partially directed paths in the wedge defined by $X, Y \geq 0$ is*

$$\lim_{n\to\infty} b_n^{1/n} = (1+\sqrt{2}) = \mu. \tag{2.2}$$

This result can be used to determine the growth constants of partially directed walks in the wedges $\mathcal{V}_p$. We first prove existence of the growth constants.

LEMMA 2.2. *For any given $p \in (0, \infty)$ the following limit exists:*

$$\lim_{n\to\infty} v_{p,n}^{1/n} = \mu_p^v \tag{2.3}$$

The limit satisfies

$$\mu_p^v \leq \mu = (1+\sqrt{2}) \tag{2.4}$$

PROOF. We have that $v_{p,n} \leq c_n$. Hence, if the above limits exist, we must have $\mu_p^v \leq \mu = (1+\sqrt{2})$.

To show existence, we prove that the sequences are super-multiplicative. Take any walk counted by $v_{p,n}$ and append a horizontal step, and any walk counted by $v_{p,m}$. This gives a walk of $n+m+1$ steps that lies within $\mathcal{V}_p$, and so is counted by $v_{p,n+m+1}$. Hence $v_{p,n}v_{p,m} \leq v_{p,n+m+1}$. A standard result (Fekete's lemma) on super-additive sequences (which we can apply by taking logarithms) then implies that μ_p^v exists. ☐

LEMMA 2.3. *For any given $p \in (0,\infty)$ we have*

$$(b_n)^N \leq v_{p,(\lceil n/p\rceil+nN+N)} \qquad \textit{and hence} \tag{2.5}$$

$$\lim_{n\to\infty}(b_n)^{1/n} = \mu \leq \mu_p^v. \tag{2.6}$$

PROOF. Take any walk counted by b_n. By prepending $\lceil n/p\rceil+1$ horizontal steps, this walk will fit inside the wedge, $\mathcal{V}_p$. Now append another horizontal step and a walk counted by b_n — repeat this until there are N walks counted by b_n. This gives a walk counted by $v_{p,(\lceil n/p\rceil+nN+N)}$. Thus we have the first inequality. Taking logs and dividing by $(\lceil n/p\rceil+nN+N)$ gives

$$\frac{N}{(\lceil n/p\rceil+nN+N)}\log b_n \leq \frac{1}{(\lceil n/p\rceil+nN+N)}\log v_{p,(\lceil n/p\rceil+nN+N)} \tag{2.7}$$

Take the limit as $N\to\infty$ to obtain

$$\frac{1}{n+1}\log b_n \leq \log \mu_p^v. \tag{2.8}$$

Next, take the limit as $n\to\infty$ to complete the proof. □

By combining the above lemmas we can prove that the growth constant for partially directed paths is independent of the wedge angle.

THEOREM 2.4. *For any given $p\in(0,\infty)$*

$$\mu_p^v = \mu_p^w = \mu = (1+\sqrt{2}). \tag{2.9}$$

This shows that the dominant asymptotic behaviour of the number of walks is independent of the wedge angle. Below, we show that the leading sub-dominant behaviour is also independent of the wedge angle (*ie* for $p \geq 1$).

3. Functional equations for walks in wedges

Consider a model of partially directed paths in a symmetric wedge as illustrated in Figure 2. If p is an integer or a rational number, then the path may touch vertices in the lines $Y=\pm pX$. These vertices are *visits* in the lines $Y=\pm pX$. In the event that p is an irrational number such visits cannot occur, however the path may approach arbitrarily close to the boundary lines (for large enough X-ordinate). In this paper we shall only consider the simplest version of this model, and we assume that p is a positive integer. Even in this case the model is apparently intractable, and we have only found the generating functions when $p=1$.

We will derive a functional equation satisfied by the generating function of partially directed paths in $\mathcal{V}_p$ (those illustrated in Figure 2)), by finding a recursive construction, similar to those in [4, 6] (and elsewhere).

Let x be the generating variable for horizontal edges in the path and let y be the generating variable for vertical edges in the path. Introduce generating variables a and b to be conjugate to the distances between the last vertex in the path and the line $Y=+pX$ and the line $Y=-pX$ respectively. The generating function of the paths are now denoted by $g_p(a,b;x,y)\equiv g_p(a,b)$ where the variables x and y are suppressed.

It turns out that the construction and resulting functional equation is simplified by considering only those partially directed walks that are either a single vertex (no edges) or end in a horizontal step. Let $f_p(a,b;x,y)\equiv f_p(a,b)$ be the generating function of such paths. It is simply related to $g_p(a,b)$ via

$$f_p(a,b) = 1 + x(ab)^p g_p(a,b). \tag{3.1}$$

We now obtain a functional equation satisfied by f_p by recursively constructing the paths column-by-column. Each path is either a single vertex, or can be constructed from a shorter path by appending either a horizontal step, or a sequence of up steps followed by a horizontal step, or a sequence of down steps followed by a horizontal step. Consider a path counted by $f_p(a,b)$, and see Figure 3.

- Appending a single horizontal step to its end increases the distance of the end point from both wedge boundary lines by p. Hence the generating function of paths with a horizontal edge appended is $x(ab)^p f_p(a,b)$.

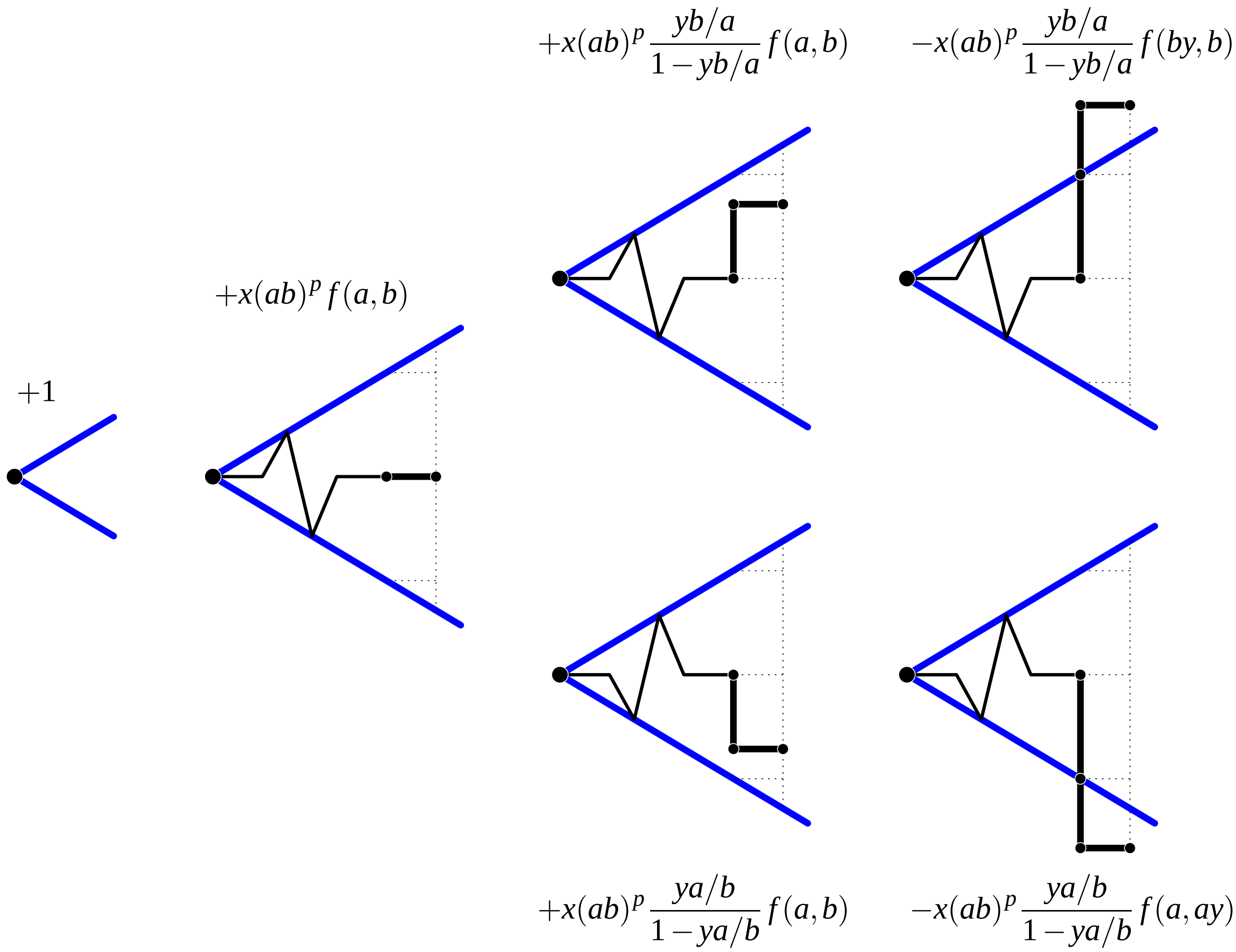

FIGURE 3. Constructing partially directed walks in the wedge $\mathcal{V}_p$. Every walk is either a single vertex, or can be obtained from a shorter walk by appending a horizontal edge (left), or a run of north steps and a horizontal edge or a run of south steps and a horizontal edge (centre-top and -bottom). Care must be taken to not step outside the wedge when appending north or south steps (right-top and -bottom).

- Appending an up step to the end of such a path increases the number of vertical steps by 1, increases the distance from the line $Y = -pX$ by 1 and decreases the distances from the line $Y = +pX$ by 1. Hence such a path has generating function $y(b/a) f_p(a,b)$. Hence appending some positive number of up steps gives $\frac{yb/a}{1-yb/a} f_p(a,b)$. Appending a horizontal step to the end of such a path gives (by the above reasoning) $x(ab)^p \frac{yb/a}{1-yb/a} f_p(a,b)$.
- Similarly appending some positive number of down steps followed by a horizontal step gives $x(ab)^p \frac{ya/b}{1-ya/b} f_p(a,b)$.

Unfortunately, when appending up or down steps it is possible that the resulting path will step outside of the wedge. Hence we must subtract off the contributions from such paths (Figure 3 right-top and -bottom).

- Consider a path that ends at a distance h_+ from the line $Y = +pX$. It we append more than h_+ up steps to the path then it will leave the wedge. We can decompose the resulting path into the original path with exactly h_+ up steps appended, and an "overhanging" Γ shaped path which is a sequence of some positive number of up steps and a horizontal step (see Figure 3 top-right).

Appending exactly h_+ up steps to the path increases the distance from $Y = -pX$ by h_+, decreases the distance from $Y = +pX$ to zero. This gives the generating function $f_p(by, b)$. The overhanging piece is (by the reasoning above) enumerated by $x(ab)^p \frac{yb/a}{1-yb/a}$.

Hence the g.f. of walks that leave the wedge is given by $x(ab)^p \frac{yb/a}{1-yb/a} f(by, b)$.

- Similarly when appending too many down steps we obtain configurations counted by $x(ab)^p \frac{ya/b}{1-ya/b} f_p(a, ay)$.

Using the above construction we arrive at the following theorem

PROPOSITION 3.1. *The generating function $f_p(a, b; x, y) \equiv f_p(a, b)$ of partially directed walks ending in a horizontal step in the wedge $\mathcal{V}_p$ satisfies the following functional equation:*

$$\begin{aligned} f_p(a,b) &= 1 + x(ab)^p f_p(a,b) \\ &\quad + x(ab)^p \frac{yb/a}{1-yb/a} \left(f_p(a,b) - f_p(by,b)\right) \\ &\quad + x(ab)^p \frac{ya/b}{1-ya/b} \left(f_p(a,b) - f_p(a,ay)\right) \end{aligned} \tag{3.2}$$

The generating function of all partially directed walks in $\mathcal{V}_p$ is given by

$$g_p(a,b) = x^{-1}(ab)^{-p}\left(f_p(a,b) - 1\right) \tag{3.3}$$

We now turn to the problem of solving this functional equation.

4. Solving the functional equation

At first sight, one might try to solve equation (3.2) by the iteration method used in [4], however the coefficients of the equation are singular when $a = by$ and $b = ay$. Multiplying both sides of the equation by $(a - by)(b - ay)$ gives a non-singular equation, however when we set $a = by$ or $b = ay$ the equation reduces to a tautology.

Instead we apply a variation of the kernel method, which we call *the iterated kernel method*. This appears to be similar in flavour to the "obstinate kernel method" used by Bousquet-Mélou [6, 7] and the iterated method used by Bousquet-Mélou and Petkovšek in [8]. We have so far only been able to apply this method when $p = 1$.

4.1. Iterated kernel method for $\mathcal{V}_1$. We restrict ourselves to $p = 1$. We write $f_1(a, b) \equiv F(a, b)$. Collecting all the $F(a, b)$ terms together on the left-hand side of the equation gives the *kernel form* of the equation:

$$K(a,b)\,F(a,b) = X(a,b) + Y(a,b)\,F(a,ya) + Z(a,b)\,F(yb,b), \tag{4.1}$$

where the functions $K(a, b)$, $X(a, b)$, $Y(a, b)$ and $Z(a, b)$ are given by

$$K(a,b) = (xy^2a^2 - xa^2 - y)b^2 + (1+y^2)ab - ya^2, \tag{4.2a}$$

$$X(a,b) = (b-ya)(a-yb), \tag{4.2b}$$

$$Y(a,b) = -xya^2b(a-yb), \tag{4.2c}$$

$$Z(a,b) = -xyab^2(b-ya). \tag{4.2d}$$

The function $K(a, b)$ is called the *kernel* of the equation. Note that the equation is symmetric under interchange of a and b:

$$F(a,b) = F(b,a) \qquad K(a,b) = K(b,a) \qquad X(a,b) = X(b,a) \qquad Y(a,b) = Z(b,a). \tag{4.3}$$

We solve equation (4.1) by substituting an infinite number of pairs of a and b values that set the kernel $K(a, b)$ to zero. When $p = 1$, the kernel is quadratic function of a and b and we can explicitly write down (simple) expressions for its zeros. This is not generally true for larger values of p and so the general p case appears intractable.

Let $\beta_\pm(a; x, y) \equiv \beta_\pm(a)$ be the zeros of $K(a, b)$ with respect to b. Hence

$$K(a, \beta_\pm(a)) = 0. \tag{4.4}$$

Thus, setting $b = \beta_\pm(a)$ removes $F(a,b)$ from equation (4.1). This is the key idea behind the "kernel method" which has been used to solve equations of this type (see [2] for example).

Unfortunately in this case, removing the kernel reduces the recurrence to an equation containing terms $F(a,ya)$ and $F(y\beta_\pm(a),\beta_\pm(a))$, which we cannot use immediately to solve for $F(a,b)$. Similar situations have been studied before using the "obstinate kernel method" ([6, 7] for example).

The method we use appears to be similar to the obstinate kernel method, except that instead of finding a finite number of pairs of values of a and b to set the kernel to zero we must use an infinite sequence of pairs. In this way, our "iterated kernel method" is related both to the kernel method and perhaps also to the iterative scheme used in [4].

The roots $\beta_\pm(a)$ can be determined explicitly:

$$\beta_\pm(a) = \frac{a}{2}\left(\frac{1+y^2 \pm \sqrt{(1-y^2)(1-4xya^2-y^2)}}{y+xa^2-xy^2a^2}\right). \tag{4.5}$$

Define the two roots:

$$\beta_1(a) \equiv \beta_-(a) = ya + O(xy^2a^3), \tag{4.6}$$

$$\beta_{-1}(a) \equiv \beta_+(a) = a/y + O(xy^{-2}a). \tag{4.7}$$

as power series in a. Later we require our solution to be a formal power series in t (after setting $x = y = t$) and one can confirm that $\beta_1(a)$ defines a formal power series in t.

Since a is a variable, we are able to substitute something else for it; substituting $a \mapsto \beta_1(a)$ into equation (4.4) gives

$$K(\beta_1(a), \beta_1(\beta_1(a)) = 0. \tag{4.8}$$

Hence the pair $(a,b) = \big(\beta_1(a), \beta_1(\beta_1(a))\big)$ also sets the kernel to zero. We can continue in this way. Hence we need to define the repeated composition of $\beta_1(a)$ with itself:

$$\beta_n(a) = \beta_1^{(n)}(a) = \underbrace{\beta_1 \circ \beta_1 \circ \ldots \circ \beta_1}_{n}(a). \tag{4.9}$$

Note that

$$\beta_{-1} \circ \beta_1(a) = \beta_1 \circ \beta_{-1}(a) = a, \qquad \text{and} \tag{4.10}$$

$$\beta_n(a) = ay^n + O(xy^{n+1}a^3). \tag{4.11}$$

There is no finite value of $n > 0$ such that $\beta_n = \beta_0$. We also define $\beta_0(a) = a$.

These observations are enough to iterate the functional equation to find a solution. Set $b = \beta_1(a)$ in equation (4.1), and set $a = \beta_n(a)$ for any finite $n \geq 0$. Then since $K(\beta_n(a), \beta_{n+1}(a)) = 0$, we have:

$$\begin{aligned} F(\beta_n(a), y\beta_n(a)) = &-\left[\frac{X(\beta_n(a),\beta_{n+1}(a))}{Y(\beta_n(a),\beta_{n+1}(a))}\right] \\ &-\left[\frac{Z(\beta_n(a),\beta_{n+1}(a))}{Y(\beta_n(a),\beta_{n+1}(a))}\right] F(y\beta_{n+1}(a),\beta_{n+1}(a)) \end{aligned} \tag{4.12}$$

We can simplify the above by defining

$$\begin{aligned} &\mathcal{F}_n(a) = F(\beta_n(a), y\beta_n(a)) = F(y\beta_n(a), \beta_n(a)), \\ &\mathcal{X}_n(a) = -\left[\frac{X(\beta_n(a),\beta_{n+1}(a))}{Y(\beta_n(a),\beta_{n+1}(a))}\right], \text{ and } \qquad \mathcal{Z}_n(a) = -\left[\frac{Z(\beta_n(a),\beta_{n+1}(a))}{Y(\beta_n(a),\beta_{n+1}(a))}\right], \end{aligned} \tag{4.13}$$

where we have made use of the symmetry $F(a,b) = F(b,a)$. While this symmetry is not essential, it does make the solution substantially simpler (see [29]). Instead of exploiting this symmetry we could iterate again to find $F(\beta_n(a), y\beta_n(a))$ in terms of $F(\beta_{n+2}(a), y\beta_{n+2}(a))$.

Equation (4.12) may be written as

$$\mathcal{F}_n(a) = \mathcal{X}_n(a) + \mathcal{Z}_n(a)\mathcal{F}_{n+1}(a). \tag{4.14}$$

Starting at $n = 0$, this can be iterated to get a series solution for $\mathcal{F}_0(a)$:

$$F(a, ya) = \mathcal{F}_0(a) = \sum_{n=0}^{\infty} \mathcal{X}_n(a) \prod_{k=0}^{n-1} \mathcal{Z}_k(a), \tag{4.15}$$

where we have *assumed* that the above sum converges (we will show that this is the case). This also gives $F(yb, b)$:

$$F(yb, b) = F(b, yb) = \mathcal{F}_0(b) = \sum_{n=0}^{\infty} \mathcal{X}_n(b) \prod_{k=0}^{n-1} \mathcal{Z}_k(b). \tag{4.16}$$

This allows us to write down the solution for $F(a, b)$:

$$f_p(a, b) = \frac{X(a, b)}{K(a, b)} + \frac{Y(a, b)}{K(a, b)} \sum_{n=0}^{\infty} \mathcal{X}_n(a) \prod_{k=0}^{n-1} \mathcal{Z}_k(a) + \frac{Z(a, b)}{K(a, b)} \sum_{n=0}^{\infty} \mathcal{X}_n(b) \prod_{k=0}^{n-1} \mathcal{Z}_k(b). \tag{4.17}$$

Of course, the above "solution" still contains many complicated algebraic functions in the form of the $\beta_n(a)$. It is quite surprising (at least to the authors!) that these functions can be drastically simplified.

4.2. An explicit expression for $f_1(1, 1)$. It is quite surprising that while $\beta_n(a)$ is (upon superficial inspection for small n) very complicated, its reciprocal appears relatively simple. Examining equations (4.2a) and (4.5) one obtains

$$\frac{1}{\beta_1(a)} + \frac{1}{\beta_{-1}(a)} = \frac{1 + y^2}{y} \frac{1}{a}. \tag{4.18}$$

Substituting $a = \beta_{n-1}(a)$ in the above, and using equation (4.10) leads to the following three term recurrence for β_n:

$$\frac{1}{\beta_n} = \frac{1 + y^2}{y} \frac{1}{\beta_{n-1}} - \frac{1}{\beta_{n-2}}. \tag{4.19}$$

Since β_0 is the identity, and β_1 is given explicitly by $\beta_-(a)$ in equation (4.5), the recurrence above can be iterated to get a solution for $\beta_n(a)$:

$$\frac{1}{\beta_n(a)} = \frac{y(1 - y^{2n})}{y^n(1 - y^2)} \frac{1}{\beta_1(a)} - \frac{y^2(1 - y^{2n-2})}{y^n(1 - y^2)} \frac{1}{a}. \tag{4.20}$$

By using the expressions for $X(a, b)$, $Y(a, b)$ and $Z(a, b)$ in equation (4.2) to determine $\mathcal{X}_n(a, b)$ and $\mathcal{Z}_k(a, b)$, one obtains

$$F(a, ya) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{\beta_{n+1} - y\beta_n}{xya\beta_n\beta_{n+1}} \right] \prod_{k=0}^{n-1} \left(\frac{\beta_{k+1} - y\beta_k}{\beta_k - y\beta_{k+1}} \right). \tag{4.21}$$

Substituting the expression for $\beta_n(a)$ given in equation (4.20) and simplifying gives:

$$F(a, ya) = \left[\frac{1}{xya^2} - \frac{1}{xa\beta_1} \right] \sum_{n=0}^{\infty} (-1)^n y^{n(n+1)} \left(\frac{1}{xya^2} - \frac{1}{xa\beta_1} - 1 \right)^n. \tag{4.22}$$

By defining

$$Q(a; x, y) = \left(\frac{1}{xa^2} - \frac{y}{xa\beta_1} - y \right). \tag{4.23}$$

the above expression for $F(a, ya)$ can be further simplified to

$$F(a, ya) = \left[1 + \frac{Q(a; x, y)}{y} \right] \sum_{n=0}^{\infty} (-1)^n y^{n^2} Q(a; x, y)^n. \tag{4.24}$$

Using the $a \leftrightarrow b$ symmetry of $F(a, b)$, we can get a similar expression for $F(yb, b)$, and so finally $F(a, b)$.

$$f(a,b) = \frac{X(a,b)}{K(a,b)} + \frac{Y(a,b)}{K(a,b)}\left(1+\frac{Q(a)}{y}\right)\sum_{n\geq 0}(-1)^n Q(a)^n y^{n^2} + \frac{Z(a,b)}{K(a,b)}\left(1+\frac{Q(b)}{y}\right)\sum_{n\geq 0}(-1)^n Q(b)^n y^{n^2} \tag{4.25}$$

We can reduce the above equation by considering only the number of walks of length n (by setting $a = b = 1, x = y = t$):

Proposition 4.1. *The generating function of partially directed walks ending in a horizontal step in the wedge $\mathcal{V}_1$ is*

$$f_1(1,1) = \frac{1-t}{1-2t-t^2} - \frac{1-t^2-\sqrt{(1-t^2)(1-5t^2)}}{1-2t-t^2}\sum_{n=0}^{\infty}(-1)^n t^{n^2} Q(1;t,t)^n, \tag{4.26}$$

where t counts the number of edges and

$$Q(1;t,t) = (1-3t^2-\sqrt{(1-t^2)(1-5t^2)})/2t. \tag{4.27}$$

The generating function of all paths in $\mathcal{V}_1$ is then found using equation (3.1):

$$g_1(1,1) = \frac{1+t}{1-2t-t^2} - \frac{1-t^2-\sqrt{(1-t^2)(1-5t^2)}}{t(1-2t-t^2)}\sum_{n=0}^{\infty}(-1)^n t^{n^2} Q(1;t,t)^n. \tag{4.28}$$

Remark. Note that $F(a, ya)$ counts all partially directed paths in the wedge $\mathcal{V}_1$ whose last vertex ends in the line $Y = -pX$. Further, the generating function $Q(a; x, y)/y$ counts the number of partially directed paths starting at the origin, lying on or above the line $Y = -pX$ and whose last vertex lies in the line $Y = -pX$. Hence $Q(a)/y$ counts a very similar set of paths to $F(a, ya)$, except that the paths counted by Q are not confined by the line $Y = pX$.

In light of the above interpretation of the function Q, we expended considerable effort to uncover a more direct combinatorial derivation of the alternating sum in equation (4.24). There appears to be some inclusion-exclusion process underlying this, but unfortunately we have not made progress in this respect.

4.3. Asymptotics for $p = 1$. The asymptotics of the number of partially directed paths in the symmetric wedge with $p = 1$ can be analysed by examining the singularities of the generating function $g_1(1, 1)$ in equation (4.28). Singularities arise either as zeros of the factor $(1 - 2t - t^2)$ in equation (4.28), or as singularities in $\sqrt{(1-t^2)(1-5t^2)}$, or as singularities in the series $\sum_{n=0}^{\infty}(-1)^n t^{n^2} Q(1;t,t)^n$.

An examination of $g_1(1, 1)$ shows that it has simple poles at the solution of $(1 - 2t - t^2) = 0$, or when $t = -1 \pm \sqrt{2}$. We note that $\sqrt{(1-t^2)(1-5t^2)}$ has branch-points (square root singularities) at $t = \pm 1$ and again at $t = \pm 1/\sqrt{5}$. The series $\sum_{n=0}^{\infty}(-1)^n t^{n^2} Q(1;t,t)^n$ is a Jacobi θ-function and it is convergent inside the unit circle except at singularities of $Q(1; t, t)$; that is, when $t = \pm 1/\sqrt{5}$.

The dominant singularity is the simple pole at $\sqrt{2} - 1$, while the next sub-dominant contributions to the asymptotics will be given by the singularities at $t = \pm 1/\sqrt{5}$. These two sub-dominant singularities will give a parity effect. The contributions from these singularities allow us to write down the asymptotic form of $v_{1,n}$.

Proposition 4.2. *The number of paths in the wedge $\mathcal{V}_1$ is asymptotic to*

$$v_{1,n} = A_0\left(1+\sqrt{2}\right)^n + \frac{5^{n/2}}{(n+1)^{3/2}}\Big(A_1 + (-1)^n A_2 + O(1/n)\Big). \tag{4.29}$$

Where the constants are

$$A_0 = 0.27730985348603118827\ldots, \tag{4.30a}$$

$$A_1 = 3.71410486533662324953\ldots, \tag{4.30b}$$

$$A_2 = 0.20697997020804157910\ldots. \tag{4.30c}$$

Note that the above result implies that walks in the wedge $\mathcal{V}_p$ have the same dominant asymptotic behaviour as walks with no bounding wedge (see equation (1.2)). Since the number of walks in any wedge $\mathcal{V}_p$ for $1 \le p < \infty$ is bounded between the number of walks in $\mathcal{V}_1$ and partially directed walks with no bounding wedge, we have the following result:

Corollary 4.3. *The number of partially directed walks in the wedge $\mathcal{V}_p$, $c_n^{(p)}$ obeys the following inequality*

$$0.2773\ldots \le \lim_{n\to\infty} \frac{c_n^{(p)}}{(1+\sqrt{2})^n} \le (1+\sqrt{2})/2 = 1.2071\ldots \tag{4.31}$$

for any $1 \le p < \infty$.

5. Conclusions

In this paper we have proved that partially directed paths in the wedges $\mathcal{V}_p$ all grow with the same exponential growth rate $1+\sqrt{2}$ independent of p. Additionally we have found generating functions for partially directed paths in the wedge $\mathcal{V}_1$ using a variation of the kernel method. From this generating functions we have computed the asymptotics of the number of paths.

Curiously the number of paths in the symmetric wedge, $\mathcal{V}_1$, has the same leading asymptotic behaviour as partially directed paths with no bounding wedge. Because of this, we are able to determine the leading asymptotic behaviour of paths in the wedges $\mathcal{V}_p$ for all $p \ge 1$.

Acknowledgements

EJJvR is supported by a grant from National Sciences and Engineering Research Council of Canada. AR was partially supported by funding from the Australian Research Council and the ARC Centre of Excellence for Mathematics and Statistics of Complex Systems. We would also like to thank the anonymous referees for their helpful suggestions.

References

[1] André, D. Solution directe du problème résolu par M. Bertrand. *Comptes Rendus Acad. Sci. Paris 105* (1887), 436–437.
[2] Banderier, C., Bousquet-Mélou, M., Denise, A., Flajolet, P., Gardy, D., and Gouyou-Beauchamps, D. Generating functions for generating trees. *Discrete Math. 246* (2002), 29–55.
[3] Barcucci, E., Pergola, E., Pinzani, R., and Rinaldi, S. A bijection for some paths on the slit plane. *Adv. in Appl. Math. 26* (2001), 89–96.
[4] Bousquet-Mélou, M. A method for the enumeration of various classes of column-convex polygons. *Discrete Math. 154* (1996), 1–25.
[5] Bousquet-Mélou, M. Walks on the slit plane: other approaches. *Adv. in Appl. Math. 27* (2001), 243–288.
[6] Bousquet-Mélou, M. Counting walks in the quarter plane. In *Mathematics and Computer Science: Algorithms, trees, combinatorics and probabilities*, Trends in Mathematics. Birkhauser, 2002, pp. 49–67.
[7] Bousquet-Mélou, M. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels. *Electronic J. Combinatorics 9*, 2 (2003), R19.
[8] Bousquet-Mélou, M., and Petkovšek, M. Walks confined in a quadrant are not always D-finite. *Theoretical Computer Science 307*, 2 (2003), 257–276.
[9] Bousquet-Mélou, M., and Schaeffer, G. Walks on the slit plane. *Probab. Theory Relat. Fields 124* (2002), 305–344.
[10] Brak, R., Owczarek, A., Rechnitzer, A., and Whittington, S. A directed walk model of a long chain polymer in a slit with attractive walls. *J. Phys. A: Math. Gen. 38* (2005), 4309–4325.
[11] Cardy, J. L. Conformal invariance and surface critical behavior. *Nuc. Phys. B. 240* (1984), 514–532.
[12] DiMarzio, E. A., and Rubin, R. J. Adsorption of a chain polymer between two plates. *J. Chem. Phys. 55* (1971), 4318–4336.
[13] Duchon, P. On the enumeration of generation of generalized Dyck words. *Discrete Math. 225* (2000), 121–135.
[14] Fayolle, G., Iasnogorodski, R., and Malyshev, V. *Random walks in the quarter plane: algebraic methods, boundary value problems and applications.* No. 40 in Applications of Mathematics. Springer Verlag, Berlin, 1999.
[15] Gessel, I. M., and Zeilberger, D. Random walk in a Weyl chamber. *Proc. Amer. Math. Soc. 115* (1992), 27–31.
[16] Grabiner, D. J. Random walk in an alcove of an affine weyl group, and non-colliding random walks on an interval. *Journal of Combinatorial Theory, Series A 97*, 2 (2002), 285–306.
[17] Kesten, H. On the number of self-avoiding walks. *J. Math. Phys. 4* (1963), 960–959.
[18] Kesten, H. On the number of self-avoiding walks ii. *J. Math. Phys. 5* (1964), 1128–1137.
[19] Krattenthaler, C. Asymptotics for random walks in alcoves of affine Weyl groups. *Arxiv preprint math.CO/0301203* (2003).
[20] Madras, N., and Slade, G. *The Self-Avoiding Walk.* Birkhauser, London, 1993.

[21] McCrae, W. K., and Whipple, F. J. W. Random paths in two and three dimensions. *Proc. Roy. Soc. Edinburgh 60* (1940), 281–298.
[22] Napper, D. H. *Polymeric Stabilization of Colloidal Dispersions*. Academic Press, London, 1983.
[23] Niederhausen, H. Lattice paths between diagonal boundaries. *Electronic J. Combinatorics 5* (1998), R30.
[24] Sato, M. Generating functions for the number of lattice paths restricted by two parallel hyperplanes. *Journal of Statistical Planning and Inference 34*, 2 (1993), 251–257.
[25] van Rensburg, E. J. J. Forces in Motzkin paths in a wedge. To appear in J. Phys. A: Math. Gen.
[26] van Rensburg, E. J. J. Adsorbing bargraph paths in a q-wedge. *J. Phys. A: Math. Gen. 38* (2005), 8505–8525.
[27] van Rensburg, E. J. J. Square lattice directed paths adsorbing on the line $y = qx$. *J. Stat. Mech.: Theo. and Exp.* (2005), P09010.
[28] van Rensburg, E. J. J., and Le, Y. Forces in square lattice directed paths in a wedge. *J. Phys. A: Math. Gen. 38* (2005), 8493–8503.
[29] van Rensburg, E. J. J., Prellberg, T., and Rechnitzer, A. Partially directed paths in a wedge. Submitted to JCTA. Also available at math.CO/0609834, 2006.
[30] Vanderzande, C. *Lattice Models of Polymers*, vol. 11 of *Cambride Lecture Notes in Physics*. Cambridge University Press, 1998.

Department of Mathematics and Statistics, York University, 4700 Keele Street, Toronto, Ontario, M3J 1P3, Canada
E-mail address: `rensburg@yorku.ca`

School of Mathematical Sciences, Queen Mary College, Univ of London, Mile End Road, London E1 4NS, UK
E-mail address: `t.prellberg@qmul.ac.uk`

Department of Mathematics University of British Columbia, Vancouver, BC V6T 1Z2, Canada.
E-mail address: `andrewr@math.ubc.ca`