Formal Power Series and Algebraic Combinatorics
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Noncommutative Monomial Symmetric Functions.

Lenny Tevlin

Abstract. This presentation will introduce noncommutative analogs of monomial symmetric functions (and their dual, forgotten symmetric functions). In analogy to the classical theory, expansion of ribbon Schur functions in this basis in nonnegative. Moreover, one can define fundamental noncommutative symmetric functions by analogy with quasi-symmetric theory. The expansion of ribbon Schur functions in this basis is also nonnegative. The availability of monomial basis allows one to prove a noncommutative Cauchy identity as well as study a noncommutative pairing implied by Cauchy identity.

Résumé. Cette présentation fera découvrir les analogues non-commutatives des fonctions symétriques monomiales et leurs duales, fonctions symétriques "forgotten". De façon identique a la théorie classique, le développement des fonctions Schur rubans dans cette base est non-négatif. Aussi on peut introduire des fonctions fondamentales symétriques comme dans la théorie quasi-symétrique. Le développement des fonctions Schur rubans dans cette base est aussi non-négatif. On peut ainsi demontrer une identité de Cauchy non-commutative et analyser le couplage non-commutatif qui en derive.

Contents

1. Introduction and Notations.

1.1. Compositions and Partitions. All definitions in this and the following sections are standard. I will use the notation from [**2**] for commutative symmetric functions and those of [**1**] for noncommutative ones.[1]

Let $I = (i_1, \ldots, i_n)$ be a composition, i.e. an ordered set of positive integers $(i_1, \ldots, i_n)$, called parts of the composition I. The sum of all components of the composition, its weight, is denoted by $|I|$, i.e.

$$|I| = i_1 + \ldots + i_n$$

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[1]With one exception. The fundamental quasi-symmetric function is denoted F_I in [**1**] and L_I in [**7**]. I will use L^I for the noncommutative version and reserve F^I for a noncommutative forgotten symmetric function.

and the number of parts in the composition – by $\ell(I)$.

For a composition I define a **reverse** composition $\overline{I} = (i_n, \ldots, i_1)$.

For two compositions $I = (i_1, \ldots, i_{r-1}, i_r)$ and $J = (j_1, j_2, \ldots, j_s)$ define two operations $\cdot$ and $\triangleright$:

$$I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s),\ \ell(I \triangleright J) = \ell(I) + \ell(J) - 1 \tag{1.1}$$

$$\text{and } I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s),\ \ell(I \cdot J) = \ell(I) + \ell(J) \tag{1.2}$$

Parts of the composition $\widetilde{I}$ **conjugate** to a composition I can be read from the diagram of the composition I from left to right and from bottom to top.

A partition is a composition with weakly decreasing parts, i.e.

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \text{ with } \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$$

The number of times an integer i occurs in a partition $\boldsymbol{\lambda}$ is denoted by $\mathtt{m_i}(\boldsymbol{\lambda})$.

The **reverse refinement order** for compositions is defined as follows. Let $I = (i_1, \ldots, i_n), J = (j_1, \ldots, j_k)$, $|J| = |I|$

$$J \preceq I$$

if J can be obtained from I, i.e. every part of J can be obtained from parts of I: $j_s = i_{r_s} + \ldots + i_{r_s+p}$ for all s and some r_s and p.

1.2. Quasideterminants. Noncommutative monomial symmetric functions will be defined in terms of quasideterminants. The definition of a quasideterminant for an arbitrary matrix was given in [**1**]. In general quasideterminant is not polynomial in its entries. All the matrices that come up in what follows will be almost-triangular with constants above the main diagonal. Moreover, in principle an $n \times n$ matrix has n^2 quasideterminants, which can be calculated with respect to any element of the matrix. In what follows by quasideterminant I will always mean the quasideterminant with respect to the element in the lower left corner (see below).

Consider a quasideterminant of an $n \times n$ almost triangular matrix. Such quasideterminant is polynomial in its entries and according to Proposition 4.7 of [**5**] can be written as:

$$Q_n(\mathbb{B}) \equiv Q_n(b_1, \ldots, b_{n-1}) = \begin{vmatrix} a_{11} & b_1 & 0 & \ldots & \ldots & \ldots & \ldots \\ a_{21} & a_{22} & b_2 & 0 & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \ldots & a_{jj} & b_j & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1\ 1} & a_{n-1\ 2} & \ldots & \ldots & \ldots & \ldots & b_{n-1} \\ \boxed{a_{n1}} & a_{n2} & \ldots & a_{nj} & \ldots & \ldots & a_{nn} \end{vmatrix} =$$

$$= \sum_{n \geq j_1 > \ldots > j_k > 1} (-1)^{k+1} a_{nj_1} b^{-1}_{j_1-1} a_{j_1-1\ j_2} b^{-1}_{j_2-1} a_{j_2-1\ j_3} \ldots b^{-1}_{j_k-1} a_{j_k-1\ 1} \tag{1.3}$$

where a_{ij} are free noncommutative elements and b_j's belong to some commutative field $\mathbb{B}$ and commute with all a_{ij}.

1.3. Noncommutative Symmetric Functions. I will think of **Sym** as a free associative algebra generated by an infinite sequence $\Psi_k, k \geq 1$ over a commutative field. Ψ_k's will be referred to as noncommutative power sums. Ψ_k is the notation used for power sum of the first kind in [**1**] and my use of the same notation is for convenience. Hopefully this will not be misleading. (I will make use of one type of power sums only and will therefore refer to Ψ_k simply as power sums.)

As one particular realization of noncommutative power sums, one can consider a most direct noncommutative analog of regular power sums. For an infinite collection of non-commuting variables $(z_1, z_2, \ldots)$ one can build symmetric powers sums

$$\Psi_k = \sum_i z_i^k$$

For any composition I define a **product**

DEFINITION 1.1.

$$\Psi^I = \prod_{k=1}^{\ell(I)} \Psi_{i_k}$$

Then using the results of [**1**], one can define elementary and complete symmetric functions as the following quasi-determinants:

$$n\Lambda_n = (-1)^{n-1} \begin{vmatrix} \Psi_1 & 1 & 0 & \dots & 0 & 0 \\ \Psi_2 & \Psi_2 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \dots & \dots & \dots & \Psi_2 & n-1 \\ \boxed{\Psi_n} & \dots & \dots & \dots & \Psi_2 & \Psi_1 \end{vmatrix}$$

and

$$nS_n = \begin{vmatrix} \Psi_1 & -(n-1) & 0 & \dots & 0 & 0 \\ \Psi_2 & \Psi_1 & -(n-2) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{n-1} & \dots & \dots & \dots & \Psi_2 & -1 \\ \boxed{\Psi_n} & \dots & \dots & \dots & \Psi_2 & \Psi_1 \end{vmatrix}$$

For a composition $I = (i_1, \dots, i_n)$ define products of

complete symmetric functions	$S^I = S_{i_1} S_{i_2} \dots S_{i_n}$
elementary symmetric functions	$\Lambda^I = \Lambda_{i_1} \Lambda_{i_2} \dots \Lambda_{i_n}$

Ribbon Schur functions are defined as

$$R_I = (-1)^{k-1} \begin{vmatrix} S_{i_k} & 1 & \dots & 0 & 0 \\ & \dots & & & \\ S_{i_3+\dots+i_k} & \dots & S_{i_3} & 1 & 0 \\ S_{i_2+\dots+i_k} & \dots & S_{i_2+i_3} & S_{i_2} & 1 \\ \boxed{S_{i_1+\dots+i_k}} & \dots & S_{i_1+i_2+i_3} & S_{i_1+i_2} & S_{i_1} \end{vmatrix}$$

Furthermore, the algebra **Sym** affords an involution:

$$\omega(\Psi_k) = (-1)^{k-1}\Psi_k$$
$$\omega(S_k) = \Lambda_k$$
$$\omega(R_I) = R_{\widetilde{I}}$$

2. Results.

In Section 3 I introduce three new linear bases in **Sym**. The first basis **monomial** symmetric functions, denoted by M^I seems to play in the noncommutative theory a role similar to that of the monomial symmetric functions in the commutative one.

The two functions enjoy the following relationship,

$$\widetilde{m}_{\boldsymbol{\mu}} = \sum_{\mathfrak{S}_n} m_I,$$

where $\widetilde{m}_{\boldsymbol{\mu}}$ is the augmented monomial symmetric function as in Exercise 10, §6, Ch. I of [**2**], m_I denotes the commutative image of M^I, the sum is over all distinct permutations of composition I, and $\boldsymbol{\mu}$ is the partition obtained from I.

The second basis is that of **forgotten** symmetric functions denoted by F^I.

Complete symmetric functions continue to be worthy of their name just as in the commutative case where h_n is a sum of all monomial functions of the same degree. The noncommutative version of this relationship is

$$S_n = \sum_{|I|=n} M^I$$

In fact a stronger statement is true:

$$F^I = \sum_{J \preceq I} M^J,$$

where the sum is over all compositions that are less fine than I. (A complete symmetric function S_n is a forgotten function corresponding to the composition 1^n.)

The third basis is that of **fundamental symmetric functions** L^I by analogy with Gessel's fundamental quasi-symmetric functions [**8, 7, 1**]

$$L^I = \sum_{J \succeq I} M^I$$

In the following Section 4, I will study recursions and multiplication of these newly introduced functions. Interestingly, products of noncommutative monomial and fundamental symmetric functions are nonnegative.

Transitions between different bases are studied in Section 5. It appears that ribbon Schur functions are nonnegative both in the monomial and fundamental bases as will be discussed in 5.4 and 5.5 respectively.

I will derive a noncommutative Cauchy identity in Section 6, which will lead to a noncommutative pairing, as discussed later in the same section.

3. Monomial, Forgotten, and Fundamental Noncommutative Symmetric Functions.

Define **noncommutative monomial symmetric function** corresponding to a composition $I = (i_1, \ldots, i_n)$ as a quasideterminant of an n by n matrix:

DEFINITION 3.1.

$$nM^I \equiv nM^{(i_1,\ldots,i_n)} = (-1)^{n-1} \begin{vmatrix} \Psi_{i_n} & 1 & 0 & \ldots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & n-1 \\ \boxed{\Psi_{i_1+\ldots+i_n}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

where n is the length of I. In particular

$$\Lambda_n = M^{1^n}$$

where Λ_n is an elementary symmetric function introduced in [**1**].

Also define **noncommutative forgotten symmetric function** as an n by n quasideterminant:

DEFINITION 3.2.

$$nF^I \equiv n\phi^{(i_1,\ldots,i_n)} = \begin{vmatrix} \Psi_{i_n} & -(n-1) & 0 & \ldots & 0 & 0 \\ \Psi_{i_{n-1}+i_n} & \Psi_{i_{n-1}} & -(n-2) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{i_2+\ldots+i_n} & \ldots & \ldots & \ldots & \Psi_{i_2} & -1 \\ \boxed{\Psi_{i_1+\ldots+i_n}} & \ldots & \ldots & \ldots & \Psi_{i_1+i_2} & \Psi_{i_1} \end{vmatrix}$$

In particular

$$F^{1^n} = S_n$$

where S_n a homogeneous symmetric function as in [**1**].

As noted above (see (1.3)) it follows from Proposition 4.7 of [**5**] that all M^I and F^I are polynomial in power sums.

Notice that if involution ω is extended by linearity (compare Prop. 3.9 in [**1**])

$$\omega(\Psi^I) = (-1)^{|I|-\ell(I)}\Psi^{\bar{I}}$$

then

PROPOSITION 3.1.

$$\omega(M^I) = (-1)^{|I|-\ell(I)}F^{\bar{I}},$$

where $\bar{I}$ is the reversed composition.

By analogy with the quasi-symmetric theory [**8, 7, 1**]), define **fundamental** noncommutative symmetric function as

DEFINITION 3.3.
$$L^I = \sum_{J \succeq I} M^I$$

These functions behave nicely under the involution ω:

PROPOSITION 3.2.
$$\omega\left(L^I\right) = L^{\widetilde{I}}$$

4. Recursion Relations, Pieri formulas, and Multiplicative Structure.

In this section I will consider multiplication rules for newly introduced functions that they inherit from the multiplication rule of power sums (1.1).

4.1. Monomial and Forgotten. The definition of the monomial functions (3.1) implies a linear relationship between monomial functions obtained from the same composition due to Theorem 1.8 of [**6**].

This relationship can be stated as follows (and can be considered as a generalization of formulas (31) in Proposition 3.3 in [**1**]):

PROPOSITION 4.1. *Newton-type relations.*

$$nM^{i_1,\dots,i_n} = \Psi_{i_1} M^{i_2,\dots,i_n} - \Psi_{i_1+i_2} M^{i_3,\dots,i_n} + \dots + (-1)^{s-1}\Psi_{i_1+\dots+i_s} M^{i_{s+1},\dots,i_n} + \dots + (-1)^n \Psi_{i_1+\dots+i_n} \tag{4.1}$$

$$n\phi^{i_1,\dots,i_n} = \phi^{i_1,\dots,i_{n-1}}\Psi_{i_n} + \phi^{i_1,\dots,i_{n-2}}\Psi_{i_{n-1}+i_n} + \dots + \phi^{i_1+\dots+i_s}\Psi_{i_{s+1},\dots,i_n} + \dots + \Psi_{i_1+\dots+i_n} \tag{4.2}$$

Moreover, Newton-type relationships above imply rules of multiplication by a power sum of an arbitrary monomial function on the left and forgotten on the right:

PROPOSITION 4.2. *Pieri-like formula for monomials and forgotten.*
Let I be a composition with $\ell(I) = n$, then

$$\Psi_r \cdot M^I = (n+1) M^{(r)\cdot I} + n\ M^{(r)\triangleright I} \tag{4.3}$$

$$F^I \cdot \Psi_r = (n+1)\phi^{I\cdot(r)} - n\ F^{I\triangleright(r)} \tag{4.4}$$

More generally, the product of two monomial symmetric functions has the following expansion:

PROPOSITION 4.3.

$$M^J \cdot M^I = \sum_{K \preceq J,\ |K|=|J|} \binom{\ell(I)+\ell(K)}{\ell(J)} M^{K\cdot I} + \binom{\ell(I)+\ell(K)-1}{\ell(J)} M^{K\triangleright I}, \tag{4.5}$$

where the sum is over all compositions preceding J in the reverse refinement order. (Binomial coefficients with negative entries are taken to be zero.)

PROPOSITION 4.4.

$$F^I \cdot F^J = \sum_{K \preceq J} (-1)^{\ell(K)-\ell(J)+1} \binom{\ell(I)+\ell(K)}{\ell(J)} F^{I\cdot K} + \sum_{K \preceq J} (-1)^{\ell(K)-\ell(J)} \binom{\ell(I)+\ell(K)-1}{\ell(J)} F^{I\triangleright K} \tag{4.6}$$

In addition, generalizing formula (30) of Proposition 3.3 in [**1**], given a composition $I = (i_1, \dots, i_n)$ the following bilinear relations are true:

PROPOSITION 4.5.
$$\sum_{s=0}^{n} (-1)^{n-s} F^{i_1,\dots,i_s} M^{i_{s+1},\dots,i_n} = \sum_{s=0}^{n} (-1)^{n-s} M^{i_1,\dots,i_s} F^{i_{s+1},\dots,i_n} = 0$$

4.2. Fundamental. So far I have not been able to obtain a formula for multiplication of fundamental corresponding to arbitrary compositions. The following partial results can nevertheless be useful:

PROPOSITION 4.6.

$$L^n \cdot L^J = \sum_{M \succeq J} \binom{n+\ell(J)-1}{\ell(M)} L^{n\cdot M} + \binom{n+\ell(J)-1}{\ell(M)-1} L^{n \triangleright M}$$

and dually

$$L^I \cdot L^{1^n} = \sum_{K \preceq I} \binom{\ell(\widetilde{I})+n-1}{\ell(\widetilde{K})} L^{K \triangleright 1^n} + \binom{\ell(\widetilde{I})+n-1}{\ell(\widetilde{K})-1} L^{K\cdot 1^n}$$

For two arbitrary compositions the following result seems to follow from numerous examples:

CONJECTURE 4.1.

$$L^I \cdot L^J = \sum_{K \preceq I,\ M \succeq J} c_{K\cdot M} L^{K\cdot M} + c_{K \triangleright M} L^{K \triangleright M}$$

with all coefficients nonnegative integers.

5. Transition between Different Bases.

It turns out that transition matrices between noncommutative monomial and forgotten function are specially simple, i.e. uni-triangular with respect to the reverse refinement order. It is necessary, however, to first revisit quasideterminants and their properties first.

5.1. An Identity between Quasideterminants of Almost-Triangular Matrices. The property of triangularity follows from a general identity for quasideterminants of almost-triangular matrices.[2] This identity relates a quasideterminant of an almost-triangular matrix with given entries on the off-diagonal to a sum of quasideterminants with the same off-diagonal entries but in reversed order and opposite sign.

Let an operator T_j act on $Q_n(\mathbb{B})$ by simultaneously removing $(j+1)^{\text{th}}$th column and j^{th} row (the column and row that intersect at the off-diagonal element b_j). (The resulting $(n-1)\times(n-1)$ matrix is filled in with the first $(n-2)$ entries of $\mathbb{B}$.) Further, for a composition $J = (j_1, j_2, \dots, j_k)$ define

$$T_J = \prod_{s=1}^{k} T_{j_s}, \text{ with } k = \ell(J)$$

Take $\mathbb{B} = \mathbb{N}$. Then the following identity is true:

PROPOSITION 5.1.

$$\frac{1}{n} Q_n(-(n-1), -(n-2), \dots, -1) = \sum_J \frac{(-1)^{n-k-1}}{n-k} T_J Q_n(1, 2, \dots, n-1), \tag{5.1}$$

where the sum is over all subsets $J \subseteq [1, 2, \dots, n-1]$.

[2] I believe this identity is new.

Example 5.1. Consider a four by four quasideterminant $Q_4(-3,-2,-1)$ and its expansion:

$$Q_4(-3,-2,-1)=\frac{1}{4}\begin{vmatrix} a_{11} & -3 & 0 & 0\\ a_{21} & a_{22} & -2 & 0\\ a_{31} & a_{32} & a_{33} & -1\\ \boxed{a_{41}} & a_{42} & a_{43} & a_{44}\end{vmatrix}=$$

$$=\left(-\frac{1}{4}T_{\emptyset}+\frac{1}{3}\left(T_1+T_2+T_3\right)-\frac{1}{2}\left(T_1T_2+T_1T_3+T_2T_3\right)+T_1T_2T_3T_4\right)Q_4(1,2,3)=$$

$$=-\frac{1}{4}\begin{vmatrix} a_{11} & 1 & 0 & 0\\ a_{21} & a_{22} & 2 & 0\\ a_{31} & a_{32} & a_{33} & 3\\ \boxed{a_{41}} & a_{42} & a_{43} & a_{44}\end{vmatrix}+\frac{1}{3}\begin{vmatrix} a_{21} & 1 & 0\\ a_{31} & a_{33} & 2\\ \boxed{a_{41}} & a_{43} & a_{44}\end{vmatrix}+\frac{1}{3}\begin{vmatrix} a_{11} & 1 & 0\\ a_{31} & a_{32} & 2\\ \boxed{a_{41}} & a_{42} & a_{44}\end{vmatrix}+\frac{1}{3}\begin{vmatrix} a_{11} & 1 & 0\\ a_{21} & a_{22} & 2\\ \boxed{a_{41}} & a_{42} & a_{43}\end{vmatrix}-$$

$$-\frac{1}{2}\begin{vmatrix} a_{31} & 1\\ \boxed{a_{41}} & a_{44}\end{vmatrix}-\frac{1}{2}\begin{vmatrix} a_{12} & 1\\ \boxed{a_{41}} & a_{43}\end{vmatrix}-\frac{1}{2}\begin{vmatrix} a_{11} & 1\\ \boxed{a_{41}} & a_{42}\end{vmatrix}+a_{41}$$

5.2. Noncommutative Monomial and Forgotten Symmetric Functions. If $a_{kj}=\Psi_{i_{n-k+1}+\ldots+i_{n-j+1}}$, where $(i_1,\ldots,i_n)$ are parts of the composition I then (5.1) implies

Proposition 5.2.

$$F^I=\sum_{J\preceq I}M^J, \tag{5.2}$$

where the sum is over compositions in the reverse refinement order.

And conversely,

$$M^I=\sum_{J\preceq I}(-1)^{\ell(I)-\ell(J)}F^J, \tag{5.3}$$

Example 5.2. Continuing Example 5.1, consider an expansion of $F^{2,2,1,3}$. Therefore let $I=(2,2,1,3)$ and $a_{kj}=\Psi_{i_{n-k+1}+\ldots+i_{n-j+1}}$, then

$$\frac{1}{4}\begin{vmatrix} \Psi_3 & -3 & 0 & 0\\ \Psi_4 & \Psi_1 & -2 & 0\\ \Psi_6 & \Psi_3 & \Psi_2 & -1\\ \boxed{\Psi_8} & \Psi_5 & \Psi_4 & \Psi_2\end{vmatrix}=-\frac{1}{4}\begin{vmatrix} \Psi_3 & 1 & 0 & 0\\ \Psi_4 & \Psi_1 & 2 & 0\\ \Psi_6 & \Psi_3 & \Psi_2 & 3\\ \boxed{\Psi_8} & \Psi_5 & \Psi_4 & \Psi_2\end{vmatrix}+\frac{1}{3}\begin{vmatrix} \Psi_4 & 1 & 0\\ \Psi_6 & \Psi_2 & 2\\ \boxed{\Psi_8} & \Psi_4 & \Psi_2\end{vmatrix}+\frac{1}{3}\begin{vmatrix} \Psi_3 & 1 & 0\\ \Psi_6 & \Psi_3 & 2\\ \boxed{\Psi_8} & \Psi_5 & \Psi_2\end{vmatrix}+$$

$$+\frac{1}{3}\begin{vmatrix} \Psi_3 & 1 & 0\\ \Psi_4 & \Psi_1 & 2\\ \boxed{\Psi_8} & \Psi_5 & \Psi_4\end{vmatrix}-\frac{1}{2}\begin{vmatrix} \Psi_6 & 1\\ \boxed{\Psi_8} & \Psi_2\end{vmatrix}-\frac{1}{2}\begin{vmatrix} \Psi_4 & 1\\ \boxed{\Psi_8} & \Psi_4\end{vmatrix}-\frac{1}{2}\begin{vmatrix} \Psi_3 & 1\\ \boxed{\Psi_8} & \Psi_5\end{vmatrix}+\Psi_8,$$

i.e.

$$F^{2,2,1,3}=M^{2,2,1,3}+M^{2,2,4}+M^{2,3,3}+M^{4,1,3}+M^{2,6}+M^{4,4}+M^{5,3}+M^8$$

5.3. Noncommutative Monomial and Power Sums. Consider $J=(j_1,\ldots,j_s)\preceq I=(i_1,\ldots,i_n)$, i.e.

$$J=(i_1+\ldots+i_{p_1},i_{p_1+1}+\ldots+i_{p_2},\ldots,i_{p_{k-1}+1}+\ldots+i_{p_k},\ldots,i_{p_s}+\ldots+i_n)$$

for some nonnegative $p_1,\ldots,p_s$. Set $p_0=0$. Then the formula for transition from monomial to power sum basis can be written as

Proposition 5.3.

$$M^I=\sum_{J\preceq I}\frac{(-1)^{\ell(I)-\ell(J)}}{\prod_{k=0}^{s-1}(\ell(I)-p_k)}\Psi^J$$

Conversely,

Proposition 5.4.

$$\Psi^I=\sum_{J\preceq I}\prod_{k=1}^{\ell(J)}(\ell(J)-k+1)^{p_k-p_{k-1}}M^J$$

Combining the expansion of monomial functions in power sums and power sums in complete (and other bases) from [**1**], one can obtain expansion monomial functions in complete (and other bases).

5.4. Ribbon Schur Functions and Noncommutative Monomials. Given the importance of Kostka numbers, it is natural to consider an expansion of ribbon Schur functions in monomials

$$R_I = \sum_{J,\ |J|=|I|} K_{IJ} M^J,$$

where K_{IJ} can be called the noncommutative Kostka numbers by analogy with the classical case. Based on a fair number of examples the following can be stated as a conjecture

Conjecture 5.3. *K_{IJ} are **nonnegative integers**.*

The fact that expansion coefficients are integers is obvious from the definition of ribbon Schur functions and the fact that product of M^Is is integral in the monomial basis. It is the fact that these integers are nonnegative that is missing a proof.

For certain types of compositions this conjecture can be confirmed by explicit expressions.

Proposition 5.5. *Let $k \geq r$, then*

$$R_{k1^r} = \binom{k+r-1}{r} \sum_{|I|=k} M^{I\cdot 1^r} \tag{5.4}$$

Proposition 5.6.

$$R_{1^k n} = \sum_{|J|=k} \sum_{|I|=n} \binom{\ell(I)+\ell(J)-1}{k} M^{J\cdot I} + \binom{\ell(I)+\ell(J)-2}{k} M^{J\triangleright I} \tag{5.5}$$

5.5. Ribbon Schur Functions and Fundamental Noncommutative Symmetric Functions. Ribbon Schur functions seem to have an expansion in fundamental noncommutative function with nonnegative coefficients as well, i.e. if

$$R_I = \sum_{J,|J|=|I|} G_{IJ} L^J$$

then

Conjecture 5.4. *G_{IJ} are **nonnegative integers**.*

This conjecture is obviously stronger then Conjecture 5.3. As in the case of noncommutative Kostka numbers only the nonnegativity requires proof.

Again, this can be confirmed by explicit formulas in special cases. For ribbon Schur functions labeled by compositions with two and three parts respectively, one obtains

Example 5.5.

$$R_{nm} = S_n S_m - S_{n+m} = L^n \cdot L^n - L^{n\triangleright m} \overset{\text{by (4.6)}}{=} \sum_{J_1 \succeq (m)} \binom{n+m-1}{\ell(J_1)} L^{n\cdot J_1} + \binom{n+m-1}{\ell(J_1)-1} L^{n\triangleright J_1} - L^{n\triangleright m} =$$
$$= \sum_{J_1 \succeq (m)} \binom{n+m-1}{\ell(J_1)} L^{n\cdot J_1} + \sum_{J_1 \succ (m)} \binom{n+m-1}{\ell(J_1)-1} L^{n\triangleright J_1}$$

Example 5.6.

$$R_{knm} = \sum_{J_1 \succeq (m)} \sum_{J_2 \succeq n\cdot J_1} \binom{n+m-1}{\ell(J_1)} \binom{k+n+\ell(J_1)-1}{\ell(J_2)} L^{k\cdot J_2} +$$
$$+ \sum_{J_1 \succeq (m)} \sum_{J_2 \succ n\cdot J_1} \binom{n+m-1}{\ell(J_1)} \binom{k+n+\ell(J_1)-1}{\ell(J_2)-1} L^{k \triangleright J_2} +$$
$$+ \sum_{J_1 \succeq (m)} \left[\binom{n+m-1}{\ell(J_1)} \binom{k+n+\ell(J_1)-1}{\ell(J_1)} - \binom{k+n+m-1}{\ell(J_1)} \right] L^{k \triangleright n \cdot J_1} +$$
$$+ \sum_{J_1 \succ (m)} \sum_{J_2 \succeq n \triangleright J_1} \binom{n+m-1}{\ell(J_1)-1} \binom{k+n+\ell(J_1)-2}{\ell(J_2)} L^{k\cdot J_2} +$$
$$+ \sum_{J_1 \succ (m)} \sum_{J_2 \succ n \triangleright J_1} \binom{n+m-1}{\ell(J_1)-1} \binom{k+n+\ell(J_1)-2}{\ell(J_2)-1} L^{k \triangleright J_2} +$$
$$+ \sum_{J_1 \succ (m)} \left[\binom{n+m-1}{\ell(J_1)-1} \binom{k+n+\ell(J_1)-2}{\ell(J_1)-1} - \binom{k+n+m-1}{\ell(J_1)-1} \right] L^{k \triangleright n \triangleright J_1}$$

with all coefficients nonnegative.

6. A Noncommutative Cauchy Identity and Pairing.

Equipped with noncommutative monomial and fundamental symmetric functions, I'm ready to state the noncommutative version of the Cauchy identity.

Proposition 6.1.

$$\sum_I M^I(X) S^I(A) = \sum_J L^J(X) R_J(A).$$

Using the noncommutative Cauchy identity as a starting point, define the noncommutative pairing

Definition 6.1.

$$\langle M^I | S^J \rangle = \delta_{IJ},$$

i.e. declare monomial symmetric functions to be dual to complete.
Then fundamental symmetric functions are dual to ribbon Schur:

Proposition 6.2.

$$\langle L^I | R_J \rangle = \delta_{IJ}$$

One of the pleasing properties of this pairing is that

Proposition 6.3. *The involution ω is an isometry of the pairing (6.1).*

One can deduce values of this pairing between different functions making use of transition formulas between different bases (see Sec. 5) and applying involution ω.
To write down the formula for pairing between power sums, which is particularly interesting as they are not orthogonal in contrast to the classical theory, I need to recall some more definitions from [**1**]. Denote the last part of a composition $I = (i_1, \ldots, i_k)$ by

$$lp(I) = i_k$$

and let J be a composition such that $J \succeq I$. Let then $J = (J_1, ..., J_m)$ be the unique decomposition of J into compositions $(J_i)_{i=1,m}$ such that $|J_p| = i_p, p = 1, ..., m$. Define

$$lp(J, I) = \prod_{i=1}^{m} lp(J_i)$$

Then

PROPOSITION 6.4.

$$\langle \Psi^I | \Psi^J \rangle = \sum_{J \preceq M \preceq I} (-1)^{\ell(M)-\ell(J)} lp(M,J) \prod_{k=1}^{\ell(M)} (\ell(M)-k+1)^{p_k - p_{k-1}},$$

where p_k *are such that for each* M

$$M = (i_1 + \ldots + i_{p_1}, i_{p_1+1} + \ldots + i_{p_2}, \ldots, i_{p_{k-1}+1} + \ldots + i_{p_k}, \ldots, i_{p_s} + \ldots + i_n)$$

In particular

$$\langle \Psi^I | \Psi^I \rangle = \left(\prod_{k=1}^{\ell(I)} i_k \right) \ell(I)! = sp(I) \tag{6.1}$$

The last notation is also from [**1**].

COROLLARY 6.2. Since the involution is an isometry of the scalar product

$$\langle \Psi^{\overline{I}} | \Psi^{\overline{J}} \rangle = (-1)^{\ell(I)-\ell(J)} \langle \Psi^I | \Psi^J \rangle$$

7. Some Noncommutative Identities.

In the Exercise 10, Ch. I, §5 of [**2**], it is shown that

$$\sum_{|\boldsymbol{\lambda}|=n} X^{\ell(\boldsymbol{\lambda})-1} m_{\boldsymbol{\lambda}} = \sum_{k=0}^{n-1} s_{n-k,1^k} (X-1)^k \tag{7.1}$$

This identity has the following noncommutative analog:

PROPOSITION 7.1.

$$\sum_{|I|=n} X^{\ell(I)-1} M^I = \sum_{k=0}^{n-1} R_{1^k,n-k} (X-1)^k$$

In some sense this is a generalization of Corollary 3.14 of [**1**]

$$\Psi_n = \sum_{k=0}^{n-1} (-1)^k R_{1^k,n-k}$$

as Proposition 7.1 reduces to the above at $X = 0$.

Another identity, a way of writing (56) from [**1**] or, a noncommutative version of Ex. 4, Ch. I, §5 of [**2**].

PROPOSITION 7.2.

$$(S_1)^n = \sum_{|I|=n} R_I = \sum_{|I|=n} \prod_{k=1}^{\ell(I)} (\ell(I)-k+1)^{i_k} M^I$$

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9. Conclusions.

In conclusion, this presentation introduces what seems to be an adequate noncommutative analog of monomial symmetric functions. The noncommutative monomial symmetric functions share a lot of properties with their classical analogs, the most important of which are the integrality and (conjectured) positivity of the ribbon Schur functions in this basis.

Ribbon Schur functions are also (conjecturally) nonnegative in the basis of noncommutative fundamental symmetric functions and dual to these functions with respect to the pairing between noncommutative monomial and complete symmetric functions.

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Physics Department, Yeshiva University, 500 West 185th Street, New York, N.Y. 10033, USA

E-mail address: tevlin@yu.edu