On the structure of regular B_2 crystals

V.I.Danilov*, A.V.Karzanov†, and G.A.Koshevoy‡

1 Introduction

For simply-laced Kac-Moody algebras g, Stembridge [8] proposed a 'local' axiomatization of crystal graphs of representations of $U_q(g)$. In fact, because of an important result in [3] an essential part in studying such crystals should have been carried out for the simplest nontrivial case $g = sl(3)$. Our paper [1] gives a combinatorial construction that describes the formation of any crystal of representations of $U_q(sl(3))$.

In this paper we attempt to carry out a similar programme for the algebra $sp(4)$. Because of [3], the Stembrindge axioms for the case of A_2 and our axioms for the B_2-case give a 'local' axiomatization doubly laced algebras, a doubly laced algebra is an algebra all regular rank 2 subalgebras are of type $A_1 \times A_1$, A_2, or B_2.

At the end of [8], Stembridge conjectured a list of relations between crystal operations and Sternberg proved this conjecture in [7]. We follow the ideology of [1] and propose axioms of monotonicity and commutativity for decorated edge-2-colored graphs which characterize the crystals of representations of $U_q(sp(4))$, regular crystal graphs of B_2-type. Specifically, an R-graph is an edge-colored graph which fulfil our Axioms K1-K5, and our main result states that regular crystal graphs of B_2-type are R-graphs and vice versa. In particular, our axiom K5 refines Sternberg's B_2-type relations by involving a certain labeling on the crystal edges. Moreover, we give a direct combinatorial construction of such crystal graphs using a general operation on graphs from [1]. On this way we introduce a new model for representations of $U_q(sp(4))$ (in 7-dimensional space) which does not exploit Young diagrams. We would like to mention that our model is not polyhedral, that means the following: due to this model, a regular crystal graph

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of B_2-type is located (at integer points) of the union of five 4-dimensional polyhedra in $\mathbb{R}^7$, but itself the crystal is not a convex set. This model has two projections on $\mathbb{R}^4$, which are the Littelmann cones, related by the proper piece-wise linear relations. Thus, our model might be seen as the proper graph of this correspondence between the Littelmann cones. (Another crossing model is developed in [2] for regular A_n-crystals, which might be seen as a proper graph for the piece-wise linear relations between the Berenstein-Zelevinsky-Littelmann cones.)

The structure of this paper is the following. Section 2 is devoted to axioms. In Section 3 we formulate the main result and give a constructive characterization of the crystal graphs in question. In Section 4 we introduce a so-called *crossing model* for crystals of B_2-type. In Section 5 we prove that certain intervals of the admissible configurations on this model are just crystals of representations of $U_q(sp(4))$. In Section 6 we prove that the graphs generated by the intervals on the crossing model are essentially the same as those constructed in Section 3.

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2 B_2-type edge-two-colored graphs

Here we construct decorated edge-two-colored directed graphs related to the B_2-type Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. Of our interest will be those graphs that match the list of axioms below (or a part of this list).

We consider a digraph $G = (V, A_1 \coprod A_2)$ with the vertex set V and the edges set partitioned into two subsets A_1 and A_2. For convenience we refer to the edges in A_1 as being colored in color 1, say *red*, and the edges in A_2 as being colored in color 2, say *green*. The first color is managed by the first column of A and the second one by the second column.

The first axiom concerns the structure of the monochromatic graphs (V, A_1) and (V, A_2), and states that they are constituted as the disjoint union of finite monochromatic strings. Specifically,

(K1) For $i = 1, 2$, each (weakly) connected component of (V, A_i) is a finite simple (directed) path, i.e., a sequence of the form $(v_0, e_1, v_1, \dots, e_k, v_k)$, where $v_0, v_1, \ldots, v_k$ are distinct vertices and each e_i is an edge going from v_{i-1} to v_i.

In particular, each vertex v has at most one outgoing 1-colored edge and at most one incoming 1-colored edge, and similarly for 2-colored edges. For brevity, we refer to a *maximal* monochromatic path in K, with color i on the edges, as an *i-string*. The i-string passing through a given vertex v (possibly consisting of the only vertex v) is denoted by $P_i(v)$, its part from the first vertex to v by $P_i^{\rm in}(v)$, and its part from v to the last vertex by $P_i^{\rm out}(v)$. The lengths of $P_i^{\rm in}(v)$ and of $P_i^{\rm out}(v)$ (i.e., the numbers of edges in these paths) are denoted by $t_i(v)$ and $h_i(v)$, respectively. The second axiom tells how these lengths can change when one traverses an edge of the other color.

Besides colors, the edges of G are endowed with *labels*, and for this reason, we say that G is a *decorated graph*. More precisely, the red edges have labels 0 or 1 and the green edges have labels 0 or $\frac{1}{2}$ or 1.

The second axioms indicates how the labels are related to the "local" structure of the graph. Consider a red edge (x, y) connecting a vertex x to a vertex y (using the operator notation for Kashiwara crystals, one can write $y = F_1 x$), and consider the strings $P_2(x)$ and $P_2(y)$. This gives us the tuple $(t_2(x), h_2(x), t_2(y), h_2(y))$, and in following axiom we pose some regularity conditions on these tuples.

(K2) For each 1-colored edge e going from x to y, there hold $t_2(x) - t_2(y) \geq 0$ and $h_2(y) - h_2(x) \geq 0$, and there holds $(t_2(x) - t_2(y)) + (h_2(y) - h_2(x)) = 1$, the label being assigned to e is equal to $h_2(y) - h_2(x)$.

For each 2-colored edge e' going from p to q, and, for the pair of strings $P_1(p)$ and $P_1(q)$, there hold $t_1(p) - t_1(q) \geq 0$ and $h_1(q) - h_1(p) \geq 0$, and there holds $\frac{t_1(p)-t_1(q)}{2} + \frac{h_1(q)-h_1(p)}{2} = 1$, the label assigned to e' is equal to $\frac{h_1(q)-h_1(p)}{2} \in \{0, \frac{1}{2}, 1\}$.

One can see that the decorated edge-colored graphs which satisfy the axioms $(K1)$-$(K2)$ are semi-normal crystals due to Kashiwara [5]. In fact, let us define the weight function $wt : V \to \mathbb{Z}^2$ by the rule

$$w\colon x \to (h_1(x) - t_1(x), h_2(x) - t_2(x)).$$

Then, the functions t_i, h_i and wt define a semi-normal crystal.

The graphs which meet axioms (K1)-(K2) are called *K-graphs*. We make a category of such graphs by setting morphisms as follows:

a mapping $f : C_1 \to C_2$ of K-graphs is a *morphism* if there holds $f(F_i(b)) = F_i(f(b))$ for each b such that there is an edge of the form $b \to F_i b$, and if the labels on the edges $b \to F_i b$ an $f(b) \to F_i(f(b))$ coincide.

Axioms K1 and K2 are motivated by the Kashiwara definition of crystals. The following axioms were found experimentally and their justification is done by Theorem 1 in which we assert that regular crystal graphs of B_2-type characterized by the whole list of axioms.

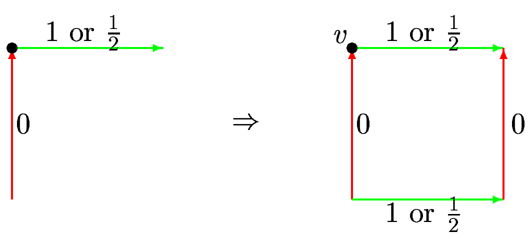
The third axiom is

(K3) For any i and any string P_i, the labels on consecutive edges e_1 and e_2 (where e_2 follows e_1) do not decrease.

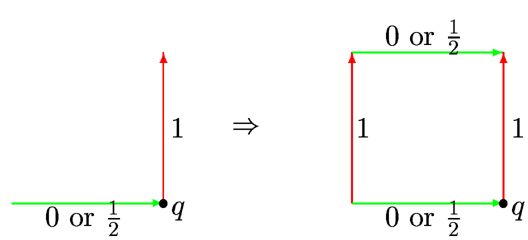
Then each red string has a unique vertex (possibly the beginning or ending one) at which labels switch from 0 to 1, we call this vertex *critical*.

The next axiom presents important commutation between special pairs of green and red edges.

(K4) Suppose a red edge enters a vertex v and has label a and let a green edge leave v and have label b. Then there holds $b \neq a$, and if $a < b$, then the commutative diagram takes place:



Similarly, suppose a green edge enters a vertex q and has label b and a red edge leaves q and has label a. Then there holds $b \neq a$, and if $a > b$, then the commutative diagram takes place:



The objects of a category of decorated edge-colored graphs which meet axioms $K1$-$K4$ are called *S-graphs* of B_2-type. The definition of the tensor product of crystal graphs in [5] might be applied for S-graphs. One can show that S-graphs form a tensor category indeed.

In order to get a filling how these axioms could 'work', we prove the following property of green strings.

Lemma 1. In any S-graph each green string either has one critical vertex, where label 0 is switched to 1, or one edge with label $\frac{1}{2}$, called the *critical edge*.

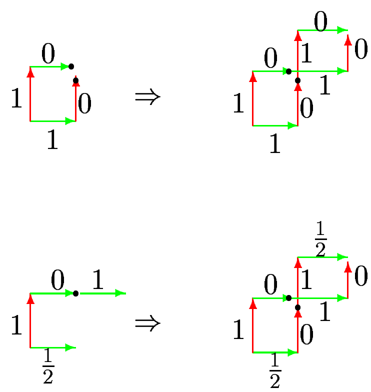
Proof. Consider a green edge labeled $\frac{1}{2}$ on a green-colored string of an S-graph. Suppose that a red edge labeled 1 emanates from the end point of this edge. Then the red string passing through this red edge is infinite. This follows from Axiom K4 and Axiom K2. Analogously the same holds if a red edge with the label 0 ends at the starting point of the green edges labeled $\frac{1}{2}$. Thus, an 'entering' red edge to a green edge labeled $\frac{1}{2}$ has to have the label 1, and a 'leaving' red edge has to have the label 0.

Now using this property of red edges we prove the claim. Suppose two green edges, both labeled $\frac{1}{2}$, are lying on a green string. Then, because of Axiom K3, there is a vertex on this string which is a common vertex for a pair of green edges labeled $\frac{1}{2}$. Hence, from Axiom K2 we conclude that there exists at least one red edge ingoing to such a vertex, and there is at least one outgoing red edge from this vertex. Then from the above property of red edges ingoing and outgoing from the end points of green edges with the label $\frac{1}{2}$, we have that the ingoing edge has the label 1 and the outgoing edge has the label 0. That contradicts to the monotonicity Axiom K3. Q.E.D.

2.1 R-graphs of B_2-types

S-graphs might have branchings and the category of S-graphs is larger than the category of crystal graphs of representations of $U_q(sp(4))$. In order to get the desired category (and avoid branchings) we introduce the following axiom.

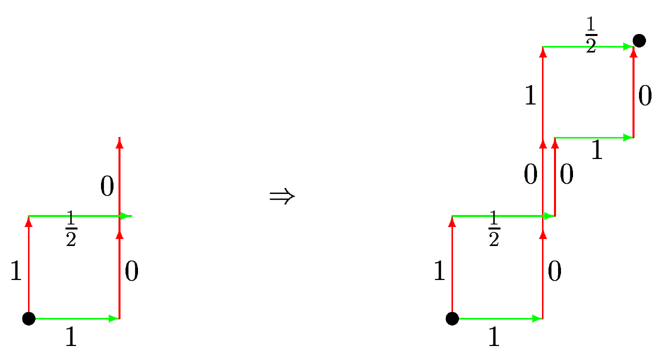
(K5) The implications illustrated on the following Pictures 1, 2, 3, and 4 have to hold.



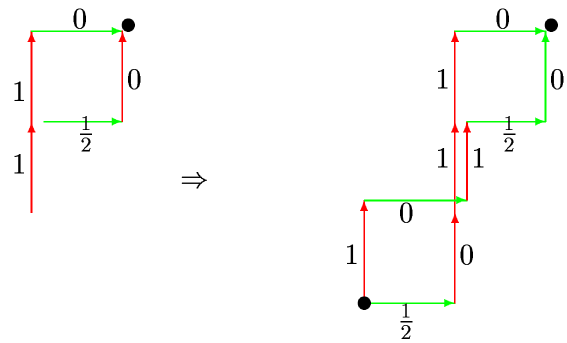
Picture 1.

These two relations have the following interpretation: the local situation in an S-graph on the left hand side implies the commutative diagram on the right hand side. The dual local situations to the left hand sides (reversing the edges and changing the labels $a \to 1 - a$) imply the same right hand side diagrams. (If the labels are ignored, one gets the degree 4 relations due to Sternberg [7]).

Next are other two relations (again the local situation on the left hand side implies the commutative squares on the right-hand side (degree 5 relations if the labels are ignored).

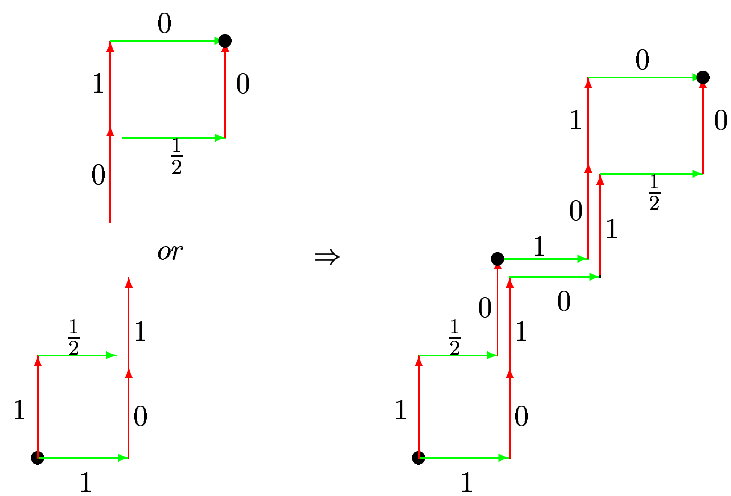


Picture 2.



Picture 3.

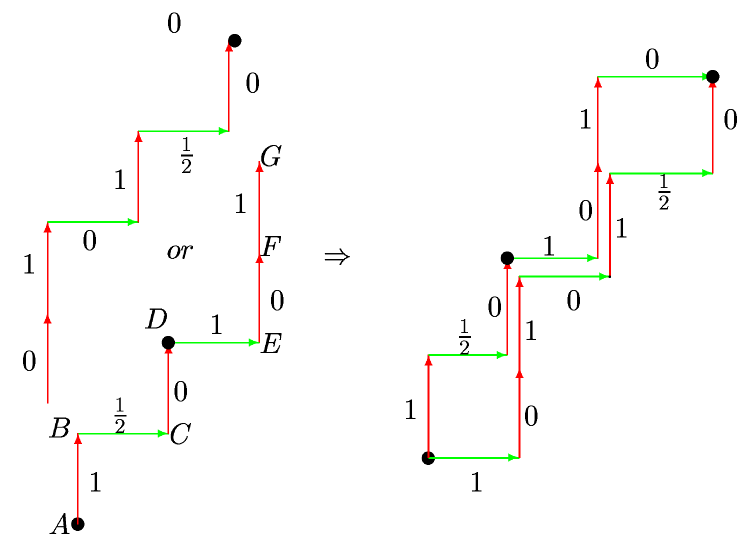
The final picture illustrates the Verma relation of degree 7, though depicted in the form somewhat different from that of the custom Verma relation of B_2-type. (The custom relation is obtained if one draws two commutative diagrams which follow from Axiom K4.)



Picture 4.

An S-graph of B_2-type in which the above relations hold is called an *R-graph* of B_2-type.

In that follows it will be of use the following two implications for B_2-type Verma relations, which follow from implications from Pictures 2, 3 and 4.



Picture 4a.

We claim the following

Theorem 1. *A connected R-graph of B_2-type is a crystal graph of an irreducible finite dimensional representation of $U_q(sp(4))$, and vise versa.*

3 Construction of R-graphs of B_2-type

Here we give a direct construction of such graphs.

Definition. Given an R-graph G, we define another labeled digraph $\hat{G} = (\hat{V}, \hat{E}, X : \hat{V} \to \mathbb{Z}_+^2)$ which we call the *view from the sky* (along red strings) on G:

the set of vertices, $\hat{V}$, of $\hat{G}$ is constituted from red strings of G, each vertex $\hat{v}$ is attributed a label being a pair of numbers $(X(\hat{v}), Y(\hat{v}))$ ($X+Y$ is the length of the corresponding red string and X is the number of edges labeled 0 on this string);

the set of edges, $\hat{E}$, is formed by the following rule: the vertices $\hat{v}$ and $\hat{v}'$ joined by an edge going from $\hat{v}$ to $\hat{v}'$ and labeled either 1, or $\frac{1}{2}$, or 0, if there exists such a green edge which joins points on the red strings which correspond to $\hat{v}$ and $\hat{v}'$.

Now we are going to give a combinatorial characterization of the view from the sky on an R-graph. A significance of this characterization is due to that any R-graph is determined by its view from the sky.

The following operation introduced in [1] will be of use.

Consider arbitrary graphs or digraphs $G = (V, E)$ and $H = (V', E')$. Let S be a distinguished subset of vertices of G, and T a distinguished subset of vertices of H. Take $|T|$ disjoint copies of G, denoted as G_t ($t \in T$), and $|S|$ disjoint copies of H, denoted as H_s ($s \in S$). We glue these copies together in the following way: for each $s \in S$ and each $t \in T$, the vertex s in G_t is identified with the vertex t in H_s. The resulting graph consisting of $|V||T| + |V'||S| - |S||T|$ vertices and $|E||T| + |E'||S|$ edges is denoted by $(G, S) \bowtie (H, T)$.

In our case the role of G and H is played by 2-colored digraphs $\hat{K}(H, 0)$ and $\hat{K}(0, A)$ depending on parameters $H, A \in \mathbb{Z}_+$ (it will be clear later that these digraphs are the views from the sky on the crystals of irreducible representations of $U_q(sp(4))$ with the highest weight $H\lambda_1$ and $A\lambda_2$, respectively, where λ_1 and λ_2 denote the fundamental weights for the B_2-type Cartan matrix).

Lemma 2. Let G be an R-graph. Then any vertex $\hat{v}$ of $\hat{G}$ has at most three ingoing edges and at most three outgoing edges; the ingoing edges have different labels and the outgoing edges have different labels; there are no parallel edges in $\hat{G}$[1].

For a proof of this Lemma we use the following properties, which follow from the "local commutative diagrams" illustrated on Pictures 1–4 and the commutative squares figured in Axiom K4.

(*) If a vertex v is located at least two red edges above a critical vertex on $P_1(v)$ or below the critical vertex on $P_1(v)$, that is $t_1(v) \geq X + 2$ or $t_1(v) \leq X - 2$, respectively ($(X, Y) = (t_1(*), h_1(*))$, where we let $*$ to denote the critical point on $P_1(v)$). Then a pair of the red and green edges ingoing in v form a commutative square, and a pair of the red and green edges outgoing from v form a commutative square.

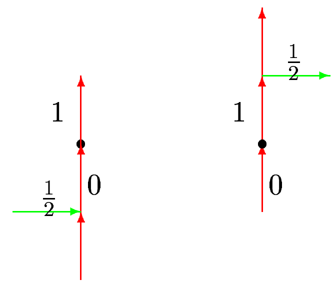
If a vertex v is such that $t_1(v) = X + 1$ (one edge above the critical), then the pair of ingoing red and green edges form a commutative square.

If a vertex v is such that $t_1(v) = X - 1$ (one edge below the critical), then the pair of outgoing red and green edges form a commutative square.

From Lemma 1 we have the following property.

(**) For each red string P, there can be at most two green strings having critical edge e (i.e., labeled $\frac{1}{2}$) such that e enters or leaves a vertex in P, and if this is the case, then the picture is as follows

[1] Edges $\hat{v}$ and $\hat{v}'$ are parallel if the join the same vertices.



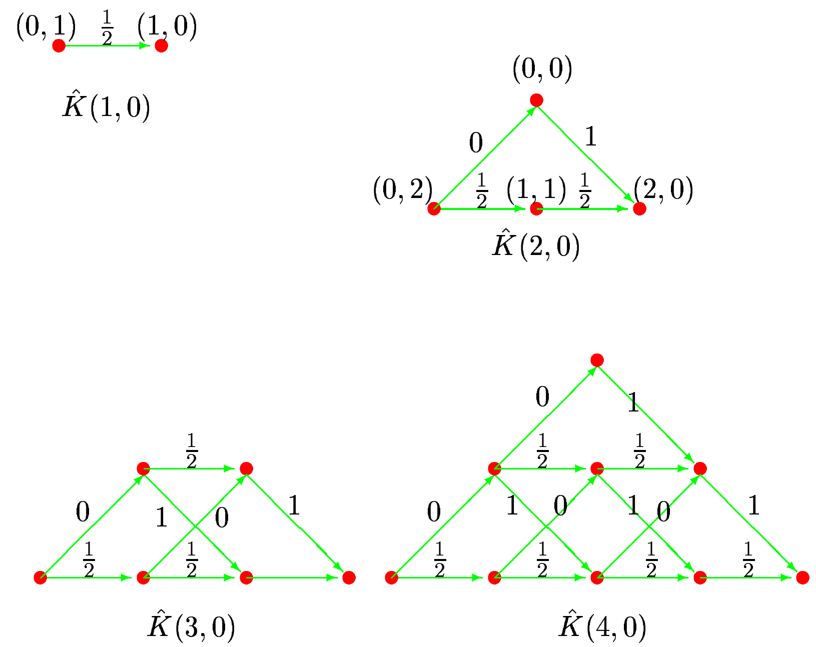
Picture 5.

That is, either e enters P one red edge before the critical point or leaves P one red edge after this point.

It will be useful to color edges in the view from the sky graph $\hat{G}$ in two colors, green and blue. We color the edges labeled $\frac{1}{2}$ in green. An edge $\hat{e}$ labeled 0 (or 1) is colored in blue if $|X(\hat{v})-X(\hat{v}')| = 1$ holds, and $\hat{e}$ is colored in green if either $|X(\hat{v}) - X(\hat{v}')| = 2$ or $|X(\hat{v}) - X(\hat{v}')| = 2$ hold.

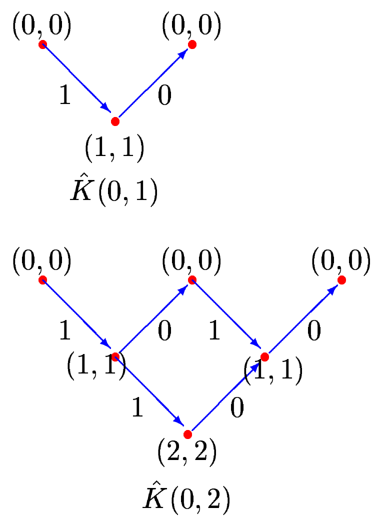
3.1 Graphs $K(H, A)$

The above-mentioned graph $K(H, 0)$ is an R-graph of B_2-type which has a red string of the form $(0, H)$ (with H red edges labeled 1 and no edges labeled 0) outgoing from the minimal vertex and no green edge outgoing from this vertex. These graphs have the "view from the sky" (depending on the parity of H) for $H = 1, 2, 3, 4$ as depicted below. One can realize how to construct such a view for an arbitrary H: the building blocks are the graph $\hat{K}(2, 0)$ (the view from the sky on the right-hand side relation from Picture 1 and two rhombuses from the graph $\hat{K}(3, 0)$, these rhombuses are view from the sky the relations from Picture 2 and 3. The rule for travelling along the red edges follows from the commutativity Axiom K3. We define the set of *distinguished vertices* in $\hat{K}(H, 0)$ to consist of the vertices at the ground floor, or 'zero etage', in the corresponding triangles and trapezoids; by an analogy with [1] we call this set the *diagonal*. Note that the red paths at the diagonal are of the form $(l, H - l)$, $l = 0, \ldots, H$, and the red paths at the etage k are of the form $(l, H - 2k - l)$, $l = 0, \ldots, H - 2k$. One can see that $\hat{K}(H, 0)$ has no blue edges.



Picture 6.

In its turn, the R-graph of B_2-type $K(0, A)$ is an R-graph which has no red edge outgoing from the minimal vertex, and the green string beginning at it consists of A edges labeled 1. The graph $\hat{K}(0, A)$ has no green edges. Therefore the structure of such a graph is forced by the A_2-type Verma relation presented at Picture 1. This graph has the form of a triangular grid, and we depict two examples with $A = 1$ and $A = 2$ in Picture 7. The distinguished vertex subset in the graph $\hat{K}(0, A)$, *the diagonal*, is defined to be constituted by the unique vertices of the degenerated red paths from the top etage.



Picture 7.

Definition. We define $K(H, A)$ to be the R-graph for which the view from the sky is of the form $\hat{K}(H, A) = \hat{K}(H, 0) \bowtie \hat{K}(0, A)$ as a digraph with labeled edges (the distinguished subsets in the graphs $\hat{K}(H, 0)$ and $\hat{K}(0, A)$ are the diagonals as described above). The labels at the vertices of $\hat{K}(H, A)$ are set by the following rule: if a vertex $\hat{v}$ belongs to a copy of $\hat{K}(H, 0)$, then we set $(\tilde{X}(\hat{v}), \tilde{Y}(\hat{v})) = (X(\hat{v}), Y(\hat{v}))$, where $(X(\hat{v}), Y(\hat{v}))$ is the label on $\hat{v}$ in $\hat{K}(H, 0)$; if $\hat{v}$ belongs to a copy of $K(0, A)$ being attached to the l-th distinguished point of $K(H, 0)$, then $(\tilde{X}(\hat{v}), \tilde{Y}(\hat{v})) = (X(\hat{v}) + l, Y(\hat{v}) + H - l)$, where $(X(\hat{v}), Y(\hat{v}))$ is the label on $\hat{v}$ in $\hat{K}(0, A)$.

Remark. If we regard $\hat{K}(H, A)$ as edge green-blue colored graph, and will take into account the rule of changing of the vertices labels along these labeled edges, then the labels on the vertices are determined if we set the label $(0, H)$ to the source vertex of $\hat{K}(H, A)$.

Proposition 1. Any connected R-graph of B_2-type takes the form $K(H, A)$ with H and A being the lengths of the red and green strings, respectively, emanating from the source vertex.

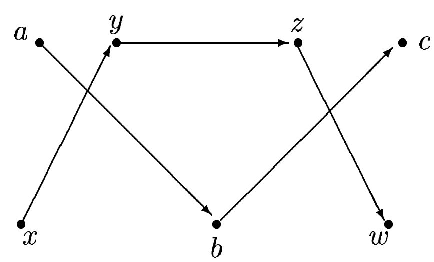
The rest of the paper is devoted to a proof that the graph $K(H, A)$ is the crystal graph of the irreducible representation of $U_q(sp(4))$ with the highest weight $H\lambda_1 + A\lambda_2$. To prove this, we will use a new model related to B_2-type crystals.

4 Crossings model for B_2-type crystals

Here we present a model of the *free regular crystal* K^∞ of B_2-type. That is an edge-2-colored graph with infinite monochromatic strings passing through each vertex, and we will show that each B_2-type crystal for irreducible representation can be obtained as an interval of K^∞.

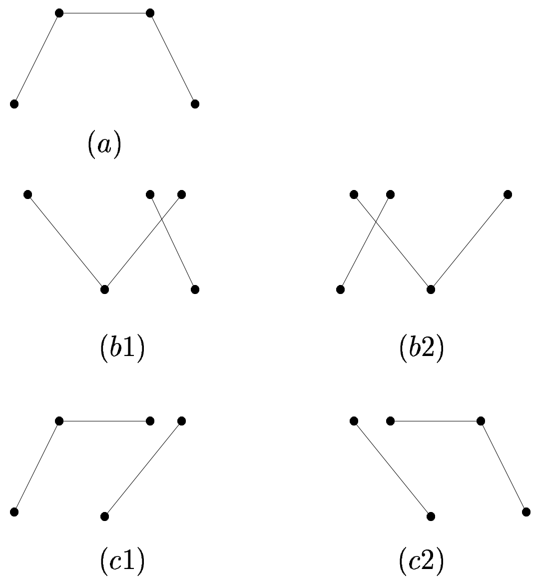
The vertices of K^∞ are the functions on the vertex-set the diagram depicted on Picture 8, for which the inequalities indicated by arrows are imposed, namely: $f(a) \geq f(b) \geq f(c)$ and $f(x) \geq f(y) \geq f(z) \geq f(w)$. Also it is required that the following parity conditions hold: $f(y) + f(z) \in 2\mathbb{Z}$ and $f(a)$, $f(c) \in 2\mathbb{Z}$[2], and that at least *three* of the above inequalities turn into equalities. The possible combinations of equalities are given in Picture 9.

Such functions are called *admissible configurations.*



Picture 8.

[2]One may consider an equivalent model, with odd $f(a)$, $f(c) \in 2\mathbb{Z}+1$ and half-integer weights (related to half-integer cells in the corresponding Young diagram).



Picture 9.

Now we define the structure of a crystal on this set of admissible configurations f by explaining to which admissible configuration f goes under the action of the operations that we denote by F_1 and F_2 as before. That is, the red edges of the crystal take the form $(f, F_1 f)$ and the green edges take the form $(f, F_2 f)$.

The operation F_1 "moves" f to an admissible configuration f' which can differ from f at exactly one of the points x, b, w. Specifically,

if $f(w) < f(z)$, then F_1 increases f by one at the vertex w: $f'(w) = f(w) + 1$;

if $f(w) = f(z)$ and $f(b) < f(a)$, then F_1 increases f by one at b: $f'(b) = f(b) + 1$;

and if $f(w) = f(z)$ and $f(b) = f(a)$, then F_1 increases f by one at x: $f'(x) = f(x) + 1$.

In its turn, $f' = F_2 f$ differs from f at one or two points among a, y, z, c, and F_2 either increases f by 2 at one point, or increases f by 1 at y and at z. More precisely,

if $f(a) - f(b) < f(a) - f(c)$, then $f'(c) = f(c) + 2$;

if $f(a) - f(b) \geq f(b) - f(c)$ and $f(y) + 2 \leq f(x)$, then $f'(y) = y + 2$;

if $f(a) - f(b) \geq f(b) - f(c)$ and $f(y) + 1 = f(x)$, then $f'(y) = f(y) + 1$ and $f'(z) = f(z) + 1$;

if $f(a) - f(b) \geq f(b) - f(c)$, $f(y) = f(x)$, and $f(z) < f(y)$, then $f'(z) = f(z) + 2$;

and if $f(a) - f(b) \geq f(b) - f(c)$, $f(y) = f(x)$, and $f(z) = f(y)$, then $f'(a) = f(a) + 2$.

It is easy to see that these operations preserve the admissibility.

Remark. The definition of an admissible configuration might be easy extended to $\mathbb{R}$-valued functions (of course, the evenness condition disappears) and the definition of the operations $F_i^{\alpha}(f)$, $i = 1, 2$, are defined by a clear modification of the above operations.

Definition. An admissible configuration is called a *fat vertex* if $f(a) = f(b) = f(c) \in 2\mathbb{Z}$ and $f(x) = f(y) = f(z) = f(w)$.

The subset $B(H, A) \subset K^{\infty}$ of configurations which satisfy the restrictions $f(c) \geq 0$, $f(w) \geq 0$, $f(a) \leq A$, $f(x) \leq H$ is called the *interval* of weight $A/2\lambda_2 + H\lambda_1$ ($A \in 2\mathbb{Z}$).

Note that the interval 'join' the fat vertex $\mathbf{0}$ and the fat vertex $f(a) = f(b) = f(c) = A$ and $f(x) = f(y) = f(z) = f(w) = H$, and the whole rectangle of the fat vertices $0 \leq f(a) = f(b) = f(c) \leq A$ and $0 \leq f(x) = f(y) = f(z) = f(w) \leq H$ belong to this interval. Also note that the interval joining a fat vertex $f(a) = f(b) = f(c) = A'$ and $f(x) = f(y) = f(z) = f(w) = H'$ and the fat vertex $f(a) = f(b) = f(c) = A + A'$ and $f(x) = f(y) = f(z) = f(w) = H + H'$, A', $A \in 2\mathbb{Z}$, H, $H' \in \mathbb{Z}$, A, $H \geq 0$, is isomorphic to the interval of weight $A/2\lambda_2 + H\lambda_1$.

We define the operations F_1 and F_2 on the interval $B(H, A)$ as they are defined in K^{∞} with the following modification at the final cases: if in the last case of the definition of F_1, one has $f(x) = H$, then we set $F_1 f := f$ (but do not draw a loop in the graph); and if in the last case of the definition of F_2, one has $f(a) = A$, then we set $F_2(f) = f$. Accordingly, the reverse operation E_2 does not act if it would result in the value on c below zero, and similarly for E_1 and w.

One can check that this model gives the inclusion

$$B(H, A) \subset B(H, 0) \bowtie B(0, A),$$

where the distinguished subsets in $B(H, 0)$ and $B(0, A)$ are constituted by the fat points.

Using Littelmann's path model [6], we prove that the interval $B(H, A) \subset K^{\infty}$ is a crystal graph of the irreducible representation of $sp(4)$ with weight $\frac{A}{2}\lambda_2 + H\lambda_1$. That is there holds

Theorem 2. *For any $A \in 2\mathbb{Z}_+$ and $H \geq 0$, the interval $B(H, A)$ is the crystal graph of a regular representation of the rank 2 algebra of B_2-type and vice versa.*

Remark. We want to stress the following aspect of the crossing model. From this model one can see that a crystal graph of a regular representation of B_2-type (a similar structure holds for other types algebras) is located on a union of 5 polyhedra, and it is not a polyhedron itself. There are two projections of this union of polyhedra to the Littelmann cones, and these cones have to be related via the specific piece-wise linear transformation. Thus, the crossing model captures the global non-convex structure of regular crystals, and the language of the Littelmann cones describes the projections of this non-convex structure.

In view of Theorem 2, Theorem 1 would follow from from the following

Theorem 3. There holds $B(H, A) = K(H, A/2)$.

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