

# Classifying lattice walks restricted to the quarter plane

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ABSTRACT. This work considers lattice walks restricted to the quarter plane, with steps taken from a set of cardinality three. We present a complete classification of the generating functions of these walks with respect to the classes algebraic, transcendental holonomic and non-holonomic. The principal results are a new algebraic class related to Kreweras' walks; two new non-holonomic classes; and enumerative data on some other classes. These results provide strong evidence for conjectures which use combinatorial criteria to classify the generating functions all nearest neighbour walks in the quarter plane.

## Introduction

The interest in an enumerative approach to lattice walks under various different types of restrictions has risen recently, (see [4, 6, 8, 21]) and there is increased need for global approaches and results. A few such studies have been performed. For example, in the case of the one dimensional lattice walks, Flajolet and Banderier [1] examine the nature of their generating functions, and provide general results of asymptotic analysis. For two dimensional walks in the quarter plane, we find a collection of case analyses [6, 8, 13, 19], and the goal here is to try to determine more general characterizations of these walks, based on the nature of their generating functions. Essentially we are interested to know which walks have holonomic generating functions, that is, when does the generating function satisfy systems of independent linear differential equations with polynomial coefficients. The answer has important repercussions for sequence generation, asymptotics, amongst other enumerative questions.

Unfortunately we do not completely succeed in giving a characterization of all walks, but we do uncover some interesting patterns, and present variety of applications of the kernel method. We derive enumerative information to support a new conjecture on the combinatorial conditions required for a class of walks to possess holonomic (or, D-finite) generating functions.

This extended abstract does not contain all of the calculations nor proofs. A full version of this work is available [16].

## 1. Walks and their generating functions

**1.1. Next nearest neighbour walks.** The walks of interest here are known as *next nearest neighbour walks*. Precisely, they use movements on the integer lattice where each step is from some fixed set  $\mathcal{Y} \subseteq \{\pm 1, 0\}^2 \setminus \{(0, 0)\}$ , which we also specify by the compass directions  $\{\mathbf{N}, \mathbf{NE}, \dots, \mathbf{W}, \mathbf{NW}\}$ . Such a set  $\mathcal{Y}$  is called a *step set*. Here we shall consider nearest neighbour walks exclusively, unless explicitly mentioned otherwise. A *walk in the quarter plane* is a sequence of steps  $w$  in  $\mathcal{Y}^*$ ,  $w = w_1, w_2, \dots, w_n$ , such that for each  $k \leq n$ , the vector sum  $(x_k, y_k) = \sum_{i=1}^k w_i$  satisfies  $x_k \geq 0, y_k \geq 0$ , that is, it remains in the first quadrant. We shall denote the set of all valid walks with steps from  $\mathcal{Y}$  by  $\mathcal{L}(\mathcal{Y})$ . We can consider this as a formal language (in the sense of theoretical computer science) over the alphabet  $\mathcal{Y}$  with the horizontal and vertical conditions as prefix conditions on any word in the language.

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2000 *Mathematics Subject Classification*. Primary 82B41; Secondary 05A15.

*Key words and phrases*. enumeration, lattice walks, D-finite functions, holonomic functions.

**1.2. Complete and counting generating functions.** Fix some step set  $\mathcal{Y}$ . We associate to  $\mathcal{L}(\mathcal{Y})$  two power series:  $W_{\mathcal{Y}}(t)$  a counting (ordinary, univariate) generating function and a complete (multivariate) generating function. The series  $W_{\mathcal{Y}}(t)$ , is a formal power series where the coefficient of  $t^n$  is the number of walks of length  $n$ . The complete generating function  $Q_{\mathcal{Y}}(x, y; t)$  encodes more information. The coefficient of  $x^i y^j t^n$  in  $Q_{\mathcal{Y}}(x, y; t)$  is the number of walks of length  $n$  ending at the point  $(i, j)$ . Remark that the specialization  $x = y = 1$  in the complete generating function is precisely  $W_{\mathcal{Y}}(t)$ , i.e.  $Q_{\mathcal{Y}}(1, 1; t) = W_{\mathcal{Y}}(t)$ . In both cases, when  $\mathcal{Y}$  is clear it is often dropped as an index.

In part, our interest in the complete generating function stems from the fact that if it is in a particular functional class, then generally so is  $W_{\mathcal{Y}}$ . Furthermore, we can determine a useful functional equation that it satisfies. The *fundamental equation* satisfied by a complete generating function is determined from the recursive definition that a walk of length  $n$  is a walk of length  $n - 1$  plus a step. For a walk ending on the  $x$ - or  $y$ -axis, it is possible that not all of the directions from  $\mathcal{Y}$  will be permissible for the next step, and thus we subtract out these sub-series.

**Definition 1.1 Fundamental equation.** The fundamental equation of the complete generating function of walks with step set  $\mathcal{Y}$  is given by<sup>1</sup>

$$(1.1) \quad Q(x, y; t) = 1 + \sum_{(i,j) \in \mathcal{Y}} t x^i y^j Q(x, y; t) - t \bar{y} \sum_{i: (i,-1) \in \mathcal{Y}} x^i Q(x, 0; t) - t \bar{x} \sum_{j: (-1,j) \in \mathcal{Y}} y^j Q(0, y; t) + \chi[(-1, -1) \in \mathcal{Y}] t \bar{x} \bar{y} Q(0, 0; t).$$

**1.3. Classifying formal series.** These two series fall into the following classes of functions. Let  $\underline{x} = x_1, x_2, \dots, x_n$ . A multivariate generating function  $G(\underline{x})$  is *algebraic* if there exists a multivariate polynomial  $P(\underline{x}, y)$  such that  $P(\underline{x}, G(\underline{x})) = 0$ . Flajolet [11] summarizes a number of criteria which imply the transcendence of a series. One which we shall use here is a consequence of his Theorem D: If the Taylor coefficient of  $z^n$  of a function  $f(z)$  (analytic at the origin) is asymptotically equivalent to  $\gamma \beta^n n^r$ , and further if  $r$  is irrational or a negative integer; or if  $\beta$  is transcendental; or if  $\gamma \Gamma(r + 1)$  transcendental, then  $f(z)$  is transcendental.

**Definition 1.2 holonomic function.** A multivariate series  $G(\underline{x})$  is *holonomic* if the vector space generated by its partial derivatives (and their iterates), over rational functions of  $\underline{x}$  is finite dimensional. This is equivalent to the existence of  $n$  partial differential equations of the form

$$0 = p_{0,i} f(\underline{x}) + p_{1,i} \frac{\partial f(\underline{x})}{\partial x_i} + \dots + p_{d_i,i} \frac{\partial^{d_i} f(\underline{x})}{(\partial x_i)^{d_i}},$$

for  $i$  satisfying  $1 \leq i \leq n$ , and where the  $p_{j,i}$  are all polynomials in  $\underline{x}$ . Holonomic functions are also known as *D-finite* functions.

Algebraic functions are always holonomic, but as  $\exp(x)$  is holonomic and transcendental, we see this containment is strict. The closure properties of these two classes are presented by Stanley [20, Ch. 6]. In particular, here we shall use the fact that if  $F(\underline{x})$  is holonomic with respect to the  $x_i$ , and if the algebraic substitution  $x_i = y_i(z_1, \dots, z_k) \equiv y_i$  makes sense as a power series substitution, then  $f(y_1, \dots, y_n)$  is holonomic with respect to the  $z_i$  variables.

The goal of this work is to determine when  $W_{\mathcal{Y}}$  and  $Q_{\mathcal{Y}}$  are holonomic, algebraic, or neither. The next two theorems are a model example of the kind of result we aspire to emulate, from the classification point of view.

**THEOREM 1.1** (Half-plane condition [1]). *Let  $\mathcal{Y}$  be a subset of  $\{\pm 1, 0\}^2$ . The complete generating series  $Q(x, y; t)$  for walks that start at  $(0, 0)$ , take their steps in  $\mathcal{Y}$  and stay in a half plane is algebraic.*

A step set  $\mathcal{Y}$  is *y-axis symmetric* (resp. *x-axis symmetric*) if  $(i, j) \in \mathcal{Y}$  implies that  $(i, -j) \in \mathcal{Y}$  (resp.  $(-i, j) \in \mathcal{Y}$ ).

**THEOREM 1.2** (Bousquet-Mélou, Petkovšek [5, 6]). *Let  $\mathcal{Y}$  be a finite subset of  $\{\pm 1, 0\} \times \mathbb{Z} \setminus \{(0, 0)\}$  that is symmetric with respect to the y-axis. Then the complete generating function  $Q(x, y; t)$  for walks that start from  $(0, 0)$ , take their steps in  $\mathcal{Y}$ , and stay in the first quadrant is holonomic.*

<sup>1</sup> $\chi[P] = 1$  if  $P$  is true and 0 otherwise;  $\bar{x} = \frac{1}{x}$ ;  $\bar{y} = \frac{1}{y}$

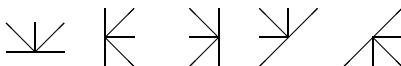


FIGURE 1. Singular walks are subsets of these sets of directions

For example,  $\mathcal{Y} = \{E, NW, SW\}$  satisfies the conditions of Theorem 1.2, and thus  $G_{\mathcal{Y}}(x, y; t)$  is a holonomic function. Naturally, an analogous result for step sets from  $\mathbb{Z} \times \{\pm 1, 0\} \setminus \{(0, 0)\}$  which are  $x$ -axis symmetric is also true.

**1.4. Combinatorial operations and conditions.** Here we introduce two combinatorial operations which act on the step sets and which play an important role. First, we have **reflect**, which switches the coordinates,  $\text{reflect}(x, y) = (y, x)$ , effectively flipping each step across the line  $x = y$ . This operator preserves both algebraicity and holonomy, since it amounts to a simple variable switch in the complete generating function.

A second useful operator is **rev**, which switches the direction of a step,  $\text{rev}(x, y) = (-x, -y)$ . For example,  $\text{rev}(N) = S$ , and  $\text{rev}(SW) = NE$ . The reverse of a set  $\mathcal{Y}$ ,  $\text{rev}(\mathcal{Y})$ , is the set resulting when **rev** is applied to each element of  $\mathcal{Y}$ . The reverse of a walk  $w = w_1 w_2 \dots$ , is the application of **rev** to each step:  $\text{rev}(w) = \text{rev}(w_1) \text{rev}(w_2) \dots$ . This action can also be viewed as a reflection in the line  $x = -y$ .

The reverse of a step set may result in a step set which has only the trivial walk in the quarter plane. Aside from these, the effect of this operation on the generating function can be determined, but as it is much less direct than in the **reflect** case, it is not entirely clear if it preserves either holonomy or algebraicity. Evidence would seem to indicate that in the case when the walk is non-trivial, that it does preserve both properties. In Section 6 we will give a better intuition as to why this could be the case.

**1.5. Classifying step sets of cardinality three.** In order to classify all of the walks with step sets of cardinality three, it is not necessary to consider all  $\binom{8}{3} = 56$  possibilities. Any step set which is a subset of  $\{SE, S, SW, W, NW\}$  has no valid walk in the quarter plane. These are 10 in total. Reflecting a step set in the line  $x = y$  yields a class of walks in bijection. There are only four step sets which are invariant under this action, leaving 21 pairs of duplicates with respect to this action, and thus, there are 25 classes of walks, up to symmetry in the line  $x = y$ . We can reduce this even further by determining other bijective classes. We shall soon show that there are 11 non-bijective classes of walks over all, and all of their generating functions can be classified.

**1.6. Singular walks.** Each step set is governed by at most two inequalities. In many cases one of the inequalities implies the other, or one is trivial. For example, the set  $\{N, NE, E\}$  only satisfies trivial inequalities, and the vertical constraint on the walks  $\{N, NE, S\}$  is implied by the horizontal. These are both examples of *singular* step sets.

**Definition 1.3 Singular step set.** A step set is *singular* if it is a subset of any of the following sets:

- (1)  $\mathcal{A} = \{W, NW, N, NE, E\}$ ,  $\text{reflect}(\mathcal{A})$ ,  $\text{rev}(\mathcal{A})$ ,  $\text{reflect}(\text{rev}(\mathcal{A}))$ ;
- (2)  $\mathcal{B} = \{NE, N, NW, W, SW\}$ ,  $\text{reflect}(\mathcal{B})$ .

These sets are pictured in vector format Figure 1.

**PROPOSITION 1.1.** *If  $\mathcal{Y}$  is a singular step set, then the complete generating function  $Q_{\mathcal{Y}}(x, y; t)$  is an algebraic function.*

The algebraicity of the counting generating function for walks with a singular step set can be deduced from the observation that the quarter plane condition of singular walks is equivalent to the half plane condition, and these walks are algebraic by Theorem 1.1. Furthermore, it is simple to construct a simple pushdown automaton which recognizes the language of the walk. The language is thus unambiguously context-free and it follows that the counting generating function is algebraic [9], and are easy to determine.

**PROOF.** In order to prove the algebraicity of the complete generating function, we give an explicit, unambiguous grammar satisfied by the walks. A well-chosen weighting on the certain variables gives a system of algebraic equations satisfied by the complete generating function, and thus proves the algebraicity.

If the walk is singular, then there is one governing inequality. Without loss of generality suppose it is the horizontal condition. Divide the directions of the step set into three subsets: Let  $A = \{a_i\}_1^k = \{(j, 1) \in \mathcal{Y}\}$ ,

$B = \{b_i\}_1^l = \{(j, -1) \in \mathcal{Y}\}$ ,  $C = \{c_i\}_1^m = \{(j, 0) \in \mathcal{Y}\}$ . Thus, the vertical condition is given by  $\sum_{i=1}^k \#a_i \geq \sum_{j=1}^l \#b_j$ .

If  $\mathcal{Y}$  is singular, then  $\mathcal{L}(\mathcal{Y})$  is generated by  $W$  in the following grammar, which assures that the number of  $A$ s is always greater than the number of  $B$ s for any prefix. The class  $W$  is decomposed by the last step up at a certain height.

$$\begin{array}{lcl} W & \rightarrow & (MA)^*M \\ A & \rightarrow & a_1 | \dots | a_k \\ C & \rightarrow & c_1 | \dots | c_m \end{array} \quad \begin{array}{lcl} M & \rightarrow & \epsilon | CM | AMBM \\ B & \rightarrow & b_1 | \dots | b_l \end{array}$$

There is a direct correspondence between these grammars and the functional equations satisfied by the generating function [9, 12]. Each step is weighted by a monomial corresponding to its direction, namely  $(i, j)$  is assigned to  $x^i y^j t$ . This gives a solvable algebraic system for the complete generating system:

$$\left\{ \begin{array}{l} S(x, y, t) = \frac{M(x, y, t)^2 A(x, y, t)}{1 - M(x, y, t) A(x, y, t)}; \\ M(x, y, t) = 1 + C(x, y, t)M(x, y, t) + A(x, y, t)B(x, y, t)M(x, y, t)^2; \\ A(x, y, t) = \sum_{i=1}^k a_i(x, y, t); \quad B(x, y, t) = \sum_{i=1}^l b_i(x, y, t); \quad C(x, y, t) = \sum_{i=1}^m c_i(x, y, t). \end{array} \right.$$

We can then solve for  $Q_{\mathcal{Y}}(x, y; t) = S(x, y, t)$ . □

To illustrate the process from the proof of Proposition 1.1, we determine algebraic equations satisfied by the generating system for walks given by  $\{\text{NE}, \text{SW}, \text{N}\}$ . Here, the horizontal constraint implies the vertical, and we set  $A = \{\text{NE}\}$ ,  $B = \{\text{SW}\}$ , and  $C = \{\text{N}\}$ . The walks are generated by  $S$  in the following grammar:

$$S \rightarrow (MA)^*M \quad M = \epsilon | CM | AMBM$$

In this case, the algebraic system is determined from the three substitutions  $A(x, y, t) = xyt$ ,  $B(x, y, t) = \bar{x}yt$ , and  $C(x, y, t) = xt$  into the above system. The solution  $S(x, y; t)$  of this system gives an expression for  $Q(x, y; t)$ :

$$Q(x, y; t) = S(x, y, t) = -\frac{-1 + yt + \sqrt{1 - 2yt + t^2 y^2 - 4t^2}}{t \left( 2t - yx + y^2 xt + yx \sqrt{1 - 2yt + t^2 y^2 - 4t^2} \right)}.$$

In total there are 18 singular step sets. However, the walks which are true two dimensional walks are governed by one of the following four inequality types, and they are all isomorphic to one of four different classes (Numbers refer to Table 1):

$$(1) \quad A, B, C \geq 0, \quad (2) \quad A \geq B, C \geq 0, \quad (3) \quad A + B \geq C, \quad (4) \quad A \geq B + C.$$

**1.7. Non-singular Walks.** The remaining non-singular walks, (seven in total) break down as follows. (Again, the numbers refer to those in Table 1.)

Remark that step sets #5 and #6 are reverses of each other. Step set #5 is known as Kreweras' walks after Kreweras' study [14]. The algebraicity of  $W_5(t)$  is surprising and is well studied. Step set #6 is examined in the next section, by applying the same algebraic kernel method [6] which gives the effective algebraicity results for step set #5. The set #7 is examined in detail in Section 3. Step sets #8 and #9 satisfy the criteria given in Theorem 1.2, and are thus holonomic, and one can exploit their symmetry in the kernel method to determine explicit expressions for  $Q(x, y; t)$ . As we shall see from calculations in Section 4, these walks are not algebraic. The final two, step sets #10 and #11, are not holonomic, and a proof is presented in Section 5.

## 2. The Algebraic Kernel Method

The technique we shall apply here, in brief, uses the fundamental equation Eq. (1.1), under different specializations of  $x$  and  $y$  which fix the coefficient of  $Q(x, y; t)$  in this equation, given by  $K_r := 1 - t \sum_{(i,j) \in \mathcal{Y}} x^i y^j$ . The coefficient in this form is called rational kernel, and is denoted  $K_r(x, y)$ . The approach known as the *kernel method*, which has demonstrated a growing popularity in combinatorics, see [5, 18], generally finds specializations of  $y$  as functions of  $x$  which annihilate a polynomial form of the kernel (here,  $xyK_r(x, y)$  would be appropriate), and the problem is reduced to a simpler one. However, Bousquet-Mélou, in her study of Kreweras' walks [6] takes a slightly different approach, and instead finds a group of actions on the pair  $(x, y)$  which fixes the *rational kernel*  $K_r(x, y)$ . This group shall be known as the *group of the walk*.

	$\mathcal{Y}$	Counting GF	Complete GF $Q_k(x, y; t) = \sum a_{ij}(n)x^i y^j t^n$	cf.
1		$(1-3t)^{-1}$	$(1-t(x+y+xy))^{-1}$	§ 1.6
2		$\frac{-4t+1-\sqrt{-8t^2+1}}{4t(3t-1)}$	$-\frac{-1+yt+\sqrt{1-2yt+t^2y^2-4t^2}}{t(2t-yx+y^2xt+yx\sqrt{1-2yt+t^2y^2-4t^2})}$	§ 1.6
3		$\frac{-3t+1-\sqrt{-3t^2-2t+1}}{2t(3t-1)}$	$-\frac{-1+\sqrt{1-4xt^2-4x^2t^2}}{t(2xt-y+y\sqrt{1-4xt^2-4x^2t^2})(1+x)}$	§ 1.6
4		$\frac{-2t+1-\sqrt{-8t^2+1}}{2t(3t-1)}$	$-\frac{-1+\sqrt{1-4t^2-4xt^2}}{t(2t+2xt-y+y\sqrt{1-4t^2-4xt^2})}$	§ 1.6
5		$\frac{T(1-t)+2t(T-1)\sqrt{1-T^2}}{tT(3t-1)}$	$\frac{xy-R(x,t)-R(y,t)}{xy-t(x+y+x^2y^2)}$	§ 2.2, [6]
6		$\frac{(T^2t+T-2t)\sqrt{(1-T)(1+T^2/4+T^3/4)}+T+Tt}{tT(3t-1)}$	$\frac{xy-S(x,t)-S(y,t)}{xy-t(x^2y+xy^2+1)}$	§ 2.3
7		$\frac{1-t-\sqrt{(1+t)(1-3t)}}{2t^2}$	$\frac{(i+1)(j+1)(i+j+2)n!}{\left(\frac{n-i-2j}{3}\right)!\left(\frac{n-i+j}{3}+1\right)!\left(\frac{n+2i+j}{3}+2\right)!}$	§ 3
8		Theorem 4.1 (holonomic/transcendental)		§ 1.7
9		Theorem 4.2 (holonomic/transcendental)		§ 1.7
10		Theorem 5.1 (Not holonomic)		§ 5
11		Theorem 5.2 (Not holonomic)		§ 5

$T \equiv T(t)$  be the power series in  $t$  defined by  $T = t(2 + T^3)$

$$U = 1 - xT(1 + T^3/4) + x^2T^2/4$$

$$R(x, t) = \frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{T} - \frac{1}{x}\right) \sqrt{1 - xT^2}$$

$$S(x, t) = \left(\frac{-2x}{T} \left(1 - \frac{T^2}{2x}\right) + \frac{1}{tx}\right) \frac{\sqrt{T}}{2} + \left(1 - tx - \frac{t}{x^2}\right) \frac{x}{2t}$$

TABLE 1. Generating function data for walks whose step set is of cardinality three

These specializations generate more functional equations from which we can deduce information such as the algebraicity of  $Q(x, y; t)$ , or potentially even explicit generating functions.

**2.1. The group of the walk.** To each non-singular walk we associate a group of actions which fix the rational kernel. For the walks we consider here, it will always be a dihedral group, generated by two involutions. We give an explicit definition for this particular case.

**Definition 2.1** *The group of the walk* ( $G(\mathcal{Y})$ ). Let  $\mathcal{Y}$  be a fixed step set which is not singular. The *group of the walk*  $\mathcal{Y}$ , denoted  $G(\mathcal{Y})$ , is the group of transformations which map  $\mathbb{R}^2$  to itself generated by  $\tau_x$  and  $\tau_y$  defined as follows. Set

$$K(x, y) := xyK_r(x, y) = xy - t \sum_{(i,j) \in \mathcal{Y}} x^{i+1}y^{j+1} = a(x)ty^2 + b(x, t)y + c(x)t,$$

and define  $\tau_y$  as the transformation which maps a pair  $(x, y)$  to  $(x, a(x)^{-1}c(x)\bar{y})$ . Switch the roles of  $x$  and  $y$ , and likewise define  $\tau_x$ .

The generators for *all* step sets with finite groups (up to symmetry in the line  $x = y$ ) are given in Table 2. We can see that these generators fix the rational kernel: If  $y = Y_0(x)$  and  $y = Y_1(x)$  are the two roots of the quadratic polynomial in  $y$  given by  $xyK_r(x, y) = 0$ , then  $\tau_x(x, y) = (x, \frac{Y_1(x)Y_0(x)}{y})$ . We can write

$K_r = ta(x) \left(1 - \frac{Y_0}{y}\right) \left(y - Y_1\right)$ , and thus

$$\begin{aligned} K_r(\tau_x(x, y)) &= K_r\left(x, \frac{Y_1Y_0}{y}\right) = ta(x) \left(1 - \frac{Y_0y}{Y_0Y_1}\right) \left(\frac{Y_0Y_1}{y} - Y_1\right) \\ &= ta(x) \left(1 - \frac{y}{Y_1}\right) Y_1 \left(\frac{Y_0}{y} - 1\right) \\ &= K_r(x, y). \end{aligned}$$

However, this group is not the full group of transformations which fix the kernel of a given stepset. Remark, for example, the transformation  $(x, y) \mapsto (\bar{x}, \bar{y})$  does not appear in the table, but clearly fixes several rational kernels.

$\mathcal{Y}$	Generators
The Group is $D_2$	
	$\tau_x(x, y) = (\bar{x}, y), \tau_y(x, y) = (x, \bar{y})$
	$\tau_x(x, y) = (\bar{x}, y), \tau_y(x, y) = (x, \frac{x}{y(x^2+1)})$
	$\tau_x(x, y) = (\bar{x}, y), \tau_y(x, y) = (x, \frac{x}{y(x^2+1+x)})$
	$\tau_x(x, y) = (\bar{x}, y), \tau_y(x, y) = (x, \frac{x^2+1}{y(x^2+1+x)})$
	$\tau_x(x, y) = (\bar{x}, y), \tau_y(x, y) = (x, \frac{x^2+1}{xy})$
	$\tau_x(x, y) = (\bar{x}, y), \tau_y(x, y) = (x, \frac{x^2+1+x}{xy})$
	$\tau_x(x, y) = (\bar{x}, y), \tau_y(x, y) = (x, \frac{x^2+1+x}{y(x^2+1)})$
The Group is $D_3$	
	$\tau_x(x, y) = (\frac{1}{xy}, y), \tau_y(x, y) = (x, \frac{1}{xy})$
	$\tau_x(x, y) = (y\bar{x}, y), \tau_y(x, y) = (x, x\bar{y})$
The Group is $D_4$	
	$\tau_x(x, y) = (\frac{1}{xy^2}, y), \tau_y(x, y) = (x, \frac{1}{xy})$
	$\tau_x(x, y) = (\bar{x}y^2, y), \tau_y(x, y) = (x, x\bar{y})$

TABLE 2. The generators of the group of the step set in finite cases for all nearest neighbour walks.

In a later section we shall propose a conjecture for conditions on  $\mathcal{Y}$  which assure finiteness of the group generated by  $\tau_x$  and  $\tau_y$ .

This approach is inspired by the Galois automorphisms and the group of the random walk describe by Fayolle *et al.* in [10], and their treatment of the random walks. Here we shall apply this method to step set #6, making use of some of the intermediary results of Bousquet-Mélou [6], from her study of step set #5.

**2.2. Kreweras' Walks.** The step set  $\mathcal{Y} = \{\text{NE}, \text{S}, \text{W}\}$  is interesting since its generating functions are algebraic, but  $\mathcal{L}(\mathcal{Y})$  is *not* context-free. (One can demonstrate this with the pumping lemma). As we mentioned above, it has been studied in several different contexts, and just recently a direct (i.e. combinatorial) explanation of its algebraicity has been offered by Bernardi [2]. He demonstrates a bijection with a family of planar maps. Here we shall stick to just reporting enumerative data. In the next section, however, we follow the same method as Bousquet-Mélou in [6], in order to determine a new family of walks with an algebraic generating function, namely,  $\text{rev}(\mathcal{Y})$ . It remains to be seen if the techniques of Bernardi can be applied to  $\text{rev}(\mathcal{Y})$  to explain algebraicity in that case.

The fundamental equation expresses the complete generating function for walks given by the set  $\{\text{NE}, \text{S}, \text{W}\}$  in terms of the walks which returns to the  $x$ -axis. Theorem 1 of [6] gives an explicit formulation of this generating function.

**THEOREM 2.1** (Bousquet-Mélou [6]). *Let  $T \equiv T(t)$  be the power series in  $t$  defined by  $T = t(2 + T^3)$ . The generating function for Kreweras' walks ending on the  $x$ -axis is*

$$Q_5(x, 0, t) = \frac{1}{tx} \left( \frac{1}{2t} - \frac{1}{x} - \left( \frac{1}{T} - \frac{1}{x} \right) \sqrt{1 - xT^2} \right).$$

We have that  $Q_5(y, 0; t) = Q_5(0, y; t)$  and thus we have the following expression for the complete generating function:

$$Q_5(x, y; t) = \frac{xy - tQ_5(x, 0; t) - tQ_5(0, y; t)}{xy - t(x + x + x^2y^2)}.$$

**2.3. Reverse Kreweras walks.** A second non-singular algebraic class is obtained from the reverse of the Kreweras' walks. The language given by  $\mathcal{L}(\mathcal{Y})$  is similarly not context-free. As Bousquet-Mélou suggests [3], the algebraic kernel method can be applied in a straightforward manner to this set. This results in explicit information about step set #6, given by  $\{\text{N}, \text{E}, \text{SW}\}$ .

We have the following fundamental equation:

$$Q_6(x, y; t) = 1 + t(x + y + \bar{x}y) Q_6(x, y; t) - \frac{t}{xy} (Q_6(0, y; t) + Q_6(x, 0; t) - Q_6(0, 0; t)).$$

We will now drop the index, rearrange the terms, and re-write this

$$(2.1) \quad xyK_rQ(x, y; t) = xy - t(R_0(x) + R_0(y) - R_{00}(t)),$$

where the kernel of the fundamental equation is  $K_r = (1 - t(x + y + \bar{x}\bar{y}))$ ,  $R_0(x) = Q_6(x, 0; t) = Q_6(0, x; t)$  (by symmetry of the step set), and  $R_{00}(t) = Q_6(0, 0; t)$ . First, we remark that since the reverse Kreweras walks are reverses of Kreweras walks, the walks of these two types that return to the origin are in bijection: simply reverse the order of the path, and take the steps in the opposite direction. Thus,  $R_{00}(t)$  is a known algebraic power series which we can deduce from Theorem 2.1. The  $x$  in the denominator of the expression for  $Q_5(x, 0; t)$  is a removable singularity, and hence we can determine  $R_{00} = Q_5(0, 0; t)$  by taking the limit as  $x$  tends to 0. This gives the expression

$$R_{00}(t) = \frac{4T - T^2}{8t}, \quad T = t(2 + T^3).$$

Next, we determine the group of the walk. It is generated by  $\tau_y : (x, y) \mapsto (x, \frac{1}{xy})$  and  $\tau_x : (x, y) \mapsto (\frac{1}{xy}, y)$ . We remark that this is the same as the group of the walk for the Kreweras' walks. Thus, we have that  $G(\mathcal{Y})$  is  $D_3$ , the dihedral group on six elements. We apply the invariance to obtain the following equalities

$$(2.2) \quad K_r(\bar{x}, \bar{y}) = K_r(\bar{x}, xy) = K_r(xy, \bar{y}),$$

and note that  $K_r(\bar{x}, \bar{y})$  is the rational kernel for Kreweras' walks. Denote this function by  $\bar{K}_r \equiv K_r(\bar{x}, \bar{y})$ . Next, generate three equations substituting different values for  $x$  and  $y$  (using Eq. (2.2)), into Eq. (2.1):

$$\begin{aligned} \bar{x}\bar{y}\bar{K}_rQ(\bar{x}, \bar{y}; t) &= \bar{x}\bar{y} - tR_0(\bar{x}) - tR_0(\bar{y}) + tR_{00}(t); \\ y\bar{K}_rQ(\bar{x}, xy; t) &= y - tR_0(\bar{x}) - tR_0(xy) + tR_{00}(t); \\ x\bar{K}_rQ(xy, \bar{y}; t) &= x - tR_0(xy) - tR_0(\bar{y}) + tR_{00}(t). \end{aligned}$$

We form a composite equation taking the sum of the first two equations and subtracting the third:

$$(2.3) \quad \bar{x}\bar{y}R(\bar{x}, \bar{y}) + yQ(\bar{x}, xy) - xQ(xy, \bar{y}) = \frac{1}{\bar{K}_r} (\bar{x}\bar{y} + y - x - 2tR_0(\bar{x}) + tR_{00}(t)).$$

The explicit expressions for  $Y_0$  and  $Y_1$  are

$$Y_0(x) = \frac{1 - t\bar{x} - \sqrt{(1 - t\bar{x})^2 - 4t^2x}}{2tx} \quad \text{and} \quad Y_1(x) = \frac{1 - t\bar{x} + \sqrt{(1 - t\bar{x})^2 - 4t^2x}}{2tx}.$$

We make direct use of some of Bousquet-Mélou's intermediary calculations: Set  $\Delta$  to the common discriminant  $(1 - t\bar{x})^2 - 4t^2x$  of  $Y_0$  and  $Y_1$ . The series  $\Delta$  factors into three power series, respectively in  $C[x][[t]]$ ,  $C[[t]]$ ,  $C[\bar{x}][[t]]$ , which we shall call  $\Delta_+(x)$ ,  $\Delta(t)$ , and  $\Delta_-(\bar{x})$ . This is an instance of a canonical factorization of a power series which is useful in many enumeration problems. She determined that  $\Delta_-(\bar{x}) = 1 - \bar{x}(T(1 + T^3/4) + \bar{x}T^2/4)$ , with  $T$  the unique power series in  $t$  defined by  $T = t(2 + T^3)$ .

The partial fraction expansion of  $\bar{K}_r^{-1}$  is given by

$$\frac{1}{\bar{K}_r} = \frac{1}{(1 - Y_0\bar{y})(y - Y_1)} = \frac{1}{\sqrt{\Delta(x)}} \left( \sum_{n \geq 0} \bar{y}^n Y_0^n + \sum_{n \geq 1} y^n Y_1^{-n} \right),$$

Next we extract the constant term with respect to  $y$  from both sides of Eq. (2.3), and express the result using  $Q_d(x)$ , the generating function for walks that end on the diagonal. This gives a new equation:

$$(2.4) \quad -xQ_d(x) = \sqrt{\Delta(x)}^{-1} \left( 2Y_0 - x - 2tR_0(\bar{x}) + tR_{00}(t) \right),$$

We omit the details, but we can isolate sub-series of negative  $x$  powers from either side of Eq. (2.4). From this we compute the explicit expression given in the following theorem. We can indeed conclude the algebraicity of  $Q(x, 0; t)$  from this expression. Using this we reconstruct an expression for the complete generating function demonstrating its algebraicity.

**THEOREM 2.2.** *Let  $\mathcal{Y} = \{\mathbf{N}, \mathbf{E}, \mathbf{SW}\}$ . Then the generating function for walks that return to the  $x$ -axis is given by*

$$2Q_6(x, 0; t) = Q_6(0, 0; t) + \left( \frac{-2x}{Tt} \left( 1 - \frac{T^2}{2x} \right) + \frac{1}{tx} \right) \sqrt{U} + \left( 1 - tx - \frac{t}{x^2} \right) xt^{-2}$$

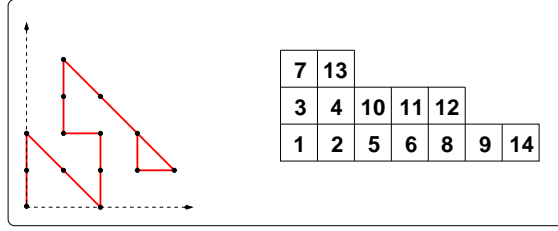


FIGURE 2. A walk of step set  $\mathcal{Y} = \{\text{N}, \text{SE}, \text{W}\}$  and its corresponding Young tableau

where  $T$  is the power series in  $t$  defined by  $T = t(2 + T^3)$ , and  $U$  is the power series in  $t$  defined by  $U(x) = 1 - xT(1 + T^3/4) + x^2T^2/4$ , and  $Q_6(0, 0; t) = (4T - T^2)/8t$ .

**THEOREM 2.3.** *Let  $\mathcal{Y} = \{\text{N}, \text{E}, \text{SW}\}$ . Then the complete generating function for walks with steps from  $\mathcal{Y}$  is given by*

$$Q_6(x, y; t) = \frac{xy - S(x, t) - S(y, t)}{xy - t(x^2y + xy^2 + 1)},$$

where  $S(x, t)$  is given by  $Q_6(x, 0; t)$  from the previous theorem.

### 3. An interesting case

The walks given by steps in  $\mathcal{Y} = \{\text{N}, \text{SE}, \text{W}\}$  are an interesting example because the nature of the complete and the counting generating functions differ. Regev [19] proved that the number of such walks of length  $n$  are equal to the  $n^{\text{th}}$  Motzkin numbers, thus admitting the counting generating function

$$W_7(t) = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t^2}.$$

We can deduce this, and an expression for the complete generating function, by exploiting a bijective correspondence between these walks and standard Young tableaux of height at most 3. Standard Young tableaux are labelled Ferrer's diagrams of a partition such that the boxes are labelled in a strictly increasing manner from left to right and from bottom to top. We construct a standard Young tableau of height three of size  $n$  from a walk  $w = w_1, w_2, \dots, w_n$  as follows. If  $w_i = \text{N}$  (resp.  $\text{SE}, \text{W}$ ), then place label  $i$  is in the next available space to the right on the bottom row (resp. second, top). The final tableau is increasing from left to right by construction, and the prefix condition  $\#\text{N} \geq \#\text{SE} \geq \#\text{W}$  ensures that it is increasing along the columns. Figure 2 gives an example of such a correspondence.

There is a well-known formula for counting standard Young tableaux, known as the hook formula [20]. We apply this formula to count the number of tableaux of form  $(n_1, n_2, n_3)$  from which we deduce the number of walks of length  $n$  that end at  $(i, j)$ . The number of such tableaux are

$$a(n_1, n_2, n_3) = (n_1 - n_2 + 1)(n_2 - n_3 + 1)(n_1 - n_3 + 2) \frac{(n_1 + n_2 + n_3)!}{(n_1 + 2)!(n_2 + 1)!n_3!}.$$

Now, the total length of the corresponding walk is the size of the tableau,  $n = n_1 + n_2 + n_3$  and to end at  $(i, j)$ , we have  $i = n_2 - n_3$  and  $j = n_1 - n_2$ . Thus, when we make the substitution,

$$a_{ij}(n) = \frac{(i+1)(j+1)(i+j+2)n!}{\left(\frac{n-i-2j}{3}\right)! \left(\frac{n-i+j+3}{3}\right)! \left(\frac{n+2i+j+6}{3}\right)!}.$$

We can verify that this is  $P$ -recursive [15], and thus the complete generating function is holonomic.

### 4. Explicit calculations for two holonomic classes

In the next example we use the kernel method to determine enumerative results for walks with steps from  $\{\text{N}, \text{SE}, \text{SW}\}$  and  $\{\text{W}, \text{NE}, \text{SE}\}$ . These results allow us to conclude that the series, although holonomic, are transcendental. The property we exploit is their axis symmetry.

Set  $\mathcal{Y} = \{\text{N}, \text{SE}, \text{SW}\}$ . The functional equation for  $Q_{\mathcal{Y}} = Q$  is

$$(4.1) \quad K(x, y)Q(x, y; t) = xy - (x^2 + 1)tQ(x, 0; t) - tQ(0, y; t) + tQ(0, 0; t),$$



with kernel  $K(x, y) = xy(1 - ty - t(x + \bar{x})\bar{y})$ . Let  $Y_1(x)$  satisfy  $K(x, Y_1(x)) = 0$ , and vanish at  $t = 0$ . We generate two equations based upon the observations that  $K(\bar{x}, y) = K(x, y)$  and  $Y_1(\bar{x}) = Y_1(x)$ :

$$\begin{aligned} 0 &= xY_1 - (x^2 + 1)tQ(x, 0; t) - tQ(0, Y_1; t) + tQ(0, 0; t) \\ 0 &= \bar{x}Y_1 - (\bar{x}^2 + 1)tQ(\bar{x}, 0; t) - tQ(0, Y_1; t) + tQ(0, 0; t) \end{aligned}$$

The difference of these two equations yields a simpler third equation,

$$(4.2) \quad (x - \bar{x})Y_1 = tQ(x, 0; t)(x^2 + 1) - tQ(\bar{x}, 0; t)(\bar{x}^2 + 1).$$

As we did in the case of the reverse Kreweras walks, we view both sides of the equation as elements of  $Q[x, \bar{x}][[t]]$ , and perform sub-series isolation. The details are omitted here, but are provided in the full version of this work.

**THEOREM 4.1.** *Let  $\mathcal{Y} = \{\mathbf{N}, \mathbf{SE}, \mathbf{SW}\}$ . Then the complete generating function for walks with steps from  $\mathcal{Y}$  is given by*

$$Q_8(x, y; t) = \frac{xy - H(x, t) - M(y, t) + H(M(y, t), t)}{xy - txy^2 - t(x^2 + 1)}$$

where  $H(x, t)$  is the power series given by  $[x^k t^{2n-1}]H(x, t) = \binom{n+1}{(n+k+1)/2} \frac{k}{n+1}$  when  $k \equiv n \pmod{2}$  and 0 otherwise; and  $M(y, t) = \frac{1}{2t}y - ty^2 - \sqrt{y^2 - 2y^3t + t^2y^4 - 4t^2} = \bar{y}t + t^2 + (y + \bar{y}^3)t^3 + (y^2 + 3\bar{y}^2)t^4 + O(t^5)$ .

A similar calculation gives the following theorem.

**THEOREM 4.2.** *Let  $\mathcal{Y} = \{\mathbf{NE}, \mathbf{SE}, \mathbf{W}\}$ . Then the generating function for walks with steps from  $\mathcal{Y}$  returning to the  $y$ -axis is given by*

$$R(y, t) = Q_9(0, y; t) = \frac{(y^2 - 1)S(y, t)}{y^2 + 1}$$

where the non-zero coefficients of  $S(y, t)$  are given by  $[y^{n-2k} t^{2n-1}]S(y, t) = 2 \binom{n}{k} \binom{2n-2}{n-1} \frac{1}{n}$ , for integer  $0 \leq k \leq n/2$ .

The complete generating function is given by

$$Q_9(x, y; t) = \frac{xy - T(x - R(T, t)) - tyR(y, t)}{xy - tx^2(y^2 + 1) - ty}$$

where  $T = \frac{(x-t) - \sqrt{(x-t)^2 - 4t^2x^4}}{tx^2}$ .

**4.1. A word about transcendence.** Suppose, for the purpose of illustrating a contradiction, that  $Q_8(x, y; t)$  were algebraic. Were it so, this implies the algebraicity of  $Q_8(x, 0; t)$  and  $Q_8(0, 0; t)$ , which then in turn implies the algebraicity of  $H(1, t)$  from the statement of the theorem. However, if  $H(1, t) = \sum a(n)t^{2n-1}$ , we can show that asymptotically, as  $n$  tends to infinity  $a(n)$  tends to  $\frac{8^n \sqrt{2}}{4\pi n^2}$ . Thus by Criterion D of [11], the series  $H(1, t)$  is transcendental. Thus, we have established a contradiction, and  $Q_8(x, y; t)$  is not algebraic. Recall, however, that it is holonomic, by Theorem 1.2.

## 5. Walks which are not holonomic

Knowing that walks in the half plane and slit planes are algebraic, and considering that walks in the quarter plane are ultimately relatively simple objects— they are governed by only two inequalities, it may be surprising that there are classes with generating functions which are not holonomic. However, it is already known that Bousquet-Mélou and Petkovšek's knight's walks [8] ( $\mathcal{Y} = \{(2, -1), (-1, 2)\}$ ) are not holonomic, and, in fact, evidence would seem to indicate that many walks in the quarter plane are not holonomic.

The final two walks we consider have non-holonomic complete generating functions. This non-holonomy is established in [17], and is based on a similar problem of self-avoiding walks in wedges [21]. The general argument is to show that there are an infinite number of singularities in the counting generating function. This is similar to the argument of Bousquet-Mélou and Petkovšek, whose proof also use an adaptation of a kernel method argument. However, for these walks, it seems that there is a more direct application of the group of the walk. Specifically, we use the fact that the group of the walk is infinite to demonstrate the infinite set of singularities.

From [17] we pull the following two results which completes our classification. In the next section we present a small summary of the argument.

**THEOREM 5.1.** *Neither the complete generating function  $Q_{10}(x, y; t)$ , nor the counting generating function  $W_{10}(t) = Q_{10}(1, 1; t)$  of nearest-neighbour walks in the first quadrant with steps from  $\{\text{NE}, \text{SE}, \text{NW}\}$  are holonomic functions with respect to their variable sets.*

**THEOREM 5.2.** *Neither the complete generating function  $Q_{11}(x, y; t)$ , nor the counting generating function  $W_{11}(t) = Q_{11}(1, 1; t)$  of nearest-neighbour walks in the first quadrant with steps from  $\{\text{NE}, \text{SE}, \text{N}\}$  are holonomic functions with respect to their variable sets.*

**5.1. The iterated kernel method and the step set  $\mathcal{Y} = \{\text{NE}, \text{SE}, \text{NW}\}$ .** To begin, we recall the fundamental equation of these walks,

$$Q_{10}(x, y; t) = 1 + txyQ_{10}(x, y; t) + tx\bar{y}(Q_{10}(x, y; t) - Q_{10}(x, 0; t)) + ty\bar{x}(Q_{10}(x, y; t) - Q_{10}(0, y; t)),$$

and its kernel form,

$$(5.1) \quad (xy - tx^2y^2 - tx^2 - ty^2) Q(x, y) = xy - tx^2Q(x, 0) - ty^2Q(y, 0).$$

Here, for brevity we did write  $Q(x, y; t)$  as  $Q(x, y)$ , and have used the  $x \leftrightarrow y$  symmetry to rewrite  $Q(0, y)$  as  $Q(y, 0)$ . There are two solutions for the kernel  $K(x, y) = xy - tx^2y^2 - tx^2 - ty^2$  as a function of  $y$

$$(5.2) \quad Y_{\pm 1}(x) = \frac{x}{2t(1+x^2)} \left( 1 \mp \sqrt{1 - 4t^2(1+x^2)} \right).$$

Since we can show that

$$Y_{+1}(Y_{-1}(x)) = x \quad \text{and} \quad Y_{-1}(Y_{+1}(x)) = x,$$

if we write  $Y_n(x) = (Y_1 \circ)^n(x)$ , and likewise for  $Y_{-n}$ , we have that the set  $\{Y_n | n \in \mathbb{Z}\}$  forms an infinite group, under the operation  $Y_n \circ Y_m = Y_{n+m}$ , with identity  $Y_0 = x$ . We can also show the useful relation,

$$\frac{1}{Y_1(x)} + \frac{1}{Y_{-1}(x)} = \frac{1}{tx},$$

and its iterated equivalent,

$$(5.3) \quad \frac{1}{Y_n(x)} = \frac{1}{tY_{n-1}} - \frac{1}{Y_{n-2}(x)}.$$

By construction we have  $K(x, Y_1(x)) = 0$ , thus substituting  $y = Y_1(x)$  into Eq. (5.1) gives (after a little tidying)

$$Q(x, 0) = \frac{Y_1(x)}{x} \frac{1}{t} - \frac{Y_1(x)^2}{x^2} Q(Y_1(x), 0).$$

Now we substitute  $x = Y_n(x)$  into this equation to obtain

$$Q(Y_n(x), 0) = \left( \frac{Y_{n+1}(x)}{Y_n(x)} \right) \frac{1}{t} - \left( \frac{Y_{n+1}(x)}{Y_n(x)} \right)^2 Q(Y_{n+1}(x), 0).$$

Using this expression for  $Q(Y_n(x), 0)$  for various  $n$ , we can iteratively generate a new expression for  $Q(x, 0)$ :

$$Q(x, 0) = \frac{1}{x^2t} \sum_{n=0}^{N-1} (-1)^n Y_n(x) Y_{n+1}(x) + (-1)^N \left( \frac{Y_N(x)}{x} \right)^2 Q(Y_N(x), 0).$$

Since  $Y_n(x) = xt^n + o(xt^n)$  we have that  $\lim_{N \rightarrow \infty} Y_N(x) = 0$  as a formal power series in  $t$  and consequently

$$(5.4) \quad Q(x, 0) = \frac{1}{x^2t} \sum_{n \geq 0} (-1)^n Y_n(x) Y_{n+1}(x).$$

We now specialize the fundamental equation:

$$(5.5) \quad (1 - 3t) Q(1, 1; t) = 1 - 2tQ(1, 0; t) = 1 - 2 \sum_{n \geq 0} (-1)^n Y_n(1) Y_{n+1}(1).$$

The result follows from the following lemma, which is proved in [17]

**LEMMA 5.3.** *Suppose  $q_c$  is a zero of  $\bar{Y}_N(q) := Y_N(1; \frac{q}{1+q^2})^{-1}$ , and that  $q_c \neq 0$ . Then*

- (1) *For all  $k \neq N$ ,  $\bar{Y}_k(q_c) \neq 0$ ; and*
- (2) *The function  $Q(1, 0; \frac{q}{1+q^2})$  has a pole at the  $q = q_c$ .*

We now have all the components in place to prove the main result.

PROOF OF THEOREM 5.1. The function  $Q(1, 1; \frac{q}{1+q^2})$  has a set of poles given by the zeroes of the  $\bar{Y}_n(q)$ , by the preceding lemma. The set of such poles form an infinite set. Thus,  $Q(1, 1, \frac{q}{1+q^2})$  is not holonomic. For a multivariate series to be holonomic, any of its algebraic specializations must be holonomic, and as  $Q(1, 1, \frac{q}{1+q^2})$  is an algebraic specialization of both  $Q(x, y; t)$  and  $Q(1, 1; t)$ , neither of these two functions are holonomic either.  $\square$

A natural extension of this work would be to complete the story, and use the functional equations to find enumerative data.

5.1.1. *What happens with the method when the generating function IS holonomic?* This is a natural question, and it speaks to robustness of the method. We can in general follow the steps to express the counting generating function as a sum of products of  $Y_n$ . However, when the group of the walk is finite, the  $Y_n$  form a finite group, and thus this sum does not converge as power series.

The difficulty in using this method arises in showing that the singularities do not cancel.

## 6. Prospects for general criteria

The group of the walk is useful for determining additional equations in the the kernel method. However, an examination of all nearest neighbour walks and their groups reveals a surprising fact about when this group is finite. Table 2 lists all nearest neighbour walks with finite groups (up to symmetry in the line  $x = y$ )

PROPOSITION 6.1. *Consider nearest neighbour walks in the quarter plane, and further suppose  $\mathcal{Y}$  is not singular. Then  $G(\mathcal{Y})$  is finite if and only if one of the following is true:*

- (1)  $\mathcal{Y}$  is  $x$ - or  $y$ - axis symmetric;
- (2)  $\mathcal{Y} = \text{rev}(\mathcal{Y})$ ;
- (3)  $\mathcal{Y} = \text{reflect}(\text{rev}(\mathcal{Y}))$
- (4)  $\mathcal{Y}$  is either Kreweras or  $\text{rev}(\text{Kreweras})$

The following conjecture is true in the case of  $|\mathcal{Y}| = 3$ , and all other known cases.

CONJECTURE 1. *Suppose the step set group  $\mathcal{Y}$  is not singular. Then  $W_{\mathcal{Y}}$  is holonomic if and only if the group of its step set is finite.*

In fact, in view of the details behind the iterated kernel method which we used to prove the non-holonomy of the two final step sets, this seems to be a reasonable conjecture indeed.

Combining the above proposition and conjecture, we have potential conditions on  $\mathcal{Y}$  for it to have a holonomic counting generating function.

CONJECTURE 2. *The generating function  $W_{\mathcal{Y}}$  is holonomic if and only if at least one of the following holds:*

- $\mathcal{Y}$  is singular;
- $\mathcal{Y}$  is  $x$ - or  $y$ - axis symmetric;
- $\mathcal{Y} = \text{rev}(\mathcal{Y})$  (path reversibility);
- $\mathcal{Y} = \text{reflect}(\text{rev}(\mathcal{Y}))$ .

This conjecture also implies that  $\text{rev}$  preserves holonomy. If  $K_r(x, y)$  is the kernel related to  $\mathcal{Y}$ , then  $K_r(\bar{x}, \bar{y})$  is the kernel related to  $\text{rev}(\mathcal{Y})$ . By taking the viewpoint that  $\bar{x}$  and  $\bar{y}$  are mere variables, we see that if the  $\mathcal{Y}$ -step set group is finite, then so is the  $\text{rev}(\mathcal{Y})$ -step set as the groups will be the same.

We note that step set #10, whose generating function is not holonomic, does have some symmetry— $\mathcal{Y} = \text{reflect}(\mathcal{Y})$ , but this is not sufficient to offer holonomy. It is worth noting that the other two classes with this same symmetry are Kreweras and reverse Kreweras.

If true, this conjecture would respond positively to a conjecture of Gessel on the nature of the walks given by the step set  $\mathcal{Y} = \{\text{N, SE, S, NW}\}$ [7]. Indeed, this step set seems resists most classical approaches and the holonomy of its counting generating function, and its complete generating function, is a subject of active debate. The present evidence seems to imply that the counting generating function is holonomic, but that the complete generating function is not. Indeed it is on account of this case that we do not propose similar conjectures for the complete generating functions.

**6.1. Future Work.** We would like to characterize the classification combinatorially. First, there is possibly a better definition for the group of the walk, namely the group of all actions that fix the rational kernel. Having resolved that, several natural generalizations are possible, such as what is a meaningful definition for group in 3 dimensions? Does an analogous definition still yield dihedral groups? What about step sets which are not of the nearest neighbour type; what kinds of groups come out of that? We are also currently investigating a similar classification for different wedge shapes, such as  $1/8$ -plane and  $3/4$ -plane. Fayolle et al. give a characterization for when the group is finite. Can this be translated into straightforward combinatorial terms? Are these walks all at least differentiably algebraic?

**6.2. Acknowledgments.** This work was completed, in part, during an NSERC (Natural Sciences and Engineering Research Council of Canada) Postdoc at LABRI, Université Bordeaux I, under the guidance of Mireille Bousquet-Mélou.

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