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Increasing and Decreasing Sequences in Fillings of Moon Polyominoes

Martin Rubey

Abstract. We present an adaptation of *jeu de taquin* for arbitrary fillings of moon polyominoes. Using this construction we show various symmetry properties of such fillings taking into account the lengths of longest increasing and decreasing chains. In particular, we prove a conjecture of Jakob Jonsson. We also relate our construction to the one recently employed by Christian Krattenthaler, thus generalising his results.

Résumé. Nous proposons une variante du *jeu de taquin* pour des remplissages de polyominos 'lunaires'. Nous montrons quelques propriétés de symétrie de tels remplissages concernant les longueurs des chaînes croissantes et décroissantes plus longues. Alors, nous donnons une preuve bijective d'une conjecture de Jakob Jonsson. En plus, nous relions notre construction avec celle utilis récemment par Christian Krattenthaler.

1. Introduction

Recently, a great variety of authors became interested in symmetry properties of the number of fillings of certain shapes taking into account the lengths of the longest increasing and decreasing chains. This topic comes about also in a different guise, namely in terms of crossings and nestings of partitions. Some recent papers are [**5, 9, 10, 11, 12**].

Our main goal is to confirm Jakob Jonsson's Conjecture [**9**], which is Theorem 5.2 of this article. The proof is surprisingly simple, especially taking into account the complicated arguments originally needed to prove a special case. Although not completely bijective, the key construction is an adaptation of *jeu de taquin* to so-called moon polyominoes, see Definition 2.2. Similar to *jeu de taquin* it turns out that the order of carrying out the basic operations of our construction is irrelevant.

Apart from proving the above mentioned conjecture we relate the bijection used in this article to the one used by Christian Krattenthaler in [**12**]. We would also like to mention the series of papers [**1, 2, 13**] studying non-crossing and non-nesting partitions in Coxeter groups. In a forthcoming article we will show that the bijection presented here can be modified to work for the setting described in these papers.

This abstract is a shortened version of [**15**]. Due to space restrictions we have to omit some proofs, which can however be found in the full version.

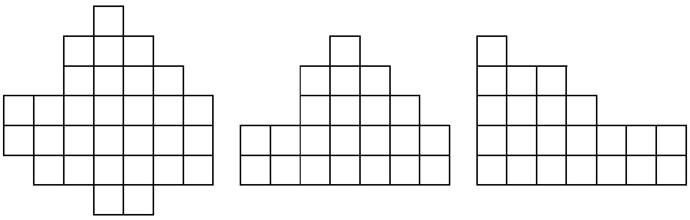

Figure 1. a moon-polyomino, a stack-polyomino and a Ferrers diagram

Key words and phrases. jeu de taquin, evacuation, growth diagrams, moon polyominoes.

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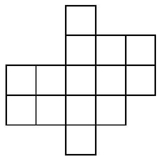
2. Definitions

DEFINITION 2.1. A *polyomino* is a finite subset of $\mathbb{Z}^2$, where we regard an element of $\mathbb{Z}^2$ as a box. A *column* of a polyomino is the set of boxes along a vertical line, a *row* is the set of boxes along a horizontal line. The polyomino is *convex*, if for any two boxes in a column, the elements of $\mathbb{Z}^2$ in-between are also boxes of the polyomino, and for any two boxes in a row, the elements of $\mathbb{Z}^2$ in-between are also boxes of the polyomino. It is *intersection-free*, if any two columns are *comparable*, i.e., the set of row-coordinates of boxes in one column is contained in the set of row-coordinates of boxes in the other. Equivalently, it is intersection-free, if any two rows are comparable.

For example, the polyomino



is convex, but not intersection free, since the first and the last columns are incomparable.

DEFINITION 2.2. A *moon polyomino* is a convex, intersection-free polyomino. A *stack polyomino* is a moon-polyomino if all columns start at the same level. A *Ferrers diagram* is a stack-polyomino with weakly increasing row widths $\lambda_1, \lambda_2, \dots, \lambda_n$, reading rows from bottom to top.

REMARK. We alert the reader that we are using 'French' notation for Ferrers diagrams.

In the following we will consider 'fillings' of such polyominoes with natural numbers, satisfying various conditions.

DEFINITION 2.3. An arbitrary *filling* of a polyomino is an assignment of natural numbers to the boxes of the polyomino. In a 0-1-*filling* we restrict ourselves to the numbers 0 and 1. A *standard filling* has the additional constraint that in each column and in each row there is exactly one entry 1, whereas a *partial filling* has at most one entry 1 in each column and in each row.

In the figures, we will usually omit zeros, and in 0-1-fillings we will replace ones by crosses for æsthetic reasons. For other fillings, we will refer to the number in a box usually as the *multiplicity* of an entry.

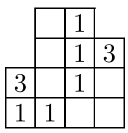
In this article we are mainly interested in the lengths of certain chains in such fillings.

DEFINITION 2.4. A *north-east chain*, or short *ne-chain* of length k in an arbitrary filling of a moon polyomino is a sequence of k non-zero entries, such that each entry is strictly to the right and strictly above the preceding entry in the sequence. Furthermore, we require that the smallest rectangle containing all entries of the sequence is completely contained in the moon polyomino. Similarly, in a *south-east chain*, for short *se-chain*, each entry is strictly to the right and strictly below the preceding entry.

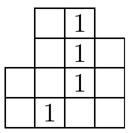
NE-chains and *SE-chains* may have entries in the same column and in the same row. For these kinds of chains, each entry contributes its size to the length of a sequence, i.e., a *NE-chain* of length k is a sequence of entries, such that each entry is weakly to the right and weakly above the preceding entry in the sequence, and the sum of the entries equals k.

For 0-1-fillings we also define *nE-chains* and *sE-chains*, where we allow an entry of the sequence to be in the same column as its predecessor, but not in the same row. Similarly, entries of *Ne-chains* and *Se-chains* are allowed to be in the same row, but not in the same column.

For example, consider the following two fillings:



and



The length of the longest ne-chain in the filling on the left is three, whereas the length of the longest se-chain is two. The lengths of the longest NE- and SE-chains are six and five respectively.

The lengths of the longest nE-, Ne-, sE-, and Se-chains in the 0-1-filling on the right are four, two, three and one respectively.

Finally, we need the notion of a partition:

Definition 2.5. A *partition* is a weakly decreasing sequence of natural numbers, which are called its *parts*. The *length* of a partition is the number of its parts, the *size* of a partition is the sum of its parts. A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is *contained* in another partition $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ if $l \le m$ and $\lambda_i \le \mu_i$ for all $i \le l$.

The union of λ and μ, denoted $\lambda \cup \mu$, is the partition $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_k)$ with $k = \max(l, m)$ and $\kappa_i = \max(\lambda_i, \mu_i)$ for all $i \le k$, where we set $\lambda_i = 0$ for $i > l$ and $\mu_i = 0$ for $i > m$.

The *transpose* or *conjugate* of a partition λ is defined as $\lambda^t = (\mu_1, \mu_2, \dots, \mu_m)$, where $m = \lambda_1$ and μ_i is the number of parts in λ greater than or equal to i.

The *transpose* of a sequence of partitions $P = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^n)$ is the sequence of partitions P^t obtained by transposing each individual partition.

Remark. We remind the reader that each Ferrers shape (in French notation) corresponds to a partition λ, setting λ_i to the length of the i^{th} row from bottom to top. Using this correspondence, the transpose of a partition can be obtained by reflecting the corresponding Ferrers shape about the main diagonal.

3. Growth Diagrams and the Robinson-Schensted-Knuth Correspondence

Sergey Fomin's growth diagrams together with Marcel Schützenberger's *jeu de taquin* [**8, 14, 16**] will be the central tools in this article. Although the contents of this section is well known, we reproduce it here for the convenience of the reader. Some additional information and more references can be found in [**12**, Sections 2 and 4].

3.1. Local Rules.

Consider a rectangular polyomino with a partial filling, as, for example, in Figure 3.a where we have replaced zeros by empty boxes and ones by crosses. Using the following construction we will inductively label the corners of each box with a partition, starting from the bottom left corner.

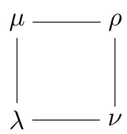


Figure 2. a cell of a growth diagram

First, we attach the empty partition $\emptyset$ to the corners along the lower and the left border. Suppose now that we have already labelled all the corners of a square except the top right with partitions λ, μ and ν, as in Figure 2. We compute ρ as follows:

- F1 Suppose that the square does not contain a cross, and that $\lambda = \mu = \nu$. Then set $\rho = \lambda$.
- F2 Suppose that the square does not contain a cross, and that $\mu \ne \nu$. Then set $\rho = \mu \cup \nu$.
- F3 Suppose that the square does not contain a cross, and that $\lambda \subset \mu = \nu$. Then we obtain ρ from μ by adding 1 to the $i + 1^{\text{st}}$ part of μ, given that λ and μ differ in the i^{th} part.
- F4 Suppose that the square contains a cross. This implies that $\lambda = \mu = \nu$ and we obtain ρ from λ by adding 1 to the first part of λ.

The important fact is, that this process is invertible: given the labels of the corners along the upper and right border of the diagram, we can reconstruct the complete growth diagram as well as the entries of the squares. To this end, suppose that we have already labelled all the corners of a square except the bottom left with partitions μ and ν and ρ, as in Figure 2. We compute λ and the entry of the square as follows:

- B1 If $\mu = \nu = \rho$ we set $\lambda = \rho$ and leave the square empty.
- B2 If $\mu \ne \nu$ we set $\lambda = \mu \cap \nu$ and leave the square empty.
- B3 If $\mu = \nu \subset \rho$ and μ and ρ differ in the i^{th} part for $i \ge 2$, we obtain λ from μ by deleting 1 from the $i - 1^{\text{st}}$ part of μ and leave the square empty.
- B4 If $\mu = \nu \subset \rho$ and μ and ρ differ in the first part we set $\lambda = \mu$ and mark the square with a cross.

a.

b.

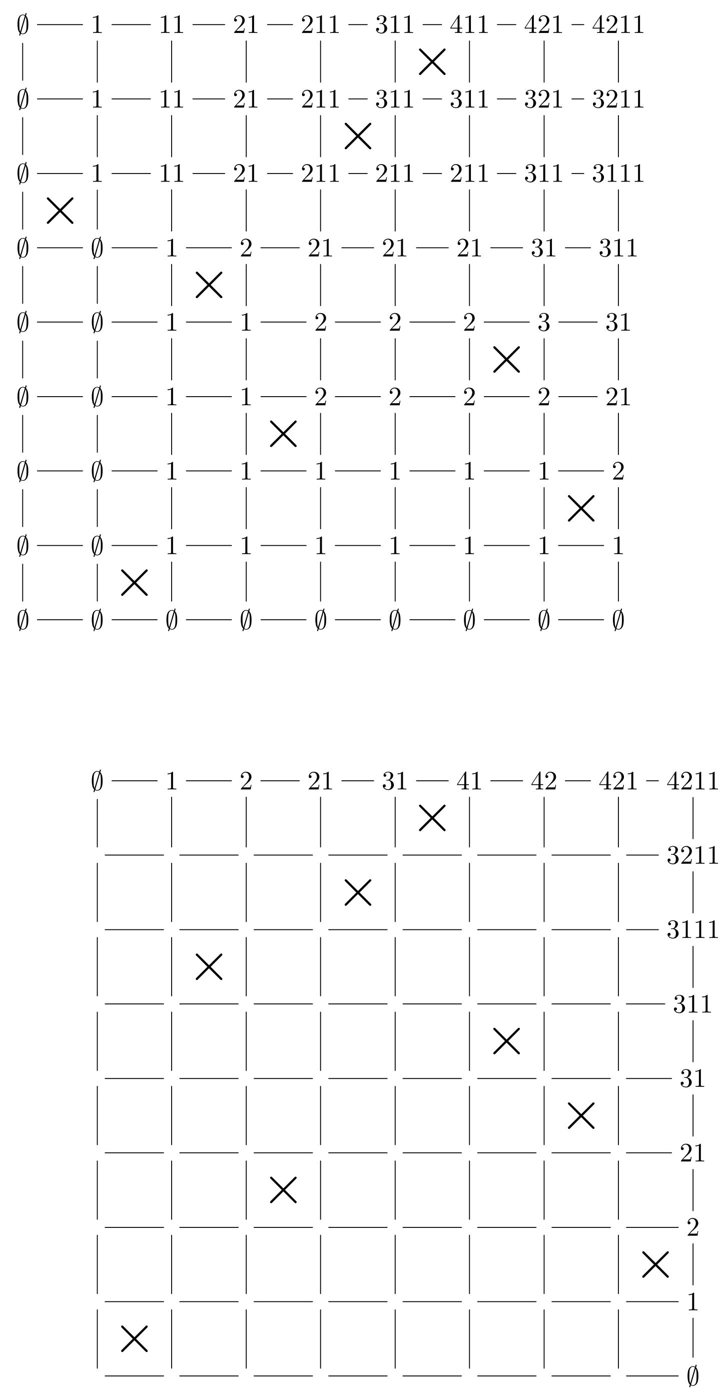


FIGURE 3. a. a growth diagram b. *jeu de taquin* on the upper border

3.2. The Robinson-Schensted Correspondence and Greene's Theorem. In the case of a standard filling of a square, the sequence of partitions $\emptyset = \mu^0, \mu^1, \dots, \mu^n$ along the upper border of the growth diagram corresponds to a standard Young tableau Q as follows: we put the entry i into the cell by which μ^{i-1} and μ^i differ. Similarly, the sequence of partitions $\emptyset = \lambda^0, \lambda^1, \dots, \lambda^n$ along the right border of the diagram corresponds to a standard Young tableau P of the same shape as Q.

Furthermore, the filling itself defines a permutation π. For example in Figure 3.a we have $\pi = 6, 1, 5, 3, 7, 8, 4, 2$ and

$$(P, Q) = \left(\begin{array}{|c|c|c|c|} \hline 6 \\ \cline{1-1} 5 \\ \cline{1-2} 3 & 7 \\ \hline 1 & 2 & 4 & 8 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline 8 \\ \cline{1-1} 4 \\ \cline{1-2} 2 & 7 \\ \hline 1 & 3 & 5 & 6 \\ \hline \end{array} \right).$$

It is well known that Q is simply the recording and P the insertion tableau produced by the Robinson-Schensted correspondence, applied to the permutation π.

Since the partitions along the upper and right border of a growth diagram determine the filling and vice versa, the following definition will be useful:

Definition 3.1. Let π be a standard filling of a square polyomino and consider the corresponding growth diagram. Suppose that the corners along the right border are labelled with a sequence of partitions P, and along the upper border with a sequence of partitions Q. We then say, that π *corresponds* to the pair (P, Q).

For our purposes it is of great importance that the partitions appearing in the corners of a growth diagram also have a 'global' description. This is called Greene's Theorem:

Theorem 3.2. [**16**, Theorem A.1.1.1] *Suppose that a corner c of the growth diagram is labelled by the partition λ. Then, for any integer k, the maximal cardinality of the union of k north-east chains situated in the rectangular region to the left and below of c is equal to $\lambda_1 + \lambda_2 + \cdots + \lambda_k$. Similarly, the maximal cardinality of the union of k south-east chains situated in the rectangular region to the left and below of c is equal to $\mu_1 + \mu_2 + \cdots + \mu_k$, where μ is the transpose of λ.*

3.3. Variations of the Robinson-Schensted Correspondence. In the following, we extend the construction described in the previous Section to arbitrary fillings of rectangular polyominoes. To begin with, in the case of partial fillings only terminology changes: Instead of a pair of standard Young tableaux (P, Q) we now obtain a pair of so-called partial Young tableaux, i.e., semi-standard Young tableaux with all entries distinct.

For an arbitrary filling, we construct a new diagram with more rows and columns, and place entries which are originally in the same column or row in different columns and rows in the larger diagrams. A similar strategy is applied to entries larger than one. More precisely, we proceed as follows:

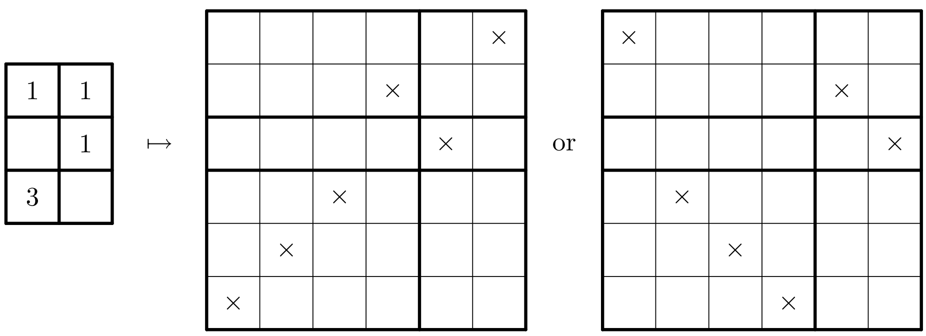


Figure 4. Separating entries of an arbitrary filling using RSK or dual RSK'

Each row and each column of the original diagram is replaced with as many rows and columns in the new diagram as it contains entries, counting multiplicities. Then, for each row and for each column of the original diagram we place the entries into the new diagram as a north-east chain. An example can be found in Figure 4, the result being the left of the two blown-up diagrams. Note that this process preserves the length of the NE- and se-chains.

Given a filling π we can apply the rules F1 to F4 to the transformed diagram and obtain a pair of sequences of partitions (P, Q). It is well known that the pair (P, Q) coincides with the result of applying the usual 'Robinson-Schensted-Knuth', short RSK correspondence, to π.

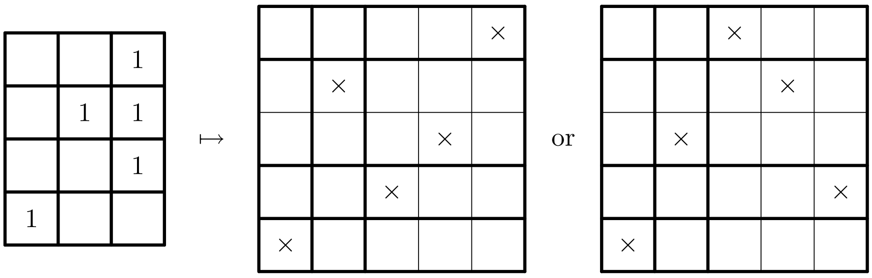


FIGURE 5. Separating entries of a 0-1-filling using dual RSK or RSK'

There is another obvious possibility to separate the entries of an arbitrary filling. Instead of placing the entries into the new diagram as a north-east chain, we could also arrange them in a south-east chain, thus preserving the length of ne- and SE-chains. An example for this transformation is given in Figure 4, the result being the right of the two blown-up diagrams. In this case, the corresponding sequences of partitions (P, Q) are the result of the dual RSK' correspondence, also known as the 'Burge' correspondence.

If we restrict ourselves to 0-1-fillings, we can also transform multiple entries of a column of the original diagram into a north-east chain and multiple entries of a row into a south-east chain. We would thus obtain the so-called dual RSK correspondence. In this case, the lengths of nE- and Se-chains are preserved, as can be seen from the example on the left of Figure 5.

As a last possibility, again for 0-1-fillings, we can transform multiple entries of a column of the original diagram into a south-east chain and multiple entries of a row into a north-east chain, obtaining the 'Robinson-Schensted-Knuth-prime' correspondence, short RSK', which preserves the lengths of Ne- and sE-chains. This is shown on the right of Figure 5.

4. Variations on Jeu de Taquin

Our second tool, *jeu de taquin*, can also be conveniently described with growth diagrams, albeit in a different form. Consider a weakly increasing sequence of partitions $P = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^n)$ where λ^{i-1} and λ^i differ in size by at most one for $i \in \{1, 2, \dots, n\}$. To this sequence, we associate $\overline{jdt}(P) = (\emptyset = \mu^0, \mu^1, \dots, \mu^n)$, with the same property as follows:

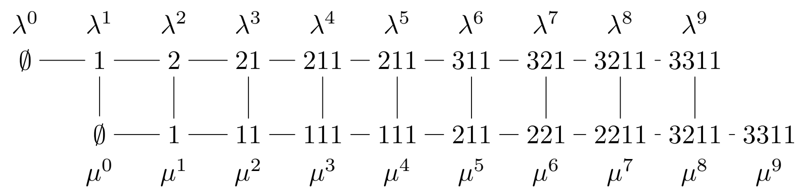


FIGURE 6. Jeu de Taquin

If $\lambda^1 = \emptyset$, we set $\mu^i = \lambda^{i+1}$ for $i < n$ and $\mu^n = \lambda^n$. Otherwise, let $\mu^0 = \emptyset$. Suppose that we have already constructed μ^{i-1} for some $i < n$. Then we distinguish three cases: if $\lambda^{i+1} = \lambda^i$, then we set $\mu^i = \mu^{i-1}$.

If ν is the only partition that contains μ^{i-1} and is contained in λ^{i+1}, we set $\mu^i = \nu$. Otherwise, there will be exactly one such partition different from λ^i, and we set μ^i equal to this partition. Finally, we set $\mu^n = \lambda^n$.

An example for this algorithm can be found in Figure 6. Note that, obviously, this algorithm is invertible. To obtain the traditional form of *jeu de taquin*, which we will denote with $jdt(P)$, we just need to drop the final partition of $\overline{jdt}(P)$.

We can combine growth diagrams as introduced in Section 3.1 and *jeu de taquin* to obtain an interesting bijection on fillings of rectangular polyominoes:

DEFINITION 4.1. Let π be a partial filling of a rectangular polyomino and let Δ be the associated growth diagram. Let $j(\Delta)$ be the growth diagram having the same sequence of partitions along the right border as Δ, whereas the sequence of partitions along the upper border is obtained by applying $\overline{jdt}$ to the corresponding sequence of Δ. Finally, apply the backward rules B1 to B4 to obtain the remaining partitions and the entries of the squares. Let $j(\pi)$ be the filling associated to $j(\Delta)$.

An example of this transformation can be found in Figure 3. Note that, again, this transformation is invertible.

It turns out that *jeu de taquin* is also intimately connected to the growth diagrams as defined in the previous section. The following proposition is a consequence of [**16**, Corollary A.1.2.6], as pointed out in the proof of [**16**, A.1.2.10]:

PROPOSITION 4.2. *Let π be a partial filling of a rectangular polyomino, and let Q be the sequence of partitions in the top row of the associated growth diagram. Let ω be the filling obtained from π by deleting the first column of the polyomino, and let R be the sequence of partitions in the top row of the associated growth diagram. Then $R = jdt(Q)$.*

Before applying this proposition to our situation, we need another definition:

DEFINITION 4.3. Two growth diagrams of the same size are *Knuth equivalent* if the partitions labelling the corners along the right border are the same. They are *dual Knuth equivalent* if the partitions labelling the corners along the top border are the same. We use the same terminology for fillings of rectangular polyominoes.

COROLLARY 4.4. *Consider the filling π' defined by columns $i+1, i+2, \dots, i+k$, $i \geq 1$, of a filling π of a rectangular polyomino. Then π' is dual Knuth equivalent to the filling defined by columns $i, i+1, \dots, i+k-1$ of $j(\pi)$. Furthermore, the filling defined by rows $i, i+1, \dots, i+k$ of π is Knuth equivalent to the filling defined by the same rows of $j(\pi)$.*

PROOF. To obtain the sequence of partitions along the upper border of π', we only have to delete the first i columns of the growth diagram and take the first k partitions labelling the upper border. By Proposition 4.2, this is equivalent to applying jdt i times to the sequence of partitions along the upper border of π and keeping only the first k partitions. Obviously, deleting the first column of π, and then the first $i-1$ columns of the resulting filling is the same as deleting i columns at once.

To prove the second statement, note first that the sequence of partitions P along the right border of π and $j(\pi)$ are the same by definition. To obtain the sequence of partitions of the filling defined by rows $i, i+1, \dots, i+k$ in π or $j(\pi)$, we can apply jdt $i-1$ times to P and finally drop all but the first $k+1$ partitions. □

In particular, if the entries in columns $i+1, i+2, \dots, i+k$ of π form a, say, south-east chain, the same is true for the entries in columns $i, i+1, \dots, i+k-1$ of $j(\pi)$, since this is the case if and only if the sequence of partitions along the top border of the restricted filling is $\emptyset, 1, 11, 111, \dots$.

Similarly, if the entries in rows $i, i+1, \dots, i+k-1$ form, for example, a north-east chain, the same is true for the entries in the same rows of $j(\pi)$, since this is the case if and only if the sequence of partitions along the right border of the restricted filling is $\emptyset, 1, 2, 3, \dots$.

To apply the transformation j to a rectangular diagram with an arbitrary filling π, we first separate the entries using one of the methods described in Section 3.3. Then we apply the transformation j to the new diagram as many times as there are entries in the first column of π counting multiplicities. Finally we shrink the diagram back again, such that column i of the transformed diagram contains as many entries, counting multiplicities, as column $i+1$ of the original diagram, and the last column of the transformed diagram contains as many entries, counting multiplicities, as the first column of the original diagram.

Note that, due to Corollary 4.4 the final step is well defined. For example, if we use Burge's method to separate the entries, in each set of columns that yields a single column in the shrunk diagram, the entries form a south-east chain. The same is true for each set of rows that yields a single row in the shrunk diagram.

Intuitively, we are pushing the entries in the first column towards the end. Note that, unfortunately, in general the transformation j does not preserve the number of entries of a given size, if we are using one of the first two methods of Section 3.3 to separate the entries of the diagram. For example, using Burge's method, $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline & 1 \\ \hline \end{array}$ is mapped to $\begin{array}{|c|c|}\hline 2 & \\ \hline & 1 \\ \hline \end{array}$. However, there is a notable exception to this failure: 0-1-fillings where each non-zero entry is the only one in its row or column are mapped to 0-1-fillings with the same restriction. Of course, if we use the method corresponding to RSK' or dual RSK, this is also the case.

5. Increasing and Decreasing Subsequences in Fillings of Moon Polyominoes

In this section we will apply the transformation j defined in Definition 4.1 to moon polyominoes, thus proving a conjecture of Jakob Jonsson [**9, 10**].

DEFINITION 5.1. The *content* of a moon polyomino is the sequence of column heights, in decreasing order.

For example, the content of the moon polyomino at the left of Figure 1 is $(7, 6, 5, 4, 3, 3, 2)$, while the content of the other two polyominoes in the same figure is $(5, 4, 4, 3, 2, 2, 2)$.

THEOREM 5.2. *Consider* 0-1*-fillings of a given moon polyomino with exactly m_i non-zero entries in row i, such that the length of the longest north-east chain equals k. Then the number of these fillings does not depend on the order of the columns, given that the resulting polyomino is again a moon polyomino. Furthermore, if we disregard the number of entries in row i, the number of fillings depends only on the content of the moon polyomino.*

Special cases of this theorem were proved by Jakob Jonsson and Volkmar Welker [**9, 10**] and by Christian Krattenthaler [**12**]. More precisely, in [**10**] the special case of stack polyominoes is proved, using a very different method. In [**12**] the special case of Ferrers shapes is dealt with. For the connection between [**12**] and our method, see Section 6.

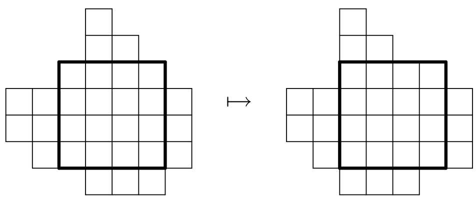
We prove this theorem in two steps. First we show that the transformation j from Definition 4.1 can be used to prove an analogous result about arbitrary fillings. In a second step we show that this implies the theorem above, albeit in a non-bijective fashion. Thus, the problem of finding a completely bijective proof of Theorem 5.2 remains open. However, it appears that this problem is difficult: only very recently, Sergi Elizalde [**7**] solved the first non-trivial case, which is $k = 2$, but only for Ferrers diagrams.

PROPOSITION 5.3. *Consider arbitrary fillings of a given moon polyomino, where the sum of the entries in row i equals m_i, the length of the longest ne-chain equals k and the length of the longest SE-chain equals l.*

Then the number of these fillings does not depend on the order of the columns, given that the resulting polyomino is again a moon polyomino. Furthermore, if we disregard the number of entries in row i, the number of fillings depends only on the content of the moon polyomino.

Similarly, we can fix the length of the longest NE- and the longest se-chain. If we restrict ourselves to 0-1*-fillings, we can fix the length of the longest nE- and the longest Se-chain, or, alternatively, the length of the longest Ne- and the longest sE-chain.*

PROOF. We first show that reordering the columns of the moon polyomino such that the result is again a moon polyomino does not change the number of fillings in question. It suffices to show this in the following special case: let c be any column of the moon polyomino that is contained in one of the columns to its right. Consider the largest rectangle completely contained in the moon polyomino that has the same height as c. Then moving the first column of this rectangle to its end does not change the number of fillings. For example, we could modify a moon polyomino as follows:



We now apply the following bijective transformation to the filling of the moon polyomino: all the entries outside of the rectangle stay as they are, whereas we apply the transformation j to the entries within the rectangle.

Obviously, the sum of the entries in each row remains the same. Furthermore, due to Corollary 4.4, this transformation preserves the length of the longest chains.

To proof the second claim, we first sort the columns according to their height, using the transformation just described, in decreasing order. This is possible, because moon polyominoes are intersection-free.

Suppose now that we want to preserve the length of nE- and Se-chains. We then reflect the polyomino about the line $x = y$, to obtain a stack polyomino. Note that this reflection transforms nE- into Ne-chains and Se- into sE-chains.

Now we sort the columns of the resulting stack polyomino according to height, preserving the maximum lengths of Ne- and sE-chains, and obtain a Ferrers shape.

Reflecting this shape again about the line $x = y$ we obtain a Ferrers shape with the same content as the original moon polyomino, such that both the length of the longest nE-chain and the length of the longest Se-chain are preserved.

The other three cases are dealt with similarly. □

Unfortunately, the proof above does not work for Theorem 5.2. As we have observed before, the transformation j does not preserve the number of entries of a given size. However, we can use simple facts about simplicial complexes and the Stanley-Reisner ring to prove the result.

Proof of Theorem 5.2. Consider the simplicial complex Δ of 0-1 fillings of the moon polyomino, having at most m_i non-zero entries in row i and whose length of the longest north-east chain is at most k.

The Stanley-Reisner ring of Δ is the polynomial ring having variables x_{ij} for each square (i, j) in the moon polyomino, modulo the relations

$$\{x_{i_0 j_0} x_{i_2 j_2} \dots x_{i_k j_k} = 0 : (i_0 j_0), (i_2 j_2), \dots, (i_k j_k) \text{ is a north-east chain in } \Delta\}. \tag{1}$$

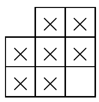
Thus, there is an obvious bijection between monomials in this ring and arbitrary fillings of the moon polyomino satisfying the restrictions of the theorem.

Similarly, we can consider the simplicial complex Δ' of 0-1 fillings of the transformed moon polyomino, having at most m_i non-zero entries in row i and whose length of the longest north-east chain is at most k.

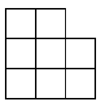
Lemma 5.3 tells us that the number of monomials of given degree in the Stanley-Reisner ring corresponding to Δ is the same as the number of monomials of the same degree in the Stanley-Reisner ring corresponding to Δ'. That is, the Hilbert functions of the two rings must be the same. Thus the corresponding simplicial complexes must have the same f-vector, which is equivalent to the claim of the theorem. □

Remark. Note that in Theorem 5.2 we cannot restrict the length of the longest SE-chain instead, not even for stack polyominoes. Although the set of 0-1-fillings whose longest SE-chain has length at most l is still a simplicial complex, there is no longer a bijection between the monomials of the associated Stanley-Reisner ring and arbitrary fillings of the moon polyomino satisfying the appropriate restrictions. The reason is that the relations in (1) do not exclude chains containing multiple entries.

Indeed, consider the following filling of a stack polyomino:

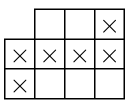


Its longest SE-chain has length 3. However, there is no such filling with seven non-zero entries of the stack polyomino

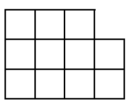


.

Similarly, we cannot preserve simultaneously the length of the longest ne- and se-chain, at least not if we insist on preserving the number of entries in each row. For example,



is a filling with longest north-east chain having length two, and longest south-east chain having length one. On the other hand, there is no such filling of the polyomino



.

As we hinted at before, it would be interesting to have a completely bijective proof of Theorem 5.2. We believe that this may well be accomplished using a modification of the Backelin-West-Xin-transformation introduced in [**3**]. Note that results similar to ours were obtained by Anna de Mier[**6**] using this transformation. For additional information, see [**4, 12**].

To conclude this section, we would like to point out a beautiful feature of the transformation j as applied in the proof of Lemma 5.3:

PROPOSITION 5.4. *Applications of j to different maximal rectangles of a moon-polyomino commute with each other.*

PROOF. Since the proof is quite lengthy, we omit it here. □

6. Evacuation and Jeu de Taquin for Stack Polyominoes

In this section we relate our bijection to evacuation, and thereby to the construction employed by Christian Krattenthaler [**12**] to prove Theorem 5.2 and 5.3 for the special case of Ferrers diagrams. We refer the reader to Christian Krattenthalers article for more on this subject.

Let us first recall the definition of evacuation. Given a weakly increasing sequence of partitions $P = (\emptyset = \lambda^0, \lambda^1, \dots, \lambda^n)$, we construct the *evacuated* sequence of partition $ev(P) = (\emptyset = \mu^0, \mu^1, \dots, \mu^n)$ as follows: We set $\mu^n = \lambda^n$, and then μ^{n-i} equal to the last partition of $jdt\,(\cdots jdt(P))$, where we apply jdt i times.

Regarding evacuation, we recall the following two important facts:

FACT 6.1 (Theorem A 1.2.10 and Corollary A 1.2.11 of [**16**]). *If π corresponds to (P,Q) then the filling obtained from π by rotation about $180°$ corresponds to $(ev(P), ev(Q))$.*

If π corresponds to (P,Q) then the filling obtained by reversing the order of the columns of π corresponds to $(P^t, ev(Q)^t)$.

Christian Krattenthaler [**12**] used the following bijection on Ferrers shapes:

DEFINITION 6.2. Let π be a filling of a Ferrers shape and Δ the associated growth diagram. Let $e(\Delta)$ be the growth diagram obtained from Δ by transposing all the partitions along the top and right border and applying the backward rules B1 to B4 to obtain the remaining partitions and the entries of the squares. Let $e(\pi)$ be the filling associated to $e(\Delta)$.

In this section we show that the growth-diagram bijections used by Christian Krattenthaler are to evacuation what our transformation j is to *jeu de taquin*. To this end we extend the notion of growth diagrams introduced in Section 3 to stack polyominoes. For brevity, we will describe our construction in terms of Greene's Theorem 3.2.

We label the corners of a stack polyomino with two partitions each:

- an *upper partition*, which is given by applying Greene's Theorem to the rectangular region below and to the left of the corner, as wide as the row just above the corner and
- a *lower partition*, which is given by applying Greene's Theorem to the rectangular region below and to the left of the corner, as wide as the row just below the corner.

Of course, if the rows just below and just above the corner are left justified, the two partitions are the same. In this case we will only indicate one partition. In particular, for Ferrers shapes the construction above coincides with the obvious extension of growth diagrams as presented in Section 3 and introduced by Sergey Fomin and Tom Roby [**8, 14**]. An example of such a generalised growth diagram is given in Figure 7.

Similar to the growth diagrams for rectangular shapes we have the following proposition:

PROPOSITION 6.3. *The sequences of partitions along the borders of a generalised growth diagram determine its entries.*

PROOF. Suppose we have reconstructed the growth diagram up to its i^{th} row, counted from the top, including the sequence of upper partitions along the bottom of this row. If the following row starts at the same column, lower and upper partitions coincide and we proceed using the usual backward rules B1 to B4 as given in Section 3 to obtain the entries of the row and the sequence of upper partitions labelling its bottom corners. Otherwise, it is necessary to reconstruct the sequence of lower partitions labelling the bottom of the i^{th} row first. As the following fact shows, this can be done. □

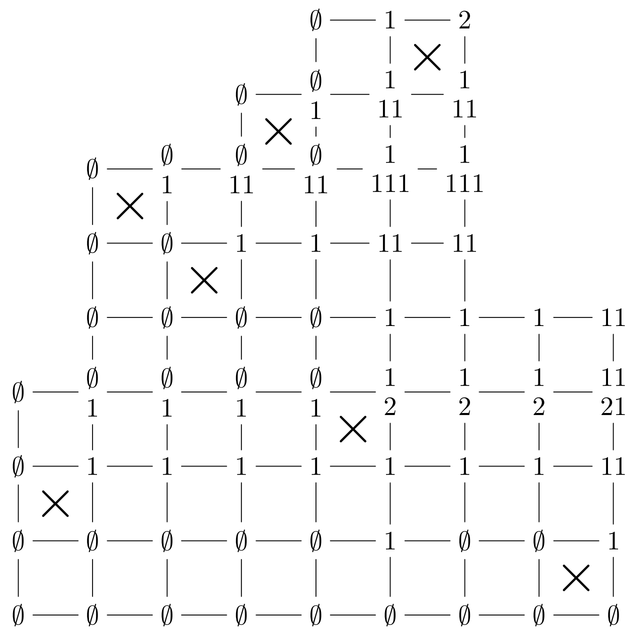


FIGURE 7. a growth diagram for a stack polyomino

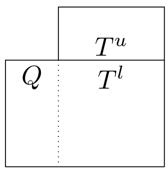


FIGURE 8. reconstructing T^l from Q and T^u

FACT 6.4. *In the situation of Figure 8, T^l is determined by T^u and the final partition in Q.*

PROOF. Observe that by Proposition 4.2 T^u can be obtained from (Q, T^l) by applying jdt as many times as there are partitions in Q. It is easy to see using growth diagrams that this implies the uniqueness of T^l. □

It remains to state precisely in which way we apply *jeu de taquin* to a given filling. The bijection we will relate to e is defined as follows:

DEFINITION 6.5. Let π be a filling of a Ferrers shape F. Apply j^{-1} to move the second column to the first position, then the third column to the first position and so on, to obtain the filling $j^*(\pi)$.

Finally, we can state and prove the main theorem of this section:

THEOREM 6.6. *Let π be a filling of a Ferrers shape F. Let $e(\pi)^r$ be the filling obtained by reflecting $e(\pi)$ about a vertical line. Then $j^*(\pi) = e(\pi)^r$.*

PROOF. We first recall that the statement is well known for rectangular shapes F: let the sequence of partitions labelling the right corners of π be P and let Q be the sequence of partitions labelling the top corners. Then, by Proposition 4.2, applying j^* to π amounts to applying evacuation on Q and leaving P unchanged.

Finally, by Fact 6.1, reversing the order of the columns of $j^*(\pi)$ amounts to applying evacuation on both P and $ev(Q)$ and transposing the resulting tableaux. Thus, in this case π^r corresponds to tableaux P^t and $ev(ev(Q))^t = Q^t$, which are by definition the tableaux corresponding to $e(\pi)$.

To prove the general case, we use the notion of generalised growth diagrams and show that the partitions labelling the corners along the borders of $j^*(\pi)$ are the same as those of $e(\pi)^r$.

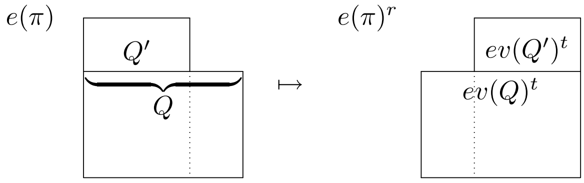


FIGURE 9. corresponding rows of $e(\pi)$ and $e(\pi)^r$

We first describe the partitions labelling the corners of $e(\pi)^r$. Let Q_i be the sequence of partitions just above the i^{th} row of $e(\pi)$. Let S_i^u be the sequence of upper partitions just above the i^{th} row of $e(\pi)^r$ and S_i^l the sequence of lower partitions. Then, $S_i^l = ev(Q_i)^t$. Furthermore, if, counting from the top, the i^{th} and $(i-1)^{\text{st}}$ row of $e(\pi)$ have the same length, $S_i^u = S_i^l = ev(Q_i)^t$. Otherwise, let Q_i' be the sequence of partitions consisting of the first l elements of Q_i, where l is the length of the $(i-1)^{\text{st}}$ row. We then have $S_i^u = ev(Q_i')^t$.

In Figure 9 the correspondence just described is shown schematically. For the proof, we only need to appeal to Fact 6.1.

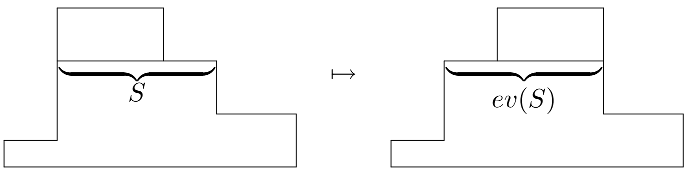


FIGURE 10. applying j^{-1} to a block of columns

We now turn to the partitions on the left border of $j^*(\pi)$. Consider the effect of moving a block of columns of the same height to the left, as sketched in Figure 10. We remark that subsequent moves of necessarily smaller columns from the right to the left do not affect the partitions labelling the corners along the border of columns already moved.

Thus, suppose that the corners along the top border of the largest rectangle containing the last column are labelled with a sequence of partitions S and suppose that the last l columns of π are of the same height. By definition, the first l lower partitions in $j^*(\pi)$ in the same row are just the first l elements of $ev(S)$. Taking into account what we have shown before, this coincides with the labelling obtained by reversing $e(\pi)$.

It is easy to see that the partitions labelling the right border of $e(\pi)^r$ and $j^*(\pi)$ coincide as well. □

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Universität Wien
E-mail address: martin.rubey@univie.ac.at
URL: http://mat.univie.ac.at/~rubey