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An Edge-Signed Generalization of Chordal Graphs, Free Multiplicities on Braid Arrangements, and Their Characterizations

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Abstract. In this article, we propose a generalization of the notion of chordal graphs to signed graphs, which is based on the existence of a perfect elimination ordering for a chordal graph. We give a special kind of filtrations of the generalized chordal graphs, and show a characterization of those graphs. Moreover, we also describe a relation between signed graphs and a certain class of multiarrangements of hyperplanes, and show a characterization of free multiarrangements in that class in terms of the generalized chordal graphs, which generalizes a well-known result by Stanley on free hyperplane arrangements. Finally, we give a remark on a relation of our results with a recent conjecture by Athanasiadis on freeness characterization for another class of hyperplane arrangements.

Résumé. Dans cet article, nous proposons une généralisation de la notion des graphes triangulés à graphes signés, qui est basé sur l'existence d'un ordre d'élimination simplicial à un graphe triangulé. Nous donnons un genre spécial de filtrations des graphes triangulés généralisés, et montrons une caractérisation de ces graphes. De plus, nous décrivons aussi une relation entre graphes signés et une certaine classe de multicompositions d'hyperplans, et montrons une caractérisation de multicompositions libres dans cette classe en termes des graphes triangulés généralisés, qui généralise un résultat célèbre de Stanley sur compositions libres d'hyperplans. Finalement, nous donnons une remarque sur une relation de nos résultats avec une conjecture récente d'Athanasiadis sur une caractérisation du freeness d'une autre classe de compositions d'hyperplans.

Keywords: hyperplane arrangement, free arrangement, chordal graph, signed graph, characterization

1 Introduction

Let V^ℓ be an ℓ-dimensional vector space over a field $\mathbb{K}$ of characteristic zero. A *hyperplane arrangement* $\mathcal{A}$ (or simply an *arrangement*) is a finite collection of affine hyperplanes in V^ℓ. In this article any arrangement $\mathcal{A}$ is assumed, unless otherwise specified, to be *central*, *i.e.*, each hyperplane in $\mathcal{A}$ contains the origin. A *multiplicity* on an arrangement $\mathcal{A}$ is a map $m : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, and a pair $(\mathcal{A}, m)$ is called a *multiarrangement*. Then an arrangement is a multiarrangement $(\mathcal{A}, m)$ with m constantly equal to one, that is also called a *simple arrangement*. Theory of (multi)arrangements is an intersecting area of geometry, algebra and combinatorics (see *e.g.*, Orlik-Terao (1992)). For example, some associated combinatorial objects such as *intersecting lattices* and *characteristic polynomials* of an arrangement reflect properties of the arrangement. On the other hand, *braid arrangements*, or more generally *Coxeter arrangements*, are fundamental objects in the theory of arrangements that are closely related to root systems of finite Coxeter groups (see *e.g.*, Saito (1975)).

One of the aims of this article is to give insight into *freeness property* of multiarrangements, that is one of the most active topics in theory of (multi)arrangements, in a combinatorial viewpoint specified below. To define the freeness property, we need some definitions and notations. Let $\{x_1, \dots, x_\ell\}$ denote a basis for the dual vector space $V^* = (V^\ell)^*$ of V^ℓ, and let $S = \mathrm{Sym}(V^*) \simeq \mathbb{K}[x_1, \dots, x_\ell]$ be the symmetric algebra on V^*. Let $\mathrm{Der}_{\mathbb{K}}(S) = \bigoplus_{i=1}^{\ell} S \cdot \partial_{x_i}$ denote the S-module of $\mathbb{K}$-linear derivations of S. For each hyperplane $H \in \mathcal{A}$ in a given multiarrangement $(\mathcal{A}, m)$, fix a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. Then $(\mathcal{A}, m)$ is called *free* if the *logarithmic derivation module* $D(\mathcal{A}, m)$ of $(\mathcal{A}, m)$ defined by

$$D(\mathcal{A}, m) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot {\alpha_H}^{m(H)} \text{ for all } H \in \mathcal{A}\} \tag{1}$$

is a free S-module (of rank ℓ). Moreover, since for any free $(\mathcal{A}, m)$ the S-module $D(\mathcal{A}, m)$ has a free basis $\{\theta_1, \dots, \theta_\ell\}$ such that each θ_i is *homogeneous of degree* $\deg(\theta_i)$ (*i.e.*, $\theta_i = \sum_{j=1}^{\ell} f_{i,j} \partial_{x_j}$ with every $f_{i,j}$ either zero or homogeneous of degree $\deg(\theta_i)$), and in that case the multiset of the degrees $\deg(\theta_i)$ is independent of the choice of the basis, we define the *exponents* of a free multiarrangement $(\mathcal{A}, m)$ by

$$\exp(\mathcal{A}, m) = \{\deg(\theta_1), \dots, \deg(\theta_\ell)\} \quad \text{(as a multiset)}. \tag{2}$$

When an arrangement $\mathcal{A}$ is fixed, we say that a multiplicity m on $\mathcal{A}$ is *free* if $(\mathcal{A}, m)$ is a free multiarrangement. It is very difficult to determine free (multi)arrangements in general, and derivation modules and free (multi)arrangements in some special cases have been well studied in preceding works. For instance, see Abe-Terao-Wakefield (2008, 2007); Saito (1975, 1980); Solomon-Terao (1998); Terao (2002).

In this article, we deal with multiplicities on a *braid arrangement* $\mathcal{A}_\ell$ (equivalently, the *Coxeter arrangement of type* A_ℓ) that is an arrangement in $V^{\ell+1}$ defined by

$$\mathcal{A}_\ell = \{H_{ij} = \{x_i - x_j = 0\} \mid i, j \in \{1, 2, \dots, \ell+1\}, i \neq j\} \ . \tag{3}$$

More generally, a *Coxeter arrangement* is the arrangement of all reflecting hyperplanes of a finite Coxeter group. There have been several preceding results on free multiplicities on braid arrangements and Coxeter arrangements (see *e.g.*, Abe (2007, preprint 2008); Abe-Yoshinaga (preprint 2007); Saito (1975, 1980); Solomon-Terao (1998); Terao (2002); Yoshinaga (preprint 2007)). In particular, a characterization of free multiplicities m on Coxeter arrangements of the form $m(H) = c + \delta_H$, with c a constant and $\delta_H \in \{0, 1\}$ for every $H \in \mathcal{A}$, has been obtained by combining results of Abe-Yoshinaga (preprint 2007); Solomon-Terao (1998); Terao (2002); Yoshinaga (2002). In this article, we consider any multiplicity m on braid arrangements of another form $m(H) = 2k + \delta_H$, with $k \in \mathbb{Z}_{>0}$ a constant and $\delta_H \in \{-1, 0, 1\}$ for every $H \in \mathcal{A}$. Motivations of focusing on such situations are explained in the next paragraph. We parameterize such a multiplicity m by using a signed graph with $\ell+1$ vertices in the following manner (see *e.g.*, Diestel (2006) for graph-theoretic terminology): For a signed graph $G = (V, E)$ with $V = \{v_1, \dots, v_{\ell+1}\}$ and

$E = E_+ \cup E_-$ (disjoint union), define an auxiliary map $m_G : \mathcal{A}_\ell \to \{-1, 0, 1\}$ by

$$m_G(H_{ij}) = \begin{cases} 1 & \text{if } v_i v_j \in E_+ \ ; \\ -1 & \text{if } v_i v_j \in E_- \ ; \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $v_i v_j$ denotes an unordered pair of v_i and v_j, and put $m = 2k + m_G$. (In this article, every graph is finite, simple, and undirected, unless otherwise specified.) One of the main theorems in this article gives a characterization of free multiplicities on $\mathcal{A}_\ell$ of the above type in terms of a certain property of the corresponding signed graph that will be described below. More precisely, our result shows that, under a certain technical and not essential condition, a multiplicity $m = 2k + m_G$ is free if and only if G is *signed-eliminable* in the sense specified in Section 2. See Theorem 2 for the precise statement of the result. The notion of signed-eliminable graphs is a generalization of chordal graphs to signed graphs, and the above result is also a signed-graphic generalization of Stanley's well-known result (Stanley (1972)) on a characterization of free graphic arrangements in terms of chordal graphs. Moreover, Theorem 2 also shows that, if a multiplicity $m = 2k + m_G$ is free, then its exponents are determined by certain quantities associated to the signed graph G. A main ingredient of the proof of the result is a characterization of signed-eliminable graphs in terms of excluded subgraphs, that is another main contribution of this article. See Theorem 1 for the precise statement.

The main motivations of studying multiplicities of the above form $m(H) = 2k + \delta_H$, $\delta_H \in \{-1, 0, 1\}$, on a braid arrangement are as follows. First, it is known that a kind of duality exists between multiplicities $2k + \delta_H$ and $2k - \delta_H$, with all $\delta_H \in \{0, 1\}$, not only for braid arrangements but also for Coxeter arrangements of other types (Abe (preprint 2008)). The authors had guessed that such a duality would extend to more general cases $\delta_H \in \{-1, 0, 1\}$; the work in this article on the case of braid arrangements is the first step to a study of the case of general Coxeter arrangements. Secondly, from the viewpoint of Stanley's freeness characterization based on (non-signed) chordal graphs, it is reasonable to expect that extending non-signed graphs to signed graphs gives a natural generalization of Stanley's theory, and the corresponding multiplicities are actually of the above type. Moreover, it will be mentioned in Section 5 that our study on the above multiplicities is closely related to a conjecture by Athanasiadis (Athanasiadis (2000)) on freeness characterization for another class of arrangements.

This article is organized as follows. In Section 2, we give a definition of signed-eliminable graphs and introduce some related objects. We also show two inductive properties of signed-eliminable graphs that play key roles of the proof of our main theorems. Moreover, we present some quantities associated to signed-eliminable graphs that are main ingredients of the description of exponents of free multiarrangements of the above type. In Section 3, we state one of the two main theorems of this article that characterizes signed-eliminable graphs in terms of excluded subgraphs (Theorem 1). We also show an outline of the proof and some key lemmas that would be of independent interest. Moreover, we present some simpler characterizations of signed-eliminable graphs in certain subclasses as easy consequences of our result. In Section 4, we state an aforementioned result on free multiplicities on braid arrangements, that is another main theorem of this article (Theorem 2), and show an outline of the proof. As an application, we also describe characteristic polynomials of free multiarrangements of the above type (Corollary 5). Finally, in Section 5, we give a remark on a relation of our result with a conjecture by Athanasiadis (Athanasiadis (2000)) on freeness characterization for another class of arrangements. More precisely, our result is applied to prove one direction of Athanasiadis's Conjecture (the sufficiency of Athanasiadis's conditions for the freeness) in a more general setting than that in the statement of the conjecture.

2 Signed-Eliminable Graphs

First, we give a definition of signed-eliminable graphs:

Definition 1 *For a signed graph $G = (V, E)$ and a bijection $\nu : V \to \{1, 2, \ldots, |V|\}$ (in this article, such a map ν is referred to as an* ordering *on G), we say that ν is a* signed-elimination ordering *(or an* SEO *in short) if for any triple (u, v, w) of vertices of G such that $\nu(u) < \nu(w) > \nu(v)$, and for each $\sigma \in \{+, -\}$, the following two conditions are satisfied:*

(E1) *If $uw \in E_\sigma$ and $vw \in E_\sigma$, then $uv \in E_\sigma$.*

(E2) *If $uv \in E_\sigma$ and $vw \in E_{-\sigma}$, then $uw \in E_\sigma$.*

We say that G is signed-eliminable *(or* SE *in short) if an SEO on G exists.*

See Figure 1 for the conditions (E1) and (E2), where single and duplicated edges represent edges with different signs. Note that, owing to a well-known characterization of chordal graphs in terms of vertex elimination orderings (see *e.g.*, Fulkerson-Gross (1965)), both subgraphs $G_+ = (V, E_+)$ and $G_- = (V, E_-)$ of an SE graph are chordal. In particular, SE graphs with either $E_+ = \emptyset$ or $E_- = \emptyset$ are nothing but chordal graphs. Two examples of SE graphs are given in Figure 2. An SEO for the graph in the left-hand side is given by $w \mapsto 1$ and $v_i \mapsto i + 1$. On the other hand, for the graph in the right-hand side, an SEO is given by $w_i \mapsto i$ and $v_i \mapsto i + 2$.

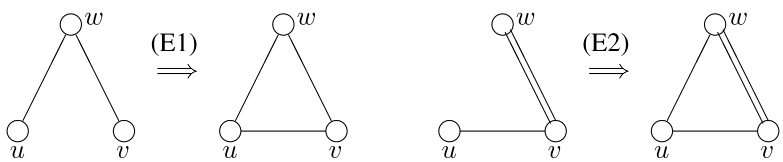

Fig. 1: Condition for signed-eliminable graphs

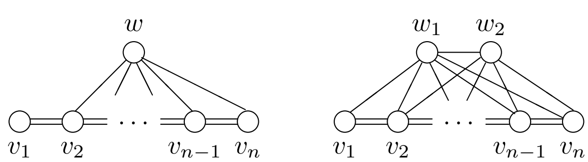

Fig. 2: Examples of SE graphs

Remark 1 *The SEOs are also characterized in the following manner: We assign weights $\omega(uv)$ to pairs of vertices u, v of a signed graph G by the rule $\omega(uv) = \pm 1$ and 0 if $uv \in E_\pm$ and $uv \notin E$, respectively. Then an ordering ν on G is an SEO if and only if, for any triple (u, v, w) with $\nu(u) < \nu(w) > \nu(v)$ and either $uw \in E$ or $vw \in E$, if $a \le b \le c$ are three weights $\omega(uv)$, $\omega(vw)$, and $\omega(uw)$ in nondecreasing order, then we have $b = \omega(uv)$. This property plays a key role in our characterization of free multiplicities of the above type.*

We summarize some fundamental properties of SE graphs. For any SEO ν on an SE graph G, the restriction of ν to an induced subgraph of G gives an SEO on that subgraph. Hence the class of SE graphs is closed under taking induced subgraphs. On the other hand, a signed graph is SE if and only if every connected component of the graph is SE. Moreover, SE graphs have the following inductive properties. To explain the properties, we introduce the following terminology:

Definition 2 *Let G be a signed graph. Then a vertex v of G is called* signed-simplicial *if the following two conditions are satisfied:*

(S1) *For each $\sigma \in \{+,-\}$, $N_{G_\sigma}[v] = \{u \in V \mid uv \in E_\sigma\} \cup \{v\}$ is a clique in G_σ (i.e., v is a simplicial vertex of the graph G_σ).*

(S2) *For each $\sigma \in \{+,-\}$, $uw \in E_{-\sigma}$ and $wv \in E_\sigma$ imply $uv \in E_{-\sigma}$.*

Moreover, let $S(G)$ denote the set of the signed-simplicial vertices of G.

Then we have the following result:

Lemma 1 *If G is an SE graph with an SEO ν, then the vertex v of G with $\nu(v) = |V|$ is signed-simplicial in G (and by the aforementioned property, the restriction of ν to the induced subgraph $G \setminus \{v\}$ of G with vertex set $V \setminus \{v\}$ is an SEO on the subgraph). Conversely, if G is a signed graph, $v \in S(G)$ and ν is an SEO on $G \setminus \{v\}$, then the unique extension $\overline{\nu}$ of ν to G with $\overline{\nu}(v) = |V|$ is also an SEO on G.*

This inductive property plays a central role in the proof of our characterization of SE graphs. Moreover, Lemma 1 implies that an SEO of any SE graph is found by a greedy algorithm, namely:

Lemma 2 *We consider the following inductive algorithm for a signed graph G: If $S(G)$ is empty then halt, otherwise let v_n be a vertex in $S(G)$, where $n = |V|$, and proceed the algorithm for $G \setminus \{v_n\}$. Then G is an SE graph if and only if the algorithm does not halt until the graph becomes empty, and the ordering ν on G with $\nu(v_i) = i$ determined in this way is an SEO if G is an SE graph.*

In contrast to the above inductive property with respect to vertices, the next property of SE graphs is inductive with respect to edges. We introduce the following notion:

Definition 3 *Let G be an SE graph with an SEO ν. For each $0 \le k \le |V|$, let $G^{(k)}$ denote the subgraph of G with the same vertex set V and an edge set consisting of all edges uv of G such that $\nu(u) \le k$ and $\nu(v) \le k$. Then we say that a sequence $G'_0 = G^{(k-1)}, G'_1, \dots, G'_r = G^{(k)}$ of subgraphs of G is a k*-th signed-eliminable filtration *(or a k*-th SE filtration *in short) of G if each G'_i $(1 \le i \le r)$ is obtained from G'_{i-1} by adding one edge and ν is also an SEO on G'_i. Moreover, we refer to a concatenation of k-th SE filtrations for all $1 \le k \le |V|$ as a* complete signed-eliminable filtration *(or a* complete SE filtration *in short) of G.*

Then we have the following property, that plays a significant role in our characterization of free multiplicities of the above type:

Proposition 1 *Any SE graph has a complete SE filtration.*

We give an outline of construction of complete SE filtrations. For an SE graph G with an SEO ν, let $v \in V$ with $\nu(v) = |V|$. Then we define a binary relation $\prec$ on $N_G(v) = N_G[v] \setminus \{v\}$ by $u \prec w$ if and only if $uw \in E_\sigma$, $uv \in E_\sigma$ and $wv \in E_{-\sigma}$ for some $\sigma \in \{+,-\}$. It is shown that the transitive closure $\prec'$ of $\prec$ is a partial order on $N_G(v)$, and that for a maximal element u of $N_G(v)$ with respect to $\prec'$, the subgraph of G obtained by deleting the edge uv is also an SE graph with the same SEO ν. Thus repetition of this process gives a desired filtration of G.

For example, for the SE graph G in the left-hand side of Figure 2 with the SEO ν specified above, the unique $(n+1)$-th SE filtration of G is given by first adding the edge wv_n and then adding the edge $v_{n-1}v_n$, therefore a complete SE filtration is also inductively obtained. On the other hand, for the other graph G in Figure 2 with the above SEO ν, the unique $(n+2)$-th SE filtration of G is given by adding three edges w_1v_n, w_2v_n, and $v_{n-1}v_n$ in this order, therefore a complete SE filtration is also inductively obtained.

In the last of this section, we introduce the following quantities associated to any SE graph that will be used to describe the exponents of free multiplicities of the above type. Let G be an SE with an SEO ν. Then we define $d_\sigma^{(\nu)}(i) \in \mathbb{Z}_{\geq 0}$ for each $1 \leq i \leq |V|$ and $\sigma \in \{+,-\}$ by

$$d_\sigma^{(\nu)}(i) = |\{u \in V \mid \nu(u) \leq i \text{ and } u_i u \in E_\sigma\}| \ , \tag{5}$$

where $u_i \in V$ such that $\nu(u_i) = i$. Moreover, for each i, put

$$\widetilde{\deg}_i = \widetilde{\deg}_i(G) = d_+^{(\nu)}(i) - d_-^{(\nu)}(i) \ . \tag{6}$$

For example, for an SE graph G in Figure 3 and an SEO ν in the left-hand side of Figure 3, we have $(d_+^{(\nu)}(i), d_-^{(\nu)}(i)) = (0,0)$, $(1,0)$, $(0,0)$, and $(1,1)$ for each $i = 1,\dots,4$, respectively. On the other hand, for an SEO μ in the right-hand side of Figure 3, we have $(d_+^{(\mu)}(i), d_-^{(\mu)}(i)) = (0,0)$, $(0,0)$, $(1,1)$, and $(1,0)$ for each $i = 1,\dots,4$, respectively. Now we see that the multisets of the pairs $(d_+^{(\nu)}(i), d_-^{(\nu)}(i))$ in the first case and of the pairs $(d_+^{(\mu)}(i), d_-^{(\mu)}(i))$ in the second case coincide with each other. This phenomenon is not just an accident, namely we have the following property (that coincides with Theorem 4 of Rose (1970) in the special case of non-signed graphs):

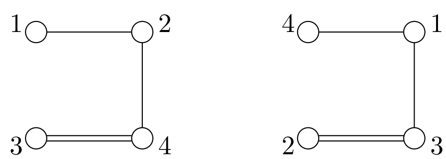

Fig. 3: Two SEOs on the same SE graph

Proposition 2 *For any SE graph G, the multiset of the pairs $(d_+^{(\nu)}(i), d_-^{(\nu)}(i))$, $1 \leq i \leq |V|$, does not depend on the choice of an SEO ν on G. In particular, the multiset $\widetilde{\deg}(G)$ of the values $\widetilde{\deg}_i(G)$ is also independent on the choice of ν.*

Note that we always have $\widetilde{\deg}_1 = 0$ in the above setting.

3 Characterization of Signed-Eliminable Graphs

In this section, we give a characterization of SE graphs. To state the characterization, we need some more definitions:

Definition 4 *We say that a sequence $(v_1, v_2, \ldots, v_n; w)$ of vertices with $n \geq 3$ is a σ-*mountain*, where $\sigma \in \{+, -\}$ (or simply a* mountain*), if $v_i v_{i+1} \in E_{-\sigma}$ for $1 \leq i \leq n-1$, $wv_i \in E_\sigma$ for $2 \leq i \leq n-1$, and any other pair of vertices is not joined by an edge (see the left-hand side of Figure 4).*

Definition 5 *We say that a sequence $(v_1, v_2, \ldots, v_n; w_1, w_2)$ of vertices with $n \geq 2$ is a σ-*hill*, where $\sigma \in \{+, -\}$ (or simply a* hill*), if $v_i v_{i+1} \in E_{-\sigma}$ for $1 \leq i \leq n-1$, $w_1 w_2 \in E_\sigma$, $w_1 v_i \in E_\sigma$ for $1 \leq i \leq n-1$, $w_2 v_i \in E_\sigma$ for $2 \leq i \leq n$, and any other pair of vertices is not joined by an edge (see the right-hand side of Figure 4).*

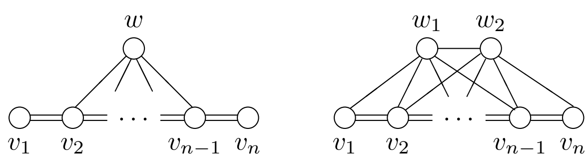

Fig. 4: Examples of non-SE graphs

A direct verification shows that neither a mountain nor a hill is an SE graph.

Definition 6 *We refer to an induced path $uvwx$ in a signed graph G with $uv \in E_\sigma$, $vw \in E_{-\sigma}$, and $wx \in E_\sigma$, where $\sigma \in \{+, -\}$, as an* alternating 4-path *in G.*

Then the characterization is given by the following theorem:

Theorem 1 *Let G be a signed graph. Then G is an SE graph if and only if all of the following three conditions are satisfied:*

(C1) *Both G_+ and G_- are chordal (i.e., having no induced cycle of length ≥ 4).*

(C2) *For any alternating 4-path $uvwx$ in G with $uv \in E_\sigma$, we have either $uw \in E_\sigma$ and $ux \in E_\sigma$, or $ux \in E_\sigma$ and $vx \in E_\sigma$.*

(C3) *G contains no mountain and no hill as an induced subgraph.*

An easy argument shows that the "only if" part of Theorem 1 holds, thus the nontrivial part of the theorem is to show that G is an SE graph if the conditions (C1)–(C3) are satisfied. Moreover, since the conditions (C1)–(C3) are closed under taking an induced subgraph, the proof can be proceeded by induction on $|V|$. Note that the case $|V| \leq 3$ is trivial, since every signed graph with at most three vertices is an SE graph. Thus, by the properties mentioned in Section 2, it suffices to show that $S(G) \neq \emptyset$ if G is connected, $E_+ \neq \emptyset$, $E_- \neq \emptyset$, and $S(G \setminus \{v\}) \neq \emptyset$ for every vertex v of G.

We explain an observation for the proof. For any graph G and a subset V' of the vertex set V of G, let $G|_{V'}$ denote the induced subgraph of G with vertex set V'. For a signed graph G, two subsets V', V'' of V with $V' \subset V''$, and $\sigma \in \{+, -\}$, let $W = \mathrm{cl}_\sigma(V'; V'')$ be the union of vertex sets of the connected components of $G_\sigma|_{V''}$ that have nonempty intersection with V', and define

$$\overline{W} = \overline{\mathrm{cl}}_\sigma(V'; V'') = \{v \in V'' \mid N_{G_{-\sigma}}[v] \cap \mathrm{cl}_\sigma(V'; V'') \neq \emptyset\} \tag{7}$$

(note that $W \subset \overline{W}$). Then it is shown that, if the condition (C2) is satisfied and every connected component of $G_\sigma|_W$ contains at least two vertices, then any simplicial vertex of the "closure" $G|_{\overline{W}}$ of V'

relative to V'' is contained in the "interior set" W of $G|_{\overline{W}}$ and is also simplicial in $G|_{V''}$. Owing to this fact, by choosing an appropriate vertex v of G, we can restrict possibilities of the simplicial vertices of the subgraph $G \setminus \{v\}$ (that exist by the induction hypothesis). This is a main tool of our proof, and a somewhat lengthy graph-theoretic argument enables us to find a desired simplicial vertex of the graph G. For the details, see a forthcoming full version of the article, or its preliminary version (Nuida (preprint 2007)).

We also present two lemmas for the proof that would be of independent interest:

Lemma 3 *Let G be a chordal graph and $V' \subset V$ a clique of G with $V' \neq V$. Then there is a vertex $v \in V \setminus V'$ such that $N_G[v]$ is a clique of G.*

Lemma 4 *Let G be a connected SE graph such that $E_+ \neq \emptyset$ and $E_- \neq \emptyset$. Then G has a signed-simplicial vertex v such that $N_{G_+}[v] \neq \{v\}$ and $N_{G_-}[v] \neq \{v\}$.*

In the last of this section, we state some special cases of our characterization:

Corollary 1 *A signed graph G with four vertices is SE if and only if one of the following conditions is satisfied:*

(FV1) *Either G_+ or G_- has a vertex of degree three.*

(FV2) *Both G_+ and G_- are chordal, G is not a mountain, and G has no alternating 4-path.*

Corollary 2 *Let G be a signed graph that is chordal (as a non-signed graph). Then G is SE if and only if both conditions (C2) and (C3) are satisfied.*

Corollary 3 *Let G be a signed graph with independence number $\alpha(G) \leq 2$ (i.e., every induced subgraph of G with three vertices has an edge). Then G is SE if and only if the condition (C2) and the following two conditions are satisfied:*

(I1) *Both G_+ and G_- have no cycle of length four or five that is an induced cycle in G.*

(I2) *G contains no hills with five or six vertices as an induced subgraph.*

Corollary 4 *Let G be a signed graph that is a complete graph (as a non-signed graph). Then G is SE if and only if for each $\sigma \in \{+,-\}$, G_σ contains, as an induced subgraph, neither a simple path with four vertices, nor a pair of two disjoint edges that are not joined by an edge in G_σ.*

4 Freeness Characterization of Multiplicities $2k + m_G$

Now we come back to the study of free multiarrangements mentioned in the Introduction. The full statement of our characterization of those free multiarrangements is the following:

Theorem 2 *Let $\mathcal{A} = \mathcal{A}_\ell$ denote the braid arrangement in $V^{\ell+1}$ as in the Introduction, let G be a signed graph with vertex set $V = \{v_1, \dots, v_{\ell+1}\}$, and let m_G be the map defined in (4). Let $k, n_1, \dots, n_{\ell+1}$ be nonnegative integers. Let m be a multiplicity on $\mathcal{A}$ defined by $m(H_{ij}) = 2k + n_i + n_j + m_G(H_{ij})$ for each $H_{ij} \in \mathcal{A}$, and put $N = (\ell+1)k + \sum_{i=1}^{\ell+1} n_i$. Assume that one of the following three conditions is satisfied:*

(a) $k > 0$.

(b) *$E_- = \emptyset$.*

(c) *$E_+ = \emptyset$ and $m(H_{ij}) > 0$ for all $H_{ij} \in \mathcal{A}$.*

Then $(\mathcal{A}, m)$ is free if and only if G is an SE graph. Moreover, if it is free, then the exponents of $(\mathcal{A}, m)$ are determined by

$$\exp(\mathcal{A}, m) = (0, N + \widetilde{\deg}_2, \ldots, N + \widetilde{\deg}_{\ell+1}) \tag{8}$$

where $\widetilde{\deg}_i$ is the quantity associated to G defined in (6).

Note that, in the case that $E_- = \emptyset$ and $k = n_1 = \cdots = n_{\ell+1} = 0$, the multiarrangement in Theorem 2 coincides with a graphic arrangement mentioned in the Introduction. Thus Theorem 2 is a generalization of Stanley's aforementioned characterization of free graphic arrangement (see Stanley (1972)).

We explain an outline of the proof of Theorem 2. First, for the "if" part, suppose that G is an SE graph with an SEO ν. By an appropriate permutation of coordinates, we assume without loss of generality that $\nu(v_i) = i$ for every i. We proceed the proof by induction on ℓ, and the case $\ell \leq 2$ follows from the result of Wakamiko (2007). For the case $\ell > 2$, Proposition 1 implies that G has a complete SE filtration $G'_0, G'_1, \ldots, G'_r = G$ corresponding to the SEO ν. We show by induction on i that the multiarrangement $(\mathcal{A}^{(i)}, m^{(i)})$ corresponding to each SE graph G'_i is free. For the step from G'_{i-1} to G'_i, let $v_j v_k$ denote the edge added to G'_{i-1} in this step, where $j < k$. Let $\mathcal{A}' = \{H \cap H_{jk} \mid H \in \mathcal{A}^{(i)} \setminus \{H_{jk}\}\}$, which is an arrangement in an ℓ-dimensional space, and let $(\mathcal{A}', m')$ be a certain special multiarrangement (called the *Euler restriction* of $(\mathcal{A}^{(i)}, m^{(i)})$ to H_{jk}) obtained by a result of Abe-Terao-Wakefield (2008). Then it follows from results of Abe-Terao-Wakefield (2008) that $(\mathcal{A}', m')$ is a multiarrangement of the form in Theorem 2 corresponding to an induced subgraph of G and the restriction of ν to this subgraph, therefore the first induction hypothesis implies that $(\mathcal{A}', m')$ is free. Now owing to Addition-Deletion Theorem (Theorem 0.8 of Abe-Terao-Wakefield (2008)), freeness of $(\mathcal{A}^{(i-1)}, m^{(i-1)})$ implies freeness of $(\mathcal{A}^{(i)}, m^{(i)})$, therefore the claim follows from the second induction hypothesis. Thus the "if" part is proved. Moreover, the description of the exponents is also obtained by the same argument in parallel. Note that existence of complete SE filtrations plays a key role in this proof, but the characterization of SE graphs has not yet appeared.

On the other hand, for the "only if" part, we show that $(\mathcal{A}, m)$ is not free if G is not an SE graph. Then by the characterization of SE graphs (Theorem 1), one of the three conditions (C1)–(C3) is not satisfied, namely we are in one of the following situations:

- Either G_+ or G_- has an induced cycle of length ≥ 4.
- G contains an alternating 4-path $uvwx$ with $uv \in E_\sigma$ such that either $ux \notin E_\sigma$, or $uw \notin E_\sigma$ and $vx \notin E_\sigma$.
- G contains a mountain or a hill as an induced subgraph.

Owing to Lemma 3.8 in Abe (2006), it suffices to show that the multiarrangement corresponding to the subgraph of G specified in the above conditions is not free. This is done by a case-by-case argument based on Addition-Deletion Theorem and other preceding results of Abe-Terao-Wakefield (2008), Abe-Terao-Wakefield (2007), and Wakamiko (2007). Thus the "only if" part is proved. For the details, see the full version of this article (Abe-Nuida-Numata (2009)).

We would like to summarize here again that the complete SE filtration plays a key role in the proof of the “if” part of Theorem 2, while the proof of the “only if” part requires our characterization of SE graphs given in Theorem 1.

In the last of this section, we give a remark on characteristic polynomials of the above multiarrangements that is an easy consequence of Theorem 2. Characteristic polynomials $\chi(\mathcal{A}, m, t)$ of multiarrangements $(\mathcal{A}, m)$ are defined by Abe-Terao-Wakefield (2007), and in that article a factorization theorem of $\chi(\mathcal{A}, m, t)$ is proved. It is difficult to compute the polynomial $\chi(\mathcal{A}, m, t)$ in general. However, if $(\mathcal{A}, m)$ is a free multiarrangement, then the computation becomes easy owing to the factorization theorem. Thus by Theorem 2, we obtain the following result on characteristic polynomials of the above multiarrangements:

Corollary 5 *Let $(\mathcal{A}, m)$ be the same multiarrangement as in Theorem 2 corresponding to a signed graph G. Let $\widetilde{m}$ be another multiplicity on $\mathcal{A}$ such that $\widetilde{m}(H_{ij}) = 2k + n_i + n_j - m_G(H_{ij})$ for each $H_{ij} \in \mathcal{A}$. Then, in the case $k > 0$, $(\mathcal{A}, m)$ is free if and only if $(\mathcal{A}, \widetilde{m})$ is free. Moreover, if G is an SE graph, then we have*

$$\chi(\mathcal{A}, m, t) = t \prod_{i=2}^{\ell+1} (t - N - \widetilde{\deg}_i) \tag{9}$$

and

$$\chi(\mathcal{A}, \widetilde{m}, t) = t \prod_{i=2}^{\ell+1} (t - N + \widetilde{\deg}_i) \ . \tag{10}$$

5 Conjecture of Athanasiadis

In this section, we explain a relation of our result with a conjecture of Athanasiadis (see Athanasiadis (2000)) on graphic characterization of free arrangements in another class. Here we consider a non-central affine arrangement in $V^{\ell+1}$ that consists of affine hyperplanes defined by $x_i - x_j = h$ $(1 \le i < j \le \ell+1)$, where $h \in \mathbb{Z}$, $-k - \varepsilon(i,j) \le h \le k + \varepsilon(j,i)$, $k \in \mathbb{Z}_{\ge 0}$ is a constant, and $\varepsilon(i,j) \in \{0,1\}$. Such an arrangement is called a *deformation of the Coxeter arrangement* and was first systematically investigated by Stanley (1996). These arrangements have been extensively studied by several persons such as Athanasiadis (1996, 1998, 2000), Edelman-Reiner (1996), Postnikov-Stanley (2000), and Yoshinaga (2004). In particular, Athanasiadis (Athanasiadis (1996)) introduced a description of the above arrangement in terms of a *directed* graph $G = (V, E)$ with vertex set $V = \{v_1, \ldots, v_{\ell+1}\}$. For such a graph G, define $\varepsilon(i,j) = 1$ if $(v_i, v_j) \in E$ and $\varepsilon(i,j) = 0$ if $(v_i, v_j) \notin E$, where (v_i, v_j) denotes an arrow from v_i to v_j. Note that every affine arrangement of the above form is parameterized in this manner. Let $\mathcal{A}_G$ denote the arrangement corresponding to G. Then Athanasiadis (Athanasiadis (1996)) gives a splitting formula of the characteristic polynomial of $\mathcal{A}_G$ in the case that G satisfies the following two conditions:

(A1) For every triple v_h, v_i, v_j of vertices of G with $i < h$ and $j < h$, $(v_i, v_j) \in E$ implies either $(v_i, v_h) \in E$ or $(v_h, v_j) \in E$.

(A2) For every triple v_h, v_i, v_j of vertices of G with $i < h$ and $j < h$, we have $(v_i, v_j) \in E$ if $(v_i, v_h) \in E$ and $(v_h, v_j) \in E$.

Moreover, he also gave a conjecture (Conjecture 6.6 in Athanasiadis (2000)) that in the case $k = 0$, the conditions (A1) and (A2) would be necessary and sufficient for the “coning” $c\mathcal{A}_G$ of $\mathcal{A}_G$ to be a free arrangement. We mention that one direction of Athanasiadis's Conjecture is proved in a more general setting by applying our results in the previous sections. Namely, we have the following result:

Theorem 3 *In the above setting, where we do not assume $k = 0$, the coning $c\mathcal{A}_G$ of $\mathcal{A}_G$ is free if G satisfies the conditions (A1) and (A2).*

We give an outline of the proof of Theorem 3. Let H_∞ be the infinity hyperplane of the coning $c\mathcal{A}_G$ of $\mathcal{A}_G$. Let $(\mathcal{A}'', m_{H_\infty})$ denote the *Ziegler restriction* of $c\mathcal{A}_G$ (see Ziegler (1989)) defined by $\mathcal{A}'' = \{H \cap H_\infty \mid H_\infty \neq H \in c\mathcal{A}_G\}$ and $m_{H_\infty}(X) = |\{H \in c\mathcal{A}_G \setminus \{H_\infty\} \mid H \cap H_\infty = X\}|$ for each $X \in \mathcal{A}''$. Then it is shown that $(\mathcal{A}'', m_{H_\infty})$ is of the form in Theorem 2 corresponding to a signed graph G', and the conditions (A1) and (A2) for G imply that G' is an SE graph. Thus $(\mathcal{A}'', m_{H_\infty})$ is free by Theorem 2. Now owing to Theorem 2.2 of Yoshinaga (2004), an argument based on induction on ℓ implies that $c\mathcal{A}_G$ is free. Thus Theorem 3 is proved. For the details, see the full version of this article (Abe-Nuida-Numata (2009)).

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Unital versions of the higher order peak algebras

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Abstract. We construct unital extensions of the higher order peak algebras defined by Krob and the third author in [Ann. Comb. 9 (2005), 411–430], and show that they can be obtained as homomorphic images of certain subalgebras of the Mantaci-Reutenauer algebras of type B. This generalizes a result of Bergeron, Nyman and the first author [Trans. AMS 356 (2004), 2781–2824].

Résumé. Nous construisons des extensions unitaires des algèbres de pics d'ordre supérieur définies par Krob et le troisième auteur dans [Ann. Comb. 9 (2005), 411–430], et nous montrons qu'elles peuvent être obtenues comme images homomorphes de certaines sous-algèbres des algèbres de Mantaci-Reutenauer de type B. Ceci généralise un résultat dû à Bergeron, Nyman et au premier auteur [Trans. AMS 356 (2004), 2781–2824].

Keywords: Descent algebras, Noncommutative symmetric functions, Peak algebras

1 Introduction

A *descent* of a permutation $\sigma \in \mathfrak{S}_n$ is an index i such that $\sigma(i) > \sigma(i+1)$. A descent is a *peak* if moreover $i > 1$ and $\sigma(i) > \sigma(i-1)$. The sums of permutations with a given descent set span a subalgebra of the group algebra, the *descent algebra* Σ_n. The *peak algebra* $\mathring{\mathcal{P}}_n$ of $\mathfrak{S}_n$ is a subalgebra of its descent algebra, spanned by sums of permutations having the same peak set. This algebra has no unit. Descent algebras can be defined for all finite Coxeter groups [19]. In [2], it is shown that the peak algebra of $\mathfrak{S}_n$ can be naturally extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of the hyperoctahedral group B_n.

The direct sum of the peak algebras turns out to be a Hopf subalgebra of the direct sum of all descent algebras, which can itself be identified with $\mathbf{Sym}$, the Hopf algebra of noncommutative symmetric functions [9]. As explained in [5], it turns out that a fair amount of results on the peak algebras can be deduced from the case $q = -1$ of a q-identity of [11]. Specializing q to other roots of unity, Krob and the third author introduced and studied *higher order peak algebras* in [12]. Again, these are non-unital, and it is natural to ask whether the construction of [2] can be extended to this case.

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We will show that this is indeed possible. We first construct the unital versions of the higher order peak algebras by a simple manipulation of generating series. We then show that they can be obtained as homomorphic images of the *Mantaci-Reutenauer algebras* of type B. Hence no Coxeter groups other than B_n and $\mathfrak{S}_n$ are involved in the process; in fact, the construction is related to the notion of superization, as defined in [16], rather than to root systems or wreath products.

2 Notations and background

2.1 Noncommutative symmetric functions

We will assume familiarity with the notations of [9] and with the main results of [12]. We recall a few definitions for the convenience of the reader.

The Hopf algebra of noncommutative symmetric functions is denoted by $\mathbf{Sym}$, or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet A. Linear bases of $\mathbf{Sym}_n$ are labelled by compositions $I = (i_1, \ldots, i_r)$ of n (we write $I \vDash n$). The noncommutative complete and elementary functions are denoted by S_n and Λ_n, and $S^I = S_{i_1} \cdots S_{i_r}$. The ribbon basis is denoted by R_I. The *descent set* of I is $\mathrm{Des}(I) = \{i_1,\ i_1+i_2, \ldots, i_1+\cdots+i_{r-1}\}$. The *descent composition* of a permutation $\sigma \in \mathfrak{S}_n$ is the composition $I = D(\sigma)$ of n whose descent set is the descent set of σ.

Recall from [8] that for an infinite totally ordered alphabet A, $\mathbf{FQSym}(A)$ is the subalgebra of $\mathbb{C}\langle A\rangle$ spanned by the polynomials

$$\mathbf{G}_\sigma(A) = \sum_{\mathrm{std}(w)=\sigma} w, \tag{1}$$

that is, the sum of all words in A^n whose standardization is the permutation $\sigma \in \mathfrak{S}_n$. The noncommutative ribbon Schur function $R_I \in \mathbf{Sym}$ is then

$$R_I = \sum_{\mathrm{D}(\sigma)=I} \mathbf{G}_\sigma\,. \tag{2}$$

This defines a Hopf embedding $\mathbf{Sym} \to \mathbf{FQSym}$. The Hopf algebra $\mathbf{FQSym}$ is self-dual under the pairing $(\mathbf{G}_\sigma\,,\, \mathbf{G}_\tau) = \delta_{\sigma,\tau^{-1}}$ (Kronecker symbol). Let $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$, so that $\{\mathbf{F}_\sigma\}$ is the dual basis of $\{\mathbf{G}_\sigma\}$. The *internal product* $*$ of $\mathbf{FQSym}$ is induced by composition $\circ$ in $\mathfrak{S}_n$ in the basis $\mathbf{F}$, that is,

$$\mathbf{F}_\sigma * \mathbf{F}_\tau = \mathbf{F}_{\sigma\circ\tau} \quad \text{and} \quad \mathbf{G}_\sigma * \mathbf{G}_\tau = \mathbf{G}_{\tau\circ\sigma}\,. \tag{3}$$

Each subspace $\mathbf{Sym}_n$ is stable under this operation, and anti-isomorphic to the descent algebra Σ_n of $\mathfrak{S}_n$. For $f_i \in \mathbf{FQSym}$ and $g \in \mathbf{Sym}$, we have the splitting formula

$$(f_1 \ldots f_r) * g = \mu_r \cdot (f_1 \otimes \cdots \otimes f_r) *_r \Delta^r g\,, \tag{4}$$

where μ_r is r-fold multiplication, and Δ^r the iterated coproduct with values in the r-th tensor power.

2.2 The Mantaci-Reutenauer algebra of level 2

We denote by $\mathbf{MR}$ the free product $\mathbf{Sym} \star \mathbf{Sym}$ of two copies of the Hopf algebra of noncommutative symmetric functions [14]. That is, $\mathbf{MR}$ is the free associative algebra on two sequences (S_n) and $(S_{\bar n})$ $(n \geq 1)$. We regard the two copies of $\mathbf{Sym}$ as noncommutative symmetric functions on two auxiliary alphabets: $S_n = S_n(A)$ and $S_{\bar{n}} = S_n(\bar{A})$. We denote by $F \mapsto \bar{F}$ the involutive automorphism which exchanges S_n and $S_{\bar{n}}$. The bialgebra structure is defined by the requirement that the series

$$\sigma_1 = \sum_{n\geq 0} S_n \text{ and } \bar{\sigma}_1 = \sum_{n\geq 0} S_{\bar{n}} \tag{5}$$

are grouplike. The internal product of $\mathbf{MR}$ can be computed from the splitting formula and the conditions that σ_1 is neutral, $\bar{\sigma}_1$ is central, and $\bar{\sigma}_1 * \bar{\sigma}_1 = \sigma_1$.

In [15], an embedding of $\mathbf{MR}$ in the Hopf algebra $\mathbf{BFQSym}$ of free quasi-symmetric functions of type B (spanned by colored permutations) is described. Under this embedding, left $*$-multiplication by $\Lambda_n = \mathbf{G}_{n\,n-1\ldots 2,1}$ corresponds to right multiplication by $n\,n-1\ldots 2,1$ in the group algebra of B_n. This implies that left $*$-multiplication by λ_1 is an involutive anti-automorphism of $\mathbf{BFQSym}$, hence of $\mathbf{MR}$.

2.3 Noncommutative symmetric functions of type B

The hyperoctahedral analogue $\mathbf{BSym}$ of $\mathbf{Sym}$, defined in [6], is the right $\mathbf{Sym}$-module freely generated by another sequence $(\tilde{S}_n)$ $(n \geq 0,\ \tilde{S}_0 = 1)$ of homogeneous elements, with $\tilde{\sigma}_1$ grouplike. This is a coalgebra, but not an algebra. It is endowed with an internal product, for which each homogeneous component $\mathbf{BSym}_n$ is anti-isomorphic to the descent algebra of B_n.

3 Solomon descent algebras of type B

3.1 Descents in B_n

The hyperoctahedral group B_n is the group of signed permutations. A signed permutation can be denoted by $w = (\sigma, \epsilon)$ where σ is an ordinary permutation and $\epsilon \in \{\pm 1\}^n$, such that $w(i) = \epsilon_i \sigma(i)$. If we set $w(0) = 0$, then, $i \in [0, n-1]$ is a descent of w if $w(i) > w(i+1)$. Hence, the descent set of w is a subset $D = \{i_0, i_0 + i_1, \ldots, i_0 + i_1 + \cdots i_{r-1}\}$ of $[0, n-1]$. We then associate to D a so-called type-B composition (a composition whose first part can be zero) $(i_0 - 0, i_1, \ldots, i_{r-1}, n - i_{r-1})$. The sum of all signed permutations whose descent set is contained in D is mapped to $\tilde{S}^I := \tilde{S}_{i_0} S^{I'}$ by Chow's anti-isomorphism [6], where $I' = (i_1, \ldots, i_r)$.

3.2 Noncommutative supersymmetric functions

An embedding of $\mathbf{BSym}$ as a sub-coalgebra and sub-$\mathbf{Sym}$-module of $\mathbf{MR}$ can be deduced from [14]. To describe it, let us define, for $F \in \mathbf{Sym}(A)$,

$$F^\sharp = F(A|\bar{A}) = F(A - q\bar{A})|_{q=-1} \tag{6}$$

(the supersymmetric version of F). The superization of $F \in \mathbf{Sym}(A)$ can also be given by

$$F^\sharp = F * \sigma_1^\sharp\,. \tag{7}$$

Indeed, $\sigma_1^\sharp$ is grouplike, and for $F = S^I$, the splitting formula gives

$$(S_{i_1} \cdots S_{i_r}) * \sigma_1^\sharp = \mu_r[(S_{i_1} \otimes \cdots \otimes S_{i_r}) * (\sigma_1^\sharp \otimes \cdots \otimes \sigma_1^\sharp)] = S^{I\sharp}\,. \tag{8}$$

We have

$$\sigma_1^\sharp = \bar{\lambda}_1\sigma_1 = \sum \Lambda_{\bar{i}} S_j \, . \tag{9}$$

The element $\bar{\sigma}_1$ is central for the internal product, and

$$\bar{\sigma}_1 * F = \bar{F} = F * \bar{\sigma}_1 \, . \tag{10}$$

Hence,

$$\bar{\sigma}_1 * \sigma_1^\sharp = \lambda_1\bar{\sigma}_1 =: \sigma_1^\flat \, . \tag{11}$$

The basis element $\tilde{S}^I$ of $\mathbf{BSym}$, where $I = (i_0, i_1, \dots, i_r)$ is a type B-composition, can be embedded as

$$\tilde{S}^I = S_{i_0}(A) S^{i_1 i_2 \cdots i_r}(A|\bar{A}) \, . \tag{12}$$

We will identify $\mathbf{BSym}$ with its image under this embedding.

3.3 A proof that $\mathbf{BSym}$ is $*$-stable

We are now in a position to understand why $\mathbf{BSym}$ is a $*$-subalgebra of $\mathbf{MR}$. The argument will be extended below to the case of unital peak algebras. Let $F, G \in \mathbf{Sym}$. We want to understand why $\sigma_1 F^\sharp * \sigma_1 G^\sharp$ is in $\mathbf{BSym}$. Using the splitting formula, we rewrite this as

$$\mu[(\sigma_1 \otimes F^\sharp) * \Delta\sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G^\sharp_{(1)})(F^\sharp * \sigma_1 G^\sharp_{(2)}). \tag{13}$$

We now only have to show that each term $F^\sharp * \sigma_1 G^\sharp_{(2)}$ is in $\mathbf{Sym}^\sharp$. We may assume that $F = S^I$, and for any $G \in \mathbf{Sym}$,

$$S^{I\sharp} * \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S^\sharp_{i_1} \otimes \cdots \otimes S^\sharp_{i_r}) * (\sigma_1 G^\sharp_{(1)} \otimes \cdots \otimes \sigma_1 G^\sharp_{(r)})] \tag{14}$$

so that it is sufficient to prove the property for $F = S_n$. Now,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= (\bar{\lambda}_1 \sigma_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_1 * \sigma_1 G^\sharp_{(1)})(\sigma_1 G^\sharp_{(2)}) \\ &= \sum_{(G)} (\bar{\sigma}_1 * \lambda_1 * \sigma_1 G^\sharp_{(1)}) \cdot \sigma_1 \cdot G^\sharp_{(2)} \end{aligned} \tag{15}$$

Now,

$$\lambda_1 * \sigma_1 G^\sharp_{(1)} = (\lambda_1 * G^\sharp_{(1)})(\lambda_1 * \sigma_1) = (\lambda_1 * G^\sharp_{(1)})\lambda_1, \tag{16}$$

since λ_1 is an anti-automorphism. We then get

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= \sum_{(G)} (\bar{\sigma}_1 * ((\lambda_1 * G^\sharp_{(1)})\lambda_1) \cdot \sigma_1 \cdot G^\sharp_{(2)} \\ &= \sum_{(G)} (\bar{\sigma}_1 * \lambda_1 * G^\sharp_{(1)}) \cdot (\bar{\sigma}_1 * \lambda_1)\sigma_1 \cdot G^\sharp_{(2)} \\ &= \sum_{(G)} (\bar{\lambda}_1 * G^\sharp_{(1)}) \cdot \sigma_1^\sharp \cdot G^\sharp_{(2)} \end{aligned} \tag{17}$$

Now, the result will follow if we can prove that $\bar{\lambda}_1 * G^\sharp$ is in $\mathbf{Sym}^\sharp$ for any $G \in \mathbf{Sym}$.

For $G = S^I$,

$$\bar{\lambda}_1 * S^{I\sharp} = \lambda_1 * \bar{\sigma}_1 * S^I * \sigma_1^\sharp = \lambda_1 * S^I * \bar{\sigma}_1 * \sigma_1^\sharp = \lambda_1 * S^I * \sigma_1^\flat \,. \tag{18}$$

Since left $*$-multiplication by λ_1 in an anti-automorphism, we only need to prove that $\lambda_1 * S_n^\flat$ is of the form $G^\sharp$. And indeed,

$$\begin{aligned} \lambda_1 * S_n^\flat &= \sum_{i+j=n} \lambda_1 * (\Lambda_i S_{\bar{j}}) \\ &= \sum_{i+j=n} (\lambda_1 * S_{\bar{j}})(\lambda_1 * \Lambda_i) \\ &= \sum_{i+j=n} \Lambda_{\bar{j}} S_i = S_n^\sharp \,. \end{aligned} \tag{19}$$

This concludes the proof that $\mathbf{BSym}$ is a $*$-subalgebra of $\mathbf{BFQSym}$.

4 Unital versions of the higher order peak algebras

As shown in [5], much of the theory of the peak algebra can be deduced from a formula of [11] for $R_I((1-q)A)$, in the special case $q = -1$. In [12], this formula was studied in the case where q is an arbitrary root of unity, and higher order analogs of the peak algebra were obtained. In [2], it was shown that the classical peak algebra can be extended to a unital algebra, which is obtained as a homomorphic image of the descent algebra of type B. In this section, we construct unital extensions of the higher order peak algebras.

Let q be a primitive r-th root of unity. All objects introduced below will depend on q (and r), although this dependence will not be made explicit in the notation. We denote by θ_q the endomorphism of $\mathbf{Sym}$ defined by

$$\tilde{f} = \theta_q(f) = f((1-q)A) = f(A) * \sigma_1((1-q)A) \,. \tag{20}$$

We denote by $\mathring{\mathcal{P}}$ the image of θ_q and by $\mathcal{P}$ the right $\mathring{\mathcal{P}}$-module generated by the S_n for $n \geq 0$. Note that $\mathring{\mathcal{P}}$ is by definition a left $*$-ideal of $\mathbf{Sym}$.

Theorem 4.1 *$\mathcal{P}$ is a unital $*$-subalgebra of* $\mathbf{Sym}$*. Its Hilbert series is*

$$\sum_{n\geq 0} \dim \mathcal{P}_n t^n = \frac{1}{1 - t - t^2 - \cdots - t^r} \,. \tag{21}$$

Proof – Since the internal product of homogeneous elements of different degrees is zero, it is enough to show that, for any $f, g \in \mathbf{Sym}$, $\sigma_1 \tilde{f} * \sigma_1 \tilde{g}$ is in $\mathcal{P}$. Thanks to the splitting formula,

$$\begin{aligned} \sigma_1 \tilde{f} * \sigma_1 \tilde{g} &= \mu[(\sigma_1 \otimes \tilde{f}) * \sum_{(g)} \sigma_1 \tilde{g}_{(1)} \otimes \sigma_1 \tilde{g}_{(2)}] \\ &= \sum_{(g)} (\sigma_1 \tilde{g}_{(1)})(\tilde{f} * \sigma_1 \tilde{g}_{(2)}) \,. \end{aligned} \tag{22}$$

Thus, it is enough to check that $\tilde{f} * \sigma_1 \tilde{h}$ is in $\mathring{\mathcal{P}}$ for any $f, h \in \mathbf{Sym}$. Now,

$$\tilde{f} * \sigma_1 \tilde{h} = f * \sigma_1((1-q)A) * \sigma_1 \tilde{h}\,, \tag{23}$$

and since $\mathring{\mathcal{P}}$ is a $\mathbf{Sym}$ left $*$-ideal, we only have to show that $\sigma_1((1-q)A) * \sigma_1 \tilde{h}$ is in $\mathring{\mathcal{P}}$. One more splitting yields

$$\begin{aligned} \sigma_1((1-q)A) * \sigma_1 \tilde{h} &= (\lambda_{-q}\sigma_1) * \sigma_1 \tilde{h} \\ &= \mu[(\lambda_{-q} \otimes \sigma_1) * \sum_{(h)} \sigma_1 \tilde{h}_{(1)} \otimes \sigma_1 \tilde{h}_{(2)}] \\ &= \sum_{(h)} (\lambda_{-q} * \sigma_1 \tilde{h}_{(1)})(\sigma_1 \tilde{h}_{(2)}) \\ &= \sum_{(h)} (\lambda_{-q} * \tilde{h}_{(1)}) \lambda_{-q} \sigma_1 \tilde{h}_{(2)} \end{aligned} \tag{24}$$

(since left $*$-multiplication by λ_{-q} is an anti-automorphism, namely the composition of the antipode and q^{degree}). The first parentheses $(\lambda_{-q} * \tilde{h}_{(1)})$ are in $\mathring{\mathcal{P}}$ since it is a left $*$-ideal. The middle term is $\sigma_1((1-q)A)$, and the last one is in $\mathring{\mathcal{P}}$ by definition.

Recall from [12, Prop. 3.5] that the Hilbert series of $\mathring{\mathcal{P}}$ is

$$\sum_{n \geq 0} \dim \mathring{\mathcal{P}}_n t^n = \frac{1 - t^r}{1 - t - t^2 - \ldots - t^r}\,. \tag{25}$$

From [12, Lemma 3.13 and Eq. (3.9)], it follows that $S_n \notin \mathring{\mathcal{P}}$ if and only if $n \equiv 0 \mod r$, so that the Hilbert series of $\mathcal{P}$ is

$$\sum_{n \geq 0} \dim \mathcal{P}_n t^n = \frac{1}{1 - t - t^2 - \ldots - t^r}\,. \tag{26}$$

■

5 Back to the Mantaci-Reutenauer algebra

The above proofs are in fact special cases of a master calculation in the Mantaci-Reutenauer algebra, which we carry out in this section.

Let q be an arbitrary complex number or an indeterminate, and define, for any $F \in \mathbf{MR}$,

$$F^\sharp = F * \sigma_1(A - q\bar{A}) = F * \sigma_1^\sharp\,. \tag{27}$$

Since $\sigma_1^\sharp$ is grouplike, it follows from the splitting formula that

$$F \mapsto F^\sharp \tag{28}$$

is an automorphism of $\mathbf{MR}$ for the Hopf structure. In addition, it is clear from the definition that it is also a endomorphism of left $*$-modules. We refer to it as the $\sharp$ *transform*.

We now define

$$\mathring{\mathcal{Q}} = \mathbf{MR}^\sharp, \tag{29}$$

the image of the $\sharp$ transform. Since the latter is an endomorphism of Hopf algebras and of left $*$-modules, $\mathring{\mathcal{Q}}$ is both a Hopf subalgebra of $\mathbf{MR}$ and a left $*$-ideal. When q is a root of unity, its image under the specialization $\bar{A} = A$ is the non-unital peak algebra $\mathring{\mathcal{P}}$ of Section 4 (and for generic q, it is $\mathbf{Sym}$).

Let $\mathcal{Q}$ be the right $\mathring{\mathcal{Q}}$-module generated by the S_n, for all $n \geq 0$. Clearly, the identification $\bar{A} = A$ maps $\mathcal{Q}$ onto $\mathcal{P}$, the unital peak algebra of Section 4.

Theorem 5.1 *$\mathcal{Q}$ is a $*$-subalgebra of $\mathbf{MR}$, containing $\mathring{\mathcal{Q}}$ as a left ideal.*

Proof – Let $F, G \in \mathbf{MR}$. As above, we want to show that $\sigma_1 F^\sharp * \sigma_1 G^\sharp$ is in $\mathcal{Q}$. Using the splitting formula, we rewrite this as

$$\mu[(\sigma_1 \otimes F^\sharp) * \Delta\sigma_1 \Delta G^\sharp] = \sum_{(G)} (\sigma_1 G^\sharp_{(1)})(F^\sharp * \sigma_1 G^\sharp_{(2)}) \tag{30}$$

and we only have to show that each term $F^\sharp * \sigma_1 G^\sharp_{(2)}$ is in $\mathring{\mathcal{Q}}$. We may assume that $F = S^I$, where I is now a bicolored composition, and for any $G \in \mathbf{MR}$,

$$S^{I\sharp} * \sigma_1 G^\sharp = \sum_{(G)} \mu_r[(S^\sharp_{i_1} \otimes \cdots \otimes S^\sharp_{i_r}) * (\sigma_1 G^\sharp_{(1)} \otimes \cdots \otimes \sigma_1 G^\sharp_{(r)})] \tag{31}$$

so that it is sufficient to prove the property for $F = S_n$ or $S_{\bar{n}}$. Now,

$$\begin{aligned} \sigma_1^\sharp * \sigma_1 G^\sharp &= (\bar{\lambda}_{-q}\sigma_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\bar{\lambda}_{-q} 1 * \sigma_1 G^\sharp_{(1)})(\sigma_1 G^\sharp_{(2)}) \\ &= \sum_{(G)} (\bar{\lambda}_{-q} * G^\sharp_{(1)}) \cdot \sigma_1^\sharp \cdot G^\sharp_{(2)} \end{aligned} \tag{32}$$

which is in $\mathring{\mathcal{Q}}$, since it is a subalgebra and a left $*$-ideal, and similarly,

$$\begin{aligned} \bar{\sigma}_1^\sharp * \sigma_1 G^\sharp &= (\lambda_{-q}\bar{\sigma}_1) * \sigma_1 G^\sharp \\ &= \sum_{(G)} (\lambda_{-q} * \sigma_1 G^\sharp_{(1)})(\bar{\sigma}_1 \bar{G}^\sharp_{(2)}) \\ &= \sum_{(G)} (\lambda_{-q} * G^\sharp_{(1)}) \cdot \bar{\sigma}_1^\sharp \cdot \bar{G}^\sharp_{(2)} \end{aligned} \tag{33}$$

is also in $\mathring{\mathcal{Q}}$. ∎

The various algebras introduced in this paper and their interrelationships are summarized in the following diagram.

$$\begin{array}{ccccccc} \mathring{\mathcal{Q}} & \subseteq & \mathcal{Q} & \subseteq & \mathbf{MR} & \subseteq & \mathbf{BFQSym} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathring{\mathcal{P}} & \subseteq & \mathcal{P} & \subseteq & \mathbf{Sym} & \subseteq & \mathbf{FQSym} \end{array} \tag{34}$$

Note that in the special case $q=-1$, by the results of Section 3.3, $\mathcal{Q}_n$ is the (Solomon) descent algebra of B_n, $\mathcal{Q}$ is isomorphic to $\mathbf{BSym}$, and $\mathcal{P}$ is the unital peak algebra of [2].

6 Further developments

6.1 Inversion of the generic $\sharp$ transform

For generic q, the endomorphism (27) of $\mathbf{MR}$ is invertible; therefore

$$\mathring{\mathcal{Q}} \sim \mathbf{MR}. \tag{35}$$

The inverse endomorphism of $\mathbf{MR}$ arises from the transformation of alphabets

$$A \mapsto (q\bar{A}+A)/(1-q^2), \tag{36}$$

which is to be understood in the following sense:

$$\sigma_1\left(\frac{q\bar{A}+A}{1-q^2}\right) := \prod_{k\geq 0} \sigma_{q^{2k+1}}(\bar{A})\sigma_{q^{2k}}(A)\,. \tag{37}$$

Indeed,

$$\begin{aligned}\sigma_1\left(\frac{q\bar{A}+A}{1-q^2}\right) * \sigma_1(A-q\bar{A}) &= \prod_{k\geq 0} \sigma_{q^{2k+1}}(\bar{A}-qA)\sigma_{q^{2k}}(A-q\bar{A})\\ &= \prod_{k\geq 0} \lambda_{-q^{2k+2}}(A)\sigma_{q^{2k+1}}(\bar{A})\lambda_{-q^{2k+1}}(\bar{A})\sigma_{q^{2k}}(A)\\ &= \sigma_1(A)\,.\end{aligned} \tag{38}$$

By normalizing the term of degree n in (37), we obtain B_n-analogs of the q-Klyachko elements defined in [9]:

$$K_n(q;A,\bar{A}) := \prod_{i=1}^{n}(1-q^{2\,i})S_n\left(\frac{q\bar{A}+A}{1-q^2}\right) = \sum_{I\vDash n} q^{2\,\mathrm{maj}(I)}R_I(q\bar{A}+A)\,. \tag{39}$$

This expression can be completely expanded on signed ribbons. From the expression of R_I in $\mathbf{FQSym}$, we have

$$R_I(\bar{A}+A) = \sum_{C(\sigma)=I} \mathbf{G}_\sigma(\bar{A}+A) \tag{40}$$

where $\bar{A}+A$ is the ordinal sum. If we order $\bar{A}$ by

$$\bar{a}_1 < \bar{a}_2 < \ldots < \bar{a}_k < \ldots \tag{41}$$

then, arguing as in [16], we have

$$\mathbf{G}_\sigma(\bar{A}+A) = \sum_{\mathrm{std}(\tau,\epsilon)=\sigma} \mathbf{G}_{\tau,\epsilon} \tag{42}$$

so that

$$R_I(\bar{A}+A) = \sum_{\rho(\mathrm{J})=I} R_{\mathrm{J}} \tag{43}$$

where for a signed composition $\mathrm{J} = (J, \epsilon)$, the unsigned composition $\rho(\mathrm{J})$ is defined as the shape of $\mathrm{std}(\sigma, \epsilon)$, where σ is any permutation of shape J.

Replacing $\bar{A}$ by $q\bar{A}$, one obtains the expansion of the q-Klyachko elements of type B:

$$K_n(q; A, \bar{A}) = \sum_{\mathrm{J}} q^{\mathrm{bmaj}(\mathrm{J})} R_{\mathrm{J}} \tag{44}$$

where

$$\mathrm{bmaj}(\mathrm{J}) = 2\,\mathrm{maj}(\rho(\mathrm{J})) + |\epsilon|\,, \tag{45}$$

where $|\epsilon|$ is the number of minus signs in ϵ.

For example,

$$K_2(q) = R_2 + q^2\, R_{\bar{2}} + q^2\, R_{11} + q^3\, R_{1\bar{1}} + q\, R_{\bar{1}1} + q^4\, R_{\bar{1}\bar{1}}. \tag{46}$$

$$\begin{aligned} K_3(q) = R_3 &+ q^3\, R_{\bar{3}} + q^4\, R_{21} + q^5\, R_{2\bar{1}} + q^2\, R_{\bar{2}1} + q^7\, R_{\bar{2}\bar{1}} + q^2\, R_{12} + q^4\, R_{1\bar{2}} \\ &+ q\, R_{\bar{1}2} + q^5\, R_{\bar{1}\bar{2}} + q^6\, R_{111} + q^7\, R_{11\bar{1}} + q^3\, R_{1\bar{1}1} + q^8\, R_{1\bar{1}\bar{1}} \\ &+ q^5\, R_{\bar{1}11} + q^6\, R_{\bar{1}1\bar{1}} + q^4\, R_{\bar{1}\bar{1}1} + q^9\, R_{\bar{1}\bar{1}\bar{1}}. \end{aligned} \tag{47}$$

This major index of type B is the flag major index defined in [1].

Following [1] and considering the signed composition (where ϵ is encoded as boolean vector for readability)

$$\mathrm{J} = (2, 1, 1, \bar{3}, \bar{1}, \bar{2}, 4, \bar{1}, 2, 2) = (2113124122, 0000111111000010000) \tag{48}$$

we can take the smallest permutation of shape $(2, 1, 1, 3, 1, 2, 4, 1, 2, 2)$, which is

$$\alpha = 1\,5\,4\,3\,2\,6\,9\,8\,7\,11\,10\,12\,13\,16\,15\,14\,18\,17\,19 \tag{49}$$

sign it according to ϵ, which yields

$$1\,5\,4\,3\,\bar{2}\,\bar{6}\,\bar{9}\,\bar{8}\,\bar{7}\,\overline{11}\,10\,12\,13\,16\,\overline{15}\,14\,18\,17\,19 \tag{50}$$

whose standardized is

$$8\,11\,10\,9\,1\,2\,5\,4\,3\,6\,12\,13\,14\,16\,7\,15\,18\,17\,19 \tag{51}$$

and has shape $\rho(\mathrm{J}) = (2, 1, 1, 3, 1, 6, 3, 2)$. The major index of $\rho(\mathrm{J})$ is 55, the number of minus signs in ϵ is 7, so $\mathrm{bmaj}(\mathrm{J}) = 2 \times 55 + 7 = 117$.

The major index of type B can be read directly on signed compositions without reference to signed permutations as follows: one can get $\rho(\mathrm{J})$ by first adding the absolute values of two consecutive parts if the left one is signed and the second one is not, then remove the signs and proceed as before.

A different solution consists in reading the composition from right to left, then associate weight 0 (resp. 1) to the rightmost part if it is positive (resp. negative) and then proceed left by adding 2 to the weight if the two parts are of the same sign and 1 if not. Finally, add up the product of the absolute values of the parts with their weight.

For example, with the same J as above we have the following weights:

$$\begin{aligned} \mathrm{J} =& (2, 1, 1,\ \bar{3}, \bar{1}, \bar{2}, 4, \bar{1}, 2, 2) \\ \text{weights}: & 14\ 12\ 10\ \ 9\ \ 7\ \ 5\ \ 4\ \ 3\ \ 2\ \ 0 \end{aligned} \tag{52}$$

so that we get $2 \cdot 14 + 1 \cdot 12 + 1 \cdot 10 + 3 \cdot 9 + 1 \cdot 7 + 2 \cdot 5 + 4 \cdot 4 + 1 \cdot 3 + 2 \cdot 2 + 2 \cdot 0 = 117$.

This technique generalizes immediately to colored compositions with a fixed number c of colors $0, 1, \ldots, c-1$: the weight of the rightmost cell is its color and the weight of a part is equal to the sum of the weight of the next part and the unique representative of the difference of the colors of those parts modulo c belonging to the interval $[1, c]$.

6.2 Generators and Hilbert series

For $n \geq 0$, let

$$S_n^{\pm} = S_n(A) \pm S_n(\bar{A})\,, \tag{53}$$

and denote by $\mathcal{H}_n$ the subalgebra of $\mathbf{MR}$ generated by the $S_k^{\pm}$ for $k \leq n$. For $n \geq 0$, we have

$$(S_n^{\pm})^{\sharp} \equiv (1 \mp q^n) S_n^{\pm} \mod \mathcal{H}_{n-1}\,, \tag{54}$$

so that the $(S_n^{\pm})^{\sharp}$ such that $1 \mp q^n \neq 0$ form a set of free generators in $\mathbf{MR}^{\sharp}$.

Conjecture 6.1 *If r is odd, a basis of $\mathbf{MR}^{\sharp}$ will be parametrized by colored compositions such that parts of color 0 are not $\equiv 0 \mod r$ and parts of color 1 are arbitrary. The Hilbert series is then*

$$H_r(t) = \frac{1-t^r}{1-2(t+t^2+\cdots+t^r)}. \tag{55}$$

If r is even, there is the extra condition that parts of color 1 are not $\equiv r/2 \mod r$. The Hilbert series is then

$$H_r(t) = \frac{1-t^r}{1-2(t+t^2+\cdots+t^r)+t^{r/2}}\,. \tag{56}$$

For example,

$$H_2(t) = 1+t+2\,t^2+4\,t^3+8\,t^4+16\,t^5+32\,t^6+64\,t^7+128\,t^8+O\left(t^9\right) \tag{57}$$

$$H_3(t) = 1+2\,t+6\,t^2+17\,t^3+50\,t^4+146\,t^5+426\,t^6+1244\,t^7+3632\,t^8+O\left(t^9\right) \tag{58}$$

$$H_4(t) = 1+2\,t+5\,t^2+14\,t^3+38\,t^4+104\,t^5+284\,t^6+776\,t^7+2120\,t^8+O\left(t^9\right) \tag{59}$$

If these conjectures are correct, the Hilbert series of the right $\mathbf{MR}^{\sharp}$-modules generated by the S_n are respectively

$$\frac{1}{1-2(t+t^2+\ldots+t^r)}\,, \tag{60}$$

or

$$\frac{1}{1-2(t+t^2+\ldots+t^r)+t^{r/2}}\,. \tag{61}$$

according to whether r is odd or even.

The cases $r=1$ and $r=2$ are easily proved as follows. Assume first that $q=1$. Set

$$f = 1+(\sigma_1^+)^{\sharp} = (\sigma_1+\lambda_{-1})(A-\bar{A})\,, \tag{62}$$

$$g = (\sigma_1^-)^{\sharp} - 1 = (\sigma_1-\lambda_{-1})(A-\bar{A})\,. \tag{63}$$

Then, $f^2 = g^2+4$, so that

$$f = 2\left(1+\frac{1}{4}g^2\right)^{\frac{1}{2}} \tag{64}$$

which proves that the $(S_n^+)^{\sharp}$ can be expressed in terms of the $(S_m^-)^{\sharp}$.

Similarly, for $q=-1$, one can express

$$f = \sum_{n\geq 1}(S_{2n}^+)^{\sharp} + \sum_{n\geq 0}(S_{2n+1}^-)^{\sharp} \tag{65}$$

in terms of

$$g = \sum_{n\geq 1} (S_{2n}^{-})^{\sharp} + \sum_{n\geq 0} (S_{2n+1}^{+})^{\sharp} \tag{66}$$

since, as is easily verified,

$$(f+2)^2 = g^2 + 4\,,\ i.e.,\, f = -2 + 2\left(1 + \frac{1}{4}g^2\right)^{\frac{1}{2}}\ . \tag{67}$$

Apparently, this approach does not work anymore for higher roots of unity.

7 Appendix: monomial expansion of the $(1-q)$-kernel

The results of [16, 7] allow us to write down a new expansion of $S_n((1-q)A)$, in terms of the monomial basis of [4]. The special case $q=1$ gives back a curious expression of Dynkin's idempotent, first obtained in [3].

Let σ be a permutation. We then define its *left-right minima* set $\mathrm{LR}(\sigma)$ as the values of σ that have no smaller value to their left. We will denote by $\mathrm{lr}(\sigma)$ the cardinality of $\mathrm{LR}(\sigma)$. For example, with $\sigma = 46735182$, we have $\mathrm{LR}(\sigma) = \{4, 3, 1\}$, and $\mathrm{lr}(\sigma) = 3$.

Let us now decompose $S_n((1-q)A)$ on the monomial basis $\mathbf{M}_\sigma$ (see [4]) of $\mathbf{FQSym}$. Thanks to the Cauchy formula of $\mathbf{FQSym}$ [7], we have

$$S_n((1-q)A) = \sum_{\sigma} \mathbf{S}^{\sigma}(1-q)\mathbf{M}_\sigma(A), \tag{68}$$

where $\mathbf{S}$ is the dual basis of $\mathbf{M}$. Given the transition matrix between $\mathbf{M}$ and $\mathbf{G}$, we see that

$$\mathbf{S}^{\sigma} = \sum_{\tau\leq\sigma^{-1}} \mathbf{F}_\tau, \tag{69}$$

where $\leq$ is the right weak order, e.g., $\mathbf{S}^{312} = \mathbf{F}_{123}+\mathbf{F}_{213}+\mathbf{F}_{231}$. Thanks to [16], we know that $\mathbf{F}_\sigma(1-q)$ is either $(-q)^k$ if $\mathrm{Des}(\sigma) = \{1, \ldots, k\}$ or 0 otherwise. Let us define *hook permutations* of hook k the permutations σ such that $\mathrm{Des}(\sigma) = \{1, \ldots, k\}$. Now, $\mathbf{S}^{\sigma}(1-q)$ amounts to compute the list of *hook permutations* smaller than σ. Note that hook permutations are completely characterized by their left-right minima. Moreover, if τ is smaller than σ in the right weak order, then $\mathrm{LR}(\tau) \subset \mathrm{LR}(\sigma)$.

Hence all hook permutations smaller than a given permutation σ belong to the set of hook permutations with left-right minima in $\mathrm{LR}(\sigma)$. Since by elementary transpositions decreasing the length, one can get from σ to the hook permutation with the same left-right minima and then from this permutation to all the others, we have:

Theorem 7.1 *Let n be an integer. Then*

$$S_n((1-q)A) = \sum_{\sigma\in\mathfrak{S}_n} (1-q)^{\mathrm{lr}(\sigma)}\mathbf{M}_\sigma. \tag{70}$$

∎

In the particular case $q=1$, we recover a result of [3]:

$$\Psi_n = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)=1}} \mathbf{M}_\sigma, \tag{71}$$

where Ψ_n is the noncommutative power sum associated with Dynkin's idempotent [11, Prop. 5.2].

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Growth function for a class of monoids

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Abstract. In this article we study a class of monoids that includes Garside monoids, and give a simple combinatorial proof of a formula for the formal sum of all elements of the monoid. This leads to a formula for the growth function of the monoid in the homogeneous case, and can also be lifted to a resolution of the monoid algebra. These results are then applied to known monoids related to Coxeter systems: we give the growth function of the Artin-Tits monoids, and do the same for the dual braid monoids. In this last case we show that the monoid algebras of the dual braid monoids of type A and B are Koszul algebras.

Résumé. Nous étudions une classe de monoïdes incluant les monoïdes de Garside, et donnons une preuve combinatoire simple d'une formule pour la somme formelle de leurs éléments. Cela mène à une formule pour la fonction de croissance du monoïde dans le cas homogène, et peut être aussi relevé en une résolution de l'algèbre de monoïdes. Ces résultats sont ensuite appliqués aux monoïdes liés aux systèmes de Coxeter: nous donnons la fonction de croissance des monoïdes d'Artin-Tits ainsi que des monoïdes duaux ; pour ces derniers nous montrons que leur algèbre de monoïde en types A et B est une algèbre de Koszul.

Keywords: monoid, growth function, Garside group, resolution, Koszul algebra

Introduction

We consider left cancellative monoids M that are generated by their atoms S, and such that if a subset of S admits a common right multiple, then it actually admits a least common multiple.

These monoids include trace monoids, for which there exists a nice combinatorial theory due to Viennot [23]. Our first result (Theorem 2) generalizes one of the proofs of Viennot for the formal sum of elements a monoid. When the monoid is homogeneous with respect to its set of atoms S, then we have immediately that the growth function of the monoid (i.e. the generating function according to the length of elements as words in S) is the inverse of a polynomial. We will apply this formula to Artin-Tits monoids, and more generally it applies to all Garside monoids [9].

The combinatorial proof, which is a actually a sign reversing involution, has an interpretation as a resolution of $\mathbb{Z}$ as a $\mathbb{Z}M$-module, where $\mathbb{Z}M$ stands for the monoid algebra of M. Another resolution can be deduced from this one, and in turn this new resolution gives another formula for the growth generating function of the monoid. We use this reduced resolution in the case of the *dual braid monoids* defined by Bessis in the types A and B; for a particular choice of the reduced resolution in these cases, we will show that the monoid algebras $\mathbb{Z}M$ are *Koszul algebras* [19, 11].

We now give an outline of the paper. In Section 1 we define the class of monoids we study, give formulas for the formal sum of their elements (Theorem 2) and the growth functions of such monoids, and give interpretations of these results as resolutions of the corresponding monoid algebras. In Section 2, we explain how these results apply to both trace monoids and Garside monoids. The following two sections apply the results of Section 1 to two families of Garside monoids related to irreducible finite Coxeter groups. In Section 3 we give the growth functions of the corresponding Artin-Tits monoids. In Section 4, we also give the corresponding growth functions for the dual braid monoids, and show that in type A and B the corresponding monoid algebras are Koszul algebras.

1 Growth function and exact resolution

1.1 Monoids

A monoid $(M, \cdot)$ is a set M together with an internal law $\cdot$ that is associative and such that there exists an identity element 1. A subset $S \subset M$ is a *generating set* if every element of M can be written as a product of elements of S.

Let S be a set, and R a collection of pairs (w, w') (called *relations*), where w and w' are words in S. We say that $\langle\, S \,|\, R \,\rangle$ is a *presentation* of the monoid M if M is isomorphic to $S^*/\ll R \gg$, where $\ll R \gg$ is the congruence generated by R. The presentation is said to be *homogeneous* if all relations of R are composed of two words of equal length. Given a generating set S of M, the *length* of an element $m \in M$ is the smallest number of generators needed to write it. We will write $|m|_S$ for this length, and we note that this length is additive if M admits an homogeneous presentation.

An element a is an *atom* of M if $a \neq 1$, and if $a = bc$ implies $b = 1$ or $c = 1$; a monoid is *atomic* if it is generated by its set of atoms, and if in addition every element m possesses a finite number of different decompositions as a product of atoms. It is easy to see that an atomic monoid has the property that $a \neq 1$ and $b \neq 1$ imply that $ab \neq 1$.

We note $\mathbb{Z}M$ the monoid algebra of M, whose elements are formal linear combinations of elements of M with coefficients in $\mathbb{Z}$; we note also $\mathbb{Z}\langle\langle\, M \,\rangle\rangle$ the algebra of formal *infinite* such linear combinations. The product of $\sum_m c_m m$ and $\sum_m d_m m$ is in both cases given by $\sum_m e_m m$ where $e_m = \sum_{ab=m} c_a d_b$: the product is well defined if the sum is finite, which is the case when M is atomic.

1.2 Main result

In all this work, we consider monoids $\mathbb{M}$ with a finite generating set $\mathcal{S}$ satisfying the following properties: $\mathbb{M}$ is atomic, left-cancellative (if $a,\, u,\, v \in \mathbb{M}$ are such that $au = av$, then $u = v$) and verifies that if a subset of $\mathcal{S}$ has a right common multiple, then it has a least right common multiple.

Lemma 1. *For such a monoid, if $J \subset \mathcal{S}$ is such that J has a common multiple, then a least common multiple (lcm) exists and is unique.*

We will call *cliques* the subsets of $\mathcal{S}$ having a common multiple, and let $\mathcal{J}$ be the set of all cliques; if J is a clique, we note M_J its unique least common multiple, and let m_J be the length of M_J. Then we have our first result:

Theorem 2. *Let $M, \mathcal{S}$ be as above. Then the following identity holds in $\mathbb{Z}\langle\langle\, M \,\rangle\rangle$:*

$$\left(\sum_{J\in\mathcal{J}} (-1)^{|J|} M_J\right) \cdot \left(\sum_{m'\in M} m'\right) = 1_M \tag{1.1}$$

As an important corollary, we get the following:

Corollary 3 (Bronfman '01)**.** *Given $M, \mathcal{S}$ as in the above theorem, suppose also that M admits a homogeneous presentation $\langle\, \mathcal{S} \mid \mathcal{R}\,\rangle$. Then its growth function is equal to :*

$$G_M(t) = \sum_{m \in M} t^{|m|_{\mathcal{S}}} = \left[\sum_{J \in \mathcal{J}} (-1)^{|J|} t^{m_J} \right]^{-1} \tag{1.2}$$

of Corollary 3. Admitting a homogeneous presentation is equivalent to the fact that the length according to $\mathcal{S}$ is additive, which means that the application $\sum_m c_m m \mapsto \sum_m c_m t^{|m|_{\mathcal{S}}}$ is a homomorphism from $\mathbb{Z}\langle\langle\, M\,\rangle\rangle$ to $\mathbb{Z}[[t]]$, the ring of power series with integer coefficients. It is indeed well defined because there is a finite number of elements of M of a given length. We can apply this homomorphism to both sides of the above theorem, which finishes the proof. $\square$

of Theorem 2. For every element $m \in M$, let us define $\mathcal{J}(m) \subseteq \mathcal{J}$ to be the subsets J of $\mathcal{S}$ such that every element s of J divides m; by the lcm property of M, we have that there exists a subset $J_m \subseteq \mathcal{S}$, such that $\mathcal{J}(m)$ consists exactly of the subsets of J_m.

From now on we fix a total order $<$ on the set of generators $\mathcal{S}$. Let us fix any $m \neq 1$. Clearly J_m is not empty in this case, and so we can define $s(m)$ as the maximal (for the order $<$) element of J_m. Define the involution Φ_m on $\mathcal{J}(m)$ as follows: $\Phi_m(J) = J \triangle \{s(m)\}$ where $\triangle$ denotes the symmetric difference $A \triangle B = (A \cup B) \setminus (A \cap B)$. The application Φ_m is simply the classical involution on the subsets of J_m; since Φ_m changes the parity of $|J|$, we have obviously

$$\sum_{J \subseteq J_m} (-1)^{|J|} = 0. \tag{1.3}$$

Note that this sum is 1 if we take $m = 1$, since there is only one term corresponding to the empty set. Now $J \in \mathcal{J}(m)$ means precisely that M_J divides m, that is there exists m' such that $M_J m' = m$: such an element m' is uniquely determined by the cancellability property. Therefore Equation (1.3) can be rewritten equivalently as

$$\sum_{\substack{(J,m') \in \mathcal{J} \times M \\ M_J m' = m}} (-1)^{|J|} = \begin{cases} 0 & \text{if } m \neq 1; \\ 1 & \text{if } m = 1. \end{cases} \tag{1.4}$$

But this quantity is precisely the coefficient c_m of m in the left term of Equation (1.1) written in the form $\sum_m c_m m$, and so this proves Theorem 2. $\square$

1.3 Posets

We refer to [22, ch. 3] for standard notions about posets. Given a locally finite poset $(P, \leq)$ (i.e all intervals have a finite number of elements), the Möbius function can be defined inductively on all pairs $x \leq z$ by

$$\mu(x,x) = 1, \;\; \mu(x,z) = \sum_{x \leq y < z} \mu(x,y) \;\; \text{for} \;\; x < z \tag{1.5}$$

Now consider a monoid M (as in Paragraph 1.2 with the divisibility relation $\preceq$. It forms a locally finite poset P_M as is readily checked, so it has a Möbius function; it has also a smallest element 1, and we write

$\mu(m) = \mu(1,m)$. In this poset, atoms of the monoids become atoms of the poset (i.e. elements that cover 1), and lcms become joins. We will use this in Section 4 to compute the growth functions of dual braid monoids of type A and B in particular, since the interval $[1, M_S]$ in P_M for these monoids are noncrossing partitions.

Note that one can identify the algebra $\mathbb{Z}\langle\langle\, M\,\rangle\rangle$ with the incidence algebra $I(P_M)$. From this we know that $\zeta_M = \sum_{m\in M} m \in \mathbb{Z}\langle\langle\, M\,\rangle\rangle$ has for inverse in $\mathbb{Z}\langle\langle\, M\,\rangle\rangle$ the function $\sum_m \mu(m)m$, so that Theorem 2 is actually a manner of computing the Möbius function of this poset, related to the crosscut theorem of Rota [21].

1.4 An exact resolution

In this paragraph we give resolutions that generalize the one in [14] which concerned trace monoids: let M, S be as at the beginning of Paragraph 1.2, $A = \mathbb{Z}M$ be the monoid algebra of M. Let $B = \mathbb{Z}\mathcal{J}$ be the free module with basis $\mathcal{J}$, and B_n be the submodule with basis $\mathcal{J}_n$ the cliques of cardinal n. Consider then $C_n = B_n \otimes_{\mathbb{Z}} A$ the free (right) A-module with basis $\mathcal{J}_n$. Now we fix a total order $<$ on $\mathcal{S}$, and we write cliques as words $s_1 \dots s_n$ where $s_i < s_{i+1}$ for all i. For two cliques $J \subset J'$, we also let $\delta_J^{J'\setminus J}$ be the element of M such that $M_J \delta_J^{J'\setminus J} = M_{J'}$; it is well defined thanks to the cancellability property. We define an A-module homomorphism $d_n : C_n \to C_{n-1}$ by

$$d_n(s_1 \dots s_n \otimes 1) = \sum_{i=1}^{n} (-1)^{n-i} s_1 \dots \hat{s_i} \dots s_n \otimes \delta^{s_i}_{s_1 \dots \hat{s_i} \dots s_n} \tag{1.6}$$

We define also $\epsilon : A \to \mathbb{Z}$ by $\epsilon(m) = 0$ if $m \neq 1$ and $\epsilon(1) = 1$, so that we have the following sequence of A-modules and A-homomorphism (where we let $k = |S|$):

$$0 \longrightarrow C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \cdots\cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 = A \xrightarrow{\epsilon} \mathbb{Z} \tag{1.7}$$

Theorem 4. *The complex* (1.7) *is a resolution of* $\mathbb{Z}$ *as an* A*-module.*

We recall that this means that the sequence is exact, i.e. we have to check that $Im(d_n) = Ker(d_{n-1})$ for all n.

Proof. Let $J = s_1 \dots s_n$ be a clique, then one checks first that $d_{n-1} \circ d_n = 0$ for any n. Indeed the computation gives $d_{n-1} \circ d_n(J \otimes 1) = \sum_{i<j} (-1)^{i+j-1} J_{i,j} \otimes \left(\delta^{s_j}_{J_{i,j}} \delta^{s_i}_{J_i} - \delta^{s_i}_{J_{j,i}} \delta^{s_j}_{J_j}\right)$, where we let $J_{i_1,\dots,i_t}$ be the clique obtained by removing the generators $s_{i_1}, \dots, s_{i_t}$ from J. Now the difference in the second term is 0 since both terms are equal to $\delta^{s_i,s_j}_{J_{i,j}}$.

So we have $Im(d_n) \subseteq Ker(d_{n-1})$, and to check the reverse inclusion, we define a $\mathbb{Z}$-homomorphism $i_{n+1} : C_n \to C_{n+1}$ in the following way: let $J \otimes m \in C_n$, with $J = s_1 \dots s_n$, and consider the subset of S consisting of divisors of $M_J m$ that are greater than s_n; call this set $\mathcal{E}(J, m)$. If $\mathcal{E}(J, m)$ is empty, set $i_{n+1}(J \otimes m) := 0$; otherwise, let s_{n+1} be the maximum element of $\mathcal{E}(J, m)$ for the order $<$, and define m_1 by $\delta_J^{s_{n+1}} m_1 = m$; then set $i_{n+1}(s_1 \dots s_n \otimes m) := s_1 \dots s_n s_{n+1} \otimes m_1$. One can then check that $i_{n-1} \circ d_{n-1} + d_n \circ i_n = 1$ for all n in a similar manner to [14], where 1 is the identity on C_{n-1}. This shows that $Ker(d_{n-1}) \subseteq Im(d_n)$ and concludes the proof. □

Now we show how this resolution gives a proof of Theorem 2:

of Theorem 2. Define the $\mathbb{Z}$-module $C(m) = \oplus_n C_{n,m}$ by letting the basis of $C_{n,m}$ be the elements $J \otimes m_1$ such that $|J| = n$ and $M_J m_1 = m$ in M. Then the functions d_n and i_{n+1} map $C(m)$ to itself as is immediately checked, so we obtain for every $m \in M$ an exact sequence of free $\mathbb{Z}$-modules:

$$0 \longrightarrow C_{k,m} \xrightarrow{d_k} C_{k-1,m} \xrightarrow{d_{k-1}} \cdots\cdots \xrightarrow{d_2} C_{1,m} \xrightarrow{d_1} C_{0,m} \xrightarrow{\epsilon} \mathbb{Z}_m \tag{1.8}$$

We have that $dim_{\mathbb{Z}} C_{n,m}$ is the number of pairs $(J, m_1) \in \mathcal{J} \times M$ such that $|J| = n$ and $M_J m_1 = m$; furthermore, $dim_{\mathbb{Z}} \mathbb{Z}_m$ is equal to 1 if $m = 1$ and 0 otherwise. Taking the Euler-Poincaré characteristic of the resolution (1.8) gives us then Equation (1.4), which has been shown to be equivalent to Theorem 2. □

Reduced resolution: Given a total order on S as above, introduce now the set $\mathcal{J}_< \subseteq \mathcal{J}$ of *order compatible cliques*: these are the cliques $s_1 \ldots s_n$ such that for all i we have that s_i is the largest divisor of $M_{s_1,\ldots,s_i}$ for the order $<$. We will write OC for order compatible.

Lemma 5. *A clique $J = s_1 \ldots s_n$ is OC if and only if for all $t \leq n$ and all sequences of indices $1 \leq i_1 < \cdots < i_t \leq n$ we have that s_{i_t} is the maximal divisor of $M_{s_{i_1},\ldots,s_{i_t}}$.*

Proof. The condition is clearly sufficient; now if $J = s_1 \ldots s_n$ is OC and $1 \leq i_1 < \cdots < i_t \leq n$, we have the inequalities $s_{i_t} \leq maxdiv(M_{s_{i_1},\ldots,s_{i_t}}) \leq maxdiv(M_{s_1,s_2,\ldots,s_{i_t}}) = s_{i_t}$. So all inequalities are in fact equalities and the lemma is proved. □

Corollary 6. *If J is an OC clique then every subset of J is also an OC clique.*

Now let $\widetilde{C}_i$ be the A-submodule of C_i with basis the OC cliques of size i. By the last corollary, the derivations d_i are well defined when restricted to these submodules, so we have a complex:

$$0 \longrightarrow \widetilde{C}_k \xrightarrow{d_k} \widetilde{C}_{k-1} \xrightarrow{d_{k-1}} \cdots\cdots \xrightarrow{d_2} \widetilde{C}_1 \xrightarrow{d_1} \widetilde{C}_0 = A \xrightarrow{\epsilon} \mathbb{Z} \tag{1.9}$$

Proposition 7. *The complex* (1.9) *is an exact resolution of $\mathbb{Z}$ by A-modules.*

Proof. We check that the homotopy i_{n+1} is still well defined when restricted to the $\mathbb{Z}$-module $\widetilde{C}_n$, which will prove the proposition. Suppose $J = s_1 \ldots s_n$ is an OC clique, $m \in M$, and that the maximal element s_{n+1} among the divisors of $M_J m$ is greater than s_n. Then, if s divides $M_{s_1,\ldots,s_{n+1}}$, it divides also $M_J m$, and thus the greatest of these divisors is s_{n+1}; this shows that $s_1 \ldots s_{n+1}$ is an OC clique, and thus that the function i_{n+1} is well defined. So now the same proof as the one of Theorem 4 can be applied, and the result follows. □

These modules were already considered in [8][Section 4], but with a different resolution.

Proposition 8. *Theorem 2 and its corollary hold if the sum is restricted to $\mathcal{J}_<$ (for any given total order $<$ on S.)*

The proof mimics the alternative proof of Theorem 2 above. We will use this proposition and the resolution in Section 4.

2 Application to some classes of monoids

We give in this section some examples of monoids that satisfy the conditions of Theorem 2.

2.1 Trace monoids

Trace monoids (also called *heaps of pieces monoids, Cartier-Foata monoids* or *free partially commutative monoids*) are defined by the presentation $M = \langle S \,|\, ab = ba \text{ if } (a,b) \in I\rangle$, where S is a finite set of generators and I is a symmetric and antireflexive relation on $S \times S$ called the commutation relation. In [23], elements of M are interpreted as heaps of pieces

At the very beginning, the aim of the work presented here was to generalize the results of [23]. It is indeed a special case of our Theorem 2: in trace monoids, for a subset J of $\mathcal{S}$, only two disjoint cases can occur: either all elements of J commute, and their product is clearly their least common multiple; or there exist two elements of J which do not commute, and J does not admit a common multiple.

The first case corresponds to what is called cliques in the trace monoid literature, from which we borrowed our terminology in our more general setting. It is then straightforward that the set of all least common multiples of cliques corresponds exactly to the set of heaps of pieces of height at most one.

This work applies too to divisibility monoids which are a natural generalization of trace monoids, studied in [10, 16].

2.2 Artin-Tits monoids

The Artin-Tits monoids are a generalization of both trace monoids and braid monoids (which are extensively studied in Section 3). Given a finite set S and a symmetric matrix $\mathbb{M} = (m_{s,t})_{s,t\in S}$ such that $m_{s,t} \in \mathbb{N} \cup \{\infty\}$ and $m_{s,s} = 1$, the *Artin-Tits monoid* M associated to S and $\mathbb{M}$ has the following presentation:

$$M = \langle s \in S |\ \underbrace{sts\ldots}_{m_{s,t} \text{ terms}} = \underbrace{tst\ldots}_{m_{s,t} \text{ terms}} \quad \text{if } m_{s,t} \neq \infty\rangle \tag{2.1}$$

An Artin-Tits monoid is clearly homogeneous, has the left and right cancellation property (see Michel, Proposition 2.4 of [17]) and has the least common multiple property (see Brieskorn and Saito, Proposition 4.1 of [7]). So in this case also our main Theorem and its corollary apply.

The *Coxeter group* associated to an Artin-Tits monoid is defined as the quotient of the latter by the relations $s^2 = 1$ for any $s \in S$. In other words, the Coxeter Group W is defined by the following presentation :

$$W = \langle s \in S \,|\, s^2 = 1 \text{ and } \underbrace{sts\ldots}_{m_{s,t} \text{ termes}} = \underbrace{tst\ldots}_{m_{s,t} \text{ terms}} \quad \text{if } m_{s,t} \neq \infty\rangle.$$

An Artin-Tits monoids is called *spherical* if and only if its Coxeter group is finite. For example, the only trace monoids that are spherical are the ones whose every elements commute. More generally, every subset of generators of a spherical Artin-Tits monoid admit a lcm. In this case the set $\mathcal{J}$ of Theorem 2 and of Corollary 3 is naturally the set of all subsets of S.

2.3 Garside monoids

In [9], Dehornoy and Paris generalize spherical Artin-Tits groups as follows:

Definition 9. *A* Garside monoid *is an atomic left cancellative monoid M, such that any two elements have left and right lcm. We require besides that M admits a* Garside element Δ*: this means an element whose sets of left and right divisors coincide, and such that this set is finite and generates M.*

A Garside monoid fitted with the set S of its atoms satisfies the conditions of the main theorem. Furthermore, as for spherical Artin-Tits monoids, all subsets of atoms of a Garside monoids have a lcm and so the set $\mathcal{J}$ is the set of every subsets of S.

3 Spherical Artin-Tits monoids

We study in this section the combinatorics of the classical braid monoid introduced by Artin and of some of its generalizations, namely the classical braid monoids of types B and D. All these monoids are spherical Artin-Tits monoids and hence some Garside monoids.

3.1 Coxeter groups

Before going further, let us just mention some points about finite Coxeter groups. A Coxeter group is said to be *irreducible* if there does not exist two disjoint subsets S_1 and S_2 of S such that $S = S_1 \cup S_2$ and such that any $s_1 \in S_1$ commutes with any $s_2 \in S_2$. The irreducible finite Coxeter groups are completely classified (see [13]). This section is devoted to the three infinite families A_n, B_n and D_n and more precisely to the corresponding Artin monoids. We compute their growth functions by applying Theorem 2; this boils down to describing how to compute lcms in such monoids.

For $X = A_n, B_n, D_n$, we write the corresponding growth function of the Artin-Tits monoid $G_X(t) = \frac{1}{H_X(t)}$, where H_X is the polynomial $\sum_J (-1)^{|J|} t^{m_J}$, in which the sum is over all subsets J of generators and m_J is the length of the lcm M_J of J. We describe in the following such lcms.

3.2 Type A

The Artin monoid $\mathcal{A}(A_n)$ is in fact the classical braid monoid on $n+1$ strands. Hence, it admits the following presentation:

$$\mathcal{A}(A_n) = \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \leq 2\rangle.$$

We denote $\Sigma_n = \{\sigma_1, \ldots, \sigma_n\}$ the set of Artin generators. To compute the lcm of a subset J of Σ_n, let us consider a partition $J = J_1 \cup \cdots \cup J_p$ such that any σ_i and σ_j in J belong to the same block of this partition if and only if $j = i \pm 1$.

We set $\Delta_{\{\sigma_j, \sigma_{j+1}, \ldots, \sigma_{j+i}\}} = (\sigma_j)(\sigma_{j+1}\sigma_j)\ldots(\sigma_{j+i}\ldots\sigma_{j+1}\sigma_j)$, then M_J is equal to $\Delta_{J_1}\ldots\Delta_{J_p}$ and $m_J = \sum_{i=1}^{p}(|J_i|(|J_i|+1)/2)$.

In this case, no explicit formula is known for H_{A_n} but the form of the lcms leads easily to the following recurrence relation:

$$H_{A_n}(t) = \sum_{i=1}^{n}(-1)^{i+1}t^{i(i-1)/2}H_{A_{n-i}}(t) + (-1)^n t^{n(n+1)/2}.$$

3.3 Type B

The Artin monoid $\mathcal{A}(B_n)$ of type B is the monoid whose set of generators is $\Sigma_n = \{\sigma_1, \ldots, \sigma_n\}$ and which is submitted to the following relations:

$$\sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_2\sigma_1\sigma_2\sigma_1, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \text{ for } i \geq 2 \quad \text{and} \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \leq 2.$$

The elements of this monoid are classically represented as positive braids whose second strand is not braided.

Similarly to Paragraph 3.2, for $J \subset \{\sigma_1, \ldots, \sigma_n\}$, we write $J = J_1 \cup \ldots \cup J_p$, where the properties satisfied by this partition are the same as those given above. Because of the particular role of σ_1, three different cases have to be considered to compute the lcm of J. Either $\sigma_1 \notin J$ or $\sigma_1 \in J$ and $\sigma_2 \notin J$ and in these cases $M_J = \Delta_{J_1} \ldots \Delta_{J_p}$ just as before. Now if $\sigma_1, \sigma_2 \in J$, without loss of generality we assume that $\sigma_1 \in J_1$, then $M_J = \widetilde{\Delta_{J_1}} \Delta_{J_2} \ldots \Delta_{J_p}$, where $\widetilde{\Delta_{J_1}} = (\sigma_1\sigma_2 \ldots \sigma_m)^{|J_1|}$ with σ_m the maximal element of J_1 for the classical ordering $\sigma_1 < \sigma_2 < \ldots < \sigma_n$ of Σ_n.

The expression of lcms enable to obtain the following recurrence relation for H_{B_n}, for $n \geq 1$ (with the convention $H_{B_0}(t) = 1$):

$$H_{B_n}(t) = \sum_{i=1}^{n} (-1)^{i+1} t^{i(i-1)/2} H_{B_{n-i}}(t) + (-1)^n t^{(n)^2}.$$

3.4 Type D

The Artin monoid $\mathcal{A}(D_n)$ of type D is the monoid whose set of generators is $\Sigma_n = \{\tau, \sigma_1, \ldots, \sigma_{n-1}\}$ and submitted to the following relations:

$$\tau\sigma_2\tau = \sigma_2\tau\sigma_2, \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i \geq 2, \tag{3.1}$$
$$\tau\sigma_i = \sigma_i\tau \text{ for } i \neq 2 \qquad \text{and} \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \leq 2. \tag{3.2}$$

In [1], Allcock introduced a representation in terms of braids on some orbifolds of the elements of this monoid.

Let $J \subset \Sigma_n$, because of the symmetric role of τ and σ_1 we have to study two cases depending on either at most one of them belongs to J or both of them. Without loss of generality, we assume that only σ_1 belongs to J, then $M_J = \Delta_{J_1} \ldots \Delta_{J_p}$, where the J_i as defined in Paragraph 3.2 (it suffices to replace each occurrence of σ_1 in M_J by τ to deal with the symmetric case). If τ and σ_1 belong both to J, we moreover assume that $\sigma_1 \in J_1$, then $M_J = \widetilde{\Delta_{J_1}} \Delta_{J_2} \ldots \Delta_{J_p}$, where $\widetilde{\Delta_{J_1}} = (\tau\sigma_1\sigma_2 \ldots \sigma_m)^{|J_1|}$ with σ_m the maximal element of J_1 for the classical ordering $\sigma_1 < \sigma_2 < \ldots < \sigma_n$ of Σ_n.

Once again, this leads to the following recurrence relation for the denominator of the generating function of $\mathcal{A}(D_n)$, for $n \geq 2$ (by convention $H_{B_0}(t) = 1$ and $H_{B_1}(t) = 1$):

$$H_{D_n}(t) = \sum_{i=1}^{n-1} (-1)^{i+1} t^{i(i-1)/2} H_{D_{n-i}}(t) + (-1)^{n-1} 2t^{(n)(n-1)/2} + (-1)^n t^{(n)(n-1)}.$$

4 Dual braid monoids

4.1 Definition

We defined Coxeter systems in paragraph 2.2. Let T be the set of reflections of W, i.e. the set $T = \{wsw^{-1}, s \in \mathcal{S}\}$; T is obviously a generating set for W, and we let $\ell_T(w) = k$ where k is the minimal number of reflections $t_i \in T$ such that $w = t_1 \cdot t_k$; the function ℓ_T is then invariant under conjugation, that is we have $\ell_T(w) = \ell_T(zwz^{-1})$ for any two elements $w, z \in W$. Then one defines a partial order on W by setting $w \leq_T z$ if $\ell_T(w) + \ell_T(w^{-1}z) = \ell_T(z)$. A Coxeter element is an element c of W which is the product of the Coxeter generators S in any order; one can show that any two Coxeter elements are conjugate in W. Given a Coxeter element $c \in W$, one defines a poset $NC(W,c) = [1,c]$ with respect to the partial order $\leq_T$. Since ℓ_T is invariant under conjugation and any two Coxeter elements are conjugate, we have that the isomorphism type of $NC(W,c)$ does not depend on the particular c chosen, and we will just write $NC(W)$. We refer to [2] and the references therein for more information about this topic.

Bessis [5] showed that one can define a certain *dual braid monoid* for every poset, with generating set in bijection with T, which is a Garside monoid such that the lattice of elements dividing the Garside element is isomorphic to the lattice $NC(W)$. As shown in Section 1.3, we need only this lattice to compute the growth function of the monoid. We refer the reader to [5] for the general definition of the monoid, and to [18] for explicit presentations in classical types.

Note that the values $\sum_{rk(x)=k} \mu(x)$ of the Möbius functions of the posets $NC(W)$ have already been computed in general, so by the results of Subsection 1.3, all growth functions of the dual braid monoids can be obtained. What we will do here is to find first a combinatorial proof of this result in type A and B, and then verify that the resolution (1.9) we obtain shows that the corresponding algebras of the corresponding dual braid monoids are in fact Koszul algebras (Paragraph 4.5). The combinatorial objects that we will deal with are *noncrossing alternating forests*, which we now study.

4.2 Noncrossing alternating forests and unary binary trees

Consider n points aligned horizontally, labeled $1,2,\dots,n$ from left to right. We identify pairs pairs (i,j), $i<j$, with arcs joining i and j above the horizontal line. Two arcs (i,j) and (k,l) are *crossing* if $i<k<j<l$ or $k<i<l<j$.

Definition 10. *A* noncrossing alternating forest *on n points is a set of noncrossing arcs on $[\![1,n]\!]$ such that at every vertex i, all the arcs are going in the same direction (to the right or to the left).*

It is easily seen that these conditions determine forests in the graph-theoretical sense, that is the arcs cannot form a cycle.We define $\mathcal{NCAF}(n,k)$ as the set of noncrossing alternating forests on n points with k arcs, and in this subsection we will determine bijectively their cardinality denoted $NCAF(n,k)$.

We will actually define a bijection with unary binary trees, by which we mean rooted plane trees all of whose vertices have $0,1$ or 2 sons. It is well known that such trees with m vertices are counted by the Motzkin number M_{m-1} (cf. [22]) and that they are in bijection with Motzkin paths with $m-1$ steps: these are paths in $\mathbb{N}^2$ from $(0,0)$ to $(m-1,0)$, with allowed steps $(1,1)$, $(1,0)$ and $(1,-1)$. The bijection consists of a prefix traversal of the tree, as shown by the dotted line around the tree on the left of Figure 1; for every left son (respectively right son, resp. single son) encountered for the first time, we draw an up step (resp. a down step, resp. a horizontal step). Under this bijection, unary vertices correspond to horizontal steps; by the cyclic lemma, it is then easy to show that:

Proposition 11. *The number of unary binary trees with m vertices and p binary vertices is given by*

$$\frac{1}{m}\binom{m}{m-2p-1,p,p+1} = \frac{(m-1)!}{(m-2p-1)!p!(p+1)!}$$

Suppose we have just one connected component in a noncrossing alternating forest, i.e $k=n-1$: we obtain the noncrossing alternating trees introduced in [12], where a bijection with binary trees with n leaves was given. We recall this bijection: given a noncrossing alternating tree on $n \geq 2$ points, there is necessarily an edge between 1 and n. Destroying that arc, we get two smaller noncrossing alternating

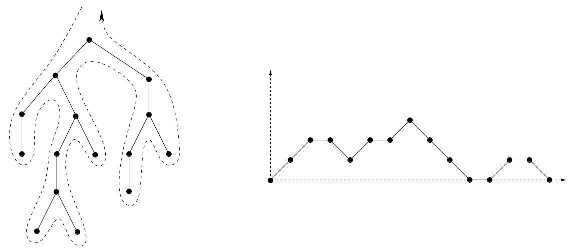

Fig. 1: A unary binary tree and the corresponding Motzkin path.

trees, on i and $n-i$ points say. By induction, we can attach a binary tree to each of these smaller trees; let T_1 and T_2 be these two trees respectively, and create a new root (corresponding to the deleted arc) with left subtree T_1 and right subtree T_2. The inverse bijection is immediate.

We can generalize this bijection as follows:

Theorem 12. *There is a bijection between unary binary trees with $n+k-1$ vertices and k binary vertices, and noncrossing alternating forests on n points with k arcs.*

Proof. Let us be given a noncrossing alternating forest on n points with k arcs; for each of the $n-k$ components, we apply the bijection for noncrossing trees described above, keeping the labels on the leaves. So we have a collection C of binary trees, such that each integer $[\![1,n]\!]$ appears exactly once as the label of a leaf. Let T be the tree containing the label 1, and let m be such that $1,\ldots,m$ label leaves of T, but $m+1$ does not; let T' be the tree containing the label $m+1$. We then form a new tree T_1 by transforming the leaf labeled m in a unary vertex (still labeled m), whose attached subtree is T'. We now remove T and T' from C and replace them by T_1; we can now repeat the same operation, and we do it until C has just one element, which is a unary binary tree with $n-k-1$ unary vertices.

Conversely, given a unary binary tree with $k-1$ unary vertices and n leaves, we make a prefix traversal of the tree, and we label only unary vertices and leaves (thus leaving binary vertices unlabeled). Then we cut every edge stemming from a unary vertex, which gives us a forest of k binary trees labeled on leaves: we apply to each of them the bijection for noncrossing trees (using as point set the labels of the leaves), thereby obtaining the desired noncrossing forest. $\square$

The bijection is illustrated on Figure 2, in which $n=10$ and $k=5$. From Proposition 11, we have the immediate corollary:

Corollary 13. *The number of noncrossing alternating forests on n points with k arcs is given by*

$$NCAF(n,k) = \frac{(n-1+k)!}{(n-1-k)!k!(k+1)!}$$

4.3 Type A

In type A, the poset $NC(W)$ is isomorphic to the noncrossing partition lattice $NC^A(n)$, which we describe. A set partition of $[n]$ is *noncrossing* if it does not have two blocks B, B' and elements $i,j \in B$ and $k,l \in B'$ such that $i<k<j<l$. Let $NC^A(n)$ be the poset of noncrossing partitions of size n ordered by refinement.

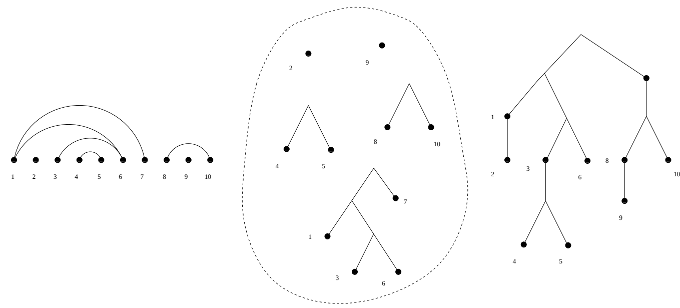

Fig. 2: Bijection between unary binary trees and noncrossing alternating forests

We now need to compute joins of cliques in this poset; we will use here a certain order on atoms to restrict to certain order compatible cliques (see Section 1). The atoms of $NC^A(n)$ are the partitions with one block of size 2 and all other blocks are singletons, and we identify these atoms with arcs (i, j) between the points labeled i and j if n points horizontally aligned and labelled from 1 to n are given. Now we define the following order on atoms:$(i, j) < (k, l)$ if $l - k > j - i$, or if $l - k = j - i$ and $i < k$; the important point is that if an arc contains another arc, then it is bigger.

Consider a clique of size two $\{(i, j), (k, l)\}$. If $i < k \leq j < l$, then the join of these elements is the partition with one non-singleton block $\{i, j, k, l\}$; but (i, l) is smaller in the poset than this partition, and bigger than both (i, j) and (k, l) for the order $<$, so the clique cannot be OC. Now it can be shown that all other size 2 cliques are OC, and that OC cliques of size k are precisely the elements of $\mathcal{NCAF}(n, k)$; the join of such an OC-clique is simply the partition whose blocks are the labels of each tree in the forest. For the element of $\mathcal{NCAF}(10, 2)$ on the left of Figure 2, the noncrossing partition has blocks $\{1, 3, 6, 7\}, \{2\}, \{4, 5\}, \{8, 10\}$ and $\{9\}$.

From this, Proposition 11 and 11 we have that the growth function of the dual braid monoid of type A is given by

$$G_A(t) = \left(\sum_{k=0}^{n} (-1)^k \frac{(n-1+k)!}{(n-1-k)!k!(k+1)!} \right)^{-1}$$

This answers a conjecture of Krammer [15, Exercise 17.37].

4.4 Type B

Here the poset $NC(W)$ is isomorphic to the type B noncrossing partitions $NC^B(n)$, which is defined as the subposet of $NC^A(2n)$ formed by partitions of $\{1, 2, \cdots, n, -1, -2, \cdots, -n\}$ that are invariant under the bijection $i \mapsto -i$. We note $((i_1, \ldots, i_t))$ the partition with non singleton blocks $\{i_1, \ldots, i_t\}$ and $\{-i_1, \ldots, -i_t\}$. There are n^2 atoms in the poset $NC^B(n)$: n with exactly one non singleton block $[i] := \{i, -i\}$, and $n(n-1)$ of the type $((i, j))$ and $((i, -j))$ where $1 \leq i < j \leq n$. Consider now as before n labeled points aligned horizontally: we identify the atoms $[i]$ with the points, and $((i, j))$ and $((i, -j))$ with arcs between i and j to which we assign respectively a positive and a negative sign.

Now we consider any linear order that extends the following partial order defined by Blass and Sagan [6]: an atom –identified with a positive or negative arc, or a negative vertex– is bigger than another if it strictly contains it, and a positive arc is bigger than the same arc with negative sign. By extending the analysis of [6] which focused on the top element $\{1, 2, \cdots, n, -1, -2, \cdots, -n\}$, we can show that the OC cliques of size k can be constructed in two ways:

- Pick an element of $\mathcal{NCAF}(n,k)$; then either choose any of the k arcs and assign a negative sign to this arc and all arcs above it, or assign all arcs positive signs.
- Pick an element of $\mathcal{NCAF}(n,k-1)$, either choose any of the $k-1$ arcs and assign both a positive and a negative sign to it, or choose any of the n points and mark it negatively. In both cases, assign a negative sign to all arcs that contain the chosen arc or point and assign a positive sign to all other arcs.

In both cases one checks that the corresponding join of atoms is of rank k exactly in the poset. From their description above one has immediately that there are $(k+1)NCAF(n,k)+(n+k-1)NCAF(n,k-1)$ OC-cliques of size k, so we get that the growth function $G_B(t)$ for the dual braid monoid of type B is given by

$$G_B(t) = \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k-1}{k} t^k \right)^{-1}$$

Remark: for W of type D_n, the poset $NC(W)$ is isomorphic to the type D noncrossing partitions $NC^D(n)$ defined in [4]; we did not find a similar order on atoms as described in types A and B in order to compute the growth function. Note that the order described in [6] cannot be used, since it is applied to a certain poset of [20] that has been since shown to be different from the poset $NC(D_n)$.

4.5 Koszul algebras

Let A be a finitely generated graded algebra $A = \oplus_{i\geq 0} A_i$,of the form $A = \mathbb{Z} < x_1, \ldots, x_k > /I$ for an homogeneous ideal I, . A is said to be a *Koszul algebra* if $\mathbb{Z}$ admits a free resolution of A-modules, such that the matrices of all linear maps in the resolution have coefficients in A_1 (the resolution is then called *linear*) [19, 11].

Now, given a homogeneous monoid M with atoms S verifying the conditions of Section 1, the algebra $\mathbb{Z}M$ is graded. In the resolutions (1.7) and (1.9), the entries of the matrices are (up to sign) the elements $\delta^{s_i}_{J_i}$, which are the elements x in M such that $M_{J-\{s_i\}}x = M_J$, and the component A_1 of the algebra is $\mathbb{Z}\mathcal{S}$. For the orders on atoms defined for dual monoids in type A and B, our analysis of OC cliques J show that $\delta^{s_i}_{J_i} = s_i$: indeed we showed that such cliques have joins of rank k in the poset, which means that in the monoid the lcm is of length $|J|$ precisely. The resolution (1.9) is thus linear, and we have:

Theorem 14. *The monoid algebras of the dual braid monoids of type A and type B are Koszul algebras.*

By the general theory of Koszul algebras, they possess graded dual algebras called *Koszul duals*, whose homogeneous components have the dimensions of the modules $\widetilde{C}_i$ in a linear resolution; in type A for instance, we have that this dual algebra is finite dimensional, and has a basis given by noncrossing alternating forests, the number of arcs determining the grading. It would be interesting to investigate the structure of these algebras, and generalize this to all finite Coxeter groups.

A promising way is certainly to investigate the descending chains for the EL-labeling of $NC(W)$ defined in [3] and relate them to the OC cliques we described in type A and B: we can prove for instance that they are identical in type A, but differ in type B.

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Universal cycles for permutation classes

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Abstract. We define a universal cycle for a class of n-permutations as a cyclic word in which each element of the class occurs exactly once as an n-factor. We give a general result for cyclically closed classes, and then survey the situation when the class is defined as the avoidance class of a set of permutations of length 3, or of a set of permutations of mixed lengths 3 and 4.

Résumé. Nous définissons un cycle universel pour une classe de n-permutations comme un mot cyclique dans lequel chaque élément de la classe apparaît une unique fois comme n-facteur. Nous donnons un resultat général pour les classes cycliquement closes, et détaillons la situation où la classe de permutations est définie par motifs exclus, avec des motifs de taille 3, ou bien à la fois des motifs de taille 3 et de taille 4.

Keywords: Restricted permutations. Universal cycles. Eulerian circuits. DeBruijn graphs.

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1 Definitions and initial observations

Given a family C of combinatorial objects that can be represented as sequences of length n, a universal cycle for such a family is a sequence, whose length n factors (read cyclically) represent all the elements of C without repetition. De Bruijn sequences ([3]) are the most well known such universal cycles, but their study was extended to other combinatorial families by Chung, Diaconis and Graham in [2]. Among the classes they considered was the set of all permutations of an n set. In the present work, we consider a notion of universal cycles for permutation pattern classes.

To each sequence $s = a_1, a_2, \ldots, a_k$ of distinct values, we associate a permutation $\tau \in S_k$ called its pattern or type, by choosing $\tau(i) < \tau(j)$ if and only if $a_i < a_j$. We will use the notation $\tau = \mathrm{pat}(s)$ to express this relationship. Our interest is in observing the presence or absence of patterns in longer permutations; thus, regarding a permutation $\pi \in S_n$ as a sequence of n elements, $\pi(1), \pi(2), \ldots, \pi(n)$, we will say that π *contains* τ if there is a selection of indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\mathrm{pat}(\pi(i_1), \pi(i_2), \ldots, \pi(i_k)) = \tau$. If π does not contain τ then it is said to *avoid* τ.

We will write $\mathrm{Av}(\tau)$ for the *avoidance class* of all permutations (of any length) which avoid τ, and use the obvious notation $\mathrm{Av}_n(\tau) = \mathrm{Av}(\tau) \cap S_n$. We extend this to multiple restrictions: $\mathrm{Av}(\tau_1, \tau_2, \ldots, \tau_r) = \mathrm{Av}(\tau_1) \cap \mathrm{Av}(\tau_2) \cap \ldots \cap \mathrm{Av}(\tau_r)$.

For example the permutation $\pi = 2314$ contains 123, as one of its subsequences is 234 and $\mathrm{pat}(234) = 123$. But 2314 avoids both 132 and 312, so we will say that $\pi \in \mathrm{Av}_4(132, 312)$.

Let $\mathcal{A} = \mathrm{Av}(\tau_1, \ldots, \tau_r)$ be an avoidance class, and let m be the size of $\mathcal{A}_n$. Then a sequence of integers $c_1, c_2, \ldots, c_m$ is a *universal cycle* for $\mathcal{A}_n$ if each of the m substrings $c_j, c_{j+1}, \ldots, c_{j+n-1}$ (taking the subscripts modulo m) has a distinct element of $\mathcal{A}_n$ as its pattern.

For instance, the sequence $1, 6, 7, 8, 4, 3, 2, 5$ is a universal cycle for the class $\mathrm{Av}_4(132, 312)$, as can be verified by taking each of the 8 substrings $1678, 6784, \ldots, 2516, 5167$, reducing each one to a permutation, and checking that these are exactly the eight permutations of length four which avoid both 132 and 312.

Suppose that such a universal cycle $c_1 \ldots c_m$ contains, in order, substrings of patterns $\pi_1, \ldots, \pi_m$. Then it is evident that a permutation $\pi_j = a_1, a_2, \ldots, a_n$ cannot be followed by an arbitary permutation $\pi_{j+1} = b_1, b_2, \ldots, b_n$. Specifically, the overlapping parts of the two permutations, $a_2, \ldots, a_n$ and $b_1, \ldots, b_{n-1}$, must have the same pattern, say $\sigma \in S_{n-1}$.

This suggests that a useful tool for constructing universal cycles would be a deBruijn-type graph, in which a vertex corresponding to the overlapping part, σ, would be visited by an incoming edge π_j and an outgoing edge π_{j+1}. A universal cycle $c_1, \ldots, c_m$ of $\mathcal{A}_n$ would thus trace an Eulerian circuit in a directed graph $G(\mathcal{A}_n)$ containing m edges, this graph being a subgraph of $G(S_n)$, which is a regular graph on $(n-1)!$ vertices, each having indegree n and outdegree n, for a total of $n!$ directed edges.

The graph $G(S_4)$ is depicted in figure 1. The vertices are labelled with the six elements of S_3. The letter labelling each edge can be read in conjunction with the label of its trailing vertex to produce a 4-permutation by appending the letter to the vertex label and then taking the pattern of the resulting 4-sequence, using the order $a < 1 < b < 2 < c < 3 < d$. For instance the edge labelled c from 123 to 132 corresponds to the 4-permutation 1243, whose 3-element prefix has pattern 123, and whose 3-element suffix has pattern 132.

As is well known, the necessary and sufficient conditions for the existence of an Eulerian circuit in a graph are that the graph be connected and that the indegrees and outdegrees match at each vertex, making it easy in general to test for the existence of an Eulerian circuit. However, the existence of Eulerian circuits does not guarantee that a universal cycle can be constructed. For instance, the deBruijn

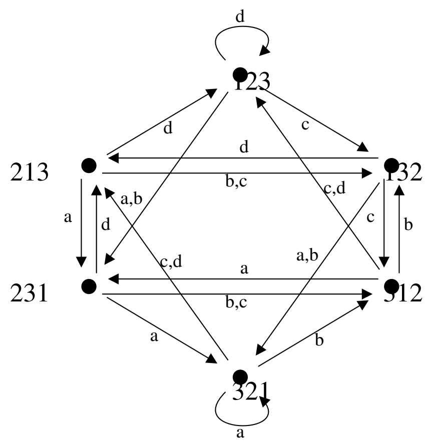

Fig. 1: The directed graph $G(S_4)$.

graph for $\mathrm{Av}_4(132, 213)$ (figure 6) has an Eulerian circuit which visits the directed edges in the sequence 1234,2341,3412,4231,3421,4321,4312,4123. However, if this were to correspond to a universal cycle $c_1, c_2, \ldots, c_8$, the second, fifth and seventh edges in the list would, respectively, force $c_2 > c_5$, $c_5 > c_8$ and $c_8 > c_2$, a contradiction. In fact, no Eulerian tour of the graph for $\mathrm{Av}_4(132, 213)$ is realizable with a sequence of values.

We will distinguish these two cases by saying that the class $\mathrm{Av}_4(132, 312)$ is *value cyclic*, but that $\mathrm{Av}_4(132, 213)$ is merely *pattern cyclic*, i.e. that its associated directed graph is Eulerian. An Eulerian cycle in the graph of a pattern cyclic class can be realized by a sequence of values if and only if the order relations implied by the individual edges form a directed acyclic graph, and thus can be extended to a partial order, as then any extension to a total order will provide a realisation of a universal cycle.

In [2], it is shown that S_n (the avoidance class of the empty set) is value cyclic for all n, and indeed conjectured that a universal cycle can always be constructed using just $n + 1$ different values; this conjecture was proven in [9].

It is instructive to consider the graphs $G(S_n)$ and $G(S_{n+1})$ together. The first of these graphs has vertex set S_{n-1} and edge set S_n, while the second has vertex set S_n and edge set S_{n+1}. Now let $\pi = a_1, a_2, \ldots, a_{n+1}$ be any permutation belonging to S_{n+1} and let $\rho_1 = \mathrm{pat}(a_1, \ldots, a_n)$ and $\rho_2 = \mathrm{pat}(a_2, \ldots, a_{n+1})$ be the pattern types of its n-prefix and n-suffix respectively.

Finally let $\sigma = \mathrm{pat}(a_2, \ldots, a_n)$, from which it is apparent that σ is simultaneously the pattern type for the $(n-1)$-suffix of ρ_1 and for the $(n-1)$-prefix of ρ_2. In graph terms, this means that π is an edge leading from ρ_1 to ρ_2 in $G(S_{n+1})$, while σ is a vertex in $G(S_n)$ which has ρ_1 leading in and ρ_2 leading out. The option in $G(S_n)$ to follow the edge ρ_1 by ρ_2 thus corresponds to an option in $G(S_{n+1})$ to move from the vertex ρ_1 to ρ_2; specifically, any Eulerian circuit of $G(S_n)$ corresponds exactly to a Hamiltonian tour of $G(S_{n+1})$.

For a given set $T = \tau_1, \tau_2, \ldots, \tau_r$ of forbidden patterns, we will be interested in two questions. Is it true that $S_n(T)$ is pattern cyclic for all values of n? Is it true that $S_n(T)$ is value cyclic for all values of n?

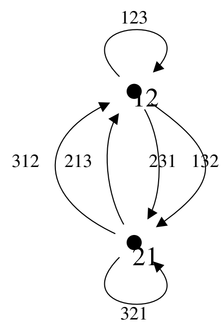

Fig. 2: The directed graph $G(S_3)$.

We begin by settling the pattern-cyclic question in the affirmative for a special infinite set of avoidance classes. We will say that a set of permutations is cyclically closed if, given any permutation in the set, its head-to-tail shift is also in the set. Thus, for instance, $1243, 2431, 4312, 3124$ is a cyclically closed set.

Proposition 1 *If the set of forbidden patterns T is cyclically closed, and $G(\mathrm{Av}_n(T))$ is connected, then $\mathrm{Av}_n(T)$ is pattern cyclic.*

Proof: First note that if T is cyclically closed, then $\mathrm{Av}(T)$ is also cyclically closed. Then note that $\pi = x, a_1, a_2, \ldots, a_n$ belongs to $\mathrm{Av}_{n+1}(T)$ if and only if $\sigma = a_1, a_2, \ldots, a_n, x$ does as well. This means that $a_1, a_2, \ldots, a_n$, as a vertex in $G(\mathrm{Av}_n(T))$, has the same number of incoming edges as outgoing edges. Together with the connectedness condition, this establishes that $G(\mathrm{Av}_n(T))$ is Eulerian. □

2 Classes defined by restrictions of length 3

For each n, $\mathrm{Av}_n(12)$ (and, symmetrically, $\mathrm{Av}_n(21)$) contains only a single permutation, so the first non-trivial restrictions are of length 3. We therefore begin by considering all possible sets of restrictions comprised of permutations from S_3.

Consider $T \subset S_3$. The pattern-cyclic question asks whether $G(\mathrm{Av}_n(T))$ has a universal cycle for all $n \geq 3$; in order to be able to settle this in the affirmative, we must, to begin, have a universal cycle for $n = 3$, and therefore it is necessary that $G(\mathrm{Av}_3(T))$ be Eulerian. The graph of $G(S_3)$, given in Figure 2 allows us to identify possible sets of edges whose removal would leave the graph connected and Eulerian. There are certain symmetries which allow us to reduce the number of cases we need to consider. In particular, the reversal of a class $\mathrm{Av}_n(T)$ in which each element of $\tau = a_1, \ldots, a_k$ of T is replaced by a $\tau^r = a_k, \ldots, a_1$ is pattern or value cyclic exactly $\mathrm{Av}_(T)$ is, because it suffices simply to reverse any given cycle, and a similar property holds for complements $\tau^c = k+1-a_1, \ldots, k+1-a_k$. These two symmetries are visible in the symmetric construction of Figure 1, in which reflection in a horizontal mirror corresponds to complementation, while reflection in a vertical mirror corresponds to reverse complemenent, and rotation through 180 degrees corresponds to reversal. However, the third operation which preserves the enumeration of pattern classes, inverse, does not preserve cyclic properties; for instance, $\mathrm{Av}_3(132, 312)$ is value cyclic, but $\mathrm{Av}_3(132, 231)$ is not even pattern cyclic.

Up to symmetry, there are six non-empty cases, which are displayed in the table below. Recall that the case for the empty set was shown to be value-cyclic in [2]. Also, we don't consider any case which involves both loops, as avoiding both 123 and 321 leaves empty permutation classes for large enough n, by a famous theorem of Erdős and Szekeres[5].

Class	Size of Class	Properties
$\mathrm{Av}_n(123)$	$C_n = (2n)!/n!(n+1)!$	not cyclic: {c,d}213{a,b,c}
$\mathrm{Av}_n(132, 312)$	2^{n-1}	value cyclic
$\mathrm{Av}_n(132, 213)$	2^{n-1}	pattern cyclic
$\mathrm{Av}_n(132, 312, 123)$	n	pattern cyclic
$\mathrm{Av}_n(132, 213, 123)$	Fibonacci	not cyclic: {d}231{a,b}
$\mathrm{Av}_n(132, 213, 321)$	n	value cyclic

The easiest cases to dispose of are those in which the pattern-class property fails to hold in general, as it suffices to give a counterexample. We have provided one in each case, in the form of a vertex (in each case, in $G(\mathrm{Av}_4(T))$) together with its attendant directed edges. Here, our notation is slightly different from that used in Figure 1, as we here label both incoming and outgoing edges relative to the vertex shown. Thus, in $\mathrm{Av}_4(123)$, we can prepend either a 3 or a 4 to the permutation 213, to produce 3214 and 4213 respectively, both of these being in the avoidance class. But we can add any of the three symbols 1, 2 or 3 at the end, to produce 3241, 3142 or 2143. Thus the indegrees and outdegrees fail to match, and $G(\mathrm{Av}_4(123))$ is not Eulerian. (Indeed, this example readily extends to show that $G(\mathrm{Av}_n(123))$ for $n \geq 4$: the permutation $n-2, n-3, \ldots, 1, n-1$ can be extended only by prepending an n or an $n-1$, but at the other end can be extended by appending anything from 1 to $n-1$.)

Now consider $\mathrm{Av}_n(132, 213, 321)$, which is the cyclically closed set consisting of $123\ldots n$ and its cyclic shifts. This is automatically pattern cyclic by Proposition 1, and indeed a universal cycle can be obtained from the sequence $1, 2, 3, \ldots, n$ itself, so the class is value cyclic. This is, up to symmetry, the only cyclically-closed class of size n, because for large n any non-empty class must contain either the all-increasing or the all-decreasing permutation, and thus, if cyclically closed, be a superset of $\mathrm{Av}_n(132, 213, 321)$ or of its reverse.

It is easy to check that the permutations belonging to $\mathrm{Av}_n(132, 312, 123)$ are those in which $n-1, n-2, \ldots, 2, 1$ form a decreasing subsequence (and the largest value, n, can be inserted into this in any one of n places). Each sequence in $\mathrm{Av}_{n-1}(132, 312, 123)$ has outdegree one (by adding in each case a new smallest element), except for the all-decreasing sequence, which has outdegree two, because one can add a new smallest element (creating a loop back to the same vertex) or a new largest element. And, symmetrically, each vertex has indegree one (by prepending a new largest element) except for the all-decreasing sequence, to which either n or $n-1$ can be prepended. The graph therefore consists of a single directed cycle, with a loop added at one vertex. It is easy to see that this class is not value cyclic, because a universal cycle would consist of only n values, $a_1, \ldots, a_n$, and each of the permutations, being of length n, would imposes a total order on this cycle; only for the all-descending permutation and the one which follows it, $n-1, \ldots, 1, n$, do these total orders coincide.

Proposition 2 *The class* $\mathrm{Av}(132, 312)$ *is value cyclic for all* n.

Proof: There are 2^{n-1} permutations in $\mathcal{A}_n = \mathrm{Av}_n(132, 312)$, which are constructed as follows. Assign a bit freely to each position from 2 to n. Now, beginning on the right, replace the 1s sequentially by $n, n-1, n-2, \ldots$ to form an *upper sequence*, and replace the 0s sequentially by $1, 2, 3\ldots$ to form a *lower sequence*. Arriving at the (unlabelled) first position, assign it the sole remaining value, which could be viewed as belonging to either the upper sequence or the lower sequence.

It is clear that these permutations belong to the avoidance class: in any subsequence x, y, z, the element z must be either the largest of the three elements (if it belongs to the upper sequence), or the smallest (if it belongs to the lower sequence).

Now note that if σ follows π in an Eulerian tour of $G(\mathcal{A}_n)$, then σ is obtained from π by deleting the first element and appending either a new maximal or a new minimal element. In terms of the bitstring, this means deleting the first bit and adding a new bit, either a 1 or a 0, to the end. So, if each edge were labelled not by a permutation but by its corresponding bitstring (and each vertex, therefore, labelled, by the bitstring which corresponds to the tail of each of its incoming edges and the head of each of its outgoing edges), the resulting graph would simply be the usual deBruijn graph on binary words. The graph $G(\mathcal{A}_n)$ is thus isomorphic to the usual deBruijn graph on bitstrings of length $n-1$.

Any Eulerian tour of the usual deBruijn graph thus corresponds to an Eulerian tour of $G(\mathcal{A}_n)$. (In general a deBruijn graph has a large number of Eulerian circuits, $2^{2^{n-2}-(n-1)}$. This formula is well-known and was rediscovered at least once; for an interesting history lesson see [3, 4, 6].) Moreover, any deBruijn cycle (obtained from an Eulerian circuit by writing down the new bits in order) can be converted into a universal cycle for $\mathcal{A}_n$ as follows.

The deBruijn cycle contains a unique run of $n-1$ 1s. Let $m = 2^{n-2}$, which is half the length of the sequence. Write the numbers $m, m-1, m-2, \ldots, 3, 2, 1$, in order, below the 0s in the cycle, beginning immediately after this run of 1s (and ending immediately before it). Likewise, write the numbers $m+1, m+2, \ldots, 2m-1, 2m$ in order below the 1s in the cycle, beginning immediately after the unique run of $n-1$ 0s (and ending again immediately before it).

Here is an example for $n = 5$:

```
0  0  0  0  1  1  1  1  0  0  1  0  1  1  0  1
4  3  2  1  9 10 11 12  8  7 13  6 14 15  5 16
```

Any consecutive run of n values from this (cyclic) sequence will reduce to a permutation in the class $\mathrm{Av}_n(132, 312)$. For the n values cannot include numbers corresponding to all of the $n-1$ consecutive 0s as well as 1s both before and after them; therefore those corresponding to 1s must all be (large and) increasing. Similarly those corresponding to 0s must be (small and) decreasing. As each bitstring of length $n-1$ occurs uniquely in the deBruijn cycle, each of the 2^{n-1} possible 01-codes for positions $2, 3, \ldots, n$ occurs. The bit in the first position might be either a 1 or a 0 (and is dependent on the choice of deBruijn cycle), but regardless the value at this position will be smaller than all the later values corresponding to 1s and larger than all those corresponding to 0s.

This construction shows that $\mathrm{Av}_n(132, 312)$ is value cyclic for all n. $\square$

The final entry in the table is for $\mathrm{Av}_n(132, 213)$, which is the well-known class of *layered permutations* (see [1]), those consisting of an initial increasing run containing all the largest elements in the permutation, followed by another increasing run containing all the largest remaining elements, and so on. The graph $G(\mathrm{Av}_n(132, 213))$ is regular of indegree 2 and outdegree 2, because any layered permutation on $n-1$ elements can be extended at the front end in one of two ways, by prepending a new first entry into the first layer, or by creating a new layer by prepending n, and likewise can be extended at the back end in either of two ways.

3 Classes defined by restrictions of lengths 3 and 4

In this section we look at pattern classes which are defined by avoiding at least one restriction of length 3 and at least one of length 4. In order to have a hope of being pattern cyclic for all n, such a class must first of all be pattern cycle at length 3, which means that the 3-restrictions must be of one of the forms considered in the previous section. (The table in that section may be a convenient reference.)

To these we wish to add some 4-restrictions, which we should do in such a way as to produce an Eulerian graph at length 4. To assist with this selection process, Figures 3, 4, 5, 6, 7, and 8 show the length-4 graphs for the various sets of 3-restrictions.

Here is our table for mixed groups of restrictions from S_3 and S_4, again taking advantage of symmetries and including only those for which $G(\mathrm{Av}_3(T))$ and $G(\mathrm{Av}_4(T))$ are Eulerian. We have only indicated the class sizes where Eulerian cycles exist.

Class	Size of Class	Properties
$\mathrm{Av}_n(123, 3142, 3412)$	$2^n - n$	value cyclic
$\mathrm{Av}_n(123, 3142, 2413)$		not cyclic: {c,d,e}4312{a,b}
$\mathrm{Av}_n(123, 3142, 3421, 4312)$		not cyclic: {}4213{a,c}
$\mathrm{Av}_n(123, 2143, 3412)$		not cyclic: {e}4132{a,b}
$\mathrm{Av}_n(123, 2143, 2413)$		not cyclic: {c,d,e}4312{a,b}
$\mathrm{Av}_n(123, 2143, 3421, 4312)$		not cyclic: {}4213{a}
$\mathrm{Av}_n(132, 312, 1234)$	$\binom{n}{2} + 1$	pattern cyclic
$\mathrm{Av}_n(132, 312, 3241, 2314)$	2(n-1)	pattern cyclic
$\mathrm{Av}_n(132, 312, 3241, 2314, 1234)$		not cyclic: {c}2134{}
$\mathrm{Av}_n(132, 213, 1234)$		not cyclic: {d,e}4123{a}
$\mathrm{Av}_n(132, 213, 3412, 4231)$	2(n-1)	pattern cyclic
$\mathrm{Av}_n(132, 213, 4321)$	$\binom{n}{2} + 1$	pattern cyclic
$\mathrm{Av}_n(132, 213, 3412, 4231, 1234)$		not cyclic: {e}4123{}
$\mathrm{Av}_n(132, 213, 3412, 4231, 4321)$		not cyclic: {}4312{c}
$\mathrm{Av}_n(132, 213, 123, 3412)$	n	pattern cyclic

There are no lines in the table corresponding to the two classes of size n in the previous section. The graphs of these classes, seen for $n = 4$ in Figure 5 and 8, each consist (for all n) of a single cycle. Therefore no addition of any further restrictions (of any length n) could leave these graphs Eulerian (or even connected) except for the removal of the loop. This, however, would eventually lead to empty pattern classes by the Erdős-Szekeres theorem.

The addition of the restriction 3412 to the avoidance class $\mathrm{Av}(132, 213, 123)$ leaves a similar set of graphs which are simply n-cycles (and so are not subject to further modification). The n-permutations belonging to $\mathrm{Av}_n(132, 213, 123, 3412)$ consist of the all-decreasing permutation, plus those permutations obtained from it by a single adjacent transposition. It is pattern cyclic: from the all-decreasing sequence, transition to $n, n-1, \ldots, 4, 3, 1, 2$, while from every other permutation in the class, transition by appending a new smallest element.

The class $\mathrm{Av}(132, 213, 4321)$ is interesting. As we saw in the previous section, $\mathrm{Av}(132, 213)$ is the class of layered permutations, and each has indegree 2 and outdegree 2, the two options being to extend the final layer, or to add a new layer. The addition of the restriction 4321 modifies the class by restricting

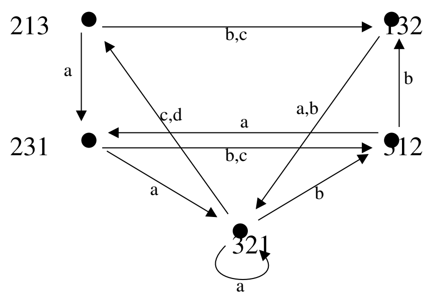

Fig. 3: The directed graph $G(\mathrm{Av}_4(123))$.

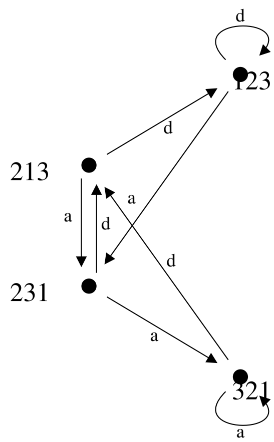

Fig. 4: The directed graph $G(\mathrm{Av}_4(132, 312))$.

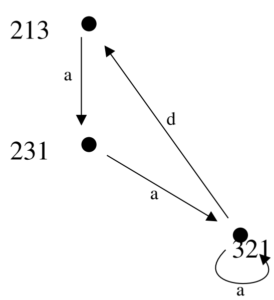

Fig. 5: The directed graph $G(\mathrm{Av}_4(132, 312, 123))$.

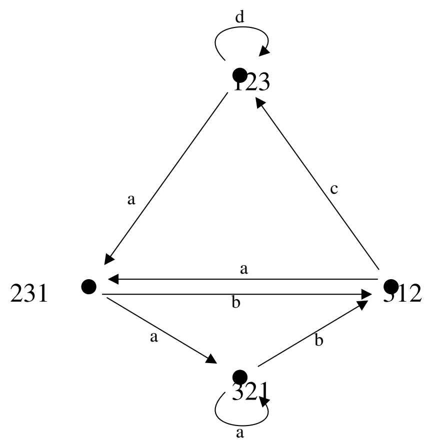

Fig. 6: The directed graph $G(\mathrm{Av}_4(132, 213))$.

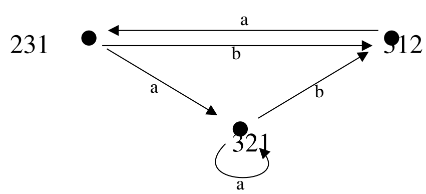

Fig. 7: The directed graph $G(\mathrm{Av}_4(132, 213, 123))$.

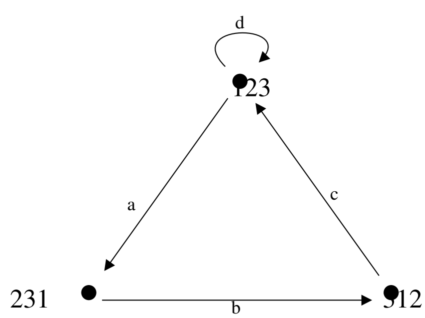

Fig. 8: The directed graph $G(\mathrm{Av}_4(132, 213, 321))$.

the number of layers to at most 3, so that in the new class a permutation with one or two layers has indegree and outdgree 2, while a permutation with three layers has outdegree only 1.

This observation generalizes to the following simple proposition, which gives us an infinite family of pattern cyclic, but non-cyclically-closed, avoidance classes.

Proposition 3 *For all $j \geq 3$, the class* $\mathrm{Av}_n(132, 213, j \ldots 321)$ *is pattern cyclic for all n.*

Proof: The class consists of layered permutations with fewer than j layers. An $(n-1)$-permutation with $j-1$ layers will have indegree and outdegree 1, while one with $j-2$ or fewer layers will have indegree and outdegree 2. □

The class $\mathrm{Av}_n(132, 213, 3412, 4231)$ is the union of two sets of permutations, those of form $(j+1, j+2, \ldots, n, j, j-1, \ldots, 1)$ for some j, and those of form $(n, n-1, \ldots, j+1, 1, 2, \ldots, j)$ for some j. Each permutation in the class has outdegree 1 – if it is of the first type, then only a new minimum element can be added, while if it is of the second type, only an element just larger than the final element can be added. Thus they are easily cycled by running through the permutations of the first kind, always adding a new smallest element, until arriving at the all-descending permutation, and then switching to the other type, adding new elements between j and $j+1$ until arriving at the all-increasing permutation. It is easy to see that this class is not value cyclic.

The class $\mathrm{Av}_n(132, 312, 3241, 2314)$ is very similar; the permutations in this class are those of the form $(j+1, j+2, \ldots, n, j, j-1, \ldots, 1)$ or of form $(j, j-1, \ldots, 1, j+1, j+2, \ldots, n)$. This class is also not value cyclic, as can be seen by examining the case for $n = 4$.

The permutations in the class $\mathrm{Av}_n(132, 312, 1234)$ are built by taking the elements $n-2, n-3, \ldots, 2, 1$ in a descending sequence, and then inserting $n-1$ and n somewhere along the sequence, with $n-1$ on the left. These thus correspond to all the 2-element subsets of n. They can be cycled by taking the large element $n-2$ off the front, putting a new small element at the end, and, when $n-1$ reaches the front, managing the separation between $n-1$ and n.

The final example is $\mathrm{Av}_n(123, 3142, 3412)$. The permutations in this class are in fact similar to those in the class $\mathrm{Av}_n(132, 312)$ studied in the previous section.

Proposition 4 *The class* $\mathrm{Av}(123, 3142, 3412)$ *is value cyclic for all n.*

Proof: Beginning with a bitstring of length n, replace all the 1s in the string with a decreasing *upper sequence*, $n, n-1, n-2, \ldots, j+1$, then replace all the 0s with a decreasing *lower sequence*, $j, j-1, \ldots, 1$. This gives a priori 2^n permutations, but there is an overcount of n because the all-decreasing permutation is constructed by all of the $n+1$ bitstrings of the special form $111\ldots11000\ldots00$. To see that the class is pattern-cyclic, observe that an ordinary vertex (i.e. with a bitstring not in the special form) has indegree 2 and outdegree 2, by prepending or appending a bit, while the special vertex $n-1, n-2, \ldots, 2, 1$ has indegree and outdegree n, because we can choose any of its n forms and append a 1 (not a 0 as this would simply loop back to the special vertex; the correct way to follow the loop is to append a 1 to the all-1s string). To express the same thing at the level of permutations, if both an upper and a lower sequence are really present, either can be extended, giving indegree and outdegree 2, while the all-descending sequence can be preceded or followed by a new element of any value.

We will show how to select an Eulerian cycle in $G(\mathrm{Av}_n(123, 3142, 3412))$ in a careful way which allows the construction of a universal cycle. The special vertex $n-1, \ldots, 1$ has n incoming edges, namely the loop $n, \ldots, 1$ (which can be coded in $n+1$ different ways as a bitstring) and the $n-1$

permutation corresponding to bitstrings of the form 01^k0^{n-1-k} for k ranging from 1 to $n-1$. Note that k cannot be zero, because this would be the all-zero bitstring, one of the possible codings for the loop. Similarly, the special vertex has n outgoing edges, the loop plus the $n-1$ edges of the form $1^k0^{n-1-k}1$ for k ranging from 0 to $n-2$. As far as the rules by which permutations may succeed permutations in Eulerian cycles, any incoming edge may be followed by any outgoing edge, but now we will insist that an incoming edge 01^k0^{n-1-k} be followed by an outgoing edge $1^k0^{n-1-k}1$ with the same value of k (in permutation terms this means that the element removed from the front of the permutation is immediately reattached at the end).

Thus every incoming edge has a corresponding outgoing edge, for k between 2 and $n-1$. This leaves the incoming 01^{n-1} and the outgoing $0^{n-1}1$, which will be matched with one another, with the loop-edge $n, \ldots, 1$ intervening. Now we will construct an Eulerian circuit, beginning with the loop-edge. Write down 0^n (one of the codes for the loop-edge), and then append a 1 to move on to edge $0^{n-1}1$, and continue to construct an Eulerian circuit, appending a bit each time, taking care that each time the special vertex is visited, the succession rule for bitstrings is followed.

This assures that the final edge in our Eulerian circuit will be the one corresponding to the bitstring 01^{n-1}. Add one more 1 at the end, so that now we have a long bitstring of length 2^n, beginning with n zeroes and ending with n ones. Replace the zeroes, from left to right, with the decreasing values $2^{n-1}, \ldots, 1$, and replace the ones, from left to right, with the decreasing values $2^n-n, \ldots, 2^{n-1}-n+1$. Noting that the n zeroes at the beginning and the n ones at the end have thus been assigned the same values, identify them to form a cycle. Using the construction, it is easy to verify that this is a universal cycle for the given pattern class. □

We observe an interesting phenomenon in our table, and in the table for length-3 permutations given in the previous section. When a class failed to be pattern cyclic for all n, we were always able to give a counterexample of the shortest possible length. This leads us to the following tentative conjecture.

Conjecture 5 *Let T be a set of patterns including some of length k and possibly some of shorter lengths. If $\mathrm{Av}_k(T)$ and $\mathrm{Av}_{k+1}(T)$ are pattern (value) cyclic, then $\mathrm{Av}_n(T)$ is pattern (value) cyclic for all n.*

Another potentially interesting question is the one settled in [9] for the case of S_n, namely for value-cyclic classes, what is the minimum number of distinct values necessary to construct a universal cycle? That is, given n, what is the least c_n such that there is a sequence of positive integers bounded above by c_n that represents the class?

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Brauer-Schur functions

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A new class of functions is studied. We define the Brauer-Schur functions $B_\lambda^{(p)}$ for a prime number p, and investigate their properties. We construct a basis for the space of symmetric functions, which consists of products of p-Brauer-Schur functions and Schur functions. We will see that the transition matrix from the natural Schur function basis has some interesting numerical properties.

Keywords: Schur function, compound basis, transition matrix

1 Introduction

Let V denote the space of polynomials with infinitely many variables:

$$V = \mathbb{Q}[t_j; j \geq 1] = \bigoplus_{n=0}^{\infty} V_n,$$

where V_n is the subspace of homogeneous polynomials of degree n, with $\deg t_j = j$. The Schur functions form a basis for V. For a partition λ of n, the Schur function $S_\lambda(t)$ indexed by λ is defined by

$$S_\lambda(t) = \sum_\rho \chi_\rho^\lambda \frac{t_1^{m_1} t_2^{m_2} \cdots}{m_1! m_2! \cdots} \quad \in V_n.$$

Here the summation runs over all partitions $\rho = (1^{m_1} 2^{m_2} \cdots)$ of n, and the integer χ_ρ^λ is the irreducible character of λ of the symmetric group S_n, evaluated at the conjugacy class ρ. The "original" (symmetric) Schur function is recovered by putting

$$t_j = \frac{1}{j}(x_1^j + x_2^j + \cdots).$$

It is known that these Schur functions are ortho-normal with respect to the inner product

$$\langle F, G \rangle = F(\partial) G(t)|_{t=0},$$

where $\partial = (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \cdots)$.

In this paper we will consider yet another basis for V, which we call the compound basis. Our new basis comes from modular representations of the symmetric group at characteristic p. We will simply replace the character χ_ρ^λ by the p-Brauer character φ_ρ^λ. It is natural that the decomposition matrices play an essential role in the argument. The aim of this note is to investigate the transition matrices between Schur function basis and our compound basis.

For $p = 2$ the compound basis was introduced in (1) in connection with the basic representation of the affine Lie algebra of type $A_1^{(1)}$. In this case Schur's Q-functions are used. However that basis cannot be defined for odd primes p. Instead we consider here the functions $B_\lambda^{(p)}(t)$, which we call the "Brauer-Schur functions".

Throughout the note, $P(n)$ always denotes the set of partitions of n, and P denotes the set of all partitions.

2 The Symmetric Functions $B_\lambda^{(p)}$

We introduce a new family of symmetric functions. It has an origin in the modular representations of the symmetric groups (6). Let p be a fixed prime number. A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ is said to be p-regular if there are no parts satisfying $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1}$. The set of p-regular partitions of n is denoted by $P^p(n)$. A partition $\rho = (1^{m_1}2^{m_2}\cdots)$ is said to be p-class regular if $m_p = m_{2p} = \cdots = 0$. The set of p-class regular partitions of n is denoted by $P_p(n)$. For example, a partition is 2-regular if it is strict, and 2-class regular if it is odd. If $p = 3$ and $n = 4$, then $P^3(4) = \{4, 31, 2^2, 21^2\}$ and $P_3(4) = \{4, 1^4, 2^2, 21^2\}$.
For $\lambda \in P^p(n)$, we define the Brauer-Schur function $B_\lambda^{(p)}(t)$ indexed by λ as follows.

$$B_\lambda^{(p)}(t) = \sum_{\rho\in P_p(n)} \varphi_\rho^\lambda \frac{t_1^{m_1}t_2^{m_2}\cdots}{m_1!m_2!\cdots} \quad \in V_n,$$

where φ_ρ^λ is the irreducible Brauer character for the symmetric group S_n of characteristic p corresponding to λ, evaluated at the conjugacy class ρ. These functions form a basis for the space $V_n^{(p)} = V^{(p)} \cap V_n$, where

$$V^{(p)} = \mathbb{Q}[t_j; j \geq 1, j \not\equiv 0 \,(\mathrm{mod}\, p)].$$

The p-decomposition matrix records the relation between ordinary irreducible characters and Brauer characters. Given a Schur function $S_\lambda(t)$, define the p-reduced Schur function $S_\lambda^{(p)}(t)$ by "killing" all variables $t_p, t_{2p}, \cdots$;

$$S_\lambda^{(p)}(t) = S_\lambda(t)|_{t_{jp}=0}.$$

By definition, the p-decomposition matrix $D_n^{(p)} = D_n = (d_{\lambda\mu})$ is given by

$$S_\lambda^{(p)}(t) = \sum_{\mu\in P^p(n)} d_{\lambda\mu} B_\mu^{(p)}(t)$$

for $\lambda \in P(n)$. It is known that the entries $d_{\lambda\mu}$ satisfies the properties; $d_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$, $d_{\lambda\mu} = 0$ unless $\mu \geq \lambda$ and $d_{\lambda\lambda} = 1$. Here " $\geq$ " denotes the dominance order.
We define an inner product $\langle\ ,\ \rangle$ on V_n by $\langle F(t), G(t)\rangle := F(\partial)G(t)|_{t=0}$, where $\partial = (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}\cdots)$. In contarst with the Schur functions which are ortho-normal with respect to this inner product, our

p-Brauer-Schur functions are not orthogonal in general. Therefore we need the dual basis for the p-Brauer-Schur functions $B_\lambda^{(p)}(t)$. To this end we introduce another symmetric functions $\widetilde{B_\lambda^{(p)}(t)}$ indexed by $\lambda \in P^p(n)$ as follows.

$$\widetilde{B_\lambda^{(p)}}(t) = \sum_{\rho \in P_p(n)} \widetilde{\varphi_\rho^\lambda} \frac{t_1^{m_1} t_2^{m_2} \cdots}{m_1! m_2! \cdots} \quad \in V_n,$$

where $\widetilde{\varphi_\rho^\lambda}$ are the entries of the matrix

$$\widetilde{\Psi}_n := {}^t D_n D_n \Psi_n$$

with $\Psi_n = (\varphi_\rho^\lambda)_{\lambda \in P^p(n), \rho \in P_p(n)}$. Then the orthogonality of the Brauer characters implies

$$\langle B_\lambda^{(p)}(t), \widetilde{B_\mu^{(p)}}(t) \rangle = \langle \widetilde{B_\lambda^{(p)}(t)}, B_\mu^{(p)}(t) \rangle = \delta_{\lambda\mu}.$$

It is known that

$$B_\lambda^{(p)}(t) = S_\lambda^{(p)}(t) = S_\lambda(t)$$

for a p-core λ, and hence, it is a homogeneous τ-function for the p-reduction of KP hierarchy (9; 4).

Here these functions are being expressed in terms of the "Sato variables" $t = (t_1, t_2, \ldots)$ appearing in the theory of soliton equations. However, for the description and the proof of our formula, it is sometimes more convenient to use the "original" variables of the symmetric functions, i.e., the "eigenvalues" $x = (x_1, x_2, \ldots)$. The variables are connected by the formula

$$t_j = \frac{1}{j}(x_1^j + x_2^j + \cdots).$$

We will denote by $B_\lambda^{(p)}(x)$ etc. when the functions are expressed in terms of variables x.

First we notice the following Cauchy identity.

Proposition 2.1

$$\sum_{\lambda \in P^p} B_\lambda^{(p)}(px) \widetilde{B_\lambda^{(p)}}(y) = \prod_{i,j} \frac{1 - x_i^p y_j^p}{(1 - x_i y_j)^p},$$

where $(px) := (\underbrace{x_1, \ldots x_1}_{p}, \underbrace{x_2, \ldots x_2}_{p}, \ldots)$.

Proof: It is known that the Schur functions form a selfdual basis for the space V with respect to the inner product $\langle\ ,\ \rangle$. Hence we have the well-known Cauchy identity

$$\sum_{\lambda \in P} S_\lambda(x) S_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \exp(\sum_{i,j} t_i {y_j}^i).$$

Our functions $B_\lambda^{(p)}(t)$ and $\widetilde{B_\lambda^{(p)}}(t)$ are dual bases for the p-reduced space $V^{(p)}$. Therefore we have

$$\sum_{\lambda \in P^p} B_\lambda^{(p)}(px) \widetilde{B_\lambda^{(p)}}(y) = \exp(\sum_{j \geq 1} \sum_{n \not\equiv 0 (mod\ p)} p t_n y_j^n).$$

The right-hand side equals

$$\begin{aligned}\exp(\sum_{j\geq 1}\sum_{n\not\equiv 0(mod\ p)} pt_n y_j^n) &= \exp(\sum_{j\geq 1}\sum_{n\geq 1} pt_n y_j^n - \sum_{j\geq 1}\sum_{n\geq 1} pt_{pn} y_j^{pn}) \\ &= \prod_{i,j}\frac{1-x_i^p y_j^p}{(1-x_i y_j)^p}.\end{aligned}$$

□

3 Compound Basis

We begin with three bijections among sets of partitions. We here remark that these bijections can be defined for any nutural number p. The first bijection is

$$\psi^{(p)} : P(n) \longrightarrow \bigcup_{n_1+pn_2=n} P^p(n_1)\times P(n_2),$$

defined by $\lambda \longmapsto (\lambda^{r(p)}, \lambda^{d(p)})$. Here the multiplicities $m_i(\lambda^{r(p)})$ and $m_i(\lambda^{d(p)})$ of $i \geq 1$ are given respectively by

$$m_i(\lambda^{r(p)}) = \begin{cases} k & \text{if} \quad m_i(\lambda)\equiv k_{\neq 0} \pmod p \\ 0 & \text{if} \quad m_i(\lambda)\equiv 0 \pmod p, \end{cases}$$

and

$$m_i(\lambda^{d(p)}) = \begin{cases} \frac{m_i(\lambda)-k}{p} & \text{if} \quad m_i(\lambda)\equiv k_{\neq 0} \pmod p \\ \frac{m_i(\lambda)}{p} & \text{if} \quad m_i(\lambda)\equiv 0 \pmod p. \end{cases}$$

For example, if $p=3$ and $\lambda = (5^4 4^6 2^{11} 1^2)$, then $\lambda^{r(3)} = (52^2 1^2)$, $\lambda^{d(3)} = (54^2 2^3)$.
In view of this bijection, we can define the function, for a prime p and $\lambda \in P(n)$,

$$B^{(p)}_{\lambda^{r(p)}}(t) S_{\lambda^{d(p)}}(t_{(p)}), \qquad t_{(p)} = (t_p, t_{2p}, t_{3p}, \cdots).$$

These functions are linearly independent, and therefore, form a basis for the space V_n. We call $\{B^{(p)}_{\lambda^{r(p)}}(t) S_{\lambda^{d(p)}}(t_{(p)}); \lambda \in P(n)\}$ the "p-compound basis" for V_n.
The second bijection reads

$$\pi^{(p)} : P(n) \longrightarrow \bigcup_{n_1+pn_2=n} P_p(n_1)\times P(n_2), \;\; \lambda \longmapsto (\lambda^{o(p)}, \lambda^{e(p)}),$$

where $\lambda^{o(p)}$ is obtained by removing all parts from λ which are multiples of p, and $\lambda^{e(p)} := (1^{m_p} 2^{m_{2p}} 3^{m_{3p}} \ldots)$ if $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \ldots)$. For example, if $p = 3$ and $\lambda = (7^4 6^3 4^5 3\, 2^6 1^2)$, then $\lambda^{o(p)} = (7^4 4^5 2^6 1^2)$, $\lambda^{e(p)} = (2^3 1)$.

The last bijection is called the Glaisher map. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a p-reguler partition. Write each part as $\lambda_i = p^{a_i} q_i$ with $(p, q_i) = 1$. Let $\mu(i)$ be the rectanguler partition of λ_i given by $\mu(i) = (q_i, \ldots, q_i)$

with length p^{a_i}. Suppose that $q_{j_1} \geq \ldots \geq q_{j_l}$. Let $\tilde{\lambda}$ be the vertical concatenation $(\mu(j_1), \ldots, \mu(j_l))$, which is p-class reguler. Then the bijection $\gamma : P^p(n) \longrightarrow P_p(n)$ is defined by $\lambda \longmapsto \tilde{\lambda}$. For example, if $p = 3$ and $\lambda = (6^2 5^3 431)$, then $\tilde{\lambda} = (5^3 42^6 1^4)$.
By composing these three bijections, we can define the map

$$\Phi^{(p)} : P(n) \longrightarrow P(n),\ \ \Phi^{(p)}(\lambda) := {\pi^{(p)}}^{-1}(\gamma \otimes id)(\psi^{(p)}(\lambda)).$$

For example, here is a table of the case $p = 3$ and $n = 6$.

$P(n)$	$\longrightarrow$	$P^p(n_1) \times P(n_2)$	$\longrightarrow$	$P_p(n_1) \times P(n_2)$	$\longrightarrow$	$P(n)$
6	$\longmapsto$	$6, \emptyset$	$\longmapsto$	$2^3, \emptyset$	$\longmapsto$	2^3
51	$\longmapsto$	$51, \emptyset$	$\longmapsto$	$51, \emptyset$	$\longmapsto$	51
42	$\longmapsto$	$42, \emptyset$	$\longmapsto$	$42, \emptyset$	$\longmapsto$	42
41^2	$\longmapsto$	$41^2, \emptyset$	$\longmapsto$	$41^2, \emptyset$	$\longmapsto$	41^2
3^2	$\longmapsto$	$3^2, \emptyset$	$\longmapsto$	$1^6, \emptyset$	$\longmapsto$	1^6
321	$\longmapsto$	$321, \emptyset$	$\longmapsto$	$21^4, \emptyset$	$\longmapsto$	21^4
$2^2 1^2$	$\longmapsto$	$2^2 1^2, \emptyset$	$\longmapsto$	$2^2 1^2, \emptyset$	$\longmapsto$	$2^2 1^2$
31^3	$\longmapsto$	$3, 1$	$\longmapsto$	$1^3, 1$	$\longmapsto$	31^3
21^4	$\longmapsto$	$21, 1$	$\longmapsto$	$21, 1$	$\longmapsto$	321
2^3	$\longmapsto$	$\emptyset, 2$	$\longmapsto$	$\emptyset, 2$	$\longmapsto$	6
1^6	$\longmapsto$	$\emptyset, 1^2$	$\longmapsto$	$\emptyset, 1^2$	$\longmapsto$	3^2

For a pair (n_1, n_2) with $n_1 + pn_2 = n$, we call the set ${\psi^{(p)}}^{-1}(P^p(n_1) \times P(n_2))$ the "(n_1, n_2)-block". Looking at the table above, we notice the following relations for lengths.

Proposition 3.1

$$\begin{aligned}(i)\quad & \sum_{\lambda \in P(n)} \ell(\lambda) = \sum_{\lambda \in P(n)} (\ell(\lambda^{r(p)}) + p\ell(\lambda^{d(p)})) = \sum_{\lambda \in P(n)} (\ell(\lambda^{o(p)}) + \ell(\lambda^{e(p)})) \\ & = \sum_{\lambda \in P(n)} (\ell(\widetilde{\lambda^{r(p)}}) + \ell(\lambda^{e(p)})), \\ (ii)\quad & \frac{\ell(\widetilde{\lambda^{r(p)}}) - \ell(\lambda^{r(p)})}{p-1} = \ell((\Phi^{(p)}(\lambda))^{d(p)}).\end{aligned}$$

4 Transittion Matrices

We investigate the transition matrix between two bases. Let $A_n^{(p)} := (a_{\lambda\mu})_{\lambda,\mu \in P(n)}$ be defined by

$$S_\lambda(t) = \sum_{\mu \in P(n)} a_{\lambda\mu} B^{(p)}_{\mu^{r(p)}}(t) S_{\mu^{d(p)}}(t_{(p)}),\ \ \lambda \in P(n).$$

We see that the transition matrix $A_n^{(p)}$ is an integral matrix, and that the determinant of $A_n^{(p)}$ has a combinatorial interpretation. The definition of $a_{\lambda\mu}$ is rewritten as

$$S_\lambda(px) = \sum_{\mu \in P(n)} a_{\lambda\mu} B^{(p)}_{\mu^{r(p)}}(px) S_{\mu^{d(p)}}(x^p),$$

where $(x^p) := (x_1^p, x_2^p, \ldots)$.

Proposition 4.1

$$\sum_{\lambda \in P} S_\lambda(px) S_\lambda(y) = \sum_{\lambda \in P} B^{(p)}_{\lambda^{r(p)}}(px) S_{\lambda^{d(p)}}(x^p) \widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) S_{\lambda^{d(p)}}(y^p).$$

Proof: By looking at the Cauchy identity for the Schur functions, we see that

$$\sum_{\lambda \in P} S_\lambda(x^p) S_\lambda(y^p) = \prod_{i,j} \frac{1}{1 - x_i^p y_j^p}.$$

Hence, from Proposition 2.1, we have

$$\begin{aligned} & \sum_{\lambda \in P} B^{(p)}_{\lambda^{r(p)}}(px) S_{\lambda^{d(p)}}(x^p) \widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) S_{\lambda^{d(p)}}(y^p) \\ = \quad & \sum_{\mu \in P^p} B^{(p)}_{\mu}(px) \widetilde{B^{(p)}_{\mu}}(y) \sum_{\nu \in P} S_\nu(x^p) S_\nu(y^p) \\ = \quad & \prod_{i,j} \frac{1 - x_i^p y_j^p}{(1 - x_i y_j)^p} \times \prod_{i,j} \frac{1}{1 - x_i^p y_j^p} \\ = \quad & \prod_{i,j} \frac{1}{(1 - x_i y_j)^p}. \end{aligned}$$

□

Theorem 4.2 *The entries $a_{\lambda\mu}$ are integers given by*

$$a_{\lambda\mu} = \langle \widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) S_{\lambda^{d(p)}}(y^p),\ S_\mu(y) \rangle.$$

Proof: We have

$$\sum_{\lambda \in P} S_\lambda(px) S_\lambda(y) = \sum_{\lambda \in P} B^{(p)}_{\lambda^{r(p)}}(px) S_{\lambda^{d(p)}}(x^p) \widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) S_{\lambda^{d(p)}}(y^p).$$

Taking the inner product $\langle\ ,\ \rangle$ with $S_\mu(y)$, we obtain

$$S_\mu(px) = \sum_{\lambda \in P} \langle \widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) S_{\lambda^{d(p)}}(y^p), S_\mu(y) \rangle\ B^{(p)}_{\lambda^{r(p)}}(px) S_{\lambda^{d(p)}}(x^p).$$

Thus we see that

$$a_{\lambda\mu} = \langle \widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) S_{\lambda^{d(p)}}(y^p), S_\mu(y) \rangle.$$

Here we use the following formula of plethysm, which is found in (3).

$$S_{\lambda^{d(p)}}(x^p) = \sum_{\mu \in P} c^{\mu}_{\lambda^{d(p)}p} S_\mu(x), \quad c^{\mu}_{\lambda p} \in \mathbb{Z}.$$

Also, by definition, we have

$$\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) = \sum_{\mu \in P(n)} d_{\mu\lambda^{r(p)}} S^{(p)}_\mu(y).$$

This shows that

$$\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) = \sum_{\mu \in P(n)} d_{\mu\lambda^{r(p)}} S_\mu(y).$$

From the orthonormality of the Schur functions, the assertion holds. □

Here we give the matrix $A^{(p)}_n$ for the case $p = 3$ and $n = 5$. Columns are labeled by $(\mu^{r(3)}, \mu^{d(3)})$.

$$A^{(3)}_5 = \begin{array}{c|ccccccc} & (5,\emptyset) & (2^21,\emptyset) & (41,\emptyset) & (32,\emptyset) & (31^2,\emptyset) & (2,1) & (1^2,1) \\ \hline (5) & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ (41) & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ (32) & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ (31^2) & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ (2^21) & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ (21^3) & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ (1^5) & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}.$$

If we expand the Schur function $S_\lambda(x)$ and the p-Brauer-Schur function $B^{(p)}_\lambda(x)$ in terms of the power sum symmetric functions, then we have

$$S_\lambda(x) = \sum_{\rho \in P(n)} Y_{\lambda,\rho} p_\rho(x), \;\; B^{(p)}_\lambda(x) = \sum_{\rho \in P_p(n)} B^{(p)}_{\lambda,\rho} p_\rho(x).$$

Put

$$Y_n := (Y_{\lambda,\rho})_{\lambda,\rho \in P(n)}, \quad B^{(p)}_n := (B^{(p)}_{\lambda,\rho})_{\lambda \in P^p(n), \rho \in P_p(n)}.$$

We will remark that

$$B^{(p)}_n = \Psi_n Z_n^{-1},$$

where $Z_n := \mathrm{diag}(z_\rho; \rho \in P_p(n))$ with $z_\rho := 1^{m_1} 2^{m_2} \cdots m_1! m_2! \cdots$ for $\rho = (1^{m_1} 2^{m_2} \cdots)$.

Now we are going to discuss determinants of the transition matrices. Let the symbol "det" mean the absolute value of the determinant. By a standard argument, we have

$$1 = (\det Y_n)^2 \prod_{\rho \in P(n)} z_\rho.$$

Here $M(W, U)$ denotes the transition matrix from the basis W to the basis U for V_n. We compute

$$\begin{aligned} \det M(S(px), B^{(p)}(px)S(x^p)) &= \det M(S(px), p(px)) \det M(p(px), B^{(p)}(px)S(x^p)) \\ &= \det M(S(x), p(x)) \det M(p(px), B^{(p)}(px)S(x^p)) \\ &= \det M(S(x), p(x)) \det M(p(px), p(x)) \det M(p(x), B^{(p)}(px)S(x^p)). \end{aligned}$$

Also, for $\lambda \in P^p(n_1), \mu \in P(n_2)$, we write

$$\begin{aligned} B_\lambda^{(p)}(px)S_\mu(x^p) &= \sum_{\rho \in P_p(n_1), \sigma \in P(n_2)} B_{\lambda,\rho}^{(p)} Y_{\mu,\sigma} p_\rho(px) p_\sigma(x^p) \\ &= \sum_{\rho \in P_p(n_1), \sigma \in P(n_2)} B_{\lambda,\rho}^{(p)} Y_{\mu,\sigma} p^{\ell(\rho)} p_\rho(x) p_{p\sigma}(x), \end{aligned}$$

where $p\sigma := (p\sigma_1, p\sigma_2, \ldots)$. This shows that the matrix

$$M(B^{(p)}(px)S(x^p), p(x))$$

is block diagonal and each block is indexed by the pair (n_1, n_2) with $n_1 + pn_2 = n$.

Proposition 4.3

$$\det M(B^{(p)}(px)S(x^p), p(x)) = \prod_{n_1 + pn_2 = n} (\det B_{n_1}^{(p)})(\det Y_{n_2})(\det L_{n_1}).$$

where $L_n = \operatorname{diag}(p^{\ell(\rho)}; \rho \in P_p(n))$.

There is a compact formula for the elementary divisors of the Cartan matrix $C_n = {}^t D_n D_n$ (10):

$$\{p^{\frac{\ell(\tilde{\lambda}) - \ell(\lambda)}{p-1}}; \lambda \in P^p(n)\}.$$

Clearly,

$$\det C_n = \prod_{\lambda \in P^p(n)} p^{\frac{\ell(\tilde{\lambda}) - \ell(\lambda)}{p-1}}.$$

Proposition 4.4

$$\prod_{\rho \in P_p(n)} z_\rho = (\det \Psi_n)^2 \times \prod_{\lambda \in P^p(n)} p^{\frac{\ell(\tilde{\lambda}) - \ell(\lambda)}{p-1}}.$$

Our main theorem involves an interesting combinatorial fact.

Theorem 4.5

$$\det A_n^{(p)} = p^T.$$

where

$$T = \sum_{\lambda \in P^p(n)} \frac{\ell(\tilde{\lambda}) - \ell(\lambda)}{p-1} = \sum_{\lambda \in P(n)} \ell(\lambda^{d(p)}),$$

which is the sum of the number of parts of multiples of p *in the partitions of* n.

Proof: We recall

$$\begin{aligned} \det A_n^{(p)} &= \det M(S(px), B^{(p)}(px)S(x^p)) \\ &= \det M(S(x), p(x)) \det M(p(px), p(x)) \det M(p(x), B^{(p)}(px)S(x^p)). \end{aligned}$$

By Propositions 4.3 and 4.4, we see that

$$\begin{aligned}
&\det M(p(x), B^{(p)}(px)S(x^p)) \\
=\ & \left(\prod_{n_1+pn_2=n} (\det B_{n_1})(\det Y_{n_2})(\det L_{n_1})\right)^{-1} \\
=\ & \prod_{n_1+pn_2=n} \left((\det \Psi_{n_1}) \left(\prod_{\sigma\in P_p(n_1)} z_\sigma^{-1}\right)(\det Y_{n_2}) \left(\prod_{\rho\in P(n_1)} p^{\ell(\rho)}\right)\right)^{-1} \\
=\ & \prod_{n_1+pn_2=n} \left(\left(\prod_{\rho\in P_p(n_1)} p^{\ell(\rho)}\right)\left(\prod_{\sigma\in P_p(n_1)} z_\sigma^{-1}\right)\right)^{-1} \\
&\times \quad \left(\left(\prod_{\rho\in P_p(n_1)} z_\rho^{-1}\right)\left(\prod_{\lambda\in P^p(n_1)} p^{\frac{\ell(\tilde\lambda)-\ell(\lambda)}{p-1}}\right)\right)^{1/2}\left(\prod_{\rho\in P(n_2)} z_\rho\right)^{1/2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\det A_n^{(p)} \quad = \quad & \prod_{\rho\in P(n)} z_\rho^{-1/2} \times \prod_{\rho\in P(n)} p^{\ell(\rho)} \\
& \times \prod_{n_1+pn_2=n} \left(\left(\prod_{\rho\in P_p(n_1)} p^{\ell(\rho)}\right)\left(\prod_{\sigma\in P_p(n_1)} z_\sigma^{-1}\right)\right)^{-1} \\
& \quad \times \quad \left(\left(\prod_{\rho\in P_p(n_1)} z_\rho^{-1}\right)\left(\prod_{\lambda\in P^p(n_1)} p^{\frac{\ell(\tilde\lambda)-\ell(\lambda)}{p-1}}\right)\right)^{1/2}\left(\prod_{\rho\in P(n_2)} z_\rho\right)^{1/2}.
\end{aligned}$$

Paying attention to the bijection $\pi^{(p)}$ and the relation

$$z_\lambda = p^{\ell_p(\lambda)} z_{\lambda^{o(p)}} z_{\lambda^{e(p)}},$$

where $\ell_p(\lambda)$ denotes the number of parts of multiples of p in the partition λ, we notice that

$$\left(\prod_{\rho\in P(n)} z_\rho^{-1/2}\right)\times\left(\prod_{n_1+pn_2=n}\prod_{\sigma\in P_p(n_1)} z_\sigma\right)\times\left(\prod_{n_1+pn_2=n}\prod_{\rho\in P_p(n_1)} z_\rho^{-1/2}\right)\times\left(\prod_{n_1+pn_2=n}\prod_{\rho\in P(n_2)} z_\rho^{1/2}\right)$$

is equal to $(p^T)^{-1/2}$. Next, we look at

$$\prod_{\rho\in P(n)} p^{\ell(\rho)} \times \prod_{n_1+pn_2=n}\left(\prod_{\sigma\in P_p(n_1)} p^{-\ell(\sigma)}\right)\times\prod_{n_1+pn_2=n}\left(\prod_{\lambda\in P^p(n_1)} p^{\frac{\ell(\tilde\lambda)-\ell(\lambda)}{p-1}}\right)^{1/2}.$$

Through the bijection $\pi^{(p)}$, we have

$$\prod_{\rho\in P(n)} p^{\ell(\rho)} \times \prod_{n_1+pn_2=n} \left(\prod_{\sigma\in P_p(n_1)} p^{-\ell(\sigma)} \right) = p^T.$$

Also, from Proposition 3.1 (ii), we obtain

$$\prod_{n_1+pn_2=n} \left(\prod_{\lambda\in P^p(n_1)} p^{\frac{\ell(\tilde{\lambda})-\ell(\lambda)}{p-1}} \right)^{1/2} = \left(\prod_{\tau\in P(n)} p^{\ell(\tau^{d(p)})} \right)^{1/2} = (p^T)^{1/2}.$$

Hence, $\det A_n^{(p)} = (p^T)^{-1/2}(p^T)(p^T)^{1/2} = p^T$. □

For example, we have $\det A_5^{(3)} = 9 = 3^2$. Here is a small list of T for $p = 3$.

n	1	2	3	4	5	6	7	$8\cdots$
T	0	0	1	1	2	5	7	$11\cdots$

We also see that the elementary divisors of $A_n^{(p)}$ coincide with

$$\{p^{\frac{\ell(\tilde{\mu})-\ell(\mu)}{p-1}}; \mu = \lambda^{r(p)}, \lambda \in P(n)\}.$$

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The Hiring Problem and Permutations

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The *hiring problem* has been recently introduced by Broder et al. in last year's ACM-SIAM Symp. on Discrete Algorithms (SODA 2008), as a simple model for decision making under uncertainty. Candidates are interviewed in a sequential fashion, each one endowed with a quality score, and decisions to hire or discard them must be taken on the fly. The goal is to maintain a good rate of hiring while improving the "average" quality of the hired staff.

We provide here an alternative formulation of the hiring problem in combinatorial terms. This combinatorial model allows us the systematic use of techniques from combinatorial analysis, e. g., generating functions, to study the problem.

Consider a permutation $\sigma : [1, \dots, n] \to [1, \dots, n]$. We process this permutation in a sequential fashion, so that at step i, we see the score or quality of candidate i, which is actually her face value $\sigma(i)$. Thus $\sigma(i)$ is the rank of candidate i; the best candidate among the n gets rank n, while the worst one gets rank 1. We define *rank-based* strategies, those that take their decisions using only the relative rank of the current candidate compared to the score of the previous candidates. For these strategies we can prove general theorems about the number of hired candidates in a permutation of length n, the time of the last hiring, and the average quality of the last hired candidate, using techniques from the area of analytic combinatorics. We apply these general results to specific strategies like hiring above the best, hiring above the median or hiring above the mth best; some of our results provide a complementary view to those of Broder et al., but on the other hand, our general results apply to a large family of hiring strategies, not just to specific cases.

Keywords: On-line decision making, secretary problem, hiring problem, permutations, generating functions, analytic combinatorics.

1 Introduction

The *hiring problem* has been recently introduced by Broder et al. (1) as a simple model for decision making under uncertainty, closely related to the well-known *secretary problem* (see, for instance (3) and the references therein). In the hiring problem, a growing company interviews and decides whether to hire applicants in a sequential manner. In its simplest formulation, the candidate that the company interviews

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at step i has a quality score Q_i, where the Q_i's are i.i.d. random variables, with common distribution Unif$(0, 1)$. Then, according to the company's hiring strategy, candidate i is either hired or discarded. The paper by Broder et al. studied two natural strategies which, on rather intuitive grounds, should lead to an increasingly improved quality of the company's staff, while maintaining some balance with the speed at which the company hires applicants: the first strategy is *hiring above the mean* and the second strategy is *hiring above the median*. As their names indicate, in *hiring above the mean* an applicant is hired if and only if her score is at least equal to the mean score of the currently hired applicants, whereas in *hiring above the median*, an applicant is hired if and only if her score is at least equal to the median score of the current employees. The paper also considered the strategies *hiring above a threshold* and *hiring above the maximum*, where candidate i is hired if and only if $Q_i \geq \tau$ for some prespecified τ, or $Q_i > \max\{Q_1, \ldots, Q_{i-1}\}$, respectively.

In this paper, we provide an alternative formulation of the hiring problem in combinatorial terms; its main virtue being that it opens the door for the application of a vast and rich array of powerful techniques coming from the combinatorial camp. We by no means claim that the model that we propose here is superior to the original model, but on the contrary, that it nicely complements the original model by providing a different point of view which may prove useful in investigating the hiring problem and its many natural extensions. In particular, the combinatorial viewpoint introduced here allows us to obtain several powerful and generic results (Theorems 1 to 3) about the number of hired candidates and other relevant parameters for large families of hiring strategies, in particular, those which base their decisions solely on the *relative* rank of a candidate compared to the ranks of previous candidates.

Consider a permutation $\sigma : [1, \ldots, n] \to [1, \ldots, n]$. We process this permutation in a sequential fashion, so that at step i, we see the score or quality of candidate i, which is actually her face value $\sigma(i)$. You may think of $\sigma(i)$ as the rank of candidate i; the best candidate among the n gets rank n, while the worst one gets rank 1. In this light, the model is very natural (see also the discussion in (1)); it's not so natural to take the face value $\sigma(i)$ as an absolute measure of the candidate's quality. For similar reasons, if the Q_i's in the original hiring model are seen as relative ranks the choice of the uniform distribution in $(0, 1)$ is perfectly justifiable, but we think that it's more debatable to see them as an absolute measure of quality; for instance, it could be more natural to assume that the Q_i's are i.i.d. normal random variables with common Gaussian distribution $\mathcal{N}(\mu, \nu^2)$.

As in the original model, at step i, we must decide then whether we hire the ith candidate or not. The decision must be made based upon the values $\sigma(1)$, …, $\sigma(i)$ seen so far, and a candidate i can be hired only at step i, if at all. No information about the future is known, not even the length of the permutation σ. If we denote by $\mathcal{H}_i(\sigma)$ the set of candidates (their indices) hired up to step i when processing permutation σ, then the rules above formally translate to: 1) $\mathcal{H}_i(\sigma) \subseteq \{1, \ldots, i\}$ (no future candidates can be hired); 2) $\mathcal{H}_i(\sigma) \setminus \{i\} = \mathcal{H}_{i-1}(\sigma)$ (no past candidates can be hired)[(i)]; and 3) $\mathcal{H}_i(\sigma) = \mathcal{H}_i(\sigma')$ for any two permutations σ and σ' as long as $\sigma(j) = \sigma'(j)$ for all j, $1 \leq j \leq i$ (decisions must be made without knowledge of the future). We call $\mathcal{H}_n(\sigma)$ the *hiring set* of permutation σ and simplify the notation to $\mathcal{H}(\sigma)$.

Actually, since the future is not known, we should consider that we are given the ranks of candidates relative to those of past candidates, rather than the actual values $\sigma(i)$. For instance, while processing some sequence of candidates, we could get the information that the candidate #11 ranks the third best if

[(i)] This condition can be substituted by $\mathcal{H}_i(\sigma) \setminus \{i\} \subseteq \mathcal{H}_{i-1}(\sigma)$ if we want to introduce firing strategies, so that at each step one or more currently hired candidates can be fired.

compared with the 10 previously seen candidates (this only implies $\sigma(11) \leq n-2$). This is properly captured by the notion of *rank-based strategies* that we present in Section 3.

Once the concept of the hiring set of a permutation has been introduced, some questions immediately come to mind: about its size, which we will denote $h(\sigma)$, and about other parameters like, for instance, the "time" of the last hiring $L(\sigma)$ or the score of the last hired candidate $r(\sigma)$.

Of course, our main concern is the expected value of these parameters on random permutations, e. g., if h_n is the size of the hiring set of a random permutation of size n, we want to obtain $\mathbb{E}\{h_n\}$. We shall consistently use the same letters for parameters in permutations and for random variables, like $A(\sigma)$ and A_n. We note here that if the hiring strategy itself were randomized, for example "Pessimizing Inc." hires candidate i with probability $\propto 1/\sigma(i)$, then the hiring set would actually be a probability measure over all subsets of $\{1,\ldots,n\}$, but all the definitions that we shall see here can be easily generalized to cope with these strategies as well.

Last but not least, we shall look at what happens in the asymptotic regime, i. e., when $n \to \infty$ and after a suitable scaling of the random variable of interest. As we shall shortly see, this provides the bridge between the original continuous model of Broder et al. and the discretized combinatorial version introduced here.

On the other hand, our model keeps the potential for extensions intact, and its generalization for multisets is both natural and immediate.

2 Simple strategies

Let us first consider hiring above a threshold τ. For simplicity, we assume $\tau \in \mathbb{Z}$. Then $\mathcal{H}_i(\sigma) = \{j \mid 1 \leq j \leq i \text{ and } \sigma(j) \geq \tau\}$, and $\mathcal{H}(\sigma) = \{1 \leq j \leq n \mid \sigma(j) \geq \tau\}$. Hence, the size h_n of the hiring set for any permutation is $n+1-\tau$. For the asymptotic regime, it is useful to consider $\tau = \alpha \cdot n + o(n)$ for some $0 < \alpha \leq 1$, for otherwise almost all candidates would be hired. Then

$$\frac{\mathbb{E}\{h_n\}}{n} = \frac{n+1-\tau}{n} = 1-\alpha+o(1).$$

The rank r_n of the last hired candidate in a random permutation of size n is any number from τ to n with identical probability, hence

$$\mathbb{E}\{r_n\} = \sum_{j=\tau}^{n} \frac{j}{n+1-\tau} = \frac{1}{(n+1-\tau)}\left(\frac{n(n+1)}{2} - \frac{\tau(\tau-1)}{2}\right) \sim n\frac{1+\alpha}{2} + o(n).$$

Therefore the normalized distance to the maximum rank (the *gap*) is on average $\mathbb{E}\{g_n\} = 1-\mathbb{E}\{r_n\}/n \sim (1-\alpha)/2 + o(1)$ (cf. (1)). Other parameters of this hiring strategy can be easily analyzed as well.

Let us now consider the other simple strategy already studied by Broder et al., hiring above the maximum. This strategy leads to a very well known and throughly studied parameter in random permutations: left-to-right maxima (see (5) and references therein). An element $\sigma(i)$ is called a left-to-right maximum if it is larger than all preceding elements, i. e., $\sigma(j) < \sigma(i)$ for all $j < i$. Obviously, $\mathcal{H}(\sigma)$ is exactly the set of positions of the left-to-right maxima in σ. It is well known that $\mathbb{E}\{h_n\} = \ln n + O(1)$, so that the size of the hiring set is exponentially small compared to the set of interviewed candidates. We don't give here additional details about this strategy, as it turns out to be a particular case (when $m = 1$) of the strategy that we examine in Section 4.

3 A general framework for rank-based strategies

In this section we develop a generic analysis of the size of the hiring set and other parameters in *rank-based* hiring strategies.

A rank-based strategy is one where each decision (hire or discard) is taken solely on the basis of the rank of the current candidate relative to the rank of the previous interviewed candidates. That is, the actual face value $\sigma(i)$ of the current candidate i is not relevant, only its position among the previous $i-1$ candidates. Rank-based strategies are natural and they adequately modelize constraints in some situations, for example, when there are no mechanisms for quality measurement in absolute terms.

In particular, it would be debatable that any absolute rank $\sigma(i)$ is actually available at step i; it is more reasonable to assume that the given permutation is unknown until the very last candidate is interviewed; what we keep at each step is the relative ordering of the candidates seen so far. This assumption is common, for instance, in the standard secretary problem, where only the relative ranks of the candidates are available as they are successively examined (3).

Given a permutation σ of length n and i, $1 \le i \le n$, let $\rho_i(\sigma)$ be the permutation of length i that we obtain by relabelling the initial prefix of length i in σ in such a way that we preserve the relative ordering. For instance, $\rho_1(25341) = 1$, $\rho_3(25341) = 132$ and $\rho_4(25341) = 1423$. Another notation that we shall define now, but use later is $\sigma \circ j$. Given a permutation σ of size n and a value j, $1 \le j \le n+1$, we denote by $\sigma \circ j$ the permutation of size $n+1$ which results after relabelling $j, j+1, \ldots, n$ in σ as $j+1, \ldots, n+1$ and appending j to the end. For example $3241 \circ 3 = 42513$ and $213 \circ 4 = 2134$.

Definition 1 *A hiring strategy is* rank-based *if and only if for all permutations σ and all i, $1 \le i \le |\sigma|$,*

$$\mathcal{H}_i(\sigma) = \mathcal{H}(\rho_i(\sigma)).$$

Hiring above the maximum, above the median, above some other quantile, and above the mth best in the current staff (see Section 4) are all rank-based hiring strategies. Hiring above a threshold or above the mean are not. We will concentrate on rank-based hiring strategies for the rest of this section.

In order to investigate the average size of the hiring set in a random permutation, we introduce the bivariate generating function (2)

$$H(z,u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)}, \tag{1}$$

where $\mathcal{P}$ denotes the set of all permutations. If we take derivates of H w.r.t. u and set $u = 1$ we obtain the generating functions of the moments of h_n, e. g.,

$$h(z) = \frac{\partial}{\partial u} H(z,u)\bigg|_{u=1} = \sum_{\sigma \in \mathcal{P}} h(\sigma) \frac{z^{|\sigma|}}{|\sigma|!}$$

Hence $\mathbb{E}\{h_n\} = [z^n]h(z)$.

Theorem 1 *Let $H(z,u)$ be the generating function defined by* (1). *Let $X(\sigma)$ denote the number of ranks j, $1 \le j \le |\sigma|+1$, such that a candidate with score j will be hired if interviewed right after σ, that is, $X(\sigma)$ is the number of scores j such that $\mathcal{H}(\sigma \circ j) = \mathcal{H}(\sigma) \cup \{|\sigma|+1\}$.*

Then

$$(1-z)\frac{\partial}{\partial z}H(z,u) - H(z,u) = (u-1)\sum_{\sigma\in\mathcal{P}} X(\sigma)\frac{z^{|\sigma|}}{|\sigma|!}u^{h(\sigma)}.$$

Proof: See Appendix A. □

Each different hiring strategy will be characterized by its corresponding definition of $X(\sigma)$; for instance, hiring above the maximum has $X(\sigma) = 1$ for all σ, since there is only one score for which we will hire a candidate coming after σ, namely, if the candidate has relative rank $|\sigma| + 1$.

Other interesting quantities can be analyzed in a similar vein. For instance, let $L(\sigma)$ denote the index of the last hired candidate in σ, that is, $L(\sigma) = \max\{i : i \in \mathcal{H}(\sigma)\}$, with the convention $L(\emptyset) = 0$. Then $L(\sigma \circ j) = L(\sigma)$ if the $(|\sigma| + 1)$th candidate is not hired, and $L(\sigma \circ j) = |\sigma| + 1$ otherwise. Letting

$$L(z,u) = \sum_{\sigma\in\mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!}u^{L(\sigma)},$$

the recurrence for $L(\sigma)$ translates to

$$(1-z)\frac{\partial L}{\partial z} - L(z,u) = u\sum_{\sigma\in\mathcal{P}} X(\sigma)\frac{(zu)^{|\sigma|}}{|\sigma|!} - \sum_{\sigma\in\mathcal{P}} X(\sigma)\frac{z^{|\sigma|}}{|\sigma|!}u^{L(\sigma)}, \tag{2}$$

with $X(\sigma)$ as before.

We now introduce a natural restriction on hiring strategies, which will allow us to obtain further general results. To begin with, we define the indicator $X_j(\sigma)$, so that $X_j(\sigma) = 1$ if a candidate with score j is hired after σ and $X_j(\sigma) = 0$ otherwise. Notice that $X(\sigma) = \sum_{1\le j\le|\sigma|+1} X_j(\sigma)$.

Definition 2 *A hiring strategy is* pragmatic *if and only if the following two conditions hold:*

1. *For all σ and all j, $X_j(\sigma) = 1$ implies $X_{j'}(\sigma) = 1$ for all $j' \ge j$.*
2. *For all σ and all j, $X(\sigma \circ j) \le X(\sigma) + X_j(\sigma)$.*

The first condition simply states that whenever a strategy would hire a candidate with score j, it would hire a candidate with a higher score. The second condition bounds the rate at which the strategy hires. In particular, the potential for hiring $X(\cdot)$ doesn't change if no new candidate gets hired. Pragmatic hiring strategies exclude pathological cases such as "hire any candidate that is interviewed at some step which is a multiple of 100, discard otherwise" (because of condition #2) or "hire any candidate whose relative score is better than that of an even number of previously interviewed candidates" (because of condition #1). Hiring above the median, above some quantile and above the mth best (Section 4) are all pragmatic.

Theorem 2 *For any pragmatic hiring strategy and any permutation σ, $\mathcal{H}(\sigma)$ contains at least the $X(\sigma)$ best candidates of σ, that is, the candidates with scores $|\sigma|$, $|\sigma| - 1, \ldots, |\sigma| + 1 - X(\sigma)$.*

Proof: See Appendix A. □

Let $r(\sigma)$ denote the absolute score of the last hired candidate in a permutation σ, and let $g(\sigma) = 1 - r(\sigma)/|\sigma|$ denote the *gap* (1).

Theorem 3 *For any pragmatic hiring strategy,*

$$\mathbb{E}\{g_n\} = \frac{1}{2n}(\mathbb{E}\{X_n\} - 1),$$

where $\mathbb{E}\{X_n\} = [z^n]\sum_{\sigma\in\mathcal{P}} X(\sigma)z^{|\sigma|}/|\sigma|!$.

Proof: See Appendix A. □

4 Hiring for the elite (above the mth best)

In this strategy we have an additional parameter m. A candidate i is hired if her score is better than the score of one of the m best currently employed candidates. In other words, if $E_{i-1}(\sigma)$ is the subset of currently hired elements before step i with the m highest scores, and $\sigma(i)$ is greater than the minimum score in $E_{i-1}(\sigma)$, then i is hired.

Note that i will become immediately part of the "elite" of the m best employees, and the element ℓ with the minimum score in $E_{i-1}(\sigma)$ will be removed from the "elite", that is, it will not be in $E_i(\sigma)$. Fortunately for ℓ, he will be still hired. Note also that for $m = 1$ this strategy is simply hiring above the maximum.

For this strategy we have $X(\sigma) = |\sigma| + 1$ if $|\sigma| < m$ since any value j will be hired after processing σ, as long as an elite of m employees hasn't built up yet. Once $|\sigma| \geq m$, we will have $h(\sigma) \geq m$ and a value j will be hired if and only if it is larger than the smallest score in the elite. Since the (relative) scores of the elite of σ must consist of $|\sigma|$, $|\sigma| - 1$, …, $|\sigma| - m + 1$ there are exactly m values for a newcomer to be hired, namely, if $j \in \{|\sigma| + 1, \ldots, |\sigma| - m + 2\}$ then the last candidate of $\sigma \circ j$ will be hired. Hence, $X(\sigma) = m$ if $|\sigma| \geq m$.

The right hand side of Theorem 1 is then

$$(u-1)\Bigg(1 + 2zu + 3z^2u^2 + \cdots + mz^{m-1}u^{m-1} + m\,H(z,u) - m(1 + zu + z^2u^2 + \cdots + z^{m-1}u^{m-1})\Bigg).$$

Plugging the expression above back into Theorem 1 and rearranging, we finally have

$$\begin{aligned}(1-z)\frac{\partial}{\partial z}H(z,u) - (mu - m + 1)H(z,u) = \\ (u-1)(1 + 2zu + \cdots + mz^{m-1}u^{m-1}) \\ - m(u-1)(1 + zu + \cdots + z^{m-1}u^{m-1}).\end{aligned} \quad (3)$$

For $m = 1$, the differential equation above reduces to

$$(1-z)\frac{\partial}{\partial z}H^{(1)}(z,u) - uH^{(1)}(z,u) = 0,$$

whose solution is

$$H^{(1)}(z,u) = \left(\frac{1}{1-z}\right)^u = \sum_{\substack{n\geq 0\\ k\geq 0}} c_{n,k}\frac{z^n}{n!}u^k,$$

as we additionally impose $H^{(1)}(z,1) = 1/(1-z)$ and $H^{(1)}(0,u) = 1$. Here, we use the superscript to make the dependence on m explicit.

The coefficients $c_{n,k} = [z^n u^k]H^{(1)}(z,u)$ are the well-known unsigned Stirling numbers of the first kind (4), also known as Stirling cycle numbers, and denoted $\left[{n \atop k}\right]$. The Stirling cycle number $\left[{n \atop k}\right]$ is the number of permutations of size n that contain exactly k cycles, and it turns out to coincide with the number of permutations of size n that have exactly k left-to-right maxima (5).

The solution for general m is

$$H^{(m)}(z,u) = \frac{1}{(mu-m+1)\cdot(mu-m)\cdots(mu-1)}\left(\left(\frac{1}{1-z}\right)^{mu-m+1} P_m(u,z) + \frac{1}{(1-z)^m}Q_m(z,u)\right), \tag{4}$$

where $P_m(u,z)$ and $Q_m(z,u)$ are polynomials in z and u.

If we differentiate w.r.t. u and set $u=1$, we obtain the generating function of the expected values

$$h^{(m)}(z) = \sum_{n\geq 0}\mathbb{E}\left\{h_n^{(m)}\right\}z^n = \frac{\partial}{\partial u}H^{(m)}(z,u)\bigg|_{u=1} = m\frac{\ln\left(\frac{1}{1-z}\right)}{1-z} - \frac{p_m(z)}{1-z},$$

with $p_m(z)$ a polynomial of degree $m-1$.

Hence $\mathbb{E}\left\{h_n^{(m)}\right\} = mH_n + O(1)$, where $H_n = \sum_{1\leq k\leq n}(1/k)$ denotes the nth harmonic number. We keep here the usual notation H_n for harmonic numbers despite the possible confusion with the hiring set parameters. Since $H_n = \ln n + \gamma + O(n^{-1})$, where $\gamma = 0.577\ldots$ is Euler's gamma constant, we conclude that $\mathbb{E}\left\{h_n^{(m)}\right\} = m\ln n + O(1)$. So the size of the hiring set is, for any fixed m, exponentially smaller than the set of interviewed candidates.

Since we have an explicit form for $H^{(m)}(z,u)$, much more information about $h_n^{(m)}$ can be extracted. In particular,we have

$$\mathbb{E}\left\{u^{h_n^{(m)}}\right\} = [z^n]H^{(m)}(z,u) \sim A_m(u)\cdot n^{m(u-1)}\cdot\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

uniformly in a complex neighborhood of $u=1$, for some analytic $A_m(u)$, so it follows by application of Hwang's quasi-powers theorem (2) that $h_n^{(m)}$ converges to a normal distribution. More precisely,

$$\frac{h_n^{(m)} - m\ln n}{\sqrt{m\ln n}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1). \tag{5}$$

Also, since $\mathbb{E}\left\{X_n\right\} = m$ if $n\geq m$, Theorem 3 yields for this strategy $\mathbb{E}\left\{g_n^{(m)}\right\} = (m-1)/2n$, if $n\geq m$.

We now consider the behavior of this strategy as m varies (notice that the results that we have discussed above hold only for fixed m). To this end we introduce

$$\mathsf{H}(z,u,v) = \sum_{m\geq 1} v^m H^{(m)}(z,u),$$

with $H^{(m)}(z,u)$ the generating function that we have studied in the preceding paragraphs.

If we set $\mathsf{h}(z,v) = (\partial \mathsf{H}/\partial u)|_{u=1}$, the coefficient $[z^n v^m]\mathsf{h}(z,v)$ is the quantity we seek, the expected size of the hiring set when the size of the elite is m. Multiplying by v^m and summing over all $m \geq 1$, the differential equation (3) translates into a corresponding differential equation for $\mathsf{H}(z,u,v)$

$$(1-z)\frac{\partial}{\partial z}\mathsf{H}(z,u,v) - \mathsf{H}(z,u,v) - (u-1)v\frac{\partial}{\partial v}\mathsf{H}(z,u,v) = (1-u)\frac{v^2}{(1-v)^2}\ln\left(\frac{1}{1-zuv}\right).$$

Similarly, differentiating w.r.t. u and setting $u=1$ the equation above we get an ordinary differential equation for $\mathsf{h}(z,v)$

$$(1-z)\frac{\partial}{\partial z}\mathsf{h}(z,v) - \mathsf{h}(z,v) - \frac{v}{(1-z)(1-v)^2} = \left(\frac{v}{1-v}\right)^2\left(\frac{1}{1-zv}\right), \tag{6}$$

since $\mathsf{H}(z,1,v) = \frac{v}{(1-z)(1-v)}$.

The solution for this equation gives (a detailed derivation can be found in Appendix A)

$$\mathsf{h}(z,v) = \frac{v\ln\frac{1}{1-z}}{(1-z)(1-v)^2} - \frac{v\ln\frac{1}{1-zv}}{(1-z)(1-v)^2} \tag{7}$$

as we impose $\mathsf{h}(0,v) = 0$.

The last step is to extract the coefficients of $\mathsf{h}(z,v)$, whose details are also given in Appendix A. For $m \geq n$, we obviously have $\mathbb{E}\left\{h_n^{(m)}\right\} = n$. For $m \leq n$ we have $\mathbb{E}\left\{h_n^{(m)}\right\} = m(H_n - H_m + 1)$, so $\mathbb{E}\left\{h_n^{(m)}\right\} \sim m\ln\left(\frac{n}{m}\right) + m + O(1)$, for $n, m \to \infty$.

5 Hiring above the median (and other quantiles)

Hiring above the median means that candidate i is hired if and only if her score $\sigma(i)$ is larger than the rth best score of the candidates hired so far, with $r = \lfloor (h_{i-1}(\sigma)+1)/2 \rfloor$.

Since this strategy is rank-based, it is not difficult to see that if the hiring set has size $k = 2t$ at some given moment then there are $t+1$ possible relative scores that will be hired in the next step, while if the hiring set has size $k = 2t+1$ then the number of relative scores that would be hired in the next step is also $t+1$. That means that $X(\sigma) = \lceil (h(\sigma)+1)/2 \rceil$. Coping with the ceilings is quite hard, so we will consider instead what happens with $X'(\sigma) = (1+h(\sigma))/2$ and $X''(\sigma) = (3+h(\sigma))/2$, which provide lower and upper bounds, respectively.

By the same token, hiring above other quantiles, say hiring above $(1-a)h(\sigma)$, with $0 < a < 1$, can be analyzed in the same way. We should have then $X(\sigma) = \lceil a \cdot (h(\sigma)+1) \rceil$. In general, for $X(\sigma) = a \cdot h(\sigma) + b$ and $0 < a < 1$, we have

$$(1-z)\frac{\partial H}{\partial z} - au(u-1)\frac{\partial H}{\partial u} - (1+b(u-1))H(z,u) = 0,$$

with the additional conditions $H(z,1) = 1/(1-z)$ and $H(0,u) = 1$. The solution turns out to be

$$H(z,u) = u^{-b/a} \frac{1}{1-z} \left(\frac{1}{1 - \frac{u-1}{u(1-z)^a}} \right)^{b/a}, \tag{8}$$

which can be readily checked (and even found!) with any reasonable computer algebra system. From this closed form we can find the successive factorial moments. It suffices to differentiate r times and set $u = 1$:

$$\mathbb{E}\left\{h_n^{\underline{r}}\right\} = [z^n] \left. \frac{\partial^r H(z,u)}{\partial u^r} \right|_{u=1},$$

where $X^{\underline{r}} = X(X-1)\cdots(X-r+1)$ denotes the rth falling factorial (4). In appendix A we show that

$$\mathbb{E}\left\{h_n^{\underline{r}}\right\} = \Theta(n^{ra}). \tag{9}$$

The expected size of the hiring set can also be obtained if we consider the differential equation satisfied by the corresponding generating function $h(z)$, namely,

$$(1-z)\frac{d}{dz}h - (1+a)h = \frac{b}{1-z}.$$

This is a simple linear first-order ordinary differential equation whose solution is

$$h(z) = \frac{b}{a}\frac{1}{1-z}\left(\frac{1}{(1-z)^a} - 1\right).$$

since $h(0) = 0$. This coincides with what we get if we differentiate $H(z,u)$ as given by (8) and set $u = 1$. The extraction of coefficients is straightforward:

$$[z^n]h(z) = \frac{b}{a}\left(\binom{n+a}{a} - 1\right) = \frac{b}{a}\frac{n^a}{\Gamma(1+a)}\left(1 + O\left(\frac{1}{n}\right)\right).$$

In particular, for $a = 1/2$ (hiring above the median) we get $\mathbb{E}\{h_n\} = \Theta(\sqrt{n})$ and for "hire A, move B" (see (1)) we have $a = 1 - B/A$; thus $\mathbb{E}\{h_n\} = \Theta(n^{1-B/A})$. Loosely speaking, when the size of the hiring set reaches k we have interviewed $n = \Theta(k^{A/(A-B)})$ candidates (compare with the results in (1)).

On the other hand $\mathbb{E}\{X_n\} = a\mathbb{E}\{h_n\} + o(\mathbb{E}\{h_n\})$, thus

$$\mathbb{E}\{g_n\} = \Theta(n^{a-1}).$$

For the particular case of hiring above the median, when $a = 1/2$, we have $\mathbb{E}\{g_n\} = \Theta(1/\sqrt{n})$.

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A Proofs

Proof (of Theorem 1): We can write $h(\sigma) = 0$ if σ is the empty permutation and $h(\sigma \circ j) = h(\sigma) + X_j(\sigma)$, where

$$X_j(\sigma) = \begin{cases} 1, & \text{if the last candidate of } \sigma \circ j \text{ is hired,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, if $\mathcal{P}_n$ denotes the set of permutations of size n,

$$\begin{aligned} H(z,u) &= \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_n} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} = 1 + \sum_{n>0} \sum_{1 \le j \le n} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma \circ j|}}{|\sigma \circ j|!} u^{h(\sigma \circ j)} \\ &= 1 + \sum_{n>0} \sum_{1 \le j \le n} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{h(\sigma) + X_j(\sigma)} = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{h(\sigma)} \sum_{1 \le j \le n} u^{X_j(\sigma)}. \end{aligned}$$

Since $X_j(\sigma)$ is either 0 or 1 for all j and all σ, we have

$$\sum_{1 \le j \le n} u^{X_j(\sigma)} = (|\sigma| + 1 - X(\sigma)) + uX(\sigma),$$

where $X(\sigma) = \sum_{1 \le j \le |\sigma|+1} X_j(\sigma)$. Note that $X(\sigma)$ is the number of relative scores such that a candidate with such a score would be hired right after processing σ.

Hence,

$$H(z,u) = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{h(\sigma)} \Big((|\sigma| + 1 - X(\sigma)) + uX(\sigma) \Big).$$

Taking derivatives w.r.t. z,

$$\begin{aligned} \frac{\partial}{\partial z} H(z,u) &= \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} \Big((|\sigma| + 1 - X(\sigma)) + uX(\sigma) \Big) \\ &= \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z \frac{d}{dz} z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} + (u-1) \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} X(\sigma) \\ &= z \frac{\partial}{\partial z} H(z,u) + H(z,u) + (u-1) \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} X(\sigma). \end{aligned}$$

After reorganizing the terms in the equation above and simplifying, we obtain the statement of the theorem. $\square$

Proof (of Theorem 2): The proof is by induction on the length n of the permutation σ. If $n = 0$ then $\mathcal{H}(\sigma) = \emptyset$ and indeed it contains the $X(\sigma)$ best candidates in the (empty) permutation σ.

Consider now $\sigma' = \sigma \circ j$. By the inductive hypothesis $\mathcal{H}(\sigma)$ contains the best $X(\sigma)$ candidates, with relative scores $\{|\sigma| - X(\sigma) + 1, \ldots, |\sigma|\}$. Since the strategy is pragmatic, only candidates with relative rank between $|\sigma| + 2 - X(\sigma)$ and $|\sigma| + 1$ will be hired. If the last candidate with relative score j is hired

then he is among the best $X(\sigma)+1$ candidates of $\mathcal{H}(\sigma')$. As $X(\sigma') \leq X(\sigma)+1$, it follows that $\mathcal{H}(\sigma')$ contains at least the best $X(\sigma')$ candidates of σ'. On the contrary, if the last candidate were not hired then the relative scores of the best $X(\sigma)$ candidates in $\mathcal{H}(\sigma)$ increase all by one. Hence, in $\mathcal{H}(\sigma')$ we have at least $X(\sigma)$ candidates with scores in $\{|\sigma|+2-X(\sigma),\ldots,|\sigma|\}$. To conclude the proof it is enough to notice that, for any pragmatic strategy, $X(\sigma \circ j) = X(\sigma)$ if the last candidate with score j was not hired.
$\square$

Proof (of Theorem 3): The last hired candidate must have an absolute score in $\{|\sigma|+1-X(\sigma),\ldots,|\sigma|\}$ because of Theorem 2. For a random permutation, all these $X(\sigma)$ scores are equally likely, hence for a random permutation of size n we have

$$\begin{aligned}\mathbb{E}\{r_n\} = \mathbb{E}\left\{\sum_{k=n-X(\sigma)+1}^{n} \frac{k}{X(\sigma)}\right\} &= \mathbb{E}\left\{\frac{1}{X(\sigma)}\left(\frac{n(n+1)}{2} - \frac{(n-X(\sigma))(n+1-X(\sigma))}{2}\right)\right\} \\ &= \mathbb{E}\left\{n+\frac{1}{2}-\frac{1}{2}X(\sigma)\right\} = n+\frac{1}{2}-\frac{\mathbb{E}\{X_n\}}{2}.\end{aligned}$$

Finally, $\mathbb{E}\{g_n\} = 1-n^{-1}\mathbb{E}\{r_n\} = (\mathbb{E}\{X_n\}-1)\,/\,2n$. $\square$

Proof (of Equation (7) and coefficient $[z^n v^m]\mathsf{h}(z,v)$): We start with the linear differential equation satisfied by $h(z,v)$ (Equation (6))

$$(1-z)\frac{\partial}{\partial z}\mathsf{h}(z,v) - \mathsf{h}(z,v) = \frac{v}{1-z}\frac{1}{(1-v)^2} - \frac{v^2}{(1-v)^2}\frac{1}{1-zv}.$$

Multiplying through by the integrating factor $1-z$ and integrating with respecth to z gives

$$(1-z)\mathsf{h}(z,v) = \frac{v}{(1-v)^2}\ln\left(\frac{1}{1-z}\right) - \frac{v}{(1-v)^2}\ln\left(\frac{1}{1-zv}\right) + c(v),$$

for some unknown function $c(v)$.

Using the initial condition $\mathsf{h}(0,v)=0$, we find that $c(v)=0$ for any v. Hence

$$\begin{aligned}[z^n v^m]\mathsf{h}(z,v) &= [z^n v^m]\frac{1}{1-z}\left(\frac{v}{(1-v)^2}\ln\left(\frac{1}{1-z}\right) - \frac{v}{(1-v)^2}\ln\left(\frac{1}{1-zv}\right)\right) \\ &= [z^n]\left(\frac{1}{1-z}\ln\left(\frac{1}{1-z}\right)[v^m]\frac{v}{(1-v)^2} - \frac{1}{1-z}[v^m]\frac{v}{(1-v)^2}\ln\left(\frac{1}{1-zv}\right)\right) \\ &= m[z^n]\frac{1}{1-z}\ln\left(\frac{1}{1-z}\right) - [z^n]\frac{1}{1-z}\sum_{k=1}^{m}\left(\frac{mz^k}{k} - z^k\right)\end{aligned}$$

Extracting the coefficient of z^n above is now easy,

$$\begin{aligned}[z^n v^m]\mathsf{h}(z,v) &= mH_n - \Big(m\sum_{k=1}^{m}\frac{1}{k}[z^{n-k}]\frac{1}{1-z} - \sum_{k=1}^{m}[z^{n-k}]\frac{1}{1-z}\Big)\\ &= mH_n - m\sum_{k=1}^{\min(m,n)}\frac{1}{k} + \sum_{k=1}^{\min(m,n)} 1\\ &= \begin{cases} mH_n - mH_m + m, & \text{if } m \le n,\\ n, & \text{if } m > n.\end{cases}\end{aligned}$$

□

Proof (of Equation 9): Our starting point is

$$H(z,u) = u^{-b/a}\frac{1}{1-z}\left(\frac{1}{1-\frac{u-1}{u(1-z)^a}}\right)^{b/a}.$$

It suffices to differentiate r times and set $u = 1$ to obtain the generating function of the rth factorial moments of h_n:

$$\mathbb{E}\{h_n^{\underline{r}}\} = [z^n]h_r(z)$$

with

$$h_r(z) = \left.\frac{\partial^r H(z,u)}{\partial u^r}\right|_{u=1}.$$

We have thus

$$h_r(z) = \gamma_r\sum_{j=0}^{r}\frac{(-1)^{r-j}}{(1-z)^{ja+1}}\binom{r}{j},$$

where γ_r is a polynomial of degree r in $x = b/a$. Extracting coefficients

$$[z^n]h_r(z) = \gamma_r\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\binom{ja+n}{n}.$$

Hence we get that, asymptotically as $n \to \infty$,

$$[z^n]h_r(z) \sim \gamma_r\frac{n^{ra}}{\Gamma(ra+1)}.$$

□

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Matroid Polytopes and Their Volumes

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Abstract. We express the matroid polytope P_M of a matroid M as a signed Minkowski sum of simplices, and obtain a formula for the volume of P_M. This gives a combinatorial expression for the degree of an arbitrary torus orbit closure in the Grassmannian $Gr_{k,n}$. We then derive analogous results for the independent set polytope and the associated flag matroid polytope of M. Our proofs are based on a natural extension of Postnikov's theory of generalized permutohedra.

Résumé. On exprime le polytope matroïde P_M d'un matroïde M comme somme signée de Minkowski de simplices, et on obtient une formule pour le volume de P_M. Ceci donne une expression combinatoire pour le degré d'une clôture d'orbite de tore dans la Grassmannienne $Gr_{k,n}$. Ensuite, on deduit des résultats analogues pour le polytope ensemble indépendant et pour le polytope matroïde drapeau associé a M. Nos preuves sont fondées sur une extension naturelle de la théorie de Postnikov de permutoèdres généralisés.

Keywords: Matroid; generalized permutohedron; matroid polytope; Minkowski sum; mixed volume; flag matroid.

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1 Introduction

The theory of matroids can be approached from many different points of view; a matroid can be defined as a simplicial complex of independent sets, a lattice of flats, a closure relation, etc. A relatively new point of view is the study of matroid polytopes, which in some sense are the natural combinatorial incarnations of matroids in algebraic geometry and optimization. Our paper is a contribution in this direction.

We begin with the observation that matroid polytopes are members of the family of *generalized permutohedra* (14). With some modifications of Postnikov's beautiful theory, we express the matroid polytope P_M as a signed Minkowski sum of simplices, and use that to give a formula for its volume $\mathrm{Vol}\,(P_M)$. This is done in Theorems 2.5 and 3.3. Our answers are expressed in terms of the beta invariants of the contractions of M.

Formulas for $\mathrm{Vol}\,(P_M)$ were given in very special cases by Stanley (17) and Lam and Postnikov (11), and a polynomial time algorithm for finding $\mathrm{Vol}\,(P_M)$ was constructed by de Loera et. al. (6). One motivation for this computation is the following. The closure of the torus orbit of a point p in the Grassmannian $Gr_{k,n}$ is a toric variety X_p, whose degree is the volume of the matroid polytope P_{M_p} associated to p. Our formula allows us to compute the degree of X_p combinatorially.

One can naturally associate two other polytopes to a matroid M: its independent set polytope and its associated flag matroid polytope. By a further extension of Postnikov's theory, we also write these polytopes as signed Minkowski sums of simplices and give formulas for their volumes. This is the content of Sections 4 and 5.

Throughout the paper we assume familiarity with the basic concepts of matroid theory; for further information we refer the reader to (13).

2 Matroid Polytopes are Generalized Permutohedra

A *generalized permutohedron* is a polytope whose inequality description is of the following form:

$$P_n(\{z_I\}) = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = z_{[n]}, \sum_{i \in I} t_i \geq z_I \text{ for all } I \subseteq [n]\}$$

where z_I is a real number for each $I \subseteq [n] := \{1, \ldots, n\}$, and $z_\emptyset = 0$. Different choices of z_I can give the same generalized permutohedron: if one of the inequalities does not define a face of $P_n(\{z_I\})$, then we can increase the value of the corresponding z_I without altering the polytope. When we write $P_n(\{z_I\})$, we will always assume that the z_Is are all chosen minimally; *i.e.*, that all the defining inequalities are tight.

The *Minkowski sum* of two polytopes P and Q in $\mathbb{R}^n$ is defined to be $P+Q = \{p+q \,:\, p \in P, q \in Q\}$. We say that the *Minkowski difference* of P and Q is $P - Q = R$ if $P = Q + R$.[(i)] The following lemma shows that generalized permutohedra behave nicely with respect to Minkowski sums.

Lemma 2.1 $P_n(\{z_I\}) + P_n(\{z'_I\}) = P_n(\{z_I + z'_I\})$.

[(i)] We will only consider Minkowski differences $P - Q$ such that Q is a Minkowski summand of P. More generally, the Minkowski difference of two arbitrary polytopes P and Q in $\mathbb{R}^n$ is defined to be $P - Q = \{r \in \mathbb{R}^n \mid r + Q \subseteq P\}$ (14). It is easy to check that $(Q + R) - Q = R$, so the two definitions agree in the cases that interest us. In this paper, a signed Minkowski sum equality such as $P - Q + R - S = T$ should be interpreted as $P + R = Q + S + T$.

Let Δ be the standard unit $(n-1)$-simplex

$$\begin{aligned}\Delta &= \{(t_1,\ldots,t_n)\in\mathbb{R}^n \,:\, \sum_{i=1}^n t_i = 1, t_i \geq 0 \text{ for all } 1\leq i\leq n\}\\ &= \operatorname{conv}\{e_1,\ldots,e_n\},\end{aligned}$$

where $e_i = (0,,\ldots,0,1,0,\ldots,0)$ with a 1 in its ith coordinate. As J ranges over the subsets of $[n]$, let Δ_J be the face of the simplex Δ defined by

$$\Delta_J = \operatorname{conv}\{e_i \,:\, i\in J\} = P_n(\{z(J)_I\})$$

where $z(J)_I = 1$ if $I \supseteq J$ and $z(J)_I = 0$ otherwise. Lemma 2.1 gives the following proposition.

The next two propositions are due to Postnikov in the case $y_I \geq 0$.

Proposition 2.2 *(14, Proposition 6.3) For any $y_I \geq 0$, the Minkowski sum $\sum y_I\Delta_I$ of dilations of faces of the standard $(n-1)$-simplex is a generalized permutohedron. We can write*

$$\sum_{A\subseteq E} y_I\Delta_I = P_n(\{z_I\}),$$

where $z_I = \sum_{J\subseteq I} y_J$ for each $I\subseteq[n]$.

Proposition 2.3 *Every generalized permutohedron $P_n(\{z_I\})$ can be written uniquely as a signed Minkowski sum of simplices, as*

$$P_n(\{z_I\}) = \sum_{I\subseteq[n]} y_I\Delta_I$$

where $y_I = \sum_{J\subseteq I}(-1)^{|I|-|J|}z_J$ for each $I\subseteq[n]$.

Proof: First we need to separate the right hand side into its positive and negative parts. By Proposition 2.2,

$$\sum_{I\subseteq[n]\,:y_I<0}(-y_I)\Delta_I = P_n(\{z_I^-\}) \text{ and } \sum_{I\subseteq[n]\,:y_I\geq 0} y_I\Delta_I = P_n(\{z_I^+\})$$

where $z_I^- = \sum_{J\subseteq I\,:y_J<0}(-y_J)$ and $z_I^+ = \sum_{J\subseteq I\,:y_J\geq 0} y_J$. Now $z_I + z_I^- = z_I^+$ gives

$$P_n(\{z_I\}) + \sum_{I\subseteq[n]\,:y_I<0}(-y_I)\Delta_I = \sum_{I\subseteq[n]\,:y_I\geq 0} y_I\Delta_I,$$

as desired. Uniqueness is clear. $\square$

Let M be a matroid of rank r on the set E. The *matroid polytope* of M is the polytope P_M in $\mathbb{R}^E$ whose vertices are the indicator vectors of the bases of M. The known description of the polytope P_M by inequalities makes it apparent that it is a generalized permutohedron:

Proposition 2.4 *(19) The matroid polytope of a matroid M on E with rank function r is $P_M = P_E(\{r - r(E-I)\}_{I\subseteq E})$.*

The *beta invariant* (5) of M is a non-negative integer given by

$$\beta(M) = (-1)^{r(M)} \sum_{X \subseteq E} (-1)^{|X|} r(X)$$

which stores significant information about M; for example, $\beta(M) = 0$ if and only if M is disconnected and $\beta(M) = 1$ if and only if M is series-parallel. If

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)} = \sum_{i,j} b_{ij} x^i y^j$$

is the *Tutte polynomial* (20) of M, then $\beta(M) = b_{10} = b_{01}$ for $|E| \geq 2$.

Our next results are more elegantly stated in terms of the *signed beta invariant* of M, which we define to be

$$\widetilde{\beta}(M) = (-1)^{r(M)+1} \beta(M).$$

Theorem 2.5 *Let M be a matroid of rank r on E and let P_M be its matroid polytope. Then*

$$P_M = \sum_{A \subseteq E} \widetilde{\beta}(M/A)\, \Delta_{E-A}. \tag{1}$$

Proof: By Propositions 2.3 and 2.4, $P_M = \sum_{I \subseteq E} y_I \Delta_I$ where

$$\begin{aligned}
y_I &= \sum_{J \subseteq I} (-1)^{|I|-|J|} (r - r(E-J)) = -\sum_{J \subseteq I} (-1)^{|I|-|J|} r(E-J) \\
&= -\sum_{E-J \supseteq E-I} (-1)^{|E-J|-|E-I|} (r(E-J) - r(E-I)) \\
&= -\sum_{X \subseteq I} (-1)^{|X|} (r(E-I \cup X) - r(E-I)) \\
&= -\sum_{X \subseteq I} (-1)^{|X|} r_{M/(E-I)}(X) = \widetilde{\beta}(M/(E-I))
\end{aligned}$$

as desired. □

Example 2.6 *Let M be the matroid on $E = [4]$ with bases $\{12, 13, 14, 23, 24\}$; its matroid polytope is a square pyramid. Theorem 2.5 gives $P_M = \Delta_{234} + \Delta_{134} + \Delta_{12} - \Delta_{1234}$, as illustrated in Figure 1. The dotted lines in the polytope $\Delta_{234} + \Delta_{134} + \Delta_{12}$ are an aid to visualize the Minkowski difference.*

One way of visualizing the Minkowski sum of two polytopes P and Q is by grabbing a vertex v of Q and then using it to "slide" Q around in space, making sure that v never leaves P. The region that Q sweeps along the way is $P + Q$. Similarly, the Minkowski difference $P - R$ can be visualized by picking a vertex v of R and then "sliding" R around in space, this time making sure that no point in R ever leaves P. The region that v sweeps along the way is $P - R$. This may be helpful in understanding Figure 1.

Some remarks about Theorem 2.5 are in order.

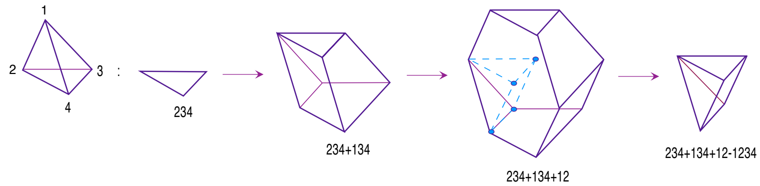

Fig. 1: A matroid polytope as a signed Minkowski sum of simplices.

- Generally most terms in the sum of Theorem 2.5 are zero. The nonzero terms correspond to the *coconnected flats* A, which we define to be the sets A such that M/A is connected. These are indeed flats, since contracting by them must produce a loopless matroid.

- A matroid and its dual have congruent matroid polytopes, and Theorem 2.5 gives different formulas for them. For example $P_{U_{1,3}} = \Delta_{123}$ while

  $P_{U_{2,3}} = \Delta_{12} + \Delta_{23} + \Delta_{13} - \Delta_{123}$.

- The study of the subdivisions of a matroid polytope into smaller matroid polytopes, originally considered by Lafforgue (10), has recently received significant attention (1; 2; 7; 15). Speyer conjectured (15) that the subdivisions consisting of series-parallel matroids have the largest number of faces in each dimension and proved this (16) for a large and important family of subdivisions: those which arise from a tropical linear space. The important role played by series-parallel matroids is still somewhat mysterious. Theorem 2.5 characterizes series-parallel matroids as those whose matroid polytope has no repeated Minkowski summands. It would be interesting to connect this characterization to matroid subdivisions; this may require extending the theory of mixed subdivisions to signed Minkowski sums.

- Theorem 2.5 provides a geometric interpretation for the beta invariant of a matroid M in terms of the matroid polytope P_M. In Section 5 we see how to extend this to certain families of Coxeter matroids. This is a promising point of view towards the notable open problem (4, Problem 6.16.6) of defining useful enumerative invariants of a Coxeter matroid.

3 The Volume of a Matroid Polytope

Our next goal is to present an explicit combinatorial formula for the volume of an arbitrary matroid polytope. Formulas have been given for very special families of matroids by Stanley (17) and Lam and Postnikov (11). Additionally, a polynomial time algorithm for computing the volume of an arbitrary matroid polytope was recently discovered by de Loera et. al. (6). Let us say some words about the motivation for this question.

Consider the Grassmannian manifold $Gr_{k,n}$ of k-dimensional subspaces in $\mathbb{C}^n$. Such a subspace can be represented as the rowspace of a $k \times n$ matrix A of rank k, modulo the left action of GL_k which does not change the row space. The $\binom{n}{k}$ maximal minors of this matrix are the *Plücker coordinates* of the subspace, and they give an embedding of $Gr_{k,n}$ as a projective algebraic variety in $\mathbb{CP}^{\binom{n}{k}-1}$.

Each point p in $Gr_{k,n}$ gives rise to a matroid M_p whose bases are the k-subsets of n where the Plücker coordinate of p is not zero. Gelfand, Goresky, MacPherson, and Serganova (9) first considered the stratification of $Gr_{k,n}$ into *matroid strata*, which consist of the points corresponding to a fixed matroid.

The torus $\mathbb{T} = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by $(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1x_1, \ldots, t_nx_n)$ for $t_i \neq 0$; this action extends to an action of $\mathbb{T}$ on $Gr_{k,n}$. For a point $p \in Gr_{k,n}$, the closure of the torus orbit $X_p = \overline{\mathbb{T} \cdot p}$ is a toric variety which only depends on the matroid M_p of p, and the polytope corresponding to X_p under the moment map is the matroid polytope of M_p (9). Under these circumstances it is known (8) that the volume of the matroid polytope M_p equals the degree of the toric variety X_p as a projective subvariety of $\mathbb{CP}^{\binom{n}{k}-1}$:

$$\mathrm{Vol}\, P_{M_p} = \deg X_p.$$

Therefore, by finding the volume of an arbitrary matroid polytope, one obtains a formula for the degree of the toric varieties arising from arbitrary torus orbits in the Grassmannian.

To prove our formula for the volume of a matroid polytope, we first recall the notion of the *mixed volume* $\mathrm{Vol}\,(P_1, \ldots, P_n)$ of n (possibly repeated) polytopes $P_1, \ldots, P_n$ in $\mathbb{R}^n$. All volumes in this section are normalized with respect to the lattice generated by $e_1 - e_2, \ldots, e_{n-1} - e_n$ where our polytopes live; so the standard simplex Δ has volume $1/(n-1)!$.

Proposition 3.1 *(12) Let n be a fixed positive integer. There exists a unique function* $\mathrm{Vol}\,(P_1, \ldots, P_n)$ *defined on n-tuples of polytopes in $\mathbb{R}^n$, called the* mixed volume *of $P_1, \ldots, P_n$, such that, for any collection of polytopes $Q_1, \ldots, Q_m$ in $\mathbb{R}^n$ and any nonnegative real numbers $y_1, \ldots, y_m$, the volume of the Minkowski sum $y_1Q_1 + \cdots + y_mQ_m$ is the polynomial in $y_1, \ldots, y_m$ given by*

$$\mathrm{Vol}\,(y_1Q_1 + \cdots + y_mQ_m) = \sum_{i_1, \ldots, i_n} \mathrm{Vol}\,(Q_{i_1}, \ldots, Q_{i_n}) y_{i_1} \cdots y_{i_n}$$

where the sum is over all ordered n-tuples $(i_1, \ldots, i_n)$ with $1 \leq i_r \leq m$.

Proposition 3.1 still holds if some of the y_is are negative as long as the expression $y_1Q_1 + \cdots + y_mQ_m$ still makes sense, as stated in the following Proposition.

Proposition 3.2 *If $P = y_1Q_1 + \cdots + y_mQ_m$ is a* signed *Minkowski sum of polytopes in $\mathbb{R}^n$, then*

$$\mathrm{Vol}\,(y_1Q_1 + \cdots + y_mQ_m) = \sum_{i_1, \ldots, i_n} \mathrm{Vol}\,(Q_{i_1}, \ldots, Q_{i_n}) y_{i_1} \cdots y_{i_n}$$

where the sum is over all ordered n-tuples $(i_1, \ldots, i_n)$ with $1 \leq i_r \leq m$.

Proof: We first show that

$$\mathrm{Vol}\,(A - B) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathrm{Vol}\,(A, \ldots, A, B, \ldots, B) \tag{2}$$

when B is a Minkowski summand of A in $\mathbb{R}^n$. Let $A - B = C$. By Proposition 3.1, for $t \geq 0$ we have that

$$\mathrm{Vol}\,(C + tB) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{Vol}\,(C, \ldots, C, B \ldots, B) t^k =: f(t)$$

and we are interested in computing $\mathrm{Vol}\,(C) = f(0)$. Invoking Proposition 3.1 again, for $t \geq 0$ we have that

$$\mathrm{Vol}\,(A+tB) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{Vol}\,(A,\ldots,A,B,\ldots,B)t^k =: g(t). \qquad (3)$$

But $A + tB = C + (t+1)B$ and therefore $g(t) = f(t+1)$ for all $t \geq 0$. Therefore $g(t) = f(t+1)$ as polynomials, and $\mathrm{Vol}\, C = f(0) = g(-1)$. Plugging into (3) gives the desired result.

Having established (2), separate the given Minkowski sum for P into its positive and negative parts as $P = Q - R$, where $Q = x_1Q_1 + \cdots + x_rQ_r$ and $R = y_1R_1 + \cdots + y_sR_s$ with $x_i, y_i \geq 0$. For positive t we can write $Q + tR = \sum x_iQ_i + \sum ty_jR_j$, which gives two formulas for $\mathrm{Vol}\,(Q + tR)$.

$$\begin{aligned}\mathrm{Vol}\,(Q+tR) &= \sum_{k=0}^{n} \binom{n}{k} \mathrm{Vol}\,(Q,\ldots,Q,R,\ldots,R)t^k \\ &= \sum_{\substack{1\leq i_a \leq r \\ 1 \leq j_b \leq s}} \mathrm{Vol}\,(Q_{i_1},\ldots,Q_{i_{n-k}},R_{j_1},\ldots,R_{j_k}) x_{i_1}\cdots x_{i_{n-k}} y_{j_1}\cdots y_{j_k} t^k\end{aligned}$$

The last two expressions must be equal as polynomials. A priori, we cannot plug $t = -1$ into this equation; but instead, we can use the formula for $\mathrm{Vol}\,(Q - R)$ from (2), and then plug in coefficient by coefficient. That gives the desired result. □

Theorem 3.3 *If a connected matroid M has n elements, then the volume of the matroid polytope P_M is*

$$\mathrm{Vol}\, P_M = \frac{1}{(n-1)!} \sum_{(J_1,\ldots,J_{n-1})} \widetilde{\beta}(M/J_1)\widetilde{\beta}(M/J_2)\cdots\widetilde{\beta}(M/J_{n-1}),$$

summing over the ordered collections of sets $J_1, \ldots, J_{n-1} \subseteq [n]$ such that, for any distinct $i_1, \ldots, i_k$, $|J_{i_1} \cap \cdots \cap J_{i_k}| < n - k$.

Proof: Postnikov (14, Corollary 9.4) gave a formula for the volume of a (positive) Minkowski sum of simplices. We would like to apply his formula to the signed Minkowski sum in Theorem 2.5, and Proposition 3.2 makes this possible. □

In Theorem 3.3, the hypothesis that M is connected is needed to guarantee that the matroid polytope P_M has dimension $n-1$. More generally, if we have $M = M_1 \oplus \cdots \oplus M_k$ then $P_M = P_{M_1} \times \cdots \times P_{M_k}$ so the ($(n-k)$-dimensional) volume of P_M is $\mathrm{Vol}\, P_M = \mathrm{Vol}\, P_{M_1} \cdots \mathrm{Vol}\, P_{M_k}$.

4 Independent Set Polytopes

In this section we show that our analysis of matroid polytopes can be carried out similarly for the *independent set polytope* I_M of a matroid M, which is the convex hull of the indicator vectors of the independent sets of M. The inequality description of I_M is known to be:

$$I_M = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \,:\, x_i \geq 0 \text{ for } i \in [n], \sum_{i\in A} x_i \leq r(A) \text{ for all } A \subseteq E\}. \qquad (4)$$

The independent set polytope of a matroid is not a generalized permutahedron. Instead, it is a *Q-polytope*; *i.e.*, a polytope of the form

$$Q_n(\{z_J\}) = \{(t_1, \dots, t_n) \in \mathbb{R}^n : t_i \geq 0 \text{ for all } i \in [n], \sum_{i \in J} t_i \leq z_J \text{ for all } J \subseteq [n]\}$$

where z_J is a non-negative real number for each $J \subseteq [n]$. We can also express these polytopes as signed Minkowski sums of simplices, though the simplices we use are not the Δ_Js, but those of the form

$$\begin{aligned} D_J &= \operatorname{conv}\{0,\ e_i\ :\ i \in J\} \\ &= Q_n(\{d(J)_I\}) \end{aligned}$$

where $d(J)_I = 0$ if $I \cap J = \emptyset$ and $d(J)_I = 1$ otherwise.

The following lemmas on Q-polytopes are proved in a way analogous to the corresponding lemmas for generalized permutahedra as was done in Section 2, and we leave them to the reader.

Lemma 4.1 $Q_n(\{z_J\}) + Q_n(\{z'_J\}) = Q_n(\{z_J + z'_J\})$

Proposition 4.2 *For any* $y_I \geq 0$ *we have*

$$\sum_{I \subseteq [n]} y_I D_I = Q_n(\{z_J\})$$

where $z_J = \sum_{I : I \cap J \neq \emptyset} y_I$.

Proposition 4.3 *Every Q-polytope* $Q_n(\{z_J\})$ *can be written uniquely*[(ii)] *as a signed Minkowski sum of* D_I*s as*

$$Q_n(\{z_J\}) = \sum_{I \subseteq [n]} y_I D_I,$$

where

$$y_J = -\sum_{I \subseteq J} (-1)^{|J|-|I|} z_{[n]-I}.$$

Proof: We need to invert the relation between the y_Is and the z_Js given by $z_J = \sum_{I : I \cap J \neq \emptyset} y_I$. We rewrite this relation as

$$z_{[n]} - z_J = \sum_{I \subseteq [n]-J} y_I$$

and apply inclusion-exclusion. As in Section 2, we first do this in the case $y_I \geq 0$ and then extend it to arbitrary Q-polytopes. $\square$

Theorem 4.4 *Let* M *be a matroid of rank* r *on* E *and let* I_M *be its independent set polytope. Then*

$$I_M = \sum_{A \subseteq E} \widetilde{\beta}(M/A)\, D_{E-A}$$

where $\widetilde{\beta}$ *denotes the signed beta invariant.*

(ii) assuming $y_\emptyset = 0$

The great similarity between Theorems 2.5 and 4.4 is not surprising, since P_M is the facet of I_M which maximizes the linear function $\sum_{i\in E} x_i$, and Δ_I is the facet of D_I in that direction as well. In fact we could have first proved Theorem 4.4 and then obtained Theorem 2.5 as a corollary.

Theorem 4.5 *If a connected matroid M has n elements, then the volume of the independent set polytope I_M is*

$$\operatorname{Vol} I_M = \frac{1}{n!} \sum_{(J_1,\ldots,J_n)} \widetilde{\beta}(M/J_1)\widetilde{\beta}(M/J_2)\cdots\widetilde{\beta}(M/J_n)$$

where the sum is over all $n-$tuples $(J_1,\ldots,J_n)$ of subsets of $[n]$ such that, for any distinct $i_1,\ldots,i_k$, we have $|J_{i_1}\cap\cdots\cap J_{i_k}| \leq n-k$.

Notice that by Hall's marriage theorem, the condition on the J_is is equivalent to requiring that $(E-J_1,\ldots,E-J_n)$ has a system of distinct representatives (SDR); that is, there are $a_1\in E-J_1,\ldots,a_n\in E-J_n$ with $a_i\neq a_j$ for $i\neq j$.

Proof: By Theorem 4.4 and Proposition 3.1 it suffices to compute the mixed volume $\operatorname{Vol}(D_{A_1},\ldots,D_{A_n})$ for each n-tuple $(A_1,\ldots,A_n)$ of subsets of $[n]$. Bernstein's theorem (18) tells us that $\operatorname{Vol}(D_{A_1},\ldots,D_{A_n})$ is the number of isolated solutions in $(\mathbb{C}-\{0\})^n$ of the system of equations:

$$\begin{cases} \beta_{1,0}+\beta_{1,1}t_1+\beta_{1,2}t_2+\cdots+\beta_{1,n}t_n &= 0 \\ \beta_{2,0}+\beta_{2,1}t_1+\beta_{2,2}t_2+\cdots+\beta_{2,n}t_n &= 0 \\ & \vdots \\ \beta_{n,0}+\beta_{n,1}t+\beta_{n,2}t_2+\cdots+\beta_{n,n}t_n &= 0 \end{cases}$$

where $\beta_{i,0}$ and $\beta_{i,j}$ are generic complex numbers when $j\in A_i$, and $\beta_{i,j}=0$ if $j\notin A_i$.

This system of linear equations will have one solution if it is non-singular and no solutions otherwise. Because the $\beta_{i,0}$ are generic, such a solution will be non-zero if it exists. The system is non-singular when the determinant is non-zero, and by genericity that happens when $(A_1,\ldots,A_n)$ has an SDR.

We conclude that $\operatorname{Vol}(D_{E-J_1},\ldots,D_{E-J_n})$ is 1 if $(E-J_1,\ldots,E-J_n)$ has an SDR and 0 otherwise, and the result follows. □

Example 4.6 *Let I_M be the independent set polytope of the uniform matroid $U_{2,3}$. We have $I_M = D_{12}+D_{23}+D_{13}-D_{123}$. Theorem 4.5 should confirm that its volume is $\frac{5}{6}$; let us carry out that computation.*

The coconnected flats of M are $1,2,3$ and $\emptyset$ and their complements are $\{23,13,12,123\}$. We need to consider the triples of coconnected flats whose complements contain an SDR. Each one of the 24 triples of the form (a,b,c), where $a,b,c\in[3]$ are not all equal, contributes a summand equal to 1. The 27 permutations of triples of the form $(a,b,\emptyset)$, contribute a -1 each. The 9 permutations of triples of the form $(a,\emptyset,\emptyset)$ contribute a 1 each. The triple $(\emptyset,\emptyset,\emptyset)$ contributes a -1. The volume of I_M is then $\frac{1}{6}(24-27+9-1)=\frac{5}{6}$.

5 Truncation Flag Matroids

We will soon see that any flag matroid polytope can also be written as a signed Minkowski sum of simplices Δ_I. We now focus on the particularly nice family of *truncation flag matroids*, introduced by Borovik, Gelfand, Vince, and White, where we obtain an explicit formula for this sum.

The *strong order* on matroids is defined by saying that two matroids M and N on the same ground set E, having respective ranks $r_M < r_N$, are *concordant* if their rank functions satisfy that $r_M(Y) - r_M(X) \leq r_N(Y) - r_N(X)$ for all $X \subset Y \subseteq E$. (4).

Flag matroids are an important family of *Coxeter matroids* (4). There are several equivalent ways to define them; in particular they also have an algebro-geometric interpretation. We proceed constructively. Given pairwise concordant matroids $M_1, \ldots, M_m$ on E of ranks $k_1 < \cdots < k_m$, consider the collection of flags $(B_1, \ldots, B_m)$, where B_i is a basis of M_i and $B_1 \subset \cdots \subset B_m$. Such a collection of flags is called a *flag matroid*, and $M_1, \ldots, M_m$ are called the *constituents* of $\mathcal{F}$.

For each flag $B = (B_1, \ldots, B_m)$ in $\mathcal{F}$ let $v_B = v_{B_1} + \cdots + v_{B_m}$, where $v_{\{a_1,\ldots,a_i\}} = e_{a_1} + \cdots + e_{a_i}$. The *flag matroid polytope* is $P_{\mathcal{F}} = \operatorname{conv}\{v_B : B \in \mathcal{F}\}$.

Theorem 5.1 *(4, Cor 1.13.5) If $\mathcal{F}$ is a flag matroid with constituents $M_1, \ldots, M_k$, then $P_{\mathcal{F}} = P_{M_1} + \cdots + P_{M_k}$.*

As mentioned above, this implies that every flag matroid polytope is a signed Minkowski sum of simplices Δ_I; the situation is particularly nice for truncation flag matroids, which we now define.

Let M be a matroid over the ground set E with rank r. Define M_i to be the rank i *truncation* of M, whose bases are the independent sets of M of rank i. One easily checks that the truncations of a matroid are concordant, and this motivates the following definition of Borovik, Gelfand, Vince, and White.

Definition 5.2 *(3) The flag $\mathcal{F}(M)$ with constituents $M_1, \ldots, M_r$ is a flag matroid, called the* truncation flag matroid *or* underlying flag matroid *of M.*

Our next goal is to present the decomposition of a truncation flag matroid polytope as a signed Minkowski sum of simplices. For that purpose, we define the *gamma invariant* of M to be $\gamma(M) = b_{20} - b_{10}$, where $T_M(x,y) = \sum_{i,j} b_{ij} x^i y^j$ is the Tutte polynomial of M.

Proposition 5.3 *The gamma invariant of a matroid is given by*

$$\gamma(M) = \sum_{I \subseteq E} (-1)^{r-|I|} \binom{r - r(I) + 1}{2}.$$

Unlike the beta invariant, the gamma invariant is not necessarily nonnegative. In fact its sign is not simply a function of $|E|$ and r. For example, $\gamma(U_{k,n}) = -\binom{n-3}{k-1}$, and $\gamma(U_{k,n} \oplus C) = \binom{n-2}{k-1}$ where C denotes a coloop.

As we did with the beta invariant, define the *signed gamma invariant* of M to be $\widetilde{\gamma}(M) = (-1)^{r(M)} \gamma(M)$.

Theorem 5.4 *The truncation flag matroid polytope of M can be expressed as:*

$$P_{\mathcal{F}(M)} = \sum_{I \subseteq E} \widetilde{\gamma}(M/I) \Delta_{E-I}.$$

Proof: By Theorems 2.5 and 5.1, $P_{\mathcal{F}(M)}$ is

$$\sum_{i=1}^{r} P_{M_i} = \sum_{i=1}^{r} \sum_{I \subseteq E} \sum_{J \subseteq I} (-1)^{|I|-|J|} (i - r_i(E-J)) \Delta_I,$$

where $r_i(A) = \min\{i, r(A)\}$ is the rank function of M_i. Then

$$\begin{aligned} P_{\mathcal{F}(M)} &= \sum_{I \subseteq E} \left[\sum_{J \subseteq I} (-1)^{|I|-|J|} \sum_{i=r(E-J)+1}^{r} (i - r(E-J)) \right] \Delta_I \\ &= \sum_{I \subseteq E} \left[\sum_{J \subseteq I} (-1)^{|I|-|J|} \binom{r - r(E-J)+1}{2} \right] \Delta_I \\ &= \sum_{I \subseteq E} \left[\sum_{X \subseteq I} (-1)^{|X|} \binom{r_{M/(E-I)} - r_{M/(E-I)}(X)+1}{2} \right] \Delta_I \\ &= \sum_{I \subseteq E} \widetilde{\gamma}(M/(E-I)) \Delta_I \end{aligned}$$

as desired. □

Corollary 5.5 *If a connected matroid M has n elements, then*

$$\operatorname{Vol} P_{\mathcal{F}(M)} = \frac{1}{(n-1)!} \sum_{(J_1,\ldots,J_{n-1})} \widetilde{\gamma}(M/J_1)\widetilde{\gamma}(M/J_2)\cdots\widetilde{\gamma}(M/J_{n-1}),$$

summing over the ordered collections of sets $J_1, \ldots, J_{n-1} \subseteq [n]$ such that, for any distinct $i_1, \ldots, i_k$, $|J_{i_1} \cap \cdots \cap J_{i_k}| < n-k$.

Proof: This follows from Proposition 3.2 and Theorem 5.4. □

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Riffle shuffles of a deck with repeated cards

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We study the Gilbert-Shannon-Reeds model for riffle shuffles and ask 'How many times must a deck of cards be shuffled for the deck to be in close to random order?'. In 1992, Bayer and Diaconis gave a solution which gives exact and asymptotic results for all decks of practical interest, e.g. a deck of 52 cards. But what if one only cares about the colors of the cards or disregards the suits focusing solely on the ranks? More generally, how does the rate of convergence of a Markov chain change if we are interested in only certain features? Our exploration of this problem takes us through random walks on groups and their cosets, discovering along the way exact formulas leading to interesting combinatorics, an 'amazing matrix', and new analytic methods which produce a completely general asymptotic solution that is remarkable accurate.

Keywords: card shuffling, lumping of Markov chains, Poisson summation

1 Introduction

A basic question in scientific computing is 'How many times must an iterative procedure be run?'. A basic answer is 'It depends.'. In this paper we study the mixing properties of the Gilbert-Shannon-Reeds model [19, 21] for riffle shuffling a deck of n cards and ask how many times the deck must be shuffled for the cards to be in close to random order. Our answer depends not only on the metric we use to measure distance to uniformity, but also on the particular properties of the deck that are of interest.

To be precise, we consider a 'deck' to be a multiset of n cards. We shuffle the deck by first cutting it into two piles according to the binomial distribution, and then riffling the piles together by successively dropping cards from either pile with probability proportional to the size. This process defines a measure, denoted $Q_2(\sigma)$, on the symmetric group $\mathcal{S}_n$. Repeated shuffles are defined by *convolution powers*

$$Q_2^{*k}(\sigma) = \sum_{\omega \cdot \tau = \sigma} Q_2(\tau) Q_2^{*(k-1)}(\omega). \tag{1}$$

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This shuffling model, which accurately models how most people actually shuffle a deck of cards, was introduced by Gilbert and Shannon [19] and independently by Reeds [21].

Bayer and Diaconis [3] generalized this to a-shuffles, which is the natural extension to shuffling with a hands: the deck is cut into a packets by multinomial distribution and cards are successively dropped from packets with probability proportional to packet size. Letting $Q_a(\sigma)$ denote this measure, they show that convolution of general a-shuffles is as nice as possible, namely

$$Q_a * Q_b = Q_{ab}. \tag{2}$$

Thus it is enough to study a single a-shuffle of the deck.

To that end, denote the *uniform distribution* by $U = U(\sigma)$. For a deck with n distinct cards, $U = 1/n!$, and for a more general deck with D_1 1's, D_2 2's, up to D_m m's, we have $U = 1/\binom{D_1+\cdots+D_m}{D_1,\ldots,D_m}$. There are several ways to measure the distance between Q_a and U, though for the purposes of this paper we restrict our attention to total variation distance and separation distance.

The *total variation distance* is defined by

$$\|Q_a - U\|_{TV} = \max_{\text{subsets} A} |Q_a(A) - U(A)| = \frac{1}{2}\sum_{\sigma} |Q_a(\sigma) - U(\sigma)|. \tag{3}$$

In general, the formulas for $Q_a(\sigma)$ may be quite complicated, making calculations of total variation intractable. Therefore we will also consider the *separation distance* defined by

$$\mathrm{SEP}(a) = \max_{\sigma} 1 - \frac{Q_a(\sigma)}{U(\sigma)}. \tag{4}$$

Here, only a single probability needs to be computed, though as we shall see even that can be difficult. From the formulas above, one can easily see that separation provides an upper bound for total variation, which makes separation a good measure to use when total variation becomes too complicated to compute.

In widely cited works, Aldous [2] and Bayer and Diaconis [3] show that $\frac{3}{2}\log_2(n) + c$ shuffles are necessary and sufficient to make the total variation distance small, while $2\log_2(n) + c$ shuffles are necessary and sufficient to make separation small. These results, however, look at all aspects of a permutation, i.e. consider a deck with distinct cards. In many card games, only certain aspects of the permutation matter. For instance, in Baccarat, suits are irrelevant and all 10's and picture cards are equivalent, and in ESP card guessing experiments, a Zener deck of 25 cards with each of 5 symbols repeated five times is used. It is natural, therefore, to ask how many shuffles are required in these situations, and so we consider a deck to have repeated cards.

Many results are known for how long it takes certain features of a permutation, e.g. longest cycle, descent structure, etc, to become random; for a thorough treatment of such results see [11]. The particular problem we address in this paper was first addressed by Conger and Viswanath [8, 9] who derive remarkable numerical procedures giving useful answers for cases of practical interest.

In this paper, we present many of our main results from [**?**], giving exact formulae and asymptotics for a deck of n cards with D_1 cards labelled 1, D_2 cards labelled 2, …, D_m cards labelled m. Our results are proved from the deck starting 'in order', i.e. with 1's on top through m's at the bottom. In Section 2, we show that the processes we study are Markov by framing the problem in the context of random walks on cosets. We derive a formula for the transition matrix following a single card in Section 3, and show that this matrix shares many properties with Holte's 'Amazing Matrix' [20]. In Section 4, we consider a general deck, limiting our metric to the separation distance, and derive new formulae and asymptotic approximations which we unify into our 'rule of thumb' formula. Section 5 shows that our results depend on the initial configuration of the deck, a fact also observed by Conger and Viswanath [8, 9, **?**]. This extended abstract contains precise statements of our main results along with the main ideas of the proofs; for full details see [**?**].

2 Random walks on Young subgroups

In this section, we reformulate shuffling in terms of random walks on a finite group, so that our investigation of particular properties of a deck becomes a quotient walk on Young subgroups of $\mathcal{S}_n$.

Let G be a finite group, and let Q be a probability on G, i.e. $Q(g) \geq 0$ and $\sum_{g \in G} Q(g) = 1$. Take a *random walk on* G by repeatedly choosing elements independently from G with probability Q, say $g_1, g_2, g_3, \ldots$, and, beginning with the identity element 1_G, multiply on the left by g_i. This generates the following sequence of elements, the left walk,

$$1_G,\ g_1,\ g_2g_1,\ g_3g_2g_1,\ \ldots.$$

By inspection, the chance that the walk is at g after k steps is given by convolution formula (1) $Q^{*k}(g)$, where $Q^0(g) = \delta_{1_G,g}$.

To focus on certain aspects of the walk, we choose a subgroup and consider the *quotient walk* as follows. Let $H \leq G$ be a subgroup of G, and let X denote the set of left cosets of H in G, i.e. $X = G/H = \{xH\}$. The quotient walk on X is derived from the left walk on G by reporting the coset to which the current position of the walk belongs. This defines a Markov chain on X with transition matrix given by

$$K(x, y) = Q(yHx^{-1}) = \sum_{h \in H} Q(yhx^{-1}). \tag{5}$$

Note that K is well-defined (i.e. independent of the choice of coset representatives) and doubly stochastic. Thus the uniform distribution on $X, U = |H|/|G|$, is a stationary distribution for K. The following result, showing that powers of K correspond precisely to convolving and taking cosets, is intuitively obvious with a straightforward proof.

Proposition 2.1 *For Q a probability distribution on a finite group G and K as defined in (5), we have*

$$K^l(x, y) = Q^{*l}(yHx^{-1}).$$

We may identify permutations in $\mathcal{S}_n$ with arrangements of a deck of n cards by setting $\sigma(i)$ to be the label of the card at position i from the top. For instance, the permutation 2 1 4 3 is associated with four cards where "2" is on top, followed by "1", followed by "4", and finally "3" is on the bottom. Therefore the random walk on $\mathcal{S}_n$ with probability Q_2 corresponds precisely to riffle shuffles of a deck of n distinct cards. If we consider the first D_1 cards to be labelled "1", the next D_2 cards to be labelled "2", and so on up to the last D_m cards labelled "m", then this corresponds precisely to the coset space of a Young subgroup,

$$X = \mathcal{S}_n / \left(\mathcal{S}_{D_1} \times \mathcal{S}_{D_2} \times \cdots \times \mathcal{S}_{D_m}\right).$$

Thus Proposition 2.1 shows that the processes studied in the body of this paper are Markov chains.

3 A new 'amazing' matrix

Suppose the ace of spades is on the bottom of a deck of n cards. How many shuffles does it take until this one card is close to uniformly distributed on $\{1, 2, \ldots, n\}$? We analyze this problem by writing down the transition matrix following a single card through an otherwise indistinguishable deck.

Proposition 3.1 *Let $P_a(i,j)$ be the chance that the card at position i moves to position j after an a-shuffle. For $1 \le i, j \le n$, $P_a(i,j)$ is given by*

$$\frac{1}{a^n}\sum_{k=1}^{a}\sum_{r=l}^{u}\binom{j-1}{r}\binom{n-j}{i-r-1}k^r(a-k)^{j-1-r}(k-1)^{i-1-r}(a-k+1)^{(n-j)-(i-r-1)}$$

where r ranges from $l = \max(0, (i+j)-(n+1))$ to $u = \min(i-1, j-1)$.

Proof: Consider the number of ways that an inverse a-shuffle can bring the card at position j to position i. First, the card at position j must have come from some pile, say k, $1 \le k \le a$. Say r of the cards above this came from piles 1 to k, and so the remaining $j-1-r$ came from piles $k+1$ to a. Those r cards all must appear before the card at position j in $\binom{j-1}{r}$ ways. This leaves $i-1-r$ cards below position j which came from piles 1 to $k-1$ in $\binom{n-j}{i-r-1}$ ways, and the remaining cards must be from piles k to a. □

For example, the $n \times n$ transition matrices for $n = 2, 3$ are given below.

$$\frac{1}{2a}\begin{pmatrix} a+1 & a-1 \\ a-1 & a+1 \end{pmatrix} \qquad \frac{1}{6a^2}\begin{pmatrix} (a+1)(2a+1) & 2(a^2-1) & (a-1)(2a-1) \\ 2(a^2-1) & 2(a^2+2) & 2(a^2-1) \\ (a-1)(2a-1) & 2(a^2-1) & (a+1)(2a+1) \end{pmatrix}$$

These matrices share many properties, given in Proposition 3.2, with the 'amazing matrix' discovered by Holte [20] in his study of the 'carries process' of ordinary addition. Diaconis and Fulman [12] show that Holte's matrix is also the transition matrix for the number of descents in repeated a-shuffles. We have not been able to find a closer connection between the two matrices.

Proposition 3.2 *The transition matrices following a single card have the following properties:*

1. *they are* cross-symmetric, *i.e. $P_a(i,j) = P_a(n-i+1, n-j+1)$;*
2. *they are multiplicative, i.e. $P_a \cdot P_b = P_{ab}$;*
3. *the eigenvalues form the geometric series $1, 1/a, 1/a^2, \ldots, 1/a^{n-1}$;*
4. *the right eigen vectors are independent of a and have the simple form:*
   $V_m(i) = (i-1)^{i-1}\binom{m-1}{i-1} + (-1)^{n-i+m}\binom{m-1}{n-i}$ *for $1/a^m$, $m \ge 1$.*

Proof: The cross-symmetry (1) follows from Proposition 3.1, and the multiplicative property (2) follows from the shuffling interpretation and equation (2). Property (1) implies that the eigen structure is quite constrained. Properties (3) and (4) follow from results of Cuicu [7]. □

The following Corollary also follows as a special case of Theorem 2.2 in [8].

Corollary 3.3 *Consider a deck of n cards with the ace of spades starting at the bottom. The chance that the ace of spades is at position j from the top after an a-shuffle is*

$$Q_a(j) \;= P_a(n,j) \;=\; \frac{1}{a^n}\sum_{k=1}^{a}(k-1)^{n-j}k^{j-1}. \tag{6}$$

From the explicit formula, we are able to give exact numerical calculations and sharp asymptotics for any of the distances to uniformity. The results below show that $\log_2 n + c$ shuffles are necessary and sufficient for both separation and total variation (and there is a cutoff for these). This is surprising since, on the full permutation group, separation requires $2\log_2 n + c$ steps whereas total variation requires $\frac{3}{2}\log_2 n + c$. Of course, for any specific n, these asymptotic results are just indicative.

Tab. 1: Distance to uniformity for a deck of 52 cards. The upper table assumes distinct cards, and the lower table follows a single card starting at the bottom of the deck.

	1	2	3	4	5	6	7	8	9	10	11	12
TV	1.00	1.00	1.00	1.00	.924	.614	.334	.167	.085	.043	.021	.010
SEP	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.996	.931	.732	.479	.278

	1	2	3	4	5	6	7	8	9	10	11	12
TV	.873	.752	.577	.367	.200	.103	.052	.026	.013	.007	.003	.002
SEP	1.00	1.00	.993	.875	.605	.353	.190	.098	.050	.025	.013	.006

Remarks on Table 1. We use Proposition 3.1 to give exact results when $n = 52$. For comparison, the upper table gives exact results for the full deck using [3]. The lower table shows that it takes about half as many shuffles to achieve a given degree of mixing for a card at the bottom of the deck. For example, the widely cited '7 shuffles' for total variation drops this distance to $.334$ for the full ordering, but this requires only 4 shuffles to achieve a similar degree of randomness for a single card at the bottom.

For asymptotic results, we first derive an approximation to separation, which also serves as an upper bound for total variation. Finally, we derive a matching lower bound for total variation. Proofs have been omitted for brevity, but again full details are available in [**?**].

Proposition 3.4 *After an a-shuffle, the probability that the bottom card is at position i satisfies*

$$\frac{1}{a}\frac{\alpha^{n-i+1}}{1-\alpha^n} \le Q_a(i) \le \frac{1}{a}\frac{\alpha^{n-i}}{1-\alpha^{n-1}},$$

where for brevity we have set $\alpha = 1 - 1/a$. In particular, the separation distance satisfies

$$1 - \frac{n}{a}\frac{\alpha^n}{1-\alpha^n} \le \mathrm{SEP}(a) \le 1 - \frac{n}{a}\frac{\alpha^{n-1}}{1-\alpha^{n-1}}.$$

If $a = 2^{\log_2(n)+c} = n2^c$, then our result shows that the $\mathrm{SEP}(a)$ is approximately

$$1 - \frac{1}{2^c}\frac{e^{-2^{-c}}}{1-e^{-2^{-c}}},$$

and for large c this is $\approx 2^{-c-1}$. The fit to the data in Table 1 is excellent: for example after ten shuffles of a fifty-two card deck we have $2^{-c-1} = \frac{26}{1024}$ which is very nearly the observed separation distance of 0.025.

Remark 3.5 *Proposition 3.4 gives a local limit for the probability that the original bottom card is at position j from the bottom. When the number of shuffles is* $\log_2 n + c$*, the density of this (with respect to the uniform measure) is asymptotically* $z(c)e^{-j/2^c}$*, with* z *a normalizing constant (*$z(c) = 1/2^c(e^{j/2^c} - 1)$*). The result is uniform in* j *for* c *fixed,* n *large.*

Proposition 3.6 *Consider a deck of* n *cards with the ace of spades at the bottom. With* $\alpha = 1 - 1/a$*, the total variation distance for the mixing of the ace of spades after an* a*-shuffle is at most*

$$\frac{\alpha^{n+1}}{1-\alpha^n} - \frac{a\alpha^2(1-\alpha^{n-1})}{n(1-\alpha^n)} + \frac{1}{n\log(1/\alpha)}\log\left(\frac{a}{n}\frac{1-\alpha^n}{\alpha^{n+1}}\right),$$

and at least

$$\frac{\alpha^n}{1-\alpha^{n-1}} - \frac{a(1-\alpha^n)}{n\alpha(1-\alpha^{n-1})} + \frac{1}{n\log(1/\alpha)}\log\left(\frac{a}{n}\frac{1-\alpha^{n-1}}{\alpha^{n-1}}\right).$$

After $\log_2 n + c$ shuffles, that is when $a = 2^c n$, Proposition 3.6 shows that the total variation distance is approximately (with $C = 2^c$)

$$C\log\Big(C(e^{1/C}-1)\Big) + \frac{1 - C\log(e^{1/C}-1)}{(e^{1/C}-1)}.$$

Thus when c is 'large and negative,' the total variation is close to 1, and when c is large and positive, the total variation is close to 0. Thus total variation and separation converge at the same rate. This is an asymptotic result and, for example, Table 1 supports this.

Similar, but more demanding, calculations show that if the ace of spades starts at position i, and $\max(i/n, (n-i)/n) \geq A > 0$ for some fixed positive A, then $\frac{1}{2}\log_2 n$ shuffles suffice for convergence in any of the metrics. We omit further details.

4 Separation distance for the general case

A main result of Bayer and Diaconis [3] is the simple formula for an a-shuffle of a deck of n distinct cards:

$$Q_a(\sigma) = \frac{1}{a^n}\binom{n+a-r}{n}, \tag{7}$$

where $r = r(\sigma)$ is the number of rising sequences in σ, equivalently one more than the number of descents in σ^{-1}. This formula allows simple closed form expressions for a variety of distances as well as asymptotic analysis.

In this section we work with general decks containing D_i cards labelled i, $1 \le i \le m$. The formulae of this section hardly resemble the elegant expression above. Further, we only give precise formula for the least likely deck. The following lemma shows that this deck, where the separation distance is achieved, is the reverse the initial deck configuration. This is equivalent to Theorem 2.1 from [8].

Proposition 4.1 *Let D be a deck as above. After an a-shuffle of the deck with 1's on top down to m's on bottom, the least likely configuration is the reverse deck w^* with m's on top down to 1's on the bottom.*

Proof: The only cuts of the initial deck resulting in w^* are those containing no pile with distinct letters. For all such cuts, each rearrangement of the deck is equally likely to occur. □

While finding a completely general formula for $Q_a(w)$ for arbitrary w is infeasible, below we do this for w^*.

Theorem 4.2 *Consider a deck with n cards and D_i cards labeled i, $i = 1, \dots, m$. Then the separation distance after an a-shuffle of the sorted deck (1's followed by 2's, etc) is given by*

$$\mathrm{SEP}(a) = 1 - \frac{1}{a^n}\binom{n}{D_1 \dots D_m} \sum_{0=k_0<\cdots<k_{m-1}<a} (a-k_{m-1})^{D_m} \prod_{j=1}^{m-1} \left((k_j-k_{j-1})^{D_j} - (k_j-k_{j-1}-1)^{D_j}\right).$$

Proof: From the analysis in the proof of Proposition 4.1, $Q_a(w^*)$ is given by

$$Q_a(w^*) = \sum_{\substack{A_1+\cdots+A_a=n \\ A \text{ refines } D}} \frac{1}{a^n}\binom{n}{A_1, \dots, A_a} \frac{1}{\binom{n}{D_1, \dots, D_m}},$$

where 'A refines D' means there exist indices $k_1, \dots, k_{m-1}$ such that $A_1 + \cdots + A_{k_1} = D_1$ and, for $i = 2, \dots, m-1$, $A_{k_{i-1}+1} + \cdots + A_{k_i} = D_i$. Taking the k_i's to be minimal, the expression for $Q_a(w^*)$ simplifies to

$$\frac{1}{a^n} \sum_{0=k_0<\cdots<k_{m-1}<a} (a-k_{m-1})^{D_m} \prod_{j=1}^{m-1} \left((k_j-k_{j-1})^{D_j} - (k_j-k_{j-1}-1)^{D_j}\right). \tag{8}$$

The result now follows from Proposition 4.1. □

Remarks on Table 2. We calculate SEP after repeated 2-shuffles for various decks using Theorem 4.2: (blackjack) 9 ranks with 4 cards each and another rank with 16 cards; (♣♢♡♠) 4 distinct suits of 13 cards each; (A♠) the ace of spades and 51 other cards; (redblack) a two color deck with 26 of either color; and (○+≋□☆) a deck with 5 cards in each of 5 suits. The missing entries in Table 2 highlight the limitations of exact calculations using Theorem 4.2.

Remark 4.3 *Comparing the data in Table 2 for* A♠ *and* redblack *shows that these two cases are remarkably similar. Indeed, both cases exhibit the same asymptotic behavior, which is remarkable since the* A♠ *has a state space of size* 52 *while* redblack *has a state space of size around* 5×10^{14}.

Tab. 2: Separation distance for k shuffles of 52 cards.

k	1	2	3	4	5	6	7	8	9	10	11	12
BD-92	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.995	.928	.729	.478	.278
blackjack	1.00	1.00	1.00	1.00	.999	.970						
♣♢♡♠	1.00	.997	.997	.976	.884	.683	.447	.260	.140	.073		
A♠	1.00	1.00	.993	.875	.605	.353	.190	.098	.050	.025	.013	.006
redblack	.890	.890	.849	.708	.508	.317	.179	.095	.049	.025	.013	.006
○+≋□☆	1.00	1.00	.993	.943	.778	.536	.321	.177				

Now we derive a basic asymptotic tool which allows asymptotic approximations for general decks.

Proposition 4.4 *Let $m \geq 2$ and a be natural numbers, let $\xi_1, \ldots, \xi_m$ be real numbers in $[0,1]$. Let $r_1, \ldots, r_m$ be natural numbers all at least $r \geq 2$. Let*

$$S_m(a;\underline{\xi},\underline{r}) = \sum_{\substack{a_1,\ldots,a_m\geq 0\\ a_1+\ldots+a_m=a}} (a_1+\xi_1)^{r_1}\cdots(a_m+\xi_m)^{r_m}.$$

Then

$$\Big|S_m(a;\underline{\xi},\underline{r}) - \frac{r_1!\cdots r_m!}{(r_1+\ldots+r_m+m-1)!}(a+\xi_1+\ldots+\xi_m)^{r_1+\ldots+r_m+m-1}\Big|$$
$$\leq r_1!\cdots r_m! \sum_{j=1}^{m-1}\binom{m-1}{j}\Big(\frac{1}{3(r-1)}\Big)^j \frac{(a+\xi_1+\ldots+\xi_m)^{r_1+\ldots+r_m+m-1-2j}}{(r_1+\ldots+r_m+m-1-2j)!}.$$

Consider a general deck of n cards with D_i cards labelled i. We use Proposition 4.4 to find asymptotics for the separation distance given in Theorem 4.2. The following is our ‘rule of thumb.’

Theorem 4.5 *For a deck of n cards as above, suppose $D_i \geq d \geq 3$ for all $1 \leq i \leq m$. Then we have*

$$\mathrm{SEP}(a) = 1-(1+\eta)\frac{a^{m-1}}{(n+1)\cdots(n+m-1)}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\Big(1-\frac{j}{a}\Big)^{n+m-1},$$

where η is a real number satisfying

$$|\eta| \leq \Big(1+\frac{n^2}{3(d-2)(a-m+1)^2}\Big)^{m-1}-1.$$

Proof: To evaluate the expression in Theorem 4.2, we require an understanding of

$$\sum_{\substack{a_1+\ldots+a_m=a\\ a_j\geq 1}} a_m^{D_m}\prod_{j=1}^{m-1}(a_j^{D_j}-(a_j-1)^{D_j}) = \int_0^1\cdots\int_0^1 \sum_{\substack{a_1+\ldots+a_m=a\\ a_j\geq 1}} a_m^{D_m}\prod_{j=1}^{m-1}\Big(D_j(a_j-1+\xi_j)^{D_j-1}d\xi_j\Big).$$

We now invoke Proposition 4.4. Thus the above equals for some $|\theta| \leq 1$

$$\begin{aligned}&\prod_{j=1}^{m} D_j! \int_0^1 \cdots \int_0^1 \Big(\frac{(a-(m-1)+\xi_1+\ldots+\xi_{m-1})^n}{n!} + \\ &+\theta \sum_{j=1}^{m-1} \binom{m-1}{j} \Big(\frac{1}{3(d-2)}\Big)^j \frac{(a-(m-1)+\xi_1+\ldots+\xi_{m-1})^{n-2j}}{(n-2j)!}\Big) d\xi_1 \cdots d\xi_{m-1}.\end{aligned}$$

We may simplify the above as

$$\Big(1+\theta\Big\{\Big(1+\frac{n^2}{3(d-2)(a-m+1)^2}\Big)^{m-1} - 1\Big\}\Big) \frac{D_1! \cdots D_m!}{n!} \int_0^1 \cdots \int_0^1 (a-m+1+\xi_1+\cdots+\xi_{m-1})^n d\xi_1 \cdots d\xi_{m-1},$$

and evaluating the integrals above this is

$$\left(1+\theta\Big\{\left(1+\frac{n^2}{3(d-2)(a-m+1)^2}\right)^{m-1} - 1\Big\}\right) \frac{D_1! \cdots D_m!}{n!} \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} (a-j)^{n-m+1}.$$

The Theorem follows. □

For simplicity we have restricted ourselves to the case when each pile has at least three cards. With more effort we could extend the analysis to include doubleton piles. The case of some singleton piles needs some modifications to our formula, but this variant can also be worked out. Below we use our rule of thumb to calculate separation for the same decks as in Table 2.

Tab. 3: Rule of Thumb for the separation distance for k shuffles of 52 cards.

k	1	2	3	4	5	6	7	8	9	10	11	12
BD-92	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.995	.928	.729	.478	.278
blackjack	1.00	1.00	1.00	1.00	.999	.970	.834	.596	.366	.204	.108	.056
♣♢♡♠	1.00	1.00	.997	.976	.884	.683	.447	.260	.140	.073	.037	.019
redblack	.962	.925	.849	.708	.508	.317	.179	.095	.049	.025	.013	.006
○+≋□☆	1.00	1.00	.993	.943	.778	.536	.321	.177	.093	.048	.024	.012

Remarks on Table 3. The first row gives exact results from the Bayer-Diaconis formula for the full permutation group. The other numbers are from the rule of thumb. Roughly, the single card or red-black numbers suggest that half the usual number of shuffles suffice. The Black-Jack (equivalently Baccarat) numbers suggest a savings of two or three shuffles, and the suit numbers lie in between. The final row is the rule of thumb for the Zener deck with 25 cards, 5 cards for each of 5 suits.

While asymptotic, Theorem 4.5 is astonishingly accurate for decks of practical interest. For instance, comparing exact calculations in Table 2 with approximations using this rule of thumb in Table 3 shows that after only 3 shuffles, the numbers agree to the given precision. Moreover, the simplicity of the formula in Theorem 4.5 allows much further computations than are possible using the formula in Theorem 4.2.

We now give a heuristic for why our rule of thumb is numerically so accurate. For $k \geq 0$, define

$$f_k(z) = \sum_{r=0}^{\infty} r^k z^r = \frac{A_k(z)}{(1-z)^{k+1}},$$

where $A_k(z)$ denotes the k-th Eulerian polynomial. The sum over $a_1, \ldots, a_m$ appearing in our proof of Theorem 4.5 is simply the coefficient of z^a in the generating function $(1-z)^{m-1} f_{D_1}(z) \cdots f_{D_m}(z)$. Our rule of thumb may be interpreted as saying that

$$(1-z)^{m-1} f_{D_1}(z) \cdots f_{D_m}(z) \approx \frac{D_1! \cdots D_m!}{(n+m-1)!} (1-z)^{m-1} f_{n+m-1}(z). \tag{9}$$

To explain the sense in which (9) holds, note that $f_k(z)$ extends meromorphically to the complex plane, and it has a pole of order $k+1$ at $z=1$. Moreover it is easy to see that $f_k(z) - k!/(1-z)^{k+1}$ has a pole of order at most k at $z=1$. Therefore, the LHS and RHS of (9) have poles of order $n+1$ at $z=1$, and their leading order contributions match. Therefore the difference between the RHS and LHS of (9) has a pole of order at most n at $z=1$. But in fact, this difference can have a pole of order at most $n-d$ at $z=1$, and thus the approximation in (9) is tighter than what may be expected *a priori*. To obtain our result on the order of the pole, we record that one can show

$$f_k(z) = \frac{k!}{(1-z)^{k+1}} \Big(\frac{(z-1)}{\log z} \Big)^{k+1} + \zeta(-k) + O(1-z).$$

5 Gilbreath principle at work

Conger and Viswanath note that the initial configuration can affect the speed of convergence to stationary. Perhaps this is most striking in the case of Section 3 where a single card is tracked. Recall Table 1, giving calculations for the distinguished card beginning at the bottom of a deck of 52 cards. In contrast, Table 4 gives calculations for the distinguished card starting in the middle, at position 26. For the latter, both total variation and separation are indistinguishable from zero after only four shuffles.

Tab. 4: Distance to uniformity for a single card starting at the middle of a 52 card deck.

	1	2	3	4
TV	.494	.152	.001	.000
SEP	1.00	.487	.003	.000

Consider next a deck with n red and n black cards. First take the starting condition of all reds atop all blacks. If the initial cut is at n (the most likely value) then the red-black pattern is perfectly mixed after a single shuffle. More generally, the chance of the deck w resulting from a single 2-shuffle of a deck with n red cards atop n black cards is given by

$$Q_2(w) = \frac{1}{2^{2n}} \left(2^{\mathrm{h}(w)} + 2^{\mathrm{t}(w)} - 1 \right),$$

where $\mathrm{h}(w)$ is the number of red cards before the first black card and $\mathrm{t}(w)$ is the number of black cards after the final red card; see [**?**]. In particular, the total variation after a single 2-shuffle is

$$\|Q_2 - U\|_{TV} = \frac{1}{2}\left(\left(\frac{2^{n+1}-1}{2^{2n}} - \frac{1}{\binom{2n}{n}}\right) + \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left|\frac{2^i+2^j-1}{2^{2n}} - \frac{1}{\binom{2n}{n}}\right|\binom{2n-(i+j+2)}{n-(i+1)}\right) \quad (10)$$

Evaluating this formula for $2n = 52$ give a total variation of 0.579.

Now take the starting condition to alternate red black red black, etc. As motivation, we recall a popular card trick: Begin with a deck of $2n$ cards arranged alternately red, black, red, black, etc. The deck may be cut any number of times. Have the deck turned face up and cut (with cuts completed) until one of the cuts results in the two piles having cards of opposite color uppermost. At this point, ask one of the participants to riffle shuffle the two piles together. The resulting arrangement has the top two cards containing one red and one black, the next two cards containing one red and one black, and so on throughout the deck. This trick is called the Gilbreath Principle after its inventor, the mathematician Norman Gilbreath. It is developed, with many variations, in Chapter 4 of [18]. From the trick we see that beginning with an alternating deck severely limits the possibilities. Analyzing the trick reveals the following formula,

$$2^{2n}\cdot Q_2(w) = \begin{cases} 2^{n-1}+2^n & \text{if } w \text{ is the initial alternating deck,} \\ 2^{n-1} & \text{if } w \text{ can result from an odd cut,} \\ 2^n & \text{if } w \text{ can result from an even cut,} \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

where an odd (resp. even) cut refers to the parity of cards in either pile. From this we compute

$$\|Q_2 - U\|_{TV} = \frac{1}{2}\left(1 - \frac{2^n + 2^{n-1} - 1}{\binom{2n}{n}}\right), \quad (12)$$

which goes to $.5$ exponentially fast as n goes to infinity, and indeed is already .500 for $2n = 52$. In contrast, starting with reds above blacks, asymptotic analysis of (10) shows that the total variation tends to 1 after a single shuffle when n is large. Thus again an alternating start leads to faster mixing.

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A kicking basis for the two column Garsia-Haiman modules

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In the early 1990s, Garsia and Haiman conjectured that the dimension of the Garsia-Haiman module R_μ is $n!$, and they showed that the resolution of this conjecture implies the Macdonald Positivity Conjecture. Haiman proved these conjectures in 2001 using algebraic geometry, but the question remains to find an explicit basis for R_μ which would give a simple proof of the dimension. Using the theory of Orbit Harmonics developed by Garsia and Haiman, we present a "kicking basis" for R_μ when μ has two columns.

Keywords: Macdonald polynomials, Garsia-Haiman modules, combinatorial basis

1 Introduction

In 1988, Macdonald [14] found a remarkable new basis of symmetric functions in two parameters which specializes to Schur functions, complete homogeneous, elementary and monomial symmetric functions and Hall-Littlewood functions, among others. With an appropriate analog of the Hall inner product, the transformed Macdonald polynomials $\widetilde{H}_\mu(Z;q,t)$ are uniquely characterized by certain triangularity and orthogonality conditions, from which their symmetry follows. The *Kostka-Macdonald* polynomials, $\widetilde{K}_{\lambda\mu}(q,t)$, are defined by

$$\widetilde{H}_\mu(Z;q,t) = \sum_{\lambda} \widetilde{K}_{\lambda\mu}(q,t) s_\lambda(Z).$$

The Macdonald Positivity Conjecture states that $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$.

In 1993, Garsia and Haiman [6] conjectured that the transformed Macdonald polynomials could be realized as the bigraded characters for a diagonal action of S_n on two sets of variables. Moreover, they were able to show that knowing the dimension of this module is enough to determine its character. Therefore the $n!$ Conjecture, which states that the dimension of the Garsia-Haiman module is $n!$, implies the Macdonald Positivity Conjecture.

By analyzing the algebraic geometry of the Hilbert scheme of n points in the plane, Haiman [13] was able to prove the $n!$ Conjecture and consequently establish Macdonald Positivity. However, it remains an important open problem in the theory of Macdonald polynomials to prove the $n!$ Theorem directly by finding an explicit basis for the module. After reviewing Macdonald polynomials and the Garsia-Haiman modules in Section 2, we give an explicit basis for the Garsia-Haiman modules indexed by a partition with at most two columns in Section 3. A new basis for hooks is also given in Section 4.

2 Macdonald polynomials and graded $\mathcal{S}_n$-modules

We assume the definitions and notations from [15] of partitions and the classical bases for symmetric functions. So as to avoid confusions when defining various modules, we use the alphabet $Z = z_1, \ldots, z_n$ for symmetric functions. For example, the *Schur functions* shall be denoted $s_\lambda(Z)$.

2.1 Macdonald positivity

Departing slightly from Macdonald's convention of defining $P_\mu(Z; q, t)$ [14], we instead use the transformed Macdonald polynomials $\widetilde{H}_\mu(Z; q, t)$ as presented in [6].

Definition 2.1 *The transformed Macdonald polynomials $\widetilde{H}_\mu(Z; q, t)$ are the unique functions satisfying the following triangularity and orthogonality conditions:*

(i) $\widetilde{H}_\mu(Z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1-q)] \ : \ \lambda \geq \mu\}$;

(ii) $\widetilde{H}_\mu(Z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda[Z/(1-t)] \ : \ \lambda \geq \mu'\}$;

(iii) $\widetilde{H}_\mu[1; q, t] = 1$.

The square brackets in Definition 2.1 stand for *plethystic substitution*. In short, $s_\lambda[A]$ means s_λ applied as a Λ-ring operator to the expression A, where Λ is the ring of symmetric functions. For a thorough account of plethysm, see [12].

The existence of such a family of functions is a theorem, following in large part from Macdonald's original proof of existence. Once established, the symmetry of $\widetilde{H}_\mu(Z; q, t)$ follows by definition. Of particular importance are the change of basis coefficients from the transformed Macdonald polynomials to the Schur functions, defined by

$$\widetilde{H}_\mu(Z; q, t) = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q, t) s_\lambda(Z). \tag{1}$$

A priori, the $\widetilde{K}_{\lambda,\mu}(q, t)$ are rational functions in q and t with rational coefficients.

Theorem 2.2 ([13]) *We have $\widetilde{K}_{\lambda,\mu}(q, t) \in \mathbb{N}[q, t]$.*

Macdonald originally conjectured Theorem 2.2 when he introduced the polynomials in 1988. The original proof, due to Haiman in 2001, realizes $\widetilde{H}_\mu(Z; q, t)$ as the bigraded character of certain modules for the diagonal action of $\mathcal{S}_n$ on $\mathbb{Q}[X, Y]$; see sections 2.2 and 2.3. From this it follows that the character can be written as a sum of irreducible representations of $\mathcal{S}_n$ with coefficients in $\mathbb{N}[q, t]$. Under the Frobenius image, these coefficients exactly give $\widetilde{K}_{\lambda,\mu}(q, t)$. The aim of this paper is to follow this method of proof until it departs the realm of representation theory for algebraic geometry.

It is worth noting that there are now two additional proofs of Macdonald positivity, both of which utilize an expansion of Macdonald polynomials in terms of LLT polynomials conjectured by Haglund [10] and proved along with Haiman and Loehr [11]. Grojnowski and Haiman [9] have a proof using Kazhdan-Lusztig theory and the first author [3] has a purely combinatorial proof.

2.2 Garsia-Haiman modules

To define the modules mentioned in Section 2.1, we consider the diagonal action of the symmetric group $\mathcal{S}_n$ on the polynomial ring $\mathbb{Q}[X,Y] = \mathbb{Q}[x_1,\ldots,x_n;y_1,\ldots,y_n]$ permuting the x_i's and y_j's simultaneously and identically. Let the coordinates of the diagram of a partition μ of n be $\{(p_1,q_1),\ldots,(p_n,q_n)\}$, where p gives the row coordinate and q the column coordinate indexed from zero; see Figure 1.

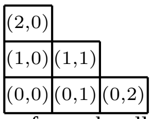

Fig. 1: The coordinates for each cell of $\mu = (3,2,1)$.

Define the polynomial $\Delta_\mu \in \mathbb{Q}[X,Y]$ by

$$\Delta_\mu(X,Y) = \det \begin{pmatrix} x_1^{p_1}y_1^{q_1} & x_2^{p_1}y_2^{q_1} & \cdots & x_n^{p_1}y_n^{q_1} \\ x_1^{p_2}y_1^{q_2} & x_2^{p_2}y_2^{q_2} & \cdots & x_n^{p_2}y_n^{q_2} \\ \vdots & \vdots & & \vdots \\ x_1^{p_n}y_1^{q_n} & x_2^{p_n}y_2^{q_n} & \cdots & x_n^{p_n}y_n^{q_n} \end{pmatrix}. \tag{2}$$

Since the bi-exponents are all distinct, Δ_μ is a non-zero homogeneous $\mathcal{S}_n$-alternating polynomial with top degree $n(\mu) = \sum_i (i-1)\mu_i$ in X and $n(\mu')$ in Y. Taking $\mu = (1^n)$ or $\mu = (n)$ gives the Vandermonde determinant in X or Y, respectively.

Let $\mathcal{I}_\mu \subset \mathbb{Q}[X,Y]$ be the ideal of polynomials φ such that

$$\varphi(\partial/\partial x_1,\ldots,\partial/\partial x_n;\partial/\partial y_1,\ldots,\partial/\partial y_n)\Delta_\mu = 0.$$

Clearly this defines an $\mathcal{S}_n$ invariant doubly homogeneous ideal. Define the *Garsia-Haiman module* $\mathcal{H}_\mu$ to be the quotient ring $\mathbb{Q}[X,Y]/\mathcal{I}_\mu$ with its natural structure of a doubly graded $\mathcal{S}_n$-module.

Garsia and Haiman [7] proved that if this module has the correct dimension (the $n!$ Conjecture), then the bi-graded character is given by the transformed Macdonald polynomial.

Theorem 2.3 ([7]) *If $\mathcal{H}_\mu$ affords the regular representation of $\mathcal{S}_n$, then the bi-graded Frobenius character, given by*

$$\mathrm{Frob}_{\mathcal{H}_\mu}(Z;q,t) = \sum_{i,j} t^i q^j \psi\left((\mathcal{H}_\mu)_{i,j}\right),$$

where ψ is the usual Frobenius map sending the Specht module S^λ to the Schur function s_λ, is equal to the transformed Macdonald polynomials $\widetilde{H}_\mu(Z;q,t)$. In particular, $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.

The following theorem is the famed $n!$ Conjecture of Garsia and Haiman [6], proved by Haiman [13] in 2001.

Theorem 2.4 ([13]) *The dimension of $\mathcal{H}_\mu$ is $n!$.*

By Theorem 2.3, Haiman's proof of the $n!$ Conjecture provided the first proof of the Macdonald positivity conjecture. Haiman's proof analyzes the isospectral Hilbert scheme of n points in a plane, ultimately showing that it is Cohen-Macaulay (and Gorenstein). As this proof uses difficult machinery in algebraic geometry, it remains an important open problem to prove Theorem 2.4 directly by finding an explicit basis for the module $\mathcal{H}_\mu$.

2.3 Orbit Harmonics

Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be sequences of distinct rational numbers. Let $(p_1, q_1), \ldots, (p_n, q_n)$ be the coordinates of the cells of μ taken in some order, recorded by the standard filling S of μ given by placing the entry i in the cell (p_i, q_i). To each S, associate the *orbit point of* S, denoted p_S, defined by

$$p_S = (\alpha_{p_1+1}, \ldots, \alpha_{p_n+1}; \beta_{q_1+1}, \ldots, \beta_{q_n+1}). \tag{3}$$

Here the shift in indices is a notational convenience. For example,

$$p_{\tiny\begin{array}{|c|c|}\hline 5&3\\\hline 6&1\\\hline 2&4\\\hline\end{array}} = (\alpha_2, \alpha_1, \alpha_3, \alpha_1, \alpha_3, \alpha_2; \beta_2, \beta_1, \beta_2, \beta_2, \beta_1, \beta_1).$$

Let $\mathcal{S}_n$ act on $\mathbb{Q}^{2n}$ by permuting the first n and second n coordinates simultaneously and identically. Let $[p_S]$ denote the ***regular*** orbit of p_S under this action. Regarding $\mathbb{Q}[X, Y]$ as the coordinate ring of $\mathbb{Q}^{2n}$, define $\mathcal{J}_\mu \subset \mathbb{Q}[X, Y]$ to be the ideal of polynomials vanishing on $[p_S]$. Define the module R_μ to be the coordinate ring of $[p_S]$, i.e. $\mathbb{Q}[X, Y]/\mathcal{J}_\mu$, with its natural $\mathcal{S}_n$ action.

Since R_μ clearly affords the regular representation, the aim is to relate this module to $\mathcal{H}_\mu$. To do this, construct the associated graded module $\mathrm{gr} R_\mu = \mathbb{Q}[X, Y]/\mathrm{gr}\mathcal{J}_\mu$. Garsia and Haiman showed that if $\mathcal{H}_\mu$ and $\mathrm{gr} R_\mu$ have the same Hilbert series, then $\mathcal{H}_\mu = \mathrm{gr} R_\mu$. While this would demonstrate the $n!$ Conjecture, the obvious problem is that one needs first to know the Hilbert series of $\mathcal{H}_\mu$, in which case the dimension can be directly calculated. The way around this problem lies in the theory of Orbit Harmonics developed by Garsia and Haiman. The main result is the following.

Theorem 2.5 ([5]) *Let Φ_μ be a basis for R_μ. Let $F_\mu(q,t) = \sum_{\varphi \in \Phi_\mu} \widehat{\varphi}(t, \ldots, t; q, \ldots, q)$, where $\widehat{\varphi}$ is the leading term of φ. If F_μ is symmetric in the following sense,*

$$\left[t^i q^j\right] \; F_\mu(q,t) \; = \; \left[t^{n(\mu)-i} q^{n(\mu')-j}\right] \; F_\mu(q,t), \tag{4}$$

then $\widehat{\Phi}_\mu = \{\widehat{\varphi} \mid \varphi \in \Phi_\mu\}$ is a basis for $\mathrm{gr} R_\mu$. Moreover, $\mathrm{gr} R_\mu \cong \mathcal{H}_\mu$ as doubly-graded $\mathcal{S}_n$ modules. In particular, $\dim \mathcal{H}_\mu = n!$.

Theorem 2.5 suggests the following strategy for constructing a basis for the Garsia-Haiman module $\mathcal{H}_\mu$. To each filling S of μ, define a polynomial $\varphi_S \in \mathbb{Q}[X, Y]$ so that the evaluation matrix $(\varphi_S(p_T))$ of polynomials on orbit points is nonsingular and the corresponding degree polynomial $F_\mu(q,t)$ is symmetric in the sense of equation (4). The remainder of this paper is devoted to carrying out this strategy in the cases when μ is a two column shape (Section 3).

3 Two columns

Throughout this section, we restrict our attention to partitions with at most two columns. Following the procedure laid out in Section 2.3, we will construct a basis for R_μ such that the degree polynomial is symmetric. Following the idea of the kicking basis for the Garsia-Procesi modules described in [8], we will construct the basis together with a linear order on fillings of μ so that the evaluation matrix has nice triangularity properties. While the Garsia-Procesi case results in an upper triangular matrix with nonzero diagonal entries, our matrix will only be block triangular with respect to the largest entry.

3.1 The kicking tree

The *kicking tree of* μ provides a nice visualization of the recursive construction of the proposed basis. Though proving that the resulting collection is a basis with symmetric Hilbert series is better done from the recursive definition, the construction is better motivated from this viewpoint.

To construct the kicking tree, entries will be added to an empty shape one at a time in all possible ways in some specified order, ultimately resulting in a total ordering for the fillings. We begin by recalling the Garsia-Procesi ordering for row-increasing tableaux [8].

Let S be a partial filling of μ with distinct entries. Define a total ordering on the rows of S containing at least one empty cell, called the *row preference order*, as follows: empty rows of length 2 from top to bottom followed by (empty) rows of length 1 from top to bottom followed by rows of length 2 with a single occupant beginning with the largest occupant. Given two rows i and j of a (partial) filling S, say that *k prefers row* j *over row* i, denoted $j \succ_k i$, if j occurs before i in the row ordering on the filling obtained by removing entries less than $k+1$ from S. For example, Figure 2 shows the ranking of the rows (on the left) for two partial fillings of $(2, 2, 2, 1, 1)$.

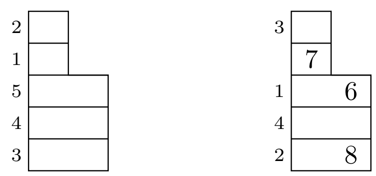

Fig. 2: The row preference order for partial fillings.

The row preference order is enough to define a total order on fillings with unsorted rows. The basic construction of the tree is to fill entries into unsorted rows one at a time according to row preference, where a row of length 2 is sorted, increasing then decreasing, as soon as it is fully occupied. The real power of the kicking tree lies in the weights assigned at each stage which we now describe.

Let S be a partial, partially sorted filling of μ with entries $n > n-1 > \cdots > k+1$. That is, each entry is assigned a row of μ, and an entry is assigned a specific column if and only if the row is fully occupied. Below S with arrows going down, place k into a row, ordered from left to right by row preference with respect to k. Label the arrow going down from S to the filling with k by

$$\prod_{j \succ_k \mathrm{row}(k)} (x_k - \alpha_j)\,.$$

If k completed a row of length 2, say with $m > k$ already in the row, then below this with arrows going down make two partial fillings: the left one having k before m and the right having m before k. Label the left branch put 1, and label the right branch

$$(y_k - \beta_1)\,.$$

If ignoring entries larger than m does not form a rectangle, then move the label from the arrow going down from S to the left-hand arrow just added, and add to the right-hand arrow

$$\prod_{\mathrm{row}(k)\succ_k i} (x_k - \alpha_i)\,.$$

The tree so constructed beginning with the empty shape μ is called the *kicking tree for* μ. For example, the kicking tree for $(2,1)$ is constructed in Figure 3. For this example, we omit vertical lines to indicate an unsorted row.

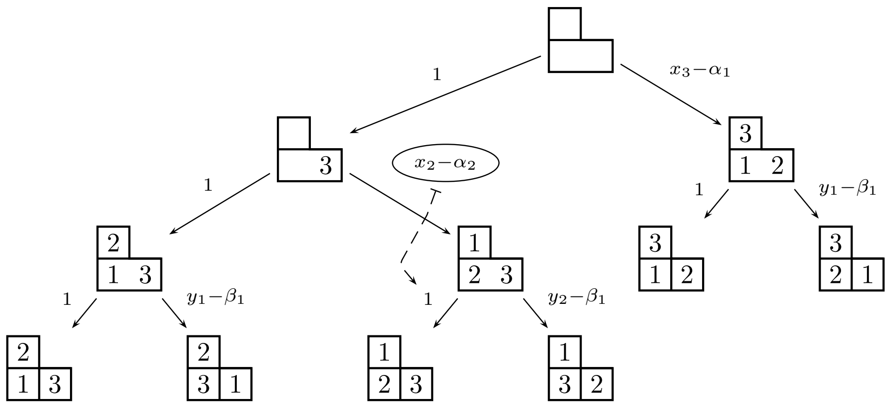

Fig. 3: The kicking tree for (2,1). Here the circled term $x_2-\alpha_2$ indicates that this term is pushed to the leftmost branch below. From left to right, the corresponding polynomials are 1 , $y_1-\beta_1$, $x_2-\alpha_2$, $y_2-\beta_1$, $x_3-\alpha_1$, $(x_3-\alpha_1)(y_1-\beta_1)$.

From the construction of the kicking tree, the product of the branch labels from a leaf S back to the empty shape μ is clearly a polynomial. The collection of polynomials for each filling of μ forms the proposed kicking basis for R_μ.

3.2 A recursive construction

In order to give an alternative recursive description of the kicking basis, we first need a bit more terminology.

For S a standard filling of size n, define $S \setminus n$ to be the standard filling of size $n-1$ obtained by removing the cell containing n and *straightening* the shape as follows. If n lies in a row of length 2, then move the remaining cell in the same row as n above rows of length 2 and below rows of length 1 and

$$\begin{array}{|c|c|}\hline 5&3\\\hline 6&1\\\hline 2&4\\\hline\end{array} \xrightarrow{S\backslash 6} \begin{array}{|c|c}\hline 1&\multicolumn{1}{|c}{}\\\hline 5&\multicolumn{1}{c|}{3}\\\hline 2&\multicolumn{1}{c|}{4}\\\hline\end{array} \xrightarrow{S\backslash 5} \begin{array}{|c|c}\hline 1&\multicolumn{1}{|c}{}\\\hline 3&\multicolumn{1}{|c}{}\\\hline 2&\multicolumn{1}{c|}{4}\\\hline\end{array} \xrightarrow{S\backslash 4} \begin{array}{|c|}\hline 1\\\hline 3\\\hline 2\\\hline\end{array} \xrightarrow{S\backslash 3} \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{S\backslash 2} \begin{array}{|c|}\hline 1\\\hline\end{array}$$

Fig. 4: An illustration of straightening after removing the largest entry.

push it to the left if necessary. Otherwise slide the cells down, preserving their order, to close the gap; see Figure 4. Notice that row dominance order commutes with straightening.

In Definition 3.1, when the largest entry of a tableau is removed and the remaining shape is straightened, the orbit point of the resulting tableau is defined using the original labelling of the rows and columns. That is, the orbit point of $S \setminus n$ is the orbit point of S with the nth and $2n$th coordinates removed. For example, in Figure 4, the orbit point of the filling of shape $(2, 1, 1)$ will be $(\alpha_2, \alpha_1, \alpha_3, \alpha_1; \beta_2, \beta_1, \beta_2, \beta_2)$.

Definition 3.1 *Define $\varphi_{\boxed{1}} = 1$. For S a standard filling of μ, $|\mu| > 1$, define φ_S recursively by*

$$\varphi_S \;=\; \varphi_{S\backslash n} \cdot \prod_{j \succ_n \mathrm{row}(n)} (x_n - \alpha_j) \cdot \begin{cases} 1 & \text{if } n \text{ at the end of row}(n) \\ (y_k - \beta_1) & \text{if } \mu = (2^b) \text{ and col}(n) = 1 \\ \dfrac{(y_k - \beta_1) \displaystyle\prod_{\mathrm{row}(k) \succ_k i} (x_k - \alpha_i)}{\displaystyle\prod_{j \succ_k \mathrm{row}(k)} (x_k - \alpha_j)} & \text{otherwise} \end{cases}$$

where k is such that row(k)=row(n).

Using the example in Figure 4, we compute

$$\varphi_{\begin{array}{|c|c|}\hline 5&3\\\hline 6&1\\\hline 2&4\\\hline\end{array}} = \overbrace{(x_6 - \alpha_3)(y_1 - \beta_1)}^{\boxed{6}} \cdot \overbrace{(y_3 - \beta_1)\left(\frac{x_3 - \alpha_1}{x_3 - \alpha_2}\right)}^{\boxed{5}} \cdot \overbrace{1}^{\boxed{4}} \cdot \overbrace{(x_3 - \alpha_2)}^{\boxed{3}} \cdot \overbrace{(x_2 - \alpha_2)}^{\boxed{2}} \cdot 1,$$

where each step in the recursion is indicated by the cell removed to obtain the given terms.

The above formula associates to each standard filling S of μ the same polynomial as the kicking tree from Section 3.1. Notice that the denominator in the last case is precisely the label which is ‘pushed down’ when constructing the kicking tree. Analyzing this statement in terms of the recursive definition yields the following result.

Proposition 3.2 *For S a standard filling of μ, φ_S is a polynomial.*

Proof: The result for $\mu = (1)$ is clear, so we proceed by induction on $n = |\mu|$. It suffices to assume k, n reside in the same length 2 row with n in column 1. We must show that each term in the denominator occurs in the numerator of $\varphi_{S\backslash n}$. The only terms that ever appear in any denominator are $x_i - \alpha_j$ where

i lies in the second column and the entry to its left is greater. In particular, if $x_k - \alpha_j$ ever occurs in a numerator in the construction of φ, it remains there through $\varphi_{S \setminus n}$. Now notice that the product outside of the brace (for k, not n) is precisely the denominator in question. $\square$

To show that these polynomials form a basis for R_μ, we show that the evaluation matrix of polynomials on orbit points is nonsingular. The argument uses a nested induction to show that the matrix is almost block triangular.

Theorem 3.3 *The $n! \times n!$ matrix $(\varphi_S(p_T))$, where S, T range over all fillings of μ, is nonsingular. In particular, the set $\{\varphi_S\}$ of polynomials associated to fillings of μ forms a basis for R_μ.*

Proof: We proceed by induction on $n = |\mu|$, the case $n = 1$ being trivial. The row preference order with respect to n makes $(\varphi_S(p_T))$ block triangular with respect to the row of n. Therefore we must show that each block, corresponding to n in a particular row, is nonsingular. If n lies in a row of length 1, this is immediate by induction, so assume n lies in a row of length 2.

For $k < n$, let T_k be a partial, partially sorted filling of μ with entries $k+1, k+2, \ldots, n$ (here n must lie in its designated row of length 2). By partially sorted, we mean that the row of each entry is determined, but the column is determined if and only if the row is fully occupied; see Figure 5 for an example. Let $\mathcal{T}_k$ be the set of standard fillings of μ which restrict to T_k on $\{k+1, \ldots, n\}$, where here again the restriction allows the column of an entry to be undetermined exactly when the other occupant of the same row is at most k; again, see Figure 5. We will show that the evaluation matrix for $\mathcal{T}_k$ is nonsingular by induction on k. As usual, the base case, $k = 1$, is trivial.

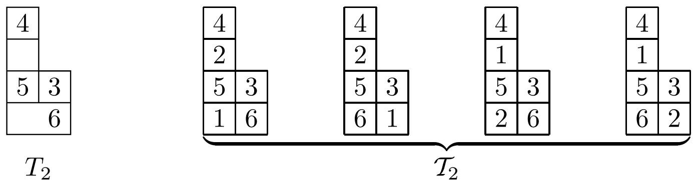

Fig. 5: An illustration of T_k and $\mathcal{T}_k$.

Restricting our attention to the set of polynomials and orbit points associated to standard fillings $S \in \mathcal{T}_k$, we put the following block ordering based on the position of k: k is the largest entry in a row of length 2 from highest row to lowest row; k lies in a row of length 1 from highest row to lowest; k lies to the left of a larger entry from largest entry to smallest; and k lies to the right of a larger entry again from smallest entry to largest. Note that the order for the first three blocks comes from the kicking tree, but the order of the fourth block is the reverse of the kicking order. By the definition of φ_S, each of the four blocks is triangular with respect to the row of k, therefore by induction each block is nonsingular since each is a fixed polynomial times the polynomials associated with $\mathcal{T}_{k-1}$ for a fixed partial filling T_{k-1}.

Also from the definition of φ_S, the first three blocks are triangular with respect to one another in the given order, and the third and fourth blocks are triangular with respect to each other as well. Moreover, for S in one of the first three cases, the monomial $(y_m - \beta_i)$ does not divide φ_S for $i = 1, 2$ and any $m \geq k$ that appears by itself in a row of length two in $\mathcal{T}_k$. Therefore the block structure of the evaluation matrix

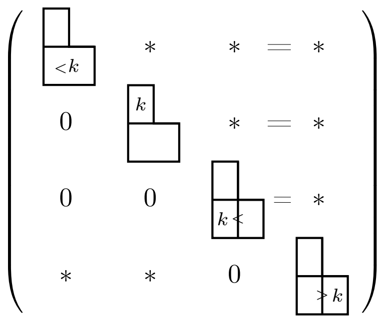

Fig. 6: Block structure of the evaluation matrix of $\mathcal{T}_k$.

is as depicted in Figure 6. Since the first two blocks are nonsingular, we may perform row reductions to eliminate the nonzero elements in the bottom block-row of the matrix. These reductions will change the bottom block-row 0 into some matrix, say M, and so by previous remarks the reductions will alter the fourth block by M as well. Hence using the row reductions from the third block to restore the 0 will also restore the fourth block. Hence the matrix can be made block triangular. Since the determinant of the matrix is the product of the determinants of the blocks, the full matrix is nonsingular. $\square$

3.3 Symmetry of the Hilbert series

Now that we have a basis for R_μ, we must show that the associated degree polynomial, denoted $F_\mu(q,t)$, is symmetric. Recall that $F_\mu(q,t)$ is given by

$$F_\mu(q,t) = \sum_{S:\mu \xrightarrow{\sim} [n]} \widehat{\varphi}_S(t,\ldots,t;q,\ldots,q), \tag{5}$$

where $\widehat{\varphi}_S$ is the highest degree term of φ_S. That is, $F_\mu(q,t)$ is the polynomial in q and t obtained by adding leading terms of the kicking basis and recording the total x degree with t and the total y degree with q.

Our aim is to show that $F_\mu(q,t)$ is symmetric, i.e.

$$F_\mu(q,t) \;=\; t^{n(\mu)}\, q^{n(\mu')}\, F_\mu(1/q,1/t). \tag{6}$$

For example, from Figure 3 we see that $F_{(2,1)}(q,t) = 1+2q+2t+qt$, which indeed exhibits the desired symmetry.

In order to establish symmetry, we will exploit a recurrence relation that follows naturally from the recursive definition of φ_S. To do this, we must first define a more general degree polynomial, denoted $J^m_{a,b}$, by

$$J^m_{a,b}(q,t) \;=\; \frac{1}{q^m} \sum_{\substack{S:\mu \xrightarrow{\sim} [n] \text{s.t.} \\ \text{for } j=0,\ldots,m-1 \\ \text{row}(n-j)=b-j, \\ \text{col}(n-j)=1}} \widehat{\varphi}_S(t,\ldots,t;q,\ldots,q), \tag{7}$$

where $a \geq b \geq m \geq 0$. Note that $J^m_{a,b}$ is a polynomial with maximum q and t exponents given by $b-m$ and $\binom{a-m}{2} + \binom{b}{2}$, respectively. Pictorially, $J^m_{a,b}$ is the degree polynomial of fillings of $(2^b, 1^{a-b})$ with the top m cells on the left-hand side of the rectangle (2^b) deleted. In particular, we have

$$J^0_{a,b}(q,t) = F_\mu(q,t). \tag{8}$$

Therefore it is enough to show that $J^m_{a,b}$ is symmetric.

Proposition 3.4 *The degree polynomials $J^m_{a,b}$ satisfy the following recurrence relations*

$$\begin{aligned} J^m_{a,b} &= [m]_t J^{m-1}_{a-1,b-1} + t^{b-m}[a-b]_t J^m_{a-1,b} + t^m [b-m]_t J^m_{a,b-1} + q[b-m]_t J^{m+1}_{a,b}, && (9)\\ J^m_{a,b} &= t^{b-m}[m]_t J^{m-1}_{a-1,b-1} + [a-b]_t J^m_{a-1,b} + q[b-m]_t J^m_{a,b-1} + t^{a-b}[b-m]_t J^{m+1}_{a,b}, && (10) \end{aligned}$$

with initial conditions

$$J^b_{a,b} = \binom{a}{b} [b]_t! \, [a-b]_t! \qquad \textit{and} \qquad J^m_{b,b} = J^0_{b,b-m},$$

where $J^m_{a,b} = 0$ unless $a \geq b \geq m \geq 0$.

The above recurrence relations follow from the recursive description in Definition 3.1. Expanding $J^m_{a,b}$ twice using both recurrence relations in Proposition 3.4 taken in one order followed by the other establishes the desired symmetry.

Theorem 3.5 *For $a \geq b \geq m \geq 0$, we have*

$$J^m_{a,b}(q,t) \;=\; t^{\binom{a-m}{2}+\binom{b}{2}} \, q^{b-m} \; J^m_{a,b}(1/q, 1/t).$$

In particular, by equation (8), Theorem 3.5 shows that the degree polynomial for the two column kicking basis is indeed symmetric. Therefore by Theorem 2.5, we have the following consequence.

Corollary 3.6 *For μ a two column partition, $\{\widehat{\varphi}_S \mid S : \mu \xrightarrow{\sim} [n]\}$ is a basis for $\mathrm{gr} R_\mu$ and so too for $\mathcal{H}_\mu$. In particular, $\dim(\mathcal{H}_\mu) = n!$ and $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$.*

4 Hooks

We next treat the case of hooks, i.e. partitions $\mu = (n-m, 1^m)$. Though there exist several known bases for Garsia-Haiman modules indexed by hooks, the first in [7] and several more in [16, 4, 2, 1]. we present this new construction because it is compatible with our two column case, i.e. the definitions of φ_S will agree on shapes of the form $(2, 1^{n-2})$, and thus suggests how to extend this approach to arbitrary shapes.

As with the two column case, we will construct a basis for R_μ such that the degree polynomial is symmetric following the idea of the kicking basis for the Garsia-Procesi modules [8]. In this case, the linear order on fillings of μ will have the property that the evaluation matrix is upper triangular with nonzero diagonal entries with respect to this basis. In the interest of brevity, we omit the direct description of the kicking tree in favor of the recursive description.

Definition 4.1 *Define* $\varphi_{\boxed{1}} = 1$*. For* S *a standard filling of* μ*,* $|\mu| = n$*, define* φ_S *by*

$$\varphi_S = \varphi_{S\setminus n} \prod_{j \succ_n \mathrm{row}(n)} (x_n - \alpha_j) \cdot \begin{cases} \displaystyle\prod_{\mathrm{col}(n) < i \leq \mu_1} (y_n - \beta_i) & \textit{if } \mu \textit{ is a single row;} \\ \displaystyle\prod_{\mathrm{col}(n) < \mathrm{col}(k)} (y_k - \beta_{l_k}) & \begin{array}{l}\textit{if } \mathrm{row}(n) > 1 \\ \textit{or } \mathrm{col}(n) > 1,\end{array} \\ \dfrac{\displaystyle\prod_{\mathrm{col}(n) < \mathrm{col}(k)} (y_k - \beta_{l_k})}{\displaystyle\prod_{j \succ_K 1} (x_K - \alpha_j) \prod_{\substack{1 \succ_j \mathrm{row}(j) \\ j > k\ \forall k \in \mathrm{row}(n)\setminus n}} (x_j - \alpha_1)} & \textit{otherwise,} \end{cases}$$

where l_k *is the maximum column index of all entries in row* 1 *larger than and to the left of* k*, and* K *is the entry in the second column of the bottom row.*

For example, we compute

$$\varphi_{\begin{smallmatrix}\boxed{1}\\\boxed{4}\\\boxed{5}\boxed{3}\boxed{6}\boxed{2}\end{smallmatrix}} = \overbrace{(y_2 - \beta_3)}^{\boxed{6}} \cdot \overbrace{\frac{(y_3 - \beta_1)(y_2 - \beta_2)}{(x_4 - \alpha_1)}}^{\boxed{5}} \cdot \overbrace{(x_4 - \alpha_1)(x_4 - \alpha_3)}^{\boxed{4}} \cdot \overbrace{\frac{(y_2 - \beta_1)}{(x_2 - \alpha_2)}}^{\boxed{3}} \cdot \overbrace{(x_2 - \alpha_2)}^{\boxed{2}} \cdot 1,$$

where each step in the recursion is indicated by the cell removed to obtain the given terms.

Both Proposition 4.2 and Theorem 4.3 are evident from the kicking tree description and are straightforward from the recursive definition.

Proposition 4.2 *For* S *a standard filling of a hook* μ*,* φ_S *is a polynomial.*

Theorem 4.3 *The* $n! \times n!$ *evaluation matrix* $(\varphi_S(p_T))$*, where* S, T *range over all fillings of* μ*, is upper triangular with nonzero diagonal entries. In particular, the set* $\{\varphi_S\}$ *forms a basis for* R_μ*.*

As before, let Φ_μ denote the kicking basis and define

$$F_\mu(q,t) = \sum_{S : \mu \xrightarrow{\sim} [n]} \widehat{\varphi}_S(t, \ldots, t; q, \ldots, q),$$

where $\widehat{\varphi}_S$ is the leading term of φ_S. Note that for a hook $\mu = (n - m, 1^m)$, the largest powers of q and t are $(n - m)(n - m - 1)/2$ and $m(m + 1)/2$, which again agree with $n(\mu')$ and $n(\mu)$, respectively.

Similar to the two column case, we can show the desired symmetry for $F_\mu(q,t)$ by defining a more general function $J_\mu(q,t)$. By deriving suitable recurrence relations for F and J in order to establish the following theorem.

Theorem 4.4 *For* μ *a hook partition, both* $F_\mu(q,t)$ *and* $J_\mu(q,t)$ *exhibit the desired symmetry. In particular, we have a basis for* $\mathcal{H}_\mu$ *of size* $n!$*, and so* $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}[q,t]$*.*

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Enumeration of alternating sign matrices of even size (quasi)-invariant under a quarter-turn rotation

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Abstract. The aim of this work is to enumerate alternating sign matrices (ASM) that are quasi-invariant under a quarter-turn. The enumeration formula (conjectured by Duchon) involves, as a product of three terms, the number of unrestrited ASm's and the number of half-turn symmetric ASM's.

Résumé. L'objet de ce travail est d'énumérer les matrices à signes alternants (ASM) quasi-invariantes par rotation d'un quart-de-tour. La formule d'énumération, conjecturée par Duchon, fait apparaître trois facteurs, comprenant le nombre d'ASM quelconques et le nombre d'ASM invariantes par demi-tour.

1 Introduction

An alternating sign matrix is a square matrix with entries in $\{-1, 0, 1\}$ and such that in any row and column: the non-zero entries alternate in sign, and their sum is equal to 1. Their enumeration formula was conjectured by Mills, Robbins and Rumsey (5), and proved by Zeilberger (9), and almost simultaneously by Kuperberg (3). Kuperberg used a bijection between the ASM's and the states of the statistical square-ice model, for which he studied and computed the partition function. He also used this method in (4) to obtain many enumeration or equinumeration results for various classes of symmetries of ASM's, most of them having been conjectured by Robbins (7). Among these results can be found the following remarkable one.

Theorem 1 (Kupeberg). *The number $A_{\mathrm{QT}}(4N)$ of ASM's of size $4N$ invariant under a quarter-turn (QTASM's) is related to the number $A(N)$ of (unrestricted) ASM's of size N and to the number $A_{\mathrm{HT}}(2N)$ of ASM's of size $2N$ invariant under a half-turn by the formula:*

$$A_{\mathrm{QT}}(4N) = A_{\mathrm{HT}}(2N)A(N)^2. \tag{1}$$

More recently, Razumov and Stroganov (6) applied Kuperberg's strategy to settle the following result, also conjectured by Robbins (7) and relative to QTASM's of odd size.

Theorem 2 (Razumov, Stroganov). *The numbers of QTASM's of odd size are given by the following formulas, where $A_{\mathrm{HT}}(2N+1)$ is the number of HTASM's of size $2N+1$:*

$$A_{\mathrm{QT}}(4N-1) = A_{\mathrm{HT}}(2N-1)A(N)^2 \tag{2}$$

$$A_{\mathrm{QT}}(4N+1) = A_{\mathrm{HT}}(2N+1)A(N)^2. \tag{3}$$

It is easy to observe (and will be proved in Section 2) that the set of QTASM's of size $4N+2$ is empty. But, by slightly relaxing the symmetry condition at the center of the matrix, Duchon introduced in (2) the notion of ASM's quasi-invariant under a quarter turn (the definition will be given in Section 2) whose class is non-empty in size $4N+2$. Moreover, he conjectured for these qQTASM's an enumeration formula that perfectly completes the three previous enumeration results on QTASM. This is the aim of this paper to establish this formula.

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Theorem 3 *The number $A_{\mathrm{QT}}(4N+2)$ of qQTASM of size $4N+2$ is given by:*

$$A_{\mathrm{QT}}(4N+2) = A_{\mathrm{HT}}(2N+1)A(N)A(N+1). \tag{4}$$

This paper is organized as follows: in Section 2, we define qQTASM's; in Section 3, we recall the definitions of square ice models, precise the parameters and the partition functions that we shall study, and give the formula corresponding to equation (4) at the level of partition functions; the Section 4 is devoted to the proofs.

2 ASM's quasi-invariant under a quarter-turn

The class of ASM's invariant under a rotation by a quarter-turn (QTASM) is non-empty in size $4N-1$, $4N$, and $4N+1$. But this is not the case in size $4N+2$.

Lemma 4 *There is no QTASM of size $4N+2$.*

Proof: Let us suppose that M is a QTASM of even size $2L$. Now we use the fact that the size of an ASM is given by the sum of its entries, and the symmetry of M to write:

$$2L = \sum_{1\le i,j\le 2L} M_{i,j} = 4 \times \sum_{1\le i,j\le L} M_{i,j} \tag{5}$$

which implies that the size of M has to be a multiple of 4. □

Duchon introduced in (2) a notion of ASM's quasi-invariant under a quarter-turn, by slightly relaxing the symmetry condition at the center of the matrix. The definition is more simple when considering the height matrix associated to the ASM, but can also been given directly.

Definition 5 *An ASM M of size $4N+2$ is said to be* quasi-invariant under a quarter-turn *(qQTASM) if its entries satisfy the quarter-turn symmetry*

$$M_{4N+2-j+1,4N+2-i+1} = M_{i,j} \tag{6}$$

except for the four central entries $(M_{2N,2N}, M_{2N,2N+}, M_{2N+1,2N}, M_{2N+1,2N+1})$ that have to be either $(0,-1,-1,0)$ or $(1,0,0,1)$.

We give below two examples of qQTASM's of size 6, with the two possible patterns at the center.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In the next section, we associate square ice models to ASM's with various types of symmetry.

3 Square ice models and partition functions

3.1 Notations

Using Kuperberg's method we introduce square ice models associated to ASM, HTASM and QTASM. We recall here the main definitions and refer to (4) for details and many examples.

Let $a \in \mathbb{C}$ be a global parameter. For x any complex number different from zero, we denote $\overline{x} = 1/x$, and we define:

$$\sigma(x) = x - \overline{x}. \tag{7}$$

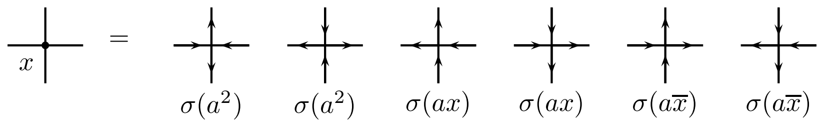
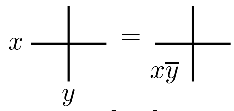


Fig. 1: The 6 possible orientations and associated weights

If G is a tetravalent graph, an *ice state* of G (we will sometimes call them configurations) is an orientation of the edges such that every tetravalent vertex has exactly two incoming and two outcoming edges.

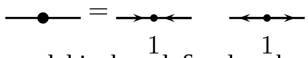
A parameter $x \neq 0$ is assigned to any tetravalent vertex of the graph G. Then this vertex gets a weight, which depends on its orientations, as shown on Figure 1.

It is sometimes easier to assign parameters, not to each vertex of the graph, but to the lines that compose the graph. In this case, the weight of a vertex is defined as:



When this convention is used, a parameter explicitly written at a vertex replaces the quotient of the parameters of the lines.

We will put a dashed line to mean that the parameter of a line is different on the sides of the dashed line. We will also use divalent vertices, and in this case the two edges have to be both in, or both out, and the corresponding weight is 1:



The partition function of a given ice model is then defined as the sum over all its states of the product of the weights of the vertices.

To simplify notations, we will denote by X_N the vector of variables $(x_1, \ldots, x_N)$. We use the notation $X \backslash x$ to denote the vector X without the variable x.

3.2 Partition functions for classes of ASM's

We give in Figures 2, 3, and 4 the ice models corresponding to the classes of ASM's that we shall study, and their partition functions. The bijection between ASM's and states of the square ice model with "domain wall boundary" is now well-known (*cf.* (4)), and the bijections for the other classes of symmetry may be easily checked in the same way.

$Z(N; x_1, \ldots, x_N, x_{N+1}, \ldots, x_{2N}) =$

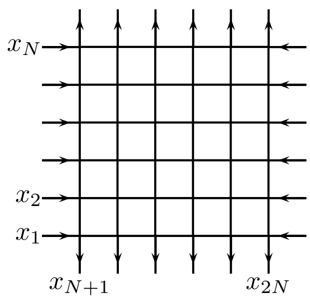


Fig. 2: Partition function for ASM's of size N

With these notations, Theorem 3 will be a consequence of the following one which addresses the concerned partition functions.

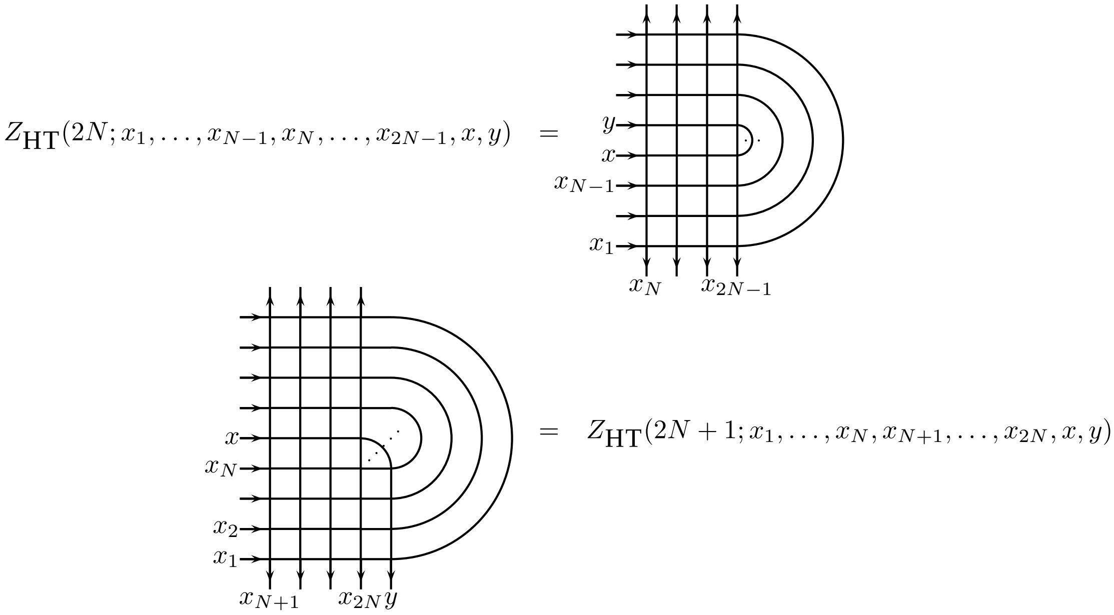


Fig. 3: Partition functions for HTASM's

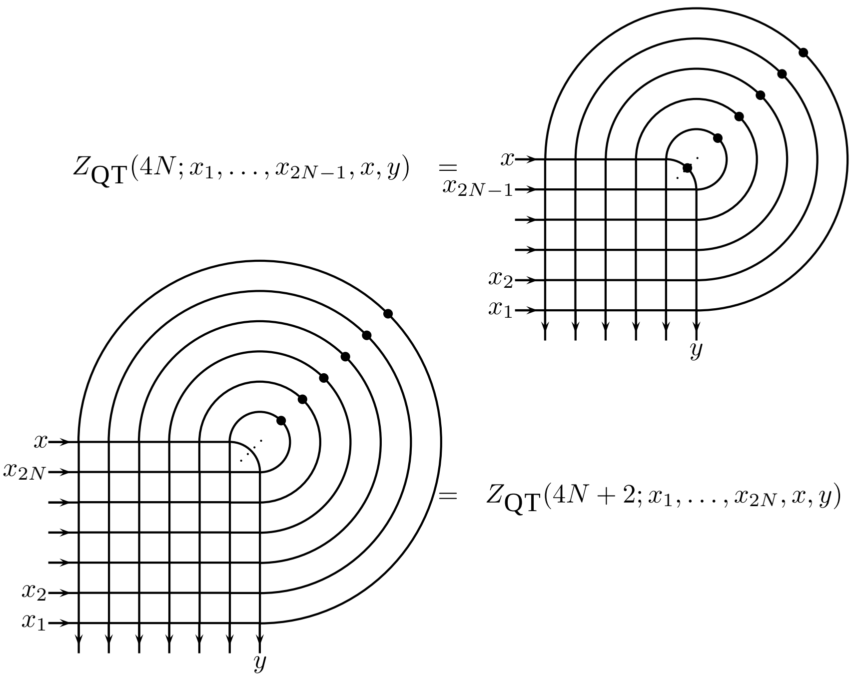


Fig. 4: Partition functions for (q)QTASM of even size

Theorem 6 *When $a = \omega_6 = \exp(i\pi/3)$, one has for $N \geq 1$:*

$$Z_{\mathrm{QT}}(4N; X_{2N-1}, x, y) = \sigma(a)^{-1} Z_{\mathrm{HT}}(2N; X_{2N-1}, x, y) Z(N; X_{2N-1}, x) Z(N; X_{2N-1}, y) \tag{8}$$

and

$$Z_{\mathrm{QT}}(4N+2; X_{2N}, x, y) = \sigma(a)^{-1} Z_{\mathrm{HT}}(2N+1; X_{2N}, x, y) Z(N; X_{2N}) Z(N+1; X_{2N}, x, y). \tag{9}$$

Equation (9) is new; equation (8) is due to Kuperberg (4) for the case $x = y$. To see that Theorem 6 implies Theorem 3, we just have to observe that when $a = \omega_6$ and all the variables equal to 1, then the weights at each vertex is $\sigma(a) = \sigma(a^2)$ thus the partition function reduces (up to multiplication by $\sigma(a)^{\text{number of vertices}}$) to the number of states.

4 Proofs

In this extended abstract, we shall only give the main ideas of the proofs. Most of them are greatly inspired from (4). To prove Theorem 6, the method is to identify both sides of equations (8) and (9) as Laurent polynomials, and to produce as many specializations of the variables that verify the equalities, as needed to imply these equations in full generality.

4.1 Laurent polynomials

Since the weight of any vertex is a Laurent polynomial in the variables x_i, x and y, the partition functions are Laurent polynomials in these variables. Moreover they are centered Laurent polynomials, *i.e.* their lowest degree is the negative of their highest degree (called the half-width of the polynomial). In order to divide by two the number of required specializations in x, we shall deal with Laurent polynomials of given parity in this variable. To do so, we group together the states with a given orientation (indicated as subscripts in the following notations) at the edge where the parameters x and y meet.

So let us consider the partition functions $Z_{\overset{\rightarrow\uparrow}{\mathrm{QT}}}(4N, X_{2N-1}, x, y)$ and $Z_{\overset{\leftarrow\downarrow}{\mathrm{QT}}}(4N, X_{2N-1}, x, y)$, respectively odd and even parts of $Z_{\mathrm{QT}}(4N; X_{2N-1}, x, y)$ in x; $Z_{\overset{\leftarrow\urcorner}{\mathrm{QT}}}(4N+2; X_{2N}, x, y)$ and $Z_{\overset{\urcorner\downarrow}{\mathrm{QT}}}(4N+2; X_{2N}, x, y)$, respectively odd and even parts of $Z_{\mathrm{QT}}(4N+2; X_{2N}, x, y)$in x; $Z_{\overset{\hookleftarrow}{\mathrm{HT}}}(2N; X_{2N-1}, x, y)$ and $Z_{\overset{\hookrightarrow}{\mathrm{HT}}}(4N; X_{2N-1}, x, y)$, respectively parts with the parity of N and of $N-1$ of $Z_{\mathrm{HT}}(4N; X_{2N-1}, x, y)$ in x; and $Z_{\overset{\leftarrow\urcorner}{\mathrm{HT}}}(2N+1; X_{2N}, x, y)$ and $Z_{\overset{\urcorner\downarrow}{\mathrm{HT}}}(2N+1; X_{2N}, x, y)$, respectively parts with the parity of $N-1$ and of N of $Z_{\mathrm{HT}}(2N+1; X_{2N}, x, y)$ in x.

With these notations, the equations (8) et (9) are equivalent to the following:

$$\begin{aligned}
\sigma(a) Z_{\overset{\rightarrow\uparrow}{\mathrm{QT}}}(4N; X_{2N-1}, x, y) &= Z_{\overset{\hookleftarrow}{\mathrm{HT}}}(2N; X_{2N-1}, x, y) Z(N; X_{2N-1}, x) Z(N; X_{2N-1}, y) && (10)\\
\sigma(a) Z_{\overset{\leftarrow\downarrow}{\mathrm{QT}}}(4N; X_{2N-1}, x, y) &= Z_{\overset{\hookrightarrow}{\mathrm{HT}}}(2N; X_{2N-1}, x, y) Z(N; X_{2N-1}, x) Z(N; X_{2N-1}, y) && (11)\\
\sigma(a) Z_{\overset{\urcorner\downarrow}{\mathrm{QT}}}(4N+2; X_{2N}, x, y) &= Z_{\overset{\urcorner\downarrow}{\mathrm{HT}}}(2N+1; X_{2N}, x, y) Z(N+1; X_{2N}, x, y) Z(N; X_{2N}) && (12)\\
\sigma(a) Z_{\overset{\leftarrow\urcorner}{\mathrm{QT}}}(4N+2; X_{2N}, x, y) &= Z_{\overset{\leftarrow\urcorner}{\mathrm{HT}}}(2N+1; X_{2N}, x, y) Z(N+1; X_{2N}, x, y) Z(N; X_{2N}) && (13)
\end{aligned}$$

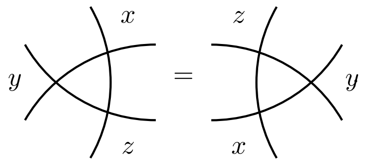
Lemma 7 *Both left-hand side and right-hand side of equations (10-13) are centered Laurent polynomials in the variable x, odd or even, of respective half-widths $2N-1$, $2N-2$, $2N$, and $2N-1$. Thus to prove each of these identities we have to exhibit specializations of x for which the equality is true, and in number strictly exceeding the half-width.*

Proof: To compute the half-width of these partition functions, just count the number of vertices in the ice models, and take care that non-zero entries of the ASM (*i.e.* the first two orientations of Figure 1) give constant weight $\sigma(a^2)$. $\square$

4.2 Symmetries

To produce many specializations from one, we shall use symmetry properties of the partition functions. The crucial tool to prove this is the Yang-Baxter equation that we recall below.

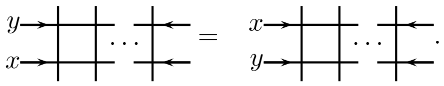
Lemma 8 [Yang-Baxter equation] *If* $xyz = \overline{a}$, *then*



(14)

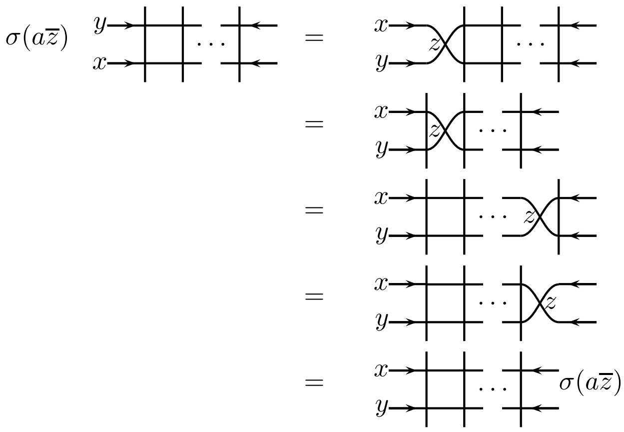
The following lemma gives a (now classical) example of use of the Yang-Baxter equation.

Lemma 9



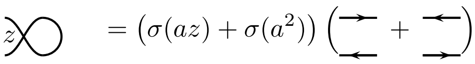
(15)

Proof: We multiply the left-hand side by $\sigma(a\overline{z})$, with $z = \overline{a}x\overline{y}$. We get



□

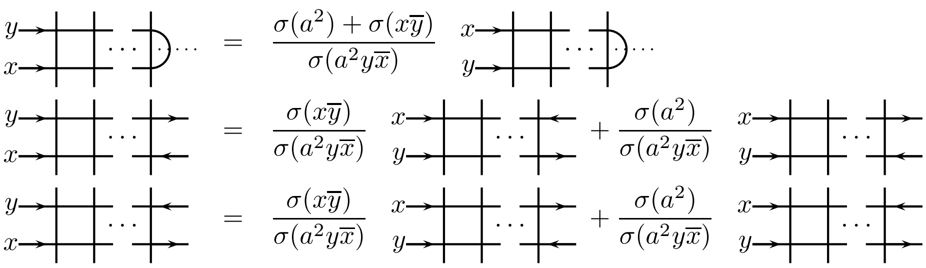
The same method, together with the easy transformation



(16)

gives the following lemma.

Lemma 10



(17)

(18)

(19)

We use Lemmas 9 and 10 to obtain symmetry properties of the partition functions, that we summarize below, where m denotes either $2N$ or $2N + 1$.

Lemma 11 *The functions $Z(N;X_{2N})$ and $Z_{\mathrm{HT}}(2N+1;X_{2N},x,y)$ are symmetric separately in the two sets of variables $\{x_i,\ i\le N\}$ and $\{x_i,\ i\ge N+1\}$, the function $Z_{\mathrm{HT}}(2N;X_{2N-1},x,y)$ is symmetric separately in the two sets of variables $\{x_i,\ i\le N-1\}$ and $\{x_i,\ i\ge N\}$, and the functions $Z_{\mathrm{QT}}(2m;X_{N-1},x,y)$ are symmetric in their variables x_i.*

Moreover, $Z_{\mathrm{QT}}(4N+2;\ldots)$ is symmetric in its variables x and y, and we have a pseudo-symmetry for $Z_{\mathrm{QT}}(4N;\ldots)$ and $Z_{\mathrm{HT}}(2N;\ldots)$:

$$Z_{\mathrm{QT}}(4N;X_{2N-1},x,y) = \frac{\sigma(a^2)+\sigma(x\overline{y})}{\sigma(a^2y\overline{x})} Z_{\mathrm{QT}}(4N;X_{2N-1},y,x) \tag{20}$$

$$Z_{\mathrm{HT}}(2N;X_{2N-1},x,y) = \frac{\sigma(a^2)+\sigma(x\overline{y})}{\sigma(a^2y\overline{x})} Z_{\mathrm{HT}}(2N;X_{2N-1},y,x). \tag{21}$$

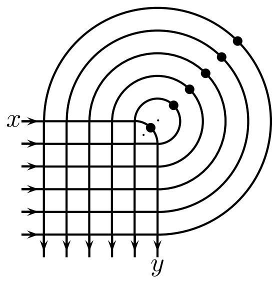
Proof: For $Z(N;\ldots)$ and $Z_{\mathrm{HT}}(m;\ldots)$, the symmetry in two "consecutive" variables x_i and x_{i+1} is a direct consequence of Lemma 9. For $Z_{\mathrm{QT}}(2m;\ldots)$, we again apply Lemma 9 together with the easy observations:

(22)

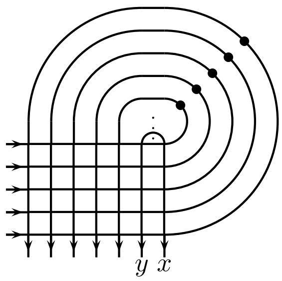
which allow us to bring the Yang-Baxter triangle through the dotted lines of Figure 4.

For the (pseudo-)symmetries in (x,y), let us deal with $Z_{\mathrm{QT}}(4N;\ldots)$, the other cases being similar or simpler. We use equation (22) to put together the lines of parameter x and y:

$Z_{\mathrm{QT}}(4N;X_{2N-1},x,y) \quad =$



$=$



and then apply Lemma 10. □

It should be clear that we have analogous properties for the even and odd parts of the partition functions. The next (and last) symmetry property was proved by Stroganov (8) (a recent and elementary proof may be found in (1)). It appears when the parameter a equals the special value $\omega_6=\exp(i\pi/3)$.

Lemma 12 *When $a=\omega_6$, the partition function $Z(N;X_{2N})$ is symmetric in* **all** *its variables.*

4.3 Specializations, recurrences

The aim of this section is to give the value of the partition functions in some specializations of the variable x or y. The first result is due to Kuperberg, the other are very similar.

Lemma 13 [specialization of Z; Kuperberg] *If we denote*

$$\begin{aligned} A(x_{N+1}, X_{2N}\backslash\{x_1, x_{N+1}\}) &= \prod_{2\le k\le N} \sigma(a x_k \overline{x}_{N+1}) \prod_{N+1\le k\le 2N} \sigma(a^2 x_{N+1}\overline{x}_k), \\ \overline{A}(x_{N+1}, X_{2N}\backslash\{x_1, x_{N+1}\}) &= \prod_{2\le k\le N} \sigma(a x_{N+1} \overline{x}_k) \prod_{N+1\le k\le 2N} \sigma(a^2 x_k\overline{x}_{N+1}), \end{aligned}$$

then we have:

$$Z(N; \mathbf{\bar{a}x_{N+1}}, X_{2N}\backslash x_1) = \overline{A}(x_{N+1}, X_{2N}\backslash\{x_1, x_{N+1}\}) Z(N-1; X_{2N}\backslash\{x_1, x_{N+1}\}) \quad (23)$$

$$Z(N; \mathbf{ax_{N+1}}, X_{2N}\backslash x_1) = A(x_{N+1}, X_{2N}\backslash\{x_1, x_{N+1}\}) Z(N-1; X_{2N}\backslash\{x_1, x_{N+1}\}). \quad (24)$$

Proof: We recall the method to prove equation (23). We observe that when $x_1 = \bar{a}x_{N+1}$, the parameter of the vertex at the crossing of the two lines of parameter x_1 and x_{N+1} is $\bar{a}$. Thus the weight of this vertex is $\sigma(a\bar{a}) = \sigma(1) = 0$ unless the orientation of this vertex is the second on Figure 1. But this orientation implies the orientation of all vertices in the row x_{N+1} and in the column x_1, as shown on Figure 5. The fixed part gives the partition function Z in size $N-1$, without parameters x_1 and x_{N+1}, and the weights of the fixed part gives the factor $\overline{A}(\ldots)$.

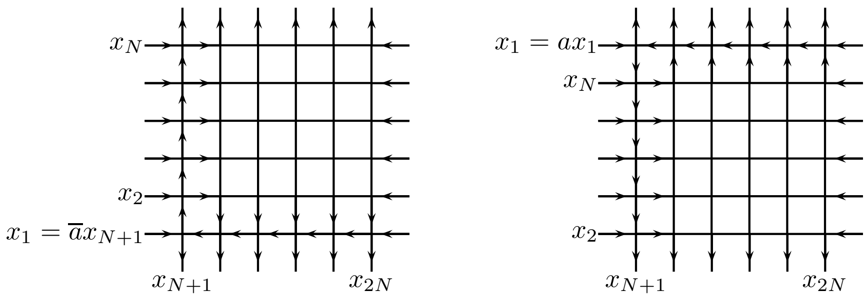


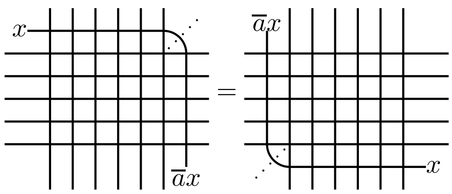
Fig. 5: Fixed edges for (23) on the left and (24) on the right

The case of (24) is similar, after using Lemma 11 to put the line x_{N+1} at the top of the grid.

□

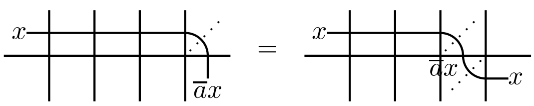
We will need the following application of the Yang-Baxter equation, which allows, under certain condition, a line with a change of parameter to go through a grid.

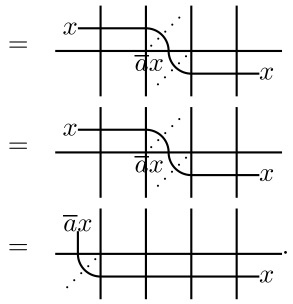
Lemma 14



(25)

Proof: We iteratively apply Lemma 8 on the rows, and row by row:





$\square$

Lemma 15 [specialization of Z_{HT}] *If we denote*

$$\begin{aligned}
A_H^1(x_1, X_{2N}\backslash x_1) &= \prod_{1\le k\le N}\sigma(a^2 x_1\overline{x}_k)\prod_{N+1\le k\le 2N}\sigma(a x_k\overline{x}_1)\\
\overline{A}_H^1(x_1, X_{2N}\backslash x_1) &= \prod_{1\le k\le N}\sigma(a^2 x_k\overline{x}_1)\prod_{N+1\le k\le 2N}\sigma(a x_1\overline{x}_k)\\
A_H^0(x_N, X_{2N-1}\backslash x_N) &= \prod_{1\le k\le N-1}\sigma(a x_k\overline{x}_N)\prod_{N\le k\le 2N-1}\sigma(a^2 x_N\overline{x}_k)\\
\overline{A}_H^0(x_N, X_{2N-1}\backslash x_N) &= \prod_{1\le k\le N-1}\sigma(a x_N\overline{x}_k)\prod_{N\le k\le 2N-1}\sigma(a^2 x_k\overline{x}_N),
\end{aligned}$$

then for $\star =$ ⤵, ↰, ⤺, ⤼ *and* $\square =$ ⤼, ⤺, ⤵, ↰ *respectively, we have*

$$Z_{\mathrm{HT}}^{\star}(2N+1; X_{2N}, x, \mathbf{ax_1}) = A_H^1(x_1, X_{2N}\backslash x_1) Z_{\mathrm{HT}}^{\square}(2N; X_{2N}\backslash x_1, x_1, x) \tag{26}$$

$$Z_{\mathrm{HT}}^{\square}(2N+1; X_{2N}, x, \mathbf{\bar{a}x_1}) = \overline{A}_H^1(x_1, X_{2N}\backslash x_1) Z_{\mathrm{HT}}^{\star}(2N; X_{2N}\backslash x_1, x, x_1) \tag{27}$$

$$Z_{\mathrm{HT}}^{\star}(2N; X_{2N-1}, x, \mathbf{ax_N}) = \sigma(a x\overline{x}_N) A_H^0(x_N, X_{2N-1}\backslash x_N) Z_{\mathrm{HT}}^{\square}(2N-1; X_{2N-1}\backslash x_N, x, x_N) \tag{28}$$

$$Z_{\mathrm{HT}}^{\square}(2N; X_{2N-1}, \mathbf{\bar{a}x_N}, y) = \sigma(a x_N\overline{y}) \overline{A}_H^0(x_N, X_{2N-1}\backslash x_N) Z_{\mathrm{HT}}^{\star}(2N-1; X_{2N-1}\backslash x_N, y, x_N) \tag{29}$$

Proof:

The proof is similar to the previous one, with the difference that before looking at fixed edges, we need to multiply the partition function by a given factor; we interpret this operation by a modification of the graph of the ice model and apply Lemma 14. It turns out that in each case, the additional factors are exactly cancelled by the weights of fixed vertices.

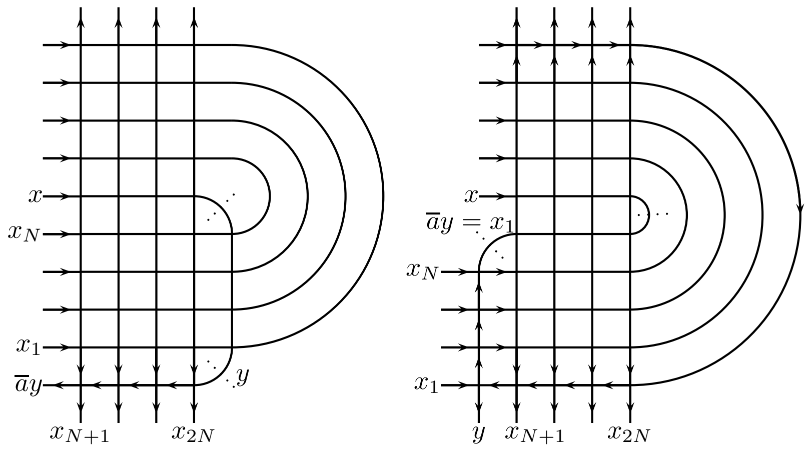
To prove (26), we multiply the left-hand side by

$$\prod_{N+1\le k\le 2N}\sigma(a^2 x_k\overline{y}),$$

which is equivalent to adding to the line of parameter y a new line $\overline{a}y$ just below the grid; the Lemma 14 transforms the graph of Figure 6(a) into the graph of Figure 6(b). When we put $y = ax_1$,we get the indicated fixed edges, which gives as partition function

$$\prod_{N+1\le k\le 2N}\sigma^2(a x_k\overline{x}_1)\prod_{1\le k\le N}\sigma(a^2 x_1\overline{x}_k) Z_{\mathrm{HT}}(2N; X_{2N}\backslash x_1, x_1, x).$$

Since $a^2 x_k\overline{y} = a x_k\overline{x}_1$, the equation simplifies. Ton conclude, we observe that if we start with an edge going out from the crossing x/x_{2N} (function $Z_{\mathrm{HT}}^{⤵}$) we get at the end the same orientation (function $Z_{\mathrm{HT}}^{⤼}$).
$\square$



(a) (b)

Fig. 6: Proof of (26)

Lemma 16 [specialization of Z_{QT}] *If we denote*

$$\begin{aligned}\overline{A}_Q(x_1, X_{m-1}\backslash x_1) &= \prod_{1\le k\le m-1} \sigma(a^2 x_k \overline{x}_1)\sigma(a x_1 \overline{x}_k),\\ A_Q(x_1; X_{m-1}\backslash x_1) &= \prod_{1\le k\le m-1} \sigma(a^2 x_1 \overline{x}_k)\sigma(a x_k \overline{x}_1),\end{aligned}$$

then for $\star =$ ⬏, ⬑, ⬎, ⬐ and $\square =$ ⬐, ⬎, ⬏, ⬑ respectively, we have:

$$Z^{\star}_{\mathrm{QT}}(2m; X_{m-1}, \bar{\mathbf{a}}\mathbf{x_1}, y) = \sigma(a x_1 \overline{y})\overline{A}_Q(x_1, X_{m-1}) Z^{\square}_{\mathrm{QT}}(2m-2; X_{m-1}\backslash x_1, y, x_1) \quad (30)$$

$$Z^{\square}_{\mathrm{QT}}(2m; X_{m-1}, x, \mathbf{a}\mathbf{x_1}) = \sigma(a x \overline{x}_1) A_Q(x_1; X_{m-1}\backslash x_1) Z^{\star}_{\mathrm{QT}}(2m-2; X_{m-1}\backslash x_1, x_1, x) \quad (31)$$

Proof: Similar to the proof of Lemma 15. □

Remark 17 *By using the (pseudo-)symmetry in (x, y), we may transform any specialization of the variable y into a specialization of the variable x. Moreover, by using Lemma 11 and (when $a = \omega_6$) Lemma 12, we obtain for Z, Z_{HT} and $Z_{\mathrm{QT}} Z$ $2N$ specializationse now have to compare them.*

4.4 Special value of the parameter a; conclusion

When $a = \omega_6 = \exp(i\pi/3)$, two new ingredients may be used. The first one is Lemma 12, as mentionned in Remark 17. The second one is that with this special value of a:

$$\sigma(a) = \sigma(a^2) \qquad \sigma(a^2 x) = -\sigma(\bar{a}x) = \sigma(a\bar{x}). \quad (32)$$

which implies that the products appearing in Lemmas 13, 15 and 16 may be written in a more compact way:

$$\begin{aligned}A(x_{N+1}, X_{2N}\backslash\{x_1, x_{N+1}\}) &= \sigma(a) \prod_{k\ne 1, N+1} \sigma(a x_k \overline{x}_{N+1}),\\ \overline{A}(x_{N+1}, X_{2N}\backslash\{x_1, x_{N+1}\}) &= \sigma(a) \prod_{k\ne 1, N+1} \sigma(a x_{N+1} \overline{x}_k),\\ A^1_H(x_1, X_{2N}\backslash x_1) &= \prod_{1\le k\le 2N} \sigma(a x_k \overline{x}_1),\end{aligned}$$

$$\begin{aligned}
\overline{A}_H^1(x_1, X_{2N}\backslash x_1) &= \prod_{1\le k\le 2N} \sigma(ax_1\overline{x}_k),\\
A_H^0(x_N, X_{2N-1}\backslash x_N) &= \prod_{1\le k\le 2N-1} \sigma(ax_k\overline{x}_N)\\
\overline{A}_H^0(x_N, X_{2N-1}\backslash x_N) &= \prod_{1\le k\le 2N-1} \sigma(ax_N\overline{x}_k),\\
\overline{A}_Q(x_1, X_{m-1}\backslash x_1) &= \prod_{1\le k\le m-1} \sigma^2(ax_1\overline{x}_k),\\
A_Q(x_1, X_{m-1}\backslash x_1) &= \prod_{1\le k\le m-1} \sigma^2(ax_k\overline{x}_1).
\end{aligned}$$

Thus we get by comparing:

$$\begin{aligned}
A(x_i, X_{2N}\backslash x_i, x)A_H^1(x_i, X_{2N}\backslash x_i) &= \sigma(ax\overline{x}_i)A_Q(x_i, X_{2N}\backslash x_i)\\
\overline{A}(x_i, X_{2N}\backslash x_i, x)\overline{A}_H^1(x_i, X_{2N}\backslash x_i) &= \sigma(ax_i\overline{x})\overline{A}_Q(x_i, X_{2N}\backslash x_i),
\end{aligned}$$

whence (10) and (11) imply that (12) and (13) are true (in size $4N+2$) for the $2N$ specializations $y = a^{(+,-)1}x_i$ ($1 \le i \le N$). It is enough to prove (13) (Laurent polynomials of half-width $2N-1$), but we still need one specialization to get (12) (half-width $2N$).

For (10) and (11), we observe the same kind of simplification

$$A(x_i, X_{2N-1}\backslash x_i)\sigma(ax\overline{x}_i)A_H^0(x_i, X_{2N-1}\backslash x_i) = \sigma(ax\overline{x}_i)A_Q(x_i, X_{2N-1}\backslash x_i),$$

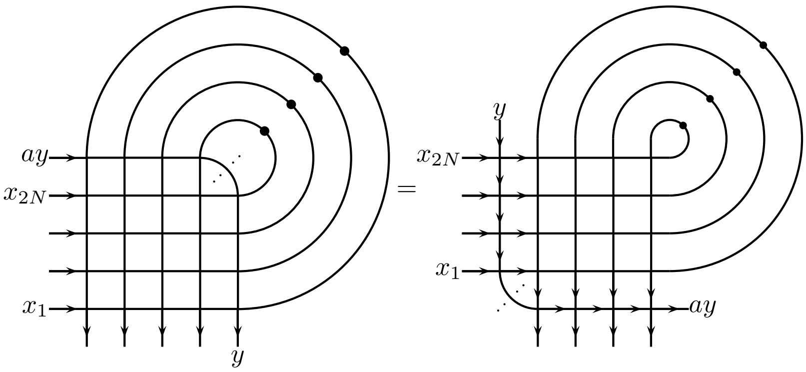
whence (13) and (12) for the size $4N-2$ imply that (10) and (11) are true for the N specializations $x = ax_i$, $N \le i \le 2N-1$. We obtain in the same way the coincidence for the N specializations $x = \overline{a}x_i$, $N \le i \le 2N-1$. Thus we have $2N$ specialiations of x: it is enough both for (10) (half-width $2N-1$), and for (11) (half-width $2N-2$).

At this point, we have *almost* proved

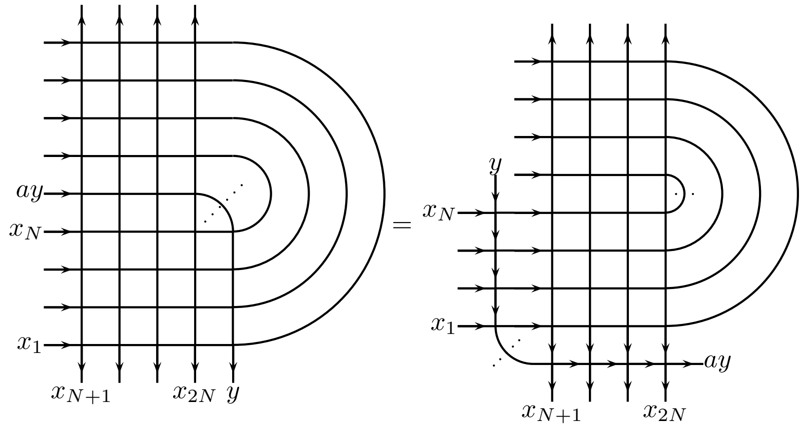
((10) and (11), size $4N$) $\Longrightarrow$ (((12) and (13), size $4N+2$) $\Longrightarrow$ ((10) and (11), size $4N+4$);

almost, because we still need *one* specialization for (12).

We get this missing speciazation, not directly for $Z_{\overrightarrow{\mathrm{QT}}}$, $Z_{\overleftarrow{\mathrm{QT}}}$, $Z_{\overrightarrow{\mathrm{HT}}}$ and $Z_{\overleftarrow{\mathrm{HT}}}$, but for the original series $Z_{\mathrm{QT}}(4N+2; X_{2N}, x, y)$ and $Z_{\mathrm{HT}}(2N+1; X_{2N}, x, y)$: indeed if we set $x = ay$ we may apply Lemma 14.



$$Z_{\mathrm{QT}}(4N+2; X_{2N}, \mathbf{ay}, y) = \sigma(a) \prod_{1\le k\le 2N} \sigma(ax_k\overline{y})\sigma(a^2y\overline{x}_k) Z_{\mathrm{QT}}(4N; X_{2N}\backslash x_{2N}, x_{2N}, x_{2N})$$



$$Z_{\mathrm{HT}}(2N+1; X_{2N}, ay, y) = \left(\prod_{1\le k\le N} \sigma(a x_k \overline{y}) \prod_{N+1\le k\le 2N} \sigma(a^2 y \overline{x}_k)\right) Z_{\mathrm{HT}}(2N; X_{2N}\backslash x_N, x_N, x_N)$$

This way, we get another point where (9) is true, and thus, because we already have (13), by difference we obtain that (12) holds for $y = \overline{a}x$.

This completes the proof of Theorem 6.

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Linear time equivalence of Littlewood–Richardson coefficient symmetry maps

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Abstract. Benkart, Sottile, and Stroomer have completely characterized by Knuth and dual Knuth equivalence a bijective proof of the Littlewood–Richardson coefficient conjugation symmetry, i.e. $c^{\lambda}_{\mu,\nu} = c^{\lambda^t}_{\mu^t,\nu^t}$. Tableau–switching provides an algorithm to produce such a bijective proof. Fulton has shown that the White and the Hanlon–Sundaram maps are versions of that bijection. In this paper one exhibits explicitly the Yamanouchi word produced by that conjugation symmetry map which on its turn leads to a new and very natural version of the same map already considered independently. A consequence of this latter construction is that using notions of Relative Computational Complexity we are allowed to show that this conjugation symmetry map is linear time reducible to the Schützenberger involution and reciprocally. Thus the Benkart–Sottile–Stroomer conjugation symmetry map with the two mentioned versions, the three versions of the commutative symmetry map, and Schützenberger involution, are linear time reducible to each other. This answers a question posed by Pak and Vallejo.

Résumé. Benkart, Sottile, et Stroomer ont complètement caractérisé par équivalence et équivalence duelle à Knuth une preuve bijective de la symétrie de la conjugaison des coefficients de Littlewood–Richardson, i.e. $c^{\lambda}_{\mu,\nu} = c^{\lambda^t}_{\mu^t,\nu^t}$. Le tableau-switching donne un algorithme par produire une telle preuve bijective. Fulton a montré que les bijections de White et de Hanlon et Sundaram sont des versions de celle bijection. Dans ce papier on exhibe explicitement le mot de Yamanouchi produit par cette bijection de conjugaison lequel à son tour conduit à une nouvelle version très naturelle de la même bijection déjà considérée indépendantement. Une conséquence de cette dernière construction c'est que en utilisant des notions de Complexité Computationnelle Relative nous pouvons montrer que cette bijection de symétrie de la conjugaison est linéairement réductible à la involution de Schützenberger et réciproquement. À cette cause la bijection de symétrie de la conjugaison de Benkart, Sottile et Stroomer avec les deux versions mentionnées, aussi bien les trois versions de la bijection de la commutativité, et la involution de Schützenberger sont linéairement réductibles à chacune de les autres. Ça répond à une question posée par Pak et Vallejo.

Keywords: Symmetry maps of Littlewood–Richardson coefficients; conjugation symmetry map; linearly time reduction of Young tableaux bijections; tableau–switching; Schützenberger involution.

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1 Introduction

Given partitions μ and ν, the product $s_\mu s_\nu$ of the corresponding Schur functions is a non-negative integral linear combination of Schur functions

$$s_\mu s_\nu = \sum_\lambda c_{\mu\,\nu}^{\lambda} s_\lambda,$$

where $c_{\mu\,\nu}^{\lambda}$ is called the Littlewood-Richardson coefficient (LiRi; Mac; Sa; St). Let λ^t denote the conjugate or transpose of the partition λ. It is obvious from the commutativity of multiplication that $c_{\mu\,\nu}^{\lambda} = c_{\nu\,\mu}^{\lambda}$, called the commutativity symmetry, and it is less obvious the conjugation symmetry $c_{\mu\,\nu}^{\lambda} = c_{\mu^t\,\nu^t}^{\lambda^t}$. As there are several Littlewood-Richardson rules to compute these numbers, the combinatorics of their symmetries is quite intriguing since in all of them the commutativity is hidden, and the conjugation is either hidden or partially hidden (BZ; KT; PV1). This is in contrast with the fact that most of the symmetries are explicitly exhibited by simple means (PV1). By "*hidden*" or "*simple*" we are referring to the computational complexity of the operations needed to reveal such symmetries. Let $\mathrm{LR}(\lambda/\mu, \nu)$ be the set of Littlewood-Richardson (LR for short) tableaux (LiRi) of shape λ/μ and content ν. Then $c_{\mu\,\nu}^{\lambda}$ counts the number of elements of this set. If one writes $c_{\mu\,\nu}^{\lambda} =: c_{\mu\,\nu\,\lambda^\vee}$, with $\lambda^\vee$ the complement partition of λ regarding some rectangle containing λ, the Littlewood-Richardson coefficients are invariant under the following action of $\mathbb{Z}_2 \times S_3$: the non–identity element of $\mathbb{Z}_2$ transposes simultaneously μ, ν and $\lambda^\vee$, and S_3 sorts μ, ν and $\lambda^\vee$ (BSS).

The Berenstein-Zelevinsky interpretation of the Littlewood-Richardson coefficients (BZ triangles for short) (BZ) manifests all the S_3-symmetries except the commutativity. Pak and Vallejo have defined in (PV1) bijections, which are explicit linear maps, between LR tableaux, Knutson-Tao hives (KT) and BZ triangles. These bijections combined with the symmetries of BZ triangles *give* all the S_3-symmetries except the commutativity. The conjugation symmetry is also hidden in BZ triangles. In (GP), it is shown that it can be revealed from a bijection between web diagrams and BZ-triangles. On the other hand, the Knutson-Tao-Woodward puzzles (KTW), the most symmetrical objects, manifest only partially the conjugation symmetry through the *puzzle duality*, *viz.* $c_{\mu\,\nu\,\lambda} = c_{\nu^t\,\mu^t\,\lambda^t}$, since the commutativity is hidden. Interestingly, as we shall see, a similar *partial conjugation symmetry*, $c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\nu^t\,\mu^t}$, is obtained on LR tableaux through a simple bijection, denoted by ⧫. In (KTW; K1; K2) bijections between hives and puzzles can be found. Recently, Purbhoo (Pu) introduced a new tool called *mosaics*, a square-triangle-rhombus tiling model with all the rhombi arranged in the shape of a Young diagram in the corners of an hexagon. Mosaics are in bijection with puzzles and with LR tableaux, and the operation *migration* on mosaics, which correspond to some sequence of *jeu de taquin* operations on LR tableaux, reveals the hidden symmetries of puzzles. The *carton rule* (TY) is a recent S_3-symmetric rule but the computational complexity of the resulted visual symmetry does not seem to be improved as it is based on non trivial properties of *jeu de taquin*.

In (PV2), a number of Young tableau commutative symmetry maps are considered and it is shown that two of them are linear time reducible to each other and to the Schützenberger involution. (Subsequently in (DK2) and in (A3) it has been shown that the two remaining ones are identical to the others.) In this paper, we consider three Young tableau conjugation symmetry maps that appeared in (W; HS; BSS; Z; A1; A2) and one shows that these three Young tableau conjugation symmetry maps and the commutative symmetry maps, considered in (PV2), are linear time reducible to each other and to the Schützenberger involution. In addition, as in the commutative case, the Young tableau conjugation symmetry maps coincide. This answers a question posed by Pak and Vallejo in (PV2).

1.1 Summary of the results

The conjugation symmetry map is a bijection (PV2)

$$\varrho : LR(\lambda/\mu, \nu) \longrightarrow LR(\lambda^t/\mu^t, \nu^t).$$

Let T be a tableau and $\widehat{T}$ its standardization. The Benkart-Sottile-Stroomer conjugation symmetry map (BSS), denoted by ϱ^{BSS}, is the bijection

$$\begin{array}{rcl} \varrho^{BSS} : LR(\lambda/\mu, \nu) & \longrightarrow & LR(\lambda^t/\mu^t, \nu^t) \\ T & \mapsto & \varrho^{BSS}(T) = [Y(\nu^t)]_K \cap [(\widehat{T})^t]_d \end{array},$$

where $[Y(\nu^t)]_K$ is the Knuth class of all tableaux with rectification the Yamanouchi tableau $Y(\nu^t)$ of shape the conjugate of ν, and $[\widehat{T}^t]_d$ is the dual Knuth class of all tableaux of shape λ^t/μ^t with Q-symbol the transpose of $\widehat{T}$. The image of T by the BSS-bijection is the unique tableau of shape λ^t/μ^t in both those two equivalence classes. Fulton showed in (F) that the White-Hanlon-Sundaram map ϱ^{WHS} (W; HS) coincides with ϱ^{BSS}. Thus $\varrho^{BSS}(T)$ can be obtained either by tableau-switching or by the White-Hanlon-Sundaram transformation ϱ^{WHS}.

Given a totally ordered finite alphabet, let σ_i denote the reflection crystal operator acting on a subword over the alphabet $\{i, i+1\}$, for all i (LS; Loth), and let $\sigma_0 = \sigma_i \cdots \sigma_j \cdots \sigma_k$ be such that $s_i \cdots s_j \cdots s_k$, with s_l the transposition $(l, l+1)$, is the longest permutation of $S_{\nu_1^t}$. The column reading word of $\varrho^{BSS}(T)$ is the Yamanouchi word of weight ν^t whose Q-symbol is the one given by the column reading word of $\widehat{T}^t$. The following transformation ϱ_3 (Z; A1; A2; ACM) makes clear the construction of that word and affords a simple way to construct $\varrho^{BSS}(T)$

$$\begin{array}{ccc} \varrho_3 : LR(\lambda/\mu, \nu) & \longrightarrow & LR(\lambda^t/\mu^t, \nu^t) \\ \scriptstyle T \text{ with word } w & & \scriptstyle \varrho_3(T) \text{ with column word } (\sigma_0 w)^{*\,\diamond} \end{array}$$

$$\mathrm{T} = \begin{array}{cccccc} & & 1 & 1 & 1 & 1 \\ & 1 & 2 & 2 & & \\ 2 & 3 & 3 & & & \end{array} \quad \underset{\text{reversal}}{\overset{e}{\longrightarrow}} \quad \mathrm{T}^e = \begin{array}{cccccc} & & 1 & 1 & 3 & 3 \\ & 2 & 2 & 2 & & \\ 3 & 3 & 3 & & & \end{array} \quad \underset{\text{of } \lambda/\mu}{\overset{\text{transposition}}{\longrightarrow}} \quad \begin{array}{ccc} & & 3 \\ & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \\ 3 & & \\ 3 & & \end{array} \quad \rightarrow \quad \begin{array}{ccc} & & 1 \\ & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & \\ 4 & & \\ 5 & & \end{array} = \varrho_3(\mathrm{T})$$

$$w = 1111221332 \quad \overset{\sigma_0}{\rightarrow} \quad \sigma_0 w = 3311222333 \quad \underset{\text{the word}}{\overset{\text{reverse}}{\rightarrow}} \quad 3332221133 \quad \rightarrow \quad \underset{\varrho_3(T) = \varrho^{BSS}(T),}{\underset{\text{column word of}}{1231231245}}$$

where $*$ denotes the dualization of a word; and $\diamond$ is the operator which transforms a Yamanouchi word of weight ν, into a Yamanouchi word of weight ν^t, by replacing the subword i^{ν_i} with $12\ldots\nu_i$, for all i. The action of the operator $\diamond$ is extended to dual Yamanouchi words by putting $(w^*)^\diamond := w^{\diamond *}$. More precisely, the $\diamond$ operator is a bijection between the Knuth classes of the Yamanouchi tableaux $Y(\nu)$ and $Y(\nu^t)$, and also between the corresponding dual Yamanouchi tableaux. The reversal e of a LR tableau can be computed by the action of σ_0 on its word. The image of a LR or dual LR tableau U under rotation of the skew-diagram by 180 degrees, with the dualization $*$ of its word is denoted by $U^\bullet$; and the image of U under the rotation and transposition of the skew-diagram, with the action of the operation $\diamond$ on its word is denoted by $U^\blacklozenge$. Again $\bullet$ and $\blacklozenge$ are involution maps. Then

$$\varrho_3(T) = T^{e\,\bullet\blacklozenge} = T^{\blacklozenge\bullet\,e} = T^{\bullet\blacklozenge\,e} \text{ and}$$

$$(\sigma_0 w)^{*\,\diamond} = (\sigma_0 w)^{\diamond\,*} = \sigma_0(w^{\diamond\,*}) \text{ is the column word of } T^{e\bullet\blacklozenge} = [Y(\nu^t)]_K \cap [(\widehat{T})^t]_d.$$

In the two next sections we shall develop the necessary machinery to show the above identities. Bijection $\blacklozenge\bullet$ appeared originally in (Z) with a different formulation. In (A1; A2) the bijection e, defined differently and based on a modified insertion, is composed with the last one to give ρ_3. Here we stress the composition of $e\,\bullet$ with $\blacklozenge$.

Following the ideas introduced in (PV2), we address, in Section 4, the problem of studying the computational cost of the conjugation symmetry map ϱ^{BSS} utilizing what is known as Relative Complexity, an approach based on reduction of combinatorial problems. To this aim we use the version ϱ_3. We consider only linear time reductions; since the bijections we consider require subquadratic time the reductions have to preserve that. Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly infinite sets of finite integer arrays, and let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map between them. We say that δ has linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A \rangle)$ for all $A \in \mathcal{A}$, where $\langle A \rangle$ is the bit–size of A. The transposition of the recording matrix of a LR tableau is the recording matrix of a tableau of normal shape. We have then a linear map τ which defines a bijection between tableaux of normal shape and LR tableaux (Lee1; Lee2; PV2; O). As the rotation map $\bullet$ and τ are linear maps, so maps of linear cost, the reversal T^e of a LR tableau T can be linearly reduced to the evacuation E of the corresponding tableau $\tau(T) = P$ of normal shape, *i.e.* $\tau(P^E) = T^{e\,\bullet}$. Additionally, in Algorithm 4.1, it is proved that the bijection $\blacklozenge$, exhibiting the symmetry $c_{\mu\,\nu\,\lambda} = c_{\lambda^t\,\nu^t\,\mu^t}$, is of linear cost. The following commutative scheme shows that the conjugation symmetry map ϱ_3, and therefore ϱ^{BSS} and ϱ^{WHS}, is linear equivalent to the Schützenberger involution or evacuation map on tableaux of normal shape,

Theorem 1.1 *The following commutative scheme holds*

$$\begin{array}{ccccc} T & \overset{e\,\bullet}{\longleftrightarrow} & T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & T^{e\bullet\blacklozenge} \\ \tau\updownarrow & & \tau\updownarrow & & \\ P & \underset{E}{\overset{evacuation}{\longleftrightarrow}} & P^E. & & \end{array}$$

Theorem 1.2 *The conjugation symmetry maps ϱ^{BSS}, ϱ^{WHS} and ϱ_3 are identical, and linear time equivalent with the Schützenberger involution E and with the reversal map e.*

We may now extend the list of linear equivalent Young tableau maps established in (PV2), Section 2, Theorem 1.

Theorem 1.3 (PV2) *The following maps are linearly equivalent:*

(1) *RSK correspondence.*
(2) *Jeu de taquin map.*

(3) *Littlewood–Robinson map.*
(4) *Tableau switching map* s.
(5) *Evacuation (Schützenberger involution)* E *for normal shapes.*
(6) *Reversal* e.
(7) *First fundamental symmetry map.*
(8) *Second fundamental symmetry map.*

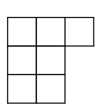
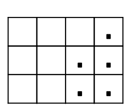
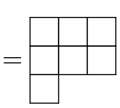
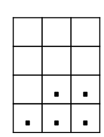
Corollary 1.1 *The following maps are linearly equivalent:*
(1) *RSK correspondence.*
(2) *Jeu de taquin map.*
(3) *Littlewood–Robinson map.*
(4) *Tableau switching map* s.
(5) *Evacuation (Schützenberger involution)* E *for normal shapes.*
(6) *Reversal* e.
(7) *First fundamental symmetry map.*
(8) *Second fundamental symmetry map.*
(9) *Third fundamental symmetry map.*
(10) ϱ^{WHS} *conjugation symmetry map.*
(11) ϱ^{BSS} *conjugation symmetry map.*
(12) ϱ_3 *conjugation symmetry map.*

In particular, first and second fundamental symmetry maps are identical (DK2); first and third fundamental symmetry maps are identical (A3); ϱ^{WHS} *and* ϱ^{BSS} *are identical conjugation symmetry maps (F), and the same happens with* ϱ^{BSS} *and* ϱ_3.

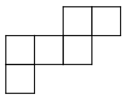
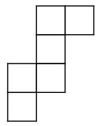
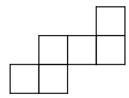
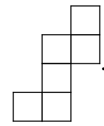
2 Preliminaries

2.1 Young diagrams and transformations

A *partition* (or normal shape) λ is a sequence of non–negative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. The number of parts is $\ell(\lambda) = \ell$ and the weight is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. (For convenience we allow zero parts.) The Young diagram of λ is the collection of boxes $\{(i,j) \in \mathbb{Z}^2 |\ 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$. The English convention is adopted in drawing such a diagram. Throughout the paper we do not make distinction between a partition λ and its Young diagram (P). Given partitions λ, μ, we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i > 0$. If (r^ℓ) is a $r \times \ell$ rectangle containing λ, the *complement* of λ regarding that rectangle is the partition $\lambda^\vee = (r - \lambda_\ell, \dots, r - \lambda_1)$. We define λ^t the *conjugation* or *transposition* of λ as the image of λ under the transposition $(i,j) \rightarrow (j,i)$. For example, let $r = 4$ and $\ell = 3$. The Young diagram of $\lambda = (3,2,2)$ and its transpose $\lambda^t = (3,3,1)$ are depicted below; and $\lambda^\vee = (2,2,1)$, $(\lambda^t)^\vee = (\lambda^\vee)^t = (3,2,0)$ are depicted by dotted boxes

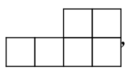
$\lambda =$ ,  $= \lambda^\vee$, $\lambda^t =$ ,  $= (\lambda^\vee)^t$.

A skew-diagram (skew-shape) λ/μ is $\{(i,j) \in \mathbb{Z}^2 |\ \ (i,j) \in \lambda, (i,j) \notin \mu\}$ the collection of boxes in λ which are not in μ. When μ is the null partition, the skew-diagram λ/μ equals the Young diagram λ. The number of boxes in λ/μ is $|\lambda/\mu| = |\lambda| - |\mu|$. The *transpose* (conjugate shape) $(\lambda/\mu)^t$ is the skew-diagram λ^t/μ^t obtained by transposing the skew-diagram λ/μ. Let $r = \lambda_1$. The *rotation* (dual shape) $(\lambda/\mu)^*$ is the image of λ/μ by rotation of 180 degrees, or the image of λ/μ under $(i,j) \longrightarrow (\ell - i + 1, r - j + 1)$. Equivalently $(\lambda/\mu)^* = \mu^\vee/\lambda^\vee$. In particular, λ^* is the skew-diagram $r^\ell/\lambda^\vee$. The dual conjugate shape $(\lambda/\mu)^\diamondsuit$ is the image of λ/μ under $(i,j) \longrightarrow (r - j + 1, \ell - i + 1)$. The map $\diamondsuit$ is the composition of the transposition with the rotation maps $\diamondsuit = *t = t*$. In particular, $\lambda^\diamondsuit = \ell^r/(\lambda^\vee)^t$. For instance, if $\mu = (2) \subset \lambda = (4,3,1)$, we have

$\lambda/\mu =$  $(\lambda/\mu)^t =$  $(\lambda/\mu)^* =$  $(\lambda/\mu)^\diamondsuit =$ .

2.2 Tableaux and words

The Littlewood–Richardson (LR for short) numbering (reading) of the boxes of a skew-diagram λ/μ is an assignment of the labels $1, 2, \ldots$ which sorts the boxes of λ/μ in increasing order from right to left along each row, starting in the top row and moving downwards; and the column LR numbering of the boxes sorts in increasing order, from right to left along each column, starting in the rightmost column and moving downwards. Analogously the reverse LR numbering and the column LR numbering of λ/μ are defined.

Example 2.1 *If* $\lambda/\mu =$ *, the LR-numbering, column LR-numbering and the corresponding reverse LR-numberings of* λ/μ *are, respectively,* $\begin{array}{cccc} & & 2 & 1\\ 6 & 5 & 4 & 3\end{array}$ $\quad\begin{array}{cccc} & & 3 & 1\\ 6 & 5 & 4 & 2\end{array}$ $\quad\begin{array}{cccc} & & 5 & 6\\ 1 & 2 & 3 & 4\end{array}$ $\quad\begin{array}{cccc} & & 4 & 6\\ 1 & 2 & 3 & 5\end{array}$.

Clearly, the column LR-numbering of λ/μ is the LR-numbering of $(\lambda/\mu)^\diamondsuit$, and the reverses of LR-numbering and column LR-numbering of λ/μ are, respectively, the LR-numbering of $(\lambda/\mu)^*$ and $(\lambda/\mu)^t$.

A *Young tableau* T of shape λ/μ is a filling of the boxes of the skew-diagram λ/μ with positive integers in $\{1, \ldots, t\}$ which is increasing in columns from top to bottom and non-decreasing in rows from left to right. When μ is the empty partition we say that T has normal shape λ. The word $w(T)$ of a Young tableau T is the sequence obtained by reading the entries of T according to its LR numbering, that is, reading right-to-left the rows of T, from top to bottom. The column word $w_{col}(T)$ is the word obtained according the column LR numbering. The weight of T is the weight of of its word. Denote by $YT(\lambda/\mu, m)$ the set of Young tableaux of shape λ/μ and weight $m = (m_1, \ldots, m_t)$.

Example 2.2 $T = \begin{array}{|c|c|c|c|c|c|} \cline{3-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 & 1 & 1 \\ \cline{2-6} \multicolumn{1}{c|}{} & 1 & 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-4} 2 & 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}$*,* $w(T) = 1111221332$ *and* $w_{col}(T) = 1112123132$.

A Young tableau with s boxes is *standard* if it is filled with $\{1, \ldots, s\}$ without repetitions. Given a tableau T of weight m, the *standardization* of T, denoted by $\widehat{T}$, is obtained by replacing, west to east, the letters 1 in T with $1, 2, \ldots, m_1$; the letters 2 with $m_1 + 1, \ldots, m_1 + m_2$; and so on. The standardization $\widehat{w}$ of a word w is defined accordingly, from right to left. For instance, the standardization of the tableau T in the previous example is $\widehat{T} = \begin{array}{|c|c|c|c|c|c|} \cline{3-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & 3 & 4 & 5 \\ \cline{2-6} \multicolumn{1}{c|}{} & 1 & 7 & 8 & \multicolumn{2}{c}{} \\ \cline{1-4} 6 & 9 & 10 & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}$, and $\widehat{w(T)} := w(\widehat{T}) = 54328711096$. If $w = w_1 w_2 \ldots w_s$ is a word and α is a permutation in the symmetric group $\mathcal{S}_s$, define $\alpha w = w_{\alpha(1)} \ldots w_{\alpha(s)}$. In the case T is standard we have $w_{col}(\widehat{T}) = rev\, w(\widehat{T}^t)$, with rev the reverse permutation.

A Young tableau T is said a *Littlewood–Richardson* (LR for short) tableau if its word, when read from the beginning to any letter, contains at least as many letters i as letters $i+1$, for all i. More generally, a word such that every prefix satisfies this property is called a *lattice permutation* or a Yamanouchi word. Notice that the column word of a LR–tableau is also a Yamanouchi word of the same weight. Denote by $LR(\lambda/\mu,\nu)$ the set of LR tableaux of shape λ/μ and weight ν. When $\mu=0$ we get the Yamanouchi tableau $Y(\nu)$, the unique tableau of shape and weight ν. In Example 2.2, T is a LR tableau with Yamanouchi word $w(T)=1111221332$ and column word $w_{col}(T)=1112123132$.

There is an one–to–one correspondence between Yamanouchi words of weight ν and standard tableaux of shape ν. Let $w=w_1w_2\cdots w_s$ be a Yamanouchi word and put the number k in the w_kth row of the diagram ν. The labels of the ith row are the k's such that $w_k=i$, thus the length is ν_i and the shape is ν. We denote this standard tableau by $U(w)$. In Example 2.2, $w=1111221332$, and $U(w)=\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&7\\ \hline 5&6&10\\ \cline{1-3} 8&9\\ \cline{1-2}\end{array}$ where the entries of the ith row are the positions of the i's in the LR reading of T.

2.3 Matrices and tableaux

Given $T\in Y(\lambda/\mu,m)$, let $M=(M_{ij})_{1\le i\le \ell(\lambda),1\le j\le t}$ be a matrix with non–negative entries such that M_{ij} is the number of $j's$ in the ith row of T, called the recording matrix of T (Lee1; Lee2; PV2). The recording matrix of a tableau of normal shape is an upper triangular matrix, and the recording matrix of an LR tableau is a lower triangular matrix. Thus we have an one–to–one correspondence between LR tableaux and tableaux of normal shape as follows. Considering T in Example 2.2, the recording matrix of T is $M=\begin{pmatrix}4&0&0\\1&2&0\\0&1&2\end{pmatrix}$. On the other hand, the transposition $M^t=\begin{pmatrix}4&1&0\\0&2&1\\0&0&2\end{pmatrix}$ encodes the tableau $B=\begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\ \hline 2&2&3\\ \cline{1-3} 3&3\\ \cline{1-2}\end{array}$ of normal shape ν and weight $\lambda-\mu$. For two Young diagrams μ and ν, define $\nu\circ\mu=(\nu_1+\mu_1,\dots,\nu_1+\mu_\ell,\nu_1,\dots,\nu_r)/(\mu_1+\nu_1,\dots,\mu_\ell+\nu_1)$, $\ell=\ell(\mu)$, $r=\ell(\nu)$. Then with $\mu=(1)$, $B\circ Y(\mu)=\begin{array}{cccccc}\cline{6-6} &&&&&\multicolumn{1}{|c|}{1}\\ \hline \multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{1}&\multicolumn{1}{c|}{2}\\ \cline{1-5} \multicolumn{1}{|c|}{2}&\multicolumn{1}{c|}{2}&\multicolumn{1}{c|}{3}\\ \cline{1-3} \multicolumn{1}{|c|}{3}&\multicolumn{1}{c|}{3}\\ \cline{1-2}\end{array}\in LR(\nu\circ\mu,\lambda)$. Given partitions λ, μ, ν such that $|\lambda|=|\mu|+|\nu|$, define $CF(\nu,\mu,\lambda)=\{B\in YT(\nu,\lambda-\mu): B\circ Y(\mu)\in \mathrm{LR}(\nu\circ\mu,\lambda)\}$ (PV2). The map $\tau: LR(\lambda/\mu,\nu)\to CF(\nu,\mu,\lambda)$ such that $\tau(M)$ is the tableau of normal shape with recording matrix M^t, where M is the recording matrix of T, is a bijection. Taking again Example 2.2, we have $\tau(T)=B$.

2.4 Rotation and transposition of LR tableaux

Given an integer i in $\{1,\dots,t\}$, let $i^*:=t-i+1$. Given a word $w=w_1w_2\cdots w_s$, over the alphabet $\{1,\dots,t\}$, of weight $m=(m_1,\dots,m_t)$, $w^*:=w_s^*\cdots w_2^*w_1^*$ is the *dual word* of w and $m^*=(m_t,\dots,m_1)$ its weight. Indeed $w^{**}=w$. A dual Yamanouchi word is a word whose dual word is Yamanouchi. Given a Young tableau T of shape λ/μ and weight $(m_1,\dots,m_t)$, $\mathrm{T}^\bullet$ denotes the Young tableau of shape $(\lambda/\mu)^*$ and weight m^*, obtained from T by replacing each entry i with i^*, and then rotating the result by 180 degrees. The word of $\mathrm{T}^\bullet$ is $w(\mathrm{T})^*$, and $\mathrm{T}^{\bullet\bullet}=\mathrm{T}$. A dual LR tableau is a tableau whose word is a dual Yamanouchi word. $\mathrm{LR}(\lambda/\mu,\nu^*)$ denotes the set of dual LR tableaux of shape λ/μ and weight ν^*, and is the image of $\mathrm{LR}((\lambda/\mu)^*,\nu)$ under the rotation map $\bullet$. Thus the rotation map $\bullet$ defines a bijection between $\mathrm{LR}((\lambda/\mu)^*, \nu^*)$ and $\mathrm{LR}(\lambda/\mu, \nu)$. Given a Yamanouchi word w of weight ν, define the standard tableau $U(w^*)$ of shape ν^* such that the label k is in row i if and only if $w_{s-i+1} = k^*$. Thus $U(w^*) = U(w)^\bullet$ and this affords a bijection between dual Yamanouchi words of weight ν^* and standard tableaux of shape ν^*. The rotation map $\bullet$ is a linear map: $M = (M_{ij})$ is the recording matrix of T if and only if the recording matrix of $T^\bullet$ is $(M_{s+1-i,t-j+1})$.

There is another natural bijection, denoted by $\blacklozenge$, between LR tableaux of conjugate weight and dual conjugate shape, see (Z; A1; A2). Given a Yamanouchi word w of weight $\nu = (\nu_1, \ldots, \nu_t)$, write $\nu^t = (\nu_1^t, \ldots, \nu_k^t)$ and observe that w is a shuffle of the words $12 \ldots \nu_i^t$ for all i, and its dual word is a shuffle of the words $t\,t-1 \cdots t-\nu_i^t+1$, for all i. Thus, we define $w^\diamond$ as the Yamanouchi word of weight ν^t obtained by replacing the subword consisting only on the letters i with the subword $12 \cdots \nu_i$, for each i. The operation $\diamond$ is defined on dual Yamanouchi words by $w^{*\diamond} := w^{\diamond *} =$, giving rise to a dual Yamanouchi word of weight ν^{*t}. The word $w^{\diamond *}$ can be obtained in just only one step: replace the subword of w consisting only on the letters i with the subword $\nu_1\, \nu_1 - 1 \cdots \nu_1 - \nu_i + 1$, for all i. Clearly, $U(w^\diamond) = U(w)^t$ is of shape ν^t, and $U(w^{*\diamond}) = U(w)^{\bullet t}$ is of shape ν^*. Given $\mathrm{T} \in \mathrm{LR}(\lambda/\mu, \nu)$ $(\mathrm{LR}(\lambda/\mu, \nu^*))$ with word w, define $\mathrm{T}^\blacklozenge$ as the LR tableau of shape $(\lambda/\mu)^\diamondsuit$ and weight ν^t obtained from T by replacing the word w with $w^\diamond$, and then rotating the result by 180 degrees and transposing. Then $\blacklozenge : \mathrm{LR}(\lambda/\mu, \nu)(\mathrm{LR}(\lambda/\mu, \nu^*)) \longrightarrow \mathrm{LR}((\lambda/\mu)^\diamondsuit, \nu^t)\mathrm{LR}((\lambda/\mu)^\diamondsuit, \nu^{*t})$ is a bijection such that $T^\blacklozenge$ has column word $w^\diamond$ and $\mathrm{T}^{\blacklozenge\blacklozenge} = \mathrm{T}$. Since $\blacklozenge\bullet = \bullet\blacklozenge$, $T^{\blacklozenge\bullet} = T^{\bullet\blacklozenge} \in \mathrm{LR}((\lambda/\mu)^t, \nu^{t*})$ $(\mathrm{LR}((\lambda/\mu)^t, \nu^t))$ has column word $w^{*\diamond}$.

Example 2.3 $\mathrm{T} = \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 1 & 2 & 2 \\ \hline 1 & 3 & & \\ \hline\end{array}$ *is a LR tableau with word* $w = 1122131$ *of weight* $\nu = (4,2,1)$. *Then*

$$\mathrm{T} = \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 1 & 2 & 2 \\ \hline 1 & 3 & & \\ \hline\end{array} \;\underset{\textit{by } w(\mathrm{T})^\diamond}{\overset{\textit{replace } w(\mathrm{T})}{\longleftrightarrow}}\; \begin{array}{|c|c|c|c|}\hline & & 2 & 1 \\ \hline & 3 & 2 & 1 \\ \hline 4 & 1 & & \\ \hline\end{array} \;\underset{\textit{transpose}}{\overset{\textit{rotate}}{\longleftrightarrow}}\; \begin{array}{|c|c|c|}\hline & 1 & 1 \\ \hline & 2 & 2 \\ \hline 1 & 3 & \\ \hline 4 & & \\ \hline\end{array} = \mathrm{T}^\blacklozenge \overset{}{\underset{\bullet}{\longleftrightarrow}} \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & 2 & 4 \\ \hline 3 & 3 & \\ \hline 4 & 4 & \\ \hline\end{array} = \mathrm{T}^{\blacklozenge\bullet}$$

. $\mathrm{T}^\blacklozenge$ *is a LR tableau with shape* $(\lambda/\mu)^\diamondsuit$ *and column word* $w^\diamond = 1212314$ *of weight* ν^t. $\mathrm{T}^{\blacklozenge\bullet}$ *is a dual LR tableau with shape* $(\lambda/\mu)^t$ *and column word* $w^{\diamond *} = 1423434$ *of weight* ν^{t*}, *where* $U(w) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & & \\ \hline 6 & & & \\ \hline\end{array}$,

$U(w^\diamond) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \hline 5 & & \\ \hline 7 & & \\ \hline\end{array} = U(w)^t$, *and* $U(w^{\diamond *}) = \begin{array}{|c|c|c|}\hline & & 1 \\ \hline & & 3 \\ \hline & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline\end{array} = U(w)^{\bullet t}$.

3 Conjugation symmetry maps

3.1 Knuth equivalence and dual Knuth equivalence

Whenever partitions $\nu \subset \mu \subset \lambda$, we say that λ/μ extends μ/ν. An *inside corner* of λ/μ is a box in the diagram μ such that the boxes below and to the right are not in μ. When a box extends λ/μ, this box is called an *outside corner*. Let T be a Young tableau and let b be an inside corner for T. A *contracting slide* (Sch; BSS) of T into the box b is performed by moving the empty box at b through T, successively interchanging it with the neighboring integers to the south and east according to the following rules: (i) if the empty box has only one neighbor, interchange with that neighbor; (ii) if it has two unequal neighbors, interchange with the smaller one; and (iii) if it has two equal neighbors, interchange with that one to the south. The empty box moves in this fashion until it has become an outside corner. This contracting slide can be reversed by performing an analogous procedure over the outside corner, called an *expanding slide*. Performing a contracting slide over each inside corner of T reduces T to a tableau T^{n} of normal shape. This procedure is known as *jeu de taquin*. T^{n} is independent of the particular sequence of inside corners used (Th), and so T^{n} is called the *rectification* of T. A word w corresponds by RSK–correspondence to a pair $(P(w), Q(w))$ of tableaux of the same shape, with $Q(w)$ standard, called the Q–symbol or recording tableau of w. Here we consider a variation of RSK-correspondence known as the Burge correspondence (B; F). Given $w = w_1 w_2 \cdots w_s$, $P(w)$ is the insertion tableau obtained by column insertion of the letters of w from left to right (F). The corresponding recording tableau $Q(w)$ is obtained by placing in $1, 2, \ldots, s$. If w is the word of T then $P(w) = T^{\mathrm{n}}$. Insertion can be translated into the language of Knuth elementary transformations. Two words w and v are said Knuth equivalent if they have the same insertion tableau. Each Knuth class is in bijection with the set of standard tableaux with shape equal to the unique tableau in that class. Two tableaux T and R are *Knuth equivalent*, written $T \equiv R$, if and only if $P(w(T)) = P(w(R))$. Equivalently, $T^{\mathrm{n}} = R^{\mathrm{n}}$, i.e. one of them can be transformed into the other one by a sequence of *jeu de taquin* slides. The insertion tableau of a Yamanouchi word w with partition weight ν, is the Yamanouchi tableau $Y(\nu)$. The recording tableau of a Yamanouchi word w is $U(w)$.

Two tableaux T and R of the same shape are *dual equivalent*, written $T \overset{d}{\equiv} R$, if any sequence of contracting slides and expanding slides that can be applied to one of them, can also be applied to the other, and the sequence of shape changes is the same for both (H; F). Dual equivalence may also be characterized by recording tableaux: $T \overset{d}{\equiv} R$ if and only if $Q(w(T)) = Q(w(R))$. Thus two tableaux of the same normal shape are dual equivalent. Let S and T be tableaux such that T extends S, and consider the set union $\mathrm{S} \cup \mathrm{T}$. The *tableau switching* (BSS) is a procedure based on *jeu de taquin* elementary moves on two alphabets that transforms $\mathrm{S} \cup \mathrm{T}$ into $\mathrm{A} \cup \mathrm{B}$, where B is a tableau Knuth equivalent to T which extends A, and A is a tableau Knuth equivalent to S. We write $\mathrm{S} \cup \mathrm{T} \xrightarrow{\mathrm{s}} \mathrm{A} \cup \mathrm{B}$. In particular, if S is of normal shape, $\mathrm{A} = \mathrm{T}^{\mathrm{n}}$, and $S = B^{\mathrm{n}}$. Switching of S with T may be described as follows: $\widehat{T}$ is a set of instructions telling where expanding slides can be applied to S. Thus switching and dual equivalence are related as below and tableaux are completely characterized by dual and Knuth equivalence.

Theorem 3.1 (H) *Let T and U be tableaux with the same normal shape and let W be a tableau which extends T.* (1) *If $T \cup W \xrightarrow{s} Z \cup X$ and $U \cup W \xrightarrow{s} Z \cup Y$, then $X \overset{d}{\equiv} Y$.*

(2) *Let $\mathcal{D}$ be a dual equivalence class and $\mathcal{K}$ be a Knuth equivalence class, both corresponding to the same normal shape. Then, there is a unique tableau in $\mathcal{D} \cap \mathcal{K}$.*

Algorithm to construct $\mathcal{D} \cap \mathcal{K}$: Let $U \in \mathcal{D}$ and let $V \in \mathcal{K}$ be the only tableau with normal shape in this class, and W any tableau that U extends: $\begin{array}{ccc} W \cup U & & W \cup X \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle s} \\ U^{\mathrm{n}} \cup Z & \rightarrow & V \cup Z. \end{array}$ Thus $X \overset{d}{\equiv} U$, $X \equiv V$, and $\mathcal{D} \cap \mathcal{K} = \{X\}$. since two words in the same Knuth class can not have the same Q–symbol.

3.2 The transposition of the rotated reversal LR tableau

Given a tableau T of normal shape, the evacuation T^{E} is the rectification of $\mathrm{T}^{\bullet}$, that is, $\mathrm{T}^{\mathrm{E}} = \mathrm{T}^{\bullet\mathrm{n}}$. T^{E} is also obtained either as the insertion tableau of the word $w(\mathrm{T})^{*}$; or according to the Schützenberger evacuation algorithm; or applying the reverse *jeu de taquin* slides to T, in the smallest rectangle containing T, to obtain T^{a} the anti-normal form T. Thus $T^{\mathrm{a}\bullet} = T^{E} = T^{\bullet\mathrm{n}}$. If w is a Yamanouchi word, by duality of Burge correspondence, $Q(w^*) = U(w)^E = U(w)^{\bullet \mathrm{n}} = U(w)^{\mathrm{a}\bullet}$. Given $w \equiv \mathrm{Y}(\nu)$, we may now define the word $w^\diamond$ as being the unique word satisfying $w^\diamond \equiv \mathrm{Y}(\nu^t)$ such that $Q(w^\diamond) = Q(w)^T = U(w)^t$. Since $(w^\diamond)^\diamond = w$, the map $w \mapsto w^\diamond$ establishes a bijection between the Knuth classes of $Y(\nu)$ and $Y(\nu^t)$. The word w^* is the unique word satisfying $w \equiv \mathrm{Y}(\nu^*)$ such that $Q(w^*) = U(w)^{\bullet \mathrm{n}}$, and $w^{\diamond *}$ is the unique word satisfying $w^{\diamond *} \equiv \mathrm{Y}(\nu^{t\,*})$ such that $Q(w^{\diamond *}) = U(w)^{E\,t}$.

Given a tableau T of any shape, the reversal T^e is the unique tableau Knuth equivalent to $\mathrm{T}^\bullet$, and dual equivalent to T (BSS). By Theorem 3.1, $T^e = [T^{\mathrm{n\,E}}]_K \cap [T]_d$, where $[\,]_K$ denotes Knuth class and $[\,]_d$ dual Knuth class. If T has normal shape, $T^E = T^e$. If $T \in \mathrm{LR}(\lambda/\mu, \nu)$, then T^e is the only tableau Knuth equivalent to $Y(\nu^*)$ and dual equivalent to T. Since crystal reflection operators, for the definition see (LS; Loth), preserve the Q–symbol, we may in the case of LR tableaux characterize explicitly the word of T^e as follows. Let w be a Yamanouchi word of weight $\nu = (\nu_1, \ldots, \nu_t)$, and let σ_i denote the reflection crystal operator acting on the subword over the alphabet $\{i, i+1\}$, for all i. If $s_{i_r} \cdots s_{i_1}$ is the longest permutation in $\mathcal{S}_t$, put $\sigma_0 := \sigma_{i_r} \cdots \sigma_{i_1}$. Then $\sigma_0 w$ is a dual Yamanouchi word of weight ν^*. Moreover, $w \equiv w'$ if and only if $\sigma_i(w) \equiv \sigma_i(w')$, and $Q(w) = Q(\sigma_i(w))$. Thus, we have proven the following

Theorem 3.2 *Let* T *be a LR tableau with shape* λ/μ *and word* w*. Then* T^e *is the dual LR tableau of shape* λ/μ *and word* $\sigma_0 w$*, and* $T^{e\,\blacklozenge\bullet}$ *is the LR tableau of shape* $(\lambda/\mu)^t$ *and column word* $(\sigma_0 w)^{\diamond\,*}$.

Corollary 3.1 $T^{e\,\blacklozenge\bullet}$ *is the unique tableau Knuth equivalent to* $Y(\nu^t)$ *and dual equivalent to* $\widehat{T}^t$.

Proof: It is enough to see that the column words of $T^{e\,\blacklozenge\bullet}$ and $\widehat{T}^t$ have the same Q–symbol. Let $\widehat{w}$ be the word of $\widehat{T}$. As $rev\,\widehat{w}$, the reverse word of $\widehat{T}$, is the column word of $\widehat{T}^t$, then $Q(rev\,\widehat{w}) = Q(\widehat{w})^{E\,t} = Q(w)^{E\,t} = Q(w^{\diamond *}) = Q(\sigma_0(w^{\diamond *})) = Q((\sigma_0 w)^{\diamond\,*})$. □

We recall that the action of crystal reflection operators on words corresponds to *jeu de taquin* slides on two-row tableaux. In particular, if w is a Yamanouchi word of weight ν and θ_i denotes the *jeu de taquin* action on the consecutive rows i and $i+1$ of $U(w)$, then $\theta_i U(w)$ is a tableau of skew-shape $(i\,i+1)\nu$ such that any two consecutive rows define a two-row tableau of normal or anti-normal shape. The labels of the j-th row of $\theta_i U(w)$ are precisely the k's such that $(\sigma_i w)_k = j$. Put $\theta_0 := \theta_{i_r} \ldots \theta_{i_1}$ with $i_r, \ldots, i_1$ as in σ_0. Thus $\theta_0 U(w) = U(w)^{\mathrm{a}}$ and $Q(\sigma_0 w) = U(w)^{\mathrm{a\,n}}$. This defines the commutative scheme

$$\begin{array}{ccccccccc} w & \longleftrightarrow & \sigma_{i_1} w & \longleftrightarrow & \sigma_{i_2}\sigma_{i_1} w & \longleftrightarrow & \cdots & \longleftrightarrow & \sigma_0 w \\ \updownarrow & & \updownarrow & & \updownarrow & & & & \updownarrow \\ U(w) & \longleftrightarrow & \theta_{i_1} U(w) & \longleftrightarrow & \theta_{i_2}\theta_{i_1} U(w) & \longleftrightarrow & \cdots & \longleftrightarrow & \theta_0 U(w). \end{array}$$

(This was the procedure in (A1).) Similarly, if $\sigma_i T$ denotes the tableau obtained by the action of σ_i on its word, we get the commutative scheme

$$\begin{array}{ccccccccc} T & \longleftrightarrow & \sigma_{i_1} T & \longleftrightarrow & \sigma_{i_2}\sigma_{i_1} T & \longleftrightarrow & \cdots & \longleftrightarrow & \sigma_0 T \\ \tau\updownarrow & & \tau\updownarrow & & \tau\updownarrow & & & & \tau\updownarrow \\ P & \longleftrightarrow & \theta_{i_1} P & \longleftrightarrow & \theta_{i_2}\theta_{i_1} P & \longleftrightarrow & \cdots & \longleftrightarrow & \theta_0 P = P^{\mathrm{a}}. \end{array}$$

Theorem 3.3 *Let* T *be a LR tableau and* $\tau(T) = P$*. Then, the following commutative scheme holds*

$$\begin{array}{ccccc} T & \overset{e}{\longleftrightarrow} & T^e & \overset{\bullet}{\longleftrightarrow} & T^{e\bullet} \\ \tau\updownarrow & & \tau\updownarrow & & \tau\updownarrow \\ P & \underset{\mathrm{a}}{\longleftrightarrow} & P^{\mathrm{a}} & \overset{\bullet}{\longleftrightarrow} & P^E. \end{array}$$

3.3 Main bijections

As already mentioned bijections ϱ^{WHS} and ϱ^{BSS} are identical. Let

$$\begin{array}{rccc} \varrho^{BSS}: & LR(\lambda/\mu,\nu) & \to & LR(\lambda^t/\mu^t,\nu^t) \\ & T & \mapsto & \varrho^{BSS}(T)=[Y(\nu^t)]_K\cap[\widehat{T}^t]_d \end{array} \qquad \text{(BSS).}$$

The image of T by the BSS-bijection is the unique tableau of shape λ^t/μ^t whose rectification is $Y(\nu^t)$ and the Q–symbol of the column reading word is $Q(T)^{Et}$. The idea behind this bijection can be told as follows: $\widehat{T}$ constitutes a set of instructions telling where expanding slides can be applied to $Y(\mu)$. Then $\widehat{T}^t$ is a set of instructions telling where expanding slides can be applied to $Y(\mu)^t$. Tableau–switching provides an algorithm to give way to those instructions:

$$\begin{array}{ccccccc} Y(\mu)\cup T & \xrightarrow[\text{of T}]{\text{standardization}} & Y(\mu)\cup\widehat{T} & \xrightarrow[\text{of }\widehat{T}]{\text{transposition}} & Y(\mu^t)\cup\widehat{T}^t & & Y(\mu^t)\cup\varrho^{BSS}(T) \\ & & & & \downarrow s & & \uparrow s \\ & & & & (\widehat{T}^t)^{\mathrm{n}}\cup Z & \mapsto & Y(\nu^t)\cup Z \end{array}.$$

Then $\varrho^{BSS}(T)\equiv Y(\nu^t)$ and $\varrho^{BSS}(T)\equiv_d \widehat{T}^t$.

Example 3.4 *Let T in $LR(\lambda/\mu,\nu)$ with $\mu=(2,1)$, $\nu=(5,3,2)$ and $\lambda=(6,4,3)$:*

$$T=\begin{array}{cccccc} & & 1&1&1&1 \\ & 1&2&2 & & \\ 2&3&3 & & & \end{array} \;\to\; \widehat{T}=\begin{array}{cccccc} & & 2&3&4&5 \\ & 1&7&8 & & \\ 6&9&10 & & & \end{array} \;\to\; \widehat{T}^t=\begin{array}{ccc} & & 6 \\ & 1 & 9 \\ 2&7&10 \\ 3&8& \\ 4&& \\ 5&& \end{array} \;\to$$

$$\begin{array}{ccc} \to Y(\mu^t)\cup\widehat{T}^t=\begin{array}{ccc} \mathbf{1}&\mathbf{1}&6 \\ \mathbf{2}&1&9 \\ 2&7&10 \\ 3&8& \\ 4&& \\ 5&& \end{array} & & \begin{array}{ccc} \mathbf{1}&\mathbf{1}&1 \\ \mathbf{2}&1&2 \\ 1&2&3 \\ 2&3& \\ 4&& \\ 5&& \end{array}=Y(\nu^t)\cup\varrho^{BSS}(T) \\ s\downarrow & & \uparrow s \\ (\widehat{T}^t)^{\mathrm{n}}\cup Z=\begin{array}{ccc} 1&6&9 \\ 2&7&10 \\ 3&8&\mathbf{1} \\ 4&\mathbf{2}& \\ 5&& \\ \mathbf{1}&& \end{array} & \longrightarrow & \begin{array}{ccc} 1&1&1 \\ 2&2&2 \\ 3&3&\mathbf{1} \\ 4&\mathbf{2}& \\ 5&& \\ \mathbf{1}&& \end{array}=Y(\nu^t)\cup Z \end{array}.$$

Let

$$\begin{array}{rccc} \varrho_3: & LR(\lambda/\mu,\nu) & \to & LR(\lambda^t/\mu^t,\nu^t) \\ & T & \mapsto & \varrho_3(T)=T^{e\bullet\blacklozenge} \\ & w & \mapsto & \sigma_0 w^{*\diamond} \end{array} \qquad \text{(Z; A1; A2).}$$

As $T^{e\bullet\blacklozenge}$ is the unique tableau Knuth equivalent to $Y(\nu^t)$ and dual equivalent to $(\widehat{T})^t$, we have

Corollary 3.2 *ϱ^{BSS} and ϱ_3 are identical bijections.*

Example 3.5 *Let T in $LR(\lambda/\mu, \nu)$ as before:*

$$\mathrm{T} = \begin{ytableau} \none & \none & 1 & 1 & 1 & 1 \\ \none & 1 & 2 & 2 \\ 2 & 3 & 3 \end{ytableau} \quad \underset{reversal}{\xrightarrow{e}} \quad \mathrm{T}^e = \begin{ytableau} \none & \none & 1 & 1 & 3 & 3 \\ \none & 2 & 2 & 2 \\ 3 & 3 & 3 \end{ytableau} \quad \underset{of\, \lambda/\mu}{\xrightarrow{transpose}} \quad \begin{ytableau} \none & \none & 3 \\ \none & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 \\ 3 \\ 3 \end{ytableau} \quad \longrightarrow \quad \begin{ytableau} \none & \none & 1 \\ \none & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 \\ 4 \\ 5 \end{ytableau} = \varrho^{BSS}(\mathrm{T})$$

$$w = 1111221332 \quad \longrightarrow \quad \sigma_0 w = 3311222333 \quad \xrightarrow{reverse} \quad 3332221133 \quad \longrightarrow \quad \underset{\varrho_3(T) = \varrho^{BSS}(T)}{\underset{\textit{column word of}}{1231231245}}$$

or

$$\mathrm{T} = \begin{ytableau} \none & \none & 1 & 1 & 1 & 1 \\ \none & 1 & 2 & 2 \\ 2 & 3 & 3 \end{ytableau} \quad \xrightarrow{e} \quad \mathrm{T}^e = \begin{ytableau} \none & \none & 1 & 1 & 3 & 3 \\ \none & 2 & 2 & 2 \\ 3 & 3 & 3 \end{ytableau} \quad \xrightarrow{\bullet} \quad \mathrm{T}^{e\bullet} = \begin{ytableau} \none & \none & \none & 1 & 1 & 1 \\ \none & \none & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 \end{ytableau} \quad \xrightarrow{\blacklozenge} \quad \mathrm{T}^{e\bullet\blacklozenge} = \begin{ytableau} \none & \none & 1 \\ \none & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 \\ 4 \\ 5 \end{ytableau}.$$

4 Computational complexity of bijection ♦ and reduction of conjugation symmetry map

We show that the computational complexity of bijection ♦ is linear on the input. We follow closely (PV2) for this section. Using ideas and techniques of Theoretical Computer Science, see (AHU; CLRS), each bijection can be seen as an algorithm having one type of combinatorial objects as *input*, and another as *output*. We define a *correspondence* as an one–to–one map established by a bijection; therefore, obviously several different defined bijections can produce the same correspondence. In this way one can think of a correspondence as a function which is computed by the algorithm, viz. the bijection. The computational complexity is, roughly, the number of steps in the bijection. Two bijections are *identical* if and only if they define the same correspondence. Obviously one task can be performed by several different algorithms, each one having its own computational complexity, see (AHU; CLRS). For example we recall that there are several ways to multiply large integers, from naive algorithms, e.g. the Russian peasant algorithm, to that ones using FFT (Fast Fourier Transform), e.g. Schönhage–Strassen algorithm; see e.g. (GG) for a comprehensive and update reference. Formally, a function f reduces linearly to g, if it is possible to compute f in time linear in the time it takes to compute g; f and g are linearly equivalent if f reduces linearly to g and vice versa. This defines an equivalence relation on functions, which can be translated into a linear equivalence on bijections.

Let $D = (d_1, \dots, d_n)$ be an array of integers, and let $m = m(D) := \max_i d_i$. The *bit–size* of D, denoted by $\langle D \rangle$, is the amount of space required to store D; for simplicity from now on we assume that $\langle D \rangle = n \lceil \log_2 m + 1 \rceil$. We view a bijection $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ as an algorithm which inputs $A \in \mathcal{A}$ and outputs $B = \delta(A) \in \mathcal{B}$. We need to present Young tableaux as arrays of integers so that we can store them and compute their bit–size. Suppose $A \in YT(\lambda/\mu; m)$: a way to encode A is through its recording matrix $(c_{i,j})$, which is defined by $c_{i,j} = a_{i,j} - a_{i,j-1}$; in other words, $c_{i,j}$ is the number of j's in the i–th row of A; this is the way Young tableaux will be presented in the input and output of the algorithms. Finally, we say that a map $\gamma : \mathcal{A} \longrightarrow \mathcal{B}$ is *size–neutral* if the ratio $\frac{\langle \gamma(A) \rangle}{\langle A \rangle}$ is bounded for all $A \in \mathcal{A}$. Throughout the paper we consider only size–neutral maps, so we can investigate the linear equivalence of maps comparing them by the number of times other maps are used, without be bothered by the timing. In fact, if we drop the condition of being size–neutral, it can happen that a map increases the bit–size of combinatorial objects, when it transforms the input into the output, and this affects the timing of its subsequent applications. Let $\mathcal{A}$ and $\mathcal{B}$ be two possibly infinite sets of finite integer arrays, and let $\delta : \mathcal{A} \longrightarrow \mathcal{B}$ be an explicit map between them. We say that δ has linear cost if δ computes $\delta(A) \in \mathcal{B}$ in linear time $O(\langle A \rangle)$ for all $A \in \mathcal{A}$. There are many ways to construct new bijections out of existing ones: we call such algorithms *circuits* and we define below several of them that we need.

∘ Suppose $\delta_1 : \mathcal{A}_1 \longrightarrow \mathcal{X}_1$, $\gamma : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ and $\delta_2 : \mathcal{X}_2 \longrightarrow \mathcal{B}$, such that δ_1 and δ_2 have linear cost, and consider $\chi = \delta_2 \circ \gamma \circ \delta_1 : \mathcal{A} \longrightarrow \mathcal{B}$. We call this circuit *trivial* and denote it by $I(\delta_1, \gamma, \delta_2)$.

∘ Suppose $\gamma_1 : \mathcal{A} \longrightarrow \mathcal{X}$ and $\gamma_2 : \mathcal{X} \longrightarrow \mathcal{B}$, and let $\chi = \gamma_2 \circ \gamma_1 : \mathcal{A} \longrightarrow \mathcal{B}$. We call this circuit *sequential* and denote it by $S(\gamma_1, \gamma_2)$.

∘ Suppose $\delta_1 : \mathcal{A} \longrightarrow \mathcal{X}_1 \times \mathcal{X}_2$, $\gamma_1 : \mathcal{X}_1 \longrightarrow \mathcal{Y}_1$, $\gamma_2 : \mathcal{X}_2 \longrightarrow \mathcal{Y}_2$, and $\delta_1 : \mathcal{Y}_1 \times \mathcal{Y}_2 \longrightarrow \mathcal{B}$, such that δ_1 and δ_1 have linear cost. Consider $\chi = \delta_2 \circ (\gamma_1 \times \gamma_2) \circ \delta_1 : \mathcal{A} \longrightarrow \mathcal{B}$: we call this circuit *parallel* and denote it by $P(\delta_1, \gamma_1, \gamma_2, \delta_2)$.

For a fixed bijection α, we say that $\sqsupset$ is an *α–based ps–circuit* if one of the following holds:

- $\sqsupset = \delta$, where δ is a bijection having linear cost.
- $\sqsupset = I(\delta_1, \alpha, \delta_2)$, where δ_1, δ_2 are bijections having linear cost.
- $\sqsupset = P(\delta_1, \gamma_1, \gamma_2, \delta_2)$, where γ_1, γ_2 are α–based ps–circuits and δ_1, δ_2 are bijections having linear cost.
- $\sqsupset = S(\gamma_1, \gamma_2)$, where γ_1, γ_2 are α–based ps–circuits.

In other words, $\sqsupset$ is an α–based ps–circuit if there is a parallel–sequential algorithm which uses only a finite number of linear cost maps and a finite number of application of map α. The α–cost of $\sqsupset$ is the number of times the map α is used; we denote it by $s(\sqsupset)$.

Let $\gamma : \mathcal{A} \longrightarrow \mathcal{B}$ be a map produced by the α–based ps–circuit $\sqsupset$. We say that $\sqsupset$ computes γ at cost $s(\sqsupset)$ of α. A map β is *linearly reducible* to α, write $\beta \hookrightarrow \alpha$, if there exist a finite α–based ps–circuit $\sqsupset$ which computes β. In this case we say that β can be computed in at most $s(\sqsupset)$ cost of α. We say that maps α and β are linearly equivalent, write $\alpha \sim \beta$, if α is linearly reducible to β, and β is linearly reducible to α. We recall, gluing together, results proved in Section 4.2 of (PV2).

Proposition 4.1 *Suppose $\alpha_1 \hookrightarrow \alpha_2$ and $\alpha_2 \hookrightarrow \alpha_3$, then $\alpha_1 \hookrightarrow \alpha_3$. Moreover, if α_1 can be computed in at most s_1 cost of α_2, and α_2 can be computed in at most s_2 cost of α_3, then α_1 can be computed in at most $s_1 s_2$ cost of α_3. Suppose $\alpha_1 \sim \alpha_2$ and $\alpha_2 \sim \alpha_3$, then $\alpha_1 \sim \alpha_3$ Suppose $\alpha_1 \hookrightarrow \alpha_2 \hookrightarrow \ldots \hookrightarrow \alpha_n \hookrightarrow \alpha_1$, then $\alpha_1 \sim \alpha_2 \sim \ldots \sim \alpha_n \sim \alpha_1$.*

We state now the computational complexity of bijection ⧫ and the reduction of conjugation symmetry map.

Algorithm 4.1 [Bijection ♦.]
Input: *LR tableau T of skew shape λ/μ, with $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$, $\mu = (\mu_1 \geq \ldots \geq \mu_n)$, and filling $\nu = (\nu_1 \geq \ldots \geq \nu_n)$, having $A = (a_{i,j}) \in M_{n\times n}(\mathbb{N})$ $(a_{i,j} = 0$ if $j > i)$ as (lower triangular) recording matrix.*

Write $\widetilde{A}$, a copy of the matrix A.
For $j := n$ down to 2 do
For $i := 1$ to n do
Begin
If $i = j$ then $\widetilde{a}_{i,i} := \widetilde{a}_{i,i} + \lambda_1 - \lambda_i$
else
If $j > i$ then $\widetilde{a}_{i,j} = 0$ else $\widetilde{a}_{i,j} := \widetilde{a}_{i,j} + \widetilde{a}_{i,j+1}$.
End

So far the computational cost is $O\left(n^2\right) = O\left(\langle A\rangle\right)$.

Set a matrix $B = (b_{i,j}) \in M_{\lambda_1\times\lambda_1}(\mathbb{N})$ such that $b_{i,j} = 0$ for all i, j.
For $i := 1$ to n do
Begin
Set $c := 0$.
For $j := 0$ to n do
Begin
$r := \widetilde{a}_{i+j,i} - a_{i+j,i}$, see Remark 4.2.
For $t := 1$ to $a_{i+j,i}$ do $b_{r+t,c+t} := b_{r+t,c+t} + 1$.
$c := c + a_{i+j,i}$.
End
End
This part has total computational cost at most equal to

$$O\left(\sum_{1\leq i.j\leq n} a_{i,j}\right) = O\left(|\lambda \setminus \mu|\right) = O\left(|\lambda| - |\mu|\right) = O\left(\langle T\rangle\right).$$

Output: *B recording matrix of the output tableau.*

Remark 4.2 *For all $1 \leq i \leq n$ and $0 \leq j \leq n - i + 1$, we have*

$$\widetilde{a}_{i+j+1,i} - \widetilde{a}_{i+j,i} \geq a_{i+j+1,i}.$$

From Theorem 3.3 and this algorithm we have

Theorem 4.3 *The conjugation symmetry maps ϱ^{BSS}, ϱ^{WHS} and ϱ_3 are identical, and linear equivalent to the Schützenberger involution E,*

$$\begin{array}{ccccc} T & \overset{e\,\bullet}{\longleftrightarrow} & T^{e\bullet} & \overset{\blacklozenge}{\longleftrightarrow} & T^{e\bullet\blacklozenge} \\ \tau\updownarrow & & \tau\updownarrow & & \\ P & \underset{E}{\overset{\text{evacuation}}{\longleftrightarrow}} & P^{E}. & & \end{array}$$

Thus conjugation symmetry maps and commutative symmetry maps are linearly reducible to each other.

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m-noncrossing partitions and *m*-clusters

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Abstract. Let W be a finite crystallographic reflection group, with root system Φ. Associated to W there is a positive integer, the generalized Catalan number, which counts the clusters in the associated cluster algebra, the noncrossing partitions for W, and several other interesting sets. Bijections have been found between the clusters and the noncrossing partitions by Reading and Athanasiadis et al.

There is a further generalization of the generalized Catalan number, sometimes called the Fuss-Catalan number for W, which we will denote $C_m(W)$. Here m is a positive integer, and $C_1(W)$ is the usual generalized Catalan number. $C_m(W)$ counts the m-noncrossing partitions for W and the m-clusters for Φ. In this abstract, we will give an explicit description of a bijection between these two sets.

The proof depends on a representation-theoretic reinterpretation of the problem, in terms of exceptional sequences of representations of quivers.

Résumé. Soit W un groupe de réflections fini et crystallographique, avec système de racines Φ. Associé à W, il y a un entier positif, le nombre de Catalan généralisé, qui compte les amas dans l'algèbre amassée associée, les partitions non-croisées de W, et plusieurs autres ensembles intéressantes. Des bijections entre les amas et les partitions non-croisées ont été données par Reading et Athanasiadis et al.

On peut encore généraliser le nombre de Catalan généralisé, obtenant le nombre Fuss-Catalan de W, que nous noterons $C_m(W)$. Ici m est un entier positif, et $C_1(W)$ est le nombre Catalan généralisé standard. $C_m(W)$ compte les partitions m-non-croisées de W et les m-amas de Φ. Dans ce résumé, nous donnerons une bijection explicite entre ces deux ensembles.

La démonstration dépend d'une réinterprétation des objets du point de vue des suites exceptionnelles de représentations de carcois.

Keywords: m-noncrossing partitions, m-clusters, Fuss-Catalan numbers

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1 Fuss-Catalan numbers

Let W be a finite reflection group, with a set of simple reflections S of cardinality n. For basic facts on reflections groups, see [Hu]. We will assume throughout that W is irreducible, that is to say, W is not the direct product of two smaller reflection groups; all our statements generalize in a completely straightforward way to the reducible case.

A *Coxeter element* for W is the product of the simple reflections of W, taken in some order. All Coxeter elements are conjugate, so they have a well-defined order, called the Coxeter number, and denoted h.

Associated to W are a collection of positive integers called its *exponents*, $e_1, \ldots, e_n$. The Fuss-Catalan number for W is given by the following formula:

$$C_m(W) = \frac{\prod_{i=1}^{n} mh + e_i + 1}{\prod_{i=1}^{n} e_i + 1}.$$

If we set $m = 1$, we get the generalized Catalan number for W.

In the case that W is the symmetric group S_{n+1}, the Coxeter element is an $n+1$-cycle, $h = n+1$, and the exponents are the numbers from 1 to n. In this case, the generalized Catalan numbers are just the usual Catalan numbers.

As we shall explain in more detail below, the Fuss-Catalan numbers count the maximal faces in the m-cluster complex associated to W and the m-noncrossing partitions for W. Bijections have been constructed between these two sets in the $m = 1$ case by Reading [Re] and Athanasiadis et al. [ABMW]. Our goal in this extended abstract is to construct a bijection for arbitrary m.

In order for m-clusters and m-noncrossing partitions to be well-defined, we do not need to assume that W is crystallographic. However, the techniques of our proof, which rely on quiver representations, do require that assumption. We will make clear at what point we have to add the crystallographic assumption.

The Fuss-Catalan numbers also arise in the study of the Shi arrangement and its generalizations (see [At]). At this point, even for $m = 1$, no type-free bijection is known from either clusters or noncrossing partitions to the regions of the Shi arrangement inside the dominant chamber (which are also counted by the generalized Catalan number).

2 Reflection group conventions

Let T be the set of all reflections for W. By definition, $T = \{wsw^{-1} \mid w \in W, s \in S\}$. Let N be the cardinality of T.

Associated to W is a *Coxeter diagram* whose vertices correspond to elements of S, and where two vertices are connected by an edge iff the corresponding simple reflections do not commute.

The Coxeter diagram of a finite reflection group is always a tree, so in particular it is a bipartite graph. Therefore, we can divide S into two parts, S^+ and S^- such that no two vertices in either part are adjacent. (This division is unique up to the labelling of the parts.) Number the reflections in S^+ as s_1 to s_r, and the reflections in S^- as s_{r+1} to s_n.

Fix the Coxeter element $c = s_1 \ldots s_n$.

For $1 \leq i \leq N$, let r_i be defined as $s_1 s_2 \ldots s_{i-1} s_i s_{i-1} \ldots s_1$, where the indexing of simple reflections is taken mod n, so that $s_{n+1} = s_1$, etc.

Each reflection in T occurs as r_i for exactly one value of i with $1 \leq i \leq N$. Define a total order on T by saying that $r_i < r_j$ iff $i < j$.

3 m-noncrossing partitions

In this section we discuss m-noncrossing partitions for a reflection group W. The definition is due to Armstrong; for further information, see [Ar1].

Define $\ell_T : W \to \mathbb{N}$ by letting $\ell_T(w)$ be the minimal length of an expression for w as a product of elements of T. (Note that this is not the classical length function for W, which would consider instead only expressions for w as a product of elements of S.) We also note, for future use, that $\ell_T(c) = n$.

We can partially order W as follows: $u <_T v$ iff there is a minimal-length expression for v as a product of elements of T which has a minimal-length expression for u as a prefix. The usual (type-free) definition of noncrossing partitions is to take $\mathrm{NC}(W)$ to be the interval from the identity element e to c in this order [BW1, Be]. The number of elements of $\mathrm{NC}(W)$ is the generalized Catalan number $C_1(W)$.

We now give an m-ified version. For $w \in W$, define a *minimal k-factorization of* w to be a k-tuple $(u_0, \dots, u_{k-1})$ of elements of W such that

$$w = u_0 \dots u_{k-1} \quad \text{and} \quad \ell_T(w) = \sum_i \ell_T(u_i).$$

We define $\mathrm{NC}^{(m)}(W)$, the m-noncrossing partitions of W to be the collection of minimal $m+1$-factorizations of c. (Note that there is a bijection from $\mathrm{NC}^{(1)}$ to NC, defined by sending (u, v) to u.)

Armstrong obtained the following enumeration of the m-noncrossing partitions.

Theorem 1 ([Ar1]) $|\mathrm{NC}^{(m)}(W)| = C_m(W)$.

4 Coloured factorizations

A *coloured factorization* of the Coxeter element c is simply an expression for c as a product of n elements of T, where each reflection has an associated *colour* in $\mathbb{Z}$. We will write the colour as a superscript in parentheses.

We define an *m-increasing coloured factorization* to be a coloured factorization whose colours are chosen from 0 to m, such that the colours appear in weakly increasing order, and among the reflections of a given colour, the order of the reflections is increasing with respect to the total order on T.

Proposition 1 *There is a bijection between m-noncrossing partitions and m-increasing coloured factorizations.*

To construct the bijection, we use the following result:

Theorem 2 ([ABW]) *Let $u \leq_T c$, with $\ell_T(u) = r$. There is a unique factorization of u as a product of r reflections $u = t_1 \dots t_r$ such that $t_1 < \dots < t_r$.*

This factorization appears in [ABW] as the set of labels on the increasing chain from e to u in an EL-labelling for NC.

We now return to the problem of constructing an m-increasing factorization of c from an m-noncrossing partition. Let $u = (u_0, \dots, u_m)$ be an m-noncrossing partition. Note that $u_i <_T c$ for all i, so Theorem 2 applies to each u_i. Let $(t_{i1}, \dots, t_{ir_i})$ be the factorization of u_i obtained from Theorem 2. The m-increasing coloured factorization associated to u is

$$(t_{01}^{(0)}, \dots, t_{0r_0}^{(0)}, t_{11}^{(1)}, \dots, t_{1r_1}^{(1)}, \dots, t_{mr_m}^{(m)}).$$

In other words, we take the factorizations of each of the u_i from Theorem 2, concatenate them, and colour the reflections corresponding to u_i with the colour i.

It is clear that this map from m-noncrossing partitions to m-increasing factorizations can be inverted, and thus defines a bijection.

5 m-clusters

Let Φ be a root system for W, with simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ corresponding (in order) to the simple reflections $s_1, \dots, s_n$. We do not (yet) assume that Φ is crystallographic. The m-coloured almost positive roots consist of m copies of $\Phi_{>0}$, each indexed by a number from 0 to $m-1$, together with a single copy of $-\Pi$, the negative simple roots. The set of m-coloured almost positive roots is denoted $\Phi^{(m)}_{\geq -1}$.

The m-cluster complex was defined by Fomin and Reading [FR] as a certain simplicial complex on this set. We will give an equivalent definition, which is due to Tzanaki [Tz] (up to some changes of convention).

Define an m-decreasing coloured factorization of c as follows:

- The colours of the reflections are integers from 0 to m.
- The colours appear in weakly decreasing order.
- Among the reflections of a fixed colour, the reflections appear in decreasing order with respect to the total order on T.
- The only reflections of colour m which are allowed are $\{r_i = s_1 s_2 \dots s_i \dots s_2 s_1 \mid 1 \leq i \leq n\}$.

There is a bijection ϕ from the set of roots $\Phi^{(m)}_{\geq -1}$, to the set of coloured reflections that can appear in an m-decreasing factorization of c. It is defined as follows:

- ϕ sends the coloured positive root $\beta^{(i)}$ to the coloured reflection $t_\beta^{(i)}$, where t_β is the reflection through the hyperplane perpendicular to β,
- ϕ sends the negative simple root $-\alpha_i$ to the reflection $r_i^{(m)}$.

Note that for $s_i \in S^-$, $\phi(-\alpha_i)$ is not the reflection s_i.

The result of Tzanaki (which generalizes a result of [BW2] in the $m = 1$ case), and which we can take as the definition of m-clusters, is the following:

Theorem 3 ([Tz]) *m-clusters can be characterized as those sets of n elements from $\Phi^{(m)}_{\geq -1}$ such that, if their corresponding reflections under ϕ are ordered in decreasing order (by colour and then with respect to the total order on T), the result is an m-decreasing factorization of c.*

The enumeration of m-clusters was carried out by Fomin and Reading:

Theorem 4 ([FR]) *The number of m-clusters for Φ is $\mathrm{NC}^{(m)}(W)$.*

We have now defined the objects which we are interested in, the m-noncrossing partitions for W and the m-clusters for Φ, and have recalled that they have the same cardinality. We will now proceed to define a bijection between them, or rather, between the m-increasing and m-decreasing factorizations of c.

6 Mutation of coloured factorizations

There is a *mutation* procedure which allows one to replace one coloured factorization of c by another. The term mutation does not come from cluster algebras, but rather from the theory of exceptional sequences. See the final section for more details and references.

For $1 \leq i \leq n-1$, define an operation μ_i on coloured factorizations as follows.

$$\mu_i(t_1^{(c_1)}, \dots, t_i^{(c_i)}, t_{i+1}^{(c_{i+1})}, \dots, t_n^{(c_n)}) = (t_1^{(c_1)}, \dots, t_{i+1}^{(c_{i+1})}, (t_{i+1}t_it_{i+1})^{(d)}, \dots, t_n^{(c_n)})$$

where $d = c_i + 1$ if $t_{i+1}t_it_{i+1} < t_i$, otherwise $d = c_i$.

Lemma 1 *The operations μ_i satisfy the braid relations, that is to say, $\mu_i\mu_{i+1}\mu_i = \mu_{i+1}\mu_i\mu_{i+1}$, and $\mu_i\mu_j = \mu_j\mu_i$ if $|i-j| \geq 2$.*

Define $\mu_{\mathrm{rev}} = \mu_1(\mu_2\mu_1)(\mu_3\mu_2\mu_1)\dots(\mu_{n-1}\mu_{n-2}\dots\mu_1)$. (Note that, since the μ_i satisfy the braid relations, there are many equivalent ways to define μ_{rev}.)

Then we have the following theorem:

Theorem 5 *μ_{rev} defines a bijection from the m-decreasing coloured factorizations of c to the m-increasing coloured factorizatons of c.*

Together with the bijections we have already established between m-clusters and m-decreasing factorizations of c, and between m-noncrossing partitions and m-increasing factorizations of c, this defines a bijection between m-clusters and m-noncrossing partitions, as desired.

7 Example: A_2, $m = 2$

In this section, we consider a small example. W is the symmetric group on 3 letters, generated by $s_1 = (12)$ and $s_2 = (23)$. Let $m = 2$.

$S^+ = \{s_1\}$, $S^- = \{s_2\}$. $c = s_1s_2 = (123)$. $h = 3$. Write t for $s_1s_2s_1 = (13)$, the unique non-simple reflection. The total order on the reflections is $s_1 < t < s_2$. Write α_1 and α_2 for the simple roots, and β for the unique non-simple positive root. The Fuss-Catalan number is 12.

In the table below, we list the twelve 2-clusters for A_2, their corresponding decreasing coloured factorizations as in Theorem 3, the result of applying $\mu_{\mathrm{rev}} = \mu_1$ to the 2-decreasing coloured factorization (which yields a 2-increasing factorization), and the corresponding 2-noncrossing partition.

$$
\begin{array}{llll}
\{\beta^{(0)},\alpha_1^{(0)}\}\to & (t^{(0)},s_1^{(0)})\to & (s_1^{(0)},s_2^{(0)})\to & (s_1s_2,e,e)\\
\{\beta^{(1)},\alpha_1^{(0)}\}\to & (t^{(1)},s_1^{(0)})\to & (s_1^{(0)},s_2^{(1)})\to & (s_1,s_2,e)\\
\{-\alpha_2,\alpha_1^{(0)}\}\to & (t^{(2)},s_1^{(0)})\to & (s_1^{(0)},s_2^{(2)})\to & (s_1,e,s_2)\\
\{\alpha_2^{(0)},\beta^{(0)}\}\to & (s_2^{(0)},t^{(0)})\to & (t^{(0)},s_1^{(1)})\to & (t,s_1,e)\\
\{\alpha_2^{(1)},\beta^{(0)}\}\to & (s_2^{(1)},t^{(0)})\to & (t^{(0)},s_1^{(2)})\to & (t,e,s_1)\\
\{\alpha_1^{(1)},\alpha_2^{(0)}\}\to & (s_1^{(1)},s_2^{(0)})\to & (s_2^{(0)},t^{(1)})\to & (s_2,t,e)\\
\{-\alpha_1,\alpha_2^{(0)}\}\to & (s_1^{(2)},s_2^{(0)})\to & (s_2^{(0)},t^{(2)})\to & (s_2,e,t)\\
\{\beta^{(1)},\alpha^{(1)}\}\to & (t^{(1)},s_1^{(1)})\to & (s_1^{(1)},s_2^{(1)})\to & (e,s_1s_2,e)\\
\{-\alpha_2,\alpha_1^{(1)}\}\to & (t^{(2)},s_1^{(1)})\to & (s_1^{(1)},s_2^{(2)})\to & (e,s_1,s_2)\\
\{\alpha_2^{(1)},\beta^{(1)}\}\to & (s_2^{(1)},t^{(1)})\to & (t^{(1)},s_1^{(2)})\to & (e,t,s_1)\\
\{-\alpha_1,\alpha_2^{(1)}\}\to & (s_1^{(2)},s_2^{(1)})\to & (s_2^{(1)},t^{(2)})\to & (e,s_2,t)\\
\{-\alpha_2,-\alpha_1\}\to & (t^{(2)},s_1^{(2)})\to & (s_1^{(2)},s_2^{(2)})\to & (e,e,s_1s_2)
\end{array}
$$

8 Positive parts

There is a subcomplex of the m-cluster complex which is called its positive part, that is, the part which does not involve any of the negative simple roots. Under the correspondence of Theorem 3, the positive m-clusters (the m-clusters in the positive part) correspond to m-decreasing factorizations of c in which no reflections with the colour m appear.

Theorem 6 ([FR]) *The number of m-clusters in the positive part of the cluster complex for Φ is:*

$$|C_{-m-1}(W)| = \frac{\prod_{i=1}^{n} mh + e_i - 1}{\prod_{i=1}^{n} e_i + 1}.$$

We can give the following description of the image of the positive m-clusters under our bijection. We use the definition of r_i from Section 2.

Theorem 7 *The image under μ_{rev} of the m-decreasing factorizations of c corresponding to positive m-clusters, consists of those m-increasing factorizations in which the coloured reflections $\{r^{(m)}_{N-i+1} \mid 1 \leq i \leq n\}$ do not appear.*

In fact, as was conjectured by Armstrong [Ar2], there is a whole family of natural bijections. In Section 2 we defined $r_1, \dots, r_N$. We will now extend that definition. For $i \geq 1$, define r_i to be the coloured reflection $(s_1 s_2 \dots s_i \dots s_1)^{(\lfloor i/N \rfloor)}$.

Totally order the coloured reflections by $r_i < r_j$ iff $i < j$. Define a decreasing coloured factorization of c to be a factorization of c into coloured reflections such that the factors are decreasing with respect to this order, and define an increasing coloured factorization of c similarly.

Then Proposition 1 can be restated as saying that m-noncrossing partitions are in bijection with increasing factorizations of c using coloured reflections from the set $\{r_1, \dots, r_{(m+1)N}\}$. Theorem 3 can be restated as saying that m-clusters are in bijection with decreasing factorizations of c using coloured reflections from the set $\{r_1, \dots, r_{mN+n}\}$, while the positive m-clusters are in bijection with decreasing factorizations using coloured reflections from the set $\{r_1, \dots, r_{mN}\}$.

We have the following generalization of Theorems 5 and 7:

Theorem 8 *For any $0 \le m$ and $0 \le k$, the image under μ_{rev} of the decreasing factorizations of c using coloured reflections $\{r_i\}$ with $1 \le i \le Nm + (k+1)n$, consists of the increasing factorizations of c using coloured reflections $\{r_i\}$ with $1 \le i \le N(m+1) + kn$.*

Other than the cases described by Theorems 5 and 7, there do not seem to be enumerative results known for these families.

9 Representation theory

In this section, we shall sketch the approach taken in our proofs of the preceding results. This approach depends heavily on the theory of quiver representations, of which we will attempt to sketch some elements. The interested reader is urged to consult [ARS, ASS] for an accessible introduction to this topic.

Assume that W is a finite, simply laced reflection group, with root system Φ. (We shall discuss more general settings at the end of the section.) Let Q be the directed graph obtained by taking the Coxeter diagram of W and orienting the edges from S_- to S_+. Fix an algebraically closed ground field k.

A *representation* V of Q is an assignment of a finite dimensional vector space V_i over k to each vertex i of Q, and a linear map V_α between the corresponding vector spaces to each arrow α of Q. A morphism from V to W is a collection of linear maps $f_i : V_i \to W_i$ which makes all squares commute. The representations of Q form an abelian category, which is denoted $\mathrm{rep}(Q)$. This category is equivalent to the category of finitely generated modules over the path algebra of Q.

If $V, W \in \mathrm{rep}(Q)$, we can define a k-vector space $\mathrm{Hom}(V, W)$. Using standard homological algebra, one can then define $\mathrm{Ext}^i(V, W)$ for $i > 0$. Note that $\mathrm{rep}(Q)$ is *hereditary*, that is to say, $\mathrm{Ext}^i(V, W) = 0$ for $i \ge 2$.

A representation of Q is called *indecomposable* if it is not the direct sum of two subrepresentations. By Gabriel's theorem, the indecomposable representations of Q are naturally in 1-1 correspondence with $\Phi_{>0}$, or, equivalently, with T. (If Q is non-Dynkin, the situation is more complex.)

Following [Cr], define an *exceptional sequence* of representations of Q to be a sequence of indecomposable representations $(F_1, \ldots, F_r)$ such that $\mathrm{Hom}(F_i, F_j) = 0 = \mathrm{Ext}^1(F_i, F_j)$ for $i < j$. (Note that this reverses the usual convention for the order of an exceptional sequence. Also, one normally must also require that $\mathrm{Ext}^1(F_i, F_i) = 0$ for all i, but this is automatic in the present setting where Q is Dynkin.) The maximal length of an exceptional sequence is n.

The notion of exceptional sequence is related to the concepts we have been discussing via the following theorem:

Theorem 9 ([IT]) *For $\beta_1, \ldots, \beta_n$ a collection of n positive roots, $(E_{\beta_1}, \ldots, E_{\beta_n})$ is an exceptional sequence iff $t_{\beta_1} \ldots t_{\beta_n} = c$.*

(This theorem is shown in [IT] in the case which we need here, when Q is Dynkin, and also when Q is affine; for arbitrary Q without oriented cycles, it is proved in [IS].)

There are well-defined mutation operations on the set of exceptional sequences of a given length. Given an exceptional sequence $(E_1, \ldots, E_n)$, which, for convenience, we assume to have maximal length, for $1 \le i \le n-1$, the operation μ_i is defined by:

$$\mu_i(E_1, \dots, E_i, E_{i+1}, \dots, E_n) = (E_1, \dots, E_{i+1}, M, \dots, E_n)$$

where M is uniquely determined by the fact that $\mu_i(E_1, \dots, E_n)$ forms an exceptional sequence.

Now, consider the collection of all factorizations of c as a product of n reflections. Clearly, there is also a mutation operation on such factorizations: just consider the mutation operation from Section 6, but ignore colour. It is a theorem of [Cr] that mutation of exceptional sequences can also be defined in more Coxeter-theoretic terms, from which it follows that mutation of exceptional sequences agrees via Theorem 9 with the mutation operation which we have just defined on factorizations of c.

In order to interpret *coloured* factorizations representation-theoretically, we must pass from $\mathrm{rep}(Q)$ to its bounded derived category $D^b(Q)$. As usual, we think of $\mathrm{rep}(Q)$ sitting inside $D^b(Q)$ in degree 0. Thanks to the shift functor [1] of $D^b(Q)$, for any indecomposable $V \in \mathrm{rep}(Q)$, we have indecomposable objects $V[i] \in D^b(Q)$ for all $i \in \mathbb{Z}$. Because $\mathrm{rep}(Q)$ is hereditary, there are no other indecomposable objects in $D^b(Q)$. Thus, there is a bijection between coloured reflections and indecomposable objects in $D^b(Q)$. The notions of exceptional sequences and mutations extend naturally to $D^b(Q)$, and these mutation operations agree precisely with those of Section 6.

Next, one has to study the special types of exceptional sequences which correspond to m-increasing and m-decreasing coloured factorizations. One has:

Proposition 2 *If $(t_1^{(c_1)}, \dots, t_n^{(c_n)})$ is a coloured factorization of c, and $(E_1, \dots, E_n)$ is the corresponding exceptional sequence in $D^b(Q)$, then the factorization is m-increasing iff:*

- *For all i, $E_i \in \mathrm{rep}(Q)[k]$ for some $0 \le k \le m$,*
- *For all $i \ne j$, $\mathrm{Ext}^k(E_i, E_j) = 0$ for $-m \le k \le 0$.*

Proposition 3 *If $(t_1^{(c_1)}, \dots, t_n^{(c_n)})$ is a coloured factorization of c, and $(E_1, \dots, E_n)$ is the corresponding exceptional sequence in $D^b(Q)$, then the factorization is m-decreasing iff:*

- *For all i, $E_i \in \mathrm{rep}(Q)[k]$ for some $0 \le k < m$ or $E_i = P[m]$ for some indecomposable projective P,*
- *For all i, j, $\mathrm{Ext}^k(E_i, E_j) = 0$ for $1 \le k \le m$.*

Note that this latter proposition is closely related to the usual approach to categorifying the m-cluster combinatorics of [FR], see [Zh, Th, Wr] and subsequent papers.

Theorem 5 is then proved by showing that μ_{rev} transforms the exceptional sequences of Proposition 3 into those of Proposition 2. Theorems 7 and 8 are proved similarly.

If W is a non-simply laced but crystallographic reflection group, then our techniques can be made to apply by a folding argument, or by working over a non-algebraically closed ground field and applying [Ri]. If W is non-crystallographic, our techniques do not apply. Note that the definition we have given of the bijection from m-increasing factorizations of c to m-decreasing factorizations of c still makes sense, but we cannot prove that it is a bijection.

There is nothing in our approach which really requires that W be finite; all we really need is the much weaker condition that Q have no oriented cycles. In this much more general setting, however, there are some additional aspects which must be taken care of. We are preparing a paper in which we will explain these extra aspects, and provide the proofs of the assertions in this extended abstract.

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Enumeration of the distinct shuffles of permutations

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A shuffle of two words is a word obtained by concatenating the two original words in either order and then sliding any letters from the second word back past letters of the first word, in such a way that the letters of each original word remain spelled out in their original relative order. Examples of shuffles of the words 1234 and 5678 are, for instance, 15236784 and 51236748. In this paper, we enumerate the distinct shuffles of two permutations of any two lengths, where the permutations are written as words in the letters $1, 2, 3, \dots, m$ and $1, 2, 3, \dots, n$, respectively.

Keywords: shuffles, permutations, enumeration, Catalan numbers, Shuffle Algebra

1 Introduction

Mathematicians have recently studied several notions of 'shuffling', including shuffling of a deck of cards (see [Aldous & Diaconis (1986)][Bayer & Diaconis (1992)] [Diaconis (1988)] [Diaconis (2002)] [Diaconis et al. (1983)] [Trefethen & Trefethen (2002)] [van Zuylen & Schalekamp (2004)]), 'shuffling' algorithms, such as the Fisher-Yates shuffle (also known as the Knuth shuffle) that generate random permutations of a finite set (see [Fisher & Yates (1948)] [Knuth (1973)] [Knuth (1998)]), and the perfect shuffle permutation (see [Diaconis et al. (1983)] [Ellis et al. (2000)] [Mevedoff & Morrison (1987)]).

We shall be interested in *shuffles of words*, where a *word* is defined to be a finite string of elements (known as *letters*) of a given set (known as an *alphabet*); in general repetitions of letters are allowed. We define the *length* of a word $u = a_1 \dots a_m$ to be $\mathfrak{l}(u) = m$ and the *support* of u to be $\mathrm{supp}(u) = \{a_1, \dots, a_m\}$. A *subword* x of a word u is defined to be a word obtained by crossing out a (possibly empty) subset of the letters of u.

For example, for the alphabet $\mathcal{A} = \{1, 2, 3, 5, 7\}$, the words $u = 25372$ and $v = 123$ have supports $\mathrm{supp}(u) = \{2, 3, 5, 7\}$ and $\mathrm{supp}(v) = \{1, 2, 3\}$, and lengths $\mathfrak{l}(u) = 5$ and $\mathfrak{l}(v) = 3$. Two subwords of u are 232 and 537.

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2 Shuffles of Words

Given two words $u = a_1 a_2 \dots a_m$ and $v = b_1 b_2 \dots b_n$ in some alphabet $\mathcal{A}$, we obtain a *shuffle* of u and v by concatenating u and v to get

$$c_1 c_2 \dots c_{m+n} = a_1 a_2 \dots a_m b_1 b_2 \dots b_n \tag{1}$$

and then permuting letters in such a way to achieve

$$w = c_{\rho(1)} c_{\rho(2)} \dots c_{\rho(m+n)}, \tag{2}$$

for some permutation $\rho \in \mathfrak{S}_{m+n}$ on $m+n$ letters satisfying the order-preserving conditions

$$\rho^{-1}(1) < \rho^{-1}(2) < \dots < \rho^{-1}(m) \tag{3}$$

and

$$\rho^{-1}(m+1) < \rho^{-1}(m+2) < \dots < \rho^{-1}(m+n). \tag{4}$$

In other words, we intersperse the letters of u with those of v to get w in such a way that the subword obtained by restricting w to the letters that came from u is simply u itself (and similarly for the subword obtained by restriction to the letters of v). Two different shuffles of the words 1234 and 5678 are, for instance, 15236784 and 51236748.

In the literature, the shuffle w is sometimes denoted by $u \sqcup\!\sqcup v$ (see [Hersh (2002)]). Since $\sqcup\!\sqcup$ depends on a choice of ρ, however, and since $u \sqcup\!\sqcup v$ sometimes denotes instead the *shuffle product* of u and v in the *shuffle algebra* (see [Reutenauer (1993)], page 24), we will use the notation $\sqcup\!\sqcup_\rho$ to avoid ambiguity. We define

$$\mathfrak{sh}(u,v) = \{u \sqcup\!\sqcup_\rho v \mid \rho \in \mathfrak{S}_{m+n} \text{ satisfies (3) and (4)}\} \tag{5}$$

to be the set of all shuffles of u with v. For ease of reference, we shall also set

$$\mathfrak{S}_{m,n} = \{\rho \in \mathfrak{S}_{m+n} \mid \rho \text{ satisfies (3) and (4)}\}. \tag{6}$$

The *shuffle algebra* $\mathfrak{A}$ (see [Crossley (2006)] [Ehrenborg (1996)] [Reutenauer (1993)]), a commutative Hopf algebra structure on the free $\mathbb{Z}$-module generated by finite words in a given alphabet $\mathcal{A}$, has as multiplication the *shuffle product* $\triangle$, which is given by

$$\triangle\,(u \otimes v) = \sum_{w \in \mathfrak{sh}(u,v)} \mu_w w \tag{7}$$

for words u and v, where

$$\mu_w = \#\{\rho \in \mathfrak{S}_{\mathfrak{l}(u),\mathfrak{l}(v)} \mid u \sqcup\!\sqcup_\rho v = w\} \tag{8}$$

is the multiplicity of w. The shuffle algebra has applications, for instance, in number theory: the multiplication of two multiple zeta values can be expressed as the sum of other multiple zeta values via a shuffle relation or a quasi-shuffle (stuffle) relation (see [Guo & Xie (2008)] [Ihara et al. (2006)]).

We can define, analogously, a shuffle of k words (or k-*shuffle*) to be a permutation of the concatenation of k words (with lengths $n_1, n_2, \ldots, n_k$) in such a way that the inverse permutation preserves order when restricted to the index subsets $[n_1], [n_1+1, n_1+n_2], \ldots, [n_1+n_2+\cdots+n_{k-1}+1, n_1+n_2+\cdots+n_k]$, where the interval notation $[n_1+1, n_1+n_2]$ denotes the set of integers from n_1+1 to n_1+n_2. A k-shuffle is also sometimes referred to as an α-*shuffle*, where $\alpha = (n_1, n_2, \ldots, n_k) \in \mathbb{P}^k$ is any k-tuple of positive integers. (But we reserve the notation $\sqcup\!\sqcup_\rho$ for 2-shuffles, as they are the main focus of our research.)

Shuffles of words arise in several contexts. For instance, given a subset

$$T = \{s_1, s_2, \ldots, s_{k-1}\} \subseteq [n-1], \tag{9}$$

it can be seen that a permutation $\tau \in \mathfrak{S}_n$ is a k-shuffle of the sets $[s_1], [s_1+1, s_2], \ldots, [s_{k-2}+1, s_{k-1}], [s_{k-1}+1, n]$ if and only if the descent set $D(\tau^{-1})$ of the inverse permutation is a subset of T (see [Stanley (1997)], page 70). Shuffles appear in the representation theory of finite groups; the left cosets of the Young Subgroup $\mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_k}$ in the Symmetric Group $\mathfrak{S}_n$ (where $n = \sum_{j=1}^{k} \alpha_j$) correspond exactly to the unique α-shuffles associated with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ (see [Stanley (1999)], page 351).

Shuffles play a role in the multiplication of fundamental quasisymmetric functions L_γ; in fact, if $u \in \mathfrak{S}_m$ and $v \in \mathfrak{S}_{[m+1,m+n]}$, then

$$L_{\mathrm{co}(u)} L_{\mathrm{co}(v)} = \sum_{w \in \mathfrak{sh}(u,v)} L_{\mathrm{co}(w)}, \tag{10}$$

where $\mathrm{co}(u)$ denotes the composition associated with the descent set $D(u)$ (see [Stanley (1999)], page 482, exercise 7.93). Moreover, shuffle posets on the words u and v can be defined by considering the set of subwords of all possible shuffles of u with v, taking u as the minimal element, v as the maximal element, and defining the cover relation to be $x \prec y$ if y can be obtained from x either by deleting one letter of u or inserting one letter of v. Greene [Greene (1988)] introduced shuffle posets, and Doran [Doran (2002)] and Hersh [Hersh (2002)] generalized them (see also [Ehrenborg (1996)] [Simion & Stanley (1999)]).

3 The Main Question

A natural question to ask is how to enumerate the *distinct* shuffles of words.

Question 1 *Given words u and v, how many distinct shuffles are there of u with v?*

Assuming m and n to be the lengths of u and v, respectively, note that if $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$, then there are $\binom{m+n}{m}$ distinct shuffles (all shuffles are distinct).

Observation 2 *For any given words u and v, we can define an equivalence relation on $\mathfrak{S}_{\mathfrak{l}(u),\mathfrak{l}(v)}$ by $\rho \sim \tau$ if $u \sqcup\!\sqcup_\rho v = u \sqcup\!\sqcup_\tau v$.*

The equivalence relation is nontrivial only when $\mathrm{supp}(u) \cap \mathrm{supp}(v) \neq \emptyset$. So one could reformulate Question 1 to ask how many different equivalence classes are induced on $\mathfrak{S}_{\mathfrak{l}(u),\mathfrak{l}(v)}$ by shuffling a given u with a given v.

In various applications of shuffles, the supports of the words are usually assumed to be disjoint, but we investigate the consequences of discarding this assumption while seeking an answer to Question 1.

We resolve this question for the important case where the words u and v are assumed to be permutations on the letters $\{1, 2, 3, \ldots, m\}$ and $\{1, 2, 3, \ldots, n\}$, respectively. Our answer is given by the following theorem, for which we shall give details in Section 5 below.

Theorem 3 *The number of distinct shuffles of a permutation $\alpha \in \mathfrak{S}_m$ with a permutation $\beta \in \mathfrak{S}_n$, with $m \leq n$, is given by the following formula:*

$$\#\mathfrak{sh}(\alpha, \beta) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\mathbf{a}=\{0=a_0<a_1<\cdots<a_{2k}<a_{2k+1}=m+1\}} (-1)^{h(\mathbf{a})} F_\sigma(\mathbf{a}), \tag{11}$$

where $\sigma = \overline{\alpha}^{-1} \circ \beta$, $\overline{\alpha} \in \mathfrak{S}_n$ is the natural extension of α, and $F_\sigma(\mathbf{a})$ is a product of determinants which enumerate the shuffles on a 'local' level.

For an explanation of the notation used and a description of the determinants involved, see Section 5 below.

4 Enumeration of the Distinct Shuffles of Permutations

We shall start by enumerating shuffles of the identity permutation with itself.

Proposition 4 *The number of distinct shuffles of the identity permutation on n letters with itself is the n^{th} Catalan number C_n, that is*

$$\#\mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n) = \frac{1}{n+1}\binom{2n}{n}. \tag{12}$$

Proof: A straightforward proof entails showing that set of shuffles of id_n with itself corresponds bijectively with the set of ballot sequences of length $2n$ (which is known to have cardinality C_n). For a given $w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n)$, simply substitute a 1 for the first occurrence of each integer between 1 and n, and a -1 for the second occurrence to get a ballot sequence of length $2n$ (that is, a sequence of n ones and n minus ones whose partial sums are all nonnegative). □

It is possible to show the following formula for the number of distinct shuffles of the identity in two different lengths.

Proposition 5 *For $m \neq n$, the number of distinct shuffles of the identity permutation on m letters with the identity permutation on n letters is given by*

$$\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{id}_n) = \sum_{r=0}^{\lfloor \frac{n-m}{2} \rfloor} (-1)^r \binom{n-m-r}{r} C_{n-r}. \tag{13}$$

We get Proposition 5 from the following recursion for shuffles of the identity in two different lengths.

Lemma 6 *For $m \neq n$, the number of distinct shuffles of the identity permutation on m elements with the identity permutation on n elements is determined by*

$$\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{id}_n) = \#\mathfrak{sh}(\mathrm{id}_{m-1}, \mathrm{id}_n) + \#\mathfrak{sh}(\mathrm{id}_m, \mathrm{id}_{n-1}). \tag{14}$$

Proof: Lemma 6 is easily verified by considering the bijection

$$\gamma : \mathfrak{sh}(\mathrm{id}_m, \mathrm{id}_n) \to \mathfrak{sh}(\mathrm{id}_{m-1}, \mathrm{id}_n) \cup \mathfrak{sh}(\mathrm{id}_m, \mathrm{id}_{n-1})$$

given by dropping the last letter of each $w \in \mathfrak{sh}(\mathrm{id}_m, \mathrm{id}_n)$ to get either $\gamma(w) \in \mathfrak{sh}(\mathrm{id}_{m-1}, \mathrm{id}_n)$ or $\gamma(w) \in \mathfrak{sh}(\mathrm{id}_m, \mathrm{id}_{n-1})$. As long as $m \neq n$, we have $\mathfrak{sh}(\mathrm{id}_{m-1}, \mathrm{id}_n) \cap \mathfrak{sh}(\mathrm{id}_m, \mathrm{id}_{n-1}) = \emptyset$. □

More generally, for $m \leq n$ and any $\alpha \in \mathfrak{S}_m$ and $\beta \in \mathfrak{S}_n$, it can be assumed without loss of generality that $\alpha = \mathrm{id}_m$, due to the following fact.

Fact 7 *For any $m \leq n$ and any $\alpha \in \mathfrak{S}_m$, $\beta \in \mathfrak{S}_n$, we have*

$$\#\mathfrak{sh}(\alpha, \beta) = \#\mathfrak{sh}(\mathrm{id}_m, (\overline{\alpha})^{-1} \circ \beta), \tag{15}$$

where $\overline{\alpha} \in \mathfrak{S}_n$ is the natural extension of α to a permutation on n letters.

Here we are simply reordering the alphabet $\mathcal{A} = [m]$ so that α now behaves like the identity permutation id_m on the reordered alphabet. It is also easy to note that $\#\mathfrak{sh}$ is symmetric: $\#\mathfrak{sh}(u, v) = \#\mathfrak{sh}(v, u)$ is true for any words u and v (they need not be permutations).

Now let the reverse permutation word $n, n-1, \ldots, 2, 1$ be denoted by rev_n. The following result can be shown via a bijective proof.

Proposition 8

$$\#\mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n) = \binom{m+n}{m} - \binom{m+n-2}{m-1}. \tag{16}$$

Proof: To verify Proposition 8, simply note that for each $w \in \mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n)$, we have either $\mu_w = 2$ or $\mu_w = 1$. (Either w has a pair of double elements, or it doesn't.)

Consider the map $\kappa : \{w \in \mathfrak{sh}(\mathrm{id}_m, \mathrm{rev}_n) \mid \mu_w = 2\} \to \mathfrak{sh}(+^{m-1}, -^{n-1})$ that sends each duplicated shuffle w to a sequence $\kappa(w) \in \mathfrak{sh}(+^{m-1}, -^{n-1})$ obtained by excising the double elements and then sending each letter from id_m to a $+$ and each letter from rev_n to a $-$. For example, for $1243321 \in \mathfrak{sh}(123, 4321)$, obtain $\kappa(1243321)$ by excising 33 to get 12421. Then replace elements with pluses and minuses to get $++---$.

Noting that $\#\mathfrak{sh}(+^{m-1}, -^{n-1}) = \binom{(m-1)+(n-1)}{m-1}$, we subtract this number of duplicates from $\binom{m+n}{m}$, the total number of shuffles, counted with multiplicity, of words of lengths m and n. □

5 The Main Theorem

Let us now enumerate the number of shuffles of the identity on m letters with any permutation $\sigma \in \mathfrak{S}_n$ (throughout, we shall assume without loss of generality that $m \leq n$). We shall first provide some terminology and motivation and then state the main theorem.

Let us call a subword x obtained from any word u *consecutive* if the letters of x appear consecutively in u. For instance, 364 is a consecutive subword of 136425. We call a shuffle $w \in \mathfrak{sh}(\mathrm{id}_n, \mathrm{id}_n)$ *indecomposable* if there is no consecutive subword w' of w such that $w' \in \mathfrak{sh}(\mathrm{id}_k, \mathrm{id}_k)$ for some $1 \leq k < n$. For ease of notation, let

$$\mathfrak{indc}(x) = \{\text{indecomposable shuffles of } x \text{ with itself}\}. \tag{17}$$

Observe that, when a shuffle w has multiplicity $\mu_w > 1$, this occurs because some consecutive subword x of σ is in fact a string of consecutive elements in the alphabet of w; we call such a subword of σ an *embedded identity subword*. On the local level we then have, embedded in w, a shuffle of the identity permutation on a consecutive subset of the intersection of the given alphabets with itself. That is,

$$w = \cdots * (x \sqcup\!\sqcup_\eta x) * \ldots \tag{18}$$

for some $\eta \in \mathfrak{S}_{\mathfrak{l}(x),\mathfrak{l}(x)}$, where $*$ denotes concatenation. We shall denote the set of embedded identity subwords of σ as

$$\mathfrak{idsub}(\sigma) = \{\text{embedded identity subwords of } \sigma\}. \tag{19}$$

If id_4 is shuffled with 52341, for example, we can obtain the shuffle

$$512342341 \in \{51 * (234 \sqcup\!\sqcup_\eta 234) * 1 \mid \eta \in \mathfrak{S}_{3,3}\}, \tag{20}$$

which has multiplicity 2 because the local shuffle $234234 \in \mathfrak{sh}(234, 234)$ is indecomposable and can be obtained in exactly two ways, whereas there are no additional ways of obtaining the global shuffle $512342341 \in \mathfrak{sh}(\mathrm{id}_4, 52341)$.

We say that a set $X = \{x_1, \ldots, x_r\}$ of embedded identity subwords of a permutation is *compatible* if the x_i have pairwise disjoint supports and if there exists some shuffle $w \in \mathfrak{sh}(\mathrm{id}_m, \sigma)$ in which each of the x_i is locally shuffled with itself. For instance, $\{23, 45\}$ is a set of compatible embedded identity subwords of 23145 because in the shuffle $1232314455 \in \mathfrak{sh}(\mathrm{id}_5, 23145)$ both 23 and 45 are locally shuffled with themselves.

Given a permutation word u and a compatible set $X = \{x_1, \ldots, x_j\}$ of embedded identity subwords of u, note that u is the concatenation $u = g_0 * x_1 * g_1 * \cdots * x_j * g_j$ for some consecutive subwords $g_0, g_1, \ldots, g_j$ of u whose supports are pairwise disjoint. We say that the g_i are the subwords of u *cut out* by the set X.

For instance, in the permutation 23145, the set $\{23, 45\}$ cuts out the subwords [], 1, and [] (where [] denotes the empty word). Likewise, for the permutation word 52341, the set $\{23, 4\}$ cuts out the subwords 5, [], and 1.

Proposition 9 *For $\sigma \in \mathfrak{S}_n$ and any $w \in \mathfrak{sh}(\mathrm{id}_m, \sigma)$, we have $\mu_w = 2^t$ for some integer $t \geq 0$, where t is the maximal number of compatible embedded identity permutation subwords in σ that are locally shuffled with themselves in w.*

To illustrate this statement, we can see that for $311223 \in \mathfrak{sh}(\mathrm{id}_3, 312)$, we have $\mu(311223) = 4$ and $t = 2$. The embedded identity subwords that are locally shuffled with themselves in 311223 are 1, 2, and 12; but $\{1, 2\}$ is the largest set of such subwords that is compatible. In general, we shall call the integer $t = \mathrm{dup}(w)$ the *number of sites of duplication* in w. Moreover, we shall set $N_t^\sigma = \#\{w \in \mathfrak{sh}(id_m, \sigma) \mid \mathrm{dup}(w) = t\}$.

We can actually enumerate $\#\mathfrak{sh}(\mathrm{id}_m, \sigma)$ by applying the Inclusion-Exclusion principle. First we take the total number of shuffles counted with multiplicity, and then alternately subtract and add the cardinalities of certain subsets counted with multiplicity until we arrive at a count of the total number of shuffles without multiplicity.

Indeed,

$$\#\mathfrak{sh}(\mathrm{id}_m, \sigma) = \binom{m+n}{m} + \sum_{j=1}^{m} (-1)^j T_j^\sigma, \tag{21}$$

where $T_j^\sigma = \sum_{t=j}^{m} \binom{t}{j} 2^{t-j} N_t^\sigma$.

Observation 10 *T_j^σ is the number of (not necessarily distinct shuffles) in $\mathfrak{sh}(\mathrm{id}_m, \sigma)$ with j or more sites of duplication, enumerated by choosing a j-element subset $X = \{x_1, \ldots, x_j\}$ of compatible embedded identity permutation subwords of σ and assuming, in turn, that each element $x_i \in X$ is shuffled locally and indecomposably with itself, then counting with multiplicity all local shuffles of each subword of id_m cut out by X with the corresponding subword of σ also cut out by X.*

That is, T_j^σ can be computed as

$$\begin{aligned} T_j^\sigma = \sum_{\text{compatible } \{x_1,\ldots,x_j\} \subseteq \mathfrak{idsub}(\sigma)} &\binom{\mathfrak{l}(f_0) + \mathfrak{l}(g_0)}{\mathfrak{l}(f_0)} \cdot \#\mathfrak{indc}(x_1) \cdot \binom{\mathfrak{l}(f_1) + \mathfrak{l}(g_1)}{\mathfrak{l}(f_1)} \cdots \\ &\cdot \#\mathfrak{indc}(x_j) \cdot \binom{\mathfrak{l}(f_j) + \mathfrak{l}(g_j)}{\mathfrak{l}(f_j)}, \end{aligned} \tag{22}$$

where the f_i and g_i are the subwords of id_m and of σ, respectively, that are cut out by the set $\{x_1, \ldots, x_j\}$. Recall that the number of local shuffles of f_i with g_i counted with multiplicity is $\binom{\mathfrak{l}(f_i)+\mathfrak{l}(g_i)}{\mathfrak{l}(f_i)}$.

In the example of $\mathfrak{sh}(\mathrm{id}_3, 312)$, we can compute

$$T_1^{312} = \binom{1}{0} \cdot C_0 \cdot \binom{3}{2} + \binom{3}{1} \cdot C_0 \cdot \binom{1}{1} + \binom{2}{2} \cdot C_0 \cdot \binom{2}{0} + \binom{1}{0} \cdot C_1 \cdot \binom{1}{1} = 8 \tag{23}$$

because we can fix first double 1's to count shuffles of the form $([] \sqcup\!\sqcup_{\rho_1} 3) * 11 * (23 \sqcup\!\sqcup_{\rho_2} 2)$, then fix double 2's to count those of the form $(1 \sqcup\!\sqcup_{\rho_3} 31) * 22 * (3 \sqcup\!\sqcup_{\rho_4} [])$, next, fix double 3's to count shuffles of the form $(12 \sqcup\!\sqcup_{\rho_5} []) * 33 * ([] \sqcup\!\sqcup_{\rho_6} 12)$, and lastly, fix the unique indecomposable shuffle of 12 with itself to count those of the form $([] \sqcup\!\sqcup_{\rho_7} 3) * 1212 * (3 \sqcup\!\sqcup_{\rho_8} [])$. Note that in each case the local identity shuffle we fix (such as 11 or 1212) is indecomposable, and so the factor C_{k-1} counts the distinct indecomposable shuffles of a local identity subword of length k with itself. Similarly,

$$T_2^{312} = \binom{1}{0} \cdot C_0 \cdot \binom{0}{0} \cdot C_0 \cdot \binom{1}{1} = 1, \tag{24}$$

as we can see by counting shuffles of the form $([] \sqcup\!\sqcup_{\rho_9} 3) * 11 * ([] \sqcup\!\sqcup_{\rho_{10}} []) * 22 * (3 \sqcup\!\sqcup_{\rho_{11}} [])$, whereas $T_3^{312} = 0$ because there is no compatible 3-subset of embedded identity permutation subwords, and so

$$\#\mathfrak{sh}(\mathrm{id}_3, 312) = \binom{3+3}{3} - 8 + 1 - 0 = 13. \tag{25}$$

We will use the notation $z_{i,j}^\sigma$ to denote the number of local shuffles counted with multiplicity of the subword a occurring between (and not including) the letters $i < j$ in id_m with the subword b occurring between the letters $i < j$ in σ. That is, if such words a and b exist, then we have $z^{\sigma}_{i,j} = \binom{\mathfrak{l}(a)+\mathfrak{l}(b)}{\mathfrak{l}(a)}$; otherwise, $z^{\sigma}_{i,j} = 0$. For example, $z^{312}_{0,2} = \binom{3}{1} = 3$, $z^{312}_{1,2} = \binom{0}{0} = 1$, and $z^{312}_{1,3} = 0$.

We use $z^{\sigma}_{i,j}$ to construct a square matrix with all ones on the subdiagonal and all zeros below the subdiagonal. For entries on or above the diagonal, $z^{\sigma}_{i,j}$ keeps track of whether or not i and j are inverted in σ, and if they are not inverted, $z^{\sigma}_{i,j}$ takes on the value of the total number of possible ways of shuffling the letters between paired occurrences of i and j, including any repeated shuffles.

By defining a matrix $Z^{\sigma}_{c,d} = [z^{\sigma}_{i,j}]_{c\le i\le d-1,\ c+1\le j\le d}$ below and taking its determinant, we are taking an alternating sum that systematically looks for compatible sets of letters (that is, compatible length 1 embedded identity subwords of σ) that occur between the letters c and d (not including c and d themselves). When the set of letters, say $\{b_1, b_2, \ldots, b_q\}$, is compatible, then we get a nonzero term of absolute value $z^{\sigma}_{c,b_1} \cdot z^{\sigma}_{b_1,b_2} \cdots z^{\sigma}_{b_q,m+1}$.

For example,

$$Z^{312}_{0,4} = \begin{pmatrix} 1 & 3 & 1 & 20 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{26}$$

For $1 \le e < f \le m$, if the word $e, e+1, \ldots, f$ is a simultaneous consecutive subword for id_m and σ, we will say that $\theta^{\sigma}(e,f)$ denotes the number of indecomposable local shuffles of the word $e, e+1, \ldots, f$ with itself; otherwise we will set $\theta^{\sigma}(e,f) = 0$. The purpose of the $y^{\sigma}_{i,j}$ below is to construct this function $\theta^{\sigma}(e,f)$ by defining a matrix $Y^{\sigma}_{e,f} = [y^{\sigma}_{i,j}]_{e\le i,j\le f-1}$ in such a way that $\theta^{\sigma}(e,f) = \det Y^{\sigma}_{e,f}$.

For example,

$$Y^{312}_{1,3} = \begin{pmatrix} C_0 & C_1 \\ 0 & 0 \end{pmatrix}, \tag{27}$$

whereas

$$Y^{312}_{1,2} = \begin{pmatrix} C_0 \end{pmatrix} \tag{28}$$

and

$$Y^{312}_{2,3} = \begin{pmatrix} 0 \end{pmatrix}. \tag{29}$$

The subsets $\mathbf{a} = \{0 = a_0 < a_1 < \cdots < a_{2k} < a_{2k+1} = m+1\} \subseteq [0, m+1]$ below determine the endpoints of the subwords $a_1 \ldots a_2, a_3 \ldots a_4$, through $a_{2k-1} \ldots a_{2k}$ of id_m, each of which has length greater than one and may possibly be an embedded identity subword for σ. The exponent $h(\mathbf{a})$ ensures the correct sign for purposes of applying the principle of Inclusion-Exclusion.

We are now ready for the main theorem.

Theorem 11 (Theorem 3, restated in detail)

$$\#\mathfrak{sh}(\mathrm{id}_m, \sigma) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{\mathbf{a}=\{0=a_0<a_1<\cdots<a_{2k}<a_{2k+1}=m+1\}} (-1)^{h(\mathbf{a})} \prod_{r=0}^{k} \det Z^{\sigma}_{a_{2r},a_{2r+1}} \prod_{s=1}^{k} \det Y^{\sigma}_{a_{2s-1},a_{2s}}, \tag{30}$$

where

$$h(\mathbf{a}) = m - \sum_{t=1}^{k} (a_{2t} - a_{2t-1}), \tag{31}$$

and we define the matrices

$$Z^{\sigma}_{c,d} = [z^{\sigma}_{i,j}]_{c \le i \le d-1,\ c+1 \le j \le d}, \tag{32}$$

with

$$z^{\sigma}_{i,j} = \begin{cases} 0, & i > j \\ 1, & i = j \\ 0, & 0 < i < j < m+1 \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j) \\ \binom{j-i-1+\sigma^{-1}(j)-\sigma^{-1}(i)-1}{j-i-1}, & 0 < i < j < m+1 \text{ and } \sigma^{-1}(i) < \sigma^{-1}(j) \\ \binom{j-1+\sigma^{-1}(j)-1}{j-1}, & i = 0,\ j < m+1 \\ \binom{m-i+n-\sigma^{-1}(i)}{m-i}, & j = m+1,\ i > 0 \\ \binom{m+n}{m}, & i = 0,\ j = m+1, \end{cases} \tag{33}$$

and the matrices

$$Y^{\sigma}_{e,f} = [y^{\sigma}_{i,j}]_{e \le i,j \le f-1}, \tag{34}$$

with

$$y^{\sigma}_{i,j} = \begin{cases} 0, & i - j > 1 \text{ or } \sigma^{-1}(i+1) \neq \sigma^{-1}(i) + 1 \\ -1, & i - j = 1 \text{ and } \sigma^{-1}(i+1) = \sigma^{-1}(i) + 1 \\ C_{j-i}, & i \le j \text{ and } \sigma^{-1}(i+1) = \sigma^{-1}(i) + 1 \end{cases} \tag{35}$$

where

$$C_{j-i} = \frac{1}{j-i+1}\binom{2(j-i)}{j-i}, \text{ the } (j-i)^{th} \text{ Catalan number.} \tag{36}$$

While equation (30) may look unwieldy, it is relatively easy to write a computer algorithm for Maple that will calculate the number of distinct shuffles of any two permutations. If at least one of the permutations has length bounded by 13, the processor on a laptop can easily handle the calculation. Examples of calculations include $\#\mathfrak{sh}(\mathrm{id}_3, 321) = 14$, $\#\mathfrak{sh}(\mathrm{id}_2, 3421) = 11$, $\#\mathfrak{sh}(2431, 1432) = 44$, $\#\mathfrak{sh}(\mathrm{id}_6, 126354) = 374$, and if $\sigma = 7, 8, 9, 10, 11, 12, 13, 1, 2, 3, 4, 5, 6 \in \mathfrak{S}_{13}$, then $\#\mathfrak{sh}(\mathrm{id}_{13}, \sigma) = 10104590$.

6 Future Directions

Open problems related to the work in this paper include the following projects:

6.1 Enumerating Distinct Shuffles of Multiset Permutations

Compute the number of distinct shuffles of any two multiset permutations; for example, $\#\mathfrak{sh}(12322, 33214)$. This is a significant generalization of the current problem, because the possible ways that duplications in such shuffles can occur are much more complicated than with ordinary permutations, and multiplicities of shuffles no longer need to be powers of 2. We believe, however, that once we can classify the types of multiplicities that can occur the problem will become tractable, and that the intuitions gained in solving the current problem will help me to reach that point.

6.2 Enumerating Distinct k-Shuffles of Permutations

Compute the number of distinct k-shuffles of k permutations of any k lengths, where k is any positive integer; for example, $\#\mathfrak{sh}(132, 231, 1324)$. This is another important generalization. Again, multiplicities need not be powers of 2; rather, they appear to be related to products of factorials, but it is not yet clear how exactly to compute them. It seems that making progress on counting shuffles of multiset permutations should give insight into what occurs with k-shuffles of ordinary permuations; observe that $\#\mathfrak{sh}(132, 231, 1324)$ is equal to $\sum_{w\in\mathfrak{sh}(132,231)} \#\mathfrak{sh}(w, 1324)$ minus a certain number of shuffles y such that $y \in \mathfrak{sh}(w, 1324) \cap \mathfrak{sh}(w', 1324)$ for some $w' \neq w \in \mathfrak{sh}(132, 231)$. Note that w and w' can be thought of as multiset permutations.

6.3 Deducing Monotonicity Results

Deduce monotonicity results for the number of distinct shuffles on permutation groups. Such results would help to clarify the meaning of the formula given in Theorem 11. For $1 \leq n \leq 6$, the minimal number of distinct shuffles of a permutation with the identity permutation of the same length is C_n, achieved by identity permutation (see Proposition 4). We conjecture that this is the case for all n.

For $n = 1, 2, 3$, the maximal number of shuffles of a permutation with the identity is achieved by the reverse permutation. For $n = 4, 5, 6$, however, the maximal number of distinct shuffles with the identity is achieved by the halfway-shifted permutations 3412, 34512, and 456123, respectively. Together, these cases give the first six terms of the sequence of maximal shuffle counts: $1, 4, 14, 54, 197, 792$ (now catalogued as sequence A145211 in the On-Line Encyclopedia of Integer Sequences; see also sequence A145208). We would like to extend this sequence and to determine whether, as we conjecture, maximality is actually achieved by the halfway-shifted permutations for all $n \geq 4$.

Indeed, we would like more generally to find a poset structure on $\mathfrak{S}_n$ for which the function $\sigma \mapsto \#sh(\mathrm{id}_n, \sigma)$ is always monotone increasing. The Bruhat order fails to provide such a structure for $n = 4, 5, 6$, but perhaps a modification of the Bruhat order would provide the desired poset structure.

6.4 Enumerating Distinct Shuffles according to Permutation Statistics

Enumerate distinct shuffles according to various permutation statistics, such as descent sets, number of inversions, or major index. Enumeration by statistics could yield insights into the above problems and refine our current results.

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The Ladder Crystal

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Abstract. In this paper, we introduce a new model of the crystal $B(\Lambda_0)$ of $\widehat{\mathfrak{sl}_\ell}$. We briefly describe some of the properties of this crystal and compare it to the combinatorial model of Misra and Miwa.

R'esum'e. Dans cette article on propose un nouveau model du cristal $B(\Lambda_0)$ de $\widehat{\mathfrak{sl}_\ell}$. On décrit bref les propriétés du cristal et on le compare avec le model combinatorial de Misra et Miwa.

Keywords: combinatorics of Young diagrams, crystals, representation theory of Hecke algebras

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1 Introduction

One description of the crystal $B(\Lambda_0)$ of $\widehat{\mathfrak{sl}_\ell}$ has as nodes ℓ-regular partitions. In this paper we give another combinatorial description of $B(\Lambda_0)$, called the ladder crystal, which we denote $B(\Lambda_0)^L$. Our crystal satisfies the following properties:

- The nodes of $B(\Lambda_0)^L$ are partitions, and there is an i-arrow from λ to μ only when the difference $\mu \setminus \lambda$ is a box of residue i.
- There exists elementary combinatorial arguments which generalize crystal theoretic results of $B(\Lambda_0)$ to $B(\Lambda_0)^L$.
- $B(\Lambda_0) \cong B(\Lambda_0)^L$ and the isomorphism is a well studied (but never before in this context) map on partitions.
- The nodes of $B(\Lambda_0)^L$ have a simple combinatorial description.

The new description of the crystal $B(\Lambda_0)$ is in many ways more important than the theorems which were proven by the existence of it. Besides the fact that it is a useful tool in proving theorems about $B(\Lambda_0)$, our new description also highlights a set of partitions (in bijection to ℓ-regular partitions), which can be interpreted in terms of the representation theory of the finite Hecke algebra $H_n(q)$.

Remark 1.0.1 *All proofs are absent from this text in the interest of space, as several of them require tedious calculations.*

1.1 Combinatorial definitions on partitions

Let λ be a partition of n (written $\lambda \vdash n$) and $\ell \geq 3$ be an integer. We will use the convention (x, y) to denote the box which sits in the x^{th} row and the y^{th} column of the Young diagram of λ. We denote the transpose of λ by λ'. Throughout this paper, all of our partitions are drawn in English notation. An *ℓ-regular partition* is one in which no part occurs ℓ or more times.

The *hook length* of the (a, c) box of λ is defined to be the number of boxes to the right and below the box (a, c), including the box (a, c) itself. It will be denoted $h^\lambda_{(a,c)}$. The *arm* of the (a, c) box of λ is defined to be the number of boxes to the right of the box (a, c), *not* including the box (a, c). It will be denoted $arm(a, c)$. Similarly, the leg is below (a, c), not including (a, c) and will be denoted $leg(a, c)$.

Remark 1.1.1 *From the definitions, it is clear that* $h^\lambda_{(a,c)} = arm(a, c) + leg(a, c) + 1$.

2 Hecke Algebras

2.1 Representation theory of $H_n(q)$

Definition 2.1.1 *For a fixed field $\mathbb{F}$ of characteristic zero and $0 \neq q \in \mathbb{F}$, the finite Hecke Algebra $H_n(q)$ is defined to be the $\mathbb{F}$-algebra generated by $T_1, ..., T_{n-1}$ with relations*

$$\begin{array}{ll} T_iT_j = T_jT_i & \textit{for } |i-j| > 1 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & \textit{for } 1 \leq i < n-1 \\ T_i^2 = (q-1)T_i + q & \textit{for } 1 \leq i \leq n-1. \end{array}$$

In this paper we will always assume that $q \neq 1$ and that $q \in \mathbb{F}$ is a primitive ℓ^{th} root of unity in a field $\mathbb{F}$ of characteristic zero (so necessarily $\ell \geq 2$).

Remark 2.1.2 *When q is specialized to 1 the Hecke algebra becomes the group algebra of the symmetric group.*

Similar to the symmetric group, a construction of the Specht module $S^\lambda = S^\lambda[q]$ exists for $H_n(q)$ (see (2)). The Specht modules need not remain irreducible when q is a primitive ℓ^{th} root of unity. Conditions for the irreducibility of these modules was conjectured by James and Mathas, and recently proven in work of Fayers (3) and Lyle (11).

All of the irreducible representations of $H_n(q)$ have been constructed when q is a primitive ℓth root of unity. These modules are indexed by ℓ-regular partitions λ, and are called D^λ. D^λ is the unique simple quotient of S^λ (see (2) for more details). In particular $D^\lambda = S^\lambda$ if and only if S^λ is irreducible and λ is ℓ-regular. For λ not necessarily ℓ-regular, S^λ is irreducible if and only if there exists an ℓ-regular partition μ so that $S^\lambda \cong D^\mu$.

3 Misra-Miwa Description of $B(\Lambda_0)$

3.1 Introduction

In this section, we recall a description of the crystal graph $B(\Lambda_0)$ currently used in the literature, first described by Misra and Miwa (13) .

3.2 Classical description of the crystal $B(\Lambda_0)$

We will assume some familiarity with the theory of crystals (see (8) for details). We will look at the crystal $B(\Lambda_0)$ of the irreducible highest weight module $V(\Lambda_0)$ of the affine Lie algebra $\widehat{\mathfrak{sl}_\ell}$ (also called the basic representation of $\widehat{\mathfrak{sl}_\ell}$). In the model of $B(\Lambda_0)$ given by Misra and Miwa, the nodes are ℓ-regular partitions. The set of nodes will be denoted $B := \{\lambda \in \mathcal{P} : \lambda \text{ is } \ell\text{-regular}\}$. We will describe the arrows of $B(\Lambda_0)$ below.

We view the Young diagram for λ as a set of boxes, with their corresponding residues $b - a \mod \ell$ written into the box (a, b). A box in λ is said to be a removable i-box if it has residue i and after removing that box the remaining diagram is still a partition. A space not in λ is an addable i-box if it has residue i and adding that box to λ yields a partition.

Example 3.2.1 *Let $\lambda = (8, 5, 4, 1)$ and $\ell = 3$. Then the residues are filled into the corresponding Young diagram as follows:*

$\lambda =$

0	1	2	0	1	2	0	1	*2*
2	0	1	2	0	*1*			
1	2	0	1	*2*				
0	*1*							
2								

Here λ has two removable 0-boxes (boxes (2,5) and (4,1)), two removable 1-boxes (boxes (1,8) and (3,4)), no removable 2-boxes, no addable 0-boxes, two addable 1-boxes (in positions (2,6) and (4,2)), and three addable 2-boxes (in positions (1,9), (3,5) and (5,1)).

For a fixed i, $(0 \leq i < \ell)$, we place $-$ in each removable i-box and $+$ in each addable i-box. The i-signature of λ is the word of $+$ and $-$'s in the diagram for λ, written from bottom left to top right. The reduced i-signature is the word obtained after repeatedly removing from the i-signature all adjacent pairs $-+$. The resulting word will now be of the form $+\cdots+++---\cdots-$. The boxes corresponding to $-$'s in the reduced i-signature are called *normal i-boxes*, and the boxes corresponding to $+$'s are called *conormal i-boxes*. $\varepsilon_i(\lambda)$ is defined to be the number of normal i-boxes of λ, and $\varphi_i(\lambda)$ is defined to be the number of conormal i-boxes. If a leftmost $-$ exists, the box corresponding to such a $-$ is called the *good i-box* of λ. If a rightmost $+$ exists, the box corresponding to such a $+$ is called the *cogood i-box*. All of these definitions can be found in Kleshchev's book (9).

Example 3.2.2 *Let $\lambda = (8, 5, 4, 1)$ and $\ell = 3$ be as above. Fix $i = 1$. The diagram for λ with removable and addable 1-boxes marked is:*

							$-$
					$+$		
			$-$				
	$+$						

The 1-signature of λ is $+-+-$, so the reduced 1-signature is $+ \quad -$ and the diagram has a good 1-box in the first row, and a cogood 1-box in the fourth row. Here $\varepsilon_1(\lambda) = 1$ and $\varphi_1(\lambda) = 1$.

We recall the action of the crystal operators on B. The crystal operator $\widetilde{e}_i : B \xrightarrow{i} B \cup \{0\}$ assigns to a partition λ the partition $\widetilde{e}_i(\lambda) = \lambda \setminus x$, where x is the good i-box of λ. If no such box exists, then $\widetilde{e}_i(\lambda) = 0$. It is clear then that $\varepsilon_i(\lambda) = max\{k : \widetilde{e}_i^k \lambda \neq 0\}$.

Similarly, $\widetilde{f}_i : B \xrightarrow{i} B \cup \{0\}$ is the operator which assigns to a partition λ the partition $\widetilde{f}_i(\lambda) = \lambda \cup x$, where x is the cogood i-box of λ. If no such box exists, then $\widetilde{f}_i(\lambda) = 0$. It is clear then that $\varphi_i(\lambda) = max\{k : \widetilde{f}_i^k \lambda \neq 0\}$.

For $i \in \mathbb{Z}/\ell\mathbb{Z}$, we write $\lambda \xrightarrow{i} \mu$ to stand for $\widetilde{f}_i \lambda = \mu$. We say that there is an i-arrow from λ to μ. Note that $\lambda \xrightarrow{i} \mu$ if and only if $\widetilde{e}_i \mu = \lambda$. A maximal chain of consecutive i-arrows will be called an i-string. We note that the empty partition $\emptyset$ is the unique highest weight node of the crystal (i.e. $\widetilde{e}_i \emptyset = 0$ for every $i \in \mathbb{Z}/\ell\mathbb{Z}$) and that $B(\Lambda_0)$ is connected. For a picture of a part of this crystal graph, see (10) for the cases $\ell = 2$ and 3.

Example 3.2.3 *Continuing with the above example, we see that $\widetilde{e}_1(8, 5, 4, 1) = (7, 5, 4, 1)$ and $\widetilde{f}_1(8, 5, 4, 1) = (8, 5, 4, 2)$. Also, $\widetilde{e}_1^2(8, 5, 4, 1) = 0$ and $\widetilde{f}_1^2(8, 5, 4, 1) = 0$. The sequence $(7, 5, 4, 1) \xrightarrow{1} (8, 5, 4, 1) \xrightarrow{1} (8, 5, 4, 2)$ is a 1-string of length 3.*

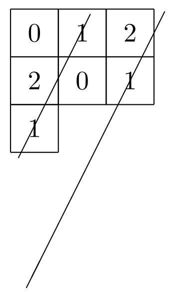
4 The Ladder Crystal: $B(\Lambda_0)^L$

4.1 Ladders

We first recall what a ladder is in regards to a partition. Let λ be a partition and let $\ell > 2$ be a fixed integer. For any box (a, b) in the Young diagram of λ, the *ladder* of (a, b) is the set of all positions (c, d) which satisfy $\frac{c-a}{d-b} = \ell - 1$ and $c, d > 0$.

Remark 4.1.1 *The definition implies that two boxes in the same ladder will share the same residue. An i-ladder will be a ladder all of whose boxes have residue i.*

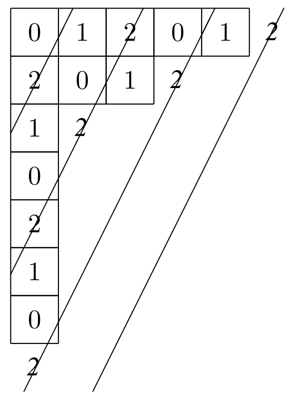
Example 4.1.2 *Let $\lambda = (3,3,1)$, $\ell = 3$. Then there is a 1-ladder which contains the boxes $(1,2)$ and $(3,1)$, and a different 1-ladder which has the box $(2,3)$ in λ and the boxes $(4,2)$ and $(6,1)$ not in λ. In the picture below, lines are drawn through the different 1-ladders.*



4.2 The ladder crystal

We will construct a new crystal $B(\Lambda_0)^L$ recursively as follows. First, the empty partition $\emptyset$ is the unique highest weight node of our crystal. From $\emptyset$, we will build the crystal by applying the operators $\widehat{f_i}$ for $0 \leq i < \ell$. We define $\widehat{f_i}$ to act on partitions, taking a partition of n to a partition of $n+1$ (or to 0) in the following manner. Given $\lambda \vdash n$, first draw all of the i-ladders of λ onto its Young diagram. Label any addable i-box with a $+$, and any removable i-box with a $-$. Now, write down the word of $+$'s and $-$'s by reading from leftmost i-ladder to rightmost i-ladder and reading from top to bottom on each ladder. This is called the *ladder i-signature* of λ. From here, cancel any adjacent $-+$ pairs in the word, until you obtain a word of the form $+\cdots+-\cdots-$. This is called the *reduced ladder i-signature of λ*. All boxes associated to a $-$ in the reduced ladder i-signature are called *ladder normal i-boxes* and all boxes associated to a $+$ in the reduced ladder i-signature are called *ladder conormal i-boxes*. The box associated to the leftmost $-$ is called the *ladder good i-box* and the box associated to the rightmost $+$ is called the *ladder cogood i-box*. Then we define $\widehat{f_i}\lambda$ to be the partition λ union the ladder cogood i-box. If no such box exists, then $\widehat{f_i}\lambda = 0$. Similarly, $\widehat{e_i}\lambda$ is the partition λ with the ladder good i-box removed. If no such box exists, then $\widehat{e_i}\lambda = 0$. We then define $\widehat{\varphi}_i(\lambda)$ to be the number of ladder conormal i-boxes of λ and $\widehat{\varepsilon}_i(\lambda)$ to be the number of ladder normal i-boxes. It is then obvious that $\widehat{\varphi}_i(\lambda) = \max\{k : \widehat{f}_i^k\lambda \neq 0\}$ and that $\widehat{\varepsilon}_i(\lambda) = \max\{k : \widehat{e}_i^k\lambda \neq 0\}$.

Example 4.2.1 *Let $\lambda = (5,3,1,1,1,1,1)$ and $\ell = 3$. Then there are four addable 2-boxes for λ. In the leftmost 2-ladder (containing box (2,1)) there are no addable (or removable) 2-boxes. In the next 2-ladder (containing box (1,3)) there is an addable 2-box in box (3,2). In the next 2-ladder (containing box (2,4)), there are two addable 2-boxes, in boxes (2,4) and (8,1). In the last drawn 2-ladder (containing box (1,6)) there is one addable 2-box, in box (1,6). There are no removable 2-boxes in λ. Therefore the ladder 2-signature (and hence reduced ladder 2-signature) of λ is $+_{(3,2)} +_{(2,4)} +_{(8,1)} +_{(1,6)}$ (Here, we have included subscripts on the $+$ signs so that the reader can see the correct order of the $+$'s). Hence $\widehat{f_2}\lambda = (6,3,1,1,1,1,1)$, $(\widehat{f_2})^2\lambda = (6,3,1,1,1,1,1,1)$, $(\widehat{f_2})^3\lambda = (6,4,1,1,1,1,1,1)$ and $(\widehat{f_2})^4\lambda = (6,4,2,1,1,1,1,1)$. $(\widehat{f_2})^5\lambda = 0$.*



Remark 4.2.2 *The weight function of this crystal is exactly the same as the weight function for $B(\Lambda_0)$. Explicitly, the weight of λ is $\Lambda_0 - \sum c_i \alpha_i$ where c_i is the number of boxes of λ with residue i (or equivalently, $\sum_i (\varphi_i - \varepsilon_i)\Lambda_i$). Throughout this paper we will suppress the weight function as it is irrelevant to the combinatorics involved.*

5 Regularization

5.1 The operation of regularization

In this section we describe a map from the set of partitions to the set of ℓ-regular partitions. The map is called regularization and was first defined by James (see (6)). For a given λ, and for a ladder $\mathcal{L}$ in λ, one can count the number of boxes $\eta_{\mathcal{L}}$ on $\mathcal{L}$. Create a new partition where on ladder $\mathcal{L}$ the top $\eta_{\mathcal{L}}$ positions in $\mathcal{L}$ have boxes and all other positions on $\mathcal{L}$ are vacant. The result is called the *regularization* of λ, and is denoted $\mathcal{R}_\ell \lambda$. It can also be thought of as pushing all boxes in each ladder of λ as far up their respective ladders as is possible. Although $\mathcal{R}_\ell$ depends on ℓ, we will usually just write $\mathcal{R}$. The following theorem contains facts about regularization originally due to James (6) (see also (7)).

Theorem 5.1.1 *Let λ be a partition. Then*

- *$\mathcal{R}\lambda$ is ℓ-regular*
- *$\mathcal{R}\lambda = \lambda$ if and only if λ is ℓ-regular.*
- *If λ is ℓ-regular and $D^\lambda \cong S^\nu$ for some partition ν, then $\mathcal{R}\nu = \lambda$.*

Regularization provides us with an equivalence relation on the set of partitions. Specifically, we say $\lambda \sim \mu$ if $\mathcal{R}\lambda = \mathcal{R}\mu$. The equivalence classes are called *regularization classes*, and the class of a partition λ is denoted $\mathcal{RC}(\lambda) := \{\mu \in \mathcal{P} : \mathcal{R}\mu = \mathcal{R}\lambda\}$.

Example 5.1.2 *Let $\lambda = (2,2,2,1,1,1)$ and let $\ell = 3$. Then $\mathcal{R}\lambda = (3,3,2,1)$. Also,*

$$\mathcal{RC}(\lambda) = \{(2,2,2,1,1,1),(2,2,2,2,1),(3,2,1,1,1,1),$$

$$(3,2,2,2),(3,3,1,1,1),(3,3,2,1)\}$$

.

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & 0 \\ \hline 1 & 2 \\ \hline 0 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \xrightarrow{\mathcal{R}} \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 0 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

6 Deregularization

The goal of this section is to provide an algorithm for finding the smallest partition in dominance order in a given regularization class. It is nontrivial to show that a smallest partition exists. We use this result to show that our new description of the crystal $B(\Lambda_0)^L$ has nodes which are smallest in dominance order in their regularization class. All of the work of this section is joint with Brant Jones of UC Davis, who gave the first definition of a locked box.

6.1 Locked Boxes

We recall a partial ordering on the set of partitions of n. For two partitions λ and μ of n, we say that $\lambda \leq \mu$ if $\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j$ for all i. This order is usually called the *dominance order*.

Finding all of the partitions which belong to a regularization class is not easy. The definition of locked boxes below formalizes the concept that some boxes in a partition cannot be moved down their ladders if one requires that the new diagram remain a partition.

Definition 6.1.1 *For a partition λ, we label boxes of λ as locked by the following procedure:*

I. *If a box x has a locked box directly above it (or is on the first row) and every unoccupied space in the same ladder as x, lying below x, has an unoccupied space directly above it then x is locked. Boxes locked for this reason are called type I locked boxes.*

II. *If a box y is locked, then every box to the left of y in the same row is also locked. Boxes locked for this reason are called type II locked boxes.*

Boxes which are not locked are called unlocked.

Remark 6.1.2 *Locked boxes can be both type I and type II.*

Example 6.1.3 *Let $\ell = 3$ and let $\lambda = (6, 5, 4, 3, 1, 1)$. Then labeling the locked boxes for λ with an L and the unlocked boxes with a U yields the picture below.*

$$\begin{ytableau} L & L & L & U & U & U \\ L & L & L & U & U \\ L & L & U & U \\ L & L & U \\ L \\ L \end{ytableau}$$

6.2 Algorithm for finding the smallest partition in a regularization class

The algorithm from here is simple. For any partition λ, to find the smallest partition (with respect to dominance order) in a regularization class we first label each box of λ as either locked or unlocked as above. Then we move all of the unlocked boxes in each ladder to the lowest unoccupied positions on their ladder. The resulting partition will be denoted $\mathcal{S}\lambda$. It is unclear that this algorithm will yield the smallest partition in $\mathcal{RC}(\lambda)$, or even that it is a partition. The following theorem resolves these issues.

Theorem 6.2.1 *$\mathcal{S}\lambda$ is the unique smallest partition in its regularization class with respect to dominance order. It can be classified as being the unique partition (in its regularization class) which has all its boxes locked.*

Example 6.2.2 *Continuing from the example above ($\lambda = (6,5,4,3,1,1)$ and $\ell = 3$), we move all of the unlocked boxes down to obtain the smallest partition in $\mathcal{RC}(\lambda)$, which is:*

$$\mathcal{S}\lambda = (3,3,2,2,2,2,2,1,1,1,1).$$

The boxes labeled L are the ones which were locked in $(6,5,4,3,1,1)$ (and did not move).

$$\begin{ytableau} L & L & L \\ L & L & L \\ L & L \\ L & L \\ L & \\ L & \\ & \\ \\ \\ \\ \\ \end{ytableau}$$

6.3 The nodes of $B(\Lambda_0)^L$ are smallest in dominance order

The nodes of $B(\Lambda_0)^L$ have been defined recursively by applying the operators $\widehat{f}_i$. We now give a simple description which determines when a partition is a node of $B(\Lambda_0)^L$.

Proposition 6.3.1 *Let λ be a partition of n. Let $\mathcal{RC}(\lambda)$ be its regularization class. Then λ is a node of $B(\Lambda_0)^L$ if and only if λ is the smallest partition in $\mathcal{RC}(\lambda)$ with respect to dominance order.*

One can view $B(\Lambda_0)$ as having nodes $\{\mathcal{RC}(\lambda) : \lambda \vdash n, n \geq 0\}$. The usual model of $B(\Lambda_0)$ takes the representative $\mathcal{R}\lambda \in \mathcal{RC}(\lambda)$, which happens to be the largest in dominance order. Here, we will take a different representative of $\mathcal{RC}(\lambda)$, the partitions $\mathcal{S}\lambda$, which are smallest in dominance order.

7 Crystal Isomorphism

7.1 The isomorphism $B(\Lambda_0) \cong B(\Lambda_0)^L$

Using the theory of locked boxes described above, we were able to prove the following theorem.

Theorem 7.1.1 *Regularization commutes with the crystal operators. In other words if $\lambda \in B(\Lambda_0)^L$ then:*

1. $(\mathcal{R} \circ \widehat{f}_i)(\lambda) = (\tilde{f}_i \circ \mathcal{R})(\lambda)$,
2. $(\mathcal{R} \circ \widehat{e}_i)(\lambda) = (\tilde{e}_i \circ \mathcal{R})(\lambda)$.

A corollary to this theorem is that the crystals are isomorphic, the isomorphism being regularization in one direction. The inverse to regularization is the map $\mathcal{S}$ described above.

Corollary 7.1.2 *The crystal $B(\Lambda_0)$ is isomorphic to $B(\Lambda_0)^L$.*

Example 7.1.3 *Let $\lambda = (2,1,1,1)$ and $\ell = 3$. Then $\mathcal{R}\lambda = (2,2,1)$. Also $\widehat{f}_2\lambda = (2,1,1,1,1)$ and $\tilde{f}_2(2,2,1) = (3,2,1)$. But $\mathcal{R}(2,1,1,1,1) = (3,2,1)$.*

$$\begin{array}{ccc} (2,1,1,1) & \xrightarrow{2} & (2,1,1,1,1) \\ \mathcal{R}\downarrow & & \downarrow\mathcal{R} \\ (2,2,1) & \xrightarrow{2} & (3,2,1) \end{array}$$

8 The Mullineux Map

The operation on the category of $H_n(q)$ modules of tensoring with the sign module (the 1-dimensional module of $H_n(q)$ where each T_i acts as -1) is a functor which takes irreducible modules to irreducible modules. For instance, when q is not a root of unity, then $S^\lambda \otimes sign = S^{\lambda'}$, where λ' is the transpose of λ. When λ is an ℓ-regular partition, and D^λ denotes the irreducible module corresponding to λ then $D^\lambda \otimes sign$ is some irreducible module $D^{m(\lambda)}$. This describes a map m between ℓ-regular partitions called the *Mullineux map*. Recent results of Fayers (4) settle a conjecture of Lyle (12) which effectively computes the Mullineux map in certain cases by means of regularization and transposition. This section will highlight the interpretation of Fayers result in terms of the ladder crystal. It should be noted that Ford and Kleshchev gave a recursive construction for computing the Mullineux map in all cases (5).

8.1 Connections with Fayers results

Since the Mullineux map is the modular analog of transposition, a natural attempt to compute $m(\lambda)$ for an ℓ-regular partition λ would be to transpose λ and then regularize. If a partition is not ℓ-regular, we could similarly guess that $m(\mathcal{R}\lambda)$ was just transposing λ and then regularize the result. This is not always the case. However, a conjecture of Lyle (12), which was proven recently by Fayers (4) gives a precise classification for when this holds. The definition below was taken from Fayers (4).

Definition 8.1.1 *An L-partition is a partition which has no box (i, j) in the diagram of λ such that $\ell \mid h^{\lambda}_{i,j}$ and either $arm(i, j) < (\ell - 1)leg(i, j)$ or $leg(i, j) < (\ell - 1)arm(i, j)$.*

Theorem 8.1.2 (Fayers (4)) *A partition is an L-partition if and only if $m(\mathcal{R}\lambda) = \mathcal{R}\lambda'$.*

It was pointed out to the author by Fayers that a classification of the nodes of $B(\Lambda_0)^L$ can be described in terms of hook lengths and arm lengths. We now include this classification.

Theorem 8.1.3 *A partition λ belongs to the crystal $B(\Lambda_0)^L$ if and only if there does not exist a box (i, j) in the Young diagram of λ such that $h^{\lambda}_{(i,j)} = \ell * arm(i, j)$.*

This classification of the partitions in $B(\Lambda_0)^L$ implies the following theorem.

Theorem 8.1.4 *All L-partitions are nodes of the crystal $B(\Lambda_0)^L$.*

It is easy to show that any partition λ for which the Specht module S^{λ} is irreducible is an L-partition. This implies the following corollary.

Corollary 8.1.5 *All partitions λ for which S^{λ} is irreducible (when q is an ℓ^{th} root of unity) are nodes of the crystal $B(\Lambda_0)^L$.*

9 Conclusion

We have built a model of the crystal $B(\Lambda_0)$ which has different partitions representing each regularization class. It has the surprising property that every partition λ for which the Specht module S^{λ} is irreducible appears. Other results relating to the representation theory of $H_n(q)$ and the crystal $B(\Lambda_0)$ can be obtained using the isomorphism between $B(\Lambda_0)$ and $B(\Lambda_0)^L$. In particular, generalizations of theorems from (1) can be proven with the use of $B(\Lambda_0)^L$. We have left these out to save space, but can be found in the authors upcoming thesis.

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Words and polynomial invariants of finite groups in non-commutative variables

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Abstract. Let V be a complex vector space with basis $\{x_1, x_2, \ldots, x_n\}$ and G be a finite subgroup of $GL(V)$. The tensor algebra $T(V)$ over the complex is isomorphic to the polynomials in the non-commutative variables $x_1, x_2, \ldots, x_n$ with complex coefficients. We want to give a combinatorial interpretation for the decomposition of $T(V)$ into simple G-modules. In particular, we want to study the graded space of invariants in $T(V)$ with respect to the action of G. We give a general method for decomposing the space $T(V)$ into simple G-module in terms of words in a particular Cayley graph of G. To apply the method to a particular group, we require a surjective homomorphism from a subalgebra of the group algebra into the character algebra. In the case of the symmetric group, we give an example of this homomorphism from the descent algebra. When G is the dihedral group, we have a realization of the character algebra as a subalgebra of the group algebra. In those two cases, we have an interpretation for the graded dimensions of the invariant space in term of those words.

Résumé. Soit V un espace vectoriel complexe de base $\{x_1, x_2, \ldots, x_n\}$ et G un sous-groupe fini de $GL(V)$. L'algèbre $T(V)$ des tenseurs de V sur les complexes est isomorphe aux polynômes à coefficients complexes en variables non-commutatives $x_1, x_2, \ldots, x_n$. Nous voulons donner une décomposition de $T(V)$ en G-modules simples de manière combinatoire. Plus particulièrement, nous étudions l'espace gradué des invariants de $T(V)$ sous l'action de G. Nous présentons une méthode générale donnant la décomposition de $T(V)$ en modules simples via certains mots dans un graphe de Cayley donné. Pour appliquer la méthode à un groupe particulier, nous avons besoin d'un homomorphisme surjectif entre une sous-algèbre de l'algèbre de groupe et l'algèbre des charactères. Pour le cas du groupe symétrique, nous donnons un example de cet homomorphisme qui provient de la théorie de l'algèbre des descentes. Pour le groupe diédral, nous avons une réalisation de l'algèbre des charactères comme une sous-algèbre de l'algèbre de groupe. Dans ces deux cas, nous avons une interprétation des dimensions graduées de l'espace des invariants en terme de ces mots.

Keywords: Invariant theory, Non-commutative variables, Symmetric group, Dihedral group, Cayley Graph, Words

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1 Introduction

Let V be a vector space over $\mathbb{C}$ with basis $\{x_1, x_2, \ldots, x_n\}$ and G a finite subgroup of $GL(V)$, then

$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots \simeq \mathbb{C}\langle x_1, x_2, \ldots, x_n \rangle$$

is the ring of non-commutative polynomials in the basis elements where we use the notation $V^{\otimes d} = V \otimes V \otimes \cdots \otimes V$. We will consider the subalgebra $T(V)^G \simeq \mathbb{C}\left\langle x_1, x_2, \ldots, x_n \right\rangle^G$ as the graded space of invariants with respect to the action of G. It is convenient to conserve the information on the dimension of each homogeneous component $\mathbb{C}\langle x_1, x_2, \ldots, x_n \rangle_d^G \simeq (V^{\otimes d})^G$ of degree d in the *Hilbert-Poincaré series*

$$P(T(V)^G) = \sum_{d \geq 0} \dim(V^{\otimes d})^G q^d.$$

Several algebraic tools allow us to study the invariants for $T(V)$ with respect to the group G. The graded character of $T(V)$ can be found in terms by what we might identify as a 'master theorem' for the tensor space,

$$\chi^{(V^{\otimes d})}(g) = tr(M(g))^d = \left[q^d\right] \frac{1}{1 - tr(M(g))q} ,$$

where $\left[q^d\right]$ represents taking the coefficient of q^d in the expression to the right and $M(g)$ is a matrix which represents the action of the group element g on a basis of V. The analogue of Molien's theorem [3] for the tensor algebra says that

$$dim\ (V^{\otimes d})^G = \left[q^d\right] \frac{1}{|G|} \sum_{g \in G} \frac{1}{1 - tr(M(g))q} .$$

In general, we can say that the invariants $T(V)^G$ are freely generated [4] by an infinite set of generators (except when G is scalar, *i*.e. when G is generated by a nonzero scalar multiple of the identity matrix) [3]. No simple general description of the invariants or the generators is known for large classes of groups and these algebraic tools do not clearly show the underlying combinatorial structure of these invariant algebras.

Our goal is to find a combinatorial method for computing the graded dimensions of $T(V)^G$. The main idea of a general theorem would be the following. To a G-module V, we associate a subalgebra of the group algebra together with a homomorphism of algebras into the ring of characters. Then we get as a consequence a combinatorial description of the invariants of $T(V)$ as words generated by a particular Cayley graph of G. To compute the coefficient of q^d in the Hilbert-Poincaré series of $T(V)^G$, it then suffices to look at the multiplicity of the trivial in $(V^{\otimes d})$. At this point, since there is not a general relation between the group algebra and the character ring, we are only able to treat some examples that we decided to present here and the method used gives rise to objects that are a priori not natural in that context. In particular, we compute the graded dimensions of $T(V)^G$ for V being the geometric module (see below) of the symmetric group and for V being any module of the dihedral group in term of words generated by a Cayley graph of G in some specific generators. The subalgebra we use in the case of the symmetric group is the Solomon's descent algebra, that will make the bridge between words in a particular Cayley graph in those generators and the decomposition of $T(V)$ into simple S_n-module. In the case of the dihedral group, we present a new non-commutative realization of the character ring as a subalgebra of the group algebra.

When the group G is generated by pseudo-reflections acting on a vector space V, then if V is simple, V is called the geometric G-module. When G is the symmetric group S_n on n letters and acts on the vector space V spanned by the vectors $\{x_1, x_2, \ldots, x_n\}$ by the permutation action then G is generated by pseudo-reflections, but is not a simple S_n-module. The space $\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle^{S_n}$ is known as the symmetric functions in non-commutative variables which was first studied by Wolf [8] and more recently by Rosas-Sagan [6]. The dimension of $(V^{\otimes d})^{S_n}$ is the number of set partitions of the numbers $\{1, 2, \ldots, d\}$ into at most n parts. If G is the symmetric group but acting on the vector space spanned by the vectors $\{x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n\}$ (again with the permutation action on the x_i) then this is also a group generated by pseudo-reflections but the invariant space $T(V)^{S_n}$ is not as well understood. The graded dimensions of the invariant space are given by the number of oscillating tableaux studied by Chauve-Goupil [1]. This interpretation for the graded dimensions has a very different nature to that of set partitions. By applying the results in this paper we find a combinatorial interpretation for the graded dimensions of these spaces, and many others, which unifies the interpretations of their graded dimensions.

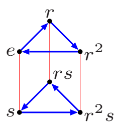
The paper is organized as follows. In section 2 we recall the definition of a Cayley graph and present a technical lemma that we will need to link the number of words of length d in a particular Cayley graph of G to some coefficients in the d-th power of a particular element of the group algebra. We will then present in section 3 the particular case of the symmetric group S_n and make explicit the result for V being the geometric S_n-module. Since the bridge between the words in the Cayley graph of S_n and the decomposition of $T(V)$ is the descent algebra, we will recall in section 3.3 some results about the Solomon's descent algebra of S_n. Section 3.6 contains some results about the invariant algebra $T(V)^{S_n}$ where we present a conjecture for a closed formula for the Hilbert-Poincaré series of $T(V)^{S_n}$, where V is the geometric S_n-module. Finally in section 4, we apply our general method in the case of the dihedral group D_m and then study in section 4.3 the particular case of the invariant algebra $T(V)^{D_m}$ when V is the geometric module and give a closed formula for the Hilbert-Poincaré series of $T(V)^{D_m}$.

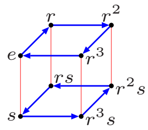
2 Cayley graph of a group G

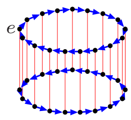
Let us recall the definition of a Cayley graph given in Coxeter [2]. A presentation of a finite group G with generating set S can be encoded by its Cayley graph. A *Cayley graph* is an oriented graph $\Gamma = \Gamma(G, S)$, having one vertex for each element of the group G and the edges associated with generators in S. Two vertices g_1 and g_2 are joined by a directed edge associated to $s \in S$ if $g_2 = g_1 s$. Then a path along the edges corresponds to a word in the generators in S. A word which *reduces to* $g \in G$ in Γ will be a path along the edges from the vertex corresponding to the identity to the one corresponding to the element g. We will denote by $w(g; d; \Gamma)$ the set of words of length d which reduce to g in Γ. We will say that a word *does not cross the identity* if it has no proper prefix which reduces to the identity.

More generally, we will consider *weighted* Cayley graphs $\Gamma(G, S)$. In other words, we will associate a weight $\omega(s)$ to each generator $s \in S$. Then we will define the *weight of a word* $w = s_1 s_2 \cdots s_r$ in the generators to be the product of the weights of the generators, $\omega(w) = \omega(s_1)\omega(s_2)\cdots\omega(s_r)$. To simplify the image, undirected edges will represent bidirectional edges and non-labelled edges will represent edges of weight one .

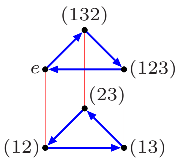
Example 2.1 *Consider the dihedral group* D_m *with presentation* $\langle s, r \mid s^2 = r^m = srsr = e\rangle$. *The Cayley graphs* $\Gamma(D_3, \{s, r\})$, $\Gamma(D_4, \{s, r\})$ *and more generally* $\Gamma(D_m, \{s, r\})$ *will look like*







Example 2.2 *The symmetric group S_n on n letters is generated by the permutations (12) and $(1\,n\,\cdots 432)$ (see [2]), hence also by the permutations (12), (132), $(1432), \ldots, (1\,n\,\cdots 432)$, written in cyclic notation. The Cayley graph $\Gamma(S_3, \{(12), (132)\})$ is*



Lemma 2.3 *Let $\Gamma = \Gamma(G, \{s_1, s_2, \ldots, s_r\})$ be a Cayley graph of G with associated weights $\omega(s_i) = \omega_i$. Then the coefficient of $\sigma \in G$ in the element $(\omega_1 s_1 + \omega_2 s_2 + \cdots + \omega_r s_r)^d$ of the group algebra $\mathbb{C}G$ is equal to*

$$\sum_{w \in w(\sigma,d;\Gamma)} \omega(w),$$

where $w(\sigma, d; \Gamma)$ is the set of words of length d which reduce to σ in Γ.

Example 2.4 *Let us consider the Cayley graph $\Gamma = (S_3, \{(12), (132)\})$ of Example 2.2. Set $a = (12)$ and $b = (132)$ to simplify. Then the table below shows that the coefficient of a specific element in $(a+b)^4$ coincides with the number of words of length three which reduce to that specific element in Γ.*

$$(a+b)^4 = \mathbf{3}\, e + \mathbf{2}\,(12) + \mathbf{3}\,(23) + \mathbf{3}\,(123) + \mathbf{2}\,(132) + \mathbf{3}\,(13)$$

e	(12)	(23)	(123)	(132)	(13)
$aaaa$	$abbb$	$aaba$	$aabb$	$abba$	$aaab$
$abab$	$bbba$	$baaa$	$baab$	$bbbb$	$abaa$
$baba$		$bbab$	$bbaa$		$babb$

3 Symmetric group S_n

We will give in that section a combinatorial way to decompose the tensor algebra on V into simple S_n-modules, for V being the geometric S_n-module, by means of words in a particular Cayley graph of S_n. We will also give a combinatorial way to compute the graded dimensions of the invariant space $T(V)^{S_n}$, which is the multiplicity of the trivial in the decomposition of $T(V)$. But first let us recall some definition and the theory of the descent algebra.

3.1 Partitions and tableaux

To fix the notation, recall the definition of a partition. A *partition* λ of a positive integer n is a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ of positive integers such that $n = |\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell$. We will write $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_\ell) \vdash n$. For example, the partitions of 3 are

$$(1,1,1) \qquad (2,1) \qquad (3).$$

It is natural to represent a partition by a diagram. The *Ferrers diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \dots \lambda_\ell)$ is the finite subset $\lambda = \{(a,b) \,|\, 0 \le a \le \ell - 1$ and $0 \le b \le \lambda_{a+1} - 1\}$ of $\mathbb{N} \times \mathbb{N}$. Visually, each element of λ corresponds to the bottom left corner of a square of dimension 1×1 in $\mathbb{N} \times \mathbb{N}$. A ***tableau*** of shape $\lambda \vdash n$, denoted $sh(t) = \lambda$, with values in $T = \{1, 2, \dots, n\}$ is a function $t : \lambda \longrightarrow T$. We can visualize it with filling each square c of a Ferrers diagram λ with the value $t(c)$. A tableau is said to be ***standard*** if its entries form an increasing sequence along each line and along each column. We will denote by $STab_n$ the set of standard tableau with n squares. For example, $STab_3$ contains the four standard tableaux

$$\begin{array}{l}\boxed{3}\\ \boxed{2}\\ \boxed{1}\end{array}\qquad \begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{3}\end{array}\qquad \begin{array}{l}\boxed{3}\\ \boxed{1}\boxed{2}\end{array}\qquad \boxed{1}\boxed{2}\boxed{3}\,.$$

The *Robinson-Schensted correspondence* is a bijection between the elements σ of the symmetric group S_n and pairs $(P(\sigma), Q(\sigma))$ of standard tableaux of the same shape, where $P(\sigma)$ is the insertion tableau and $Q(\sigma)$ the recording tableau.

3.2 Simple S_n-modules

Since the conjugacy classes in S_n are in bijection with the partitions of n, it is natural to index the simple S_n-modules by the partitions λ of n and we will denote them by V^λ. In particular, the simple S_n-module $V^{(n)}$ indexed by the partition (n) is the the trivial one. Let us consider the linear span $V = \mathcal{L}\{x_1, x_2, \dots, x_n\}$ on which S_n acts by permuting the coordinates. Then we have

$$V = \mathcal{L}\{x_1 + x_2 + x_3 + \ldots + x_n\} \oplus \mathcal{L}\{x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n\},$$

so the decomposition of V into simple S_n-modules is $V = V^{(n)} \oplus V^{(n-1,1)}$. Note that the S_n-module $V^{(n-1,1)}$ corresponds to the geometric S_n-module. Let X_n denote the set of variables $x_1, x_2, \dots, x_n$ and Y_{n-1} denote the set of variables $y_1, y_2, \dots, y_{n-1}$. If we identify $T(V)$ with $\mathbb{R}\langle X_n\rangle$, then $T(V^{(n-1,1)}) \simeq \mathbb{R}\langle X_n\rangle / \langle x_1 + x_2 + \cdots + x_n\rangle$ can be identified with $\mathbb{R}\langle Y_{n-1}\rangle$, where $y_i = x_i - x_{i+1}$ for $1 \le i \le n-1$.

3.3 Solomon's descent algebra of S_n

Surprisingly, the key to prove the general result is the theory of descent algebra of the symmetric group. Let us recall some of that theory here. Let $I = \{1, 2, \dots, n-1\}$. The descent set of $\sigma \in S_n$ is the set $Des(\sigma) = \{i \in I \,|\sigma(i) > \sigma(i+1)\}$. For $K \subseteq I$, set

$$d_K = \sum_{\substack{\sigma \in S_n \\ Des(\sigma) = K}} \sigma.$$

The Solomon's descent algebra $\Sigma(S_n)$ is a subalgebra of the group algebra $\mathbb{Z}S_n$ with basis $\{d_K | K \subseteq I\}$ [7]. For a standard tableau t of shape $\lambda \vdash n$ define

$$z_t = \sum_{\substack{\sigma \in S_n \\ Q(\sigma) = t}} \sigma,$$

where $Q(\sigma)$ corresponds to the recording tableau in the Robinson-Schenstead correspondence. Then consider the linear span $\mathcal{Q}_n = \mathcal{L}\{z_t |\ t \in STab_n\}$. Note in general that $\mathcal{Q}_n$ is not a subalgebra of $\mathbb{Z}S_n$, for $n \geq 4$. Define the descent set of a standard tableau t by $Des(t) = \{i | i+1 \text{ is above i in } t\}$. Then

$$d_K = \sum_{\substack{t \in STab_n \\ Des(t)=K}} z_t.$$

and $\Sigma(S_n) \subseteq \mathcal{Q}_n$. There is an algebra morphism $\theta : \Sigma(S_n) \to \mathbb{Z}\mathrm{Irr}(S_n)$ due to Solomon [7]. Moreover, there is a linear map [5] $\tilde{\theta} : \mathcal{Q}_n \to \mathbb{Z}\mathrm{Irr}(S_n)$ defined by $\tilde{\theta}(z_t) = \chi^{\mathrm{sh}(t)}$, and $\tilde{\theta}$ restricted to $\Sigma(S_n)$ corresponds to θ. We can observe that

$$z_{\begin{array}{|c|cccc}\hline 2 & & & & \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{n} \\ \hline \end{array}} = (12) + (132) + (1432) + \cdots + (1\,n\,\cdots 432) = d_{\{1\}}$$

hence $\theta(d_{\{1\}}) = \chi^{(n-1,1)}$.

3.4 General method for S_n

We are developing a general combinatorial method for determining the multiplicity of V^λ in $V^{\otimes d}$, when V is any S_n-module. To this end, we will consider the algebra morphism $\theta : \Sigma(S_n) \to \mathbb{Z}\mathrm{Irr}(S_n)$ of section 3.3. The next proposition says that this multiplicity is given as the sum of some coefficients in f^d, when f is an element of $\Sigma(S_n)$ such that $\theta(f) = \chi^V$.

Proposition 3.1 *Let V be an S_n-module such that $\theta(f) = \chi^V$, for some $f \in \Sigma(S_n)$. For $\lambda \vdash n$, the multiplicity of V^λ in $V^{\otimes d}$ is equal to*

$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} [z_t] f^d,$$

where $[z_t]f^d$ is the coefficient of z_t in f^d.

Although the next theorem is an easy consequence of the Lemma 2.3 and Proposition 3.1, it provides us with an interesting interpretation for the multiplicity of V^λ in the d-fold Kronecker product of a S_n-module. This multiplicity is the weighted sum of words in a particular Cayley graph of S_n which reduce to the element σ_t, where σ_t has recording tableau t of shape λ in the Robinson-Schensted correspondance. Recall that the *support* of an element f of the group algebra is defined by $\mathrm{supp}(f) = \{g \in G | [g]f \neq 0\}$.

Theorem 3.2 *Let V be an S_n-module such that $\theta(f) = \chi^V$, for some $f \in \Sigma(S_n)$. For $\lambda \vdash n$, the multiplicity of V^λ in $V^{\otimes d}$ is*

$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} \sum_{w \in w(\sigma_t, d; \Gamma)} \omega(w),$$

where σ_t is such that $Q(\sigma_t) = t$, $\Gamma = \Gamma(S_n, supp(f))$ with $\omega(\sigma) = [\sigma](f)$ for each $\sigma \in supp(f)$ and $w(\sigma_t, d; \Gamma)$ is the set of words of length d which reduce to σ_t in Γ.

3.5 Decomposition of $T(V^{(n-1,1)})$ and words in a Cayley graph of S_n

Since we are particularly interested in the geometric S_n-module, we make explicit the following two corollaries respectively of Proposition 3.1 and Theorem 3.2 needed to draw a connection between the multiplicity of V^λ in $(V^{(n-1,1)})^{\otimes d}$ and words of length d in a particular Cayley graph of S_n. To this end, we use the fact that the element $d_{\{1\}}$ of the descent algebra, which is the sum of elements of S_n having descent set $\{1\}$, is sent to $\chi^{(n-1,1)}$ under the θ morphism.

Corollary 3.3 *Let $\lambda \vdash n$. The multiplicity of V^λ in $(V^{(n-1,1)})^{\otimes d}$ is*

$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} [z_t] {d_{\{1\}}}^d.$$

Corollary 3.4 *Let $\lambda \vdash n$. The multiplicity of V^λ in $(V^{(n-1,1)})^{\otimes d}$ is equal to*

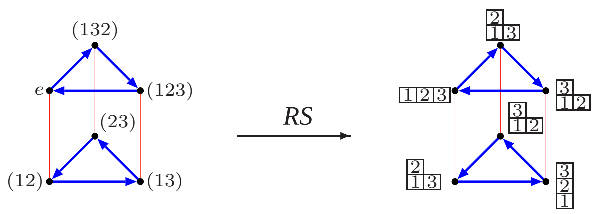
$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} |w(\sigma_t, d; \Gamma)|,$$

where $\sigma_t \in S_n$ is such that $Q(\sigma_t) = t$ and $\Gamma = \Gamma(S_n, \{(12), (132), \ldots, (1\, n \cdots 432)\})$. In particular, the multiplicity of the trivial is $|w(e, d; \Gamma)|$.

Example 3.5 *The S_3-module $(V^{(2,1)})^{\otimes 4}$ decomposes as $3\, V^{(3)} \oplus 5\, V^{(2,1)} \oplus 3\, V^{(1,1,1)}$ since*

$$\begin{aligned} {d_{\{1\}}}^4 &= 3\, d_\emptyset + 3\, d_{\{2\}} + 2\, d_{\{1\}} + 3\, d_{\{1,2\}} \\ &= 3\, z_{\begin{smallmatrix}1&2&3\end{smallmatrix}} + 3\, z_{\begin{smallmatrix}3&\\1&2\end{smallmatrix}} + 2\, z_{\begin{smallmatrix}2&\\1&3\end{smallmatrix}} + 3\, z_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}. \end{aligned}$$

These multiplicities can also be computed using Corollary 3.4 in the following way. The Cayley graph $\Gamma = \Gamma(S_3, \{(12), (132)\})$ looks like



and if we write a for (12) and b for (132) to simplify, and choose the representatives

$$\sigma_{\begin{smallmatrix}1&2&3\end{smallmatrix}} = e \qquad \sigma_{\begin{smallmatrix}3&\\1&2\end{smallmatrix}} = (23) \qquad \sigma_{\begin{smallmatrix}2&\\1&3\end{smallmatrix}} = (12) \qquad \sigma_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} = (13)$$

the multiplicities are respectively given by the cardinalities of the sets of words (see Example 2.4)

$$\begin{array}{lll} V^{(3)}: & |w(e\,, 4; \Gamma)| & = |\{aaaa, abab, baba\}| = 3, \\ V^{(2,1)}: & |w((23)\,, 4; \Gamma)| + |w((12)\,, 4; \Gamma)| & = |\{aaba, baaa, bbab\}| + |\{abbb, bbba\}| = 5, \\ V^{(1,1,1)}: & |w((13)\,, 4; \Gamma)| & = |\{aaab, abaa, babb\}| = 3. \end{array}$$

3.6 Invariant algebra $T(V^{(n-1,1)})^{S_n} \simeq \mathbb{R}\langle Y_{n-1}\rangle^{S_n}$

We have an interpretation of the invariant algebra $T(V^{(n-1,1)})^{S_n}$ in terms of words which reduce to the identity in the Cayley graph $\Gamma(S_n, \{(12), (132), \ldots, (1\,n\,\cdots\,432)\})$. As a corollary of Corollary 3.4, we can now show that the dimension of $T(V^{(n-1,1)})^{S_n}$ in each degree d, which is also the multiplicity of the trivial representation in $(V^{(n-1,1)})^{\otimes d}$, can be indexed by those precise words of length d.

Corollary 3.6 *The dimension of $((V^{(n-1,1)})^{\otimes d})^{S_n} \simeq \mathbb{R}\langle Y_{n-1}\rangle_d^{S_n}$ is equal to the number of words of length d which reduce to the identity in the Cayley graph $\Gamma(S_n, \{(12), (132), \ldots, (1\,n\,\cdots\,432)\})$.*

Example 3.7 *Consider the symmetric group S_3. Using the Reynold's operator $\sum_{\sigma\in S_n}\sigma$ acting on the monomials, a basis for the invariant space $\mathbb{R}\langle y_1, y_2\rangle_4^{S_3}$ is given by the three following polynomials*

$$y_1^2y_2^2 - y_1y_2^2y_1 - y_2y_1^2y_2 + y_2^2y_1^2,$$
$$y_1y_2y_1y_2 - y_1y_2^2y_1 - y_2y_1^2y_2 + y_2y_1y_2y_1,$$
$$2y_1^4 + y_1^3y_2 + y_1^2y_2y_1 + y_1y_2y_1^2 + 3y_1y_2^2y_1 + y_1y_2^3 + y_2y_1^3 + 3y_2y_1^2y_2 + y_2y_1y_2^2 + y_2^2y_1y_2 + y_2^3y_1 + 2y_2^4.$$

which agree with the number of words $\{aaaa, abab, baba\}$ in the letters $a = (12)$ and $b = (132)$ which reduce to the identity in the Cayley graph $\Gamma(S_3, \{(12), (132)\})$ (see Example 2.4).

Proposition 3.8 *The number of free generators of $T(V^{(n-1,1)})^{S_n}$ as an algebra are counted by the words which reduce to the identity without crossing the identity in $\Gamma(S_n, \{(12), (132), \ldots, (1\,n\,\cdots\,432)\})$.*

Example 3.9 *The number of free generators of $T(V^{(2,1)})^{S_3}$ are counted by the number of words in the following subsets of words which reduce to the identity without crossing the identity in $\Gamma(S_3, \{(12), (132)\})$*

$$\{aa\}, \{bbb\}, \{abab, baba\}, \{abbba, baabb, bbaab\}, \{abaaab, abbabb, baaaba, babbab, bbabba\}, \ldots$$

with cardinalities corresponding to the Fibonacci numbers.

We present next a conjecture for a closed formula giving the Hilbert-Poincaré series of $T(V^{(n-1,1)})^{S_n}$ which does not seem to obviously follow from our combinatorial interpretations for the dimensions.

Conjecture 3.10 *The Hilbert-Poincaré series of $T(V^{(n-1,1)})^{S_n}$ is*

$$P(T(V^{(n-1,1)})^{S_n}) = \frac{1}{1+q} + \frac{q}{1+q}\sum_{k=0}^{n-1}\frac{q^k}{(1-q)\,(1-2\,q)\cdots(1-k\,q)}.$$

4 Dihedral group D_m

The same kind of results can be observed for other finite groups, for example in the case of cyclic and dihedral groups. We will present in this section the case of the dihedral group D_m with presentation $D_m = \langle s, r \mid s^2 = r^m = srsr = e\rangle$. We will give a combinatorial way to decompose the tensor algebra on any D_m-module into simple modules by looking to words in a particular Cayley graph of D_m. The bridge between those words and the decomposition of the tensor algebra into simple modules is made possible via a subalgebra of the group algebra $\mathbb{R}D_m$ and a surjective algebra morphism from this subalgebra into the algebra of characters that we will present in next section.

4.1 Simple D_m-modules

For our purpose, let us first compute the irreducible characters of the dihedral group D_m. For $m = 2\,k$ even, there are $k+3$ simple D_m-modules (up to isomorphisms) V^{id}, V^{γ}, V^{ϵ}, $V^{\gamma\epsilon}$ and V^i, for $1 \le i \le k-1$ with associated irreducible characters

$$\begin{array}{rcl} id : D_m & \to & \mathbb{C} \\ r^\eta & \mapsto & 1 \\ s & \mapsto & 1 \\ rs & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} \gamma : D_m & \to & \mathbb{C} \\ r^\eta & \mapsto & (-1)^\eta \\ s & \mapsto & -1 \\ rs & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} \chi_i : D_m & \to & \mathbb{C} \\ r^\eta & \mapsto & 2\cos(\frac{2\pi\eta i}{m}) \\ s & \mapsto & 0 \\ rs & \mapsto & 0 \end{array}$$

$$\begin{array}{rcl} \epsilon : D_m & \to & \mathbb{C} \\ r^\eta & \mapsto & 1 \\ s & \mapsto & -1 \\ rs & \mapsto & -1 \end{array} \qquad \begin{array}{rcl} \gamma\epsilon : D_m & \to & \mathbb{C} \\ r^\eta & \mapsto & (-1)^\eta \\ s & \mapsto & 1 \\ rs & \mapsto & -1 \end{array}$$

For $m = 2\,k+1$ odd, the $k+2$ simple D_m-modules (up to isomorphisms) are V^{id}, V^{ϵ} and V^i, for $1 \le i \le k$ and the associated irreducible characters are respectively id, ϵ and χ_i. The next two propositions define the surjective algebra morphism needed to link the decomposition of $T(V)$ to words in a Cayley graph of D_m.

Proposition 4.1 *Let $y_i = r^{1-i}s + r^i$. For $m = 2k$ even, $\mathcal{Q} = \mathcal{L}\{e, r^k, rs, r^{k+1}s, y_i, y_i rs\}_{1\le i\le k-1}$ is a subalgebra of $\mathbb{Z}D_m$, and there is a surjective algebra morphism $\theta : \mathcal{Q} \to \mathbb{Z}\mathrm{Irr}(D_m)$ defined by $\theta(e) = id$, $\theta(rs) = \epsilon$, $\theta(r^k) = \gamma$, $\theta(r^{k+1}s) = \gamma\epsilon$ and $\theta(y_i) = \theta(y_i rs) = \chi_i$.*

Proposition 4.2 *Let $y_i = r^{1-i}s + r^i$. For $m = 2k+1$ odd, the linear span $\mathcal{Q} = \mathcal{L}\{e, rs, y_i, y_i rs\}_{1\le i\le k}$ is a subalgebra of $\mathbb{Z}D_m$, and there is a surjective algebra morphism $\theta : \mathcal{Q} \to \mathbb{Z}\mathrm{Irr}(D_m)$ defined by $\theta(e) = id$, $\theta(rs) = \epsilon$ and $\theta(y_i) = \theta(y_i rs) = \chi_i$.*

4.2 Decomposition of $T(V)$ and words in a Cayley graph of D_m

To simplify the notation, we will denote the subalgebras of Proposition 4.1 and 4.2 by $\mathcal{Q} = \mathcal{L}\{b_i\}_{i\in I}$, where each element b_i of the basis is sent to an irreducible character by θ and $V^{(i)}$ will denote a simple D_m-module with irreducible character $\chi^{(i)}$. As for the symmetric group, we have the following two results. Recall that $\mathrm{supp}(f) = \{g \in G | [g]f \ne 0\}$.

Proposition 4.3 *Let V be a D_m-module. If $f \in \mathcal{Q}$ is such that $\theta(f) = \chi^V$, then the multiplicity of $V^{(k)}$ in $V^{\otimes d}$ is equal to*

$$\sum_{\substack{b_i \\ \theta(b_i)=\chi^{(k)}}} [b_i] f^d.$$

Theorem 4.4 *Let V be a D_m-module. If $f \in \mathcal{Q}$ is such that $\theta(f) = \chi^V$, then the multiplicity of $V^{(k)}$ in $V^{\otimes d}$ is equal to*

$$\sum_{\substack{b_i \\ \theta(b_i)=\chi^{(k)}}} \sum_{w\in w(\sigma_i,d;\Gamma)} \omega(w),$$

where $\sigma_i \in supp(b_i)$, $\Gamma = \Gamma(D_m, supp(f))$ with $\omega(g) = [g](f)$ for each $g \in supp(f)$.

Example 4.5 *Consider the D_4-module $(2\,V^1 \oplus V^{\gamma\epsilon})^{\otimes 2}$. By Theorem 4.1 , there is a subalgebra $\mathcal{Q} = \mathcal{L}\{e, r^2, rs, r^3s, s+r, r^3+r^2s\}$ of the group algebra and $\theta : \mathcal{Q} \to \mathbb{Z}\mathrm{Irr}(D_4)$ defined by*

$$\theta(e) = id, \quad \theta(rs) = \epsilon, \quad \theta(r^2) = \gamma, \quad \theta(r^3s) = \gamma\epsilon, \quad \theta(s+r) = \theta(r^3+r^2s) = \chi_1.$$

Let $f = 2\,(r^3+r^2s)+r^3s$. Applying θ, $f^2 = 5\,e+4\,rs+4\,r^2+4\,r^3s+2\,(s+r)+2\,(r^3+r^2s)$ is sent to $(2\,\chi_1+\gamma\epsilon)^2 = 5\,id+4\,\epsilon+4\,\gamma+4\,\gamma\epsilon+2\,\chi_1+2\,\chi_1$ so the decomposition into simple modules is

$$(2\,V^1 \oplus V^{\gamma\epsilon})^{\otimes 2} = 5\,V^{id} \oplus 4\,V^{\epsilon} \oplus 4\,V^{\gamma} \oplus 4\,V^{\gamma\epsilon} \oplus 4V^1.$$

These multiplicities can also be computed using words in the Cayley graph $\Gamma = \Gamma(D_4, \{r^3, r^2s, r^3s\})$ with weights $\omega(r^3) = \omega(r^2s) = 2$ and $\omega(r^3s) = 1$. Applying Theorem 4.4, the multiplicities are

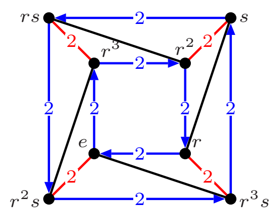
$$V^{id}: \quad \sum_{w \in w(e,2;\Gamma)} \omega(w) = \omega(aa)+\omega(cc) = 2\cdot 2+1\cdot 1 = 5$$

$$V^{\epsilon}: \quad \sum_{w \in w(rs,2;\Gamma)} \omega(w) = \omega(ba) = 2\cdot 2 = 4$$

$$V^{\gamma}: \quad \sum_{w \in w(r^2,2;\Gamma)} \omega(w) = \omega(bb) = 2\cdot 2 = 4$$

$$V^{\gamma\epsilon}: \quad \sum_{w \in w(r^3s,2;\Gamma)} \omega(w) = \omega(ab) = 2\cdot 2 = 4$$

$$V^{1}: \quad \sum_{w \in w(r,2;\Gamma)} \omega(w) + \sum_{w \in w(r^3,2;\Gamma)} \omega(w) = \omega(ca)+\omega(ac) = 1\cdot 2+2\cdot 1 = 4.$$



4.3 Invariant algebra $T(V^1)^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle^{D_m}$

We were particularly interested in studying the invariant space of the tensor algebra on the geometric representation V^1 and we have the following results. Since the dimension of $((V^1)^{\otimes d})^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle_d^{D_m}$ is equal to the multiplicity of the trivial in $(V^1)^{\otimes d} \simeq \mathbb{R}\langle x_1, x_2\rangle_d$, the following Corollary follows from Theorem 4.4 and the fact that $\theta(s+r) = \chi_1$.

Corollary 4.6 *The dimension of $((V^1)^{\otimes d})^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle_d^{D_m}$ is equal to the number of words of length d which reduce to the identity in the Cayley graph $\Gamma(D_m, \{r, s\})$.*

Proposition 4.7 *The number of free generators of $T(V^1)^{D_m}$ as an algebra are counted by the words in the Cayley graph $\Gamma(D_m, \{r, s\})$ which reduce to the identity without crossing the identity.*

Proposition 4.8 *The Hilbert-Poincaré series of $T(V^1)^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle^{D_m}$ is*

$$P(T(V^1)^{D_m}) = 1 + \frac{1}{2}\left(\frac{(2q)^m + \sum_{i=0}^{\lfloor m/2\rfloor}\left(\binom{m+1}{2i+1} - 2\binom{m}{2i}\right)(1-4q^2)^i}{\sum_{i=0}^{\lfloor m/2\rfloor}\binom{m}{2i}(1-4q^2)^i - (2q)^m}\right).$$

5 Appendix

$S_n \backslash d$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
S_3	1	0	1	1	3	5	11	21	43	85	171	341	683	1365
S_4	1	0	1	1	4	10	31	91	274	820	2461	7381	22144	66430
S_5	1	0	1	1	4	11	40	147	568	2227	8824	35123	140152	559923
S_6	1	0	1	1	4	11	41	161	694	3151	14851	71621	350384	1729091

Tab. 1: Dimension of $((V^{(n-1,1)})^{\otimes d})^{S_n} \simeq \mathbb{R}\langle Y_{n-1}\rangle_d^{S_n}$. Number of words of length d which reduce to the identity in $\Gamma(S_n, \{(12), (132), (1432), \ldots, (1\,n\,\cdots\,432)\})$.

$d \backslash S_n$	S_3	S_4	S_5	S_6
2	aa	aa	aa	aa
3	bbb	bbb	bbb	bbb
4	$aaaa$ $abab$ $baba$	$aaaa$ $abab$ $cccc$ $baba$	$aaaa$ $abab$ $cccc$ $baba$	$aaaa$ $abab$ $cccc$ $baba$
5	$aabbb$ $abbba$ $baabb$ $bbaab$ $bbbaa$	$aabbb$ $accbc$ $abbba$ $bcacc$ $baabb$ $caccb$ $bbaab$ $cbcac$ $bbbaa$ $ccbca$	$aabbb$ $accbc$ $abbba$ $bcacc$ $baabb$ $caccb$ $ddddd$ $bbaab$ $cbcac$ $bbbaa$ $ccbca$	$aabbb$ $accbc$ $abbba$ $bcacc$ $baabb$ $caccb$ $ddddd$ $bbaab$ $cbcac$ $bbbaa$ $ccbca$

Tab. 2: Words of length d in the letters $a = (12), b = (132), c = (1432), d = (15432)$ which reduce to the identity in $\Gamma(S_n, \{(12), (132), (1432), \ldots, (1\,n\,\cdots\,432)\})$.

$D_m \backslash d$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
D_3	1	0	1	1	3	5	11	21	43	85	171	341	683	1365
D_4	1	0	1	0	4	0	16	0	64	0	256	0	1024	0
D_5	1	0	1	0	3	1	10	7	35	36	127	165	474	715
D_6	1	0	1	0	3	0	11	0	43	0	171	0	683	0

Tab. 3: Dimension of $((V^1)^{\otimes d})^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle_d^{D_m}$. Number of words in the letters r and s of length d which reduce to the identity in $\Gamma(D_m, \{r, s\})$.

$d \backslash D_m$	D_3	D_4	D_5	D_6
2	ss	ss	ss	ss
3	rrr			
4	$ssss$ $rsrs$ $srsr$	$ssss$ $rsrs$ $srsr$ $rrrr$	$ssss$ $rsrs$ $srsr$	$ssss$ $rsrs$ $srsr$
5	$ssrrr$ $rrssr$ $srrrs$ $rrrss$ $rssrr$		$rrrrr$	

Tab. 4: Words of length d in the letters r and s which reduce to the identity in $\Gamma(D_m, \{r, s\})$.

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Shortest path poset of finite Coxeter groups

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Abstract. We define a poset using the shortest paths in the Bruhat graph of a finite Coxeter group W from the identity to the longest word in W, w_0. We show that this poset is the union of Boolean posets of rank absolute length of w_0; that is, any shortest path labeled by reflections $t_1, \ldots, t_m$ is fully commutative. This allows us to give a combinatorial interpretation to the lowest-degree terms in the complete **cd**-index of W.

Résumé. Nous définons un poset en utilisant le plus court chemin entre l'identité et le plus long mot de W, w_0, dans le graph de Bruhat du groupe finie Coxeter, W. Nous prouvons que ce poset est l'union de posets Boolean du même rang que la longueur absolute de w_0; ça signifie que tous les plus courts chemins, étiquetés par reflections $t_1, \ldots, t_m$ sont totalement commutatives. Ça nous permet de donner une interpretation combinatorique aux terms avec le moindre grade dans le **cd**-index complet de W.

Keywords: Coxeter group, Bruhat order, Boolean poset, complete **cd**-index.

1 Introduction

Let (W, S) be a Coxeter system, and let $T(W) = \{wsw^{-1} : s \in S, w \in W\}$ be the set of *reflections* of (W, S). The *Bruhat graph* of (W, S), denoted by $B(W, S)$ or simply $B(W)$, is the directed graph with vertex set W, and a directed edge $w_1 \to w_2$ between $w_1, w_2 \in W$ if $\ell(w_1) < \ell(w_2)$ and there exists $t \in T(W)$ with $tw_1 = w_2$. ℓ denotes the *length function* of (W, S). The edges of $B(W)$ are labeled by reflections; for instance the edge $w_1 \to w_2$ is labeled with t. The Bruhat graph of an interval $[u, v]$, denoted by $B([u, v])$, is the subgraph of $B(W)$ obtained by only considering the elements of $[u, v]$. A *path* in the Bruhat graph $B([u, v])$, will always mean a *directed* path from u to v. As it is the custom, we will label these paths by listing the edges that are used.

A *reflection ordering* $<_{T(W)} = <_T$ is a total order of $T(W)$ so that $r <_T rtr <_T rtrtr <_T \ldots <_T trt <_T t$ or $t <_T trt <_T trtrt <_T \ldots <_T rtr <_T r$ for each subgroup $W' = \langle t, r \rangle$ where $t, r \in T(W)$. Let $\Delta = (t_1, t_2, \ldots, t_k)$ be a path in $B([u, v])$, and define the *descent set* of Δ by $D(\Delta) = \{j : t_{j+1} <_T t_j\} \subset [k-1]$.

Let $w(\Delta) = x_1 x_2 \cdots x_{k-1}$, where $x_i = \mathbf{a}$ if $t_i < t_{i+1}$, and $x_i = \mathbf{b}$, otherwise. In other words, set x_i to **a** if $i \notin D(\Delta)$ and to **b** if $i \in D(\Delta)$. In [3], Billera and Brenti showed that $\sum_{\Delta \in B([u,v])} w(\Delta)$ becomes

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a polynomial in the variables **c** and **d**, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called the *complete* **cd**-*index* of $[u, v]$, and it is denoted by $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. Notice that the complete **cd**-index of $[u, v]$ is an encoding of the distribution of the descent sets of each path Δ in the Bruhat graph of $[u, v]$, and thus seems to depend on $<_T$. However, it can be shown that this is not the case. For details on the complete **cd**-index, see [3].

As an example, consider A_2, the symmetric group on 3 elements with generators $s_1 = (1 \ \ 2)$ and $s_2 = (2 \ \ 3)$. Then $t_1 = s_1 < t_2 = s_1 s_2 s_1 < t_3 = s_2$ is a reflection ordering. The paths of length 3 are: $(t_1, t_2, t_3), (t_1, t_3, t_1), (t_3, t_1, t_3)$, and (t_3, t_2, t_1), that encode to $\mathbf{a^2} + \mathbf{ab} + \mathbf{ba} + \mathbf{b^2} = \mathbf{c^2}$. There is one path of length 1, namely t_2, which encodes simply to 1. So $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \mathbf{c^2} + 1$.

Consider the paths in $B([u, v])$ of minimum length. Using these paths, we can define a ranked poset by thinking of the edges of these paths as cover relations. We call this poset $SP([u, v])$, and when the interval $[u, v]$ is the full group, we simply use the notation $SP(W)$. The rank of an element x in $SP([u, v])$ is given by its distance from u (and so if $[u, v]$ is the whole group, the rank of x is given by its distance from the identity e). Here we are interested in $SP(W)$, where W is a finite Coxeter group. To illustrate the definition consider B_2 and $SP(B_2)$ as depicted below. The rank of $SP(B_2)$ is two since that is the length of the shortest paths in $B(B_2)$.

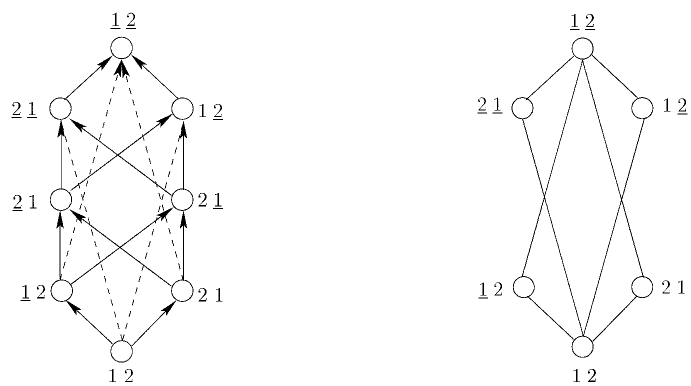

Fig. 1: $B(B_2)$ and $SP(B_2)$.

For any finite Coxeter group, there is a word w_0^W of maximal length. It is a well known fact that $\ell(w_0^W) = |T(W)|$. For any $w \in W$, one can write $t_1 t_2 \cdots t_n = w$ for some $t_1, t_2, \ldots, t_n \in T(W)$. If n is minimal, then we say that w is $T(W)$-*reduced*, and that the *absolute length* of w is n. We write $\ell_{T(W)}(w) = n$, or simply $\ell_T(w) = n$.

Notice that for $w \in W$, if $\ell_T(w) = \ell$, then $t_1 t_2 \cdots t_\ell = w$ for some reflections in $T(W)$, but this *does not* mean that $(t_1, t_2, \ldots, t_\ell)$ is a (directed) path in $B([e, w])$. Nevertheless, we will show that for finite W and $w = w_0^W$, $(t_1, t_2, \ldots, t_\ell)$ and any of its permutations $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(\ell)})$, $\tau \in A_{\ell-1}$, is a path in $B(W)$. To be more specific, we show the following theorem.

Theorem 1.1 *Let W be a finite Coxeter group and $\ell_0 = \ell_{T(W)}(w_0^W)$, the absolute length of the longest element of W. Then $SP(W)$ is isomorphic to the union of Boolean posets of rank ℓ_0.*

In Section 2 we present the proof of the theorem for the infinite families (groups of type A, B, and D and Dihedral groups). In Section 3 we discussed the validity of the Theorem for the exceptional groups. Computer search was used for F_4, H_3, H_4, and E_6, and a geometric argument was used to prove the case E_7 and E_8. We summarize the number of Boolean posets that form $SP(W)$ and the rank of $SP(W)$ for each finite Coxeter group in Table 1.

In Section 4 we discuss why Theorem 1.1 implies that the lowest-degree terms of the complete **cd**-index of W is given by $\alpha_W \widetilde{\psi}(\mathcal{B}_{\ell_0})$, where $\widetilde{\psi}(\mathcal{B}_{\ell_0})$ is the **cd**-index of the Boolean poset of rank $\ell_0 = \ell_{T(W)}(w_0^W)$, and α_W is the number of Boolean posets that form $SP(W)$.

The following lemma will be used in our proofs.

Lemma 1.2 (Shifting Lemma, [1], Lemma 2.5.1) *If $w = t_1 t_2 \cdots t_r$ is a $T(W)$-reduced expression for $w \in W$ and $1 \leq i < r$, then $w = t_1 t_2 \cdots t_{i-1}(t_i t_{i+1} t_i) t_i t_{i+2} \cdots t_r$ and $w = t_1 t_2 \cdots t_{i-2} t_i (t_i t_{i-1} t_i) t_{i+1} \cdots t_r$ are $T(W)$-reduced.*

As a consequence, there exists a $T(W)$-reduced expression for w having t_i as last reflection (or first), for $1 \leq i \leq r$. Furthermore, for any two reflections $t_i, t_j, i < j$ there exists a $T(W)$-reduced expression for w with t_i, t_j as the last two reflections (or the first two).

2 Groups of type A, B and D

2.1 *The poset* $SP(A_{n-1})$

Lemma 2.1 $\ell_{T(A_{n-1})}(w_0^{A_{n-1}}) = \lfloor \frac{n}{2} \rfloor$.

Proof: Recall that $w_0^{A_{n-1}}$ is the reverse of the identity $123 \ldots n$; that is, $w_0^{A_{n-1}} = n\ (n-1)\ (n-2) \ldots 21$. So $\ell_{T(A_{n-1})}(w_0^{A_{n-1}}) \geq \lfloor \frac{n}{2} \rfloor$ since a reflection in A_{n-1} is just a transposition, and thus cannot permute more than two elements of $[n]$ at a time.

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor = k$, let r_i be the transposition permuting i and $n+1-i$; that is, $r_i = (i \quad n+1-i)$. Notice that $r_1 \cdots r_k = w_0^{A_{n-1}}$, and so $\ell_{T(A_{n-1})}(w_0^{A_{n-1}}) \leq \lfloor \frac{n}{2} \rfloor$. □

Lemma 2.2 *For $\sigma \in A_{n-1}$, let $k = \lfloor \frac{n}{2} \rfloor$,*

$$f^A(\sigma) = \left\lfloor \frac{|\{i \in [n] \mid \sigma(i) = w_0^{A_{n-1}}(i)\}|}{2} \right\rfloor$$

and

$$g^A(\sigma) = \min\{\ell : \text{ there exists } t_1, t_2, \ldots, t_\ell \in T(A_{n-1}) \text{ with } t_1 t_2 \ldots t_\ell \sigma = w_0^{A_{n-1}}\}.$$

Then $f^A(\sigma) = i \implies g^A(\sigma) \geq \lfloor \frac{n}{2} \rfloor - i$ for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Proof: We proceed by reverse (downward) induction. The case $i = k$ holds, since g^A is a non-negative function. Suppose that the statement holds for i. We now show that it also holds for $i - 1$. Let $\sigma \in A_{n-1}$ with $f^A(\sigma) = i - 1$. Consider $t_1, t_2, \ldots, t_\ell \in T(A_{n-1})$ with $t_1 t_2 \cdots t_\ell \sigma = w_0^{A_{n-1}}$ and $\ell = g^A(\sigma)$. Notice that there exists an positive integer m with $f^A(t_{\ell-m+1} t_{\ell-m+2} \cdots t_\ell \sigma) = i$, since $f^A(t_1 t_2 \cdots t_\ell \sigma) = k$ and a reflection can fix at most two elements in their position in $w_0^{A_{n-1}}$, and so $f^A(t\tau) \leq f^A(\tau) + 1$ for $t \in T(A_{n-1})$ and $\tau \in A_{n-1}$. The last equality yields $g^A(\sigma) = \ell \geq k + m - i \geq k + 1 - i$. □

We can now show the proposition below, which gives Theorem 1.1 for type A.

Proposition 2.3 *Let $k = \lfloor \frac{n}{2} \rfloor$, and $R = \{r_1, r_2, \ldots, r_k\}$, where $r_i = (i \quad n+1-i)$ is the transposition permuting $i \in [k]$ and $n+1-i$. If $t_1 t_2 \cdots t_k = w_0^{A_{n-1}}$, then:*
(a) $\{t_1, \ldots, t_k\} = R$
(b) $t_i t_j = t_j t_i$ for all $i, j \in [k]$.
(c) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(k)})$ is a path in $B(A_{n-1})$ for all $\tau \in A_{k-1}$.

Proof: (a) Suppose that there exists $t_i \in T \setminus R$. Without loss of generality, using the Shifting Lemma, we can assume that $i = k$. Say $t_k = (m \;\; j)$ where $m < j \leq n$ and $j \neq n+1-k$. Hence $f^A(t_k) = 0$, and thus by Lemma 2.2 we have that $g^B(t_k) \geq k$. But this contradicts $t_1 t_2 \cdots t_{k-1} t_k = w_0^{A_{n-1}}$, which gives that $g^A(t_k) \leq k-1$.

(b) Notice that r_i and r_j are disjoint transpositions for $i \neq j$, and thus commute.

(c) By (b) it is enough to show that $\ell(t_1 t_2 \cdots t_m) > \ell(t_1 t_2 \cdots t_{m-1})$ for $1 < m \leq n$. To do this, we use Proposition 1.5.2 in [4]: If $w \in A_{n-1}$ then

$$\ell(w) = \mathrm{inv}(w) = |\{(i,j) \in [n] \times [n] \mid i < j, w(i) > w(j)\}|.$$

Let $w \in A_{n-1}$, if $i < j, w(i) > w(j)$ then we say that (i,j) is an *inversion pair* of w.

Suppose that $w_m = t_1 t_2 \cdots t_m$; we now compare $\mathrm{inv}(w_m)$ and $\mathrm{inv}(w_{m-1})$. By (a) we have that the t_i's are in R, so $t_m = (i \;\; n+1-i)$ for some $i \in [k]$. Moreover, $w_{m-1}(i) = i$, $w_{m-1}(n+1-i) = n+1-i$ and $w_m(l) = w_{m-1}(l)$ for all $l \in [n] \setminus \{i, n+1-i\}$. Now consider that:

1. If (l,i) is an inversion pair of w_{m-1} then $l < i$ and $w_{m-1}(l) > i$. If $w_{m-1}(l) > n+1-i$ then (l,i) and $(l, n+1-i)$ are inversion pairs of both w_{m-1} and w_m. If $w_{m-1}(l) \leq n+1-i$, then $(l, n+1-i)$ is not an inversion pair of w_{m-1}, but since $w_m(n+1-i) = i$, it is an inversion pair of w_m.

2. If $(l, n+1-i)$ in an inversion pair of w_{m-1} then $l < n+1-i$ and $w_m(l) = w_{m-1}(l) > n+1-i > i = w_m(n+1-i)$. Hence $(l, n+1-i)$ is also an inversion pair of w_m

3. If (i,l) an inversion pair of w_{m-1} then $i < l$ and $i > w_{m-1}(l)$. Since $w_m(i) = n+1-i > i > w_{m-1}(l) = w_m(l)$, (i,l) is also an inversion pair of w_m.

4. If $(n+1-i, l)$ is an inversion pair of w_{m-1} then $n+1-i < l$ and $n+1-i > w_{m-1}(l)$. If $i > w_{m-1}(l)$ then (i,l) and $(n+1-i, l)$ are inversion pairs of both w_{m-1} and w_m. If $i \leq w_{m-1}(l)$, then (i,l) is not an inversion pair of w_{m-1}, but since $w_m(i) = n+1-i$, it is an inversion pair of w_m.

Thus $\mathrm{inv}(w_m) \geq \mathrm{inv}(w_{m-1})$. To show that $\mathrm{inv}(w_m) \geq \mathrm{inv}(w_{m-1}) + 1$, consider the pair $(i \;\; n+1-i)$ which *is not* an inversion pair of w_{m-1}. But since $w_m(i) = n+1-i > i = w_m(n+1-i)$, this is an inversion pair of w_m. □

We remark that the above proposition shows that $SP(A_{n-1})$ is isomorphic to the Boolean poset of rank k. Moreover, $SP(A_{n-1})$ is the poset of subsets of R ordered by inclusion.

2.2 The poset $SP(B_n)$

We used the combinatorial description of B_n and $T(B_n)$ in [4], Section 8.1.

Recall that B_n is the group of *signed permutations*; that is, the group of permutations σ of the set $[\pm n] = \{-n, -n+1, \dots, -1, 1, 2, \dots, n-1, n\}$ with the property $\sigma(-i) = -\sigma(i)$ for all $i \in [\pm n]$. We used the notation $\underline{i}$ to denote $-i$ for $i \in [\pm n]$. We have that $w_0^{B_n} = \underline{1}\ \underline{2} \dots \underline{n}$. Further, $T(B_n) = \{(i\ \ \underline{i}) : i \in [n]\} \cup \{(i\ \ j)(\underline{i}\ \ \underline{j}) : 1 \le i < |j| \le n\}$. We call the set $\{(i\ \ \underline{i}) : i \in [n]\}$ ***reflections of type I*** and the set $\{(i\ \ j)(\underline{i}\ \ \underline{j}) : 1 \le i < |j| \le n\}$ ***reflections of type II***. We now prove the analogous versions of the propositions in Section 2.1.

Proposition 2.4 $\ell_{T(B_n)}(w_0^{B_n}) = n$.

Proof: Notice that a reflection of type II changes the sign of either zero or two elements in $[n]$ and swaps them. So at least another reflection must be used to place them back in their respective order. That is, at least two reflections of type II are needed to place two elements in $[n]$ in their positions in $w_0^{B_n}$. Hence at least $2m$ reflections of type II are needed to place $2m$ elements of $[n]$ in their position in $w_0^{B_n}$, with $0 \le m \le \lfloor \frac{n}{2} \rfloor$, and after that $n - 2m$ reflections of type I are needed to place the remaining $n - 2m$ elements in their position in $w_0^{B_n}$. So $\ell_{T(B_n)} \ge n$.

Now, notice that $(1\ \ \underline{1})(2\ \ \underline{2}) \cdots (n\ \ \underline{n}) = w_0^{B_n}$, and so $\ell_{T(B_n)}(w_0^{B_n}) \le n$. $\square$

Lemma 2.5 *For $\sigma \in B_n$, let*

$$f^B(\sigma) = |\{i \in [n] \mid \sigma(i) = w_0^{B_n}(i) = \underline{i}\}| + |\{(i,j) \in [n] \times [n], i < j \mid (\sigma(i), \sigma(j)) \in \{(j,i), (\underline{j}, \underline{i})\}|$$

and

$$g^B(\sigma) = \min\{\ell : \text{ there exists } t_1, t_2, \dots, t_\ell \text{ with } t_1 t_2 \dots t_\ell \sigma = w_0^{B_n}\}.$$

Then $f^B(\sigma) = i \implies g^B(\sigma) \ge n - i$ for $0 \le i \le n$.

Proof: We proceed by reverse induction. The case $i = n$ holds, since g^B is a non-negative function. Suppose that the statement holds for i. We now show that it also holds for $i - 1$. Let $\sigma \in B_n$ with $f^B(\sigma) = i - 1$. Consider $t_1, t_2, \dots, t_\ell \in T(B_n)$ with $t_1 t_2 \cdots t_\ell \sigma = w_0^{B_n}$ and $\ell = g^B(\sigma)$. Notice that there exists m with $f^B(t_{\ell-m+1} \cdots t_{\ell-1} t_\ell \sigma) = i$, since $f^B(t_1 t_2 \cdots t_\ell \sigma) = n$ and a reflection can fix at most one element in its position in $w_0^{B_n}$ or create a pair (i,j) that is sent to $(\underline{j}, \underline{i})$ or (j,i). The last equality yields $g^B(\sigma) = \ell \ge k + m - i \ge k + 1 - i$. $\square$

Let t_1, t_2 be two reflections of type II satisfying $\{t_1, t_2\} = \{(k\ \ \underline{l})(\underline{k}\ \ l), (k\ \ l)(\underline{k}\ \ \underline{l})\}$ for some k, l with $1 \le k < l \le n$. Then we see that $t_1 t_2(k) = t_2 t_1(k) = \underline{k}$ and $t_1 t_2(l) = t_2 t_1(l) = \underline{l}$. We call the pair t_1, t_2 a ***good pair***. Good pairs play a special role in the shortest paths in $B(B_n)$, as seen in the theorem below.

Proposition 2.6 *If $t_1 t_2 \dots t_n = w_0^{B_n}$, then:*
(a) For every $i \in [n]$ either t_i is of type I or there exists $j \in [n]$, $i \ne j$ so that t_i, t_j is a good pair.
(b) $t_i t_j = t_j t_i$ for all $i, j \in [n]$.
(c) $(t_{\tau(1)}, t_{\tau(2)}, \dots, t_{\tau(n)})$ is a path in $B(B_n)$ for all $\tau \in A_{n-1}$.

Proof: (a) Suppose that some reflection in $\{t_1, \dots, t_n\}$ is of type II, say $t_i = (k\ \ l)(\underline{k}\ \ \underline{l})$, and suppose that there is no t_j so that t_i, t_j is a good pair. Since $w_0^{B_n}(k) = \underline{k}$ and $w_0^{B_n}(l) = \underline{l}$, there must be another reflection t_m that is not disjoint from t_i. Without loss of generality, we can assume that $\{t_i, t_m\} = \{t_{n-1}, t_n\}$. Since t_{n-1}, t_n is not a good pair, then $f^B(t_{n-1}t_n) = 0$. Hence $g^B(t_{n-1}t_n) \geq n$, which contradicts $t_1 t_2 \cdots t_n = w_0^{B_n}$.

(b) Notice that since all the reflections in $t_1 \cdots t_n = w_0^{B_n}$ of type I are distinct, they commute with each other. Furthermore, if t_i, t_j are a good pair, then they also commute. We need to verify that (i) if t_i, t_j are of type II and *not* a good pair, then they are commuting reflections, and (ii) if t_i, t_j are of mixed types, then they commute. Using the Shifting Lemma again, we can assume that the reflections in both cases are t_{n-1} and t_n. Suppose that t_{n-1} and t_n do not commute. In both (i) and (ii) we see that $f^B(t_{n-1}t_n) = 0$, and so $g^B(t_{n-1}t_n) \geq n$ by Lemma 2.5, which contradicts $t_1 t_2 \cdots t_{n-1} t_n = w_0^{B_n}$.

(c) By Proposition 8.1.1 in [4], if $w \in B_n$ then

$$\ell(w) = \operatorname{inv}(w) + \operatorname{Neg}(w)$$

where

$$\operatorname{inv}(w) = \operatorname{inv}(w(1), w(2), \dots, w(n)) \quad \text{and} \quad \operatorname{Neg}(w) = -\sum_{j \in [n] : w(j) < 0} v(j).$$

For $i \in [n]$, let $w_i = t_1 t_2 \cdots t_i$. Notice that from (b) it is enough to prove that $\ell(w_m) > \ell(w_{m-1})$ for $1 < m \leq n$. We have the following cases:

1. t_m is of type I, say $t_m = (j\ \ \underline{j})$, with $j \in [n]$. (a) and (b) give that no other reflection involves the element j, and so $w_{m-1}(j) = j$. Furthermore, we have that $w_m(k) = w_{m-1}(k)$ for $k \in [n] \setminus \{j\}$. Now,

   - If (i, j) is an inversion pair of w_{m-1}, then $i < j$ and $w_{m-1}(i) > w_{m-1}(j) = j$, which gives that $w_{m-1}(i) > 0$. So $w_m(i) = w_{m-1}(i) > \underline{j} = w_m(j)$, and the pair (i, j) is also an inversion pair of w_m. Since $\operatorname{Neg}(w_m) = \operatorname{Neg}(w_{m-1}) + j$, we have that $\ell(w_{m-1}) < \ell(w_m)$.
   - If (j, i) is an inversion pair of w_{m-1}, then $j < i$ and $w_{m-1}(j) = j > w_{m-1}(i)$. Suppose that (j, i) is not an inversion pair of w_m. There are at most $j - 1$ such inversion pairs (j, i) of w_{m-1}, since $1 < w_{m-1}(i) < j$. On the other hand, notice that $\operatorname{Neg}(w_m) = \operatorname{Neg}(w_{m-1}) + j$. So

$$\begin{aligned}
\ell(w_m) - \ell(w_{m-1}) &= \operatorname{inv}(w_m) + \operatorname{Neg}(w_m) - (\operatorname{inv}(w_{m-1}) + \operatorname{Neg}(w_{m-1})) \\
&\geq \operatorname{inv}(w_{m-1}) - (j-1) + (\operatorname{Neg}(w_{m-1}) + j) - (\operatorname{inv}(w_{m-1}) + \operatorname{Neg}(w_{m-1})) \\
&\geq 1
\end{aligned}$$

2. t_m is of type II but does not change any element's signs, say $t_m = (i\ \ j)(\underline{i}\ \ \underline{j})$ with $1 \leq i < j \leq n$. Then by the same argument as in the proof of Proposition 2.3 (c), we have that $\ell(w_m) > \ell(w_{m-1})$.

3. If $t_m = (i\ \ \underline{j})(\underline{i}\ \ j)$, with $1 \leq i < j \leq n$; that is, t_m swaps i and j and changes their sign. (a) and (b) give that $(w_{m-1}(i), w_{m-1}(j)) \in \{(i, j), (j, i)\}$, $(w_m(i), w_m(j)) \in \{(\underline{j}, \underline{i}), (\underline{i}, \underline{j})\}$, and $w_{m-1}(k) = w_m(k)$ for $k \in [\pm n] \setminus \{\pm i, \pm j\}$. Then

- If (k, i) is an inversion pair of w_{m-1} then $k < i$ and either $w_{m-1}(k) > i$ or $w_{m-1}(k) > j$. In either case (k, i) is also an inversion pair of w_m since $w_m(k) = w_{m-1}(k) > 0$ and $w_m(i) < 0$. Further, $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + i + j$, and so $\ell(w_{m-1}) < \ell(w_m)$.
- If (i, k) is an inversion pair of w_{m-1} then $i < k$ and either $i > w_{m-1}(k)$ or $j > w_{m-1}(k)$. If we assume that (i, k) *is not* an inversion pair of w_m, then in the former case, there are at most $i - 1$ pairs lost, and in the latter there are at most $j - 1$. However, since $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + i + j$, we still have that $\ell(w_{m-1}) < \ell(w_m)$.
- If (j, k) is an inversion pair of w_{m-1} then $j < k$ and either $j > w_{m-1}(k)$ or $i > w_{m-1}(k)$. If we assume that (j, k) *is not* an inversion pair of w_m, then in the former case, there are at most $j - 1$ pairs lost, and in the latter there are at most $i - 1$. However, since $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + i + j$, we still have that $\ell(w_{m-1}) < \ell(w_m)$.
- If (k, j) is an inversion pair of w_{m-1} then $k < j$ and either $w_{m-1}(k) > j$ or $w_{m-1}(k) > i$. In either case (k, j) is also an inversion pair of w_m since $w_m(k) = w_{m-1}(k) > 0$ and $w_m(j) < 0$. Further, $\mathrm{Neg}(w_m) = \mathrm{Neg}(w_{m-1}) + i + j$, and so $\ell(w_{m-1}) < \ell(w_m)$.

In all cases, we have the desired result. □

The previous proposition says that $SP(B_n)$ is (isomorphic to) the union of Boolean posets of rank n, one for each set $\{t_1, t_2, \ldots, t_n\}$ with $t_1 t_2 \cdots t_n = w_0^{B_n}$. As an example, Figure 1 illustrates that $SP(B_2)$ is the union of two Boolean posets. In general, one can compute the number of Boolean posets in $SP(B_n)$.

2.2.1 Number of Boolean posets in $SP(B_n)$

Let b_n be the number of Boolean posets in B_n. We obtain a Boolean poset for each set $\{t_1, \ldots, t_n\}$ with $t_1 t_2 \cdots t_n = w_0^{B_n}$. It is easy to see that $b_1 = 1$ and $b_2 = 2$ (see Figure 1). For $n \geq 2$, notice that if $t_1 t_2 \cdots t_n(1) = \underline{1}$, then by Proposition 2.6 there are two possible cases: (i) there exists $t_j = (1\ \ \underline{1})$ or there exists a good pair of reflections of the form $(1\ \ \underline{k})(\underline{k}\ \ 1), (1\ \ k)(\underline{k}\ \ \underline{1})$. There are b_{n-1} such reflections in case (i) and $(n-1)b_{n-2}$ in case (ii). So b_n satisfies the recurrence relation

$$b_n = b_{n-1} + (n-1)b_{n-2}$$

with initial conditions $b_1 = 1$ and $b_2 = 2$. Notice that this count is the same as the number of partitions of a set of n distinguishable elements into sets of size 1 and 2.

It is easy to see that

$$b_n = 1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{j!} \prod_{i=0}^{j-1} \binom{n-2i}{2}.$$

2.3 The poset $SP(D_n)$

As in the previous section, we used the combinatorial description of D_n and $T(D_n)$ in [4] Section 8.2.

D_n $(n > 1)$ is the group of *signed permutations* with an *even* number of negative elements (e,g, $\underline{1}\ \underline{2}\ 3$ is an element in D_3 whereas $\underline{1}\ \underline{2}\ \underline{3}$ is not). Like B_n, if $\sigma \in D_n$ then $\sigma(-i) = -\sigma(i)$ for $i \in [\pm n]$. Moreover, $w_0^{D_n} = \underline{1}\ \underline{2} \ldots \underline{n}$ if n is *even* and $w_0^{D_n} = 1\ \underline{2} \ldots \underline{n}$ if n is *odd*. Further, $T(D_n) = \{(i\ \ j)(\underline{i}\ \ \underline{j}) : 1 \leq i < |j| \leq n\}$; that is, the reflections of D_n are the reflections of B_n of type II.

Proposition 2.7 *$\ell_{T(D_n)}(w_0^{D_n}) = n$ if n is even, and $\ell_{T(D_n)} = n-1$ if n is odd.*

Proof: Same as for Proposition 2.4, but only using reflections of type II. Notice that that for n even, $r_1r_1'r_2r_2'\cdots r_kr_k' = w_0^{D_n}$, where $k = n/2$ and $r_i = (2i{-}1\ \underline{2i})(\underline{2i}\ 2i{-}1)$, $r_i' = (2i{-}1\ 2i)(\underline{2i}\ \underline{2i-1})$ $1 \leq i \leq n/2$. Similarly, for n odd, we have that $t_1t_1't_2t_2'\cdots t_kt_k' = w_0^{D_n}$, where $k = (n-1)/2$ and $r_i = (2i\ \underline{2i+1})(\underline{2i+1}\ 2i)$, $r_i' = (2i\ 2i+1)(\underline{2i+1}\ \underline{2i})$, $1 \leq i \leq (n-1)/2$. □

Lemma 2.8 *For $\sigma \in D_n$, for n even, define*

$$f^D(\sigma) = |\{i \in [n] \mid \sigma(i) = w_0^{D_n}(i) = \underline{i}\}| + |\{(i,j) \in [n]\times[n], i<j \mid (\sigma(i),\sigma(j)) \in \{(j,i),(\underline{j},\underline{i})\}|$$

and for n odd, define

$$f^D(\sigma) = |\{i \in [n]\backslash\{1\} \mid \sigma(i) = w_0^{D_n}(i) = \underline{i}\}| + |\{(i,j) \in [n]\times[n], i<j \mid (\sigma(i),\sigma(j)) \in \{(j,i),(\underline{j},\underline{i})\}|.$$

Moreover, let

$$g^D(\sigma) = \min\{\ell : \text{ there exists } t_1, t_2, \ldots, t_\ell \text{ wih } t_1t_2\ldots t_\ell\sigma = w_0^{D_n}\}.$$

Then $f^D(\sigma) = i \implies g^D(\sigma) \geq m - i$ for $0 \leq i \leq m$ and $m = n$ if n is even, $m = n-1$ if n is odd.

Proof: Same as in Lemma 2.5, using only reflections of type II. □

Proposition 2.9 *Suppose that $t_1t_2\ldots t_m = w_0^{D_n}$, where $m = n$ if n is even and $m = n-1$ if n is odd. Then:*
(a) For every $i \in [m]$ there exists $j \in [m]$, $i \neq j$ so that t_i, t_j is a good pair.
(b) $t_it_j = t_jt_i$ for all $i, j \in [m]$.
(c) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(m)})$ is a path in $B(D_n)$ for all $\tau \in A_{m-1}$.

Proof: The proof for (a) and (b) is the same as in Proposition 2.6, but only using reflections of type II.
For (c), even though the length function is not the same as described in the Section 2.2, we recall that $B(D_n)$ is the induced graph of $B(B_n)$ on the elements of D_n by Proposition 8.2.6 in [4]. □

2.3.1 Number of Boolean posets in $SP(D_n)$

Let d_n be the number of Boolean posets in $SP(D_n)$ for each set $\{t_1, t_2, \ldots, t_n\} \subset T(D_n)$ with $t_1t_2\cdots t_n = w_0^{D_n}$. Counting these subsets is equivalent to counting the partitions of $[n]$, if n is even, or $[n-1]$, if n is odd, into subsets of two elements (these represents the good pairs). That is,

$$d_m = \frac{1}{\lfloor \frac{m}{2} \rfloor !} \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-2i}{2}$$

where $m = n$ if n is even, and $m = n-1$ if n is odd. Since m is even, notice that this is the same as counting the number of partitions of $[m]$ into sets of size 2.

2.4 Finite Dihedral groups

Let $I_2(m), m \geq 1$ be the dihedral group of order $2m$ with generating set $\{s_1, s_2\}$, and let $T = T(I_2(m))$ its reflection set. If n is odd, then

$$w_0^{I_2(m)} = \underbrace{s_1 s_2 s_1 \cdots s_1}_{m} = \underbrace{s_2 s_1 s_2 \ldots s_2}_{m}$$

is a reflection, and so $\ell_T(w_0) = 1$. Hence $SP(I_2(m))$ is isomorphic to the Boolean poset of rank 1, if m is odd.

The case where m is even is more interesting, as $w_0^{I_2(m)} \notin T$. We readily see that $\ell_T(w_0) = 2$, since for instance $w_0^{I_2(m)} = s_1 \underbrace{s_2 s_1 \cdots s_2}_{m-1}$. Thus $SP(I_2(m))$ is the union of Boolean posets of rank 2, if m is even.

Fix w_0 to start with s_1. We now count number of Boolean posets in $SP(I_2(m))$ for m even. This number is the same as the number of sets $\{t_1, t_2\}$ with $t_1 t_2 = w_0^{I_2(m)}$. There is one such set for each element of odd rank that starts with s_1, since for each such element t_1 there exists a unique element t_2 with $t_1 t_2 = w_0^{I_2(m)}$. Since there are $\frac{m}{2}$ such elements, there are $\frac{m}{2}$ Boolean posets in $SP(I_2(m))$.

3 Exceptional Coxeter groups

3.1 F_4, H_3, H_4, and E_6

We were able to verify through computer search that the the results in the previous sections also worked for the following exceptional groups: F_4, H_3, H_4, E_6. That is, the shortest path poset for these groups form a union of Boolean posets of rank the absolute length of the longest word w_0^W. We summarize the results in Table 1. The computer search was done using Stembridge's `coxeter` Maple package [7], and it basically consisted of finding all shortest paths and verifying the analogous of Propositions 2.3, 2.6, 2.9 for those groups; that is, that the paths are given by reflections that are fully commutative.

An interesting observation is that the 3 Boolean posets that form $SP(E_6)$ are almost disjoint, sharing only e and $w_0^{E_6}$ (the bottom and top elements of each poset).

3.1.1 E_7 and E_8

For E_7 and E_8 we were not able to verify by computer that the shortest paths form a union of Boolean posets, since it involved more computer power (or a better code) than was available to us. However, we can argue that this is indeed the case using geometric arguments. Let (W, S) be Coxeter system, and consider the *geometric representation* of W, $\sigma : W \hookrightarrow GL(V)$, where V is a vector space with basis $\Pi = \{\alpha_s \mid s \in S\}$ (Π is called the set of *simple roots*). It is shown in [6] Section 5.4 that σ is a faithful representation.

The *root system* of the Coxeter system (W, S) is the set $\Phi = \{\sigma(w)(\alpha_s) : s \in S, w \in W\}$. Let $\beta \in \Phi$, then $\beta = \sum_{s \in S} c_s \alpha_s$. It is a well-known result that either $c_s \geq 0$ or $c_s \leq 0$ for all $s \in S$. In the former case we say that β is a *positive root*, and in the latter case we say that β is *negative root*. The set of positive roots is denoted by Φ^+ and the set of negative roots is denoted by Φ^-. It is also a well known fact (Proposition 4.4.5 in [4]) that there is a bijection between the set of reflections of W, $T(W)$ and Φ^+ given by $t = wsw^{-1} \mapsto \sigma(w)(\alpha_s)$.

Finally , we shall use the fact that $\sigma(w_0^{E_n}) = -\,\mathbf{id}$, where **id** is the identity matrix of dimension n, and $n = 7, 8$. We point out that $\sigma(w_0^{E_n}) \neq -\,\mathbf{id}$, and thus $\mathrm{rank}(SP(E_6)) < 6$). For details, see [2] Chapter VI, §4.10 and §.11. With this in mind we can show

Proposition 3.1 *For E_n, where $n = 7, 8$ we have that:*
(a) $\ell_T(w_0^{E_n}) = n$.
(b) If $w_0^{E_n} = t_1 t_2 \cdots t_n$ then $t_i t_j = t_j t_i$ for all $i, j \in [n]$.
(c) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(n)})$ is a path in $B(E_n)$ for all $\tau \in A_{n-1}$.

Proof: (a) Since a reflection fixes a hyperplane, the product of k reflections fixes the intersection of the k hyperplanes that are fixed by each reflection. This intersection has codimension at most k, and so it's not empty unless $k \geq n$. In particular, $\sigma(w_0^{E_n}) = -\,\mathbf{id}$ leaves no points fixed (except for **0**) and so cannot be written as a product of fewer than n reflections; that is $\ell_T(w_0^{E_n}) \geq n$. Moreover by Carter's Lemma (Lemma 2.4.5 in [1]), we have that $\ell_T(w_0^{E_n}) \leq n$. Thus $\ell_T(w_0^{E_n}) = n$.

(b) Now consider $-\,\mathbf{id} = s_{t_1} s_{t_2} \cdots s_{t_n}$, where $\sigma(t_i) = s_{t_i}$ for $1 \leq i \leq n$ are the reflections (in V) with respect to the hyperplanes $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ that are perpendicular to the unit vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The space fixed by the product $s_{t_1} s_{t_2} \cdots s_{t_{n-1}}$ is $\mathbb{R}\mathbf{v}_n$ (since the product of everything is $-\,\mathbf{id}$) which has co-dimension $n-1$ and then by the previous argument,

$$\bigcap_{i<n} \mathcal{H}_i = \mathbb{R}\mathbf{v}_n,$$

that is, $v_n \in \mathcal{H}_i$ for all $i < n$. Hence, $\mathbf{v}_i \perp \mathbf{v}_n$, which means that t_n commutes with t_i for $i < n$. By the Shifting Lemma, we have that any two reflections t_i, t_j commute.

(c) Let $t_1 \cdots t_n = w_0^{E_n}$. We are going to show that $\ell(t_1 t_2 \cdots t_k) > \ell(t_1 t_2 \cdots t_{k-1})$ for $1 < k \leq n$. As before, let $s_{t_i} = \sigma(t_i)$ be the reflection on V corresponding to t_i about the hyperplane $\mathcal{H}_i$, and let $\mathbf{v}_i$ be the normal vector to $\mathcal{H}_i$. Since $\mathbf{v}_i \perp \mathbf{v}_j$ for all $i \neq j$, we have that $s_{t_1} s_{t_2} \cdots s_{t_{i-1}}(\alpha_i) = \alpha_i$, where $\alpha_i \in \Phi^+$ is the positive root corresponding to t_i. Thus by Proposition 4.4.6 in [4], we have that $\ell(t_1 t_2 \cdots t_i) > \ell(t_1 t_2 \cdots t_{i-1})$ for $1 \leq i \leq n$. □

As a consequence of the above theorem, $SP(E_7)$ and $SP(E_8)$ are both formed by the union of Boolean posets that share at least the bottom and top elements. We are now done with the proof of Theorem 1.1.

3.1.2 Number of Boolean posets in $SP(E_7)$ and $SP(E_8)$

To count the number of paths (chains) in $SP(E_n)$ where $n = 7, 8$ we simply count the number n-tuples of perpendicular roots, since $\sigma(w_0^{E_n}) = -\,\mathbf{id}$. Each one of these n-tuples up to signs and permutations represents a Boolean poset. Direct computation yields 135 Boolean posets in $SP(E_7)$ and 2025 Boolean posets in $SP(E_8)$. These results are included in Table 1.

Remark 3.2 *The above geometric argument can be used to obtain the results that were proven in Section 2. As was the case in our proofs, each group type requires its own argument, since $\sigma(w_0^W)$ is different for each case. However we believe that the combinatorial proofs are more appropriate for the FPSAC audience.*

4 Lowest-degree terms of the complete **cd**-index of finite Coxeter groups

For any *Eulerian poset* P, one can define the **cd**-*index* of P. This polynomial encodes the *flag* h-*vector*. The interested reader is referred to [5] for more information on the **cd**-index of Eulerian posets. Since Bruhat intervals are Eulerian and the reflection ordering has the property of having a unique chain (path in the Bruhat graph) with no descents for every interval $[u, v]$, the highest-degree terms of the complete **cd**-index coincide with the **cd**-index.

Let $\widetilde{\psi}(B_n)$ be **cd**-index of the Boolean poset $\mathcal{B}_n$ (so $\mathcal{B}_n$ is the poset of subsets of $[n]$ ordered by inclusion). We can use Theorem 5.2 in [5] to compute $\widetilde{\psi}(\mathcal{B}_n)$. First $\widetilde{\psi}(\mathcal{B}_1) = 1$ and for $n > 1$,

$$\widetilde{\psi}(\mathcal{B}_n) = \widetilde{\psi}(\mathcal{B}_{n-1}) \cdot \mathbf{c} + G(\widetilde{\psi}(\mathcal{B}_{n-1})) \tag{1}$$

where G is is the derivation $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$. In particular, we have that $\widetilde{\psi}(\mathcal{B}_2) = \mathbf{c}$, $\widetilde{\psi}(\mathcal{B}_3) = \mathbf{c^2} + \mathbf{d}$, $\widetilde{\psi}(\mathcal{B}_4) = \mathbf{c^3} + 2\mathbf{cd} + 2\mathbf{dc}$, and so on.

Propositions 2.3, 2.6 and 2.9, and the results and computer search of Section 3 give that for a finite Coxeter group W, the corresponding $SP(W)$ is the union of Boolean posets (that share at least the bottom and top elements). So any interval in $SP(W)$ belongs to a Boolean poset corresponding to a set $R = \{t_1, t_2, \ldots, t_\ell\} \subset T(W)$ with $\ell_{T(W)}(w_0^W) = \ell$ and $t_1 t_2 \cdots t_\ell = w_0^W$. Thus any interval of $SP(W)$ (thought of as paths in $B(W)$ labeled with $T(W)$, where $T(W)$ is ordered by a reflection ordering) has a unique chain (path) with empty descent set. Hence counting descent sets in the chains given by R is the same as counting the flag h-vector of the Boolean poset of rank ℓ.

As a consequence, the lowest-degree terms in the complete **cd**-index of W add up to a multiple N of the **cd**-index of the Boolean poset of ranks $\ell_{T(W)}(w_0^W)$. N is the number of Boolean posets in $SP(W)$; that is, the number of sets $\{t_1, \ldots, t_{\ell_{T(w_0^W)}}\}$ with $t_1 t_2 \cdots t_{\ell_{T(w_0^W)}} = w_0^W$. These terms can be computed using (1) and Table 1. So we have

Theorem 4.1 *Let W be a finite Coxeter group, α_W is the number of Boolean posets that form $SP(W)$ and $\ell_0 = \ell_T(w_0^W)$. Then lowest degree terms of $\widetilde{\psi}_{e,w_0^W}$ are given by $\alpha_W \widetilde{\psi}(\mathcal{B}_{\ell_0})$.*

In particular, the lowest-degree terms of $\widetilde{\psi}_{e,w_0^W}$ are minimized by $\widetilde{\psi}(\mathcal{B}_{\ell_0})$.

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Tab. 1: Finite coxeter groups W, $\mathrm{rank}(SP(W))$, and the number of Boolean posets in $SP(W)$

W	$\mathrm{rank}(SP(W))$	# of Boolean posets in $SP(W)$
A_{n-1}	$\lfloor \frac{n-1}{2} \rfloor$	1
B_n	n	b_n
D_n	n if n is even; $n-1$ if n is odd	d_n
$I_2(m)$	2 if m is even; 1 if m is odd	$\frac{m}{2}$ if m is even; 1 if m is odd
F_4	4	24
H_3	3	5
H_4	4	75
E_6	4	3
E_7	7	135
E_8	8	2025

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Automatic Classification of Restricted Lattice Walks

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We propose an *experimental mathematics approach* leading to the computer-driven *discovery* of various conjectures about structural properties of generating functions coming from enumeration of restricted lattice walks in 2D and in 3D.

Keywords: Automated guessing, lattice paths, generating functions, computer algebra, enumeration

1 Introduction

There is a strange phenomenon about the generating functions that count lattice walks restricted to the quarter plane: depending on the choice of the set $\mathfrak{S} \subseteq \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$ of admissible steps, the generating function is sometimes rational, sometimes algebraic [but not rational], sometimes D-finite [but not algebraic], and sometimes not even D-finite. This is quite in contrast to the corresponding problem in 1D, where the generating functions invariably are algebraic [3]. Much progress was made recently on understanding why this is so, and only very recently, Bousquet-Mélou and Mishna [9] have announced a classification of all the 256 possible step sets into algebraic, transcendental D-finite, and non-D-finite cases, together with proofs for the algebraic and D-finite cases and strong evidence supporting the conjectured non-D-finiteness of the others.

As usual, a power series $S(t) \in \mathbb{Q}[[t]]$ is called *algebraic* if there exists a bivariate polynomial $P(T, t)$ in $\mathbb{Q}[T, t]$ such that $P(S(t), t) = 0$, and transcendental otherwise. Also as usual, a power series $S(t)$ is called D-finite if it satisfies a linear differential equation with polynomial coefficients. (Every algebraic power series is D-finite, but not vice versa.) At first glance, it might seem easy to prove that a power series is algebraic or D-finite: just come up with an appropriate equation, and then verify that the series satisfies this equation. But as far as lattice walks are concerned, most proofs given so far are indirect in that they avoid exhibiting the equation explicitly but merely are satisfied showing its existence. This is probably so because the equations appearing in this context are often too big to be dealt with by hand.

Nevertheless, it is interesting to know the equations explicitly, because they provide a standard canonical representation for a series, from which lots of further information can be extracted in a straightforward manner. By applying a well-known technique from computer algebra (in modern fashion, cf. Section 2), we have systematically searched for differential equations and algebraic equations that the series counting the walks in the quarter plane satisfy. These are given in Section 3. We have also made a first step towards classifying walks in $\mathbb{Z}^3$ confined to the first octant (cf. Section 4) by considering all step sets $\mathfrak{S}$ with up to five elements, and

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performed a systematic search for equations of the corresponding series. More than 2000 hours of computation time have been spent in order to analyze about 3500 different sequences.

We do not provide proofs that the equations we found are indeed correct, but the computational evidence in favor of our equations is striking. We have no doubt that all the equations we found are correct. In principle, it would be possible to supplement the "automatically guessed" equations by computer proofs in a systematic fashion, using techniques that have recently been applied to some special cases [25, 24, 6]. But we found that the computational cost for performing these automated proofs would be by far higher than what was needed for the mere discovery.

2 Methodology

To study generating functions for lattice walks, we follow a classical scheme in experimental mathematics. It is based on the following steps: (S1) computation of high order expansions of generating power series; (S2) guessing differential and/or algebraic equations satisfied by those power series; (S3) empirical certification of the guessed equations (sieving by inspection of their analytic, algebraic and arithmetic properties); (S4) rigorous proof, based on (exact) polynomial computations.

In what follows, we only explain Steps (S1), (S2) and (S3). A full description of Step (S4) is given in [6]. By way of illustration, we choose an example requiring computations with human-sized outputs, namely the classical case, initially considered by Kreweras [27, 7, 8], of walks in the quarter plane restricted to the step set $\mathfrak{S} = \{\leftarrow, \nearrow, \downarrow\}$.

2.1 Basic Definitions and Facts

We focus on 2D and 3D lattice walks. The 2D walks that we consider are confined to the quarter plane $\mathbb{N}^2$, they join the origin of $\mathbb{N}^2$ to an arbitrary point $(i, j) \in \mathbb{N}^2$, and are restricted to a fixed subset $\mathfrak{S}$ of the step set $\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$. If $f(n; i, j)$ denotes the number of such walks of length n (i.e., using n steps chosen from $\mathfrak{S}$), the sequence $f(n; i, j)$ satisfies the multivariate recurrence with constant coefficients

$$f(n+1; i, j) = \sum_{(h,k)\in\mathfrak{S}} f(n; i-h, j-k) \quad \text{for} \quad n, i, j \geq 0. \tag{1}$$

Together with the appropriate boundary conditions

$$f(0; 0, 0) = 1 \quad \text{and} \quad f(n; i, j) = 0 \quad \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0,$$

the recurrence relation (1) uniquely determines the sequence $f(n; i, j)$. As is customary in combinatorics, we let

$$F(t; x, y) = \sum_{n\geq 0} \Big(\sum_{i,j\geq 0} f(n; i, j) x^i y^j \Big) t^n$$

be the trivariate generating power series of the sequence $f(n; i, j)$. As $f(n; i, j) = 0$ as soon as $i > n$ or $j > n$, the inner sum is actually finite, and so we may regard $F(t; x, y)$ as a formal power series in t with polynomial coefficients in $\mathbb{Q}[x, y]$.

Specializing $F(t; x, y)$ to selected values of x and y leads to various combinatorial interpretations. Setting $x = y = 1$ yields the power series $F(t; 1, 1)$ whose coefficients count the total number of walks with prescribed number of steps (and arbitrary endpoint); the choice $x = y = 0$ gives the series $F(t; 0, 0)$ whose coefficients count the number of walks returning to the origin; setting $x = 1$, $y = 0$ yields the power series whose coefficients count the number of walks ending somewhere on the horizontal axis, etc.

By [10, Th. 7], multivariate sequences that satisfy recurrences with constant coefficients have moderate growth, and thus their generating series are analytic at the origin. The next theorem refines this result in our context.

Theorem 1 *The following inequality holds*

$$f(n;i,j) \leq |\mathfrak{S}|^n \quad \textit{for all} \quad (i,j,n) \in \mathbb{N}^3. \tag{2}$$

In particular, the power series $F(t;0,0), F(t;1,0), F(t;0,1)$ and $F(t;1,1)$ are convergent in $\mathbb{C}[[t]]$ at $t=0$ and their radius of convergence is at least $1/|\mathfrak{S}|$.

Proof: The total number of unrestricted n-step walks starting from the origin is $|\mathfrak{S}|^n$, so the number of walks restricted to a certain region is bounded by this quantity. This implies that the coefficient of t^n in $F(t;1,1)$ is at most $|\mathfrak{S}|^n$. The bound also applies to the coefficient of t^n in $F(t;\alpha,\beta)$ for $\alpha,\beta \in \{0,1\}$, as these series count walks which are subject to further restrictions. □

2.1.1 D-finite generating series of walks are G-functions

A power series $S(t) = \sum_{n\geq 0} a_n t^n$ in $\mathbb{Q}[[t]]$ is called a G-function[(i)] if (a) it is D-finite; (b) its radius of convergence in $\mathbb{C}[[t]]$ is positive; (c) there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, the common denominator of $a_0, \ldots, a_n$ is bounded by C^n.

Examples of G-functions are the power series expansions at the origin of $\log(1-t)$ and $(1-t)^\alpha$ for $\alpha \in \mathbb{Q}$. More generally, the Gauss hypergeometric series ${}_2F_1(\alpha,\beta,\gamma;t)$ with rational parameters α,β,γ, is also a G-series [17]. A celebrated theorem of Eisenstein assures that any algebraic power series must be a G-function (if S is algebraic, there exists an integer $C \in \mathbb{N}$ such that $a_n C^{n+1}$ is an integer for all n.) The fact that G-functions arise frequently in combinatorics was recently pointed out by Garoufalidis [20].

G-functions enjoy many remarkable properties. Chudnovsky [14] proved that the minimal order differential equation satisfied by a G-series must be *globally nilpotent* (see Section 2.4.4 below for the definition and an algorithmic use of this notion). By a theorem of Katz and Honda [22, 21], the global nilpotence of a differential operator implies that all of its singular points are *regular singular* points with *rational exponents*. See also [1, 13, 17] for more details on this topic.

Theorem 2 *Let $S(t)$ be one of the power series $F(t;0,0), F(t;1,0), F(t;0,1)$ and $F(t;1,1)$. If S is D-finite, then S is a G-series. In particular, its minimal order homogeneous linear differential equation is Fuchsian and it has only rational exponents. Moreover, the coefficient sequence of $S(t)$ is asymptotically equivalent to a sum of terms of the form $\kappa\rho^n n^\alpha (\log n)^\beta$ for some constants $\kappa \in \mathbb{R}$, $\alpha \in \mathbb{Q}$, $\rho \in \overline{\mathbb{Q}}$, and $\beta \in \mathbb{N}$.*

Proof: The conditions (a) and (c) in the definition of a G-function are clearly satisfied. The only non-trivial point is the fact that the series S has a positive radius of convergence in $\mathbb{C}$. This follows from Theorem 1. The Fuchsianity of the minimal equation for S, and the rationality of its exponents, follow by combining the results by Katz, Honda and Chudnovsky cited above. The claim on the asymptotics of the coefficients of $S(t)$ is a consequence of [20, Prop. 2.5]. □

For 3D walks, the definitions are analogous. The trivariate power series $F(t;x,y)$ is simply replaced by the generating series $G(t;x,y,z) \in \mathbb{Q}[x,y,z][[t]]$ of the sequence $g(n;i,j,k)$ that counts walks in $\mathbb{N}^3$ starting at $(0,0,0)$ and ending at $(i,j,k) \in \mathbb{N}^3$. Note that the appropriate versions of Theorems 1 and 2 hold; in particular, the generating series of octant walks $G(t;1,1,1)$ is a G-series whenever it is D-finite.

2.2 Computing large series expansions

The recurrence (1) can be used to determine the value of $f(n;i,j)$ for specific integers $n,i,j \in \mathbb{N}$. Theorem 1 implies that $f(n;i,j)$ is a non-negative integer whose bit size is at most $O(n)$. If $N \in \mathbb{N}$, the values $f(n;i,j)$

(i) The usual definition is more general, the coefficients of S can be taken in an arbitrary algebraic number field. For our purposes it is sufficient and convenient to restrict to rational coefficients.

for $0 \le n, i, j \le N$ can thus be computed altogether by a straightforward algorithm that uses $O(N^3)$ arithmetic operations and $\tilde{O}(N^4)$ bit operations. (We assume that two integers of bit-size N can be multiplied in $\tilde{O}(N)$ bit operations; here, the soft-O notation $\tilde{O}(\ )$ hides logarithmic factors.) The memory storage requirement is proportional to N^3. The same is also true for the truncated power series $F_N = F(t; x, y) \bmod t^N$. For our experiments in 2D, we have chosen $N = 1000$. With this choice, the computation of the $f(n; i, j)$ is the step which consumes by far the most computation time in our calculations.[ii]

Example 1 *The Kreweras walks satisfy the recurrence*

$$f(n+1, i, j) = f(n, i+1, j) + f(n, i, j+1) + f(n, i-1, j-1) \quad \textit{for} \quad n, i, j \ge 0,$$

which allows the computation of the first terms of the series $F(t; x, y)$

$$\begin{aligned} F(t; x, y) = 1 + xyt &+ (x^2y^2 + y + x)t^2 + (x^3y^3 + 2xy^2 + 2x^2y + 2)t^3 \\ &+ (x^4y^4 + 3x^2y^3 + 3x^3y^2 + 2y^2 + 6xy + 2x^2)t^4 \\ &+ (x^5y^5 + 4x^3y^4 + 4x^4y^3 + 5xy^3 + 12x^2y^2 + 5x^3y + 8y + 8x)t^5 + \cdots \end{aligned}$$

and also the first terms of the generating series $F(t; 1, 1)$ for the total number of Kreweras walks

$$\begin{aligned} F(t; 1, 1) = 1 + t + 3t^2 + 7t^3 + 17t^4 + 47t^5 + 125t^6 + 333t^7 + 939t^8 + 2597t^9 + \\ 7183t^{10} + 20505t^{11} + 57859t^{12} + 163201t^{13} + 469795t^{14} + \cdots \end{aligned}$$

In the 3D case, the values $g(n; i, j, k)$ for $0 \le n, i, j, k \le N$ can be computed in $O(N^4)$ arithmetic operations, $\tilde{O}(N^5)$ bit operations and $O(N^4)$ memory space. In practice, we found that computing $G \bmod t^N$ with $N = 400$ is feasible.

2.3 Guessing

Once the first terms of a power series are determined, our approach is to search systematically for candidates of linear differential equations or of algebraic equations which the series may possibly satisfy. This technique is classical in computer algebra and mathematical physics, see for example [11, 31, 28]. Differential and algebraic guessing procedures are available in some computer algebra systems like Maple and Mathematica.

2.3.1 Differential guessing

If the first N terms of a power series $S \in \mathbb{Q}[[t]]$ are available, one can search for a differential equation satisfied by S at precision N, that is, for an element $\mathcal{L}$ in the Weyl algebra $\mathbb{Q}[t]\langle D_t \rangle$ of differential operators in the derivation $D_t = \frac{d}{dt}$ with polynomial coefficients in t, such that

$$\mathcal{L}(S) = c_r(t)S^{(r)}(t) + \cdots + c_1(t)S'(t) + c_0(t)S(t) = 0 \bmod t^N. \tag{3}$$

Here, the coefficients $c_0(t), \ldots, c_r(t) \in \mathbb{Q}[t]$ are not simultaneously zero, and their degrees are bounded by a prescribed integer $d \ge 0$. By a simple linear algebra argument, if d and r are chosen such that $(d + 1)(r + 1) > N$, then such a differential equation always exists. On the other side, if d, r and N are such that $(d + 1)(r + 1) \ll N$, the equation (3) translates into a highly over-determined linear system, so it has no reason to possess a non-trivial solution.

The idea is that if the given power series $S(t)$ happens to be D-finite, then for a sufficiently large N, a differential equation of type (3) (thus satisfied a priori only at precision N) will provide a differential equation

[ii] We have carried out our computations on various different machines whose main memory ranges from 8 Gb to 32 Gb and which are equipped with (multiple) processors all running at about 3GHz.

which is really satisfied by $S(t)$ in $\mathbb{Q}[[t]]$ (i.e., at precision infinity). In other words, the D-finiteness of a power series can be (conjecturally) recognized using a finite amount of information.

Given the values d, r, N, and the first N terms of the series S, a candidate differential equation of type (3) for S can be computed by Gaussian elimination in $O(N^3)$ arithmetic operations and $\tilde{O}(N^4)$ bit operations. Actually, a modular approach is preferred to a direct Gaussian elimination over $\mathbb{Q}$. Precisely, the linear algebra step is performed modulo several primes p, and the results (differential operators modulo p) are recombined over $\mathbb{Q}$ via rational reconstruction based on an effective version of the Chinese remainder theorem. (See [23] for an implementation of this technique in Mathematica.)

If no differential equation is found, this definitely rules out the possibility that a differential equation of order r and degree d exists. This does not, however, imply that the series at hand is not D-finite. It may still be that the series satisfies a differential equation of order higher than r or an equation with polynomial coefficients of degree exceeding d.

Asymptotically more efficient guessing algorithms exist, based on fast Hermite-Padé approximation [4] of the vector of (truncated) power series $[S, S', \ldots, S^{(r)}]$; they have arithmetic complexity quadratic or even softly-linear in N. Such sophisticated algorithms were not needed to obtain the results of this paper, but they have provided crucial help in the treatment of examples of critical sizes (e.g. guessing with higher values of d, r, N and/or over a parametric base field like $\mathbb{Q}(x)$ instead of $\mathbb{Q}$) needed for the proof in [6].

Example 2 (continued) *$N = 100$ terms of the generating series $F(t;1,1)$ of the total number of Kreweras walks are sufficient to conjecture that $F(t;1,1)$ is D-finite, since it verifies the differential equation $\mathcal{L}_{1,1}(F(t;1,1)) = 0 \bmod t^N$, where*

$$\begin{aligned}\mathcal{L}_{1,1} = {} & 4t^2(t+1)(3t-4)(3t-1)^3(9t^2+3t+1)D_t^4 \\ & + 2t(3t-1)^2(2916t^5 - 1296t^4 - 3564t^3 - 477t^2 - 93t + 52)D_t^3 \\ & + 3(3t-1)(29808t^6 - 26244t^5 - 28440t^4 + 2754t^3 + 431t^2 + 448t - 40)D_t^2 \\ & + 6(68040t^6 - 88452t^5 - 37206t^4 + 16758t^3 + 954t^2 + 253t - 126)D_t \\ & + 18(6480t^5 - 8856t^4 - 3078t^3 + 714t^2 + 211t + 2).\end{aligned} \tag{4}$$

Thus, with high probability, $F(t;1,1)$ verifies the differential equation $\mathcal{L}_{1,1}(F(t;1,1)) = 0$.

Sometimes (see Section 2.4.4) one needs to guess the minimal-order differential equation $\mathcal{L}_{\min}(S) = 0$ satisfied by the given generating power series. Most of the time, the choice (d, r) of the target degree and order does not lead to this minimal operator. Worse, it may even happen that the number of initial terms N is not large enough to allow the recovery of $\mathcal{L}_{\min}$, while these N terms suffice to guess non-minimal order operators. (The explanation of why such a situation occurs systematically was given in [5], for the case of differential equations satisfied by algebraic functions.) A good heuristic is to compute several non-minimal operators and to take their greatest common right divisor; generically, the result is exactly $\mathcal{L}_{\min}$.

As a final general remark, let us point out that a power series satisfies a linear differential equation if and only if its coefficients satisfy a linear recurrence equation with polynomial coefficients. A recurrence equation can be computed either from a differential equation, or it can be guessed from scratch by proceeding analogously as described above for differential equations.

Example 3 (continued) *$N = 100$ terms of the series $S(t) = F(t;1,1)$ suffice to guess that its coefficients satisfy the order-6 recurrence*

$$\begin{aligned}& 2(n+6)(n+7)(2n+13)(7n+34)u_{n+6} - (n+6)(140n^3 + 2402n^2 + 13687n + 25843)u_{n+5} \\ & + 3(28n^4 + 626n^3 + 5123n^2 + 18281n + 24070)u_{n+4} \\ & - 18(n+4)(28n^3 + 311n^2 + 897n + 304)u_{n+3} + 108(n+3)(35n^3 + 443n^2 + 1787n + 2309)u_{n+2} \\ & - 324(n+2)(7n^3 + 90n^2 + 382n + 545)u_{n+1} - 972(n+1)(n+2)(n+4)(7n+41)u_n = 0.\end{aligned}$$

2.3.2 Algebraic guessing

If the first N terms of a power series $S \in \mathbb{Q}[[t]]$ are available, one can also search for an algebraic equation satisfied by S at precision N, that is, for a bivariate polynomial $P(T,t)$ in $\mathbb{Q}[T,t]$ such that

$$P(S(t),t) = c_r(t)S(t)^r + \cdots + c_1(t)S(t) + c_0(t) = 0 \bmod t^N. \tag{5}$$

A similar discussion shows that candidate algebraic equations of type (5) for S can be "guessed" by performing either Gaussian elimination or Hermite-Padé approximation on the vector $[1, S, \ldots, S^r]$, followed by a gcd computation in $\mathbb{Q}[T,t]$ applied to two (or more) different guesses.

Example 4 (continued) $N = 100$ *terms of the series* $S(t) = F(t;1,1)$ *counting the total number of Kreweras walks suffice to guess that* $F(t;1,1)$ *is very probably algebraic, namely solution of the bivariate polynomial*

$$\begin{aligned} P_{1,1}(T,t) = {} & t^5(3t-1)^3T^6 + 6t^4(3t-1)^3T^5 + t^3(3t-1)(135t^2-78t+14)T^4 \\ & + 4t^2(3t-1)(45t^2-18t+4)T^3 + t(3t-1)(135t^2-26t+9)T^2 \\ & + 2(3t-1)(27t^2-2t+1)T + 43t^2 + t + 2. \end{aligned} \tag{6}$$

2.4 Empirical certification of guesses

Once discovered a differential equation (3) or an algebraic equation (5) that the power series $S(t)$ seems to satisfy, we inspect several properties of these equations, in order to provide more convincing evidence that they are correct. These properties have various natures: some are computational features (moderate bit sizes), others are algebraic, analytic and even arithmetic properties. We check them systematically on all the candidates; if they are verified, as in the Kreweras example, this offers striking evidence that the guessed equations are not artefacts.

2.4.1 Size sieve: Reasonable bit size

The differential equation (3) has typically much lower bit size than a differential equation produced by the same guessing procedure applied to the same order, degree and precision, but to an arbitrary series having coefficients of bit-size comparable to that of $S(t)$. A similar observation holds for the algebraic equation (5).

Example 5 (continued) *If we perturb the coefficients of* $S(t) = F(t;1,1)$ *by just adding a random integer between* -100 *and* 100 *to each of its coefficients, then the differential guessing procedures at order* $r = 4$*, degree* $d = 9$ *and precision* $N = 100$ *will either give no result (the over-determined system approach) or produce fake candidates (the Hermite-Padé approach) with polynomial coefficients in* t*, whose coefficients in* $\mathbb{Q}$ *have numerators and denominators of about 500 decimal digits each, instead of 4 digits for* $\mathcal{L}_{1,1}$.

2.4.2 Algebraic sieve: High order series matching

The equations (3) and (5) were obtained starting from N coefficients of the power series $S(t)$. They are therefore satisfied a priori only modulo t^N. We compute more terms of $S(t)$, say $2N$, and check whether the same equations still hold modulo t^{2N}. If this is the case, chances increase that the guessed equations also hold at infinite precision.

2.4.3 Analytic sieve: Singularity analysis

By Theorem 2, the minimal order operators for power series like $S(t) = F(t;0,0)$ and $S(t) = F(t;1,1)$ must have only regular singularities (including the point at infinity) and their exponents must be rational numbers.

Example 6 (continued) *The differential operator* $\mathcal{L}_{1,1}$ *is Fuchsian. Indeed, a (fully automated) local singularity analysis shows that the set of its singular points* $\left\{-1, 0, \infty, \frac{1}{3}, \frac{4}{3}, -\frac{1}{6}(1 \pm i\sqrt{3})\right\}$ *is formed solely of regular singularities. Moreover, the indicial polynomials of* $\mathcal{L}_{1,1}$ *are, respectively:* $t(t-1)(t-2)(2t-1), t(t-1)(2t+1)(t+1), (t-5)(t-1)(t-2)(t-4), (t+1)t(4t-1)(4t+1), t(t-1)(t-2)(t-4)$*, and* $t(t-2)(2t-3)(t-1)$. *Their roots are the rational exponents of the singularities.*

2.4.4 Arithmetic sieve: G-series and global nilpotence

Last, but not least, we check an arithmetic property of the guessed differential equations by exploiting the fact that those expected to arise in our combinatorial context are very special.

Indeed, by a theorem due to the Chudnovsky brothers [14], the minimal order differential operator $\mathcal{L} \in \mathbb{Q}[t]\langle D_t\rangle$ killing a G-series enjoys a remarkable arithmetic property: $\mathcal{L}$ is *globally nilpotent*. By definition, this means that for almost every prime number p (i.e., for all with finitely many exceptions), there exists an integer $\mu \geq 1$ such that the remainder of the Euclidean (right) division of $D_t^{p\mu}$ by $\mathcal{L}$ is congruent to zero modulo p [21, 16].

From a computational view-point, a fine feature is that the nilpotence modulo p is checkable. If r denotes the order of $\mathcal{L}$, let M_p be the *p-curvature matrix of* $\mathcal{L}$, defined as the $r \times r$ matrix with entries in $\mathbb{Q}(t)$ whose (i,j) entry is the coefficient of D_t^{j-1} in the remainder of the Euclidean (right) division of D_t^{p+i-1} by $\mathcal{L}$. Then, $\mathcal{L}$ is nilpotent modulo p if and only if the matrix M_p is nilpotent modulo p [16, 32].

In combination with Theorem 2, this yields a fast algorithmic filter: as soon as we guess a candidate differential equation satisfied by a generating series which is suspected to be a G-series (e.g. by $F(t;1,1)$), we check whether its p-curvature is nilpotent, say modulo the first 50 primes for which the reduced operator $\mathcal{L} \bmod p$ is well-defined. If the p-curvature matrix of $\mathcal{L}$ is nilpotent modulo p for all those primes p, then the guessed equation is, with very high probability, the correct one.

We push even further this arithmetic sieving. A famous conjecture, attributed to Grothendieck, asserts that the differential equation $\mathcal{L}(S) = 0$ possesses a basis of *algebraic solutions* (over $\mathbb{Q}(x)$) if and only if its p-curvature matrix M_p *is zero modulo* p for almost all primes p. Even if the conjecture is, for the moment, fully proved only for order one operators and partially in the other cases [13], we freely use it as an oracle to detect whether a guessed differential equation has a basis of algebraic solutions. For instance, the computation of the p-curvature of an order 11 differential operator with polynomial coefficients of degree 96 in t, was one of the key points in our discovery [6] that the trivariate generating function for Gessel walks is algebraic.

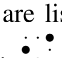
Example 7 (continued) *The* 5*-curvature matrix* $M_5(t)$ *of the differential operator* $\mathcal{L}_{1,1}$ *in* (4) *has the form* $\frac{1}{d(t)}\tilde{M}_5(t)$, *where* $d(t) = (3t-1)^7 t^6 (t+1)^5 (9t^2+3t+1)^5 (3t-4)$ *and* $\tilde{M}_5(t)$ *is a* 4×4 *matrix with polynomial entries in* $\mathbb{Q}[t]$ *of degree at most* 27*. The characteristic polynomial* χ_{M_5} *of* M_5 *reads*

$$T^4 + \frac{3\cdot 5}{2^5}\, N_3(t)\, t^5\, (3t-1)^{10}\, T^3 + \frac{3^3\cdot 5}{2^{10}}\, N_2(t)\, (3t-1)^5\, T^2 + \frac{3^5\cdot 5^2\cdot 7}{2^7}\, N_1(t)\, T + \frac{3^9\cdot 5^3\cdot 7^2}{2^3}\, N_0(t),$$

where N_0, N_1, N_2, N_3 *are irreducible polynomials in* $\mathbb{Z}[t]$, *of degree, respectively,* $21, 26, 26, 21$ *and with coefficients having at most 20 decimal digits.*

The polynomial χ_{M_5} *obviously equals* T^4 *modulo* $p = 5$, *so the* 5*-curvature of* $\mathcal{L}_{1,1}$ *is nilpotent (but not zero*[iii]*) modulo* 5*. In fact, for all the primes* $7 \leq p < 100$, *the* p*-curvature matrix of* $\mathcal{L}_{1,1}$ *is also nilpotent modulo* p*; it is even zero modulo* p*. Under the assumption that Grothendieck's conjecture is true, this indicates that* $\mathcal{L}_{1,1}$ *admits a basis of algebraic solutions, and so provides independent evidence that also* $S(t) = F(t;1,1)$ *is algebraic.*

3 Empirical Results in 2D

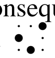
In this section, we consider the total number of walks only, i.e., the generating function $F(t;1,1)$. Because of symmetries, the 256 possible step sets give rise to 92 different sequences only. By inspection of the first $N = 1000$ terms, we found that 36 of them appear to be D-finite: 19 are algebraic and 17 are transcendental. The D-finite step sets, together with the sizes of the equations we discovered, are listed in Table 1 in the appendix. (There, and below, step sets are represented by compact pictograms, e.g.  for $\mathfrak{S} = \{\leftarrow, \nearrow, \downarrow\}$.)

[iii] Modulo 5, the curvature matrix $M_5(t)$ has T^2 as minimal polynomial.

3.1 Combinatorial Observations

Our classification matches the results of Bousquet-Mélou and Mishna [9]: for every sequence they prove D-finite our software found a recurrence and a differential equation, and whenever a series is algebraic indeed, our programs recognized it. Moreover, we found no recurrence or differential equation for any step set conjectured non-D-finite by Bousquet-Mélou and Mishna. This strengthens the evidence in favor of the conjectured non-D-finiteness of these cases.

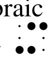
3.2 Algebraic Observations

All but two of the minimal polynomials of the algebraic series share the property that they define a curve of genus 0. As a consequence, there exists a rational parametrization in all these cases. For example, for the Kreweras step set , the minimal polynomial $P_{1,1}$ given in (6) defines a curve parameterized by

$$T(u) = \frac{(u^2+24u+151)a(u)}{(u+9)(u^2+24u+147)} \quad \text{and} \quad t(u) = \frac{2}{a(u)},$$

where $a(u) = \big(u^6+66u^5+1827u^4+27180u^3+229431u^2+1042866u+1995717\big)/(u+11)(u^2+22u+125)^2$, i.e., for these rational functions we have

$$P_{1,1}(T(u), t(u)) = 0.$$

The two algebraic series that do not admit a rational parametrization belong to the step sets  (reverse Kreweras) and  (Gessel's). Their genus is 1.

Another feature of the series which we found to be algebraic is that they all admit closed forms in terms of (nested) radical expressions. For example, for the Kreweras step set, we find that $F(t;1,1)$ is equal to

$$-\frac{1}{t} + \sqrt{\frac{\big(i-\sqrt{3}\big)\,(216t^3+1)\,(t-3t^2)^2 - 2it(36t^2-15t+1)a(t) + \big(i+\sqrt{3}\big)\,a(t)^2}{6it^3(3t-1)^3a(t)}}$$

where $i = \sqrt{-1}$ and $a(t) = \sqrt[3]{24\sqrt{3t^9(3t-1)^9\,(9t^2+3t+1)^3} - t^3(3t-1)^3\,(5832t^6+540t^3-1)}$. Such representations can be found by appealing to the built-in equation solvers of Maple and Mathematica applied to the equation $P_{1,1} = 0$. Both features are remarkable because, among all algebraic power series, only a few are rationally parameterizable or expressible in terms of radicals.

Also the transcendental D-finite series appear to have some special properties. Being D-finite, these series are annihilated by some linear differential operator

$$\mathcal{L} = c_0(t) + c_1(t)D_t + \cdots + c_r(t)D_t^r \in \mathbb{Q}[t]\langle D_t\rangle.$$

According to the DFactor command from Maple's DEtools package, all the operators can be factorized into a product of one irreducible operator of order 2 and several operators of order 1. As all the operators are globally nilpotent, so are all their factors [16, 17].

We can therefore expect that every solution of these factors can be written as a sum of terms of the form

$$R(t)^{\delta} \cdot {}_2F_1\left(\begin{matrix}\alpha \quad \beta \\ \gamma\end{matrix}\,\middle|\, Z(t)\right), \tag{7}$$

where R and Z are rational functions in $\mathbb{Q}(t)$ and $\alpha, \beta, \gamma, \delta$ are rational numbers. Indeed, Dwork [16, Item 7.4] has conjectured that any globally nilpotent second order differential equation has either algebraic solutions or is gauge equivalent to a weak pullback of a Gauss hypergeometric differential equation with rational parameters. This conjecture was disproved by Krammer [26] and recently by Dettweiler and Reiter [15]; the counter-examples given in these papers require involved tools in algebraic geometry (arithmetic triangle groups, systems associated to periods of Shimura curves, ...)

We are therefore in a win-win situation: either the second order operators appearing as factors of our operators admit only solutions which are indeed sums of terms of the form (7), or there is a simple combinatorial counter-example to Dwork's conjecture. Let us illustrate this on one of the most simple examples, the step set . We find here the differential operator

$$4(32t^2 - 12t - 1) + 4(8t - 1)(20t^2 - 3t - 1)D_t + t(4t - 1)(112t^2 - 5)D_t^2 + t^2(4t - 1)^2(4t + 1)D_t^3$$

which Maple factors into

$$\big(2(192t^3 - 56t^2 - 6t + 1) + 4(24t^2 - 1)(4t - 1)tD_t + (4t - 1)^2(4t + 1)t^2D_t^2\big)\big(1/t + D_t\big).$$

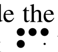
With the help of Maple's built-in differential equation solver (the dsolve command), it can be found that the differential operator gives rise to the representation

$$F(t;1,1) = -\frac{1}{4t} + \Big(1 + \frac{1}{4t}\Big)\,{}_2F_1\left(\begin{matrix}1/2 & 1/2\\ & 1\end{matrix}\,\middle|\, 16t^2\right).$$

(Incidentally, this solution can also be expressed in terms of elliptic functions.) We believe that all the transcendental D-finite generating functions for any step set admit a representation as (a nested integral of) such an expression. The solvers of Maple and Mathematica, however, are able to discover such a representation only in the simplest cases. (Note that at present, no complete algorithm is known that is capable of finding general pullback representations.)

3.3 Analytic Observations

By Theorem 2, all the coefficient sequences grow like $\kappa n^\alpha \rho^n \log(n)^\beta$ for some constants $\kappa, \rho, \alpha, \beta$ (we only care about the dominant part of their asymptotic expansions). From the differential equation or the recurrence equation, we can determine ρ, α, and β exactly as roots of characteristic polynomials and indicial equations, respectively. (See [33, 19] on how this is done.) We find that $\beta = 0$ in all cases. Knowing the recurrence, we can also compute easily tens of thousands of sequence terms. With the help of convergence acceleration techniques [12] applied to so many terms, it is possible to determine the remaining constant κ to an accuracy of thirty digits or more. With that many digits, it makes sense to search systematically for potential exact expressions of these constants using Plouffe's inverter [30] and/or algorithms like LLL and PSLQ [2]. We actually found "closed form" expressions for all these constants. They are included in Table 1 in the appendix.

By Theorem 1, the numbers ρ are bounded by the cardinality of the step set $\mathfrak{S}$. It turns out that $\rho = |\mathfrak{S}|$ unless the vector sum of the elements of the step set points outside the first quadrant. In these cases, ρ is an algebraic number of degree 2 (e.g., $\rho = 1 + 2\sqrt{2}$ for the step set ). For α, we found only non-positive numbers. Note that α being a negative integer implies that the corresponding series is transcendental [18].

All the constants κ have the form $u\rho^{e_0}\phi_1^{e_1}\phi_2^{e_2}\cdots\phi_r^{e_r}$, where the ϕ_i are usually small integers, the e_i are rational numbers, and u is $1/\pi$ if $F(t;1,1)$ is transcendental, and $1/\Gamma(\alpha + 1)$ if $F(t;1,1)$ is algebraic.

There are some cases where the ϕ_i are not integers. Among them, very strange is only the case of the step set , for which we found $r = 1$, $e_0 = 7/2, e_1 = 1/2$, $\rho = 2 + 2\sqrt{6}$ and $\phi_1 = (1137 + 468\sqrt{6})/152000$. This last number may look like a guessing artefact at first glance, but we trust in its correctness, because the number of correct digits exceeds by far the number of correct digits to be expected from an artefact.

4 Empirical Results in 3D

We have investigated walks in three dimensions confined to the first octant with step sets of up to five elements. A priori, there are 83682 such step sets, and they give rise to 3334 different sequences. Of those, we have computed the first $N = 400$ terms of the generating function $G(t;1,1,1)$ of general walks, and searched for potential differential equations, algebraic equations, and recurrence equations. We found that 134 sequences appear to be D-finite, and among those, 50 appear to be algebraic.

4.1 Combinatorial Observations

For some of the sequences, it can be realized that their D-finiteness or algebraicity is a consequence of the D-finiteness or algebraicity of a certain 2D walk. For example, the sequence corresponding to the step set

 $1, 4, 17, 75, 339, 1558, 7247, 34016, 160795, 764388, \ldots$ (A026378)

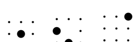
is readily seen to be D-finite, since it may be regarded as a variation of the 2D step set  in which the step $\uparrow$ appears in two copies and empty steps are allowed. (Here and below, a three dimensional step set is depicted in three separate slices: first the arrows tops of the forms $(x,y,-1)$, then $(x,y,0)$, then $(x,y,1)$. For example, the step set above is $\{(-1,0,0),\ (0,1,0),\ (1,0,0),\ (0,0,1),\ (0,1,1)\}$. The given numbers are the first coefficients in the expansion of $G(t;1,1,1)$.)

Discarding these cases from consideration, we are left with 35 different sequences whose generating series appear to be D-finite; among those, three appear to be algebraic. Their step sets are given in the appendix.

We were not able to find an equation for the step set

 $1, 1, 4, 7, 28, 70, 280, 787, 3148, 9526, 38104, \ldots$ (A149080)

which is symmetric about all three axes, not even with 800 terms instead of 400. Also the step set

 $1, 1, 4, 13, 40, 136, 496, 1753, 6256, 22912, 85216, \ldots$ (A149424)

which enjoys a rotational symmetry about the middle line of the first octant, and which may be viewed as a three dimensional analogue of Kreweras's step set, appears to be non-D-finite, even when 800 terms are taken into account.

For walks in the quarter plane, it is conjectured in [29, Section 3] that D-finiteness is preserved under reversing arrows, i.e., the generating function for a step set $\mathfrak{S}$ is D-finite if and only if the generating function for the step set $\mathfrak{S}'$ is, when $\mathfrak{S}'$ is obtained from $\mathfrak{S}$ by reversing all arrows. Our computations do not suggest that this criterion also applies in 3D. Among the 134 sequences we found D-finite, there are 42 which correspond to step sets in $\mathfrak{S}$ for whose counterpart in $\mathfrak{S}'$ we were not able to find an equation. Among those, there are some which satisfy only very large equations, so that chances are that they remain D-finite upon reversing arrows, but with equations which are too large for us to find. Others satisfy quite small equations, for example the sequence A026378 whose step set is given above.

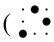
4.2 Algebraic Observations

As in the 2D case, it turns out that most of the minimal polynomials of the algebraic series define curves of genus 0, which therefore can be rationally parameterized. There are twelve cases of genus 1, these are elliptic curves. Some of them turn out to be isomorphic (over $\overline{\mathbb{Q}}$). For example, those corresponding to the cases

 $1, 1, 4, 11, 32, 110, 360, 1163, 4112, 14066, 47848, \ldots$ (A149232)

 $1, 2, 7, 27, 105, 426, 1787, 7590, 32633, 142152, 624659, \ldots$ (A150591)

 $1, 2, 10, 40, 176, 808, 3720, 17152, 81440, 384448, \ldots$ (A151023)

all have 1728 as j-invariant. They originate from the 2D Kreweras walks. Most interestingly, there are also three step sets originating from the 2D reverse Kreweras walks () for which the genus is 5 (!).

For the transcendental series, we could observe the same phenomenon as in 2D: all the operators factor as a product of a single irreducible operator of order two and several operators of order one. We therefore expect again that all these series admit a representation as a hypergeometric pullback. As an example, the generating function $G(t; 1, 1, 1)$ of the sequence

 $1, 1, 2, 4, 10, 25, 70, 196, 588, 1764, \ldots$ (A005817)

can be written in the form

$$\frac{4t+1}{2t}\,{}_2F_1\left(\left.\begin{matrix}1/2 & 1/2\\ & 3\end{matrix}\,\right|\,16t^2\right) - \frac{2t-1}{4t^3}\,{}_2F_1\left(\left.\begin{matrix}-1/2 & -1/2\\ & 2\end{matrix}\,\right|\,16t^2\right) - \frac{4t^2-2t+1}{4t^3}.$$

This representation was found by Mark van Hoeij. It is beyond the scope of the standard tools of Maple or Mathematica.

4.3 Analytic Observations

Also concerning asymptotics, similar remarks apply as in 2D. All coefficient sequences grow like $\kappa n^\alpha \rho^n$ for some constants κ, α, ρ, where ρ is an integer or an algebraic number of degree 2 and α is a non-positive number. We have not gone through the laborious task of determining the constants κ.

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Appendix

Table 1 D-finite series and their step sets in 2D. The equation sizes columns refer to (minimal) recurrence equation, differential equation, and algebraic equation, respectively. Example: The series $F(t;1,1)$ for Kreweras walks (A151265) satisfies a differential equation of order 4 with polynomial coefficients of degree 9 and an algebraic equation $P(F(t;1,1),t)=0$ for a polynomial $P(T,t)$ of degree 6 in T and 8 in t. The coefficient sequence of $F(t;1,1)$ satisfies a recurrence equation of order 6 with polynomial coefficients of degree 4. The labels used in the columns "OEIS Tag" are taken from Sloane's On-Line Encyclopedia of Integer Sequences `http://www.research.att.com/~njas/sequences/`. Constants in the asymptotics columns are abbreviated $A=1+\sqrt{2}$, $B=1+2\sqrt{2}$, $C=1+\sqrt{3}$, $D=1+2\sqrt{3}$, $E=\sqrt{6(379+156\sqrt{6})}$ (!), $F=1+\sqrt{6}$.

OEIS Tag	Steps	Equation sizes			Asymptotics	OEIS Tag	Steps	Equation sizes			Asymptotics
A000012		1, 0	1, 1	1, 1	1	A000079		1, 0	1, 1	1, 1	2^n
A001405		2, 1	2, 3	2, 2	$\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n}{\sqrt{n}}$	A000244		1, 0	1, 1	1, 1	3^n
A001006		2, 1	2, 3	2, 2	$\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{n^{3/2}}$	A005773		2, 1	2, 3	2, 2	$\frac{\sqrt{3}}{\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$
A126087		3, 1	2, 5	2, 2	$\frac{12\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^{3n/2}}{n^{3/2}}$	A151255		6, 8	4, 16	–	$\frac{24\sqrt{2}}{\pi}\frac{2^{3n/2}}{n^2}$
A151265		6, 4	4, 9	6, 8	$\frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$	A151266		7, 10	5, 16	–	$\frac{\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{3^n}{\sqrt{n}}$
A151278		7, 4	4, 12	6, 8	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(\frac{1}{4})}\frac{3^n}{n^{3/4}}$	A151281		3, 1	2, 5	2, 2	$\frac{1}{2}3^n$
A005558		2, 3	3, 5	–	$\frac{8}{\pi}\frac{4^n}{n^2}$	A005566		2, 2	3, 4	–	$\frac{4}{\pi}\frac{4^n}{n}$
A018224		2, 3	3, 5	–	$\frac{2}{\pi}\frac{4^n}{n}$	A060899		2, 1	2, 3	2, 2	$\frac{\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$
A060900		2, 3	3, 5	8, 9	$\frac{4\sqrt{3}}{3\Gamma(\frac{1}{3})}\frac{4^n}{n^{2/3}}$	A128386		3, 1	2, 5	2, 2	$\frac{6\sqrt{2}}{\Gamma(\frac{1}{2})}\frac{2^n 3^{n/2}}{n^{3/2}}$
A129637		3, 1	2, 5	2, 2	$\frac{1}{2}4^n$	A151261		5, 8	4, 15	–	$\frac{12\sqrt{3}}{\pi}\frac{2^n 3^{n/2}}{n^2}$
A151282		3, 1	2, 5	2, 2	$\frac{A^2B^{3/2}}{2^{3/4}\Gamma(\frac{1}{2})}\frac{B^n}{n^{3/2}}$	A151291		6, 10	5, 15	–	$\frac{4}{3\Gamma(\frac{1}{2})}\frac{4^n}{\sqrt{n}}$
A151275		9, 18	5, 24	–	$\frac{12\sqrt{30}}{\pi}\frac{(\sqrt{24})^n}{n^2}$	A151287		7, 11	5, 19	–	$\frac{\sqrt{8}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
A151292		3, 1	2, 5	2, 2	$\frac{\sqrt[4]{3}C^2D^{3/2}}{8\Gamma(\frac{1}{2})}\frac{D^n}{n^{3/2}}$	A151302		9, 18	5, 24	–	$\frac{\sqrt{5}}{3\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$
A151307		8, 15	5, 20	–	$\frac{\sqrt{5}}{2\sqrt{2}\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$	A151318		2, 1	2, 3	2, 2	$\frac{\sqrt{5/2}}{\Gamma(\frac{1}{2})}\frac{5^n}{\sqrt{n}}$
A129400		2, 1	2, 3	2, 2	$\frac{3\sqrt{3}}{2\Gamma(\frac{1}{2})}\frac{6^n}{n^{3/2}}$	A151297		7, 11	5, 18	–	$\frac{\sqrt{3}C^{7/2}}{2\pi}\frac{(2C)^n}{n^2}$
A151312		4, 5	3, 8	–	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	A151323		2, 1	2, 3	4, 4	$\frac{\sqrt{2}\,3^{3/4}}{\Gamma(\frac{1}{4})}\frac{6^n}{n^{3/4}}$
A151326		7, 14	5, 18	–	$\frac{2\sqrt{3}}{3\Gamma(\frac{1}{2})}\frac{6^n}{\sqrt{n}}$	A151314		9, 18	5, 24	–	$\frac{EF^{7/2}}{5\sqrt{95}\pi}\frac{(2F)^n}{n^2}$
A151329		9, 18	5, 24	–	$\frac{\sqrt{7/3}}{3\Gamma(\frac{1}{2})}\frac{7^n}{\sqrt{n}}$	A151331		3, 4	3, 6	–	$\frac{8}{3\pi}\frac{8^n}{n}$

Table 2 Conjecturally algebraic series and their step sets in 3D. Step set figures are as in Section 4. Equation sizes are as in Table 1.

First terms (OEIS Tag)		Step sets		Equation sizes		
1, 1, 4, 10, 37, 121, 451, 1639, …	(A025237)			2, 1	2, 3	2, 2
1, 1, 5, 15, 51, 199, 755, 2789, …	(A149576)			11, 22	7, 31	12, 17
1, 2, 4, 14, 46, 134, 502, 1820, …	(A149847)			8, 6	4, 16	6, 9

Table 3 Conjecturally transcendental D-finite generating series and their step sets in 3D. The equation sizes columns refer to (minimal) recurrence equations, and differential equations, respectively.

OEIS Tag	Step sets	Equation sizes		OEIS Tag	Step sets	Equation sizes	
A148060		9, 17	5, 28	A148438		7, 10	5, 17
				A149090		9, 17	5, 28
A149589		10, 21	6, 29				
A005817		2, 2	3, 4	A148005		5, 8	4, 15
A148052		7, 18	6, 27	A148068		7, 17	6, 25
A148072		12, 57	10, 69	A148162		4, 3	3, 6
A148284		14, 57	10, 71	A148331		11, 43	9, 53
A148507		4, 6	4, 11	A148525		7, 16	6, 25
A148548		7, 19	6, 28	A148689		8, 25	8, 31
A148703		4, 3	3, 6	A148790		6, 12	5, 18
A148934		5, 5	4, 11	A149279		14, 62	10, 75
A149290		11, 53	9, 61	A149363		7, 16	6, 24
A149632		7, 11	5, 16	A149713		8, 22	7, 29
A150054		12, 39	9, 52	A150370		14, 62	10, 75
A150410		4, 6	4, 11	A150471		12, 33	8, 42
A150499		14, 48	9, 61	A150764		7, 13	6, 19
A150950		8, 23	7, 29	A151053		14, 38	9, 48

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Unlabeled $(\mathbf{2}+\mathbf{2})$-free posets, ascent sequences and pattern avoiding permutations

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Abstract. We present statistic-preserving bijections between four classes of combinatorial objects. Two of them, the class of unlabeled $(\mathbf{2}+\mathbf{2})$-free posets and a certain class of chord diagrams (or involutions), already appeared in the literature, but were apparently not known to be equinumerous. The third one is a new class of pattern avoiding permutations, and the fourth one consists of certain integer sequences called *ascent sequences.*

We also determine the generating function of these classes of objects, thus recovering a non-D-finite series obtained by Zagier for chord diagrams. Finally, we characterize the ascent sequences that correspond to permutations avoiding the barred pattern $3\bar{1}52\bar{4}$, and enumerate those permutations, thus settling a conjecture of Pudwell.

Résumé. Nous présentons des bijections, transportant de nombreuses statistiques, entre quatre classes d'objets. Deux d'entre elles, la classe des EPO (ensembles partiellement ordonnés) sans motif $(\mathbf{2}+\mathbf{2})$ et une certaine classe d'involutions, sont déjà apparues dans la littérature. La troisième est une classe de permutations à motifs exclus, et la quatrième une classe de suites que nous appelons *suites à montées*.

Nous déterminons ensuite la série génératrice de ces classes, retrouvant ainsi un résultat prouvé par Zagier pour les involutions sus-mentionnées. La série obtenue n'est pas D-finie. Apparemment, le fait qu'elle compte aussi les EPO sans motif $\mathbf{2}+\mathbf{2}$ est nouveau. Finalement, nous caractérisons les suites à montées qui correspondent aux permutations évitant le motif barré $3\bar{1}52\bar{4}$ et énumérons ces permutations, ce qui démontre une conjecture de Pudwell.

Keywords: $(\mathbf{2}+\mathbf{2})$-free poset, interval order, pattern-avoidance, enumeration, ascent sequence, kernel method.

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1 Introduction

This paper presents correspondences between four seemingly unrelated structures; unlabeled $(\mathbf{2}+\mathbf{2})$-free posets on n elements, certain sequences of n nonnegative integers called *ascent sequences*, a new class of permutations on n letters, and finally certain involutions on $2n$ points.

A poset is said to be $(\mathbf{2}+\mathbf{2})$*-free* if it does not contain an induced subposet that is isomorphic to $\mathbf{2}+\mathbf{2}$, the union of two disjoint 2-element chains. Fishburn [6] showed that a poset is $(\mathbf{2}+\mathbf{2})$-free precisely when it is isomorphic to an interval order. Another characterization is that a poset is $(\mathbf{2}+\mathbf{2})$-free if and only if the collection of strict principal down-sets can be linearly ordered by inclusion [5; 4].

Our *ascent sequences* have a simple recursive definition, given in Section 2. We also define there the class of permutations we consider: they avoid a particular pattern of length three, but this type of pattern is new, in the sense that it does not admit an expression in terms of the dashed(i) patterns introduced by Babson and Steingrímsson [1]. It is our hope that the results of this paper will stimulate research into these new patterns. We show how to deconstruct these permutations element by element, and how this gives a bijection with ascent sequences. In Section 3 we perform a similar task for $(\mathbf{2}+\mathbf{2})$-free posets.

In Section 4 we present a simple algorithm that given an ascent sequence x computes what we call the modified ascent sequence, denoted $\widehat{x}$. Some of the properties of the permutation and the poset corresponding to x are more easily read from $\widehat{x}$ than from x. We also explain how to go directly from a given poset to the corresponding permutation as opposed to via the ascent sequence. As an additional application, we show that the fixed points under $x \mapsto \widehat{x}$ are in one-to-one correspondence with permutations avoiding the barred pattern $3\bar{1}52\bar{4}$. We count ascent sequences that are left unchanged by the map $x \mapsto \widehat{x}$, thus proving a conjecture of Lara Pudwell on the number of $3\bar{1}52\bar{4}$-avoiding permutations.

In Section 5 we present statistics on the objects that are preserved under the stated bijections. In Section 6, we determine the generating function of ascent sequences (and thus, of $(\mathbf{2}+\mathbf{2})$-free posets and pattern avoiding permutations), which turns out to be a rather complicated, non-D-finite series. This series has already been shown by Zagier [13] to count certain *chord diagrams*, or involutions, introduced by Stoimenow [12] to give upper bounds on the dimension of the space of Vassiliev's knot invariants of a given degree. In Section 7 we give a new proof of this result by establishing a bijection between these involutions and $(\mathbf{2}+\mathbf{2})$-free posets.

The proofs are omitted in this abstract, but can be found in the full version of the paper [2].

2 Ascent sequences and pattern avoiding permutations

Let $(x_1, \dots, x_i)$ be an integer sequence. The number of *ascents* of this sequence is

$$\mathrm{asc}(x_1, \dots, x_i) = |\{\, 1 \le j < i \,:\, x_j < x_{j+1} \,\}|.$$

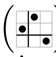
Let us call a sequence $x = (x_1, \dots, x_n) \in \mathbb{N}^n$ an *ascent sequence of length* n if it satisfies $x_1 = 0$ and $x_i \in [0, 1+\mathrm{asc}(x_1, \dots, x_{i-1})]$ for $2 \le i \le n$. For instance, (0, 1, 0, 2, 3, 1, 0, 0, 2) is an ascent sequence. The length (number of entries) of a sequence x is denoted $|x|$.

Let $\mathcal{S}_n$ be the symmetric group on n elements. Let $V = \{v_1, v_2, \dots, v_n\}$ with $v_1 < v_2 < \cdots < v_n$ be any finite subset of $\mathbb{N}$. The *standardisation* of a permutation π on V is the permutation $\mathrm{std}(\pi)$

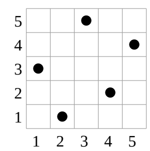
(i) Babson and Steingrímsson call these patterns "generalized" rather than "dashed", but we wish to promote a change of terminology here, since "dashed" is more descriptive.

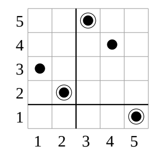
on $[n] := \{1, 2, \ldots, n\}$ obtained from π by replacing the letter v_i with the letter i. As an example, $\mathrm{std}(19452) = 15342$. Let $\mathcal{R}_n$ be the following set of permutations:

$$\mathcal{R}_n = \{\, \pi_1 \ldots \pi_n \in \mathcal{S}_n : \text{ if } \mathrm{std}(\pi_i\pi_j\pi_k) = 231 \text{ then } j \neq i+1 \text{ or } \pi_i \neq \pi_k + 1 \,\}.$$

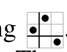
Equivalently, if $\pi_i\pi_{i+1}$ forms an ascent, then $\pi_i - 1$ is not found to the right of this ascent. This class of permutations could be more descriptively written as $\mathcal{R}_n = \mathcal{S}_n\Big($$\Big)$, the set of permutations avoiding the pattern in the diagram. Dark lines indicate adjacent entries (horizontally or vertically) whereas lighter lines indicate an elastic distance between the entries.

As illustrated here, the permutation 31524 avoids the pattern  while the permutation 32541 does not.





Consider the following three symmetries of a square: reflection in a centered vertical line, reflection in a centered horizontal line, and reflection in the diagonal $x = y$. In the context of permutations these operations are known as *reverse*, *complement* and *inverse*, respectively. Together they generate the dihedral group D_8, the symmetry group of a square. This is the symmetry of classical patterns. The dashed patterns of Babson and Steingrímsson [1] can be seen as those patterns that allow dark vertical (but not horizontal) lines in their diagram. That set of patterns is not closed under inverse: under reflection in the diagonal $x = y$ a (dark) vertical line turns into a (dark) horizontal line. Thus dashed patterns only enjoy the symmetry of a rectangle. Our patterns provide the minimal extra generality needed to contain the dashed patterns and have the full symmetry of a square.

Let us return to the set $\mathcal{R} := \cup_n \mathcal{R}_n$ of permutations avoiding . Let π be a permutation of $\mathcal{R}_n$, with $n > 0$. Let τ be obtained by deleting the entry n from π. Then $\tau \in \mathcal{R}_{n-1}$. Indeed, if $\tau_i\tau_{i+1}\tau_j$ is an occurrence of the forbidden pattern in τ (but not in π), then this implies that $\pi_{i+1} = n$. But then $\pi_i\pi_{i+1}\pi_{j+1}$ would form an occurrence of the forbidden pattern in π.

This property allows us to construct the permutations of $\mathcal{R}_n$ inductively, starting from the empty permutation and adding a new maximal value at each step. Given $\tau = \tau_1 \ldots \tau_{n-1} \in \mathcal{R}_{n-1}$, the sites where n can be inserted in τ so as to produce an element of $\mathcal{R}_n$ are called *active*. It is easily seen that the site before τ_1 and the site after τ_{n-1} are always active. The site between the entries τ_i and τ_{i+1} is active if and only if $\tau_i = 1$ or $\tau_i - 1$ is to the left of τ_i. Label the active sites, from left to right, with labels 0, 1, 2...

Our bijection Λ between permutations of $\mathcal{R}_n$ and ascent sequences of length n is defined recursively on n as follows. For $n = 1$, we set $\Lambda(1) = (0)$. Now let $n \geq 2$, and suppose that $\pi \in \mathcal{R}_n$ is obtained by inserting n in the active site labeled i of a permutation $\tau \in \mathcal{R}_{n-1}$. Then the sequence associated with π is $\Lambda(\pi) := (x_1, \ldots, x_{n-1}, i)$, where $(x_1, \ldots, x_{n-1}) = \Lambda(\tau)$.

Example 1 *The permutation $\pi = 61832547$ corresponds to the sequence $x = (0, 1, 1, 2, 2, 0, 3, 1)$, since it is obtained by the following insertions (the subscripts indicate the labels of the active sites):*

$$_0 1_1 \xmapsto{x_2=1} {}_0 1_1 2_2 \xmapsto{x_3=1} {}_0 1_1 3\ 2_2 \xmapsto{x_4=2} {}_0 1_1 3\ 2_2 4_3 \xmapsto{x_5=2} {}_0 1_1 3\ 2_2 5\ 4_3$$
$$\xmapsto{x_6=0} {}_0 6\ 1_1 3\ 2_2 5\ 4_3 \xmapsto{x_7=3} {}_0 6\ 1_1 3\ 2_2 5\ 4_3 7_4 \xmapsto{x_8=1} \quad 6\ 1\ 8\ 3\ 2\ 5\ 4\ 7.$$

Theorem 2 *The map Λ is a bijection from $\mathcal{R}_n$ to the set of ascent sequences of length n.*

The proof proceeds by induction. The key is to understand how the number of actives sites of π, and the label located just before its maximal entry, can be read in the ascent sequence.

3 Ascent sequences and unlabeled $(\mathbf{2}+\mathbf{2})$-free posets

Let $\mathcal{P}_n$ be the set of unlabeled $(\mathbf{2}+\mathbf{2})$-free posets on n elements. In this section we shall give a bijection between $\mathcal{P}_n$ and the set $\mathcal{A}_n$ of ascent sequences of length n. As in the previous section, this bijection encodes a recursive way of decomposing $(\mathbf{2}+\mathbf{2})$-free posets by removing one maximal element. This removal procedure is less elementary than in the case of permutations. Before giving these operations we need to introduce some terminology.

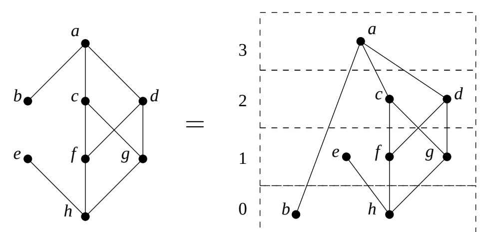
Let $D(x) = \{ y : y < x \}$ be the set of *predecessors* of x (the strict down-set of x). It is well-known—see for example Khamis [8]—that a poset is $(\mathbf{2}+\mathbf{2})$-free if and only if the set $\{D(x) : x \in P\}$ can be linearly ordered by inclusion. Let

$$D(P) = \{D_0, D_1, \ldots, D_k\}$$

with $\emptyset = D_0 \subset D_1 \subset \cdots \subset D_k$. In this context we define $D_i(P) = D_i$ and we write $\ell(P) = k$. We say the element x is at level i in P if $D(x) = D_i$ and we write $\ell(x) = i$. The set of all elements at level i we denote $L_i(P) = \{ x \in P : \ell(x) = i \} = \{ x \in P : D(x) = D_i \}$. For instance, $L_0(P)$ is the set of minimal elements. All the elements of $L_k(P)$ are maximal, but there may be maximal elements of P at level less than k. If $L_i(P)$ contains a maximal element, we say that *the level* i contains a maximal element. Let $\ell^\star(P)$ be the minimum level containing a maximal element.

Example 3

Consider the following $(\mathbf{2}+\mathbf{2})$-free poset P, which we have labeled for convenience. The diagram on the right shows the poset redrawn according to the levels of the elements. We have $D(a) = \{b, c, d, f, g, h\}$, $D(b) = \emptyset$, $D(c) = D(d) = \{f, g, h\}$, $D(e) = D(f) = D(g) = \{h\}$ and $D(h) = \emptyset$. These may be ordered by inclusion as



$$\underbrace{D(h) = D(b)}_{\ell(h)=\ell(b)=0} \subset \underbrace{D(e) = D(f) = D(g)}_{\ell(e)=\ell(f)=\ell(g)=1} \subset \underbrace{D(c) = D(d)}_{\ell(c)=\ell(d)=2} \subset \underbrace{D(a)}_{\ell(a)=3}.$$

Thus $\ell(P) = 3$. The maximal elements of P are e and a, and they lie respectively at levels 3 and 1. Thus $\ell^\star(P) = 1$. In addition, $D_0 = \emptyset$, $D_1 = \{h\}$, $D_2 = \{f, g, h\}$ and $D_3 = \{b, c, d, f, g, h\}$. With $L_i = L_i(P)$ we also have $L_0 = \{h, b\}$, $L_1 = \{e, f, g\}$, $L_2 = \{c, d\}$ and $L_3 = \{a\}$.

3.1 Removing an element from a $(\mathbf{2}+\mathbf{2})$-free poset

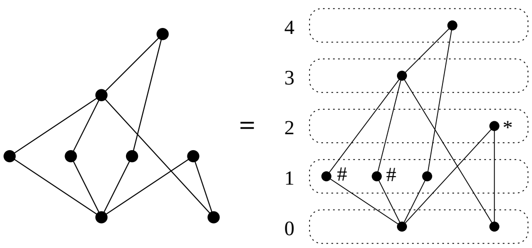
The removal operation will be the counterpart of the deletion of the last entry in an ascent sequence (or the deletion of the largest entry in a permutation of $\mathcal{R}$). Let P be a $(\mathbf{2}+\mathbf{2})$-free poset of cardinality $n \geq 2$, and let $i = \ell^\star(P)$ be the smallest level of P containing a maximal element. All the maximal elements located at level i are order-equivalent in the unlabeled poset P. We will remove one of them. Let Q be the poset that results from applying:

(Rem1) If $|L_i(P)| > 1$ then simply remove one of the maximal elements at level i.

(Rem2) If $|L_i(P)| = 1$ and $i = \ell(P)$ then remove the unique element lying at level i.

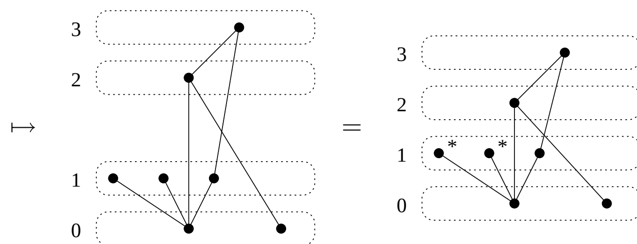
(Rem3) If $|L_i(P)| = 1$ and $i < \ell(P)$ then set $\mathcal{N} = D_{i+1}(P) \setminus D_i(P)$. Make each element of $\mathcal{N}$ a maximal element by deleting from the order all relations $x < y$ where $x \in \mathcal{N}$. Finally, remove the unique element lying at level i.

Example 4 *Let P be the unlabeled $(\mathbf{2}+\mathbf{2})$-free poset with the following Hasse diagram.*

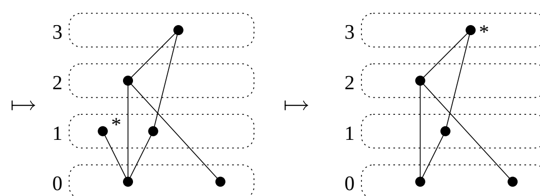


The second diagram shows the poset redrawn according to the levels of the elements. There is a unique maximal element of minimal level, which is marked with $*$, and $\ell^\star(P) = 2$. Since $2 < \ell(P)$, apply Rem3 to remove this maximal element. The elements of $\mathcal{N}$ are indicated by #'s.

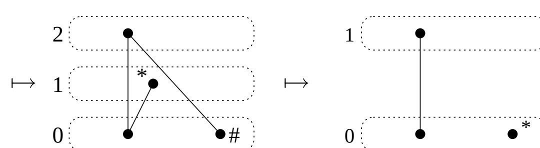
In order to delete all relations of the form $x \leq y$ where $x \in \mathcal{N}$, one deletes all edges corresponding to coverings of elements of $\mathcal{N}$, and adds an edge between the elements at level 0 and 3 to preserve their relation. Finally, one removes the element at level 2. This gives a new $(\mathbf{2}+\mathbf{2})$-free poset, with level numbers shown on the right.



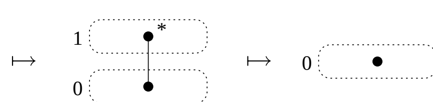
There are now two maximal elements of minimal level $\ell^\star = 1$, both marked by $*$. Remove one of them according to rule Rem1. This gives the first poset shown to the right, for which $\ell^\star$ is still 1. Apply Rem1 again to obtain the second poset on the right.



There is now a single maximal element, lying at maximal level 3, so we apply rule Rem2. In the poset thus obtained, $\ell^\star(P) = 1 < \ell(P)$ and there is a unique element at level 1, so apply Rem3. The set $\mathcal{N}$ consists of the rightmost point at level 0.



In the new poset, the star element is not alone at level 0, so apply Rem1, and finally Rem2.



We have thus reduced P to a one element poset by removing the elements in a canonical order.

3.2 From $(\mathbf{2}+\mathbf{2})$-free posets to ascent sequences

Our bijection Ψ between $(\mathbf{2}+\mathbf{2})$-free posets of cardinality n and ascent sequences of length n is defined recursively on n as follows. For $n=1$, we associate with the one-element poset the sequence (0). Now let $n \geq 2$, and suppose that the removal operation, applied to $P \in \mathcal{P}_n$, gives the poset Q. Then the sequence associated with P is $\Psi(P) := (x_1, \dots, x_{n-1}, i)$, where $i = \ell^\star(P)$ and $(x_1, \dots, x_{n-1}) = \Psi(Q)$.

For instance, the poset of Example 4 corresponds to the sequence $(0,1,0,1,3,1,1,2)$.

Theorem 5 *The map Ψ is a one-to-one correspondence between $(\mathbf{2}+\mathbf{2})$-free posets of size n and ascent sequences of length n.*

4 Modified ascent sequences and their applications

In this section we introduce a transformation on ascent sequences and show some applications. For instance, this transformation can be used to give a non-recursive description of the bijection Λ between permutations of $\mathcal{R}$ and ascent sequences. It is also useful to characterize the image by Λ of a subclass of $\mathcal{R}$, which we will enumerate in Subsection 4.4. We also describe how to transform $(\mathbf{2}+\mathbf{2})$-free posets into permutations without resorting to ascent sequences.

4.1 Modified ascent sequences

Let $x = (x_1, x_2, \dots, x_n)$ be any finite sequence of integers. Define

$$\mathfrak{asc}(x) = \big(\, i : i \in [n-1] \text{ and } x_i < x_{i+1} \,\big);$$

so $\operatorname{asc}(x) = |\mathfrak{asc}(x)|$. In terms of an algorithm we shall now describe a function from integer sequences to integer sequences. Let $x = (x_1, x_2, \dots, x_n)$ be the input sequence. Do

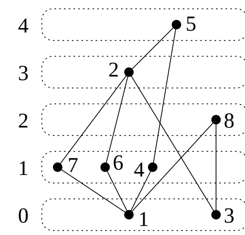
```
for i ∈ asc(x):
    for j ∈ [i − 1]:
        if x_j ≥ x_{i+1} then x_j := x_j + 1
```

and denote the resulting sequence by $\widehat{x}$. Assuming that x is an ascent sequence we call $\widehat{x}$ the *modified ascent sequence*. As an example, consider the ascent sequence $x = (0,1,0,1,3,1,1,2)$. We have $\mathfrak{asc}(x) = (1,3,4,7)$ and the algorithm computes the modified ascent sequence $\widehat{x}$ in the following steps:

$$\begin{array}{rcl} x & = & 0\ \mathbf{1}\ 0\ 1\ 3\ 1\ 1\ 2 \\ & & 0\ 1\ 0\ \mathbf{1}\ 3\ 1\ 1\ 2 \\ & & 0\ 2\ 0\ 1\ \mathbf{3}\ 1\ 1\ 2 \\ & & 0\ 2\ 0\ 1\ 3\ 1\ 1\ \mathbf{2} \\ & & 0\ 3\ 0\ 1\ 4\ 1\ 1\ 2\ =\ \widehat{x} \end{array}$$

In each step every element strictly to the left of and weakly larger than the boldface letter is incremented by one. Observe that the positions of ascents in x and $\widehat{x}$ coincide, and that the number of ascents in x (or $\widehat{x}$) is $\operatorname{asc}(x) = \operatorname{asc}(\widehat{x}) = \max(\widehat{x})$. The above procedure is easily invertible and the map $x \mapsto \widehat{x}$ is therefore injective.

The modified ascent sequence $\widehat{x}$ is related to the level distribution of the poset P associated with x. First, observe that the removal operation of Section 3.1 induces a canonical labelling of the size n poset P by elements of $[n]$: the first element that is removed gets label n, and so on. Applying this to the poset of Example 4 we get the labelling shown on the right.



The following lemma is easily proved by induction.

Lemma 6 *Let P be a $(\mathbf{2}+\mathbf{2})$-free poset equipped with its canonical labelling. Let x be the associated ascent sequence, and $\widehat{x}=(\widehat{x}_1,\dots,\widehat{x}_n)$ the corresponding modified ascent sequence. Then for all $i\le n$, the element i of the poset lies at level $\widehat{x}_i$.*

For instance, listing the elements of the poset above and their respective levels gives

$$\begin{array}{l} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8 \\ 0\ 3\ 0\ 1\ 4\ 1\ 1\ 2\ =\ \widehat{x}, \end{array}$$

where we recognize the modified ascent sequence of $(0,1,0,1,3,1,1,2)=\Psi(P)$.

4.2 From posets to permutations

The canonical labelling of the poset P can also be used to set up the bijection from $(\mathbf{2}+\mathbf{2})$-free posets to permutations of $\mathcal{R}$ without using ascent sequences. We read the elements of the poset by increasing level, and, for a fixed level, in descending order of their labels. This gives a permutation $f(P)$. In our example we get 31764825, which is the permutation of $\mathcal{R}_8$ associated with the ascent sequence $(0,1,0,1,3,1,1,2)=\Psi(P)$.

Proposition 7 *For any $(\mathbf{2}+\mathbf{2})$-free poset P equipped with its canonical labelling, the permutation $f(P)$ described above is the permutation of $\mathcal{R}$ corresponding to the ascent sequence $\Psi(P)$. In other words,*

$$\Lambda^{-1}\circ\Psi(P)=\widehat{L}_0\widehat{L}_1\dots\widehat{L}_{\ell(P)}:=\pi,$$

where $\widehat{L}_j$ is the word obtained by reading the elements of $L_j(P)$ is decreasing order. Moreover, the active sites of the above permutation are those preceding and following π, as well as the sites separating two consecutive factors $\widehat{L}_j$.

4.3 From ascent sequences to permutations, and vice-versa

By combining Lemma 6 and Proposition 7, we obtain a non-recursive description of the bijection between ascent sequences and permutations of $\mathcal{R}$. Let x be an ascent sequence, and $\widehat{x}$ its modified sequence. Take the sequence $\widehat{x}$ and write the numbers 1 through n below it. In our running example, $x=(0,1,0,1,3,1,1,2)$, this gives

$$\begin{array}{rl} \widehat{x}= & 0\ 3\ 0\ 1\ 4\ 1\ 1\ 2 \\ & 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\,. \end{array}$$

Let P be the poset associated with x. By Lemma 6, the element labeled i in P lies at level $\widehat{x}_i$. This information is not sufficient to reconstruct the poset P but it *is* sufficient to reconstruct the word $f(P)$ obtained by reading the elements of P by increasing level: Sort the pairs $\binom{\widehat{x}_i}{i}$ in ascending order with respect to the top entry and brake ties by sorting in descending order with respect to the bottom entry. In the above example, this gives

$$\begin{array}{cccccccc} 0 & 0 & 1 & 1 & 1 & 2 & 3 & 4 \\ 3 & 1 & 7 & 6 & 4 & 8 & 2 & 5\,. \end{array}$$

By Proposition 7, the bottom row, here 31764825, is the permutation $\Lambda^{-1}(x)$. We have thus established the following direct description of Λ^{-1}.

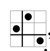
Corollary 8 *Let x be an ascent sequence. Sorting the pairs $\binom{\widehat{x}_i}{i}$ in the order described above gives the permutation $\pi = \Lambda^{-1}(x)$. Moreover, the number of entries of π between the active sites i and $i+1$ is the number of entries of $\widehat{x}$ equal to i, for all $i \geq 0$.*

The second statement gives a non-recursive way of deriving $x = \Lambda(\pi)$ (or, rather, $\widehat{x}$) from π. Take a permutation $\pi \in \mathcal{R}_n$, and indicate its actives sites. For instance, $\pi = {}_0\, 31_1 764_2 8_3 2_4 5_5$. Write the letter i below all entries π_j that lie between the active site labeled i and the active site labeled $i+1$:

$$\begin{array}{cccccccc} 3 & 1 & 7 & 6 & 4 & 8 & 2 & 5 \\ 0 & 0 & 1 & 1 & 1 & 2 & 3 & 4 \end{array} \;\rightarrow\; \text{Sort the pairs } \binom{\pi_j}{i} \text{ by increasing order of the } \pi_j \;\rightarrow\; \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 3 & 0 & 1 & 4 & 1 & 1 & 2\,. \end{array}$$

We have recovered, on the bottom row, the modified ascent sequence $\widehat{x}$ corresponding to π.

4.4 Permutations avoiding $3\bar{1}52\bar{4}$ and self modified ascent sequences

A permutation π avoids the barred pattern $3\bar{1}52\bar{4}$ if every occurrence of the (classical) pattern 231 plays the role of 352 in an occurrence of the (classical) pattern 31524. In other words, for every $i < j < k$ such that $\pi_k < \pi_i < \pi_j$, there exists $\ell \in (i, j)$ and $m > k$ such that $\pi_i \pi_\ell \pi_j \pi_k \pi_m$ is an occurrence of 31524. Note that every such permutation avoids the pattern , and thus belongs to the set $\mathcal{R}$. A conjecture concerning the enumeration of these permutations was given by Pudwell [10, p. 84]. Here, we describe the ascent sequences corresponding to these permutations via the bijection Λ from which we can settle her conjecture.

An ascent sequence x is *self modified* if it is fixed by the map $x \mapsto \widehat{x}$ defined above. For instance, $(0,0,1,0,2,2,0,3,1,1)$ is self modified. In view of the definition of the map $x \mapsto \widehat{x}$, this means that, if $x_{i+1} > x_i$, then $x_j < x_{i+1}$ for all $j \leq i$.

Proposition 9 *The ascent sequence x is self modified if and only if the corresponding permutation π avoids $3\bar{1}52\bar{4}$. In this case, $\max(x) = \operatorname{asc}(\pi) = \operatorname{rmin}(\pi) - 1$, where $\operatorname{rmin}(\pi)$ is the number of right-to-left minima of π, that is, the number of i such that $\pi_i < \pi_j$ for all $j > i$.*

Recall that $\operatorname{asc}(x) = \max(\widehat{x})$. It is not hard to see that $(x_1, \ldots, x_n)$ is a self modified ascent sequence if and only if $x_1 = 0$ and, for all $i \geq 1$, either $x_{i+1} \leq x_i$ or $x_{i+1} = 1 + \max\{x_j : j \leq i\}$. Consequently, a modified ascent sequence x with $\max(x) = k$ reads $0A_0 1A_1 2A_2 \ldots k\, A_k$, where A_i is a (possibly empty) weakly decreasing factor, and each element of A_i is less than or equal to i. This structure is the key to count these sequences, and thus permutations avoiding $3\bar{1}52\bar{4}$.

Proposition 10 *The length generating function of $3\bar{1}52\bar{4}$-avoiding permutations is $\sum_{k\geq 1} t^k/(1-t)^{\binom{k+1}{2}}$. The k-th term of this sum counts those permutations that have k right-to-left minima, or, equivalently, $k-1$ ascents. This is also the number of self modified ascent sequences of length n with largest element $k-1$.*

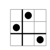
5 Statistics

We shall now look at statistics on ascent sequences, permutations and posets—statistics that we can translate between using our bijections.

Let $x = (x_1, x_2, \ldots, x_n)$ be any sequence of nonnegative integers. Let $\mathrm{last}(x) = x_n$. Define $\mathrm{zeros}(x)$ as the number of zeros in x. A *right-to-left maximum* of x is a letter with no larger letter to its right; the number of right-to-left maxima is denoted $\mathrm{rmax}(x)$. For example,

$$\mathrm{rmax}(0, 1, 0, \mathbf{2}, \mathbf{2}, 0, \mathbf{1}) = 3;$$

the right-to-left maxima are in bold. For sequences x and y of nonnegative integers, let $x \oplus y = xy'$, where y' is obtained from y by adding $1 + \max(x)$ to each of its letters, and juxtaposition denotes concatenation. For example, $(0, 2, 0, 1) \oplus (0, 0) = (0, 2, 0, 1, 3, 3)$. We say that a sequence x has k *components* if it is the sum of k, but not $k + 1$, nonempty nonnegative sequences. Note that $y \oplus z$ is a modified ascent sequence (as defined in Section 4) if and only if y and z are themselves modified ascent sequences. This is the case in the above example.

For any permutation $\pi = \pi_1 \ldots \pi_n$, the statistic $\mathrm{ldr}(\pi)$ (the *leftmost decreasing run*) is defined as the largest integer i such that $\pi_1 > \pi_2 > \cdots > \pi_i$. For permutations π and σ, let $\pi \oplus \sigma = \pi\sigma'$, where σ' is obtained from σ by adding $|\pi|$ to each of its letters. We say that π has k components if it is the sum of k, but not $k + 1$, nonempty permutations. Observe that $\pi \oplus \sigma$ avoids  if and only if both π and σ avoid it. This is the case for instance for $314265 = 3142 \oplus 21$, which corresponds to the above modified ascent sequence $(0, 2, 0, 1, 3, 3) = (0, 2, 0, 1) \oplus (0, 0)$.

For $\pi \in \mathcal{R}_n$, label the active sites with 0, 1, 2, etc. Then $b(\pi)$ denotes the label immediately to the left of the maximal entry of π.

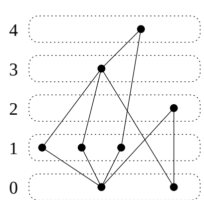
The number of minimal (resp. maximal) elements of a poset P is denoted $\min(P)$ (resp. $\max(P)$). The ordinal sum of two posets P and Q is the poset $P \oplus Q$ on the union $P \cup Q$ such that $x \leq_{P \oplus Q} y$ if $x \leq_P y$, or $x \leq_Q y$, or $x \in P$ and $y \in Q$. The definition applies to labeled or unlabeled posets. Let us say that P has k *components* if it is the ordinal sum of k, but not $k + 1$, nonempty posets. Observe that $P \oplus Q$ is $(\mathbf{2}+\mathbf{2})$-free if and only if both P and Q are $(\mathbf{2}+\mathbf{2})$-free.

Theorem 11 *Given an ascent sequence $x = (x_1, \ldots, x_n)$ with modified ascent sequence $\widehat{x}$, let P and π be the poset and permutation corresponding to x under the bijections described in Sections 2 and 3. Then*

$$\begin{aligned}(\min(P), \ell^\star(P), \ell(P), \max(P), \mathrm{comp}(P)) &= (\mathrm{zeros}(x), \mathrm{last}(x), \mathrm{asc}(x), \mathrm{rmax}(\widehat{x}), \mathrm{comp}(\widehat{x})) \\ &= \big(\mathrm{ldr}(\pi), b(\pi), \mathrm{asc}(\pi^{-1}), \mathrm{rmax}(\pi), \mathrm{comp}(\pi)\big),\end{aligned}$$

where comp *denotes the number of components of the individual structures, as defined above.*

Example 12 *Let P be the poset from Example 4 and let x and π be the corresponding ascent sequence and permutation. One checks that the above theorem holds with* $(\min(P), \ell^\star(P), \ell(P), \max(P), \mathrm{comp}(P)) = (2, 2, 4, 2, 1)$.

$P =$ 

;

$$x = (0, 1, 0, 1, 3, 1, 1, 2); \quad \pi = {}_0 3 1_1 7 6 4_2 8_3 2_4 5_5,$$
$$\widehat{x} = (0, 3, 0, 1, 4, 1, 1, 2); \quad \pi^{-1} = 27158436.$$

6 Enumeration

Theorem 13 *Let p_n be the number of $(\mathbf{2}+\mathbf{2})$-free posets of cardinality n and let $P(t) = \sum_{n\geq 0} p_n t^n$ be the associated generating function. Then*

$$P(t) \;=\; \sum_{n\geq 0} \prod_{i=1}^{n} \left(1-(1-t)^i\right).$$

This series also counts permutations of $\mathcal{R}$, and ascent sequences, by length.

To our knowledge, this result is new. El-Zahar [4] and Khamis [8] used a recursive description of $(\mathbf{2}+\mathbf{2})$-free posets, different from that of Section 3, to derive a pair of functional equations that define the series $P(t)$. However, they did not solve these equations. Haxell, McDonald and Thomasson [7] provided an algorithm, based on a complicated recurrence relation, to produce the first numbers p_n.

These numbers, and the above expression of $P(t)$, occur in the Encyclopedia of Integer Sequences as sequence A022493 [11]. But there, $P(t)$ is described as counting certain involutions, or chord diagrams [12; 13], that form the topic of Section 7. It is known [13] that

$$\frac{p_n}{n!} \sim \kappa \left(\frac{6}{\pi^2}\right)^n \sqrt{n},$$

which proves that the series $P(t)$ is not D-finite (the exponential growth constant would be algebraic).

The proof of Theorem 13 exploits the recursive structure of ascent sequences. The structure translates into a functional equation that defines a 3-variable generating function $F(t;u,v)$, which counts these sequences by length (t), ascent number (u) and last entry (v):

$$(v-1-tv(1-u))\,F(t;u,v) = (v-1)(1-tuv) - tF(t;u,1) + tuv^2F(t;uv,1).$$

The so-called *kernel method* then gives:

$$F(t;u,1) = \sum_{k\geq 1} \frac{(1-u)\,u^{k-1}(1-t)^k}{(u-(u-1)(1-t)^k)\prod_{i=1}^{k}(u-(u-1)(1-t)^i)}.$$

Observe that this expression is divergent when $u=1$. In a final step, we transform it into

$$F(t;u,1) = \sum_{n\geq 0}\sum_{\ell=0}^{n}(u-1)^{n-\ell}u^{\ell}\sum_{m=\ell}^{n}(-1)^{n-m}\binom{n}{m}(1-t)^{m-\ell}\prod_{i=m-\ell+1}^{m}\left(1-(1-t)^i\right),$$

which specializes to Theorem 13 when $u=1$.

7 Involutions with no neighbour nesting

As discussed above, the series of Theorem 13 is known to count certain involutions on $2n$ points, called *regular linearized chord diagrams* (RLCD) by Stoimenow [12]. This result was proved by Zagier [13], following Stoimenow's paper. In this section, we give a new proof of Zagier's result, by constructing a bijection between RLCDs on $2n$ points and unlabeled $(\mathbf{2}+\mathbf{2})$-free posets of size n.

Let $\mathcal{I}_{2n}$ be the collection of involutions π in $\mathcal{S}_{2n}$ that have no fixed points and for which every descent crosses the main diagonal in its dot diagram. Equivalently, if $\pi_i > \pi_{i+1}$ then $\pi_i > i \geq \pi_{i+1}$. An alternative description can be given in terms of the *chord diagram* of π, which is obtained by joining the points i and $\pi(i)$ by a chord (Figure 1, left). Indeed, $\pi \in \mathcal{I}_{2n}$ if and only if, for any i, the chords attached to i and $i+1$ are not *nested*, in the terminology used recently for matchings [3; 9]. That is, the configurations shown on the left of the rules of Fig. 2 are forbidden (but a chord linking i to $i+1$ is allowed).

Recall that a poset P is $(\mathbf{2}+\mathbf{2})$-free if and only if it is an *interval order* [5]. This means that there exists a collection of intervals on the real line whose relative order is P, under the order relation:

$$[a,b] < [c,d] \iff b < c.$$

Let $\pi \in \mathcal{I}_{2n}$ with transpositions $\{(\alpha_i, \beta_i)\}_{i=1}^n$ where $\alpha_i < \beta_i$ for all i. Define $O(\pi)$ to be the interval order (or equivalently, poset) associated with the collection of intervals $\{[\alpha_i, \beta_i]\}_{i=1}^n$.

Example 14 *Consider $\pi = 4\,5\,7\,1\,2\,8\,3\,6\,10\,9 \in \mathcal{I}_{10}$. The transpositions of π are shown in the chord diagram of Figure 1. Beneath the chord diagram is the collection of intervals that corresponds to π, and the $(\mathbf{2}+\mathbf{2})$-free poset $O(\pi)$ is illustrated on the right hand side. We have added labels to highlight the correspondence between intervals and poset elements.*

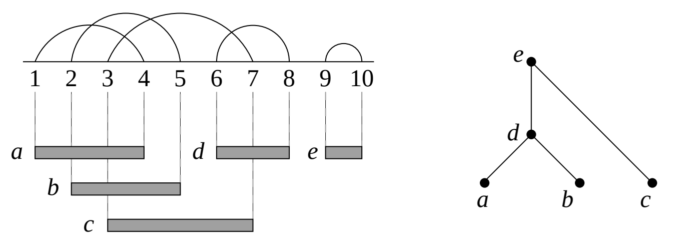


Fig. 1: An involution in $\mathcal{I}_{10}$, the corresponding collection of intervals and the associated $(\mathbf{2}+\mathbf{2})$-free poset.

Theorem 15 *The map O is a bijection between involutions of $\mathcal{I}_{2n}$ and $(\mathbf{2}+\mathbf{2})$-free posets on n elements.*

It is not very hard to prove that O is a surjection. That is, for every $(\mathbf{2}+\mathbf{2})$-free order P, one can find an involution π such that $O(\pi) = P$. The proof uses the transformations of Fig. 2. We then explain that the involution is uniquely determined by the poset.

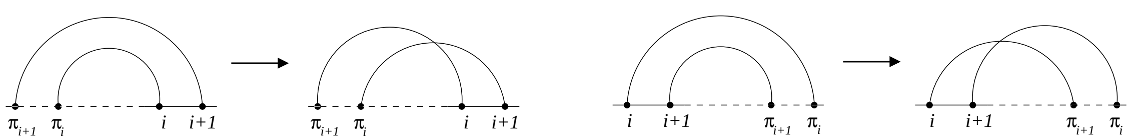


Fig. 2: Two operations on chord diagrams.

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*Application of graph combinatorics to rational identities of type A**

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Abstract. To a word w, we associate the rational function $\Psi_w = \prod (x_{w_i} - x_{w_{i+1}})^{-1}$. The main object, introduced by C. Greene to generalize identities linked to Murnaghan-Nakayama rule, is a sum of its images by certain permutations of the variables. The sets of permutations that we consider are the linear extensions of oriented graphs. We explain how to compute this rational function, using the combinatorics of the graph G. We also establish a link between an algebraic property of the rational function (the factorization of the numerator) and a combinatorial property of the graph (the existence of a disconnecting chain).

Résumé. À un mot w, nous associons la fonction rationnelle $\Psi_w = \prod (x_{w_i} - x_{w_{i+1}})^{-1}$. L'objet principal, introduit par C. Greene pour généraliser des identités rationnelles liées à la règle de Murnaghan-Nakayama, est une somme de ses images par certaines permutations des variables. Les ensembles de permutations considérés sont les extensions linéaires des graphes orientés. Nous expliquons comment calculer cette fonction rationnelle à partir de la combinatoire du graphe G. Nous établissons ensuite un lien entre une propriété algébrique de la fonction rationnelle (la factorisation du numérateur) et une propriété combinatoire du graphe (l'existence d'une chaîne le déconnectant).

Keywords: Rational functions, posets, maps

1 Introduction

A *partially ordered set (poset)* $\mathcal{P}$ is a finite set V endowed with a partial order. By definition, a word w containing exactly once each element of V is called a *linear extension* if the order of the letters is compatible with $\mathcal{P}$ (if $a \leq_{\mathcal{P}} b$, then a must be before b in w). To a linear extension $w = v_1 v_2 \dots v_n$, we associate a rational function:

$$\psi_w = \frac{1}{(x_{v_1} - x_{v_2}) \cdot (x_{v_2} - x_{v_3}) \dots (x_{v_{n-1}} - x_{v_n})}.$$

We can now introduce the main object of the paper. If we denote by $\mathcal{L}(\mathcal{P})$ the set of linear extensions of $\mathcal{P}$, then we define $\Psi_{\mathcal{P}}$ by:

$$\Psi_{\mathcal{P}} = \sum_{w \in \mathcal{L}(\mathcal{P})} \psi_w.$$

* This paper is an extended abstract of the paper on arXiv 0811.2562, which contains all detailed proofs.

1.1 Background

The linear extensions of posets contain very interesting subsets of the symmetric group: for example, the linear extensions of the poset considered in the article (3) are the permutations smaller than a permutation π for the weak Bruhat order. In this case, our construction is close to that of Demazure characters (4). S. Butler and M. Bousquet-Mélou characterize the permutations π corresponding to acyclic posets, which are exactly the cases where the function we consider is the simplest.
Moreover, linear extensions are hidden in a recent formula for irreducible character values of the symmetric group: if we use the notations of (7), the quantity $N^{\lambda}(G)$ can be seen as a sum over the linear extensions of the bipartite graph G (bipartite graphs are a particular case of oriented graphs). This explains the similarity of the combinatorics in article (6) and in this one.

The function $\Psi_{\mathcal{P}}$ was considered by C. Greene (8), who wanted to generalize a rational identity linked to Murnaghan-Nakayama rule for character values of the symmetric group. He has given in his article a closed formula for planar posets ($\mu_{\mathcal{P}}$ is the Möbius function of $\mathcal{P}$):

$$\Psi_{\mathcal{P}} = \begin{cases} 0 & \text{if } P \text{ is not connected,} \\ \prod_{y,z \in \mathcal{P}} (x_y - x_z)^{\mu_{\mathcal{P}}(y,z)} & \text{if } P \text{ is connected,} \end{cases}$$

However, there is no such formula for general posets, only the denominator of the reduced form of $\Psi_{\mathcal{P}}$ is known (see article (2)). In this paper, the first author has investigated the effects of elementary transformations of the Hasse diagram of a poset on the numerator of the associated rational function. He has also noticed, that in some case, the numerator is a Schur function (2, paragraph 4.2) (we can also find Schubert polynomials or sums of multiSchur functions).

In this paper, we obtain some new results on this numerator, thanks to a simple local transformation in the graph algebra, preserving linear extensions.

1.2 Main results

An inductive algorithm The first main result of this paper is an induction relation on linear extensions (Theorem 3.1). When one applies Ψ on it, it gives an efficient algorithm to compute the numerator of the reduced fraction of $\Psi_{\mathcal{P}}$ (the denominator is already known).

A combinatorial formula If we iterate our first main result in a clever way, we can describe combinatorially the final result. The consequence is our second main result: if we give to the graph of a poset $\mathcal{P}$ a rooted map structure, we have a combinatorial non-inductive formula for the numerator of $\Psi_{\mathcal{P}}$ (Theorem 3.7).

A condition for $\Psi_{\mathcal{P}}$ to factorize Green formula's for the function associated to a planar poset is a quotient of products of polynomials of degree 1. In the non-planar case, the denominator is still a product of degree 1 terms, but not the numerator. So we may wonder when the numerator $N(\mathcal{P})$ can be factorized.
Our third main result is a partial answer (a sufficient but not necessary condition) to this question: the numerator $N(\mathcal{P})$ factorizes if there is a chain disconnecting the Hasse diagram of $\mathcal{P}$ (see Theorem 3.8 for a precise statement). An example is drawn on figure 1 (the disconnecting chain is

$(2, 5)$). Note that we use here and in the whole paper a unusual convention: we draw the posets from left (minimal elements) to right (maximal elements).

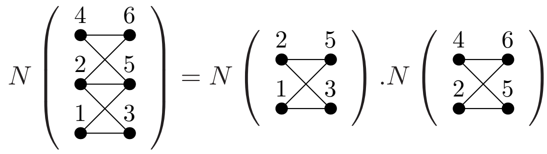

Fig. 1: Example of chain factorization

1.3 Open problems

Necessary condition for factorization The conclusion of the factorization Theorem 3.8 is sometimes true, even when the separating path is not a chain: see for example Figure 2 (the path $(5, 6, 3)$ disconnects the Hasse diagram, but is not a chain).
This equality, and many more, can be easily proved using the same method as Theorem 3.8. Can we give a necessary (and sufficient) condition for the numerator of a poset to factorize into a product of numerators of subposets? Are all factorizations of this kind?

Characterization of the numerator Let us consider a poset $\mathcal{P}$, which has only minimal and maximal elements (respectively $a_1, \ldots, a_l$ and $b_1, \ldots, b_r$). The numerator $N(\mathcal{P})$ of $\Psi_{\mathcal{P}}$ is a polynomial in $b_1, \ldots, b_r$ which degree in each variable can be easily bounded (2, Proposition 3.1). Thanks to Proposition 3.4, we see immediately that $N(\mathcal{P}) = 0$ on some affine subspaces of the space of variables. Unfortunately, these vanishing relations and its degree do not characterize $N(\mathcal{P})$ up to a multiplicative factor. Is there a bigger family of vanishing relations, linked to the combinatorics of the Hasse diagram of the poset, which characterizes $N(\mathcal{P})$?
This question comes from the following observation: for some particular posets, the numerator is a Schubert polynomial and Schubert polynomials are known to be easily defined by vanishing conditions (9).

2 Graphs, posets and rational functions

Oriented graphs are a natural way to encode information of posets. To avoid confusions, we recall all necessary definitions in paragraph 2.1. The definition of linear extensions and hence of our rational

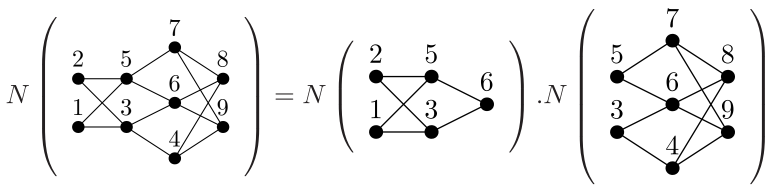

Fig. 2: An example of factorization, not contained in Theorem 3.8.

function can be easily formulated directly in terms of graphs (paragraphs 2.2 and 2.3).
We will also define some elementary removal operations on graphs (paragraph 2.4), which will be used in the next section. Due to transitivity relations, it is not equivalent to perform these operation on the Hasse diagram or on the complete graph of a poset, that's why we prefer to formulate everything in terms of graphs.

2.1 Definitions and notations on graphs

In this paper, we deal with finite *directed graphs*. So we will use the following definition of a graph G:

- A finite set of vertices V_G.
- A set of edges E_G defined by $E_G \subset V_G \times V_G$.

If $e \in E_G$, we will note by $\alpha(e) \in V_G$ the first component of e (called *origin* of e) and $\omega(e) \in V_G$ its second component (called *end* of e). This means that each edge has an orientation.
Let $e = (v_1, v_2)$ be an element of $V_G \times V_G$. Then we denote by $\overline{e}$ the pair (v_2, v_1).

With this definition of graphs, we have four definitions of injective walks on the graph.

	can not go backwards	can go backwards
closed	circuit	cycle
non-closed	chain	path

More precisely,

Definition 2.1 *Let G be a graph and E its set of edges.*

chain *A chain is a sequence of edges $c = (e_1, \ldots, e_k)$ of G such that $\omega(e_1) = \alpha(e_2)$, $\omega(e_2) = \alpha(e_3)$, $\ldots$ and $\omega(e_{k-1}) = \alpha(e_k)$.*

circuit *A circuit is a chain $(e_1, \ldots, e_k)$ of G such that $\omega(e_k) = \alpha(e_1)$.*

path *A path is a sequence $(e_1, \ldots, e_h)$ of elements of $E \cup \overline{E}$ such that $\omega(e_1) = \alpha(e_2)$, $\omega(e_2) = \alpha(e_3)$, $\ldots$ and $\omega(e_{k-1}) = \alpha(e_k)$.*

cycle *A cycle C is a path with the additional property that $\omega(e_k) = \alpha(e_1)$. If C is a cycle, then we denote by $E(C)$ the set $C \cap E$.*

In all these definitions, we add the condition that all edges and vertices are different (except of course, the equalities in the definition).

Remark 1 *The difference between a cycle and a circuit (respectively a path and a chain) is that, in a cycle (respectively in a path), an edge can appear in both directions (not only in the direction given by the graph structure). The edges, which appear in a cycle C with the same orientation than their orientation in the graph, are exactly the elements of $E(C)$.*

To make the figures easier to read, $\alpha(e)$ is always the left-most extremity of e and $\omega(e)$ its right-most one. Such drawing construction is not possible if the graph contains a circuit. But its case will not be very interesting for our purpose.

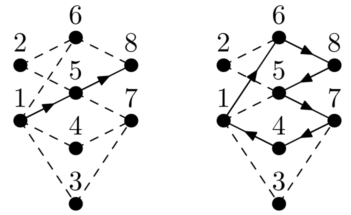

Fig. 3: Example of a chain and a cycle C (we recall that orientations are from left to right).

Example 1 *An example of graph is drawn on figure 3. In the left-hand side, the non-dotted edges form a chain c, whereas, in the right-hand side, they form a cycle C, such that $E(C)$ contains 3 edges:* $(1, 6), (6, 8)$ *and* $(5, 7)$.

The *cyclomatic number* of a graph G is $|E_G| - |V_G| + c_G$, where c_G is the number of connected components of G. A graph contains a cycle if and only if its cyclomatic number is not 0 (see (5)). If it is not the case, the graph is called *forest*. A connected forest is, by definition, a tree. Beware that, in this context, there are no rules for the orientation of the edges of a tree (often, in the literature, an oriented tree is a tree which edges are oriented from the *root* to the *leaves*, but we do not consider such objects here).

2.2 Posets, graphs, Hasse diagrams and linear extensions

In this paragraph, we recall the link between graphs and posets.

Given a graph G, we can consider the binary relation on the set V_G of vertices of G:

$$x \leq y \overset{\text{def}}{\iff} \left(x = y \text{ or } \exists\, e \in E_G \text{ such that } \left\{ \begin{array}{l} \alpha(e) = x \\ \omega(e) = y \end{array} \right. \right)$$

This binary relation can be completed by transitivity. If the graph has no circuit, the resulting relation $\leq$ is antisymmetric and, hence, endows the set V_G with a poset structure, which will be denoted $\mathtt{poset}(G)$.

The application `poset` is not injective. Among the pre-images of a given poset $\mathcal{P}$, there is a minimum one (for the inclusion of edge set), which is called Hasse diagram of $\mathcal{P}$.

The definition of linear extensions given in the introduction can be formulated in terms of graphs:

Definition 2.2 *A linear extension of a graph G is a total order $\leq_w$ on the set of vertices V such that, for each edge e of G, one has $\alpha(e) \leq_w \omega(e)$.*

The set of linear extensions of G is denoted $\mathcal{L}(G)$. Let us also define the formal sum $\varphi(G) = \sum_{w \in \mathcal{L}(G)} w$.

We will often see a total order $\leq_w$ defined by $v_{i_1} \leq_w v_{i_2} \leq_w \ldots \leq_w v_{i_n}$ as a word $w = v_{i_1} v_{i_2} \ldots v_{i_n}$.

Remark 2 *If G contains a circuit, then it has no linear extensions. Else, its linear extensions are the linear extensions of* $\mathtt{poset}(G)$*. Thus considering graphs instead of posets does not give more general results.*

2.3 Rational functions on graphs

Given a graph G with n vertices $v_1, \dots, v_n$, we are interested in the following rational function Ψ_G in the variables $(x_{v_i})_{i=1\dots n}$: $\Psi_G = \displaystyle\sum_{w \in \mathcal{L}(G)} \frac{1}{(x_{w_1} - x_{w_2}) \dots (x_{w_{n-1}} - x_{w_n})}$.

We also consider the renormalization: $N(G) := \Psi_G \cdot \displaystyle\prod_{e \in E_G} (x_{\alpha(e)} - x_{\omega(e)})$.

The following properties of $N(G)$ have been proved in (2): the value of N on forests is essential in the next section because we will compute N by induction on the cyclomatic number.

Lemma 2.1 (Pruning-invariance) *Let G be a graph with a vertex v of valence 1 and e the edge of extremity (origin or end) v. Then one has $N(G) = N\big(G \backslash \{v\}\big)$.*

Proposition 2.2 *If T is a tree, $N(T) = 1$. If F is a disconnected forest, $N(F) = 0$.*

2.4 Removing edges and vertices in graphs

The main tool of this paper consists in removing some edges of a graph G.

Definition 2.3 *Let G be a graph and E' a subset of its set of edges E_G. We will denote by $G \backslash E'$ the graph G' with*

- *the same set of vertices as G ;*
- *the set $E_{G'} := E_G \backslash E'$ as set of edges.*

Definition 2.4 *If G is a graph and V' a subset of its set of vertices V, V' has an induced graph structure: its edges are exactly the edges of G, which have both their extremities in V'.*

If $V \backslash V' = \{v_1, \dots, v_l\}$, this graph will be denoted by $G \backslash \{v_1, \dots, v_l\}$. The symbol is the same than in definition 2.3, but it should not be confusing.

3 Computation and properties of the numerator

In the previous section, we have defined a simple operation on graphs consisting in removing edges. Thanks to this operation, we will be able to construct an operator which lets invariant the formal sum of linear extensions (paragraph 3.1). Due to the definition of Ψ, this implies immediately an inductive relation on the rational functions Ψ_G (paragraph 3.2).
In paragraph 3.3, we solve the induction and obtain an additive formula for $N(G)$. But this formula has never a factorized form (even in the planar case), so we give in the last paragraph (3.4) a simple graphical condition which implies the partial factorization of $N(G)$.

3.1 Equality on linear extensions

In this paragraph, we prove an induction relation on the formal sums of linear extensions of graphs. More exactly, we write, for any graph G with at least one cycle, $\varphi(G)$ as a linear combination of $\varphi(G')$, where G' runs over graphs with a strictly lower cyclomatic number. In the next paragraphs, we will iterate this relation and apply Ψ to both sides of the equality to study Ψ_G.

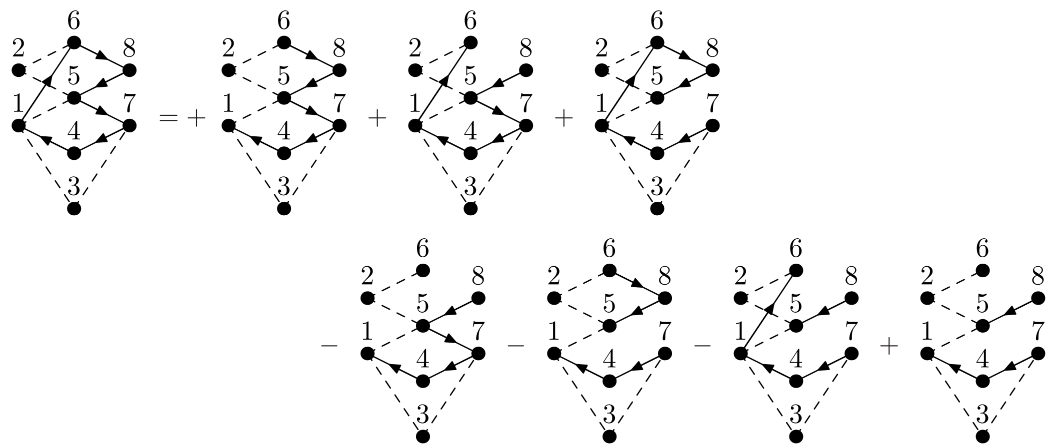

Fig. 4: Example of application of theorem 3.1

If G is a finite graph and C a cycle of G, let us denote by $T_C(G)$ the following formal alternate sum of subgraphs of G:

$$T_C(G) = \sum_{\substack{E' \subset E(C) \\ E' \neq \emptyset}} (-1)^{|E'|-1} G \backslash E'.$$

The function $\varphi(G) = \sum_{w \in \mathcal{L}(G)} w$ can be extended by linearity to the free abelian group spanned by graphs. One has the following theorem:

Theorem 3.1 *Let G be a graph and C a cycle of G then, $\varphi(G) = \varphi(T_C(G))$.*

An example is drawn on figure 4 (to make it easier to read, we did not write the operator φ in front of each graph).

Remark 3 *In the case where $E(C) = \emptyset$, this theorem says that a graph with a circuit has no linear extensions (see remark 2).*

If it is a singleton, it says that we do not change the set of linear extensions by erasing an edge if there is a chain going from its origin to its end (thanks to transitivity).

To prove Theorem 3.1, we will need the two following lemma:

Lemma 3.2 *Let $w \in \mathcal{L}(G \backslash E(C))$. There exists $E'(w) \subset E(C)$ such that*

$$\forall E'' \subset E(C), \quad w \in \mathcal{L}(G \backslash E'') \Longleftrightarrow E'(w) \subset E'' \subset E(C).$$

Proof: Left to the conscientious reader. $\square$

Lemma 3.3 *Let $w \in \mathcal{L}(G\backslash E(C))$, there exists $E'' \subsetneq E(C)$ such that $w \in \mathcal{L}(G\backslash E'')$.*

Proof: Suppose that we can find a word w for which the lemma is false. Since $w \in \mathcal{L}(G\backslash E(C))$, the word w fulfills the relations given by the edges of G, which are not in $E(C)$.
But, if $e \in E(C)$, one has $w \notin \mathcal{L}(G\backslash(E(C)\backslash\{e\}))$. That means that w does not fulfill the relation corresponding to the edge e. As w is a total order, it fulfills the opposite relation: $w \in \mathcal{L}\left[\big(G\backslash E(C)\big) \cup \{\overline{e}\}\right]$. With the same argument applied for each $e \in E(C)$, one has $w \in \mathcal{L}\left[\big(G\backslash E(C)\big) \cup \overline{E(C)}\right]$. But this graph contains a circuit, so its set of linear extension is empty. $\square$

Let us come back to the proof of Theorem 3.1. Let w be a word containing exactly once each element of V_G. We will compute its coefficient in $\varphi(G) - \varphi(T_C(G)) = \sum_{E' \subset E(C)} (-1)^{|E'|} \varphi(G\backslash E')$:

- If $w \notin \mathcal{L}(G\backslash E(C))$, its coefficient is zero in each summand.
- If $w \in \mathcal{L}(G\backslash E(C))$, thanks Lemma 3.2, we know that there exists $E'(w) \subset E(C)$ such that

$$\forall E'' \subset E(C), w \in \mathcal{L}(G\backslash E'') \Longleftrightarrow E'(w) \subset E'' \subset E(C).$$

So the coefficient of w in $\varphi(G) - \varphi(T_C(G))$ is $\displaystyle\sum_{E'(w) \subset E'' \subset E(C)} (-1)^{|E''|} = 0$ because $E'(w) \neq E(C)$ (Lemma 3.3).

3.2 Consequences on rational functions

In the previous paragraph, we have established an induction formula for the formal sum of linear extensions (Theorem 3.1). One can apply Ψ to both sides of this equality to compute $N(G)$:

Proposition 3.4 *Let G be a graph containing a cycle C. Then,*

$$N(G) = \sum_{\substack{E' \subset E(C) \\ E' \neq \emptyset}} \left[(-1)^{|E'|-1} N(G\backslash E') \prod_{e \in E'} (x_{\alpha(e)} - x_{\omega(e)}) \right].$$

By Proposition 2.2, one has $N(T) = 1$ if T is a tree and $N(F) = 0$ if F is a disconnected forest. So this Proposition gives us an algorithm to compute $N(G)$: we just have to iterate it with any cycles until all the graphs in the right hand side are forests. More precisely, if after iterating transformations of type T_C on G, we obtain the formal linear combination $\sum c_F F$ of subforests of G, then:

$$N(G) = \sum_{T \text{ subtree of } G} c_T \prod_{e \in E_G \backslash E_T} (x_{\alpha(e)} - x_{\omega(e)}).$$

In this formula, $N(G)$ appears as a sum of polynomials. So the computation of $N(G)$, using this formula, is easier than a direct application of the definition

$$N(G) = \sum_{w \in \mathcal{L}(G)} \left(\Psi_w \cdot \prod_{e \in E_G} (x_{\alpha(e)} - x_{\omega(e)}) \right),$$

where the summands may have poles.

Corollary 3.5 *For any graph G, the rational function $N(G)$ is a polynomial. Moreover, if G is disconnected, $N(G) = 0$.*

In fact, if a connected graph G is the Hasse diagram of poset, the fraction $\Psi_G = \frac{N(G)}{\prod_{e \in E_G} (x_{\alpha(e)} - x_{\omega(e)})}$ is irreducible (see (2) for a proof of this fact).

Example 2 (explicit computation) *Let $G_{2,4}$ be the graph with a set of vertices V partitioned in two subsets $V_1 = \{a_1, a_2\}$ and $V_2 = \{b_1, b_2, b_3, b_4\}$ and $E = V_1 \times V_2$ as set of edges. After iterating Theorem 3.1, we obtain the equality of Figure 5 (the operator φ has been once again omitted).*

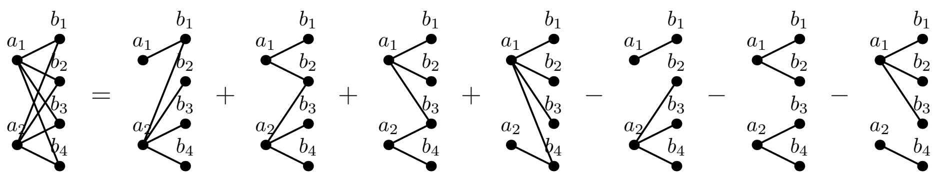

Fig. 5: Decomposition of $\varphi(G_{2,4})$.

Thus, $N(G_{2,4}) = \sum_{i=1}^{4} \left(\prod_{j<i} (b_j - a_1) \cdot \prod_{k>i} (b_k - a_2) \right)$.

3.3 A combinatorial formula for N

To compute the polynomial N of a graph G, we only have to find the coefficient of trees in a formal linear combination of forests obtained by iterating transformations T_C on G. But there are many possible choices of cycle at each step and these coefficients depend on these choices.

A way to avoid this problem is to give to G a rooted map structure M and to look at the particular decomposition $D(M)$ introduced in the paper (6, section 3). We will not describe here this particular choice of cycles (see the complete version), but we have a combinatorial description of the trees with coefficient $+1$ in $D(M)$, all other trees having 0 as coefficient.

Definition 3.1 *A (combinatorial oriented) map is a connected graph with, for each vertex v, a cyclic order on the edges whose origin or end is v. This definition is natural when the graph is drawn on a two dimensional surface (see for example (10)).*
It is more convenient when we deal with maps, to consider edges as couples of two darts (h_1, h_2), the first one of extremity $\alpha(e)$ and the second one of extremity $\omega(e)$. A rooted map is a map with an external dart h_0, that is to say a dart which do not belong to any edge, but has an extremity and a place in the cyclic order given by this extremity.

We will need the following definition:

Definition 3.2 *If T is a spanning subtree of a rooted map M, the tour of the tree T beginning at h_0 defines an order on the darts which do not belong to T. The definition is easy to understand on a figure: for example, on Figure 6, the tour is $h_1^1, h_2^1, h_1^2, h_2^2, h_1^3, h_1^4, h_2^3, h_2^4$ (see (1) for a precise definition).*

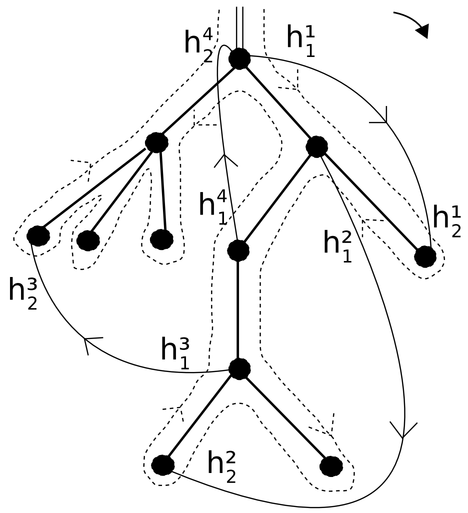

Fig. 6: Tour of a spanning tree of a rooted map.

We are now able to describe the coefficients of trees in $D(M)$:

Proposition 3.6 *Let M be a rooted map and T a spanning tree of M.*

- *If there is an edge $e = (h_1, h_2) \in M \backslash T$ such that h_2 appears before h_1 in the tour of T, then the coefficient of T in $D(M)$ is 0.*
- *Else, the coefficient of T in $D(M)$ is $+1$ (in this case, T is said to be* good*).*

For example, the spanning tree of Figure 6 is good. Note that the property of being a good spanning tree does not depend on the orientations of the edges of the tree, but only on the orientations of those which do not belong to it.

This Proposition is not very hard to prove, once we have the good definition of $D(M)$, but the latter is quite technical and requires a non-easy proof of confluence. As an immediate consequence of the proposition, we have the following formula for $N(G)$:

Theorem 3.7 *The polynomial N associated to the underlying graph G of a rooted map M is given by the following combinatorial formula:*

$$N(G) = \sum_{\substack{T \text{ good spanning} \\ \text{tree of } M}} \left[\prod_{e \in E_G \backslash E_T} \left(x_{\alpha(e)} - x_{\omega(e)} \right) \right]. \tag{1}$$

3.4 Chain factorization

In the previous paragraph, we have given an additive formula for the numerator of the reduced fraction of Ψ_P. Green formula for planar posets (see subsection 1.1) and the example of Figure 1 show that, in some cases, it can also be written as a product. In this paragraph we give a simple graphical condition on a graph G, which implies the factorization of $N(G)$.

Remark 4 *In this section, we assume that G has no circuit and no transitivity relation (an edge going from the beginning to the end of a chain). This is always true in the case of Hasse diagrams of posets so we do not lose in generality. With this assumption, if we consider a chain c, there is no extra edges between the vertices of the chain.*

Let G be a connected graph, c a chain of G, V_c the set of vertices of c (including the origin and the end of the chain) and $G_1, \ldots, G_k$ be all the connected components of $G \setminus V_c$. The complete subgraphs $\overline{G_i} = G_i \cup V_c$ (for $1 \le i \le k$) will be called regions of G. An example (with $k = 4$) is drawn on Figure 7 (we consider the chain with $V_c = \{1, 2, 13, 3, 4, 5, 6, 14\}$).

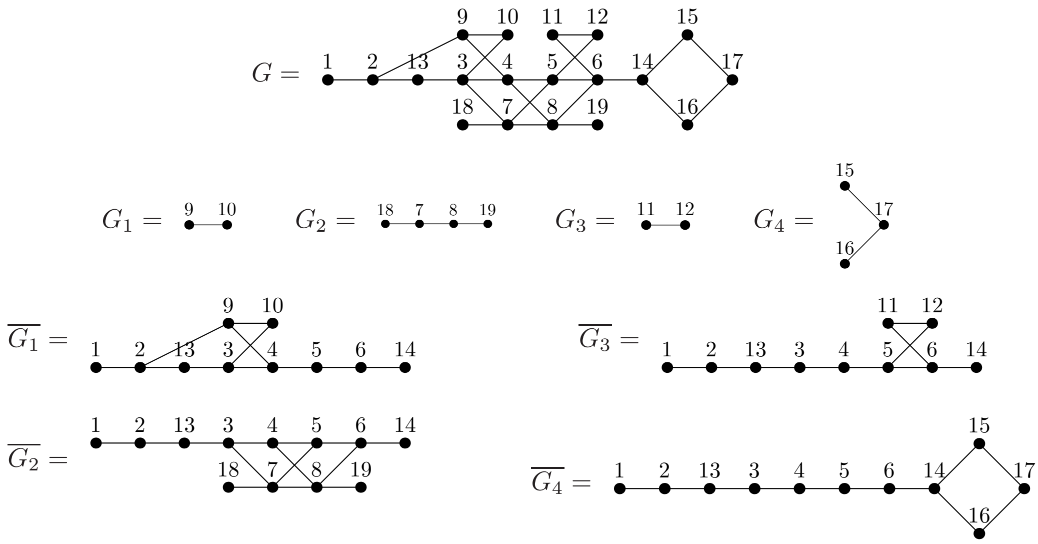

Fig. 7: A graph G with a chain c, the connected components G_i of $G \setminus V_c$ and the corresponding regions $\overline{G_i}$.

We can now state our third main result:

Theorem 3.8 *Let G be a connected graph, c a chain of G and $\overline{G_1}, \overline{G_2}, \ldots, \overline{G_k}$ be the corresponding regions of G. Then one has:*

$$N(G) = \prod_{j=1}^{k} N(\overline{G_j}).$$

In the example 7, the numerator $N(G)$ can be factorized into four non-trivial factors. This theorem is proved in the complete version of the paper. It relies on a clever application of Proposition 3.4 and is a little technical.

In the case of planar posets considered by Greene (8), this theorem explains the fact that the numerator of the associated rational function is a product of polynomials of degree 1. We can even give a new proof

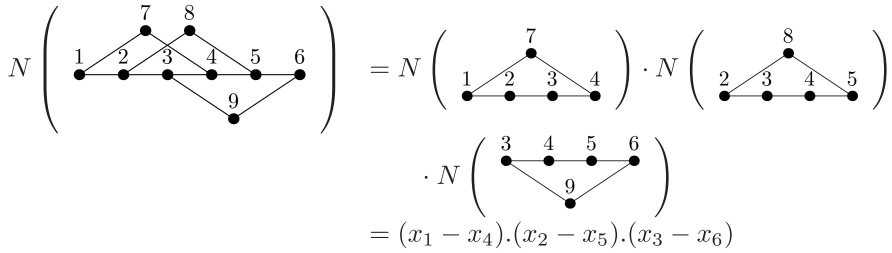

Fig. 8: A non-planar (with Greene's definition) poset for which Greene's formula is true.

of Greene's formula (stated in subsection 1.1), which works in a context a little more general than planar posets (see Figure 8).

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Quasipolynomial formulas for the Kronecker coefficients indexed by two two–row shapes (extended abstract)

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Abstract. We show that the Kronecker coefficients indexed by two two–row shapes are given by quadratic quasipolynomial formulas whose domains are the maximal cells of a fan. Simple calculations provide explicitly the quasipolynomial formulas and a description of the associated fan.

These new formulas are obtained from analogous formulas for the corresponding reduced Kronecker coefficients and a formula recovering the Kronecker coefficients from the reduced Kronecker coefficients.

As an application, we characterize all the Kronecker coefficients indexed by two two-row shapes that are equal to zero. This allowed us to disprove a conjecture of Mulmuley about the behavior of the stretching functions attached to the Kronecker coefficients.

Résumé. Nous démontrons que les coefficients de Kronecker indexés par deux partitions de longueur au plus 2 sont donnés par des formules quasipolynomiales quadratiques dont les domaines de validité sont les cellules maximales d'un éventail. Des calculs simples nous donnent une description explicite des formules quasipolynomiales et de l'éventail associé. Ces nouvelles formulas sont obtenues de formules analogues pour les coefficients de Kronecker réduits correspondants et au moyen d'une formule reconstruisant les coefficients de Kronecker à partir des coefficients de Kronecker réduits.

Une application est la caractérisation exacte de tous les coefficients de Kronecker non–nuls indexés par deux partitions de longueur au plus deux. Ceci nous a permis de réfuter une conjecture de Mulmuley au sujet des fonctions de dilatations associées aux coefficients de Kronecker.

Keywords: Kronecker coefficients, internal product of symmetric functions, Saturation properties, Representations of the symmetric group

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Introduction

A fundamental problem in algebraic combinatorics is the Clebsch-Gordan problem: given a linearly reductive group G, give a combinatorial description of the coefficients $m^{\lambda}_{\mu\nu}$ in the decomposition into irreducibles of the tensor product of two (finite-dimensional complex) irreducible representation $V_\mu(G)$ and $V_\nu(G)$:

$$V_\mu(G) \otimes V_\nu(G) \cong \bigoplus_{\lambda} m^{\lambda}_{\mu\nu} V_\lambda(G)$$

While this problem has been solved satisfactorily for the general linear group, $GL(n)$, the most elementary linear group, this is not the case for the symmetric group, S_n, the most fundamental finite group.

In the case of $GL(n)$, the coefficients $m^{\lambda}_{\mu\nu} = c^{\lambda}_{\mu\nu}$ are the well known Littlewood-Richardson coefficients. There exists several combinatorial descriptions for them. One of these descriptions was given by Berenstein and Zelevinsky (1992) that showed that $c^{\lambda}_{\mu\nu}$ counts the integral points in a well-defined family of polytopes. This initiated a series of works concerning the stretching functions associated to these coefficients that culminated with the proof by Knutson and Tao (1999) of the saturation conjecture. Finally, Rassart (2004) showed that the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ are given by polynomial functions of the parts of λ, μ and ν, on the maximal cells of a fan.

For the symmetric group S_n, the coefficients $m^{\lambda}_{\mu\nu} = g^{\lambda}_{\mu\nu}$ are called the Kronecker coefficients. Amazingly, there is no combinatorial description of these coefficients in general. Particular families have been investigated. In this paper the *Kronecker coefficients indexed by two two–row shapes* are considered. They are the coefficients $g^{\lambda}_{\mu\nu}$ such that both μ and ν have two rows. Formulas for them have already been given by Remmel and Whitehead (1994) and Rosas (2001). Recent works by Luque and Thibon (2003); Garsia et al. (2008); Brown et al. (2008) have revived the interest of obtaining better formulas for the Kronecker coefficients indexed by two two–row shapes as Hilbert series related to these coefficients have been linked to problems in quantum information theory.

New problems about the Clebsch–Gordan coefficients have been raised recently by the specialists of computational complexity. Narayanan (2006) showed that the computation of the Littlewood–Richardson coefficients is a #P–complete problem. Bürgisser and Ikenmeyer (2008) showed that the computation of the Kronecker coefficients is #P–hard. On the other hand, the saturation property implies that the non–vanishing of a Littlewood–Richardson coefficient can be decided in polynomial time (Mulmuley and Sohoni, 2005). Is it also the case for the Kronecker coefficients? This question lies at the heart of a detailed plan, *Geometric Complexity Theory*, that Mulmuley and Sohoni (2001) elaborated to prove that $P \neq NP$ over the complex numbers (an arithmetic, non–uniform version of $P \neq NP$). This lead Mulmuley (2007) to state a series of conjectures about the stretching functions associated to the Kronecker coefficients. The scarce information available about Kronecker coefficients made difficult even the experimental checking of these conjectures. By means of the formulas by Remmel and Whitehead (1994) and Rosas (2001) it was only possible to check them on large samples of Kronecker coefficients indexed by two two–row shapes (see Mulmuley, 2007).

The present article obtains a new description for the Kronecker coefficients indexed by two two–row shapes, given by quasi–polynomial functions on the chambers of fans, resembling the description of Rassart (2004) for the Littlewood–Richardson coefficients. It is efficient enough to check Mulmuley's conjectures for all Kronecker coefficients indexed by two two–row shapes (and, actually, disprove them by providing explicit counter–examples). We start our investigation by looking at Murnaghan's *reduced Kronecker coefficients* $\overline{g}^{\gamma}_{\alpha\beta}$ (Murnaghan, 1938), a related family of coefficients indexed by triples of partitions, which are stable values of stationary sequences of Kronecker coefficients. Our first result expresses the Kronecker coefficients in terms of the reduced Kronecker coefficients (Theorem 3). Exploiting the work of Rosas (2001) we are able to show that the reduced Kronecker polynomials related to the two-row family count integral points in a polygon of $\mathbb{R}^2$. From this we describe an explicit piecewise quasipolynomial formula for these reduced Kronecker coefficients. The pieces are the 26 maximal cells of a fan. Last, using our formula that recovers the Kronecker coefficients from the reduced Kronecker coefficients, we obtain, with the help of the Maple package *convex* by Franz (2006), explicit piecewise quasipolynomial formulas for the Kronecker coefficients indexed by two two-row shapes. It is given by 74 quadratic quasipolynomials whose domains are the maximal cells of a fan.

As an application, we list all Kronecker coefficients indexed by two two-row shapes that are equal to zero. This made possible the discovery of counter–examples to Mulmuley's conjectures (Briand et al., 2008). In short, the advantage of our results is that for the first time we can completely study a complete nontrivial family of the Kronecker coefficients.

The detailed proofs will be presented in a full version (Briand et al., In preparation) of this extended abstract.

1 Piecewise Quasipolynomials

We now give a more detailed description of the main result. A quasipolynomial is a function on $\mathbb{Z}^n$ given by polynomial formulas, whose domains are the cosets of a full rank sublattice of $\mathbb{Z}^n$. Remarkable examples of (univariate) quasipolynomials are the Ehrhart functions of polytopes of $\mathbb{R}^k$ with rational vertices, that count the integral points in the dilations of the polytope (see Stanley, 1997, chap. 4).

We will obtain a description for the Kronecker coefficients indexed by two two–row shapes as a function of the following kind.

Definition 1 *A* vector partition–like function *is a function ϕ on $\mathbb{Z}^n$ fulfilling the following: (i) There exists a convex rational polyhedral cone C such that ϕ is zero outside C. (ii) Inside C, the function ϕ is given by quasipolynomial formulas whose domains are (the sets of integral points of) the maximal (closed) cells of a fan $\mathfrak{F}$.*

If C and $\mathfrak{F}$ are as above and Q is the family of quasipolynomial formulas, indexed by the maximal cells of $\mathfrak{F}$, we say that the triple $(C, \mathfrak{F}, Q)$ is a presentation of ϕ as a vector partition–like function.

Remark 1 *A sum of vector partition–like functions ϕ_1, ϕ_2 is not necessarily vector partition–like. It is, however, the case when the functions admit presentations $(C, \mathfrak{F}, Q)$ and $(C', \mathfrak{F}', Q')$ with the same cone: $C = C'$.*

Examples of vector partition–like functions are the vector partition functions, whose corresponding fans are the *chamber complexes* (see Sturmfels, 1995; Brion and Vergne, 1997).

Vector partition–like functions also arise as functions counting integral solutions to some systems of linear inequalities depending on parameters. Precisely, consider a system of inequalities of the form

$$u_i(x) + c_i(h) \geq 0, \qquad i = 1, \ldots, N \tag{1}$$

where the functions u_i and c_i are integral, homogeneous linear forms on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. The unknown is x and the parameter is h. Assume that for any $h \in \mathbb{R}^n$ the set of solutions x of the system is bounded. Let $h \mapsto \phi(h)$ be the function that counts the *integral* solutions x of the system. This function ϕ

is vector partition–like. This follows from the reduction of this function to a vector partition function (see Brion and Vergne, 1997). Here the cone C in Definition 1 is the set of values of the parameter h making the system feasible.

Let ℓ be a positive integer. The function $(\lambda, \mu, \nu) \mapsto c^{\lambda}_{\mu,\nu}$ from triples of partitions with at most ℓ parts to Littlewood–Richardson coefficients is vector partition–like. This is because this function counts the integral solutions of a system of inequalities depending on parameters (the parts of the partitions) of the form (1). Indeed, such a system can be derived from the Littlewood–Richardson rule (see Mulmuley and Sohoni, 2005). Alternatively, one can use the system defining Knutson and Tao's Hive polytopes (see the exposition by Buch, 2000).

It is natural to ask if similar results also hold for the Kronecker coefficients. Let ℓ_1 and ℓ_2 be positive integers. If μ and ν are partitions of length at most ℓ_1 and ℓ_2 respectively then $g^{\lambda}_{\mu,\nu}$ can be nonzero only if λ has at most $\ell_1\ell_2$ parts. The analogous function to consider is thus $G_{\ell_1,\ell_2} : (\lambda, \mu, \nu) \mapsto g^{\lambda}_{\mu,\nu}$ defined on triples of partitions with at most $\ell_1\ell_2$, ℓ_1 and ℓ_2 parts respectively. No interpretation of the functions G_{ℓ_1,ℓ_2} as counting integral solutions to systems of inequalities of the form (1) is known. Nevertheless, very close results were obtained by Mulmuley (2007): (i) The functions G_{ℓ_1,ℓ_2} fulfill the conditions in Definition 1 with $\mathfrak{F}$ a complex of polyhedral cones instead of a fan. (ii) For any λ, μ, ν, the *stretching function* $N \in \mathbb{N} \mapsto g^{N^{\lambda}}_{N\mu,N\nu}$ is a univariate quasipolynomial. Here $N\lambda$ stands for the partitions obtained from λ by multiplying all parts by N. Combining these two results, one gets that the functions G_{ℓ_1,ℓ_2} fulfill the conditions in the definition of vector partition–like with "maximal closed cells" replaced with "open cells" in (ii).

The simplest non–trivial case is $G_{2,2}$, describing the Kronecker coefficients indexed by two two–row shapes. Even this case is somehow difficult. In this work we prove the following:

Theorem 1 *The function*

$$G_{2,2} : \ (\lambda_1, \ldots, \lambda_4, \mu_1, \mu_2, \nu_1, \nu_2) \in \mathbb{Z}^8 \mapsto g^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)}$$

is vector partition–like.

Remark 2 *A Kronecker coefficient $g^{\lambda}_{\mu,\nu}$ can be nonzero only if its three indexing partitions have the same weight. This and the formula $g^{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}_{(\mu_1,\mu_2)(\nu_1,\nu_2)} = g^{(\lambda_1-1,\lambda_2-1,\lambda_3-1,\lambda_4-1)}_{(\mu_1-2,\mu_2-2)(\nu_1-2,\nu_2-2)}$ reduce the study of $G_{2,2}$ to the study of the function*

$$(n, \gamma_1, \gamma_2, r, s) \mapsto g^{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2)}_{(n-r,r)(n-s,s)}$$

2 Murnaghan's Theorem and reduced Kronecker coefficients

In this section we introduce Murnaghan's reduced Kronecker coefficients $\overline{g}^{\gamma}_{\alpha,\beta}$. They are integers indexed by triples of partitions closely related to the Kronecker coefficients. The Kronecker coefficients indexed by two two–row shapes will be re–obtained from the reduced Kronecker coefficients indexed by two one–row shapes (Section 3) which will be easy to describe (Theorem 4 and Section 4).

The Jacobi–Trudi formula expresses the Schur functions as determinants in the complete sums h_k. When λ has at most k parts, it asserts that:

$$s_{\lambda} = \det(h_{j-i+\lambda_i})_{i,j=1,\ldots,k}$$

(where $h_k = 0$ when $k < 0$, $h_0 = 1$ and $\lambda_i = 0$ for i greater than the length of λ.)

This formula can also be applied in the case when λ is not a partition, i.e. is not nondecreasing. The functions s_λ obtained are either 0, or Schur functions up to a sign.

Let n be an integer and λ a partition. Then $|\lambda|$ stands for the sum of the parts of λ and for any integer n, we denote with $(n-|\lambda|, \lambda)$ the sequence $(n-|\lambda|, \lambda_1, \lambda_2, \ldots)$. This is a partition if and only if $n \geq |\lambda|+\lambda_1$. Last $\overline{\lambda}$ stands for the partition $(\lambda_2, \lambda_3, \ldots)$, which is obtained by removing the first part of λ.

Theorem 2 (Murnaghan (1938, 1955)) *There exists a family of nonnegative integers $(\overline{g}^{\gamma}_{\alpha,\beta})$ indexed by triples of partitions (α, β, γ) such that, for fixed partitions α and β, only finitely many terms $\overline{g}^{\gamma}_{\alpha,\beta}$ are non–zero, and for all $n \geq 0$,*

$$s_{(n-|\alpha|,\alpha)} * s_{(n-|\beta|,\beta)} = \sum_{\gamma} \overline{g}^{\gamma}_{\alpha,\beta} s_{(n-|\gamma|,\gamma)}$$

Following Klyachko (2004), we call the coefficients $\overline{g}^{\gamma}_{\alpha,\beta}$ the *reduced Kronecker coefficients*. They are called *extended Littlewood–Richardson numbers* in Kirillov (2004) because of the following property, observed first in Murnaghan (1955) and proved in Littlewood (1958): if α, β and γ are three partitions such that $|\gamma| = |\alpha| + |\beta|$ then $\overline{g}^{\gamma}_{\alpha\beta} = c^{\gamma}_{\alpha\beta}$.

Remark 3 *It follows from Murnaghan's Theorem that for fixed partitions α, β, γ, the sequence of Kronecker coefficients $g^{(n-|\gamma|,\gamma)}_{(n-|\alpha|,\alpha),(n-|\beta|,\beta)}$ (n big enough so that all three indices are partitions) is stationary with limit $\overline{g}^{\gamma}_{\alpha\beta}$.*

3 From reduced to non–reduced Kronecker coefficients

In this section we give a formula that allows us to recover the Kronecker coefficients from the reduced Kronecker coefficients, and we apply it for the Kronecker coefficients indexed by two two–row shapes.

For any infinite sequence $u = (u_1, u_2, \ldots)$ and any positive integer i we denote with $u^{\dagger i}$ the sequence obtained from u by incrementing by 1 its $i-1$ first terms and removing its i–th term, that is: $u^{\dagger i} = (u_1{+}1, u_2{+}1, \ldots, u_{i-1}{+}1, u_{i+1}, u_{i+2} \ldots)$. Partitions are identified with infinite sequences by appending trailing zeros. Under this identification, if λ is a partition then so is $\lambda^{\dagger i}$ for all i.

Theorem 3 *Let ℓ_1, ℓ_2 and n be positive integers. Let λ, μ, ν be partitions of n such that μ has length at most ℓ_1 and ν has length at most ℓ_2. Then:*

$$g^{\lambda}_{\mu\nu} = \sum_{i=1}^{\ell_1\ell_2} (-1)^{i+1} \overline{g}^{\lambda^{\dagger i}}_{\overline{\mu},\overline{\nu}} \tag{2}$$

For $\ell_1 = \ell_2 = 2$, Formula (2) applies as follows:

$$g^{(\lambda_1,\lambda_2,\lambda_3)}_{(n-r,r)(n-s,s)} = \overline{g}^{(\lambda_2,\lambda_3)}_{(r)(s)} - \overline{g}^{(\lambda_1+1,\lambda_3)}_{(r)(s)} + \overline{g}^{(\lambda_1+1,\lambda_2+1)}_{(r)(s)} \tag{3}$$

where $n = |\lambda|$, because the last expected summand $\overline{g}^{(\lambda_1+1,\lambda_2+1,\lambda_3+1)}_{(r)(s)}$ is always zero.

The reduced Kronecker coefficients that appear in this formula are all of the form $\overline{g}^{(\gamma_1,\gamma_2)}_{(r)(s)}$. These coefficients admit the following description, derived in Briand et al. (2008) from the description for the Kronecker coefficients indexed by two two–row shapes provided by Rosas (2001). An equivalent description for the reduced Kronecker coefficients indexed by two one–row shapes is given by Thibon (1991).

Theorem 4 (Briand et al. (2008)) *Let r, s and $\gamma_1 \geq \gamma_2$ be nonnegative integers and $h = (r, s, \gamma_1, \gamma_2)$. The reduced Kronecker coefficient $\overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$ counts the integral solutions to the system of inequalities $u_i(X, Y) + c_i(h) \geq 0$ for $i = 0, \ldots, 6$, where:*

$$\begin{array}{rcl} u_0(v) + c_0(h) &=& X - s \\ u_1(v) + c_1(h) &=& X - r \\ u_2(v) + c_2(h) &=& X + Y - r - s + \gamma_1 \\ u_3(v) + c_3(h) &=& Y \end{array} \qquad \begin{array}{rcl} u_4(v) + c_4(h) &=& Y - X + |\gamma| \\ u_5(v) + c_5(h) &=& -X - Y + r + s - \gamma_2 \\ u_6(v) + c_6(h) &=& X - Y - \gamma_1 \end{array} \tag{4}$$

In particular, the function $R : (r, s, \gamma_1, \gamma_2) \in \mathbb{Z}^4 \mapsto \overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$ is vector partition–like.

Theorem 4 and Formula (3) provide a piecewise quasipolynomial description for $G_{2,2}$ (see Remark 2). But the corresponding domains of quasipolynomiality obtained are neither closed, nor cones. The remainder of this work is devoted to correct this and obtain, still from Theorem 4 and Formula (3) a vector partition–like presentation for $G_{2,2}$.

The main tools are the Lemma 1, below, and an explicit vector partition–like presentation for the function R (section 4) showing that the lemma applies.

Let F_0, F_1, F_2 be the linear maps from $\mathbb{R}^5$ to $\mathbb{R}^4$ that send $(n, r, s, \gamma_1, \gamma_2)$ to $(r, s, \gamma_1, \gamma_2)$, $(r, s, n - \gamma_1 - \gamma_2, \gamma_2)$, $(r, s, n - \gamma_1 - \gamma_2, \gamma_1)$ respectively. Let T_1 and T_2 be the translations in $\mathbb{R}^4$ of vector $v_1 = (0, 0, 1, 0)$ and $v_2 = (0, 0, 1, 1)$ respectively.

Let Δ (resp. Δ') be the cone of $\mathbb{R}^5$ (resp. of $\mathbb{R}^4$) generated by all $(n, r, s, \gamma_1, \gamma_2) \in \mathbb{Z}^5$ (resp. all $(r, s, \gamma_1, \gamma_2) \in \mathbb{Z}^4$) such that the Kronecker coefficient $g_{(n-r,r)(n-s,s)}^{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2)}$ (resp. the reduced Kronecker coefficient $\overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$) is defined and positive. The explicit description of Δ is provided by Bravyi (2004) (see also the general approach by Klyachko (2004)). The cone Δ' is the image of Δ under F_0.

For $x \in \mathbb{Z}^5$ set $\chi_\Delta(x) = 1$ if $x \in \Delta$ and $\chi_\Delta(x) = 0$ otherwise. Then we can rewrite Formula 3 as follows:

$$G(x) = R \circ F_0(x) - \chi_\Delta(x) \cdot R \circ T_1 \circ F_1(x) + \chi_\Delta(x) \cdot R \circ T_2 \circ F_2(x)$$

where $G(x) = G(n, r, s, \gamma_1, \gamma_2) = g_{(n-r,r)(n-s,s)}^{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2)}$ when $(n - r, r)$, $(n - s, s)$, $(n - \gamma_1 - \gamma_2, \gamma_1, \gamma_2)$ are partitions, and $G(n, r, s, \gamma_1, \gamma_2) = 0$ otherwise.

After Remark 1, Theorem 1 will be proved if we show that all three vector partition–like functions $R \circ F_0$, $\chi_\Delta \cdot R \circ T_1 \circ F_1$ and $\chi_\Delta \cdot R \circ T_2 \circ F_2$ admit presentations with the same cone: $(\Delta, \mathfrak{F}_0, Q_0)$, $(\Delta, \mathfrak{F}_1, Q_1)$ and $(\Delta, \mathfrak{F}_2, Q_2)$.

That $R \circ F_0$ admits a presentation $(\Delta, \mathfrak{F}_0, Q_0)$ is immediate because $F_0^{-1}(\Delta') = \Delta$. To show that $\chi_\Delta \cdot R \circ T_1 \circ F_1$ and $\chi_\Delta \cdot R \circ T_2 \circ F_2$ also admit presentations with cone Δ we will need to apply two times Lemma 1 below, with $p = 5$, $q = 4$, $C = \Delta$, $C' = \Delta'$, $\phi = R$ and $F = F_i$, $v = v_i$ for $i = 1, 2$.

Given subsets A, B of $\mathbb{R}^q$ we denote with $A + B$ the set $\{a + b \,|\, a \in A, b \in B\}$. Given $v \in \mathbb{R}^q$ and I subset of $\mathbb{R}$ we denote with $I\, v$ the set $\{xv \,|\, x \in I\}$.

Lemma 1 *Let ϕ be a vector partition–like function on $\mathbb{Z}^q$ with presentation $(C', \mathfrak{F}', Q)$. Let C be a convex rational polyhedral cone of $\mathbb{R}^p$ and F an integral linear map from $\mathbb{R}^p$ onto $\mathbb{R}^q$. Let $v \in \mathbb{Z}^q$ and T be the translation of $\mathbb{R}^q$ of vector v. Let $\mathfrak{F}$ be the fan subdividing $C \cap F^{-1}(C')$, whose cells are all sets of the form $C \cap F^{-1}(\sigma')$ for σ' cell of $\mathfrak{F}'$.*

Assume that the cone $C \cap F^{-1}(C')$ is full–dimensional in $\mathbb{R}^p$. Assume also that:

(a) *Whenever H is a hyperplane separating two adjacent maximal cells σ'_1, σ'_2 of $\mathfrak{F}'$ such that $F(C)$ is not included in $H+\mathbb{R}_+v$, the following holds: The quasipolynomials $Q_{\sigma'_1}$ and $Q_{\sigma'_2}$ coincide on the integral points of the strip $H+]0;1]\,v$.*

(b) *Whenever H is a hyperplane containing a facet of C', such that $\mathbb{R}_+\,v\;+\;F(C)$ is not contained in the half–plane $H+C'$, the following holds: For all maximal cells σ' of $\mathfrak{F}'$ having a facet contained in H, the quasipolynomial $Q_{\sigma'}$ vanishes on the integral points of the strip $H+]0;1]\,v$.*

Then

(i) *The function $\phi\circ T\circ F$ is zero on the integral points of the closure of $C\setminus F^{-1}(C')$.*

(ii) *If $C\cap F^{-1}(\sigma')$ is a maximal cell of $\mathfrak{F}$ (where σ' is a maximal cell of $\mathfrak{F}'$) then $\phi\circ T\circ F$ and $Q_\sigma\circ T\circ F$ coincide on its integral points.*

Applying the lemma as indicated requires a precise description of a presentation $(\Delta', \mathfrak{F}_R, Q_R)$ of R. The next section provides such a description.

4 Formulas for the reduced Kronecker coefficients indexed by two one–row shapes

Let u_i and c_i, for $i=0,1,\ldots,6$ be the integral linear forms defined in (4). After Brion and Vergne (1997), the function ψ that associates to $y\in\mathbb{Z}^7$ the number of integral solutions of the system $u_i(X,Y)+y_i\geq 0$, $i=0,\ldots,6$ is a vector partition function. In particular, it admits a very well–described vector partition–like presentation $(C_\psi,\mathcal{F}_\psi,Q_\psi)$. The corresponding fan is the *chamber complex* of ψ, see Brion and Vergne (1997); Sturmfels (1995).

Remember (Theorem 4) that R is the function that associates the reduced Kronecker coefficient $\overline{g}_{(r)(s)}^{(\gamma_1,\gamma_2)}$ to $(r,s,\gamma_1,\gamma_2)\in\mathbb{Z}^4$. Then $R=\psi\circ c$, where c is the linear map from $\mathbb{R}^4$ to $\mathbb{R}^7$ that maps $h=(r,s,\gamma_1,\gamma_2)$ to $(c_0(h),c_1(h),\ldots,c_6(h))$. Therefore, one obtains a very explicit vector partition–like presentation $(c^{-1}(C_\psi),\mathfrak{F}_R,Q_R)$ for R by taking for $\mathfrak{F}_R$ the inverse image of $\mathfrak{F}_\psi$ under c, and for Q_R the family of functions $Q_{R,c^{-1}(\sigma)}=Q_{\psi,\sigma}\circ c$ for σ maximal cell of $\mathfrak{F}_\psi$. We present this description.

Let $h\in\mathbb{R}^7$. Denote with $\Pi(h)$ the set of real solutions of the system (4). For $i=0,1,\ldots,6$, let $L_i(h)$ be the line with equation $a_iX+b_iY+c_i(h)=0$ where $u_i(X,Y)=a_iX+b_iY$.

For any three elements i, j, k of $\{0,1,\ldots,6\}$ define:

$$f_{ijk}(h)=-\begin{vmatrix} a_i & a_j & a_k\\ b_i & b_j & b_k\\ c_i(h) & c_j(h) & c_k(h)\end{vmatrix} \tag{5}$$

Define also $f_{25}=\gamma_1-\gamma_2$ and $f_{46}=\gamma_2$. The linear form f_{25} (resp. f_{46}) is proportional to f_{25k} for all $k\neq 2,5$ (resp.: to f_{46k} for all $k\neq 4,6$) and its vanishing is the condition for the two parallel lines L_2 and L_5 (resp. L_4 and L_6) to coincide.

- **The cone** $c^{-1}(C_\psi)$ is equal to the cone Δ' introduced in Section 3. It is defined by the system of linear inequalities:

$$f_{145}\leq 0,\quad f_{045}\leq 0,\quad f_{356}\leq 0,\quad f_{035}\leq 0,\quad f_{135}\leq 0,\quad f_{25}\geq 0,\quad f_{46}\geq 0.$$

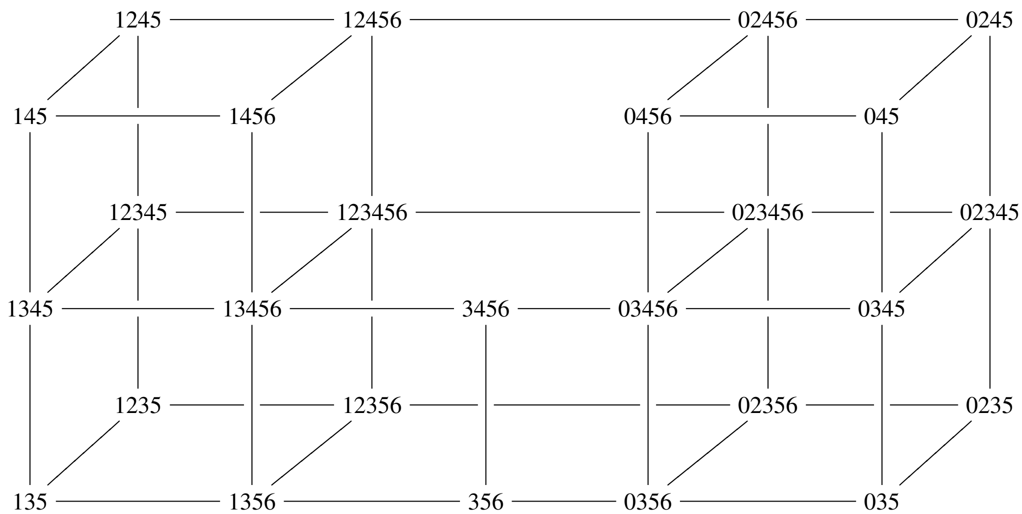

Fig. 1: The graph $\mathcal{G}$.

- **The fan $\mathfrak{F}_R$**: Let S be the locus of parameters h such that three lines $L_i(h)$, $L_j(h)$, $L_k(h)$ meet in $\Pi(h)$. The fan $\mathfrak{F}_R$ is the fan whose chambers (maximal open cells) are the connected components of $\Delta' \setminus S$. In each chamber σ the set of indices i such that $L_i(h)$ supports a side of $\Pi(h)$ is constant. Denote this set with $\mathsf{Sides}(\sigma)$. This set $\mathsf{Sides}(\sigma)$ determines σ. Therefore we denote a chamber σ with σ_I when $\mathsf{Sides}(\sigma) = I$, e.g. σ_{1245} for the chamber σ such that $\mathsf{Sides}(\sigma) = \{1, 2, 4, 5\}$. There are 26 chambers σ_I in $\mathfrak{F}_R$. The corresponding indices $I = \mathsf{Sides}(\sigma_I)$ are the vertices of the graph $\mathcal{G}$ in Figure 1. Adjacency in $\mathcal{G}$ represents adjacency in $\mathfrak{F}_R$: chambers σ_I and σ_J are adjacent (i.e. their closures have a common facet) if and only if I and J are adjacent vertices in $\mathcal{G}$. Observe that when σ_I and σ_J are adjacent then:
  - either I and J are obtained from each other by exchanging 0 and 1. Then σ_I and σ_J are separated by the hyperplane of equation $r = s$. There is $r > s$ on σ_I if $1 \in I$.
  - or one of the sets is obtained from the other by inserting a unique element. Say $J = I \cup \{j\}$ with $j \notin I$. If the elements of J are $p_1 < p_2 < \cdots < p_t$ say that the successor of p_q is p_{q+1}, for $q = 1, \ldots, t-1$, and that the successor of p_t is p_1. This defines a cyclic order on J. Let i and k be the predecessor and successor of j in this cyclic order. Then σ_I and σ_J are separated by the hyperplane of equation $f_{ijk} = 0$, and $f_{ijk} > 0$ on σ_I.
- **The quasipolynomial formulas on each maximal cell:** For simplicity we set $q_I = Q_{R,\overline{\sigma_I}}$. This is the quasipolynomial formula for R valid on the cell $\overline{\sigma_I}$ (the topological closure of the chamber σ_I). Rather than displaying explicit expressions for all quasi–polynomials q_I, it is enough to present one of them (we choose q_{135}) and display all differences $q_I - q_J$ for σ_I and σ_J adjacent. All quasi–polynomials q_I can be recovered easily from this information by chasing on the graph $\mathcal{G}$ (Figure 1),

ijk	$q_I(h) - q_J(h)$	**Values** δ **s. t.** $q_I = q_J$ **on** $f_{ijk} = \delta$
$613, 123, 134$ $603, 023, 034$	$\frac{1}{2} f_{ijk}(h)\,(f_{ijk}(h) - 1)$	$0, 1$
234	$\frac{1}{4}\,(f_{ijk}(h))^2 + \begin{cases} 0 & \text{if } f_{ijk}(h) \equiv 0 \mod 2 \\ -1/4 & \text{else.} \end{cases}$	$-1, 0, 1$
$345, 124, 561$ $024, 560$	$\frac{1}{4} f_{ijk}(h)\,(f_{ijk}(h) - 2) + \begin{cases} 0 & \text{if } f_{ijk}(h) \equiv 0 \mod 2 \\ 1/4 & \text{else} \end{cases}$	$0, 1, 2$

Tab. 1: The differences $q_I - q_J$ for σ_I and σ_J adjacent chambers of $\mathfrak{F}$.

e.g.

$$q_{1456} = (q_{1456} - q_{1\mathbf{3}456}) + (q_{13\mathbf{4}56} - q_{1356}) + (q_{135\mathbf{6}} - q_{135}) + q_{135}$$

There is:

$$q_{135}(r, s, \gamma_1, \gamma_2) = \frac{1}{2}\,(s - \gamma_2 + 1)\,(s - \gamma_2 + 2)$$

Let σ_I and σ_J be two adjacent chambers of $\mathfrak{F}$.

- If I and J are obtained from each other by exchanging 0 and 1 then $q_I = q_J$.
- If $J = I \cup \{j\}$ with $j \notin I$ then $q_I - q_J$ depend only of j and its predecessor i and successor k in J, and is as indicated in Table 1.

If σ_I and σ_J are adjacent, the quasi–polynomials q_I and q_J coincide not only on the affine hyperplane spanned by the facet $\overline{\sigma_I} \cap \overline{\sigma_J}$ but also on close parallel hyperplanes.

Proposition 1 *Let σ_I and σ_J be two adjacent chambers of $\mathfrak{F}$ such that $J = I \cup \{j\}$ with $j \notin I$. Let i and k be the predecessor and successor, respectively, of j in J.*

Then $q_I - q_J$ coincide on the affine hyperplanes $f_{ijk} = \delta$ for the values of δ given by the third column in Table 1.

Similarly, if the hyperplane H supports a facet of a maximal cell $\overline{\sigma_I}$, and this facet is contained in the border of Δ', then q_I vanishes on affine hyperplanes close and parallel to H.

Proposition 2 *Let σ_I be a chamber of $\mathfrak{F}$ and τ an external facet of $\overline{\sigma_I}$ (i.e. a facet contained in the border of Δ'). The hyperplane supporting τ admits as equation $f = 0$ where f is one of the linear forms f_{145}, f_{045}, f_{356}, f_{035}, f_{135}, f_{25}, f_{46}.*

The set of values $\delta \in \mathbb{Z}$ such that f vanishes identically on the affine hyperplane of equation $f = \delta$ is provided by Table 2.

It is immediate that $R \circ F_0$ has a vector partition–like presentations $(\Delta, \mathfrak{F}_0, Q_0)$. Propositions 1 and 2 are used to apply Lemma 1 and show that $\chi_\Delta \cdot R \circ T_1 \circ F_1$ and $\chi_\Delta \cdot R \circ T_2 \circ F_2$ have vector partition–like presentations $(\Delta, \mathfrak{F}_1, Q_1)$ and $(\Delta, \mathfrak{F}_2, Q_2)$. After Remark 1, this proves Theorem 1 and provides a way to compute a vector partition–like presentation for G and $G_{2,2}$.

Form f	**Chambers having a facet supported by $f=0$**	**Values δ such that q_I vanishes identically on $f=\delta$**
$f_{46}=\gamma_2$	$3456, 1456, 0456$	-1
$f_{25}=\gamma_1-\gamma_2$	$1245, 0245, 1235, 0235$	-1
$f_{145}=r-s-\gamma_1$	145	$1,2,3$
$f_{045}=s-r-\gamma_1$	045	$1,2,3$
$f_{356}=\lvert\gamma\rvert-r-s$	356	$1,2,3$
$f_{035}=\gamma_2-r$	035	$1,2$
$f_{135}=\gamma_2-s$	135	$1,2$

Tab. 2: The linear forms defining the facets of Δ'.

5 Formulas for the Kronecker coefficients indexed by two two–row shapes

Once the presentations $(\Delta, \mathfrak{F}_0, Q_0)$, $(\Delta, \mathfrak{F}_1, Q_1)$, $(\Delta, \mathfrak{F}_2, Q_2)$ for $R\circ F_0$, $\chi_\Delta\cdot R\circ T_1\circ F_1$ and $\chi_\Delta\cdot R\circ T_2\circ F_2$ have been determined, an explicit presentation $(\Delta, \mathfrak{F}_3, Q_3)$ for G is obtained: The cells of $\mathfrak{F}_3$ are the intersection $\sigma_0\cap\sigma_1\cap\sigma_2$ for σ_i a cell of $\mathfrak{F}_i$, $i\in\{0,1,2\}$. If $\sigma_0\cap\sigma_1\cap\sigma_2$ is a maximal cell of $\mathfrak{F}_3$ then the corresponding quasipolynomial formula for G is $Q_{0,\sigma_0}-Q_{1,\sigma_1}+Q_{2,\sigma_2}$. We computed the description for $\mathfrak{F}_3$ by using the Maple Package CONVEX by Franz (2006): it has 177 maximal cells. It turns out that on some of them G is given by the same quasipolynomial formulas, and that they can be glued together to form the maximal cells of a new fan $\mathfrak{F}_K$. In the new presentation $(\Delta, \mathfrak{F}_K, P)$ obtained for G the fan $\mathfrak{F}_K$ has only 74 maximal cells.

All 74 quasipolynomial formulas P_σ have the following form:

$$P_\sigma = 1/4\ Q_\sigma + 1/2\ L_\sigma + M_\sigma/4 \tag{6}$$

where Q_σ and L_σ are integral homogeneous polynomials in $(n, r, s, \gamma_1, \gamma_2)$ respectively quadratic and linear. The function M_σ takes integral values, fulfills $M_\sigma(0)/4 = 1$ and is constant on each coset of $\mathbb{Z}^5$ modulo the sublattice defined by $r+s\equiv n\equiv\gamma_1\equiv\gamma_2\equiv 0 \mod 2$.

Moreover, for all maximal cells σ, the functions Q_σ, L_σ are nonnegative on σ. This also holds for M_σ, for all cells σ except four. This makes specially easy studying the *support* of the Kronecker coefficients indexed by two two–row shapes. This is the set of all triples (λ,μ,ν) such that $g^{\lambda}_{\mu,\nu}>0$ and μ and ν have at most two parts.

We obtain the following result. Let $(n, r, s, \gamma_1, \gamma_2)\in\Delta$. Then $g^{(n-\gamma_1-\gamma_2,\gamma_1,\gamma_2)}_{(n-r,r)(n-s,s)}$ is zero if and only if at least one of the following five systems of conditions is fulfilled:

$$\begin{cases} n = 2\,s = 2\,r \\ \gamma_1 \text{ or } \gamma_2 \text{ odd.} \end{cases} \qquad \begin{cases} n = \max(2\,r, 2\,s, |\gamma|+\gamma_1) \\ \gamma_2 = 0 \\ r+s+\gamma_1 \text{ odd.} \end{cases} \qquad \begin{cases} n = max(2\,r, 2\,s) \\ |r-s| = 1 \\ \min(2\,r, 2\,s) \geq |\gamma|+\gamma_1 \\ \gamma_1 \text{ or } \gamma_2 \text{ even.} \end{cases}$$
$$\begin{cases} n = \max(2\,r, 2\,s) \\ \gamma_1 = \gamma_2 \\ r+s+\gamma_1 \text{ odd.} \end{cases} \qquad \begin{cases} n = |\gamma|+\gamma_1 = \max(2\,r, 2\,s) \\ r+s+\gamma_1 \text{ odd.} \end{cases} \tag{7}$$

This exhaustive description led us to a family of counterexamples for SH, a saturation conjecture formulated by Mulmuley (2007). The *stretching functions* $\widetilde{g}^{\lambda}_{\mu,\nu} : N \mapsto g^{N\lambda}_{N\mu,N\nu}$ attached to the Kronecker coefficients are quasipolynomials (Mulmuley, 2007). This means that for any fixed λ, μ, ν there exist an integer k and polynomials $p_1, p_2, \ldots, p_k$ such that for any $N \geq 1$, $\widetilde{g}^{\lambda}_{\mu,\nu}(N) = p_i(N)$ when $N \equiv i \mod k$. Mulmuley's SH conjecture stated that for any such description, $g^{\lambda}_{\mu,\nu} = 0 \Leftrightarrow F_1 = 0$. The rightmost system of conditions in (7) above provides a family of counterexamples to this conjecture (Briand et al., 2008). The discovery of these counterexamples led Mulmuley (2008) to propose a weaker form of the conjecture SH, still strong enough for the aims of Geometric Complexity Theory.

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A preorder-free construction of the Kazhdan-Lusztig representations of S_n, with connections to the Clausen representations

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Abstract. We use the polynomial ring $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ to modify the Kazhdan-Lusztig construction of irreducible S_n-modules. This modified construction produces exactly the same matrices as the original construction in [*Invent. Math* **53** (1979)], but does not employ the Kazhdan-Lusztig preorders. We also show that our modules are related by unitriangular transition matrices to those constructed by Clausen in [*J. Symbolic Comput.* **11** (1991)]. This provides a $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$-analog of results of Garsia-McLarnan in [*Adv. Math.* **69** (1988)].

Résumé. Nous utilisons l'anneau $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ pour modifier la construction Kazhdan-Lusztig des modules-S_n irreductibles dans $\mathbb{C}[S_n]$. Cette construction modifiée produit exactement les mêmes matrices que la construction originale dans [*Invent. Math* **53** (1979)], mais sans employer les préordres de Kazhdan-Lusztig. Nous montrons aussi que nos modules sont relies par des matrices unitriangulaires aux modules construits par Clausen dans [*J. Symbolic Comput.* **11** (1991)]. Ce résultat donne un $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$-analogue des résultats de Garsia-McLarnan dans [*Adv. Math.* **69** (1988)].

Keywords: Kazhdan-Lusztig, immanants, irreducible representations, symmetric group

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1 Introduction

In 1979, Kazhdan and Lusztig introduced [8] a family of irreducible modules for Coxeter groups and related Hecke algebras. These modules, which have many fascinating properties, also aid in the understanding of modules for quantum groups and other algebras. Important ingredients in the construction of the Kazhdan-Lusztig modules are the computation of certain polynomials in $\mathbb{Z}[q]$ known as *Kazhdan-Lusztig polynomials*, and the description of preorders on Coxeter group elements known as the *Kazhdan-Lusztig preorders*. These two tasks, which present something of an obstacle to one wishing to construct the modules, have become fascinating research topics in their own right. Even in the simplest case of a Coxeter group, the symmetric group S_n, the Kazhdan-Lusztig polynomials and preorders are somewhat poorly understood.

As an alternative to the "traditional" Kazhdan-Lusztig construction of type-A modules in terms of subspaces of the type-A Hecke algebra $H_n(q)$ (or of its specialization S_n), one may construct modules in terms of subspaces of a noncommutative "quantum polynomial ring" (or of its specialization $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$). Theoretically, this alternative offers no special advantage over the original construction. On the other hand, a simple modification of this alternative completely eliminates the need for the Kazhdan-Lusztig preorders in a new construction of S_n modules.

In Sections 2-3, we review essential definitions for the symmetric group, Hecke algebra, and Kazhdan-Lusztig modules. In Section 4 we review definitions related to the polynomial ring $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ and a particular $n!$-dimensional subspace of $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ called the *immanant space*. We recall the definition of the bideterminant basis of the immanant space, which gained notoriety in the work of Désarménien, Kung, and Rota [3], and Clausen's use of this basis to construct irreducible S_n-modules [2]. In Section 5, we use the basis of Kazhdan-Lusztig immanants studied in [12] to transfer the traditional Kazhdan-Lusztig representations to the immanant space.

Aspects of Clausen's work will then motivate us to modify the above representations in Section 6 and to apply vanishing properties of Kazhdan-Lusztig immanants obtained in [13]. This leads to our main result that the resulting representations, which do not rely upon the Kazhdan-Lusztig preorders, have matrices equal to those corresonding to the original Kazhdan-Lusztig representations in [8]. We finish in Section 7 by showing that the relationship between the bideterminant and Kazhdan-Lusztig immanant bases studied in [13] leads to unitriangular transition matrices relating Clausen's irreducible representations of S_n to those of Kazhdan-Lusztig. This provides an analog in $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ of the Garsia-McLarnan result [6, Thm. 5.3] relating Young's natural representations to those of Kazhdan-Lusztig in $\mathbb{C}[S_n]$.

2 Tableaux and the symmetric group

We call a weakly decreasing sequence $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ of positive integers with $\sum_{i=1}^{\ell}\lambda_i = r$ an *integer partition* of r, and we denote this by $\lambda \vdash r$ or $|\lambda| = r$. A partial ordering on integer partitions of r called *dominance order* is given by $\lambda \succeq \mu$ if and only if

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i, \text{ for all } i = 1,\ldots,\ell. \tag{1}$$

From an integer partition λ we can construct a *Ferrers diagram* which has λ_i left justified dots in row i. When we replace the dots in a diagram with $1,\ldots,r$ we have a *Young tableau* where the *shape* of the tableau is λ. An *injective tableau* is merely one in which the replacing is performed injectively, i.e. the $1,\ldots,r$ appear exactly once in the tableau. We call a tableau *column-(semi)strict* if its entries are (weakly) increasing downward in columns. A tableau is *row-(semi)strict* if entries (weakly) increase from left to right in rows. We call a tableau *semistandard* if it is column-strict and row-semistrict, and *standard* if it is semistandard and injective. We define transposition of partitions $\lambda \mapsto \lambda^\top$ (also known as conjugation) and tableaux $T \mapsto T^\top$ in a manner analogous to matrix transposition. We define a *bitableau* to be a pair of tableaux of the same shape, and say that it posesses a certain tableau property if both of its tableaux posess this property.

For each partition λ we define the *superstandard tableau* of shape λ to be the tableau $U(\lambda)$ having entries in reading order. For example,

$$U((4,2,1)) = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & & \\ 7 & & & \end{matrix}\;. \tag{2}$$

The standard presentation of S_n is given by generators $s_1,\ldots,s_{n-1}$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1,\ldots,n-1, \\ s_is_js_i &= s_js_is_j, && \text{if } |i-j| = 1, \\ s_is_j &= s_js_i, && \text{if } |i-j| \geq 2. \end{aligned} \tag{3}$$

Let S_n act on rearrangements of the letters $[n] = \{1,\ldots,n\}$ by

$$s_i \circ v_1 \cdots v_n \underset{\text{def}}{=} v_1 \cdots v_{i-1}v_{i+1}v_iv_{i+2}\cdots v_n. \tag{4}$$

For each permutation $w = s_{i_1} \cdots s_{i_\ell} \in S_n$ we define the *one-line notation* of w to be the word

$$w_1 \cdots w_n \underset{\text{def}}{=} s_{i_1} \circ (\cdots (s_{i_\ell} \circ (1 \cdots n)) \cdots). \tag{5}$$

For each $w \in S_n$ we define two tableaux, $P(w), Q(w)$ which are obtained from the *Robinson-Schensted correspondence* using row insertion to the one-line notation of w. (See, e.g., [14, Sec. 3.1].) It is well known that these tableaux satisfy $P(w^{-1}) = Q(w)$. Since $\mathrm{sh}(P(w)) = \mathrm{sh}(Q(w))$ we can define the *shape of a permutation* as $\mathrm{sh}(w) = \mathrm{sh}(P(w))$.

Given a permutation $w \in S_n$ expressed in terms of generators $w = s_{i_1} \cdots s_{i_\ell}$ we say this expression is *reduced* if w cannot be expressed as a shorter product of generators of S_n. We call the *length* of a permutation $w \in S_n$ $\ell(w) = \ell$, in the previous equation. We define the *Bruhat order* on S_n by $v \leq w$ if some (equivalently every) reduced expression for w contains a reduced expression for v as a subword (The reader is referred to [1] for more on this topic). Throughout this paper we will use w_0 to denote the unique maximal element in the Bruhat order. Multiplying a permutation on the right by w_0 also changes the bitableau of the Robinson-Schensted correspondence for that permutation. Specifically, this change can be described in terms of transposition and Schützenberger's evacuation algorithm. (See [1, Appendix].)

Lemma 2.1 *If $v \in S_n$, then $P(v) = evac(P(vw_0))^{\top}$.*

3 Kazhdan-Lusztig representations

Given an indeterminate q we define the *Hecke algebra*, $H_n(q)$, to be the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra with multiplicative identity $\widetilde{T}_e$ generated by $\{\widetilde{T}_{s_i}\}_{i=1}^{n-1}$ with relations

$$\widetilde{T}_{s_i}^2 = (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_{s_i} + \widetilde{T}_e, \qquad \text{for } i = 1, \ldots, n-1, \tag{6}$$

$$\widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j}, \qquad \text{if } |i-j| = 1, \tag{7}$$

$$\widetilde{T}_{s_i}\widetilde{T}_{s_j} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}, \qquad \text{if } |i-j| \geq 2. \tag{8}$$

We then can define $\widetilde{T}_w$ for any $w \in S_n$ by $\widetilde{T}_w = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_l}}$ where $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression. Inverses of generators are given by

$$\widetilde{T}_{s_i}^{-1} = \widetilde{T}_{s_i} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})\widetilde{T}_e = \widetilde{T}_{s_i} - q^{\frac{-1}{2}}(q-1)\widetilde{T}_e. \tag{9}$$

When $q = 1$ we see that this presentation is simply that of the group algebra, $\mathbb{C}[S_n]$.

An important involution of the Hecke algebra is the so called *bar involution*. The involution is defined as

$$\sum_w a_w \widetilde{T}_w \mapsto \overline{\sum_w a_w \widetilde{T}_w} = \sum_w \overline{a_w}\, \overline{\widetilde{T}_w} \tag{10}$$

where

$$\overline{q} = q^{-1}, \qquad \overline{\widetilde{T}_w} = \left(\widetilde{T}_{w^{-1}}\right)^{-1}. \tag{11}$$

The *Kazhdan-Lusztig basis*, $\{C_w(q) \,|\, w \in S_n\}$, is the unique basis of $H_n(q)$ such that the basis elements are invariant under the bar involution, $\overline{C}_w = C_w$ for all $w \in S_n$, and that C_w in terms of the $\{\widetilde{T}_v\}$ is given by

$$C_w = \sum_{v \leq w} \epsilon_{v,w} q_{v,w} \overline{P_{v,w}(q)}\, \widetilde{T}_v, \tag{12}$$

where $P_{v,w}(q)$ are polynomials in q of degree at most $\frac{\ell(w)-\ell(v)-1}{2}$ and where we define the convenient notation $\epsilon_{v,w} = (-1)^{\ell(w)-\ell(v)}$, $q_{v,w} = (q^{\frac{1}{2}})^{\ell(w)-\ell(v)}$. These polynomials are known as the *Kazhdan-Lusztig polynomials* and in fact belong to $\mathbb{N}[q]$.

Kazhdan and Lusztig also introduced another basis $\{C'_w(q) \,|\, w \in S_n\}$ with similar properties which we shall call the *signless Kazhdan-Lusztig basis*. $C_w(q)$ and $C'_w(q)$ are related by $C'_w(q) = \psi(C_w(q))$, where ψ is the semilinear map defined by

$$\psi : q^{\frac{1}{2}} \mapsto q^{\frac{-1}{2}} \text{ and } \widetilde{T}_w \mapsto \epsilon_{e,w}\widetilde{T}_w. \tag{13}$$

Thus $C'_w(q)$ is also bar invariant and its expression in terms of $\{\widetilde{T}_v\}$ is

$$C'_w(q) = \sum_{v \leq w} q_{v,w}^{-1} P_{v,w}(q)\widetilde{T}_v. \tag{14}$$

As a preliminary to the proof of the existence and uniqueness of their bases Kazhdan and Lusztig also defined the following function

$$\mu(u,v) \underset{\text{def}}{=} \begin{cases} \text{coefficient of } q^{(\ell(v)-\ell(u)-1)/2} \text{ in } P_{u,v}(q) & \text{if } u < v, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Note that $\mu(u,v) = 0$ if $\ell(v) - \ell(u)$ is even since $P_{u,v}(q)$ has only integer powers of q. Also, it is well known that $P_{u,v}(q) = P_{w_0uw_0,w_0vw_0}(q)$, and therefore that $\mu(u,v) = \mu(w_0uw_0, w_0vw_0)$. Kazhdan and Lusztig showed further [8, Cor. 3.2] $\mu(u,v) = \mu(w_0v, w_0u)$, even though $P_{u,v}(q)$ and $P_{w_0v,w_0u}(q)$ are not equal in general.

In the existence proof of the Kazhdan-Lusztig basis in [8, Pf. of Thm. 1.1] an expression for the action of $\widetilde{T}_{s_i}$ on the basis element $C_w(q)$ is given by

$$\widetilde{T}_{s_i} C_w(q) = \begin{cases} q^{\frac{1}{2}} C_w(q) + C_{s_iw}(q) + \displaystyle\sum_{\substack{v<w \\ s_iv<v}} \mu(v,w) C_v(q) & \text{if } s_iw > w, \\ -q^{\frac{-1}{2}} C_w(q) & \text{if } s_iw < w. \end{cases} \tag{16}$$

Along with these bases Kazhdan-Lusztig defined a preorder on S_n in order to construct representations of $H_n(q)$. This preorder, called the *left preorder*, is denoted by $\leq_L$ and is defined as the transitive closure of $\lessdot_L$ where $u \lessdot_L v$ if $C_u(q)$ has nonzero coefficient in the expression of $\widetilde{T}_w C_v(q)$ for some $w \in S_n$. It is known that $w \leq_L v$ implies $\mathrm{sh}(v) \preceq \mathrm{sh}(w)$.

We follow the desription in [7, Appendix] of the Kazhdan-Lusztig construction of an irreducible $H_n(q)$-module (S_n-module) indexed by partition $\lambda \vdash n$. Choosing tableau T of shape λ, we allow $H_n(q)$ to act by left multiplication on

$$K^\lambda \underset{\text{def}}{=} \mathrm{span}\{C_w(q) \,|\, P(w) = T\}, \tag{17}$$

regarded as the quotient $\mathrm{span}\{C_v(q) \,|\, v \leq_L w\}/\mathrm{span}\{C_v(q) \,|\, v \leq_L w, v \not\geq_L w\}$. The quotient is necessary because K^λ is not in general closed under the action of $H_n(q)$. In particular, for $\lambda \neq (n)$ we have the containments $K^\lambda \subset H_n(q)K^\lambda \subseteq K^\lambda \oplus \mathrm{span}\{C_v(q) \,|\, v \leq_L w, v \not\geq_L w\}$.

4 The polynomial ring and Clausen's representations

Let $x = (x_{i,j})$ be an $n \times n$-matrix of variables. The polynomial ring $\mathbb{C}[x]$ has a natural grading $\mathbb{C}[x] = \oplus_{r\geq 0}\mathcal{A}_r$, where $\mathcal{A}_r$ is the span of all monomials of total degree r. Further decomposing each space $\mathcal{A}_r$, we define a multigrading

$$\mathbb{C}[x] = \bigoplus_{r\geq 0} \mathcal{A}_r = \bigoplus_{r\geq 0} \bigoplus_{L,M} \mathcal{A}_{L,M}, \tag{18}$$

where $L = \{\ell(1) \leq \ldots \leq \ell(r)\}$ and $M = \{m(1) \leq \ldots \leq m(r)\}$ are r-element multisets of $[n]$, written as weakly increasing sequences, and where $\mathcal{A}_{L,M}$ is the span of monomials whose row and column indices are given by L and M, respectively. We define the *generalized submatrix* of x with respect to (L, M) by

$$x_{L,M} = \begin{bmatrix} x_{\ell(1),m(1)} & \cdots & x_{\ell(1),m(r)} \\ x_{\ell(2),m(1)} & \cdots & x_{\ell(2),m(r)} \\ \vdots & & \vdots \\ x_{\ell(r),m(1)} & \cdots & x_{\ell(r),m(r)} \end{bmatrix}. \tag{19}$$

We refer to the space

$$\mathcal{A}_{[n],[n]} = \mathrm{span}\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in S_n\}, \tag{20}$$

as the *immanant space*, and define the notation $x^{u,v} = x_{u_1,v_1} \cdots x_{u_n,v_n}$ for permutations $u, v \in S_n$.

Given subsets $I, J \subset [n]$ we define the I, J *minor* of x to be the determinant $\Delta_{I,J}(x) = \det(x_{I,J})$, and given a column-strict, semistandard bitableau (S, T) we define the *bideterminant*

$$(S \,|\, T)(x) = \Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x), \tag{21}$$

where $I_1, \ldots, I_k$ are the sets of entries in columns $1, \ldots, k$ of S and $J_1, \ldots, J_k$ are the sets of entries in columns $1, \ldots, k$ of T. For example,

$$\left(\begin{matrix} 1 & 2 & 4 \\ 3 & & \end{matrix} \,\middle|\, \begin{matrix} 1 & 3 & 4 \\ 2 & & \end{matrix}\right)(x) = \Delta_{13,12}(x)x_{2,3}x_{4,4} = x_{1,1}x_{3,2}x_{2,3}x_{4,4} - x_{1,2}x_{3,1}x_{2,3}x_{4,4}. \tag{22}$$

For each permutation w in S_n, define

$$R_w(x) \underset{\mathrm{def}}{=} (Q(w)^{\top} \,|\, P(w)^{\top})(x) \tag{23}$$

where $(P(w), Q(w))$ is the bitableau obtained from the Robinson-Schensted row insertion algorithm. With little effort one can see that each semistandard bideterminant can be viewed as a standard bideterminant of a generalized submatrix. Similarly, standard bideterminants evaluated at generalized submatrices are either zero or a semistandard bideterminant. Therefore, for multisets L, M of $[n]$ with $|L| = |M| = r$ we have that the set $\{R_w(x_{L,M}) \,|\, w \in S_r\}$ is a spanning set for the space $\mathcal{A}_{L,M}$.

A natural S_n-action on $\mathbb{C}[x]$ is given by

$$s_i \circ g(x) \underset{\mathrm{def}}{=} g(s_i x), \tag{24}$$

where $g \in \mathbb{C}[x]$ and $s_i x$ is interpreted as the product of the permutation matrix of s_i and x. Clausen [2, Thm. 8.1] constructed an irreducible S_n-module indexed by $\lambda \vdash n$ by letting $M = 1^{\lambda_1} \cdots n^{\lambda_n}$ and defining

$$B^{\lambda} \underset{\mathrm{def}}{=} \mathrm{span}\{R_w(x_{[n],M}) \,|\, P(w)^{\top} = U(\lambda)\}. \tag{25}$$

The matrix representations arising from these modules are the exactly the same as those of *Young's natural representation*. This fact follows from the isomorphism found in [10, Sec. 4.2] between bideterminants and the polytabloids, which are the basis of the irreducible S_n-modules in Young's natural representation.

5 Kazhdan-Lusztig immanants

We now define a generalization of the polynomial ring $\mathbb{C}[x]$ called the *quantum polynomial ring*, $\mathcal{A}(n;q)$. The ring $\mathcal{A}(n;q)$ is a noncommutative $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra on n^2 generators $x = (x_{1,1} \dots, x_{n,n})$ with relations (assuming $i < j$ and $k < \ell$),

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{i,\ell}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{j,k}, \\ x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}. \end{aligned} \tag{26}$$

A natural basis for the quantum polynomial ring consists of the set of monomials in lexicographic order. Analogous to the multigrading of $\mathbb{C}[x]$ is the multigrading

$$\mathcal{A}(n;q) = \bigoplus_{r \geq 0} \mathcal{A}_r(n;q) = \bigoplus_{r \geq 0} \bigoplus_{L,M} \mathcal{A}_{L,M}(n;q), \tag{27}$$

where $\mathcal{A}_r(n;q)$ is the span of all monomials of total degree r, and where $\mathcal{A}_{L,M}(n;q)$ is the span of monomials whose row and column indices are given by r-element multisets L and M of $[n]$. We again call the space $\mathcal{A}_{[n],[n]}(n;q) = \mathrm{span}\{x^{e,w} \mid w \in S_n\}$ the *immanant space* of $\mathcal{A}(n;q)$.

Define a left action of the Hecke algebra on $\mathcal{A}_{[n],[n]}(n;q)$ by

$$\widetilde{T}_{s_i} \circ x^{e,v} = x^{s_i,v} = \begin{cases} x^{e,s_iv} & \text{if } s_iv > v, \\ x^{e,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,v} & \text{if } s_iv < v. \end{cases} \tag{28}$$

Related to the bar involution on $H_n(q)$ is another *bar involution* on $\mathcal{A}_{[n],[n]}(n;q)$ defined by

$$\sum_w a_w x^{e,w} \mapsto \overline{\sum_w a_w x^{e,w}} = \sum_w \overline{a_w}\ \overline{x^{e,w}} \tag{29}$$

where

$$\overline{q} = q^{-1}, \qquad \overline{x^{e,w}} = x^{w_0,w_0w} = x_{n,w_n} \cdots x_{1,w_1}. \tag{30}$$

Lemma 5.1 *The bar involutions of (10) and (29) are compatible with the action of $H_n(q)$ on $\mathcal{A}_{[n],[n]}(n;q)$. That is,*

$$\overline{\widetilde{T}_{s_i} \circ x^{e,v}} = \overline{\widetilde{T}_{s_i}} \circ \overline{x^{e,v}} \tag{31}$$

for all $v \in S_n$.

Proof: Omitted. □

It is known that there is a unique, bar-invariant basis of $\mathcal{A}_{[n],[n]}(n;q)$ closely related to the Kazhdan-Lusztig basis of the Hecke algebra. We call the elements of this basis the *Kazhdan-Lusztig immanants* $\{\mathrm{Imm}_v(x;q)\}$. Further description of the Kazhdan-Lusztig immanants can be found in [12, Sec. 2], [4], and also [5]. For the benefit of the reader we provide a proof analogous to that in [8, Thm. 1.1].

Theorem 5.2 *For any $v \in S_n$, there is a unique element $\mathrm{Imm}_v(x;q) \in \mathcal{A}_{[n],[n]}(n;q)$ such that*

$$\overline{\mathrm{Imm}_v(x;q)} = \mathrm{Imm}_v(x;q) \tag{32}$$

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}(q) x^{e,w}, \tag{33}$$

where $Q_{v,w}(q)$ are polynomials in q of degree $\leq \frac{\ell(w)-\ell(v)-1}{2}$ if $v < w$ and $Q_{v,v}(q) = 1$.

Proof: Omitted. □

The polynomials $Q_{u,v}(q)$ in the above proof are actually the *inverse Kazhdan-Lusztig polynomials*, found in [8, Sec. 3]. They are related to the Kazhdan-Lusztig polynomials by

$$Q_{u,v}(q) = P_{w_0 v, w_0 u}(q) = P_{v w_0, u w_0}(q). \tag{34}$$

We can now describe a left action of $H_n(q)$ on the immanant space by its action on the Kazhdan-Lusztig immanants.

Corollary 5.3 *The left action of the Hecke algebra on $\mathcal{A}_{[n],[n]}(n;q)$ is described by*

$$\widetilde{T}_{s_i}\, \mathrm{Imm}_v(x;q) = \begin{cases} q^{\frac{1}{2}} \mathrm{Imm}_v(x;q) + \mathrm{Imm}_{s_i v}(x;q) + \displaystyle\sum_{\substack{w>v \\ s_i w > w}} \mu(v,w) \mathrm{Imm}_w(x;q) & \textit{if } s_i v < v, \\ -q^{\frac{-1}{2}} \mathrm{Imm}_v(x;q) & \textit{if } s_i v > v. \end{cases} \tag{35}$$

Proof: Omitted. □

A deeper connection between the Kazhdan-Lusztig immanants and the Kazhdan-Lusztig basis is evident in the $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-bilinear form on $\mathcal{A}_{[n],[n]}(n;q) \times H_n(q)$ defined by by $\langle x^{e,v}, \widetilde{T}_w \rangle = \delta_{v,w}$. Specifically, we have $\langle \mathrm{Imm}_v(x;q), C'_w(q) \rangle = \delta_{v,w}$, so the signless Kazhdan-Lusztig basis is dual to the basis of Kazhdan-Lusztig immanants.

In the following lemma we relate the definition of the left preorder in the Hecke algebra with these Kazhdan-Lusztig immanants. The results in the proof will also be essential in describing the relationship of the $H_n(q)$ representations associated with the Kazhdan-Lusztig basis and immanants.

Lemma 5.4 *Let $v, v' \in S_n$. Then $v \lessdot_L v'$ if $\mathrm{Imm}_{v'}(x;q)$ appears with nonzero coefficient in $\widetilde{T}_u \mathrm{Imm}_v(x;q)$ for some $u \in S_n$.*

Proof: Omitted. □

With Lemma 5.4 we can now express the preorder in terms of the Kazhdan-Lusztig immanants. We can now construct $H_n(q)$-modules indexed by $\lambda \vdash n$, like in [7, Appendix], with the Kazhdan-Lusztig immanants. We choose a tableau T of shape λ and allow $H_n(q)$ to act by left multiplication on

$$V^\lambda \underset{\mathrm{def}}{=} \mathrm{span}\{\mathrm{Imm}_w(x;q) \,|\, P(w)^\top = T\}, \tag{36}$$

regarded as the quotient $\mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_L w\} / \mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_L w, v \not\leq_L w\}$. The quotient is necessary because like K^λ, V^λ is not in general closed under the action of $H_n(q)$. In particular, whenever $\lambda \neq (1^n)$ we have the containments

$$V^\lambda \subset H_n(q) V^\lambda \subseteq V^\lambda \oplus \mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_L w, v \not\leq_L w\}. \tag{37}$$

6 Generalized submatrices in Kazhdan-Lusztig immanants

In [13] Rhoades and Skandera found vanishing conditions for immanants and bideterminants of matrices having repeated rows and columns. Using these results we will evaluate the Kazhdan-Lusztig immanants at generalized submatrices, similar to Clausen's construction. It turns out that if we evaluate the Kazhdan-Lusztig immanants at specific generalized submatrices we can eliminate the quotient needed in the construction of the S_n-modules.

To express the vanishing results we need to define the *column repetition partition* of an $n \times n$-matrix A by

$$\nu_{[j]}(A) \underset{\mathrm{def}}{=} (\nu_1, \ldots, \nu_k), \tag{38}$$

where k is the number of distinct columns in the $n \times j$-submatrix $A_{[n],[j]}$, and $\nu_1, \ldots, \nu_k$ are the multiplicities with which distinct columns appear, written in weakly decreasing order.

We will write $\mathrm{Imm}_w(x) = \mathrm{Imm}_w(x; 1)$ for Kazhdan-Lusztig immanants in $\mathcal{A}_{[n],[n]}$. The following vanishing results found in [13, Thms. 4.10-4.11] will be instrumental in later proofs.

Lemma 6.1 *Let $w \in S_n$ and A be an $n \times n$ matrix. If $\mathrm{sh}(w^{-1}_{[j]}) \not\succeq \nu_{[j]}(A)$ for some $1 \leq j \leq n$, then $\mathrm{Imm}_w(A) = R_w(A) = 0$.*

We can now see that the left S_n-action defined in Corollary 5.3 (setting $q = 1$) actually describes an S_n-module if we evaluate the immanants at generalized submatrices.

Theorem 6.2 *Let $\lambda \vdash n$ and set $M = 1^{\lambda_1} \cdots n^{\lambda_n}$. Define*

$$W^{\lambda} \underset{\mathrm{def}}{=} \mathrm{span}\{\mathrm{Imm}_w(x_{[n],M}) \,|\, P(w)^{\top} = U(\lambda)\}, \tag{39}$$

where $U(\lambda)$ is the superstandard tableau of shape λ. Then W^{λ} is an S_n-module.

Proof: By (37) we know that it suffices to show that $\mathrm{Imm}_v(x_{[n],M}) = 0$ for $v >_L w$ where $P(w)^{\top} = U(\lambda)$. Since $v >_L w$ then we know that $\mathrm{sh}(w) \succ \mathrm{sh}(v)$. The column multiplicity partition of $x_{[n],M}$ is $\nu(x_{[n],M}) = \lambda$. So $\mathrm{sh}(v) \prec \mathrm{sh}(w) = \nu(x_{[n],M})$. Thus $\mathrm{sh}(v) = \mathrm{sh}(v^{-1}) \not\succeq \nu(x_{[n],M})$. Therefore, by Lemma 6.1, $\mathrm{Imm}_v(x_{[n],M}) = 0$ for all $v >_L w$. $\square$

The condition for inclusion in the basis of this module is $P(w)^{\top} = U(\lambda)$ unlike the condition, $P(w)^{\top} = T$ where $\mathrm{sh}(T) = \lambda$, used in the definition of V^{λ} above. The need for the change in conditions will become clear later on in Proposition 7.2.

We would now like to show that these modules, W^{λ}, are isomorphic to the modules constructed by the action S_n on the Kazhdan-Lusztig basis. We will actually achieve this result by generalizing to the action of the Hecke algebra on the Kazhdan-Lusztig basis and immanants. We shall then show that the action of $\widetilde{T}_{s_i}$ on either basis yields equal matrices, up to ordering of the basis elements. Let $\rho_1 : S_n \to GL(K^{\lambda})$ and $\rho_2 : S_n \to GL(W^{\lambda})$ be the representations of S_n defined by letting $q = 1$ in the left actions described in (16) and Corollary 5.3, respectively.

Theorem 6.3 *Let $X_1(v)$, $X_2(v)$ be the matrices of $\rho_1(v)$, $\rho_2(v)$ with respect to the Kazhdan-Lusztig basis and the Kazhdan-Lusztig immanant basis. Then $X_1(v) = X_2(v)$.*

Proof: First, we construct K^λ as in (17) with $T = \text{evac}(U(\lambda))$. Let $B = \{v \in S_n \mid P(v) = \text{evac}(U(\lambda))\}$. From Lemma 2.1 we see that if $C_w(1)$ is a basis element of K^λ, i. e. $w \in B$, then $P(ww_0)^\top = \text{evac}(P(w)) = U(\lambda)$. Thus if $w \in Bw_0$, then $\text{Imm}_w(x_{[n],M})$ is a basis element of W^λ, as in (39). Define coefficients $a^{s_i}_{v,w}$ for each generators s_i of S_n and $v, w \in B$ so that

$$\widetilde{T}_{s_i} C_v(1) = \sum_{w \in B} a^{s_i}_{v,w} C_w(1). \tag{40}$$

Then from the proof of Lemma 5.4 we see that for all $v \in B$

$$\widetilde{T}_{s_i} \text{Imm}_{vw_0}(x_{[n],M}) = \sum_{w \in B} a^{s_i}_{v,w} \text{Imm}_{ww_0}(x_{[n],M}). \tag{41}$$

Thus $X_1(s_i) = X_2(s_i)$. Since any element of $v \in S_n$ is a product of generators we have that $X_1(v) = X_2(v)$. □

Corollary 6.4 *The modules W^λ indexed by partitions $\lambda \vdash n$ are the irreducible S_n-modules.*

This result follows immediately from the fact that the modules K^λ are the irreducible S_n-modules.

7 Transition matrices

The goal of this section is to show that the S_n representations constructed with the bideterminant basis and the Kazhdan-Lusztig immanant basis are related by unitriangular matrices. The inspiration for this result comes from the work of Garsia and McLarnan where they showed a similar relationship between the S_n representations constructed with Young's natural basis and the Kazhdan-Lusztig basis [6, Thm. 5.3]. The results of this section are also similar to the work of McDonough and Pallikaros where they found a unitriangular relationship between the $H_n(q)$ representations constructed with Specht modules and the cell modules of Kazhdan and Lusztig [11, Thm. 4.1].

For two standard tableaux S, T with $\text{sh}(S), \text{sh}(T) \vdash n$ we can define *iterated dominance of tableaux* by $S \trianglelefteq_I T$ if for all $j \in [n]$ we have $\text{sh}(T_{[j]}) \preceq \text{sh}(U_{[j]})$, where $T_{[j]}$ is the subtableau of T consisting of all entries less then or equal to j. Also we define the permutation $w_{[j]} \in S_j$ from $w \in S_n$ by arranging $1, \dots, j$ in the same relative order of the first j terms in the one line notation of w. For two standard bitableaux we define *iterated dominance of bitableaux* by componentwise iterated dominance of the tableaux. Thus we have that $(T, U) \trianglelefteq_I (T', U')$ if $T \trianglelefteq_I T'$ and $U \trianglelefteq_I U'$. Using this order on bitableaux and the Robinson-Schensted association we can define *iterated dominance of permutations* by $v \leq_I w$ if and only if $(P(v)^\top, Q(v)^\top) \trianglelefteq_I (P(w)^\top, Q(w)^\top)$ (see [13]).

The following result can be found in [9, Thm. 5.1.4 C] and is usually attributed to Schützenberger.

Lemma 7.1 *If $v \in S_n$ and $1 \leq i \leq n$ then* $\text{sh}(w_{[i]}) = \text{sh}(Q(w)^\top_{[i]})$.

The following proposition was alluded to earlier in the construction of the irreducible S_n-modules with Kazhdan-Lusztig immanants with repeated columns.

Proposition 7.2 *Let $\lambda \vdash n$ and M be defined as above. If* $\text{sh}(w) \prec \lambda$ *or if* $\text{sh}(w) = \lambda$ *and* $P(w)^\top \neq U(\lambda)$ *then* $\text{Imm}_w(x_{[n],M}) = 0$.

Proof: When $\mathrm{sh}(w) \prec \lambda$ then we see that

$$\mathrm{sh}(w^{-1}) = \mathrm{sh}(w) \prec \lambda = \nu(x_{[n],M}). \tag{42}$$

Thus by Lemma 6.1 we see that $\mathrm{Imm}_w(x_{[n],M}) = 0$. Suppose $P(w)^\top \neq U(\lambda)$. It follows that $P(w)^\top \lhd_I U(\lambda)$ since $U(\lambda)$ is maximal in iterated dominance of tableaux among all tableaux of shape λ. Since $P(w)^\top \lhd_I U(\lambda)$ then there exists an index j such that

$$\mathrm{sh}(P(w)^\top_{[j]}) \succ \mathrm{sh}(U(\lambda)_{[j]}). \tag{43}$$

Let i be the greatest index so that $\lambda_1 + \cdots + \lambda_i \leq j$. By Lemma 7.1 we can see that $\mathrm{sh}(w^{-1}_{[j]}) = \mathrm{sh}(P(w)^\top_{[j]})$. For the generalized submatrix $x_{[n],M}$ we can see that

$$\begin{aligned} \nu_{[j]}(x_{[n],M}) &= (\lambda_1, \ldots, \lambda_i, j - (\lambda_1 + \cdots + \lambda_i)) && (44)\\ &= \mathrm{sh}(U(\lambda)_{[j]}), && (45) \end{aligned}$$

since the entries of $U(\lambda)$ are in reading order. Thus after combining results we have

$$\mathrm{sh}(w^{-1}_{[j]}) = \mathrm{sh}(P(w)^\top_{[j]}) \succ \mathrm{sh}(U(\lambda)_{[j]}) = \nu_{[j]}(x_{[n],M}). \tag{46}$$

Thus $\mathrm{Imm}_w(x_{[n],M}) = 0$ if $P(w)^\top \lhd_I U(\lambda)$ by Lemma 6.1. □

An analogous result for the bideterminants has essentially the same proof.

Proposition 7.3 *Let $\lambda \vdash n$ and M be defined as above. If $\mathrm{sh}(w) \prec \lambda$ or if $\mathrm{sh}(w) = \lambda$ and $P(w)^\top \neq U(\lambda)$ then $R_w(x_{[n],M}) = 0$.*

In [13, Sec. 6] Rhoades-Skandera described a filtration of the immanant space. Define for a partition $\lambda \vdash n$ the permutation $w(\lambda)$ to be the unique element of S_n where $P(w(\lambda))^\top = Q(w(\lambda))^\top = U(\lambda)$. We then can define

$$U_\lambda(x) = \mathrm{span}\{R_v(x) \,|\, v \leq_I w(\lambda)\}. \tag{47}$$

Rhoades and Skandera showed that the space $U_\lambda(x)$ also has a spanning set of certain Kazhdan-Lusztig immanants. Specifically their result [13, Thm. 6.4] implies the following.

Lemma 7.4 *Fix a partition $\lambda \vdash n$ then*

$$U_\lambda(x) = \mathrm{span}\{R_v(x) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{\mathrm{Imm}_v(x) \,|\, \mathrm{sh}(v) \preceq \lambda\}. \tag{48}$$

We now can conclude that the modules constructed with the bideterminants and the Kazhdan-Lusztig immanants, both specialized at a generalized submatrix, are the same S_n-module.

Theorem 7.5 *Let $\lambda \vdash n$ then $B^\lambda = W^\lambda$, where B^λ is defined in (25) and W^λ is defined in (39).*

Proof: With $M = 1^{\lambda_1} \cdots n^{\lambda_n}$ we can specialize $U_\lambda(x)$ at the generalized submatrix $x_{[n],M}$ to get

$$U_\lambda(x_{[n],M}) = \mathrm{span}\{R_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{\mathrm{Imm}_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\}. \tag{49}$$

Then by Proposition 7.3

$$\mathrm{span}\{R_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{R_v(x_{[n],M}) \,|\, P(v)^\top = U(\lambda)\} = B^\lambda. \tag{50}$$

Similarly, by Proposition 7.2

$$\mathrm{span}\{\mathrm{Imm}_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{\mathrm{Imm}_v(x_{[n],M}) \,|\, P(v)^{\top} = U(\lambda)\} = W^{\lambda}. \tag{51}$$

Thus

$$U_{\lambda}(x_{[n],M}) = B^{\lambda} = W^{\lambda}. \tag{52}$$

□

By [13, Cor. 5.11], the Kazhdan-Lusztig immanant basis, evaluated at generalized submatrices, is related to the bideterminant basis by a unitriangular matrix. Specifically we have the following result.

Proposition 7.6 *Fix a permutation $v \in S_n$ and n-element multiset M of $[n]$. Define coefficients $\{d_{u,v}^{[n],M} \,|\, u \in S_n\}$ by*

$$R_v(x_{[n],M}) = \sum_{u \in S_n} d_{u,v}^{[n],M} \mathrm{Imm}_u(x_{[n],M}). \tag{53}$$

Then we have $d_{u,v}^{[n],M} = 0$ if $u \not\leq_I v$ and $d_{v,v}^{[n],M} = 1$ for all $v \in S_n$.

Proposition 7.6 describes the change of basis matrix between the bideterminants and the Kazhdan-Lusztig immanant basis of $\mathcal{A}_{[n],M}(n;q)$. The change-of-basis matrix is given by

$$\mathcal{Z} = \left[d_{u,v}^{[n],M} \right], \tag{54}$$

where the permutations are in any linear extension of the iterated dominance order. These change-of-basis matrices imply a close relationship between the S_n representations generated by the Kazhdan Lusztig immanants, evaluated at generalized submatrices, and the bideterminant representations. Let $\rho_3 : S_n \to GL(B^{\lambda})$ be the representation of S_n defined in (25) and let $X_3(v)$ be the matrix of this representation with respect to the Clausen basis. By the above argument, the matrix $X_2(v)$ defined before Theorem 6.3 is the matrix of $\rho_3(v)$ with respect to the Kazhdan-Lusztig immanant basis.

Theorem 7.7 *For all $v \in S_n$, $X_3(v) = Z^{-1} X_2(v) Z$, where Z is a unitriangular matrix.*

Proof: Let Z be the principal submatrix of the matrix $\mathcal{Z}$ (54) corresponding to rows and columns indexed by permutations u satisfying $P(u) = U(\lambda)^{\top}$. □

Since the matrix representation arising from the bideterminants is the equivalent to that of Young's natural representation then the previous theorem gives a new interpretation of Garsia and McLarnan's result [6, Thm. 5.3], in the setting of the polynomial ring.

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A max-flow algorithm for positivity of Littlewood-Richardson coefficients

Dedicated to Michael Clausen on the occasion of his 60th birthday.

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Abstract. Littlewood-Richardson coefficients are the multiplicities in the tensor product decomposition of two irreducible representations of the general linear group $\mathrm{GL}(n, \mathbb{C})$. They have a wide variety of interpretations in combinatorics, representation theory and geometry. Mulmuley and Sohoni pointed out that it is possible to decide the positivity of Littlewood-Richardson coefficients in polynomial time. This follows by combining the saturation property of Littlewood-Richardson coefficients (shown by Knutson and Tao 1999) with the well-known fact that linear optimization is solvable in polynomial time. We design an explicit *combinatorial* polynomial time algorithm for deciding the positivity of Littlewood-Richardson coefficients. This algorithm is highly adapted to the problem and it is based on ideas from the theory of optimizing flows in networks.

Résumé. Les coefficients de Littlewood-Richardson sont les multiplicités dans la décomposition du produit tensoriel de deux représentations irréductibles du groupe général linéaire $\mathrm{GL}(n, \mathbb{C})$. Ces coefficients ont plusieurs interprétations en combinatoire, en théorie des représentations et en géométrie. Mulmuley et Sohoni ont observé qu'on peut décider si un coefficient de Littlewood-Richardson est positif en temps polynomial. C'est une conséquence de la propriété de saturation des coefficients de Littlewood-Richardson (démontrée par Knutson et Tao en 1999) et le fait bien connu que la programmation linéaire est possible en temps polynomial. Nous décrivons un algorithme *combinatoire* pour décider si un coefficient de Littlewood-Richardson est positif. Cet algorithme est bien adapté au problème et il utilise des idées de la théorie des flots maximaux sur des réseaux.

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1 Introduction

The *Schur polynomials* form a $\mathbb{Z}$-basis of the ring $\Lambda \subseteq \mathbb{Z}[X_1, \dots, X_n]$ of symmetric polynomials in n variables. They are indexed by partitions λ into at most n parts, which are vectors $\lambda \in \mathbb{N}^n$ of weakly decreasing natural numbers. There are various characterizations of the Schur polynomials s_λ. The shortest one is algebraic and states that $s_\lambda = \Delta^{-1} \det[x_j^{\lambda_i+n-i}]_{1\le i,j\le n}$, where $\Delta = \prod_{i<j}(X_i - X_j)$. A

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combinatorial characterization is $s_\lambda = \sum_T X_1^{\alpha_1(T)} \cdots X_n^{\alpha_n(T)}$, where the sum is over all semistandard tableaux T of shape λ and $\alpha_i(T)$ counts the number of occurences of i in T. We note that s_λ has degree $|\lambda| := \sum_i \lambda_i$. For more information see Stanley (Sta99).

The *Littlewood-Richardson coefficients* $c_{\lambda\mu}^\nu$ are the coefficients in the expansion of the product of two Schur functions in the basis of Schur functions: $s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$, where the sum is over all partitions ν such that $|\nu| = |\lambda| + |\mu|$. The Littlewood-Richardson coefficients play an important role in various mathematical disciplines (combinatorics, representation theory, algebraic geometry). For instance, they describe the multiplicities in the tensor product decomposition of irreducible representations of the general linear group $\mathrm{GL}(n, \mathbb{C})$. Also, they determine the multiplication in the cohomology ring of the Grassmann varieties.

The well-known Littlewood-Richardson rule provides a combinatorial description of the numbers $c_{\lambda\mu}^\nu$ and leads to several algorithms for computing them, e.g., see (CS84). However, all of these algorithms take exponential time in the size of the input partitions (consisting of integers encoded in binary). Narayanan (Nar06) proved that this is unavoidable: the computation of $c_{\lambda\mu}^\nu$ is a $\#\mathbf{P}$-complete problem. Hence there does not exist a polynomial time algorithm for computing $c_{\lambda\mu}^\nu$ under the widely believed hypothesis $\mathbf{P} \neq \mathbf{NP}$. Surprisingly, as pointed out by Mulmuley and Sohoni (MS05), the positivity of $c_{\lambda\mu}^\nu$ can be decided by a polynomial time algorithm. This can be seen as follows.

Knutson and Tao (KT99) proved the following saturation property: $c_{N\lambda N\mu}^{N\nu} > 0$ implies $c_{\lambda\mu}^\nu > 0$, where N denotes a positive integer. This has implications for various, seemingly unrelated mathematical problems, see Fulton (Ful00). It also has algorithmic consequences: the Littlewood-Richardson rule implies that $\exists N\ c_{N\lambda N\mu}^{N\nu} > 0$ can be rephrased as the feasibility problem of a rational polyhedron. It is well-known that the latter can be solved in polynomial time, cf. (GLS93). Hence by the above saturation property, $c_{\lambda\mu}^\nu > 0$ can be decided in polynomial time.

In (MS05) it was asked whether there is a purely combinatorial algorithm for deciding $c_{\lambda\mu}^\nu > 0$ in polynomial time that does not use linear programming, i.e., one similar to the max-flow or weighted matching problems in combinatorial optimization. The polytopes arising in that setting are integral, i.e. all of its vertices are integral. However, the polytopes occuring in the Littlewood-Richardson situation are not integral, cf. (KTT04).

In this paper we answer the above question in the affirmative by exhibiting a combinatorial polynomial time algorithm for deciding $c_{\lambda\mu}^\nu > 0$. Our algorithm also yields a proof of the saturation property. Knutson, Tao and Woodward (KTW04) proved a conjecture by Fulton stating that $c_{\lambda\mu}^\nu = 1$ iff $c_{N\lambda N\mu}^{N\nu} = 1$ for all N. As a by-product of our developments, we obtain a new proof of this conjecture as well as a combinatorial polynomial time algorithm for deciding multiplicity freeness. (So far this works for strictly descreasing partitions λ, μ, ν only.)

Here is a rough outline of the main ideas underlying our algorithm. By the description in (KT99; Buc00), $c_{\lambda\mu}^\nu$ counts the integral hives with border labels prescribed by λ, μ, ν on the big triangle graph Δ (see §2 for the notation). We establish a bijection between the integral hives and certain integral flows on the dual graph of Δ, which we call *hive flows*. Using this, we convert the problem of deciding the positivity of $c_{\lambda\mu}^\nu$ into the problem of optimizing a certain linear function (the throughput) on the set of integral points of the polyhedron P^b of b-bounded hive flows. We solve this combinatorial optimization problem in analogy to the well-known Ford-Fulkerson algorithm (AMO93) for maximizing flows in networks. We start with the zero flow and iteratively increase the flow f by a fixed integer amount along a cycle while staying in P^b. The set of feasible directions in which to increase can be interpreted as the convex cone of feasible flows of an auxiliary network $\mathsf{RES}^b(f)$. It is essential and nontrival that one can increase at least by one unit. This key property is expressed in Theorem 11. All of this only works when f is *shattered*, but this nondegeneracy condition is easy to obtain.

In order to obtain a polynomial time algorithm we replace our algorithm by a scaled version that increases flows by integral multiples of 2^k, but several technical difficulties have to be overcome.

Due to page restrictions in this extended abstract we can only provide sketchs for some of the proofs. The symbol □ at the end of a statement indicates the complete omission of proof. For detailed arguments we refer to the diploma thesis of the second author (Ike08).

Acknowledgment We thank Fritz Eisenbrand for valuable discussions.

2 Preliminaries

2.1 Saturation Property and hive description

Let $\lambda, \mu, \nu \in \mathbb{N}^n$ be partitions such that $|\nu| = |\lambda| + |\mu|$. We start with a triangular array of vertices, $n+1$ on each side, as seen in Figure 1.

This graph is called the *big triangle graph* Δ with vertex set H. To avoid confusion with vertices in other graphs that will be introduced later, vertices in Δ are denoted by underlined capital letters ($\underline{A}$, $\underline{B}$, etc.). The vertices on the border of the big triangle graph form the set B. Denote with $\underline{T}$ the top vertex of Δ and set $H' := H \setminus \{\underline{T}\}$. The graph Δ is subdivided into n^2 small triangles whose corners are graph vertices. We call a triangle in Δ an *upright* triangle if it is of the form '△'. Otherwise ('▽') we call the triangle an *upside down* triangle. By a *rhombus* $\Diamond(\underline{A}, \underline{B}, \underline{C}, \underline{D})$ with $\underline{A}, \underline{B}, \underline{C}, \underline{D} \in H$ we mean the union of two small triangles next to each other, where $\underline{A}$ is the acute vertex of the upright triangle and $\underline{B}$, $\underline{C}$ and $\underline{D}$ are the other vertices in counter-clockwise direction (see Figure 2). Two rhombi are called *overlapping* if they share exactly one triangle.

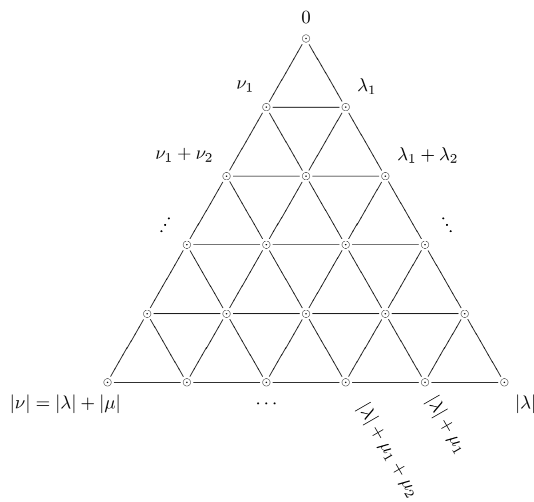

Fig. 1: The big triangle graph Δ with additional border labels resulting from partitions λ, μ and ν, ($n = 5$).

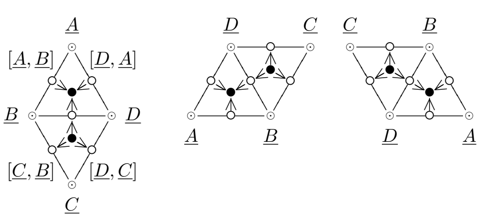

Fig. 2: Rhombus labelings in all possible ways.

Let $h \in \mathbb{R}^H$ be a labeling of the vertices of Δ with real numbers. We call h *integral* iff $h \in \mathbb{Z}^H$. The *h-slack* $\sigma(\Diamond, h)$ of a rhombus $\Diamond := \Diamond(\underline{A}, \underline{B}, \underline{C}, \underline{D})$ is defined as

$$\sigma(\Diamond, h) := \big(h(\underline{B}) + h(\underline{D})\big) - \big(h(\underline{A}) + h(\underline{C})\big)$$

(note that $\underline{A}$ and $\underline{C}$ are the acute vertices of $\Diamond$). The rhombus $\Diamond$ is called *h-flat* iff $\sigma(\Diamond, h) = 0$.

A vertex labeling $h \in \mathbb{R}^H$ is called a *hive* iff the *hive inequalities* $\sigma(\Diamond, h) \geq 0$ are satisfied for all rhombi $\Diamond$. We note that the set of hives is a convex cone.

For partitions λ, μ and ν with $|\nu| = |\lambda| + |\mu|$, let $b(\lambda, \mu, \nu) \in \mathbb{Z}^B$ denote the border with labels as in Figure 1. This vertex labeling of B is called the *target border* of λ, μ, ν. A border $b \in \mathbb{R}^B$ is called *regular* if for all border vertices $\underline{A}, \underline{B}, \underline{C} \in B$, that are consecutive vertices on the same side of the big hive triangle, we have $b(\underline{A}) + b(\underline{C}) < 2b(\underline{B})$. If λ, μ and ν are strictly decreasing partitions, then the target border $b(\lambda, \mu, \nu)$ is regular.

The following theorem is a consequence of the Littlewood-Richardson rule (KT99; Buc00; PV05).

Theorem 1 *Let λ, μ, ν be three partitions such that $|\nu| = |\lambda| + |\mu|$. Then $c^{\nu}_{\lambda\mu}$ is the number of integral hives with border labels $b(\lambda, \mu, \nu)$.*

2.2 Flows in networks

In this section we introduce basic terminology and facts about flows and augmenting-path algorithms, cf. (AMO93).

Graphs A *graph* $G = (V, E)$ consists of a finite set V of vertices and a finite set $E \subseteq \binom{V}{2}$ of edges whose elements are unordered pairs of distinct vertices. Vertices v and w are called *adjacent* if $\{v, w\} \in E$. We call a vertex v and an edge e *incident* if $v \in e$.

Flows on digraphs Given a graph $G = (V, E)$ we can assign an edge direction to each edge in E by endowing G with an *orientation function* $o\colon E \to V$ that maps each edge to one of its vertices. This turns G into a *directed graph (digraph)*. We call an edge $\{v, w\}$ *directed away from* v (or *directed towards* w) iff $o(\{v, w\}) = v$. The set of edges incident to a vertex $v \in V$ can then be divided into the set $\delta_{\text{in}}(v)$ of edges that are directed towards v and the set $\delta_{\text{out}}(v)$ of edges that are directed away from v. For a mapping $f : E \to \mathbb{R}$ we define

$$\delta_{\text{in}}(v, f) := \sum_{e \in \delta_{\text{in}}(v)} f(e) \quad \text{and} \quad \delta_{\text{out}}(v, f) := \sum_{e \in \delta_{\text{out}}(v)} f(e).$$

Definition 2 (Flow and throughput) A *flow* f on a digraph $G = (V, E, o)$ is a mapping $f\colon E \to \mathbb{R}$ which satisfies $\delta_{\text{in}}(v, f) = \delta_{\text{out}}(v, f)$ for all $v \in V$. We call $\delta(v, f) := \delta_{\text{in}}(v, f)$ the *throughput* of f in v. The flow f is called *integral* iff it only takes integral values.

We note that negative flows on edges are allowed and that therefore the flows on a digraph G form a real vector space $F(G)$.

Capacities We can assign capacities to a digraph $G = (V, E, o)$ by defining two functions $u\colon E \to [0, \infty], e \mapsto u_e$ and $l\colon E \to [-\infty, 0], e \mapsto l_e$, which we call the *upper bound* and *lower bound*, respectively. A digraph with capacities is sometimes called a *network* in the literature. We will tacitly assume that $u_e = \infty$ and $l_e = -\infty$ if no other requirement for the edge e is made.

Definition 3 (Feasible flow) Let $G = (V, E, o)$ be a digraph with capacities u and l. A flow f on G is said to be *feasible with respect to u and l* if $l_e \leq f(e) \leq u_e$ for each edge $e \in E$. The set $P_{\text{feas}}(G) \subseteq F(G)$ of feasible flows on G is said to be the *polyhedron of feasible flows on G*.

Definition 4 (Cycle) (1) A *cycle* $c = (v_1, \ldots, v_\ell, v_{\ell+1} = v_1)$ on a graph $G = (V, E)$ is a finite sequence of at least 3 vertices in V in which for all $1 \leq i < j \leq \ell$ we have that $v_i \neq v_j$ and for all $1 \leq i \leq \ell$ we have $\{v_i, v_{i+1}\} \in E$. We call $e = \{v_i, v_{i+1}\}$ the edges of c and write $e \in c$. The length $\ell(c) := \ell$ of c is defined as the number of edges in c.

(2) Suppose an orientation function o is fixed. We call $e = \{v_i, v_{i+1}\}$ a forward edge of c iff e is directed away from v_i, and a backwards edge of c otherwise. The cycle c is called *well-directed* iff all of its forward edges e satisfy $u(e) > 0$ and all of its backward edges e satisfy $l(e) < 0$.

(3) We assign to c the *cycle flow* f_c by setting $f_c(e) = 1$ for its forward edges e and $f_c(e) = -1$ for its backward edges e. All other edges carry flow zero.

To simplify notation, we will identify a cycle c with its cycle flow f_c. We remark that c is well-directed iff there is an $\varepsilon > 0$ such that εc is a feasible flow.

It is well-known and easy to see that feasible flows can be decomposed into cycles as follows.

Lemma 5 (Flow decomposition) *Given a digraph $G = (V, E, o)$ and a flow f on G. Then there exist cycles $c_1, \ldots, c_m$ on G, $m \le |E|$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_{>0}$ such that $f = \sum_{i=1}^{m} \alpha_i c_i$ and for all i and all $e \in c_i$ we have $\mathrm{sgn}(f(e)) = \mathrm{sgn}(c_i(e))$. We call α_i the* multiplicity *of the cycle c_i in the decomposition. Moreover, if f is feasible, then $c_1, \ldots, c_m$ are well-directed.* □

3 Hives and flows

We transfer the problem of finding an integral hive into the language of flows.

The graph structure We now define a bipartite planar digraph $G = (V, E, o)$, which is essentially the dual graph of Δ. The definition is similar to one in (Buc00): G has one fat black vertex in the middle of each small triangle of Δ. In addition there is one circle vertex on every triangle side (see Figure 3). We denote a circle vertex between two vertices $\underline{A}$ and $\underline{B}$ of upright triangles (read in counterclockwise direction) as $[\underline{A}, \underline{B}]$. Each fat black vertex is adjacent to the three circle vertices on the sides of its triangle. There is an additional fat black vertex o with edges from o to all circle vertices that lie on the border of the big triangle. The graph G is embedded in the plane in a way such that the top vertex $\underline{T}$ lies in the outer face, where a *face* is a region bounded by edges, including the outer, infinitely-large region.

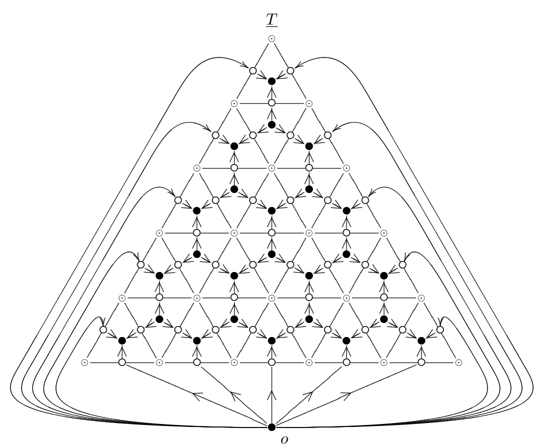

Fig. 3: The digraph G and graph Δ.

Next we assign a direction to each edge in G (see Figure 3). The edges incident to o are directed from o towards the border of the big triangle graph. The edges in an upright triangle are directed towards the incident fat black vertex, while the edges in an upside down triangle are directed towards the incident circle vertex.

Let $F = F(G)$ denote the vector space of flows on G. Note that a flow f on G is completely defined by its throughput $\delta([\underline{A}, \underline{B}], f)$ on each circle vertex $[\underline{A}, \underline{B}]$.

Winding numbers Let $\underline{A} \in H$. We define $\mathscr{N}\mathscr{W}(\underline{A})$ to be the set of circle vertices in V that lie on the northwest diagonal drawn from $\underline{A}$. This diagonal hits a border vertex $\underline{B} \in B$. Define $\mathscr{N}\mathscr{E}(\underline{A})$ to be the set of circle vertices in V that lie on the northeast diagonal drawn from that border vertex $\underline{B}$. Now define the *winding number* of a vertex $\underline{A} \in H$ with respect to a flow $f \in F$ as

$$\mathrm{wind}(\underline{A}, f) = \sum_{v \in \mathscr{N}\mathscr{W}(\underline{A})} \delta(v, f) - \sum_{v \in \mathscr{N}\mathscr{E}(\underline{A})} \delta(v, f).$$

The winding number is linear in the flow f. We note that for a cycle flow f_c, this coincides with the familiar topological notion of the winding number of the cycle c with respect to the point $\underline{A}$.

Lemma 6 *There is an isomorphism $\eta : \mathbb{R}^{H'} \to F$ between the real vector space $\mathbb{R}^{H'}$ of vertex labels in Δ, in which the top vertex $\underline{T}$ has value 0, and the real vector space F of flows on G. For $h \in \mathbb{R}^{H'}$ the flow $\eta(h)$ is defined by requiring $\delta([\underline{A}, \underline{B}], \eta(h)) = h(\underline{A}) - h(\underline{B})$. The inverse of η is given by $\eta^{-1}(f)(\underline{A}) = \mathit{wind}(\underline{A}, f)$ for $f \in F$. Both η and η^{-1} preserve integrality.* □

Hive inequalities on flows As η is an isomorphism, we can identify a flow $f \in F$ with its vertex labeling $h = \eta^{-1}(f) \in \mathbb{R}^{H'}$. We define the f-*slack* of a rhombus $\diamondsuit = \diamondsuit(\underline{A}, \underline{B}, \underline{C}, \underline{D})$ as $\sigma\big(\diamondsuit, f\big) := \sigma\big(\diamondsuit, h\big)$. Note that $\sigma\big(\diamondsuit, f\big) = \delta([\underline{D}, \underline{C}], f) - \delta([\underline{A}, \underline{B}], f) = \delta([\underline{D}, \underline{A}], f) - \delta([\underline{C}, \underline{B}], f)$. Thus the hive inequalities $\sigma\big(\diamondsuit, h\big) \geq 0$ translate into the following simpler linear inequalities

$$\delta([\underline{A}, \underline{B}], f) \leq \delta([\underline{D}, \underline{C}], f). \tag{1}$$

We call f a *hive flow* if $\eta^{-1}(f)$ is a hive. Similarly, we speak of f-flat rhombi.

4 The algorithmic idea

We now introduce the optimization problem to be solved for deciding whether a Littlewood-Richardson coefficient is positive.

Define the set $\mathscr{S} \subset V$ of *source vertices* as the set of all circle border vertices of G lying on the right or bottom border of the big triangle. Define the set $\mathscr{T} \subset V$ of *sink vertices* as the set of all circle border vertices of G lying on the left border of the big triangle. We call $\delta(f) := \sum_{s \in \mathscr{S}} \delta(s, f)$ the (global) *throughput* of f. Note that $\delta\colon F \to \mathbb{R}, f \mapsto \delta(f)$ is a linear map.

For a given border vertex labeling $b \in \mathbb{R}^B$ we define now the network G^b on the digraph G by introducing the capacities $u_{\{o,s\}} := b(\underline{A}) - b(\underline{B})$ for all $s = [\underline{A}, \underline{B}] \in \mathscr{S}$ and $l_{\{o,t\}} := b(\underline{A}) - b(\underline{B})$ for all $t = [\underline{A}, \underline{B}] \in \mathscr{T}$. We call the feasible flows of G^b b-bounded. The set of b-bounded hive flows is a polyhedron that will be denoted by P^b.

We call an edge $\{o, s\}$ *used to capacity* with respect to a flow $f \in P^b$ iff $\delta(s, f) = u_{\{o,s\}}$. Similarly, we say that the edge $\{o, t\}$ is *used to capacity* with respect to f iff $\delta(t, f) = l_{\{o,t\}}$.

The following lemma shows the significance of the *polyhedron P^b of b-bounded hive flows*.

Lemma 7 *Let $b = b(\lambda, \mu, \nu)$ be the target border of partitions λ, μ and ν with $|\nu| = |\lambda| + |\mu|$. Then*

(1) For all $f \in P^b$ we have $\delta(f) \leq |\nu|$.

(2) $c_{\lambda\mu}^{\nu}$ equals the number of integral $f \in P^b$ such that $\delta(f) = |\nu|$.

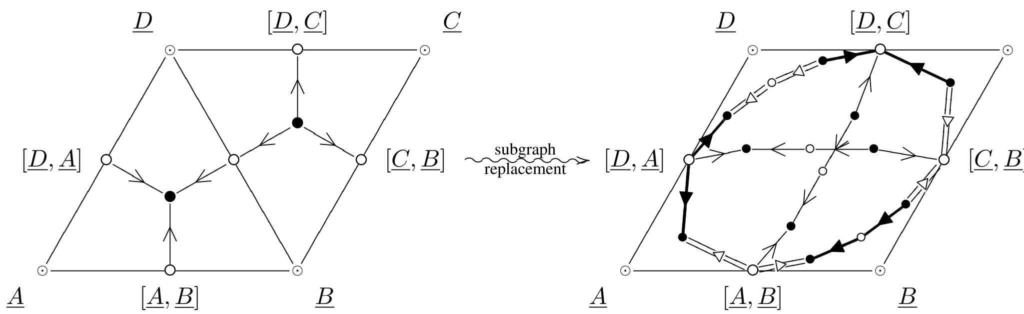

Fig. 4: The subgraph replacement for an f-flat rhombus $\Diamond(\underline{A},\underline{B},\underline{C},\underline{D})$.

Proof: (1) We have $\delta(f) = \sum_{s\in\mathscr{S}} \delta(s,f) \le |\lambda| + |\mu| = |\nu|$.

(2) According to Theorem 1, $c^{\nu}_{\lambda\mu}$ equals the number of integral hives with border labels b. Lemma 6 shows that this number equals the number of b-bounded integral hive flows with throughput $|\nu|$. □

Lemma 7 translates the problem of deciding positivity of Littlewood-Richardson coefficients to the problem of optimizing the linear function δ on the integer points of the polyhedron P^b. We will solve this combinatorial optimization problem in analogy to the well-known Ford-Fulkerson algorithm (AMO93) for maximizing flows in networks. We start with the zero flow and iteratively increase the flow f by one unit along a cycle while staying in P^b. The set of feasible directions in which to increase can be interpreted as the convex cone of feasible flows of an auxiliary network, that we introduce in the next section. We will thus be able to replace the hive inequalities (1) locally by capacity constraints.

5 The residual network

We start with a general definition. Given a polyhedron P in a real vector space V and a vector $f \in P$. We define the *cone of feasible directions $C_f(P)$ of P at f* as

$$C_f(P) := \{d \in V \mid \exists \varepsilon > 0 : f + \varepsilon d \in P\}.$$

We note that $P \cap U = (f + C_f(P)) \cap U$ for a small neighborhood U of f.

Recall that a rhombus $\Diamond$ is called *f-flat* with respect to a flow f iff $\sigma(\Diamond, f) = 0$. The flow f is called *shattered* iff there is no pair of overlapping f-flat rhombi.

Lemma 8 (Shattering) *Let λ, μ, ν be strictly decreasing partitions and let $f \in P^b$. Then one can algorithmically find a shattered integral flow $\mathrm{shatter}(f) \in P^b$ such that $\delta(\mathrm{shatter}(f)) = \delta(f)$.*

Proof: This was basically shown by Buch in his proof of the Saturation Property (Buc00). □

Fix a b-bounded shattered hive flow f. We now introduce the residual network $\mathsf{RES}^b(f)$.

The *residual digraph* $\mathsf{RES}(f)$ w.r.t. f is constructed as follows. The vertex and edge set of $\mathsf{RES}(f)$ are initially the vertex and edge set of G. Then each f-flat rhombus $\Diamond = \Diamond(\underline{A},\underline{B},\underline{C},\underline{D})$ is replaced by the digraph illustrated in Figure 4. We remove all inner vertices of $\Diamond$, keep $[\underline{A},\underline{B}]$, $[\underline{C},\underline{B}]$, $[\underline{D},\underline{C}]$ and $[\underline{D},\underline{A}]$, and add 14 vertices as in the figure. Then we add edges, some of which are fat black, some of which are fat white and some of which are normal as in the figure.

Note that in $\mathsf{RES}(f)$, the circle vertex $[\underline{B},\underline{D}]$ is no longer present. The graph $\mathsf{RES}(f)$ is still bipartite, but may not be planar.

We next define the *residual network* $\mathsf{RES}^b(f)$. For each fat black edge e we set $l_e := 0$. This enforces that a well-directed cycle can only pass such e in the direction of e (compare Definition 4). For each fat white edge e we set $u_e := 0$. This enforces that a well-directed cycle can only pass such e in the reverse direction of e.

We now introduce additional capacities that are dependent on b. For the edges $e = \{o, s\}$, $s \in \mathscr{S}$, that are used to capacity with respect to the flow f in G^b we set $u_e := 0$. Moreover, for the edges $e = \{o, t\}$, $t \in \mathscr{T}$, that are used to capacity with respect to the flow f in G^b we set $l_e := 0$. We note that the feasible flows on $\mathsf{RES}^b(f)$ form a convex cone.

The following lemma shows that feasible flows on $\mathsf{RES}^b(f)$ give the directions from $f \in P^b$ that do not point out of P^b.

Lemma 9 (Residual Correspondence) *Let f be a b-bounded shattered hive flow. There is a natural surjective linear map*

$$\gamma\colon F(\mathsf{RES}(f)) \to F(G)$$

that preserves the throughput on all circle vertices that are both in $\mathsf{RES}(f)$ and G. We have

$$P_{\text{feas}}(\mathsf{RES}^b(f)) = \gamma^{-1}(C_f(P^b)).$$

The map γ preserves integrality and the global throughput δ. □

For example, the flow $\gamma(f)(e)$ on the edge e directed away from $[\underline{A},\underline{B}]$ in G is just the sum of flows $f(e_1) + f(e_2) + f(e_3)$ of the edges e_i directed away from $[\underline{A},\underline{B}]$ in $\mathsf{RES}(f)$.

The following lemma gives an optimality criterion for optimizing the linear function δ on P^b.

Lemma 10 (Optimality Test) *Let f be a shattered b-bounded hive flow and let $\delta\colon F(G) \to \mathbb{R}$ be a linear function. Then f maximizes δ on P^b iff $\mathsf{RES}^b(f)$ has no well-directed cycle c with $\delta(\gamma(c)) > 0$.*

Proof: As δ is linear, f does not maximize δ on P^b iff there exists $d \in F$ such that $d \in P^b - f$ and $\delta(d) > 0$. Since $P^b - f$ equals $C_f(P^b)$ in a small neighborhood of 0, the latter condition is equivalent to the existence of some $d \in C_f(P^b)$ with $\delta(d) > 0$. According to Lemma 9, this is equivalent to the existence of a some $d' \in P_{\text{feas}}(\mathsf{RES}^b(f))$ with $\delta(\gamma(d')) > 0$. We now show that this is equivalent to the existence of a well-directed cycle c on $\mathsf{RES}^b(f)$ with $\delta(\gamma(c)) > 0$.

Let $d' \in P_{\text{feas}}(\mathsf{RES}^b(f))$ with $\delta(\gamma(d')) > 0$. Lemma 5 states that d' can be decomposed as $d' = \sum_{i=1}^{M} \alpha_i c_i$ where c_i are well-directed cycles on $\mathsf{RES}^b(f)$ and $\alpha_i > 0$. Thus $\delta\left(\gamma(d')\right) = \sum_{i=1}^{M} \alpha_i \delta(\gamma(c_i))$ is positive and hence there is a well-directed cycle c_i with $\delta(\gamma(c_i)) > 0$.

Conversely, if c is a well-directed cycle on $\mathsf{RES}^b(f)$ with $\delta(\gamma(c)) > 0$, then εc is a feasible flow on $\mathsf{RES}^b(f)$ for sufficiently small $\varepsilon > 0$. □

6 The main algorithm LRPA

The algorithm LRPA (Littlewood-Richardson Positivity Algorithm) is listed below.

Algorithm LRPA

Input: $\lambda, \mu, \nu \in \mathbb{N}^n$ strictly decreasing partitions with $|\nu| = |\lambda| + |\mu|$.
Output: Decide whether $c^{\nu}_{\lambda\mu} > 0$.
1: Create the regular target border b and the digraph G.
2: Start with $f \leftarrow 0$, `done` $\leftarrow$ **false**.
3: **while** not `done` **do**
4:   $f \leftarrow \mathrm{shatter}(f)$.
5:   Construct $\mathsf{RES}^b(f)$.
6:   **if** there is a well-directed cycle in $\mathsf{RES}^b(f)$ with $\delta(\gamma(c)) > 0$ **then**
7:     Find a shortest well-directed cycle c in $\mathsf{RES}^b(f)$ with $\delta(\gamma(c)) > 0$.
8:     Augment 1 unit over c: $f \leftarrow f + \gamma(c)$.
9:     // We have $f \in P^b$, due to Theorem 11.
10:   **else**
11:     `done` $\leftarrow$ **true**.
12:   **end if**
13: **end while**
14: **if** $\delta(f) = |\nu|$ **then return true**.
15: **else return false**.

The shattering in line 4 is done by the algorithm mentioned in Lemma 8. Searching for shortest well-directed cycles in line 7 with positive δ-value can be done by a variant of the well-known Bellman-Ford algorithm (CLRS01).

The most interesting property of LRPA is that *shortest* well-directed cycles on $\mathsf{RES}^b(f)$ can be used to increase $\delta(f)$ by *one unit* (see line 8) while still remaining in P^b. The reason for this is the following crucial Theorem 11.

Theorem 11 (Shortest Cycle) *Let f be a b-bounded integral shattered hive flow. Assume that c is a shortest cycle among all well-directed cycles $\tilde{c}$ on $\mathsf{RES}^b(f)$ with $\delta(\tilde{c}) > 0$. Then $f + \gamma(c) \in P^b$.*

Proof sketch: The proof is rather involved. Assume that c is a cycle on $\mathsf{RES}^b(f)$ with $\delta(c) > 0$ and $f + \gamma(c) \notin P^b$. Let $\varepsilon := \max\{\varepsilon' \in \mathbb{R} \mid f + \varepsilon'\gamma(c) \in P^b\}$ and put $g := f + \varepsilon\gamma(c)$. We can show that $\varepsilon \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. Rhombi which are not f-flat, but g-flat, are called *critical rhombi*. Since $f + \gamma(c) \notin P^b$, there is at least one critical rhombus. A *g-flatspace* is a maximal connected union of small triangles such that any rhombus contained in it is g-flat.

In the case where there exists a critical rhombus that is not overlapping with any other g-flat rhombi, we can find well-directed cycles c_1, c_2 on $\mathsf{RES}^b(f)$ that split c in the sense that $\gamma(c_1 + c_2) = \gamma(c)$ and $\ell(c_1) \leq \ell(c_2) < \ell(c)$. One of these cycles must satisfy $\delta(c_i) > 0$ and we are done, because we found a cycle with positive throughput that is shorter than c.

In the other case we have overlapping g-flat rhombi and thus g is not shattered. So we can find a chain of g-flatspaces as described in (Buc00). We get a flow Ψ on G corresponding to raising the inner vertices of the chain by one unit. Moreover $g + \varepsilon\Psi \in P^b$ and $\delta(\Psi) = 0$. Our goal is to find well-directed cycles $c_1, \dots, c_m$ such that $\gamma(\sum_i c_i) = \gamma(c) + \Psi$ with $\ell(c_i) < \ell(c)$ for all i. Then we are done, because one of these cycles must satisfy $\delta(c_i) > 0$.

We achieve this by a complete classification of the possible shapes of g-flatspaces and the possible ways in which c passes through those shapes. Having understood this, we can do local changes to c such that it decomposes into smaller cycles that have the desired property. □

Theorem 11 allows us to prove the correctness of LRPA.

Theorem 12 *If given as input three strictly decreasing partitions $\lambda, \mu, \nu \in \mathbb{N}^n$ with $|\nu| = |\lambda| + |\mu|$, then the LRPA returns true iff $c^{\nu}_{\lambda\mu} > 0$.*

Proof: Note that during the algorithm f stays integral all the time and $f \in P^b$, because shattering preserves these properties according to Lemma 8 and we have $f + \gamma(c) \in P^b$ according to Theorem 11. After the while loop, the flow f maximizes δ on P^b according to Lemma 10. Lemma 7 tells us that $c^{\nu}_{\lambda\mu} > 0$ iff $\delta(f) = |\nu|$. □

7 Polynomial running time

We briefly sketch the ideas needed to transform LRPA into a combinatorial polynomial-time algorithm.

Theorem 13 *There is a polynomial-time algorithm that decides for given partitions $\lambda, \mu, \nu \in \mathbb{N}^n$ with $|\nu| = |\lambda| + |\mu|$ whether $c^{\nu}_{\lambda\mu} > 0$. The running time is polynomial in n and $\log|\nu|$.*

Proof sketch: We first note that by a perturbation argument, the general case can be reduced to the case where all the input partitions λ, μ, ν are strictly decreasing. This is done by exhibiting a special target border $\overline{b}$ such that for any partitions $\lambda, \mu, \nu \in \mathbb{N}^n$ and N sufficiently large, but of bitsize polynomial in n, we have $c^{\nu}_{\lambda\mu} > 0$ iff $c^{\tilde{\nu}}_{\tilde{\lambda}\tilde{\mu}} > 0$, where $\tilde{\lambda} = N\lambda + \overline{\lambda}$, $\tilde{\mu} = N\mu + \overline{\mu}$, and $\tilde{\nu} = N\nu + \overline{\nu}$.

We use a scaling method similar to that described in (AMO93) in the scaling of the Ford-Fulkerson algorithm. For $z \in \mathbb{R}$ a flow f is called *z-integral* iff it only takes values that are integral multiples of z. The algorithm now works as follows:

Put $k \leftarrow \lceil \log|\nu| \rceil + 1$. We efficiently construct an initial 2^k-integral $f \in P^b$ that is shattered and has regular border.

(*) We construct a modification $\mathsf{RES}^b_{2^k}(f)$ of the residual network $\mathsf{RES}^b(f)$ that excludes certain circle border vertices. Then we search for a shortest well-directed cycle c in $\mathsf{RES}^b_{2^k}(f)$ with $\delta(\gamma(c)) > 0$. If there is no such cycle, we set $k \leftarrow k - 1$ and go to (*). Otherwise we augment 2^k units over c: $f \leftarrow f + 2^k\gamma(c)$. A variation of Theorem 11 guarantees that f is still in P^b. Moreover, by construction of $\mathsf{RES}^b_{2^k}(f)$, f has still regular border. By an auxiliary optimization procedure, we can turn f into a *shattered* 2^k-integral flow in P^b. This is a refinement of Lemma 8 for which we need regularity on the border, as long as $k > 0$. We decrease now $k \leftarrow k - 1$ and go to (*).

The algorithm terminates when $k < 0$. Its output is an integral flow $f \in P^b$ with optimal δ-value. We have $c^{\nu}_{\lambda\mu} > 0$ iff $\delta(f) = |\nu|$ by Lemma 7. The algorithm can be shown to work in polynomial time. □

8 Deciding multiplicity freeness

Let $\mathsf{RES}_\times(f)$ denote the network that results from deleting in $\mathsf{RES}^b(f)$ the vertex o and all incident edges. Note that $\mathsf{RES}_\times(f)$ is independent of b.

The proof of Theorem 11 also yields the following result.

Proposition 14 *Given a b-bounded integral shattered hive flow f and a shortest well-directed cycle c on $\mathsf{RES}_\times(f)$. Then $f + \gamma(c) \in P^b$.* □

Corollary 15 *Let f be a b-bounded integral shattered hive flow with $\delta(f) = |\nu|$. Then we have $c^\nu_{\lambda\mu} > 1$ if and only if there exists a well-directed cycle in $\mathsf{RES}_\times(f)$.*

Proof: Suppose that c is a shortest well-directed cycle in $\mathsf{RES}_\times(f)$. Proposition 14 tells us that $g := f + \gamma(c)$ lies in P^b. It is easy to see that $\gamma(c) \neq 0$ and $\delta(\gamma(c)) = 0$. Hence g is another integral flow on P^b with throughput $|\nu|$. Lemma 7 implies $c^\nu_{\lambda\mu} > 1$.

To show the converse, suppose that $c^\nu_{\lambda\mu} > 1$. Lemma 7 implies that there exists an integral flow $g \in P^b$, $g \neq f$, with $\delta(g) = |\nu|$. The flow $d := g - f$ satisfies $\delta(d) = 0$ and hence uses no circle border vertex, which means that its support lies inside Δ. By Lemma 9 there exists $d' \in P_{\text{feas}}(\mathsf{RES}^b(f))$ such that $d = \gamma(d')$. It is obvious that in fact $d' \in P_{\text{feas}}(\mathsf{RES}_\times(f))$. Decomposing d' according to Lemma 5 shows the existence of a well-directed cycle in $\mathsf{RES}_\times(f)$. □

For strictly decreasing partitions we get a new proof of Fulton's conjecture, first shown by Knutson, Tao and Woodward (KTW04).

Corollary 16 *Let λ, μ, ν be strictly decreasing partitions with $|\nu| = |\lambda| + |\mu|$. Then the following three conditions are equivalent:*

$$(1)\ c^\nu_{\lambda\mu} = 1, \quad (2)\ \exists N\ c^{N\nu}_{N\lambda N\mu} = 1, \quad (3)\ \forall N\ c^{N\nu}_{N\lambda N\mu} = 1.$$

Proof: It suffices to show the implication from (2) to (3). Suppose that $c^{N\nu}_{N\lambda N\mu} = 1$ for some N, hence $c^\nu_{\lambda\mu} > 0$ by the saturation property. Since $c^\nu_{\lambda\mu} > 1$ implies $c^{N\nu}_{N\lambda N\mu} > 1$ we must have $c^\nu_{\lambda\mu} = 1$. Let f be a $b(\lambda, \mu, \nu)$-bounded integral shattered hive flow f with $\delta(f) = |\nu|$. Corollary 15 says that $\mathsf{RES}_\times(f)$ has no well-directed cycle. Since $\mathsf{RES}_\times(f) = \mathsf{RES}_\times(N'f)$ for all N', $\mathsf{RES}_\times(N'f)$ contains no well-directed cycle as well. Corollary 15 implies now that $c^{N'\nu}_{N'\lambda N'\mu} = 1$. □

Theorem 17 *There is a polynomial-time algorithm that decides for given strictly decreasing partitions $\lambda, \mu, \nu \in \mathbb{N}^n$ with $|\nu| = |\lambda| + |\mu|$ whether $c^\nu_{\lambda\mu} = 1$. The running time is polynomial in n and $\log|\nu|$.*

Proof: Using the algorithm of Theorem 13 one can compute an integral shattered $f \in P^b$ with $\delta(f) = |\nu|$. It is easy to check in polynomial time whether $\mathsf{RES}_\times(f)$ contains a well-directed cycle. Hence the assertion follows with Corollary 15. □

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Combinatorial invariant theory of projective reflection groups

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Abstract. We introduce the class of projective reflection groups which includes all complex reflection groups. We show that several aspects involving the combinatorics and the representation theory of complex reflection groups find a natural description in this wider setting.

Résumé. On introduit la classe des groupes de réflexions projectifs, ce qui généralises la notion de groupe engendré par des réflexions. On montre que plusieurs aspects concernants la combinatoire et la théorie des representations des groupes de reflexions complèxes trouvent une description naturelle dans ce cadre plus général.

Keywords: Reflection groups, descent statistics, invariant algebras, Young tableaux.

1 Introduction

A complex reflection (or simply a reflection) is an endomorphism of a complex vector space V which is of finite order and such that its fixed point space is of codimension 1. Finite reflection groups are finite subgroups of $GL(V)$ generated by reflections. They have probably been introduced by Shephard in (16) and have been characterized by means of their ring of invariants and completely classified by Chevalley (11) and Shephard-Todd (17) in the fifties, generalizing the well-known fundamental theorem of symmetric functions. In this classification there is an infinite family $G(r,p,n)$ of irreducible reflection groups, where r, p, n are positive integers (with $r \equiv 0 \mod p$) and 34 other exceptional groups. The relationship between the combinatorics and the (invariant) representation theory of symmetric groups is fascinating from both combinatorial and algebraic points of view, and the problem of generalizing these sort of results to all reflection groups has been faced in many ways. Besides several results that holds in the full generality of reflection groups, there are some relevant generalizations which have been obtained only for the wreath product groups $G(r,n) = G(r,1,n)$ (see, e.g., (21; 22; 3; 5; 2)). Some attempts to extend these results to other reflection groups have been made, in particular for Weyl groups of type D, (see, e.g., (8; 9; 4)) though they are probably not completely satisfactory as in the case of wreath products.

In this work we introduce a new class of groups, the projective reflection groups, which are a generalization of reflection groups. We will concentrate our attention on an infinite family $G(r,p,q,n)$ of such groups (where $G(r,p,1,n) = G(r,p,n)$ in the previous notation). Fundamental in this theory is the following notion of duality: if $G = G(r, p, q, n)$ then we denote by $G^* = G(r, q, p, n)$ (where the roles of p and q have been interchanged). We note in particular that reflection groups G satisfying $G = G^*$ are exactly the wreath products $G(r, n) = G(r, 1, 1, n)$ and that in general if G is a reflection group then G^* is not. We show that much of the theory of reflection groups can be extended to projective reflection groups and that the combinatorics of a projective reflection group $G = G(r, p, q, n)$ is strictly related to the (invariant) representation theory of G^*, generalizing several known results for wreath products in a very natural way.

The paper is organized as follows. We present definitions and a characterization in terms of invariants of projective reflection groups in §2. We exploit those combinatorial aspects of these groups that we need in §3. In §4 we further consider the action of a projective reflection group on a ring of polynomials to define and study its coinvariant algebra. In §5 we analyze the structure of the irreducible representations of a projective reflection group $G(r, p, q, n)$ and we provide a combinatorial interpretation for their dimensions. In §6 we consider a decomposition of the homogeneous components of the coinvariant algebra that leads us to define the descent representations of a projective reflection group: these representations are used in §7 to describe the main new results of this paper. Here we show an explicit basis of the diagonal invariant algebra as a free module over the tensorial invariant algebra of all projective reflection groups $G(r, p, q, n)$. It is in this description that the interplay between a group G and its dual G^* attains its apex. In §8 we deduce some properties of the Kronecker coefficients of a projective reflection group that can be deduce from the main results. In §9 we extend the Robinson-Schensted correspondence on wreath products to all projective reflection groups of the form $G(r, p, q, n)$: in this general context it is not a bijection but it will be the key point to solve in §10 a problem posed by Barcelo, Reiner and Stanton on the Hilbert series of a certain diagonal invariant module twisted by a Galois automorphism.

2 Definitions and characterizations

Let V be a finite dimensional complex vector space and consider the natural map $\varphi : GL(V) \rightarrow GL(S^q(V))$, where $S^q(V)$ is the q-th symmetric power of V. We clearly have $\ker \varphi = C_q$, where C_q is the cyclic group of scalar matrices of order q generated by $\zeta_q I$, with $\zeta_q \stackrel{\text{def}}{=} e^{\frac{2\pi i}{q}}$.
Now, if $W \subseteq GL(V)$ is a finite reflection group we have $\varphi(W) \cong W/(W \cap C_q)$. In particular, if $C_q \subset W$ we have that W/C_q can be identified with a subgroup of $GL(S^q(V))$ by means of the map φ.

Definition 2.1 *Let G be a finite subgroup of $GL(S^q(V))$. We say that G is a projective reflection group if there exists a reflection group $W \subset GL(V)$ such that $C_q \subseteq W$ and $G = W/C_q$.*

Note that we obtain standard reflection groups in the case $q = 1$.

It follows from the previous definition that to classify all possible projective reflection groups we only have to describe all possible scalar subgroups of a reflection group. We know by the work of Shephard and Todd (17) that all but a finite number of irreducible reflection groups are the groups $G(r, p, n)$ that we are going to describe. If A is a matrix with complex entries we denote by $|A|$ the real matrix whose entries are the absolute values of the entries of A. The groups $G(r, n) = G(r, 1, n)$ are given by all $n \times n$ matrices satysfying the following conditions:

- the non-zero entries are r-th roots of unity;
- there is exactly one non-zero entry in every row and every column (i.e. $|A|$ is a permutation matrix).

If p divides r then the reflection group $G(r,p,n)$ is the subgroup of $G(r,n)$ given by all matrices $A \in G(r,n)$ such that $\frac{\det A}{\det |A|}$ is a $\frac{r}{p}$-th root of unity.

It is easy to characterize all possible scalar subgroups of the groups $G(r,p,n)$: in fact we can easily observe that the scalar matrix $\zeta_q I$ belongs to $G(r,p,n)$ if and only if $q|r$ and $pq|rn$.

Definition 2.2 *Let $r,p,q,n \in \mathbb{N}$ be such that $p|r$, $q|r$ and $pq|rn$. Then we let*

$$G(r,p,q,n) \stackrel{\text{def}}{=} G(r,p,n)/C_q,$$

where C_q is the cyclic group generated by $\zeta_q I$.

We observe that starting from the wreath product group $G(r,n)$ we could have done first the quotient by the subgroup C_q and then taken the subgroup of this quotient consisting of elements A satysfing $\frac{\det A}{\det |A|}$ is a $\frac{r}{p}$-th root of unity (note that this requirement would have been well-defined). We would have obtained the same group $G(r,p,q,n)$ and one of the targets of this paper is to convince the reader that these two operations of "taking subgroups" and "taking quotients" have the same dignity and for many aspects their are dual to each other. In fact, we note the symmetry on the conditions for the parameters p and q in the definition of the group $G(r,p,q,n)$. In particular if $G = G(r,p,q,n)$ then the group $G^* \stackrel{\text{def}}{=} G(r,q,p,n)$, where the roles of the parameters p and q are interchanged, is always well-defined. The classical Weyl groups of type A,B and D are respectively in this notation the groups $G(1,1,1,n), G(2,1,1,n)$ and $G(2,2,1,n)$. Note that while Weyl groups of type A and B are fixed by the $*$-operatator, Weyl groups of type D and general reflection groups are not. The main target of this work is to show that the several aspects of the invariant theory of a projective reflection group G is strongly related to and easily described by the combinatorics of G^*.

One may ask for which choice of the parameters one has $G \cong G^*$ as abstract groups.

Proposition 2.3 *Let $G = G(r,p,q,n)$, with $n \neq 2$. Then $G \cong G^*$ if and only if* $\mathrm{GCD}(\frac{rn}{pq}, \frac{r}{p}) = \mathrm{GCD}(\frac{rn}{pq}, \frac{r}{q})$.

Any finite subgroup of $GL(V)$ acts naturally on the symmetric algebra $S(V^*)$. A well-known theorem due to Chevalley and Shephard-Todd says that a finite group G of $GL(V)$ is a reflection group if and only if its invariant ring $S(V^*)^G$ is itself a polynomial algebra. Our next target is to generalize this result to the present context. We recall that a projective reflection group is equipped with an action on the symmetric power $S^q(V)$. The dual action can be extended to $S_q[V^*]$, the algebra of polynomial functions on V generated by homogeneous polynomial functions of degree q.

Theorem 2.4 *Let V be a complex vector space, $n = \dim V$, and G be a finite group of graded automorphisms of $S_q[V^*]$, the algebra generated by homogeneous polynomial functions on V of degree q. Then G is a projective reflection group if and only if the invariant algebra $S_q[V^*]^G$ is generated by n algebraically independent homogeneous elements.*

3 Statistics

In this section we introduce the main combinatorial tools of projective reflection groups that we need.
If $g \in G(r,n)$ we write $g = [\sigma; c_1, \ldots, c_n]$ if the non-zero entry in the i-th row of g is $\zeta_r^{c_i}$ and $\sigma \in S_n$ is the permutation associated to $|g|$ (i.e. $\sigma(i) = j$ if $g_{i,j} \neq 0$). We observe that g determines σ uniquely

while the integers c_i are determined only modulo r. We also note that in this notation we have that $g = [\sigma; c_1, \ldots, c_n]$ belongs to $G(r, p, n)$ if and only if $\sum c_i \equiv 0 \mod p$.
If $g \in G(r, p, q, n)$ we also write $g = [\sigma; c_1, \ldots, c_n]$ to mean that g can be represented by $[\sigma; c_1, \ldots, c_n]$ in $G(r, p, n)$ and we let

$$\begin{aligned} \mathrm{HDes}(g) &\stackrel{\mathrm{def}}{=} \{i \in [n-1] : \, c_i \equiv c_{i+1} \text{ and } \sigma_i > \sigma_{i+1}\} \\ h_i(g) &\stackrel{\mathrm{def}}{=} \#\{j \geq i : \, j \in \mathrm{HDes}(g)\} \\ k_i(g) &\stackrel{\mathrm{def}}{=} \begin{cases} [c_n]_{r/q} & \text{if } i = n \\ k_{i+1} + [c_i - c_{i+1}]_r & \text{if } i \in [n-1], \end{cases} \end{aligned}$$

where $[c]_s$ is the smallest non negative representative of the class of the integer c modulo s.
Note that these statistics do not depend on the choice of the integers $c_1, \ldots, c_n$ for representing g. For example, let $g = [27648153; 2, 3, 3, 5, 1, 7, 3, 2] \in G(6, 2, 3, 8)$. Then $\mathrm{HDes}(g) = \{2, 5\}$, $(h_1, \ldots, h_8) = (2, 2, 1, 1, 1, 0, 0, 0)$ and $(k_1, \ldots, k_8) = (18, 13, 13, 9, 5, 5, 1, 0)$.

If $q = p = 1$ these statistics give an alternative definition for the flag-major index of Adin and Roichman (see (3)) for wreath products $G(r, n)$. In fact, if we let

$$\mathrm{Des}(g) = \{i : \text{ either } [c_i]_r < [c_{i+1}]_r \text{ or } [c_i]_r = [c_{i+1}]_r \text{ and } \sigma_i > \sigma_{i+1}\},$$

then the *flag-major index* is defined as

$$\mathrm{fmaj}(g) \stackrel{\mathrm{def}}{=} r \sum_{i \in \mathrm{Des}(g)} i + \sum_i [c_i]_r$$

and we can easily verify that in this case we have $\mathrm{fmaj}(g) = \sum (r \cdot h_i(g) + k_i(g))$.

We note that if $\lambda_i(g) \stackrel{\mathrm{def}}{=} r \cdot h_i(g) + k_i(g)$ then the sequence $\lambda(g) \stackrel{\mathrm{def}}{=} (\lambda_1(g), \ldots, \lambda_n(g))$ is a partition. We may also observe that $\lambda(g)$ is such that $g = [|g|; \lambda(g)]$ and that $\lambda(g)$ is the minimal partition (with respect to containment of the corresponding Ferrers diagram) satysfing this condition.
Extending the notion of flag-major index we define the flag-major index for the projective reflection group $G(r, p, q, n)$ by $\mathrm{fmaj}(g) \stackrel{\mathrm{def}}{=} |\lambda(g)|$.

4 The coinvariant algebra

As we observed in §2 we have an action of any projective reflection group G on the algebra $S_q[V^*]$ generated by homogeneous polynomial functions on V of degree q. We denote by I_+^G the ideal of $S_q[V^*]$ generated by homogeneous elements in $S_q[V^*]^G$ of positive degree, and define the *coinvariant algebra* of G by

$$R^G \stackrel{\mathrm{def}}{=} S_q[V^*]/I_+^G.$$

The coinvariant algebra R^G is a graded representation of G and, as it was the case for reflection groups, it is isomorphic as a G-module to the group algebra $\mathbb{C}G$.

Proposition 4.1 *If G is any projective reflection group, we have an isomorphism of G-modules $R^G \cong \mathbb{C}G$.*

If the projective reflection group G is of the form $G = G(r, p, q, n)$ we can describe the coinvariant algebra in a more explicit way. It could be natural to expect a basis for the algebra R^G indexed by elements of G. As it was mentioned in the introduction, this is the first occurrence of the invariant theory of a projective reflection group G which is naturally described by its dual group G^*. Generalizing and unifying results and definitions in (1; 8; 4), we associate to any element $g \in G$ a monomial $a_g \in \mathbb{C}[X] \stackrel{\text{def}}{=} \mathbb{C}[x_1, \dots, x_n]$ of degree $\mathrm{fmaj}(g)$ in the following way

$$a_g(X) \stackrel{\text{def}}{=} \prod_i x_{|g|(i)}^{\lambda_i(g)}.$$

We denote by $S_q[X]$ the algebra of polynomials in $\mathbb{C}[X]$ generated by the monomials of degree q. Then it is not difficult to verify that

$$a_g \in S_p[X],$$

for all $g \in G(r, p, q, n)$.

Theorem 4.2 *If $G = G(r, p, q, n)$ then the set $\{a_g : g \in G^*\}$ represents a basis for R^G.*

5 The irreducible representations

In this section we describe explicitly a natural parametrization of the irreducible representations of a projective reflection group $G(r, p, q, n)$. Given a partition μ of n, the *Ferrers diagram of shape* μ is a collection of boxes, arranged in left-justified rows, with μ_i boxes in row i. We denote by $\mathrm{Fer}(r, p, n)$ the set of r-tuples of Ferrers diagrams whose shapes $(\lambda^{(0)}, \dots, \lambda^{(r-1)})$ are such that $\sum |\lambda^{(i)}| = n$ and $\sum_i i|\lambda^{(i)}| \equiv 0 \mod p$. This may recall the definition of $G(r, p, n)$ where the role of $\sum_i c_i(g)$ is played by $\sum_i i|\lambda^{(i)}|$. In an extreme parallelism with the groups $G(r, p, n)$ we have the following result.

Lemma 5.1 *Let $(\lambda^{(0)}, \dots, \lambda^{(r-1)}) \in \mathrm{Fer}(r, p, n)$ (and $q \in \mathbb{N}$ be such that $q|r$ and $pq|rn$). Then*

$$(\lambda^{(r/q)}, \dots, \lambda^{(r-1+r/q)}) \in \mathrm{Fer}(r, p, n),$$

where $\lambda^{(j)} \stackrel{\text{def}}{=} \lambda^{(j-r)}$ if $j \geq r$.

If $\mu \in \mathrm{Fer}(r, p, n)$ we denote by $\mathcal{ST}_\mu$ the set of all possible fillings of the boxes in μ with all the numbers from 1 to n appearing once, in such way that rows are increasing from left to right and columns are incresing from top to bottom in every single Ferrers diagram of μ. Moreover we let $\mathcal{ST}(r, p, n) \stackrel{\text{def}}{=} \cup_{\mu \in \mathrm{Fer}(r,p,n)} \mathcal{ST}_\mu$.
By Lemma 5.1 we have a natural action of C_q on both $\mathrm{Fer}(r, p, n)$ and $\mathcal{ST}(r, p, n)$. We denote the corresponding quotient sets by $\mathrm{Fer}(r, p, q, n)$ and $\mathcal{ST}(r, p, q, n)$. If $T \in \mathcal{ST}(r, p, q, n)$ we denote by $\mu(T)$ its corresponding shape in $\mathrm{Fer}(r, p, q, n)$ and if $\mu \in \mathrm{Fer}(r, p, q, n)$ we let $\mathcal{ST}_\mu \stackrel{\text{def}}{=} \{T \in \mathcal{ST}(r, p, q, n) : \mu(T) = \mu\}$. Finally, if $\mathrm{Fer} = \mathrm{Fer}(r, p, q, n)$, we let $\mathrm{Fer}^* \stackrel{\text{def}}{=} \mathrm{Fer}(r, q, p, n)$.

Proposition 5.2 *The irreducible representations of $G(r, p, q, n)$ are naturally parametrized by pairs (μ, ρ), where $\mu \in \mathrm{Fer}^*$ and $\rho \in (C_p)_\mu$, the stabilizer of any element in the class μ. Moreover the dimension of the irreducible representation indexed by (μ, ρ) is independent of ρ and it is equal to $|\mathcal{ST}_\mu|$.*

If ϕ is an irreducible representation of G indexed by a pair (μ, ρ) we let $\mu(\phi) \stackrel{\text{def}}{=} \mu \in \mathrm{Fer}^*$.

6 The descent representations

If M is a monomial in $\mathbb{C}[X]$ we denote by $\lambda(M)$ its *exponent partition*, i.e. the partition obtained by rearranging the exponents of M. We say that a polynomial is homogeneous of *partition degree* λ if it is a linear combination of monomials whose exponent partition is λ. If $G = G(r, p, q, n)$ and $|\lambda| \equiv 0 \mod q$, we can consider the submodule $R^G_{\trianglelefteq\lambda}$ of R^G spanned by monomials of total degree $|\lambda|$ and partition degree $\trianglelefteq\lambda$ and we can similarly define $R^G_{\triangleleft\lambda}$. Here $\trianglelefteq$ and $\triangleleft$ mean smaller and strictly smaller in the dominance order of partitions. Following and generalizing (1; 8; 4) we denote the quotient module by

$$R^G_\lambda \stackrel{\text{def}}{=} R^G_{\trianglelefteq\lambda}/R^G_{\triangleleft\lambda}.$$

We call the modules R^G_λ the *descent representations* of G. A straightforward application of Maschke's theorem implies that, for any $k \equiv 0 \mod q$, we have an isomorphism

$$\varphi : R^G_i \xrightarrow{\cong} \bigoplus_{\lambda \vdash i} R^G_\lambda$$

such that every element in $\varphi^{-1}(R^G_\lambda)$ can be represented by a homogeneous polynomial in $S_q[X]$ of partition degree λ. We recall that if $g \in G^*$ then the monomial a_g has partition degree $\lambda(g)$ and so it represents an element in $R^G_{\lambda(g)}$.

Lemma 6.1 *Let λ be a partition such that $|\lambda| \equiv 0 \mod q$. Then the set*

$$\{a_g : g \in G^* \text{ and } \lambda(g) = \lambda\}$$

is a system of representatives of a basis of R^G_λ. In particular $\dim(R^G_\lambda) = |\{g \in G^* : \lambda(g) = \lambda\}|$.

Our next target is an explicit description of the irreducible decomposition of the modules R^G_λ. We can define the statistics h_i and k_i in $\mathcal{ST}(r, p, q, n)$ similarly to the case of $G(r, p, q, n)$. Let $T \in \mathcal{ST}(r, p, q, n)$ be represented by $(T_1, \ldots, T_r)$. For $i \in [n]$ we let $c_i = j$ if $i \in T_j$.

$$\begin{aligned}
\mathrm{HDes}(T) &\stackrel{\text{def}}{=} \{i \in [n-1] : c_i = c_{i+1} \text{ and } i \text{ appears strictly above } i+1\}\\
h_i(T) &\stackrel{\text{def}}{=} \#\{j \geq i : j \in \mathrm{HDes}(T)\}\\
k_i(T) &\stackrel{\text{def}}{=} \begin{cases} [c_n]_{r/q} & \text{if } i = n\\ k_{i+1} + [c_i - c_{i+1}]_r & \text{if } i \in [n-1]\end{cases}
\end{aligned}$$

It is clear that these definitions do not depend on the choice of the representative $(T_1, \ldots, T_r)$. For example if

$$T = \left(\begin{array}{|c|c|}\hline 1 & 4\\ \hline 5 \\ \cline{1-1}\end{array},\quad \begin{array}{|c|c|}\hline 2 & 8\\ \hline 3 & 9\\ \hline\end{array},\quad \begin{array}{|c|c|}\hline 6 & 7\\ \hline\end{array}\right) \in \mathcal{ST}(3, 1, 3, 9),$$

we have $(h_1, \ldots, h_9) = (3, 3, 2, 2, 1, 1, 1, 1, 0)$ and $(k_1, \ldots, k_9) = (5, 3, 3, 2, 2, 1, 1, 0, 0)$. We define $\lambda_i(T) \stackrel{\text{def}}{=} rh_i(T) + k_i(T)$, $\lambda(T) = (\lambda_1(T), \ldots, \lambda_n(T)$ and $\mathrm{fmaj}(T) = |\lambda(T)|$.

Proposition 6.2 *Let ϕ be an irreducible representation of G. Then the multiplicity of ϕ in R^G_λ is given by*

$$\langle \chi^\phi, \chi^{R^G_\lambda} \rangle = |\{T \in \mathcal{ST}_{\mu(\phi)} : \lambda(T) = \lambda|,$$

where $\mu(\phi) \in \mathrm{Fer}^*$ is defined at the end of §5.

This proposition unifies and generalizes the corresponding coarse results of Lusztig (unpublished) and Kraśkiewicz-Weyman (14) in Type A and Stembridge (22) for reflection groups and the corresponding refined results of Adin-Brenti-Roichman (1) in Type A and B and of Bagno-Biagioli (4) for reflection groups.

7 Tensorial and diagonal actions

In this section we describe the main result of this work (Theorem 7.5) which present an explicit basis for the diagonal invariant algebra of a projective reflection group $G = G(r, p, q, n)$ (considered as a free module over the tensorial invariant algebra) in terms of the dual group G^*. This result is new also in the generality of reflection groups (see (7; 6) for related results in type A and B). Here it is really apparent that not only the combinatorics of G^* (as in the previous sections) but also its algebraic structure play a crucial role in the invariant theory of G. Let $S_q[X]^{\otimes k}$ be the k-th tensor power of the algebra of polynomials $S_q[X]$ defined in §4. On this algebra we consider the natural action of the group G^k (where the i-th coordinate of G^k acts on the i-th factor in $S_q[X]^{\otimes k}$) and of its diagonal subgroup ΔG. We are particularly interested in the corresponding invariant algebras. Every monomial in $S_q[X]^{\otimes k}$ can be described by a $k \times n$-matrix with non negative integer entries such that the sum in each row is divided by q. To any such matrix A we associate the monomial $\mathcal{X}^A \stackrel{\text{def}}{=} \prod_{i,j} x_{i,j}^{a_{i,j}}$. Here and in what follows we identify $S_q[X]^{\otimes k}$ with the algebra of polynomials $S_q[X_1, \ldots, X_k] = S_q[x_{i,j}]$ (where $i \in [k]$ and $j \in [n]$) spanned by monomials whose degree in $x_{i,1}, \ldots, x_{i,n}$ is a multiple of q for all $i \in [k]$. We refer to the algebra $S_q[X_1, \ldots, X_k]^{\Delta G}$ as the *diagonal invariant algebra of* G. It is clear that $S_q[X_1, \ldots, X_k]^{\Delta G}$ is generated by the polynomials

$$(\mathcal{X}^A)^{\#} \stackrel{\text{def}}{=} \frac{1}{|G|} \sum_{g \in \Delta G} g(\mathcal{X}^A).$$

Lemma 7.1 *Let A be a $k \times n$ matrix with row sums divided by q and let s_i be the sum of the entries in its i-th column. Then $(\mathcal{X}^A)^{\#} \neq 0$ if and only if the following two conditions are satisfied*

1. $s_i \equiv s_j$ *for all* i, j;
2. $ps_i \equiv 0$ *for all* i.

We recall that a *k-partite partition* (see (12; 13)) is a $k \times n$ matrix $A = (a_{i,j})$ with non-negative integer entries such that $a_{i,j} \geq a_{i,j+1}$ whenever $a_{h,j} = a_{h,j+1}$ for all $h < i$. We denote by $\mathcal{B}_k(r, p, q, n)$ the set of $k \times n$-matrices which are k-partite partitions, with row sums divided by q and column sums satisfying (1) and (2) in Lemma 7.1.

Corollary 7.2 *The set $\{(\mathcal{X}^A)^{\#} : A \in \mathcal{B}_k(r, p, q, n)\}$ is a basis for the diagonal invariant algebra of G.*

We recall that the algebra $S_q[X_1, \ldots, X_k]^{\Delta G}$, being Cohen-Macaulay (see (19, Proposition 3.1)), is a free module over its subalgebra $S_q[X_1, \ldots, X_k]^{G^k}$ and our next target is the description of a basis for $S_q[X_1, \ldots, X_k]^{\Delta G}$ as a free $S_q[X_1, \ldots, X_k]^{G^k}$-module.

Definition 7.3 *Let λ be a partition with n parts and $g \in G(r, p, q, n)$. We say that λ is g-compatible if $\lambda - \lambda(g)$ is a partition and $g = [|g|; \lambda]$.*

We note that in the case of the symmetric group the condition $g = [|g|; \lambda]$ in the previous definition is empty and we obtain an equivalent definition of a σ-compatible partition given in (12). The special case of the following result where G is the symmetric group is proved in (12).

Theorem 7.4 *There is a bijection between $\mathcal{B}_k(r,p,q,n)$ and $(2k)$-tuples $(g_1,\dots,g_k;\lambda^{(1)},\dots,\lambda^{(k)})$ where*

- $g_1,\dots,g_k \in G^*$ *are such that* $g_1 \cdots g_k = 1$*;*
- $\lambda^{(i)}$ *is a* g_i*-compatible partition.*

The bijection is given by

$$\Phi(g_1,\dots,g_k;\lambda^{(1)},\dots,\lambda^{(k)}) = \begin{pmatrix} \lambda_1^{(1)} & \lambda_2^{(1)} & \cdots & \lambda_n^{(1)} \\ \lambda_{\sigma_1(1)}^{(2)} & \lambda_{\sigma_1(2)}^{(2)} & \cdots & \lambda_{\sigma_1(n)}^{(2)} \\ \\ \lambda_{(\sigma_1\dots\sigma_{k-1})(1)}^{(k)} & \lambda_{(\sigma_1\dots\sigma_{k-1})(2)}^{(k)} & \cdots & \lambda_{(\sigma_1\dots\sigma_{k-1})(n)}^{(k)} \end{pmatrix},$$

where $\sigma_i = |g_i|$ and the composition of permutations is from left to right.

If $g_1,\dots,g_k \in G^*$ and $g_1 \cdots g_k = 1$ we let

$$A(g_1,\dots,g_k) \stackrel{\text{def}}{=} \Phi(g_1,\dots,g_k;\lambda(g_1),\dots,\lambda(g_k)).$$

With this terminology Theorem 7.4 can be restated as follows: if $A \in \mathcal{B}_k(r,p,q,n)$ then there exist unique $g_1,\dots,g_k \in G^*$ with $g_1 \cdots g_k = 1$ such that

$$\mathcal{X}^A = \mathcal{X}^{A(g_1,\dots g_k)} M_1(X_1) \cdots M_k(X_k),$$

where, for all $i \in [k]$, M_i is a monomial such that $\lambda(M_i) = \lambda_i(\mathcal{X}^A) - \lambda(g_i)$ is a partition whose parts are all congruent to the same multiple of r/p modulo r. Here $\lambda_i(\mathcal{X}^A)$ is the exponent partition of $\mathcal{X}^A$ with respect to the variables $x_{i,1},\dots,x_{i,n}$. This is the main point in the proof of the following result.

Theorem 7.5 *The set of polynomials*

$$\{(\mathcal{X}^{A(g_1,\dots,g_k)})^{\#} : g_1,\dots,g_k \in G^* \textit{ and } g_1 \cdots g_k = 1\},$$

is a basis for $S_q[X_1,\dots,X_k]^{\Delta G}$ as a free module over $S_q[X_1,\dots,X_k]^{G^k}$.

An immediate consequence of Theorem 7.5 is the following equality

$$\frac{\mathrm{Hilb}(S_q[X_1,\dots,X_k]^{\Delta G})(y_1,\dots,y_k)}{\mathrm{Hilb}(S_q[X_1,\dots,X_k]^{G^k})(y_1,\dots,y_k)} = \sum_{\substack{g_1,\dots,g_k \in G^*: \\ g_1 \cdots g_k = 1}} y_1^{\mathrm{fmaj}(g_1)} \cdots y_k^{\mathrm{fmaj}(g_k)}.$$

Theorem 7.5 and its proof allow us to obtain an important refinement of the previous identity. The algebra $S_q[X_1,\dots,X_k]$ is multigraded by k-tuples of partitions with at most n parts: we just say that a monomial M is homogeneous of multipartition degree $(\lambda^{(1)},\dots,\lambda^{(k)})$ if its exponent partition with respect to the variables $x_{i,1},\dots x_{i,n}$ is $\lambda^{(i)}$ for all i. If we consider the Hilbert series of the invariant algebras above with respect to this multipartition degree we obtain the following result.

Corollary 7.6 *We have*

$$\frac{\mathrm{Hilb}(S_q[X_1,\ldots,X_k]^{\Delta G})(Y_1,\ldots,Y_k)}{\mathrm{Hilb}(S_q[X_1,\ldots,X_k]^{G^k})(Y_1,\ldots,Y_k)} = \sum_{\substack{g_1,\ldots,g_k\in G^*:\\ g_1\cdots g_k=1}} Y_1^{\lambda(g_1)}\cdots Y_k^{\lambda(g_k)},$$

where $Y_i=(y_{i,1},\ldots,y_{i,n})$.

8 The Kronecker coefficients

We can use the descent representations of a projective reflection group introduced in §6 to give to the coinvariant algebra the structure of a partition-graded module. By means of this grading of the coinvariant algebra we can also decompose the algebra

$$\frac{S_q[X_1,\ldots,X_k]}{I_+^{G^k}} \cong \underbrace{R^G\otimes\cdots\otimes R^G}_{k}$$

and its diagonal invariant subalgebra

$$\left(\frac{S_q[X_1,\ldots,X_k]}{I_+^{G^k}}\right)^{\Delta G} \cong \frac{S_q[X_1,\ldots,X_k]^{\Delta G}}{J_+^{G^k}}$$

in homogeneous components whose degrees are k-tuples of partitions with at most n parts. Here $I_+^{G^k}$ and $J_+^{G^k}$ are the ideals generated by homogeneous G^k-invariant polynomials of positive degree inside $S_q[X_1,\ldots,X_k]$ and $S_q[X_1,\ldots,X_k]^{\Delta G}$ respectively.

We define the *refined fake degree polynomial* $f^{\phi}(y_1,\ldots,y_n)$ of a projective reflection group G as the polynomial whose coefficient of $y_1^{\lambda_1}\cdots y_n^{\lambda_n}$ is the multiplicity of the irreducible representation ϕ of G in R_λ^G. If $\phi_1,\ldots,\phi_k$ are k irreducible representations of G we define the *Kronecker coefficients* of G by

$$g_{\phi_1,\ldots,\phi_k} \stackrel{\mathrm{def}}{=} \frac{1}{|G|}\sum_{g\in G}\chi^{\phi_1}(g)\cdots\chi^{\phi_k}(g),$$

If $G=G(r,p,q,n)$ and $\mu_1,\ldots,\mu_k\in\mathrm{Fer}^*=\mathrm{Fer}(r,q,p,n)$, we define the *coarse Kronecker coefficients* of G by

$$g_{\mu_1,\ldots,\mu_k} \stackrel{\mathrm{def}}{=} \sum_i \sum_{\phi_i:\mu(\phi_i)=\mu_i} g_{\phi_1,\ldots,\phi_k}.$$

The following result is a consequence of (18, Theorem 5.11).

Theorem 8.1 *We have*

$$\mathrm{Hilb}\Big(\frac{S_q[X_1,\ldots,X_k]^{\Delta G}}{J_+^{G^k}}\Big)(Y_1,\ldots,Y_k) = \sum_{\phi_1,\ldots,\phi_k\in\mathrm{Irr}(G)} g_{\phi_1,\ldots,\phi_k} f^{\phi_1}(Y_1)\cdots f^{\phi_k}(Y_k)$$

where the sum is taken over all k-tuples of irreducible representations of G.

By Proposition 6.2 we deduce that Theorem 8.1 can be restated as follows

$$\mathrm{Hilb}\Big(\frac{S_q[X_1,\ldots,X_k]^{\Delta G}}{J_+^{G^k}}\Big)(Y_1,\ldots,Y_k)=\sum_{T_1,\ldots,T_k\in\mathcal{ST}^*} g_{\mu(T_1),\ldots,\mu(T_k)}Y_1^{\lambda(T_1)}\cdots Y_k^{\lambda(T_k)},$$

where $\mathcal{ST}^* \stackrel{\text{def}}{=} \mathcal{ST}(r,q,p,n)$. So, by Theorem 7.5 and Corollary 7.6 we have the following result.

Corollary 8.2 *Let $G=G(r,p,q,n)$ and $\mathcal{ST}=\mathcal{ST}(r,p,q,n)$. Then*

$$\sum_{\substack{g_1,\ldots,g_k\in G\\ g_1\cdots g_k=1}} Y_1^{\lambda(g_1)}\cdots Y_k^{\lambda(g_k)}=\sum_{T_1,\ldots,T_k\in\mathcal{ST}} g_{\mu(T_1),\ldots,\mu(T_k)}Y_1^{\lambda(T_1)}\cdots Y_k^{\lambda(T_k)}.$$

We observe that Corollary 8.2 provides us a purely combinatorial algorithm to compute the coarse Kronecker coefficients of G. This can be achieved in a way which is similar to the corresponding result for the symmetric group (see (10, §4)).

In the next section we describe a bijective proof of Corollary 8.2 in the case $k=2$.

9 The Robinson-Schensted correspondence

Recall the classical Robinson-Schensted correspondence from (20, §7.11)). This correspondence has been extended to wreath product groups $G(r,n)$ in (21) in the following way. Given $g\in G(r,n)$ and $j\in[0,r-1]$, we let $\{i_1,\ldots,i_h\}=\{l\in[n]:c_l(g)=j\}$ and we consider the two-line array $A_j=\begin{pmatrix} i_1 & i_2 & \cdots & i_h\\ \sigma(i_1) & \sigma(i_2) & \cdots & \sigma(i_h)\end{pmatrix}$, where $\sigma=|g|$, and the pair of tableaux (P_j,Q_j) obtained by applying the Robinson-Schensted correspondence to A_j. Then the Stanton-White correspondence

$$g\mapsto (P(g),Q(g))\stackrel{\text{def}}{=}\Big((P_0,\ldots,P_{r-1}),(Q_0,\ldots,Q_{r-1})\Big)$$

is a bijection between $G(r,n)$ and pairs of tableaux of the same shape in $\mathcal{ST}(r,1,n)$. Furthermore we have $\lambda(g)=\lambda(Q(g))$ and $\lambda(\bar{g}^{-1})=\lambda(P(g))$.

Now let $g\in G(r,p,q,n)$ and $\tilde{g}\in G(r,p,n)$ be a lifting of g. Then the classes in $\mathcal{ST}(r,p,q,n)$ of the tableaux $P(\tilde{g})$ and $Q(\tilde{g})$ obtained by applying the previous correspondence depend uniquely on g and not on the lifting $\tilde{g}$. Therefore one can define a map $g\mapsto(P(g),Q(g))$ which associates to any element in $G(r,p,q,n)$ a pair of tableaux in $\mathcal{ST}(r,p,q,n)$ of the same shape. The following result is the natural generalization of the Stanton-White correspondence to projective reflection groups.

Theorem 9.1 *Let P,Q be two tableaux in $\mathcal{ST}(r,p,q,n)$ of the same shape μ. Then*

$$|\{g\in G(r,p,q,n): P(g)=P \text{ and } Q(g)=Q\}|=|(C_q)_\mu|,$$

where $(C_q)_\mu$ is the stabilizer in C_q of any element in the class μ.

We observe that Theorem 9.1 provides a bijective proof that

$$|G|=\sum_{\phi\in\mathrm{Irr}(G^*)}(\dim\phi)^2,$$

since $\dim\phi=|\mathcal{ST}_{\mu(\phi)}|$ and, given $\mu\in\mathrm{Fer}$, we have $|\{\phi\in\mathrm{Irr}(G^*):\mu(\phi)=\mu\}|=|(C_q)_\mu|$.

10 Galois automorphisms

The next target is to use the theory developed in the previous sections to solve a problem posed in (5, Question 6.3). The objects of our study here are again Hilbert series of invariant algebras as in §7 but with a new ingredient given by a Galois automorphism. Given any projective reflection group G (not necessarily of the form $G(r,p,q,n)$) we consider a cyclotomic field $\mathbb{Q}[e^{\frac{2\pi i}{m}}]$ which contains the entries of the (representatives of the) elements in G. Then we observe that for any $\sigma \in \mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{m}}],\mathbb{Q})$ we have $\sigma(C_q) = C_q$ and so we can consider the group $G^\sigma \stackrel{\mathrm{def}}{=} \sigma(G)$ obtained by applying σ to the entries of the representatives of the elements of G. We observe that if $G = G(r,p,q,n)$ then, since $\sigma(\zeta_r) = \zeta_r^d$ for some d such that $GCD(r,d) = 1$, we have that $G^\sigma = G$, i.e. $\sigma \in \mathrm{Aut}(G)$. The setting is similar to that of §7 with $k = 2$: we consider the following twisted diagonal subgroup of $G \times G^\sigma$

$$\Delta^\sigma G \stackrel{\mathrm{def}}{=} \{(g, g^\sigma) : \; g \in G\},$$

where $g^\sigma \stackrel{\mathrm{def}}{=} \sigma(g)$. We recall that $G \times G^\sigma$ acts on the symmetric algebra $S_q[X_1, X_2]$ and that this algebra has a bipartition degree given by the exponent partitions in the two sets of variables. The coinvariant algebra of $R^{G\times G^\sigma}$ is canonically isomorphic to $R^G \otimes R^{G^\sigma}$ and so it also affords a bipartition degree given by

$$R^{G\times G^\sigma}_{\lambda^{(1)},\lambda^{(2)}} \cong R^G_{\lambda^{(1)}} \otimes R^{G^\sigma}_{\lambda^{(2)}}.$$

We are interested in the subalgebra of $R^{G\times G^\sigma}$ consisting of $\Delta^\sigma G$-invariants and in particular to its Hilbert series with respect to the bipartition degree defined above.

The following result was proved in (5) for reflection groups in its unrefined version (i.e. considering only the bidegree in $\mathbb{N}^2$ and not the bipartition degree).

Theorem 10.1 *Let G be any projective reflection group. Then*

$$G^\sigma(Y_1, Y_2) \stackrel{\mathrm{def}}{=} \mathrm{Hilb}\left(\frac{S_q[X_1, X_2]^{\Delta^\sigma G}}{J_+^{G\times G^\sigma}}\right)(Y_1, Y_2) = \sum_{\phi\in \mathrm{Irr}(G)} f^{\sigma\phi}(Y_1) f^{\bar{\phi}}(Y_2).$$

If G is of the form $G = G(r,p,q,n)$, by Theorem 9.1, then the polynomial $G^\sigma(Y_1, Y_2)$ takes the following simple form in terms of the dual group G^*.

Corollary 10.2 *For any projective reflection group $G = G(r,p,q,n)$ and any Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}[e^{\frac{2\pi i}{m}}]/\mathbb{Q})$ we have*

$$G^\sigma(Y_1, Y_2) = \sum_{g\in G^*} Y_1^{\lambda(g^\sigma)} Y_2^{\lambda(g^{-1})}.$$

The unrefined version of the previous corollary

$$G^\sigma(y_1, y_2) = \sum_{g\in G^*} y_1^{\mathrm{fmaj}(g^\sigma)} y_2^{\mathrm{fmaj}(g^{-1})}$$

provides an answer to (5, Question 6.3). We believe that one can generalize these facts to a multivariate setting using Corollaries 7.6 and 8.2 instead of Theorem 9.1.

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A new combinatorial identity for unicellular maps, via a direct bijective approach.

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Abstract. We give a bijective operation that relates unicellular maps of given genus to unicellular maps of lower genus, with distinguished vertices. This gives a new combinatorial identity relating the number $\epsilon_g(n)$ of unicellular maps of size n and genus g to the numbers $\epsilon_j(n)$'s, for $j < g$. In particular for each g this enables to compute the closed-form formula for $\epsilon_g(n)$ much more easily than with other known identities, like the Harer-Zagier formula. From the combinatorial point of view, we give an explanation to the fact that $\epsilon_g(n) = R_g(n)\mathrm{Cat}(n)$, where $\mathrm{Cat}(n)$ is the n-th Catalan number and R_g is a polynomial of degree $3g$, with explicit interpretation.

Résumé. On décrit une opération bijective qui relie les cartes à une face de genre donné à des cartes à une face de genre inférieur, portant des sommets marqués. Cela conduit à une nouvelle identité combinatoire reliant le nombre $\epsilon_g(n)$ de cartes à une face de taille n et genre g aux nombres $\epsilon_j(n)$, pour $j < g$. En particulier, pour tout g, cela permet de calculer la formule close donnant $\epsilon_g(n)$ bien plus facilement qu'à l'aide des autres identités connues, comme la formule d'Harer-Zagier. Du point de vue combinatoire, nous donnons une explication au fait que $\epsilon_g(n) = R_g(n)\mathrm{Cat}(n)$, où $\mathrm{Cat}(n)$ est le $n$ième nombre de Catalan et R_g est un polynôme de degré $3g$, à l'interprétation explicite.

Keywords: Polygon gluings, combinatorial identity, bijection.

1 Introduction.

A *unicellular map* is a graph embedded on a compact orientable surface, in such a way that its complement is a topological polygon. Equivalently, a unicellular map can be viewed as a polygon, with an even number of edges, in which edges have been pasted pairwise in order to create a closed orientable surface. The number of handles of this surface is called the *genus* of the map.

These objects are reminiscent in combinatorics, and have been considered by several authors, with different methods, and under different names. According to the context, unicellular maps can also be called *polygon gluings*, *one-border ribbon graphs*, or *factorisations of a cycle*. The most famous example of unicellular maps are *planar* unicellular maps, which, from Jordan's lemma, are exactly plane trees, enumerated by the Catalan numbers.

The first result in the enumeration of unicellular maps in positive genus was obtained by Lehman and Walsh [WL72]. Using a direct recursive method, relying on multivariate recurrence equations, they expressed the number $\epsilon_g(n)$ of unicellular maps with n edges on a surface of genus g as follows:

$$\epsilon_g(n) = \sum_{\gamma \vdash g} \frac{(n+1)\dots(n+2-2g-l(\gamma))}{2^{2g}\prod_i c_i!(2i+1)^{c_i}}\mathrm{Cat}(n), \tag{1}$$

where the sum is taken over partitions γ of g, c_i is the number of parts i in γ, $l(\gamma)$ is the total number of parts, and $\mathrm{Cat}(n)$ is the n-th Catalan number. This formula has been extended by other authors ([GS98]).

Later, Harer and Zagier [HZ86], via matrix integrals techniques, obtained the two following equations, known respectively as the Harer-Zagier *recurrence* and the Harer-Zagier *formula*:

$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (2n-1)(n-1)(2n-3)\epsilon_{g-1}(n-2), \tag{2}$$

$$\sum_{g\geq 0} \epsilon_g(n) y^{n+1-2g} = \frac{(2n)!}{2^n n!} \sum_{i\geq 1} 2^{i-1} \binom{n}{i-1}\binom{y}{i}. \tag{3}$$

Formula 3 has been retrieved by several authors, by various techniques. A combinatorial interpretation of this formula was given by Lass [Las01], and the first bijective proof was given by Goulden and Nica [GN05]. Generalizations were given for bicolored, or multicolored maps [Jac87, SV08].

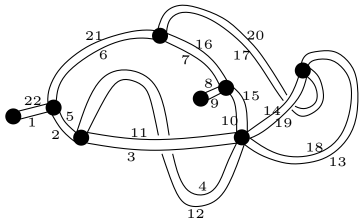
The purpose of this paper is to give a new angle of attack to the enumeration of unicellular maps, at a level which is much more combinatorial than what existed before. Indeed, until now no bijective proof (or combinatorial interpretation) of Formulas 1 and 2 are known. As for Formula 3, it is concerned with some *generating polynomial* of the numbers $\epsilon_g(n)$: in combinatorial terms, the bijections in [GN05, SV08] concern maps which are *weighted* according to their genus, by an additional coloring of their vertices, but the genus does not appear explicitely in the constructions. Moreover, these bijections concern those *weighted maps*, more than the unicellular maps themselves.

On the contrary, this article is concerned with the structure of unicellular maps themselves, at given genus. We investigate in details the way the unique face of such a map "interwines" with itself in order to create the handles of the surface. We show that, in each unicellular map of genus g, there are $2g$ "special places", which we call *trisections*, that concentrate, in some sense, the handles of the surface. Each of these places can be used to *slice* the map to a unicellular map of lower genus. Conversely, we show that a unicellular map of genus g can always be obtained in $2g$ different ways by gluing vertices together in a map of lower genus. In terms of formulas, this leads us to the new combinatorial identity:

$$\begin{aligned} 2g \cdot \epsilon_g(n) &= \binom{n+3-2g}{3}\epsilon_{g-1}(n) + \binom{n+5-2g}{5}\epsilon_{g-2}(n) + \ldots + \binom{n+1}{2g+1}\epsilon_0(n) && (4)\\ &= \sum_{p=0}^{g-1} \binom{n+1-2p}{2g-2p+1}\epsilon_p(n). && (5) \end{aligned}$$

This identity enables to compute, for each g, the closed formula giving $\epsilon_g(n)$ in terms of the n-th Catalan number much more easily than Formulas 1,2,3 (indeed, even Formula 1 has quite a big number of terms). In combinatorial terms, this enables to perform either exhaustive or random sampling of unicellular maps of given genus and size easily. When iterated, our bijection really shows that all unicellular maps can be obtained in a canonical way from plane trees by successive gluings of vertices, hence giving the first explanation to the fact that $\epsilon_g(n)$ is the product of a polynomial in n by the n-th Catalan number.

Acknowledgements. I am indebted to Olivier Bernardi, and to my advisor Gilles Schaeffer, for very stimulating discussions.



$H = [\![1, 22]\!]$

$\alpha = (1,22)(2,5)(3,11)(4,12)(6,21)(7,16)$
$(8,9)(10,15)(13,18)(14,19)(17,20)$

$\sigma = (1,5,21)(2,11,4)(3,12,18,14,10)$
$(6,16,20)(7,9,15)(8)(13,19,17)(22)$

$\gamma = \alpha\sigma = (1,2,3,\ldots,22)$

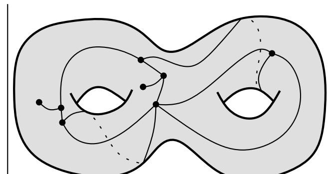


Figure 1: A unicellular map with 11 edges, 8 vertices, and genus 2: (a) ribbon graph; (b) permutations; (c) topological embedding.

2 Unicellular maps.

2.1 Permutations and ribbon graphs.

Rather than talking about topological embeddings of graphs, we work with a combinatorial definition of unicellular maps:

Definition 1. A *unicellular map* M of size n is a triple $M = (H, \alpha, \sigma)$, where H is a set of cardinality $2n$, α is an involution of H without fixed points, and σ is a permutation of H such that $\gamma = \alpha\sigma$ has only one cycle. The elements of H are called the *half-edges* of M. The cycles of α and σ are called the *edges* and the *vertices* of M, respectively, and the permutation γ is called the *face* of M.

Given a unicellular map $M = (H, \sigma, \alpha)$, its associated (multi)graph G is the graph whose edges are given by the cycles of α, vertices by the cycles of σ, and the natural incidence relation $v \sim e$ if v and e share an element. Moreover, we draw each edge of G as a *ribbon*, where each side of the ribbon represents one half-edge; we decide which half-edge corresponds to which side of the ribbon by the convention that, if a half-edge h belongs to a cycle e of α and v of σ, then h is the right-hand side of the ribbon corresponding to e, when considered entering v. Furthermore, we draw the graph G in such a way that around each vertex v, the *counterclockwise ordering* of the half-edges belonging to the cycle v is given by that cycle: we obtain a graphical object called the *ribbon graph* associated to M, as in Figure 1(a). Observe that the unique cycle of the permutation $\gamma = \alpha\sigma$ interprets as the sequence of half-edges visited when *making the tour* of the graph, keeping the graph on its left.

A *rooted* unicellular map is a unicellular map carrying a distinguished half-edge r, called the root. These maps are considered up to relabellings of H preserving the root, i.e. two rooted unicellular maps M and M' are considered the same if there exists a permutation $\pi : H \to H'$, such that $\pi(r) = r'$, $\alpha = \pi^{-1}\alpha'\pi$, and $\sigma = \pi^{-1}\sigma'\pi$. In this paper, *all* unicellular maps will be rooted, even if not stated.

Given a unicellular map M of root r and face $\gamma = \alpha\sigma$, we define the linear order $<_M$ on H by setting: $r <_M \gamma(r) <_M \gamma_2(r) <_M \ldots <_M \gamma^{2n-1}(r)$.
In other words, if we relabel the half-edge set H by elements of $[\![1, 2n]\!]$ in such a way that the root is 1 and the tour of the face is given by the permutation $(1, \ldots, 2n)$, the order $<_M$ is the natural order on the integers. However, since in this article we are going to consider maps with a fixed half-edge set, but a changing permutation γ, it is more convenient (and prudent) to define the order $<_M$ in this way.

Unicellular maps can also be interpreted as graphs embedded in a topological surface, in such a way that the complement of the graph is a topological polygon. If considered up to homeomorphism, and suitably rooted, these objects are in bijection with ribbon graphs. See [MT01], or the example of Figure 1(c). The *genus* of a unicellular map is the genus, or number of handles, of the corresponding surface. If a unicellular map of genus g has n edges and v vertices, then Euler's characteristic formula says that $v = n + 1 - 2g$. From a combinatorial point of view, this last equation can also be taken as a definition of the genus.

2.2 The gluing operation.

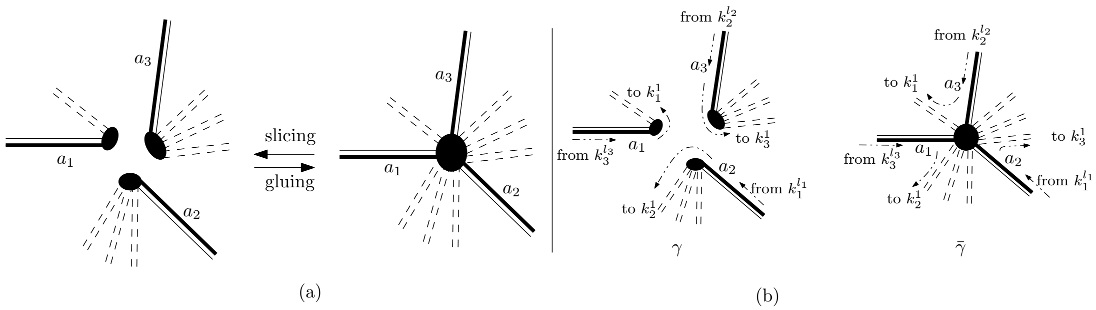


Figure 2: (a) The gluing and slicing operations. (b) The "proof" of Lemma 1.

We let $M = (H, \alpha, \sigma)$ be a unicellular map of genus g, and $a_1 <_M a_2 <_M a_3$ be three half-edges of M *belonging to three distinct vertices*. Each half-edge a_i belongs to some vertex $v_i = (a_i, h_i^1, \dots h_i^{m_i})$, for some $m_i \geq 0$. We define the permutation

$$\bar{v} := (a_1, h_2^1, \dots h_2^{m_2}, a_2, h_3^1, \dots h_3^{m_3}, a_3, h_1^1, \dots h_1^{m_1}),$$

and we let $\bar{\sigma}$ be the permutation of H obtained by deleting the cycles v_1, v_2, and v_3, and replacing them by $\bar{v}$. The transformation mapping σ on $\bar{\sigma}$ interprets combinatorially as the *gluing* of the three half-edges a_1, a_2, a_3, as shown on Figure 2(a). We have:

Lemma 1. *The map $\bar{M} := (H, \alpha, \bar{\sigma})$ is a unicellular map of genus $g + 1$. If we let $\gamma = \alpha\sigma = (a_1, k_1^1, \dots k_1^{l_1}, a_2, k_2^1, \dots k_2^{l_2}, a_3, k_3^1, \dots k_3^{l_3})$ be the face permutation of M, then the face of $\bar{M}$ is given by:*

$$\bar{\gamma} = (a_1, k_2^1, \dots k_2^{l_2}, a_3, k_1^1, \dots k_1^{l_1}, a_2, k_3^1, \dots k_3^{l_3})$$

Proof: In order to prove that M is a well-defined unicellular map, it suffices to check that its face is given by the long cycle $\bar{\gamma}$ given in the lemma. This is very easy to check by observing that the only half-edges whose image is not the same by γ and by $\bar{\gamma}$ are the three half-edges a_1, a_2, a_3, and that by construction $\bar{\gamma}(a_i) = \alpha\bar{\sigma}(a_i) = \alpha\sigma(a_{i+1}) = \gamma(a_{i+1})$. For a more "visual" explanation, see Figure 2(b).

Now, by construction, M' has two less vertices than M, and the same number of edges, so from Euler's formula it has genus $g + 1$ (intuitively, the gluing operation has created a new "handle"). □

2.3 Some intertwining hidden there, and the slicing operation.

The aim of this paper is to show that all unicellular maps of genus $g+1$ can be obtained *in some canonical way* from unicellular maps of genus g from the operation above. This needs to be able to "revert" (in some

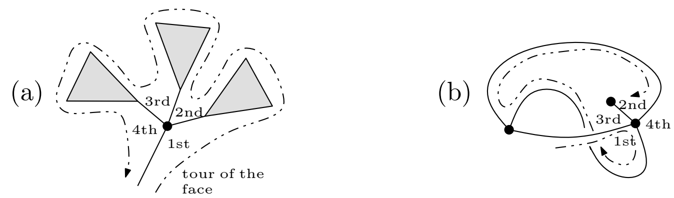


Figure 3: (a) In a plane tree, the tour of the face always visits the half-edges around one vertex in *counterclockwise* order; (b) in positive genus (here in genus 1), things can be different.

sense) the gluing operation, hence to be able to determine, given a map of genus $g+1$, which vertices may be "good candidates" to be sliced-back to a map of lower genus.

Observe that in the unicellular map $\bar{M}$ obtained after the gluing operation, the three half-edges a_1, a_2, a_3 appear *in that order* around the vertex $\bar{v}$, whereas they appear in the *inverse* order in the face $\bar{\gamma}$. Observe also that this is very different from what we observe in the planar case: if one makes the tour of a plane tree, with the tree on its left, then one necessarily visits the different half-edges around each vertex in *counterclockwise order* (see Figure 3). Informally, one could hope that, in a map of positive genus, those places where the vertex-order does not coincide with the face-order hide some "intertwining" (some handle) of the map, and that they may be used to slice-back the map to lower genus.

We now describe the *slicing* operation, which is nothing but the gluing operation, taken at reverse. We let $\bar{M} = (H, \alpha, \bar{\sigma})$ be a map of genus $g+1$, and three half-edges a_1, a_2, a_3 belonging to a same vertex $\bar{v}$ of $\bar{M}$. We say that a_1, a_2, a_3 are *intertwined* if they do not appear in the same order in $\bar{\gamma} = \alpha\bar{\sigma}$ and in $\bar{\sigma}$. In this case, we write $\bar{v} = (a_1, h_2^1, \dots h_2^{m_2}, a_2, h_3^1, \dots h_3^{m_3}, a_3, h_1^1, \dots h_1^{m_1})$, and we let σ be the permutation of H obtained from $\bar{\sigma}$ by replacing the cycle $\bar{v}$ by the product $(a_1, h_1^1, \dots h_{m_1}^1)(a_2, h_1^2, \dots h_{m_2}^2)(a_3, h_1^3, \dots h_{m_3}^3)$.

Lemma 2. *The map $M = (H, \alpha, \sigma)$ is a well-defined unicellular map of genus g. If we let $\bar{\gamma} = (a_1, k_2^1, \dots k_2^{l_2}, a_3, k_1^1, \dots k_1^{l_1}, a_2, k_3^1, \dots k_3^{l_3})$ be the unique face of $\bar{M}$, then the unique face of M is given by: $\gamma = \alpha\sigma = (a_1, k_1^1, \dots k_1^{l_1}, a_2, k_2^1, \dots k_2^{l_2}, a_3, k_3^1, \dots k_3^{l_3})$.*

The gluing and slicing operations are inverse one to the other.

Proof: The proof is the same as in Lemma 1: it is sufficient to check the expression given for γ in terms of $\bar{\gamma}$, which is easily done by checking the images of a_1, a_2, a_3. □

2.4 Around one vertex: up-steps, down-steps, and trisections.

Let $M = (H, \alpha, \sigma)$ be a map of face permutation $\gamma = \alpha\sigma$. For each vertex v of M, we let $\min_M(v)$ be the minimal half-edge belonging to v, for the order $<_M$. Equivalently, $\min_M(v)$ is the first half-edge from which one reaches v when making the tour of the map, starting from the root. Given a half-edge $h \in H$, we note $V(h)$ the unique vertex it belongs to (i.e. the cycle of σ containing it).

Definition 2. We say that a half-edge $h \in H$ is an *up-step* if $h <_M \sigma(h)$, and that it is a *down-step* if $\sigma(h) \leq_M h$. A down-step h is called a *trisection* if $\sigma(h) \neq \min_M V(h)$, i.e. if $\sigma(h)$ is not the minimum half-edge inside its vertex.

As illustrated on Figure 3, trisections are specific to the non-planar case (there are no trisections in a plane tree), and one could hope that trisections "hide" (in some sense) the handles of the surface. Before making this more precise, we state the following lemma, which is the cornerstone of this paper:

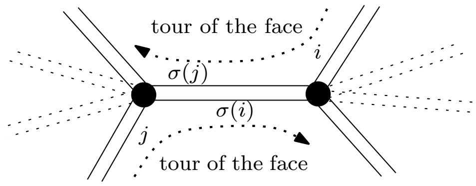


Figure 4: The main argument in the proof of the trisection lemma: the tour of the face visits i before $\sigma(i)$ if and only if it visits $\sigma(j)$ before j, unless $\sigma(i)$ or $\sigma(j)$ is the root of the map.

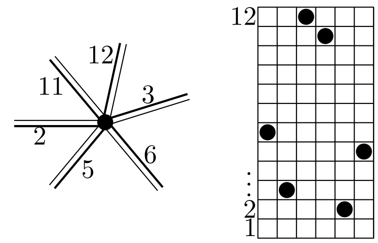


Figure 5: A vertex $(6, 3, 12, 11, 2, 5)$ in a map with 12-half-edges, and its diagram representation (the marked half-edge is 6).

Lemma 3 (The trisection lemma)**.** *Let M be a unicellular map of genus g. Then M has exactly $2g$ trisections.*

Proof: We let $M = (H, \alpha, \sigma)$, and $\gamma = \alpha\sigma$. We let n_+ and n_- denote the number of up-steps and down-steps, respectively. Then, we have $n_- + n_+ = 2n$, where n is the number of edges of M. Now, let i be a half-edge of M, and $j = \sigma^{-1}\alpha\sigma(i)$. Observe that we have $\sigma(j) = \gamma(i)$, and $\gamma(j) = \sigma(i)$. Graphically, i and j lie in two "opposite" corners of the same edge, as shown on Figure 4. On the picture, it seems clear that if we visit i before $\sigma(i)$, then we necessarily visit $\sigma(j)$ before j (except if the root is one of these four half-edges) so that, roughly, there must be almost the same number of up-steps and down-steps. More precisely, let us distinguish three cases.

First, assume that i is an up-step. Then we have $i <_M \sigma(i) = \gamma(j)$. Now, by definition of the total order $<_M$, $i <_M \gamma(j)$ implies that $\gamma(i) \leq_M \gamma(j)$. Hence, $\sigma(j) \leq_M \gamma(j)$, which, by definition of $<_M$ again, implies that $\sigma(j) \leq_M j$ (here, we have used that $\sigma(j) \neq \gamma(j)$ since α has no fixed point). Hence, if i is an up-step, then j is a down-step.

Second, assume that i is a down-step, *and* that $\gamma(j)$ is not equal to the root of M. In this case, we have $j <_M \gamma(j)$, and $\gamma(j) = \sigma(i) \leq_M i = \sigma(j)$. Hence $j <_M \sigma(j)$, and j is an up-step.

The third and last case is when i is a down step, *and* $\gamma(j)$ is the root r of M. In this case, j is the maximum element of H for the order $<_M$, so that it is necessarily a down-step.

Therefore we have proved that each edge of M (more precisely, each cycle of $\sigma^{-1}\alpha\sigma$) is associated to one up-step and one down-step, except a special one that has two down-steps. Consequently, there are exactly two more down-steps that up-steps in the map M, i.e.: $n_- = n_+ + 2$. Recalling that $n_- + n_+ = 2n$, this gives $n_- = n + 1$.

Finally, each vertex of M carries exactly one down-step which is not a trisection (its minimal half-edge). Hence, the total number of trisections equals $n_- - v$, where v is the number of vertices of M. Since from Euler's characteristic formula, v equals $n + 1 - 2g$, the lemma is proved. □

3 Making the gluing operation injective.

We have defined above an operation that glues a triple of half-edges, and increases the genus of a map. In this section, we explain that, if we restrict to *certain types* of triples of half-edges, this operation can be made reversible.

3.1 A diagram representation of vertices.

We first describe a graphical visualisation which should make the exposition more easy. Let v be a vertex of M, with a distinguished half-edge h. We write $v = (u_0, u_1, \ldots, u_m)$, with $u_0 = h$. We now consider a grid with $m + 1$ columns and $2n$ rows. Each row represents an element of H, and the rows are ordered from the bottom to the top by the total order $<_M$ (for example the lowest row represents the root). Now, for each i, inside the i-th column, we plot a point at the height corresponding to the half-edge u_i. We say that the obtained diagram is *the diagram representation of* v*, starting from* h. In other words, if we identify $[\![1, 2n]\!]$ with H via the order $<_M$, the diagram representation of v is the graphical representation of the sequence of labels appearing around the vertex v. If one changes the distinguished half-edge h, the diagram representation of v is changed by a circular permutation of its columns. Figure 5 gives an example of such a diagram (where the permutation γ is in the form $\gamma = (1, 2, 3, \ldots)$).

The gluing operation is easily visualised on diagrams. We let as before $a_1 <_M a_2 <_M a_3$ be three half-edges belonging to distinct vertices in a unicellular map M, and we let $\Delta_1, \Delta_2, \Delta_3$ be their corresponding diagrams. We now consider the three horizontal rows corresponding to a_1, a_2, and a_3: they separate each diagram Δ_i into four blocks (some of which may be empty). We give a name to each of these blocks: A_i, B_i, C_i, D_i, from bottom to top, as on Figure 6(a).

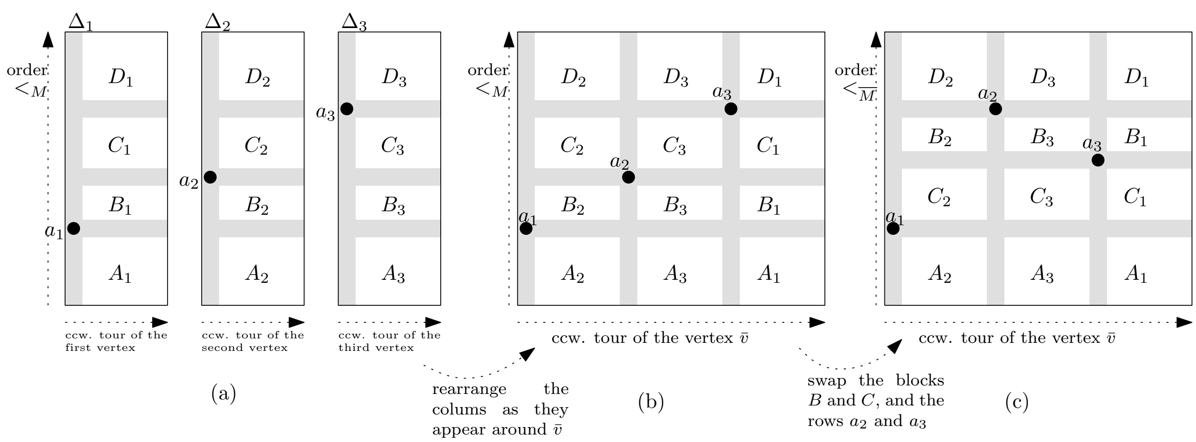


Figure 6: The gluing operation visualized on diagrams. (a) the diagrams before gluing; (b) a temporary diagram, where we the columns represent the counterclockwise turn around $\bar{v}$, but the rows still represent the original permutation γ; (c) the final diagram of the new vertex in the new map, where the rows represent the permutation $\bar{\gamma}$.

We now juxtapose $\Delta_2, \Delta_3, \Delta_1$ together, from left to right, and we rearrange the three columns containing a_1, a_2, a_3 so that these half-edges appear in that order: we obtain a new diagram (Figure 6(b)), whose columns represent the order of the half-edges around the vertex $\bar{v}$. But the *rows* of that diagram are still ordered according to the order $<_M$. In order to obtain the diagram representing $\bar{v}$ in the new map $\bar{M}$, we have to rearrange the rows according to $<_{\bar{M}}$. We let A be the union of the three blocks A_i (and similarly, we define B, C, and D). We know that the face permutation of M has the form $\gamma = (— A —, a_1, — B —, a_2, — C —, a_3, — D —)$, where by $— A —$, we mean "all the elements of A, appearing in a certain order". Now, from the expression of $\bar{\gamma}$ given in Lemma 1, the permutation $\bar{\gamma}$ is: $\bar{\gamma} = (— A —, a_1, — C —, a_3, — B —, a_2, — D —)$, where inside each block, the half-edges appear in the same order as in γ. In terms of diagrams, this means that the diagram representing $\bar{v}$ in the new map $\bar{M}$ can be obtained by *swapping* the block B with the block C, and the row corresponding to a_2 with the one corresponding to a_3: see Figure 6(c). To sum up, we have:

Lemma 4. *The diagram of the vertex $\bar{v}$ in the map $\bar{M}$ is obtained from the three diagrams $\Delta_1, \Delta_2, \Delta_3$ by the following operations, as represented on Figure 6:*
- Juxtapose $\Delta_2, \Delta_3, \Delta_1$ (in that order), and rearrange the columns containing a_1, a_2, a_3, so that they appear in that order from left to right.
- Exchange the blocks B and C, and swap the rows containing a_2 and a_3.

Observe that, when taken at reverse, Figure 6 gives the way to obtain the diagrams of the three vertices resulting from the *slicing* operation of three intertwined half-edges a_1, a_2, a_3 in the map $\bar{M}$.

3.2 Gluing three vertices: trisections of type **I**.

In this section, we let v_1, v_2, v_3 be three distinct vertices in the map M. We let $a_i := \min_M v_i$, and, up to re-arranging the three vertices, we may assume (and we do) that $a_1 <_M a_2 <_M a_3$. We let Δ_1, Δ_2, Δ_3 be the three corresponding diagrams. Since in each diagram the marked edge is the minimum in its vertex, observe that the blocks A_1, A_2, B_2, A_3, B_3, C_3 do not contain any point. We say that they are *empty*, and we note: $A_1 = A_2 = B_2 = A_3 = B_3 = C_3 = \varnothing$.

We now glue the three half-edges a_1, a_2, a_3 in M: we obtain a new unicellular map $\bar{M}$, with a new vertex $\bar{v}$ resulting from the gluing. Now, let τ be the element preceding a_3 around $\bar{v}$ in the map $\bar{M}$. Since $A_3 = B_3 = C_3 = \varnothing$, we have either $\tau \in D_3$ or $\tau = a_2$, so that in both case $a_3 <_{\bar{M}} \tau$. Moreover, a_3 in not the minimum inside its vertex (the minimum is a_1). Hence, τ is a *trisection* of the map $\bar{M}$. We let $\Phi(M, v_1, v_2, v_3) = (\bar{M}, \tau)$ be the pair formed by the new map $\bar{M}$ and the newly created trisection τ.

It is clear that given $(\bar{M}, \tau)$, we can inverse the gluing operation. Indeed, it is easy to recover the three half-edges a_1 (the minimum of the vertex), a_3 (the one that follows τ), and a_2 (observe that, since B_2 and B_3 are empty, a_2 is the smallest half-edge on the left of a_3 which is greater than a_3). Once a_1, a_2, a_3 are recovered, it is easy to recover the map M by *slicing* $\bar{v}$ at those three half-edges. This gives:

Lemma 5. *The mapping Φ, defined on the set of unicellular maps with three distinguished (unordered) vertices, is injective.*

It is natural to ask for the image of Φ: in particular, can we obtain all pairs $(\bar{M}, \tau)$ in this way ? The answer needs the following definition (see Figure 7):

Definition 3. Let $\bar{M} = (H, \alpha, \bar{\sigma})$ be a map of genus $g+1$, and τ be a trisection of $\bar{M}$. We let $\bar{v} = V(\tau)$, $b_1 = \min_{\bar{M}}(\bar{v})$, and we let Δ be the diagram representation of $\bar{v}$, starting from the half-edge b_1. We let $b_3 = \sigma(\tau)$ be the half-edge following τ around $\bar{v}$, and we let b_2 be the minimum half-edge among those which appear before b_3 around $\bar{v}$ and which are greater than b_3 for the order $<_{\bar{M}}$.

The rows and columns containing b_1, b_2, b_3 split the diagram Δ into twelve blocks, five of which are necessarily empty, as in Figure 7. We let K be second-from-left and second-from-bottom block. We say that τ is *a trisection of type* **I** is K is empty, and that τ is *a trisection of type* **II** otherwise.

The following proposition is the half way to our main result:

Proposition 1. *The mapping Φ is a bijection between the set $\mathcal{U}_g^3(n)$ of unicellular maps of genus g, with n edges, and three distinguished vertices, and the set $\mathcal{D}_{g+1}^I(n)$of unicellular maps of genus $g+1$ with n edges and a distinguished trisection of type* **I**.

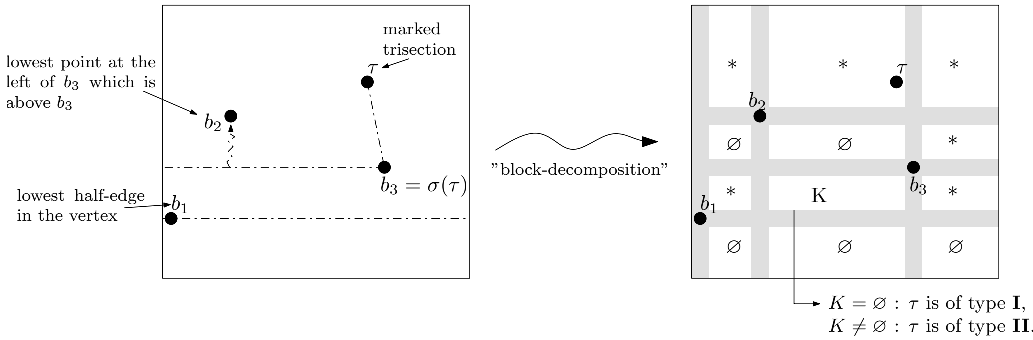


Figure 7: Trisections of type **I** and **II**.

Proof: We already know that Φ is injective.

We let M be a unicellular map of genus g with three distinguished vertices v_1, v_2, v_3, and $\bar{M}$ be the map obtained, as above, by the gluing of M by the half-edges $a_1 = \min_M v_1$, $a_2 = \min_M v_2$, $a_3 = \min_M v_3$ (we assume again that $a_1 <_M a_2 <_M a_3$). We let $\bar{\Delta}$ be the diagram representation of the new vertex $\bar{v}$ obtained from the gluing in the map $\bar{M}$, and we use the same notations for the blocks as in Section 3.1. We also let $\tau = \sigma^{-1}(a_3)$ be the created trisection, and we use the notations of Definition 3 with respect to the trisection τ, so that $b_3 = a_3$. Then, since $a_1 = \min_{\bar{M}} \bar{v}$, we have $a_1 = b_1$, and since the blocks B_2, B_3, are empty, we have $b_2 = a_2$. Hence, the block C_3 of Figure 6(c) coincides with the block K of Figure 7. Since C_3 is empty, τ is a trisection of type **I**. Therefore the image of Φ is included in $\mathcal{D}^I_{g+1}(n)$.

Conversely, let $\bar{M} = (H, \alpha, \bar{\sigma})$ be a map of genus $g+1$, and τ be a trisection of type **I** in $\bar{M}$. We let b_1, b_2, b_3 and K be as in Definition 3. First, since $b_1 <_{\bar{M}} b_3 <_{\bar{M}} b_2$, these half-edges are intertwined, and we know that the slicing of $\bar{M}$ by these half-edges creates a well-defined unicellular map M of genus g (Lemma 2). Now, if we compare Figures 7 and 6, we see that the result of the slicing is a triple of vertices v_1, v_2, v_3, such that each half-edge b_i is the minimum in the vertex v_i: indeed, the blocks A_1, A_2, A_3, B_2, B_3 are empty by construction, and the block $C_3 = K$ is empty since τ is a trisection of type **I**. Hence we have $\Phi(M, v_1, v_2, v_3) = (\bar{M}, \tau)$, so that the image of Φ exactly equals the set $\mathcal{D}^I_{g+1}(n)$. □

3.3 Trisections of type II.

Of course, it would be nice to have a similar result for trisections of type **II**. Let $\bar{M} = (H, \alpha, \bar{\sigma})$ be a map of genus $g+1$ with a distinguished trisection τ of type **II**. We let b_1, b_2, b_3 and K be as in Definition 3 and Figure 7, and we let M be the result of the slicing of $\bar{M}$ at the three half-edges b_1, b_2, b_3. If we use the notations of Figure 6, with $a_i = b_i$, we see that we obtain three vertices, of diagrams $\Delta_1, \Delta_2, \Delta_3$, such that $A_1 = A_2 = B_2 = A_3 = B_3 = \varnothing$. Hence, we know that $a_1 = \min_M(v_1)$, that $a_2 = \min_M(v_2)$, and that $a_2 < \min_M(v_3)$. Observe that, contrarily to what happened in the previous section, the block $C_3 = K$ is *not* empty, therefore a_3 is *not* the minimum inside its vertex.

Now, we claim that τ *is still a trisection* in the map M. Indeed, by construction, we know that τ belongs to D_3 (since, by definition of a trisection, it must be above a_3 in the map $\bar{M}$, and since B_3 is empty). Hence we still have $a_3 <_M \tau$ in the map M. Moreover, we have clearly $\sigma(\tau) = a_3$ in M (since

τ is the rightmost point in the blocks $C_3 \cup D_3$), and it follows that τ is a trisection in M.

We let $\Gamma(\bar{M}, \tau) = (M, v_1, v_2, \tau)$ be the 4-tuple consisting of the new map M, the two first vertices v_1 and v_2 obtained from the slicing, and the trisection τ. It is clear that Γ is injective: given $(\bar{M}, v_1, v_2, \tau)$, one can reconstruct the map $\bar{M}$ by letting $a_1 = \min v_1$, $a_2 = \min v_2$, and $a_3 = \sigma(\tau)$, and by gluing back together the three half-edges a_1, a_2, a_3. Conversely, we define:

Definition 4. We let $\mathcal{V}_g(n)$ be the set of 4-tuples (M, v_1, v_2, τ), where M is a unicellular map of genus g with n edges, and where v_1, v_2, and τ are respectively two vertices and a trisection of M such that:

$$\min_M v_1 <_M \min_M v_2 <_M \min_M V(\tau). \tag{6}$$

Given $(M, v_1, v_2, \tau) \in \mathcal{V}_G(n)$, we let $\bar{M}$ be the map obtained from the gluing of the three half-edges $\min v_1$, $\min v_2$, and $\sigma(\tau)$, and we let $\Psi(M, v_1, v_2, \tau) := (\bar{M}, \tau)$.

We can now state the following proposition, that completes Proposition 1:

Proposition 2. *The mapping Ψ is a bijection between the set $\mathcal{V}_g(n)$ of unicellular maps of genus g with n edges a distinguished triple (v_1, v_2, τ) satisfying Equation 6, and the set $\mathcal{D}^{II}_{g+1}(n)$ of unicellular maps of genus $g+1$ with n edges and a distinguished trisection of type* **II**.

Proof: In the discussion above, we have already given a mapping $\Gamma : \mathcal{D}^{II}_{g+1}(n) \to \mathcal{V}_g(n)$, such that $\Psi \circ \Gamma$ is the identity on $\mathcal{D}^{II}_{g+1}(n)$.

Conversely, let $(M, v_1, v_2, \tau) \in \mathcal{V}_g(n)$, and let $a_1 = \min v_1$, $a_2 = \min v_2$, and $a_3 = \sigma(\tau)$. By definition, we know that $a_2 < \min V(\tau)$, so that in the diagram representation of the three vertices $v_1, v_2, V(\tau)$ (Figure 6(a)) we know that the blocks A_1, A_2, A_3, B_2, B_3 are empty. Moreover, since τ is a trisection, a_3 is not the minimum inside its vertex, so the block C_3 is *not* empty. Hence, comparing Figures 6(c) and 7, and observing once again that the blocks C_3 and K coincide, we see that after the gluing, τ is a trisection of type **II** in the new map $\bar{M}$. Moreover, since the slicing and gluing operations are inverse one to the other, it is clear that $\Gamma(\bar{M}, \tau) = (M, v_1, v_2, \tau)$. Hence, $\Gamma \circ \Psi$ is the identity, and the proposition is proved. □

4 Iterating the bijection.

Of course Proposition 1 looks nicer than its counterpart Proposition 2: in the first one, one only asks to distinguish three vertices in a map of lower genus, whereas in the second one, the marked triple must satisfy a nontrivial constraint (Equation 6). In this section we will work a little more in order to get rid of this problem. We start with two definitions (observe that for $k = 3$ this is coherent with what precedes):

Definition 5. We let $\mathcal{U}^k_g(n)$ be the set of unicellular maps of genus g with n edges, and k distinct distinguished vertices, undistinguishable one from the others.

Definition 6. We let $\mathcal{D}_g(n) = \mathcal{D}^I_g(n) \cup \mathcal{D}^{II}_g(n)$ be the set of unicellular maps of genus g with n edges, and a distinguished trisection.

4.1 *Training examples: genera* 1 *and* 2.

Observe that the set $\mathcal{V}_0(n)$ is empty, since there are no trisections in a plane tree. Hence, from Proposition 2, there are no trisections of type **II** in a map of genus 1 (i.e. $\mathcal{D}^{II}_1(n) = \varnothing$). Proposition 1 gives:

Corollary 1. *The set $D_1(n)$ of unicellular maps of genus 1 with n edges and a distinguished trisection is in bijection with the set $U_0^3(n)$ of rooted plane trees with n edges and three distinguished vertices.*

We now consider the case of genus 2. Let M be a unicellular map of genus 2, and τ be a trisection of M. If τ is of type **I**, we know that we can use the application Φ^{-1}, and obtain a unicellular map of genus 1, with three distinguised vertices.

Similarly, if τ is of type **II**, we can apply the mapping Ψ^{-1}, and we are left with a unicellular map M' of genus 1, and a marked triple (v_1, v_2, τ), such that $\min_{M'} v_1 <_{M'} \min_{M'} v_2 <_{M'} \min_{M'} V(\tau)$. From now on, we use the more compact notation: $v_1 <_{M'} v_2 <_{M'} V(\tau)$, i.e. we do not write the min's anymore. The map (M', τ) is a unicellular map of genus 1 with a distinguished trisection: therefore we can apply the mapping Φ^{-1} to (M', τ). We obtain a plane tree M'', with three distinguished vertices v_3, v_4, v_5 inherited from the slicing of τ in M'; since those three vertices are undistinguishable, we can assume that $v_3 <_{M''} v_4 <_{M''} v_5$. Observe that in M'' we also have the two marked vertices v_1 and v_2 inherited from the slicing of τ in M. Moreover the fact that $v_1 <_{M'} v_2 <_{M'} V(\tau)$ in M' implies that $v_1 <_{M''} v_2 <_{M''} v_3$ in M'': indeed, the gluing operation does not modify the part of cycle γ appearing between the root and the smallest glued half-edge, so that appearing before $V(\tau)$ in M' is equivalent to appearing before v_3 in M''. Hence, we are left with a plane tree M'', with *five* distinguished vertices $v_1 <_{M''} v_2 <_{M''} v_3 <_{M''} v_4 <_{M''} v_5$. Conversely, given such a 5-tuple of vertices, it is always possible to glue the three last ones together by the mapping Φ to obtain a triple (v_1, v_2, τ) satisfying Equation 6, and then to apply the mapping Ψ to retrieve a map of genus 2 with a marked trisection of type **II**. This gives:

Corollary 2. *The set $\mathcal{D}_2^{II}(n)$ is in bijection with the set $\mathcal{U}_0^5(n)$ of plane trees with* five *distinguished vertices.*

The set $\mathcal{D}_2(n)$ of unicellular maps of genus 2 with one marked trisection is in bijection with the set $\mathcal{U}_1^3(n) \cup \mathcal{U}_0^5(n)$.

4.2 The general case, and our main theorem.

We let $p \geq 0$ and $q \geq 1$ be two integers, and $(M, v_*) = (M, v_1, \dots, v_{2q+1})$ be an element of $\mathcal{U}_p^{2q+1}(n)$. Up to renumbering the vertices, we can assume that $v_1 <_M v_2 <_M \dots <_M v_{2q+1}$.

Definition 7. We consider the following procedure:
i. Glue the three last vertices $v_{2q-1}, v_{2q}, v_{2q+1}$ together, via the mapping Φ, in order to obtain a new map M_1 of genus $p+1$ with a distinguished trisection τ of type **I**.
ii. for i from 1 to $q-1$ **do:**
Let $(v_{2q-2i-1}, v_{2q-2i}, \tau)$ be the triple consisting of the last two vertices which have not been used until now, and the trisection τ. Apply the mapping Ψ to that triple, in order to obtain a new map M_{i+1} of genus $p+i+1$, with a distinguished trisection τ of type **II**.
end for.
We let $\Lambda(M, v_*) := (M_q, \tau)$ be the map with a distinguished trisection obtained at the end of this procedure. Observe that if $q = 1$, the distinguished trisection is of type **I**, and that it is of type **II** otherwise.

The following Theorem can easily be proved from Propositions 1 and 2 by adapting the arguments we used in the particular case of genus 2:

Theorem 1 (Our main result)**.** *The application Λ defines a bijection:*

$$\Lambda : \biguplus_{p=0}^{g-1} \mathcal{U}_p^{2g-2p+1}(n) \longrightarrow \mathcal{D}_g(n).$$

In other words, all unicellular maps of genus g with a distinguished trisection can be obtained in a canonical way by starting with a map of lower genus with an odd number of distinguished vertices, and then applying once the mapping Φ, and a certain number of times the mapping Ψ.

Given a map with a marked trisection (M, τ), the converse application consists in slicing recursively the trisection τ while it is of type **II***, then slicing* once *the obtained trisection of type* **I***, and remembering all the vertices resulting from the successive slicings.*

Finally, our new identity (Equation (4)) follows from the theorem and the Trisection lemma (Lemma 3).

Further developments: - It is known that *labelled* unicellular maps are in bijection with *general* maps of the same genus (this is the Marcus-Schaeffer bijection). Hence our bijection also leads to a full description of maps of positive genus in terms of plane labelled trees with distinguished vertices.

- It is straightforward to obtain a formula analogous to (4) for the numbers $\beta_g(k, l)$ of *bipartite* unicellular maps with k white and l black vertices (just be careful to glue only vertices of the same color).

- It is possible to iterate the mapping Λ in order to obtain only *plane* trees at the end. This leads to the following formula, which interestingly reminds of Equation 1:

$$\epsilon_g(n) = \left(\sum_{0=g_0<g_1<\ldots<g_r=g} \prod_{i=1}^{r} \frac{1}{2g_i} \binom{n+1-2g_{i-i}}{2(g_i - g_{i-1})+1} \right) Cat(n).$$

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Indecomposable permutations with a given number of cycles

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Abstract. A permutation $a_1a_2\dots a_n$ is *indecomposable* if there does not exist $p < n$ such that $a_1a_2\dots a_p$ is a permutation of $\{1,2,\dots,p\}$. We compute the asymptotic probability that a permutation of $\mathbb{S}_n$ with m cycles is indecomposable as n goes to infinity with m/n fixed. The error term is $O(\frac{\log(n-m)}{n-m})$. The asymptotic probability is monotone in m/n, and there is no threshold phenomenon: it degrades gracefully from 1 to 0. When $n = 2m$, a slight majority ($51.1\dots$ percent) of the permutations are indecomposable. We also consider indecomposable fixed point free involutions which are in bijection with maps of arbitrary genus on orientable surfaces, for these involutions with m left-to-right maxima we obtain a lower bound for the probability of being indecomposable.

Résumé. Une permutation $a_1a_2\dots a_n$ est *indécomposable*, s'il n'existe pas de $p < n$ tel que $a_1a_2\dots a_p$ est une permutation de $\{1,2,\dots,p\}$. Nous calculons la probabilité pour qu'une permutation de $\mathbb{S}_n$ ayant m cycles soit indécomposable et plus particulièrement son comportement asymptotique lorsque n tend vers l'infini et que m/n est fixé. Cette valeur décroît régulièrement de 1 à 0 lorsque m/n croît, et il n'y a pas de phénomène de seuil. Lorsque $n = 2m$, une faible majorité ($51.1\dots$ pour cent) des permutations sont indécomposables. Nous considerons aussi les involutions sans point fixe indécomposables qui sont en bijection avec les cartes de genre quelconque plongées dans une surface orientable, pour ces involutions ayant m maxima partiels (ou records) nous obtenons une borne inférieure pour leur probabilité d'êtres indécomposables.

Keywords: Permutations, enumeration, asymptotics.

1 Introduction.

Indecomposable permutations (also often called connected) have been considered by many authors trying to show that they play the same role for permutations as connected graphs play in graph theory. Marshall Hall [Hal49] was probably the first to implicitly consider them while enumerating subgroups of finite index of the free group with 2 generators. They were studied in more detail 20 years later by A. Lentin[Len72] and L. Comtet [Com72] and are quoted in good place in many classical books in Combinatorics and Algorithms (see for instance [Com74], [Knu05], [GJ83], and [Sta99]). More recently, a bijection was given by P. Ossona de Mendez and P. Rosenstiehl in [dMR04] with hypermaps (or equivalently bicolored maps) is such a way that the number of cycles of the permutation is equal to the number

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of vertices of the hypermap (equivalently the number of vertices of a given color of the bipartite map). Hence in order to generate at random a hypermap with a fixed number m of vertices, a natural algorithm consists in generating permutations with m cycles until obtaining an indecomposable one, then to build the hypermap in bijection with it. The efficiency of this algorithm depends on the value of the probability for a permutation with m cycles to be indecomposable. Intuitively this probability is expected to be a decreasing function of $\frac{m}{n}$; we will prove this fact asymptotically in this paper and give a precise description of the asymptotic limit of this function when n and m tend to infinity keeping $\frac{n}{m}$ constant.

In a second part of the paper we restrict these permutations to be involutions with no fixed points and take as parameter the number of left-to-right maxima instead of the number of cycles, note that these two statistics are equal for general permutations. Similarly the above bijection associates to indecomposable involutions maps on orientable surfaces having the same number of vertices as the involution has left-to-right maxima. We obtain a lower bound on the probability for an involution with no fixed points and a given number of let-to-right maxima to be indecomposable. We use combinatorial arguments and a coding of these involutions by labeled Dyck words, often called *histoires d'Hermite*, (see [dMV94], [Dra09]).

Notation

A permutation will be denoted $a_1a_2\dots a_n$, it is called *decomposable* if there exists $p < n$ such that $a_1a_2\dots a_p$ is a permutation of $\{1, 2, \dots, p\}$, and is called *indecomposable* otherwise. Let $\mathbb{S}_n$ denote the set of permutations of $\{1, 2, \dots, n\}$. In [Com72], Comtet proved that almost all permutations of $\mathbb{S}_n$ are indecomposable, more precisely:

$$\Pr_{\mathbb{S}_n}\{\alpha \text{ indecomposable}\} = 1 - \frac{2}{n} + O(\frac{1}{n^2}).$$

The event that α is decomposable depends heavily on the number of cycles of α. The permutation with n cycles (the identity) is decomposable, and among the $\binom{n}{2}$ permutations with $n-1$ cycles (the transpositions), all but one are decomposable. At the other extreme, a permutation with only one cycle is never decomposable. Intuitively, it seems clear that a permutation with more cycles is more likely to be decomposable. In this note we prove this statement, up to lower order terms; we prove that a permutation with $n/2$ cycles is indecomposable with probability about $.5117\dots$; and for any $\mu \in (0, 1]$, we calculate the asymptotic probability that a permutation over $\{1, \dots, n\}$ with approximatively μn cycles is decomposable.

Let $\mathbb{S}_{n,m}$ denote the set of permutations of $\mathbb{S}_n$ with m cycles, $s_{n,m}$, the unsigned Stirling number of the first kind, denote the cardinality of $\mathbb{S}_{n,m}$, and $\mu = m/n$. Let $\alpha = a_1a_2\dots a_n$ denote a permutation of $\{1, 2, \dots n\}$.

2 Main result and proof overview

Theorem 1 *Let μ be a rational number less than 1. If m and n tend to infinity while keeping their ratio fixed at $m/n = \mu$, then the probability $p_{n,m}$ that a permutation of $\mathbb{S}_{n,m}$ is indecomposable tends to $p(\mu)$,*

$$p(\mu) = \frac{(e^u - 1)^2}{e^{2u}}, \tag{1}$$

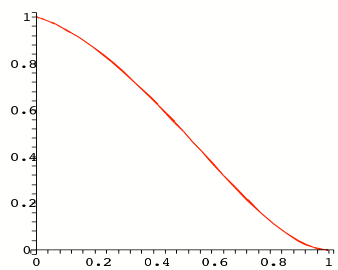

Fig. 1: Asymptotic probability $p_\infty(\mu)$ that a permutation of $\mathbb{S}_n$ with μn cycles is indecomposable, as a function of μ.

where $u > 0$ is defined implicitly by the equation

$$\mu = \frac{u}{e^u - 1}. \tag{2}$$

Moreover, $|p_{n,m} - p(\mu)| = O(\log(n-m)/(n-m))$.

The asymptotic probability of indecomposability of a permutation as a function of μ is depicted in Figure 1.

The value for $\mu = 1/2$ computed with Maple is 0.511699676. The proof of Theorem 1 follows directly from the following three lemmas. The first lemma states some simple facts and has a short proof.

Lemma 1 *If the following condition holds, then α is decomposable:*

$$(a_1 = 1) \quad \text{or} \quad (a_n = n) \tag{3}$$

If the following condition holds, then α is indecomposable:

$$(\exists i, i \le a_1 \text{ and } a_i > a_n) \tag{4}$$

Proof: If condition (3) holds then either a_1 is a permutation of $\mathbb{S}_1$ or $a_1 \dots a_{n-1}$ is a permutation of $\mathbb{S}_{n-1}$.

If α is decomposable then there exist $p < n$ such that $a_1 a_2 \dots a_p$ is a permutation of $\mathbb{S}_p$, this implies $a_n > a_i$ for all $1 \le i \le p$. Moreover all i such that $p < i \le n$ either $i \le p$ satisfies $a_i > a_1$ contradicting (4). Note that there is a simple way to represent indecomposability as a simple drawing: put n points on a horizontal segment numbered 1 to n from left to right draw a half circle from i to a_i when $a_i \ne i$ then the permutation is decomposable if and only if there is no vertical line intersecting the segment but not any of the half circles. As an example the proof of the above Lemma is illustrated on Figure 2.

□

The second Lemma will be proved in the next section using an evaluation of the asymptotics of Stirling numbers due to Moser and Wyman [MW58]

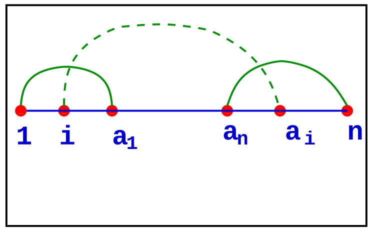

Fig. 2: Illustration of Condition 4 guaranteeing indecomposability.

Lemma 2 *Let m, n, μ, u be defined as in Theorem 1. Then the probability that a permutation of $\mathbb{S}_{n,m}$ satisfies condition (3) tends to*

$$\frac{2e^u - 1}{e^{2u}}.$$

The third lemma, is the main technical point in our paper and will be proved in a following section:

Lemma 3 *The probability that a permutation of $\mathbb{S}_{n,m}$ satisfies neither condition (3) nor condition (4) is* $O(\frac{\log(n-m)}{n-m})$.

3 Proofs

3.1 Proof of Lemma 2.

We use the inclusion-exclusion formula. The number of permutations of $\mathbb{S}_{n,m}$ such that $a_1 = 1$ is equal to $s_{n-1,m-1}$, the number of those such that $a_n = n$ is also equal to $s_{n-1,m-1}$, and the number of those such that $a_1 = 1$ and $a_n = n$ is equal to $s_{n-2,m-2}$; hence the number satisfying condition (3) is equal to

$$t_{n,m} = 2s_{n-1,m-1} - s_{n-2,m-2}.$$

Moser and Wyman ([MW58] Equation (5.7)) give the following formula for Stirling numbers of the first kind in the asymptotic regime where n and m tend to infinity such that $m/n = \mu$ is fixed:

$$s_{n,m} = b\frac{n!}{a^n\sqrt{n}}\frac{u^m}{m!}(1 + O_\mu(1/m)), \tag{5}$$

where u satisfies Equation (2) with $\mu = m/n$, $a = 1 - e^{-u}$, $b = \sqrt{\frac{u}{2\pi(ue^u - e^u + 1)}}$ (note that since $\mu < 1$, we have $u > 0$ and b is well defined,) and the constants in the $O_\mu(1/m)$ are continuous functions of μ.

Using continuity, it is easy to prove that $\frac{t_{n,m}}{s_{n,m}}$ tends to $\frac{2e^u-1}{e^{2u}}$. The details will be given in the extended version of the paper.

3.2 Proof of Lemma 3.

Let $S_{n,m}$ denote the set of permutations of $\mathbb{S}_{n,m}$ such that neither condition (3) nor condition (4) hold. We will partition the permutations of $\mathbb{S}_{n,m}$ according to their *shape*, defined below, and prove by probabilistic arguments that within each class of permutations having the same shape, the fraction of those which are in $E_{n,m}$ is negligeable.

To each permutation α in $E_{n,m}$, we associate a shape $(n_1, \ldots, n_m; p, q, b, r)$ defined as follows. $n_1 \geq n_2 \geq \cdots \geq n_m$ are the lengths of the m cycles of α; p and q are the lengths of the cycles containing 1 and n; when $p = q$, b is a boolean indicating whether 1 and n are in the same cycle; and when b is true, $r > 1$ is the smallest integer such that $\alpha^r(1) = n$. The shape of a permutation in $\mathbb{S}_{n,m}$ may be represented by a directed graph with n vertices of indegree and outdegree 1, consisting of the union of m (directed) cycles of lengths $n_1, n_2, \ldots, n_m$, and of two distinguished vertices, belonging to cycles of length not less than 2 and called the "initial"and the "last" vertices. We identify a shape and the associated graph.

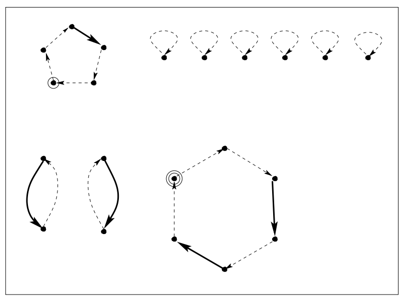

Fig. 3: The shape $(6, 5, 2, 2, 1, 1, 1, 1, 1, 1; 5, 6)$, the initial vertex is indicated by a circle and the last one by a double circle; the marked edges are in bold.

Given any shape σ, the following process defines a permutation drawn uniformly at random among the permutations of $\mathbb{S}_{n,m}$ with shape σ:

- To each undistinguished vertex, independently assign a real number drawn uniformly at random from the interval $[0, 1]$; assign 0 to the initial vertex and 1 to the last vertex.
- Give integer labels $1, 2, \ldots n$ to the n vertices of the diagram in such a way that the labels are in the same order as the reals assigned to them. This defines the permutation $a_1, a_2, \ldots, a_n$ such that the edge with head labeled i has tail labeled a_i.

Lemma 4 *In the graph representing a shape σ there exist $(n - m)/2 - 2$ edges, called* marked *edges, such that no head of a marked edge is the tail of another marked edge and such that the initial and the last vertex are neither a head nor a tail of a marked edge.*

Proof: There are m cycles, of which m_1 have length 1. In each of the cycles of length $n_i \geq 2$, we can mark at least $(n_i - 1)/2$ disjoint edges, for a total of $[(n - m_1) - (m - m_1)]/2$ marked edges. Discounting the marked edges that touch the initial or the last vertex yields the result. $\square$

Lemma 3 follows by summing Equation (6) below over all shapes.

Lemma 5 *Given a shape σ, let $s^{\sigma}_{n,m}$ and $e^{\sigma}_{n,m}$ be the number of permutations with shape σ in $\mathbb{S}_{n,m}$ and in $E_{n,m}$. Then*

$$e^{\sigma}_{n,m} \leq s^{\sigma}_{n,m} \frac{4\log((n-m-4)/2)}{n-m-4}(1+o(1)). \tag{6}$$

Proof: (of Lemma 5.) Let $\alpha = a_1, a_2, \ldots, a_n$ be a permutation of shape $\sigma = (n_1, \ldots, n_m; p, q, b, r)$ obtained by the process. We may suppose $p, q > 1$ since this means $a_1 \neq 1, a_n \neq n$. Then α in $E_{n,m}$, if for all i the following condition holds

$$\neg(i \leq a_1 \text{ and } a_i > a_n)$$

The probability of this event is less than if the condition holds only for the i corresponding to the heads of marked edges. But since the marked edges have no common end points, the conditions on each marked edges are independent. Hence an upper bound for the probability of decomposability is the ℓ-th power of the satisfaction of one of the conditions.

Let x and y be the real numbers assigned to the tails of the edges which heads are the first and the last vertex respectively. For every marked edge, the values x_i and y_i associated to its head and tail respectively are such that we do not have ($x_i < x$ and $y_j > y$).

Fix x, y; for each marked edge, the probability of the event ($x_i < x$ and $y_i > y$) is $x(1-y)$. By definition of the marked edges, the values x_i, y_i are independent, and so the probability that no (x_i, y_i) among the $(n-m-4)/2$ marked edges has ($x_i < x$ and $y_i > y$) is : $(1 - x(1-y))^{(n-m-4)/2}$. Then, denoting $\ell = (n-m-4)/2$, the proportion $\varepsilon_{n,m}$ of permutations with shape σ in $E_{n,m}$ is bounded by:

$$\varepsilon_{n,m} \leq \int_0^1 \int_x^1 (1 - x(1-y))^{\ell} dydx.$$

Using the well known inequality $1 - z \leq e^{-\frac{z}{2}}$ for $z \in [0,1]$ (with $z = x(1-y)$) we obtain:

$$\varepsilon_{n,m} \leq \int_0^1 \int_x^1 e^{-\frac{x(1-y)\ell}{2}} dydx \;=\; \int_0^1 \int_0^{1-x} e^{-\frac{xy\ell}{2}} dydx \;=\; \int_0^1 \frac{2}{x\ell}(1 - e^{-\frac{x(1-x)}{2}})dx.$$

We decompose $[0,1]$ in two intervals $[0, \frac{1}{\ell}]$ and $[\frac{1}{\ell}, 1]$. When $x \geq 1/\ell$ the function inside the integral can be bounded by $2/(x\ell)$. When $x < 1/\ell$ we use again $1 - e^{-\frac{z}{2}} \leq z$ for $z \in [0,1]$ (with $z = x(1-x)$) and write:

$$\varepsilon_{n,m} \leq \int_0^{\frac{1}{\ell}} \frac{2(1-x)}{\ell} dx + \int_{\frac{1}{\ell}}^1 \frac{2}{x\ell} dx \leq \frac{2\log\ell}{\ell} + \frac{2}{\ell^2} = \frac{2\log\ell}{\ell}(1+o(1)).$$

Substituting $\ell = (n-m-4)/2$, the lemma follows. □

4 Remarks

4.1 Numerical results

It is well-known that $(s_{n,m})$ satisfies $s_{n,p} = 0$ for $p = 0$ or $p > n$, $s_{1,1} = 1$, and:

$$s_{n,p} = s_{n-1,p-1} + (n-1)s_{n-1,p} \tag{7}$$

The numbers $c_{n,m}$ of indecomposable permutations of $\mathbb{S}_{n,m}$, can be computed by a formula similar to that giving the number of those in $\mathbb{S}_n$, (see for instance [Cor09], Proposition 2)

$$c_{n,k} = s_{n,k} - \sum_{p=1}^{n-1} \sum_{i=1}^{min(k,p)} c_{p,i} s_{n-p,k-i} \tag{8}$$

Thus the exact value of $\frac{c_{n,k}}{s_{n,k}}$ can be computed exactly by using the above formulas inductively for small n.

We have proved that the error term $|p_{n,m} - p(\mu)|$ is bounded by $O(\log(n-m)/(n-m))$. The error is actually very small. For instance we find for $n = 20$ and $n = 100$:

m/n	0.05	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.95	1
$p_{20,m}$	1	0.968	0.883	0.774	0.644	0.5	0.35	0.207	0.090	0.02	0.005	0
$p_{100,m}$	0.981	0.95	0.868	0.764	0.643	0.51	0.371	0.236	0.116	0.03	0.006	0
$p_\infty(m/n)$	0.978	0.946	0.865	0.762	0.642	0.511	0.374	0.241	0.122	0.035	0.009	0

4.2 Left-to-right maxima

A left-to-right maximum of a permutation $\alpha = a_1 \dots a_n$ is an a_j such that for any $i < j$ one has $a_j > a_i$. A classical result states that the number of permutations of $\mathbb{S}_n$ with m cycles is equal to the number of those with m left-to-right maxima. Moreover the so called *First Fundamental Transform* (see [Lot83] chap. 10) is a bijection between permutations of $\mathbb{S}_n$ which maps a permutation with m cycles to a permutation with m left-to right maxima. It is not difficult to prove (see [Cor09] Proposition 1) that the permutation is indecomposable if and only if its image under this transformation is. Hence the probabilities obtained above are also those for a permutation with m left-to-right maxima to be indecomposable.

4.3 Comments

- The majority ($51.1\dots$ percent) of permutations of $\mathbb{S}_{2m}$ with m cycles are indecomposable.

- Since there is a bijection between indecomposable permutations and hypermaps (see [dMR04]) our result shows that the probability for an ordered pair of permutations σ, α on $\mathbb{S}_n$ to generate a transitive group when σ is supposed to have m cycles is about the same as the probability for a permutation of $\mathbb{S}_{n+1,m}$ to be indecomposable. Hence this probability is about 0.511 when $n = 2m$.

- It would be interesting to know the structure of the group generated by two permutations when their number of cycles is given. When these numbers are not fixed then Dixon (see [Dix05]) proved that the probability that they generate the symmetric or alternating group is near to 1, his proof uses the fact that they generate a transitive group with probability 1. But as we saw transitivity cannot be assumed when the number of cycles is given and large.

5 Fixed point free involutions.

We now consider involutions with no fixed points (which we will call fpf-involutions for short in the sequel), that is, permutations of $\mathbb{S}_{2m}$ with m cycles, all of length 2, hence belonging the subset of $\mathbb{S}_{2m}$ for which the probability of being indecomposable is close to 0.51. However assuming that the cycles are all of length 2 implies that the probability increases to $1 - \frac{1}{m}$. This can easely be proved using the recursion formula giving the number c_m of indecomposable fpf-involutions namely :

$$c_m = (2m-1)!! - \sum_{k=1}^{m-1} c_k (2m-2k-1)!!$$

where $(2k-1)!! = \prod_{i=1}^{k}(2i-1)$ is the total number of fpf-involutions of $\mathbb{S}_{2k}$.

Ossona de Mendez and Rosenstiehl in [dMR05] gave a bijection between rooted maps on orientable surfaces with $m-1$ edges, and indecomposable fpf-involutions of $\mathbb{S}_{2m}$; in this bijection the number of vertices of the map is equal to the number of left-to-right maxima of the corresponding involution. This allows a new proof of the results in [AB00], see also: [BJ02], [Dra09].

Hence it is interesting to find the number $a_{m,k}$ of fpf-involutions of length m having k left-to-right maxima. No simple formula for these numbers are known unlike for Stirling numbers, which the statistics of the same parameter for general permutations.

Let $c_{m,k}$ be the number of indecomposable fpf-involutions of $\mathbb{S}_{2m}$ with k lef-to-right maxima; it is also the number of rooted maps with $m-1$ edges and k vertices.

In order to calculate the numbers $a_{n,k}$ and $c_{n,k}$ we use a bijection between these involutions and labeled Dyck words which will be recalled in Section 6. The values obtained allowed us to conjecture that the probability for an fpf-involution to be indecomposable increases smoothly when the number of left-to-right maxima decreases.

We are not able to prove this conjecture, but we obtain as a partial result a lower bound for the the proportion of indecomposable fpf-involutions with a given number of left-to-right maxima:

Theorem 2 *The numbers of decomposable fpf-involutions $d_{m,k} = a_{m,k} - c_{m,k}$ of $\mathbb{S}_{2m}$, having k left-to-right maxima satisfy:*

$$d_{m,k} \leq \frac{4k}{m} c_{m,k}$$

Hence the probability that a random fpf-involution of $\mathbb{S}_{2m}$ with λm left-to-right maxima is decomposable is at most $4\lambda/(1+4\lambda)$.

In Section 6 we recall the bijection between fixed point free involutions and labeled Dyck words. In Section 7 we give a sketch of the proof of Theorem 2.

6 Labeled Dyck words.

Dyck words.

We consider words over the two letters alphabet $\{a, b\}$. We denote the length of a word w by $|w|$, and the number of occurrences of the letter x by $|w|_x$. A Dyck word u is a word such that $|u|_a = |u|_b$ and $|u'|_a \geq |u'|_b$ for any of its prefixes u' (i. e. $u = u'u''$). The *height* of the occurrence of a letter x in

$w = w'xw"$ is defined as $|w'x|_a - |w'x|_b$. A Dyck word is *decomposable* if there exist two non-empty Dyck words u', $u"$ such that $u = u'u"$. It is indecomposable otherwise.

Labeling

We consider the infinite alphabet $\{a, b_0, b_1, b_2, \dots, b_i, \dots\}$, and use the notation $|w|_b = \sum_{i \geq 0} |w|_{b_i}$ allowing to define the heights of occurence of letters as above. A *labeled Dyck word* is a word f on this alphabet such that

1. Replacing every b_i for $i \geq 0$ by b in f gives a Dyck word, and
2. Every occurrence of b_i in f has height at least i.

A labeled Dyck word is decomposable if replacing every b_i for $i \geq 0$ by b gives a decomposable Dyck word. Let L_m denote the set of labeled Dyck words of length $2m$, and $L_{m,k}$ denote the subset of those having k occurrences of b_0. Let $a_{m,k}$ be the number of words of $L_{m,k}$ and $c_{m,k}$ the number of indecomposable ones. We define the polynomials A_m and C_m by :

$$A_m(x) = \sum_{k=0}^{m} a_{m,k} x^k \qquad C_m(x) = \sum_{k=0}^{m} c_{m,k} x^k \tag{9}$$

Then we have, where A_0 is set equal to 1 :

Proposition 1 *For any $m \geq 1$ the polynomials A_m and C_m satisfy the following recursion equations:*

$$C_m(x) \quad = \quad xA_{m-1}(x+1) \qquad A_m(x) \quad = \quad \sum_{k=1}^{m} C_k(x) A_{m-k}(x) \tag{10}$$

Proof: For the first equations note that an indecomposable Labeled Dyck word w is equal to avb_0 where v is obtained from a a labbeled Dyck word u by choosing a subset of occurrences of b_i and replacing them by b_{i+1}. The seoncd one follows from the fact that any labeled Dyck word is the concatenation of an indecomposable one and another labeled Dyck word (possibly empty). □

Bijection.

The following algorithm describes a well-known bijection between fpf-involutions of $\mathbb{S}_m$ and labeled Dyck words of length $2m$, such that the labeled Dyck word is indecomposable if and only if the involution is indecomposable. Less known is the fact that the number of left-to-right maxima of α is equal to the number of occurrences of b_0 in the corresponding word. It takes as input a fpf-involution $\alpha \in \mathbb{S}_{2m}$ and outputs the labeled Dyck word $f = f_1 f_2 \cdots f_{2m}$. It uses an ordered list Q.

for i = 1 to 2m do
  if $(\alpha(i) > i)$
    then { add i at the end of Q, and set $f_i = a$; }
    else { let j be the position of $\alpha(i)$ in Q; set $f_i = b_{j-1}$, and remove $\alpha(i)$ from Q; }

For instance, for the involution $\alpha = (1,4)(2,7)(3,10)(5,9)(6,8)(11,14)(12,13)$, we get

$$f = a\ a\ a\ b_0\ a\ a\ b_0\ b_2\ b_1\ b_0\ a\ a\ b_1\ b_0$$

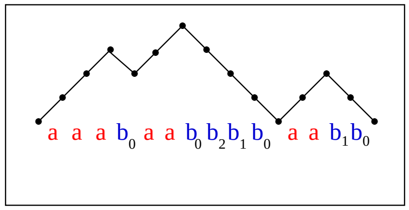

Fig. 4: Labeled Dyck word corresponding to the involution $(1,4)(2,7)(3,10)(5,9)(6,8)(11,14)(12,13)$

6.0.1 Enumeration

A consequence of this bijection is that the number of fpf-involutions of $\mathbb{S}_{2m}$ with k left-to-right maxima is $a_{m,k}$ and that of indecomposable ones is $c_{m,k}$. These numbers can be computed thanks to Equation (10). Moreover we have $a_{m,1} = c_{m,1} = (2m-3)!!$ since any fpf-involution with one left-to-right maximum is indecomposable and is equal to $(1,2m)\beta$, where β is any fpf-involutions over $2,\ldots 2m-1$ a. We also have: $a_{m,m} = C_m$ and $c_{m,m} = C_{m-1}$ where C_k denotes the k-th Catalan number since the fpf-involutions with of $\mathbb{S}_{2m}$ with m left-to-right maxima correspond to labbelled Dyck words with all the b_i equal to b_0 and the indecomposable ones to indecomposable Dyck words with the same property.

This gives some values for this numbers and allows to compare our result in Theorem 2 with values of $\frac{c_{m,k}}{a_{m,k}}$ for $m = 100$ showing an important gap except when k is close to 1 or close to m.

k/m	0.001	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1
$m/(m+4k)$	0.96	0.71	0.55	0.45	0.38	0.33	0.29	0.26	0.24	0.22	0.20
$c_{m,k}/a_{m,k}$	1	0.98	0.95	0.90	0.86	0.80	0.74	0.67	0.58	0.46	0.25

7 Sketch of proof of Theorem 2

In order to explain this proof we consider a much simple result for Dyck words which we will try to genralise for labeled Dyck words.

7.1 A simpler result

Proposition 2 *The number of decomposable Dyck words of length $2m$ is less than 4 times the number of indecomposable ones.*

Proof: There is a very simple proof since we know that the number of decomposable Dyck words of length $2m$ is $C_m - C_{m-1}$ and that of indecomposable ones is C_{m-1}. □

Note that this simple proof shows also that we could have a better result replacing 4 by 3 in the Proposition. But it is impossible to generalize this simple proof in ordrer to obtain a result for labeled Dyck words with k occurrences of b_0, since we do not know a nice formula for the number of such words of length $2m$. Hence we need another proof which do not uses any formula and which has a bijective flavour. For that we define *admissible factorisations* of decomposable Dyck words and prooced in three steps.

An *admissible factorisation* of a decomposable Dyck word w consists of a pair of words (u, v) such that $uv = w$, u ends with an a and contains a prefix which is an indecomposable (hence non empty) Dyck word. So that we can write $u = u_1 u' a$ where u_1 is an indecomposable Dyck word.

Let d_m and c_m be the number of decomposable and indecomposable Dyck words of length $2m$. Denote F_m the set of all admissible factorisations of decomposable ones and f_m the number of elements of F_m

Step 1: $md_m \leq 2f_m$

Proof: A decomposable Dyck word w writes $w = u_1 u_2 \ldots u_k$ where the u_i are indecomposable Dycl words. Let $2m_i$ denote the length of u_i for $i = 1, k$, the number of admissible factorisations of w is $m - m_1$. If this number is less than $m/2$ then the word $w = u_k u_1 u_2 \ldots u_{k-1}$ has $m - m_k$ admissible factorisations and the sum of these two numbers is greater or equal to m. Hence proving the result. □

Step 2: factorisations of indecomposable Dyck words

Consider the set F'_m of pair of words (u, v) such that uv is an indecomposable Dyck word of length $2m$ and u ends with an occurrence of a. Then the number f'_m of elements of F'_m is equal to mc_m.

Step 3: $f_m < 2f'_m$

We build a mapping Φ from the set F_m into F'_m such that each element of F'_m is the image of at most 2 elements of F_m. Let $(u_1 w' a, v) \in F_m$ be such that u_1 is an indecomposable Dyck word and $u_1 w' a v$ is a decomposable one.

- If $u_1 = ab$ then we set $\Phi((u_1 w' a, v)) = (aw'a, vb)$
- If $u_1 \neq ab$ then $u_1 = aaw_1 b w_2 b$ where w_1, w_2 are (not necessarily indecomposable) Dyck words. In that case we set $\Phi((u_1 w' a, v)) = (aw_1 a w' a, v b w_2 b)$.

It is clear that for any factorisation (u', v') in F'_m there are only two candidates (u, v) to be such that $\Phi(u, v) = (u', v')$.

Putting all together we obtain :

$$md_m \;\leq\; 2f_m \;<\; 4f'_m \;=\; 4mc_m$$

Ingredients for the generalisation

In order to prove Theorem 2 we consider the set $L_{2m,k}$ of Dyck words of length $2m$ on the alphabet with 3 letters: $\{a, b_0, b\}$, having k occurrences of b_0 and such that no occurrence of b has height less than 1. For each word $w \in L_{2m,k}$ denote $\lambda(w)$ the number of labeled Dyck words that are obtained from w by replacing the occurrences of b by a b_i. Then the numbers $c_{m,k}$ and $d_{m,k}$ in Theorem 2 are the sums of the $\lambda(w)$ for indecomposable and decomposable words of $L_{2m,k}$ respectively. Then we modify the proof in three steps above to make it work for words in $L_{2m,k}$. The main difficulty is in Step 3, since the mapping Φ has to be modified in such a way that $\Phi((u, v)) = (u', v')$ implies $\lambda(uv) \leq \frac{m}{k}\lambda(u'v')$.

The detailed construction of such a mapping Φ will be given in the extended version of this paper.

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Matrix Ansatz, lattice paths and rook placements

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Abstract. We give two combinatorial interpretations of the Matrix Ansatz of the PASEP in terms of lattice paths and rook placements. This gives two (mostly) combinatorial proofs of a new enumeration formula for the partition function of the PASEP. Besides other interpretations, this formula gives the generating function for permutations of a given size with respect to the number of ascents and occurrences of the pattern 13-2, the generating function according to weak exceedances and crossings, and the n^{th} moment of certain q-Laguerre polynomials.

Résumé. Nous donnons deux interprétations combinatoires du Matrix Ansatz du PASEP en termes de chemins et de placements de tours. Cela donne deux preuves (presque) combinatoires d'une nouvelle formule pour la fonction de partition du PASEP. Cette formule donne aussi par exemple la fonction génératrice des permutations de taille donnée par rapport au nombre de montées et d'occurrences du motif 13-2, la fonction génératrice par rapport au nombre d'éxcédences faibles et de croisements, et le $n^{\text{ième}}$ moment de certains polynômes de q-Laguerre.

Keywords: Enumeration, Permutation tableaux, Rook placements, Lattice paths

1 Introduction

In recent work of Postnikov [17], permutations were given a new description as pattern-avoiding fillings of Young diagrams. More precisely, Postnikov made a correspondence between positive Grassmann cells, pattern-avoiding fillings called Ⅎ-diagrams, and decorated permutations (which are permutations where the fixed points are bi-coloured). In particular, the usual permutations are in bijection with permutation tableaux, a subclass of Ⅎ-diagrams. Permutation tableaux have subsequently been studied by Steingrìmsson, Williams, Burstein, Corteel, Nadeau [4, 7, 8, 20], and proved to be very useful for working on permutations.

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Corteel and Williams established a link between permutation tableaux and the stationary distribution of a classical process studied in statistical physics, the Partially Asymmetric Exclusion Process (PASEP). This process is described in [8, 9]. Briefly, the stationary probability of a given state in the process is proportional to the sum of weights of permutation tableaux of a given shape. The factor behind this proportionality is the partition function, which is the sum of weights of permutation tableaux of a given half-perimeter.

An alternative way of finding the stationary distribution of the PASEP is given by the Matrix Ansatz [9]. Suppose that we have operators D and E, a row vector $\langle W|$ and a column vector $|V\rangle$ such that:

$$DE - qED = D + E, \qquad \langle W|E = \langle W|, \qquad D|V\rangle = |V\rangle, \qquad \text{and} \quad \langle W||V\rangle = 1. \tag{1}$$

Then, coding any state of the process by a word w of length n in D and E, the probability of the state w is given by $\langle W|w|V\rangle$ normalised by the partition function $\langle W|(D+E)^n|V\rangle$.

We briefly describe how the Matrix Ansatz is related to permutation tableaux [8]. First, notice that there are unique polynomials $n_{i,j} \in \mathbb{Z}[q]$ such that

$$(D+E)^n = \sum_{i,j\geq 0} n_{i,j} E^i D^j$$

This sum is called the normal form of $(D+E)^n$. It is useful since, for example, the sum of coefficients $n_{i,j}$ gives an evaluation of $\langle W|(D+E)^n|V\rangle$. Each coefficient $n_{i,j}$ is a generating function for permutation tableaux satisfying certain conditions, or equivalently, *alternative tableaux* as defined by Viennot [27].

We give here two combinatorial interpretations of the Matrix Ansatz in terms in lattice paths and rook placements, and get two semi-combinatorial proofs of the following theorem:

Theorem 1 *For any $n > 0$, we have:*

$$\langle W|(yD+E)^{n-1}|V\rangle = \tfrac{1}{y(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\sum_{j=0}^{n-k} y^j \left(\tbinom{n}{j}\tbinom{n}{j+k} - \tbinom{n}{j-1}\tbinom{n}{j+k+1} \right) \right) \left(\sum_{i=0}^{k} y^i q^{i(k+1-i)} \right).$$

The combinatorial interpretation of this polynomial, in terms of permutations, is given in Proposition 1. For $y = 1$ this specialises to:

Corollary 1 *For any $n > 0$, we have:*

$$\langle W|(D+E)^{n-1}|V\rangle = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\tbinom{2n}{n-k} - \tbinom{2n}{n-k-2} \right) \left(\sum_{i=0}^{k} q^{i(k+1-i)} \right).$$

Besides the references mentioned earlier, we have to point out an article of Williams [29], where we find the following formula for the coefficient of y^{m-1} in $\langle W|(yD+E)^n|V\rangle$:

$$E_{m,n}(q) = \sum_{i=0}^{m-1} (-1)^i [m-i]_q^n q^{mi-m^2} \left(\tbinom{n}{i} q^{m-i} + \tbinom{n}{i-1} \right). \tag{2}$$

It was obtained by enumerating ⅃-diagrams of a given shape and then computing the sum of all possible shapes. Until now it was the only known polynomial formula for the distribution of a permutation pattern of length greater than two (See Proposition 1). Although the article [29] focuses on ꓕ-diagrams, Williams and her coauthors sketched in Section 4 of [16] how this could have been done directly on permutation tableaux. Recently, Williams's formula has been obtained also by Kasraoui, Stanton and Zeng in their work on orthogonal polynomials [12]. We will show in the last section how our formula can be applied to prove and extend a conjecture presented in [29].

The polynomial $y\langle W|(yD+E)^{n-1}|V\rangle$ was already heavily studied.

Proposition 1 *For any $n \geq 1$ the following polynomials are equal:*

- $y\langle W|(yD+E)^{n-1}|V\rangle$,
- *the generating function for permutation tableaux of size n, the number of lines counted by y and the number of superfluous 1's counted by q [8, 28],*
- *the generating function for permutations of size n, the number of ascents counted by y and the number of 13-2 patterns counted by q [7, 20],*
- *the generating function for permutations of size n, the number of weak exceedances counted by y and the number of crossings counted by q [6, 20],*
- *the generating function of PDSAWs (partially directed self-avoiding walks) in the asymmetric wedge of length n where the number of descents is counted by y and the number of north steps is counted by q [23],*
- *the n^{th} moment of the Al-Salam-Chihara q-Laguerre polynomials [12, 23].*

Remark. We can view the formula in Corollary 1 as an analog of the Touchard-Riordan formula [24] for the number of matchings of $2n$ according to the number of crossings:

$$\sum_{M \text{ matching of } 2n} q^{\mathrm{cr}(M)} = \frac{1}{(1-q)^n} \sum_{k=0}^{n} (-1)^k \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) q^{\frac{k(k+1)}{2}}.$$

We remark that this formula also gives the $2n^{\text{th}}$ moment of the q-Hermite polynomials.

In [22], Penaud gave a combinatorial proof of this formula. By generalising Penaud's method we conjectured Theorem 1 and were hoping for a completely combinatorial proof thereof. However, at the time of writing the last step of this combinatorial proof is still missing.

This article is organised as follows: we first show how the Matrix Ansatz is naturally related to lattice paths. Then we give two proofs of our main Theorem, one based on lattice paths and the other one based on rook placements. We end with a discussion and some applications.

2 A first proof using lattice paths and functional equations

2.1 The Matrix Ansatz and lattice paths

We follow the ideas developed in [2, 3]. Looking for a solution of the system defined in Equation (1) we find:

Proposition 2 *Let $D = (D_{i,j})_{i,j\geq 0}$ and $E = (E_{i,j})_{i,j\geq 0}$ such that*

$$\begin{aligned} D_{i,j} &= \begin{cases} 1+\ldots+q^i & \textit{if } i \textit{ equals } j-1 \textit{ or } j, \\ 0 & \textit{otherwise,} \end{cases} \\ E_{i,j} &= \begin{cases} 1+\ldots+q^i & \textit{if } i \textit{ equals } j \textit{ or } j+1, \\ 0 & \text{otherwise,} \end{cases} \\ \langle W| &= (1,0,0,\ldots), \textit{ and} \\ |V\rangle &= (1,0,0,\ldots)^T . \end{aligned}$$

Then these matrices and vectors satisfy the Ansatz of Equation (1)*.*

We can interpret $y\langle W|(yD+E)^{n-1}|V\rangle$ as the generating polynomial of paths of length $n-1$. The weight of a path is the product of the weight of its steps and the weight of the starting and ending points. If a path starts (resp. ends) at $(0,i)$ (resp. $(n-1,i)$) the weight of the starting (resp. ending) point is W_i (resp. V_i). The weight of a step going from (x,i) to $(x+1,j)$ is $D_{i,j}+E_{i,j}$. We call i the starting height of the step. See [2, 3] for details.

Proposition 2 implies that the paths we are dealing with here are bi-coloured Motzkin paths, i.e., paths that start and end at height zero and consist of north-east, south-east and two types of east steps. Using a classical bijection we can transform these paths of length $n-1$ into Motzkin paths of length n where east steps of type 2 can not appear at height zero.

Proposition 3 $y\langle W|(yD+E)^{n-1}|V\rangle$ *is the generating polynomial of weighted bi-coloured Motzkin paths of length n such that the weight of steps starting at height i is*

- $y+yq+\ldots+yq^i = y\frac{1-q^{i+1}}{1-q}$ *for north-east steps and east steps of type 1, and*
- $1+q+\ldots+q^{i-1} = \frac{1-q^i}{1-q}$ *for south-east steps and east steps of type 2.*

This can also be done combining results in [6, 8, 20].

2.2 The proof

The method used in this subsection is inspired by an article of Penaud [22]. We extract a factor of $(1-q)^n$ from the generating polynomial of the weighted bi-coloured Motzkin paths from Proposition 3 and obtain that

$$y\langle W|(yD+E)^{n-1}|V\rangle = \frac{1}{(1-q)^n}\sum_{p\in P(n)} w(p),$$

where $P(n)$ is the set of labelled bi-coloured Motzkin paths of length n such that the weight of steps starting at height i is either

- y or $-yq^{i+1}$ for north-east steps or east steps of type 1,
- 1 or $-q^i$ for south-east steps or east steps of type 2,

and $w(p)$ is the total weight of the path.

Let $M(n)$ be the subset of the paths in $P(n)$ such that the weight of any east step and the weight of any peak (a north-east step followed by a south-east step) is neither 1 nor y. Let $\mathcal{M}_{n,k,j}$ be the number of left factors of bi-coloured Motzkin paths of length n, final height k, and with j south-east and east steps of type 1.

Lemma 1 *There is a bijection between paths in $P(n)$ and pairs of paths such that for some $k \in \{0, \ldots, n\}$*

- *the first path is a left factor of a bi-coloured Motzkin path of length n and final height k,*
- *the second path is in $M(k)$.*

In particular, we have

$$\sum_{p \in P(n)} w(p) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \mathcal{M}_{n,k,j} y^j \sum_{p \in M(k)} w(p).$$

Proof: Let p be a path in $P(n)$. We decompose p into a sequence $m_1 q_1 m_2 q_2 \ldots m_k q_k m_{k+1}$ such that

- the m_i are maximal (but possibly empty) sub-paths of p with all steps having weight 1 or y, and returning to their starting height,
- the q_i are single steps.

It follows that $q_1 q_2 \ldots q_k$ is a path in $M(k)$. Replacing in the sequence $m_1 q_1 m_2 q_2 \ldots m_k q_k m_{k+1}$ each step q_i by a north-east step, and taking into account the number of south-east steps and east steps of type 1, we obtain a path in $\mathcal{M}_{n,k,j}$ of weight y^j. $\square$

It remains to compute $\mathcal{M}_{n,k,j}$ and $M_k = \sum_{p \in M(k)} w(p)$.

Proposition 4 *The number $\mathcal{M}_{n,k,j}$ of left factors of bi-coloured Motzkin paths of length n, final height k, and with j south-east steps and east steps of type 1, is $\binom{n}{j}\binom{n}{j+k} - \binom{n}{j-1}\binom{n}{j+k+1}$.*

Proof: We note that the formula can be seen as a 2×2 determinant. By the Lindström-Gessel-Viennot lemma, this equals the number of pairs of non-intersecting lattice paths taking north and east steps from $(1, 0)$ to $(n - j, j)$ and $(0, 1)$ to $(n - j - k, j + k)$ respectively.

We transform such a pair of paths step by step into a single Motzkin path according to the following translation table:

i^{th} step of	lower path	upper path	Motzkin path
	north	north	east type 1
	east	east	east type 2
	north	east	north-east
	east	north	south-east.

It is easy to see that the condition that the two lattice paths do not intersect corresponds to the condition that the Motzkin path does not run below the x-axis. Furthermore, we see that the number of east and south-east steps equals j, the number of north steps of the lower path. $\square$

Proposition 5 *The generating polynomial M_k equals $\sum_{i=0}^{k} y^i q^{i(k+1-i)}$.*

Proof: We add an extra parameter on the paths in $M(n)$, that marks the number of steps that have a weight different from 1 and y. More precisely, the weight of steps starting at height i is

- y or $-yzq^{i+1}$ for north-east steps or east steps of type 1, and
- 1 or $-zq^i$ for south-east steps or east steps of type 2.

Let $M(z) = \sum_{n\geq 0} t^n \sum_{p\in M(n)} w(p)$. We can obtain a functional equation for $M(z)$ by considering the following decomposition: A path is either (a) empty, (b) a north-east step of weight 1, followed by a path, followed by a south east step of weight $-qyz$, followed by another path, (c) a north-east step of weight $-qz$, followed by a path, followed by a south east step of weight $-qyz$, followed by another path, (d) a north-east step of weight $-qz$, followed by a path, followed by a south east step of weight y, followed by another path, (e) a north-east step of weight 1, followed by a *non-empty* path, followed by a south east step of weight y, followed by another path, (f) an east step of type 1 followed by another path, or (g) an east step of type 2 followed by a path. The corresponding weight is (a) 1, (b) $-M(qz)qyzM(z)t^2$, (c) $qzM(qz)qyzM(z)t^2$, (d) $-qzM(qz)yM(z)t^2$, (e) $(M(qz)-1)\,yM(z)t^2$, (f) $-qyzM(z)t$, or (g) $-zM(z)t$, respectively. Thus, we have:

$$M(z) = 1 - (qyzt + zt + yt^2)M(z) + yt^2(1-qz)^2 M(z)M(qz)\ .$$

Proceeding similar to [18], we use the linearising Ansatz

$$M(z) = \frac{1}{1-z}\frac{H(qz)}{H(z)}$$

to obtain

$$H(z) - (1+yt^2)H(qz) + yt^2H(q^2z) = z\left(H(z) + (1+qy)tH(qz) + qyt^2H(q^2z)\right).$$

Solving recursively for the coefficients c_n of $H(z) = \sum_{n=0}^{\infty} c_n z^n$, we obtain a solution in terms of a basic hypergeometric series,

$$H(z) = {}_2\phi_1(-t, -tqy; t^2qy; q, z) = \sum_{n=0}^{\infty} \frac{(-t, -tqy; q)_n}{(t^2qy, q; q)_n} z^n\ .$$

Note that we are dealing with a series of the type ${}_2\phi_1(a, b; ab; q, z)$ where $a = -t$ and $b = -tqy$. In order to take the limit $z \to 1$, we need to transform using Heine's transformation

$${}_2\phi_1(a, b, ab; q, z) = \frac{(az, b; q)_\infty}{(ab, z; q)_\infty} {}_2\phi_1(a, z; az; q, b)\ .$$

We find that

$$M(z) = \frac{1}{1-az}\frac{{}_2\phi_1(a, qz; aqz; q, b)}{{}_2\phi_1(a, z, az; q, b)}$$

and therefore

$$M(1) = \frac{1}{1-a}{}_2\phi_1(a, q; aq; q, b) = \sum_{n=0}^{\infty} \frac{b^n}{1-aq^n}\ .$$

Changing back to $a = -t$ and $b = -tqy$,

$$M_k = (-1)^k[t^k]M(1) = \sum_{m+n=k} y^n q^{n(m+1)} = \sum_{i=0}^{k} y^i q^{i(k-i+1)}.$$

□

Combining the previous results, we get a proof of Theorem 1.

3 A second proof using the Matrix Ansatz and rook placements

For further details about material in this section, see [11]. One of the ideas at the origin of this proof is the following. From D and E of the Matrix Ansatz, we define new operators $\hat{D}$ and $\hat{E}$ as

$$\hat{D} = \frac{q-1}{q} D + \frac{1}{q}, \qquad \hat{E} = \frac{q-1}{q} E + \frac{1}{q}.$$

An immediate consequence is that

$$\hat{D}\hat{E} - q\hat{E}\hat{D} = \frac{1-q}{q^2}, \qquad \langle W|\hat{E} = \langle W|, \qquad \text{and} \quad \hat{D}|V\rangle = |V\rangle. \tag{3}$$

This commutation relation is somewhat simpler than the one satisfied by D and E, as it has no terms linear in $\hat{D}$ or $\hat{E}$. Moreover, we have $q(y\hat{D}+\hat{E}) + (1-q)(yD+E) = 1+y$, for any parameter y. Using this identity, we obtain the following inversion formulae between $(yD+E)^n$ and $(y\hat{D}+\hat{E})^n$:

$$(1-q)^n(yD+E)^n = \sum_{k=0}^{n} \binom{n}{k} (1+y)^{n-k}(-1)^k q^k (y\hat{D}+\hat{E})^k, \quad \text{and} \tag{4}$$

$$q^n(y\hat{D}+\hat{E})^n = \sum_{k=0}^{n} \binom{n}{k} (1+y)^{n-k}(-1)^k (1-q)^k (D+E)^k. \tag{5}$$

In particular, the first formula means that in order to compute the coefficients of the normal form of $(yD+E)^n$, it is sufficient to compute the ones of $(y\hat{D}+\hat{E})^k$ for all $0 \le k \le n$ (as taking the normal form is a linear operation).

Except for a factor $-q$, the operators $\hat{D}$ and $\hat{E}$ are also defined in [25] and [1]. In the first reference, Uchiyama, Sasamoto and Wadati used the commutation relation between $\hat{D}$ and $\hat{E}$ to find explicit matrices for these operators. They derive the eigenvalues and eigenvectors of $\hat{D}+\hat{E}$, and consequently the ones of $D+E$, in terms of orthogonal polynomials. In the second reference, Blythe, Evans, Colaiori and Essler also use these eigenvalues and obtain an integral form for $\langle W|(D+E)^n|V\rangle$. They also provide an exact integral-free formula of this quantity, somewhat complicated since it contains three summations and several q-binomial coefficients, but more general since it contains two other parameters.

In this article, instead of working on representations of $\hat{D}$ and $\hat{E}$ and their eigenvalues, we study the combinatorics of the rewriting in the normal form of $(\hat{D}+\hat{E})^n$, and more generally $(y\hat{D}+\hat{E})^n$ for some parameter y. In the case of $\hat{D}$ and $\hat{E}$, the objects that appear are the *rook placements in Young diagrams*, long-known by combinatorists since the results of Kaplansky, Riordan, Goldman, Foata and Schützenberger (see [19] and references therein). This method is described in [26], and is the same that the one leading to permutation tableaux or alternative tableaux in the case of D and E.

Definition 1 *Let λ be a Young diagram. A rook placement of shape λ is a partial filling of the cells of λ with rooks (denoted by a circle $\circ$), such that there is at most one rook per row (resp. per column).*

For convenience, we distinguish with a cross ($\times$) each cell of the Young diagram that is not below (in the same column) or to the left (in the same row) of a rook (we are using the French convention). The number of crosses is an important statistic on rook placements, which was introduced in [10] as a generalisation of the inversion number for permutations. Indeed, if λ is a square of side length n, a rook placement R with n rooks may be visualised as the graph of a permutation $\sigma \in \mathfrak{S}_n$, and in this interpretation the number of crosses in R is the inversion number of σ.

Definition 2 *The weight of a rook placement R with r rooks, s crosses and t columns is $w(R) = p^r q^s y^t$, where $p = \frac{1-q}{q^2}$.*

With the definition of rook placements and their weights we can give the combinatorial interpretation of $\langle W|(y\hat{D} + \hat{E})^n|V\rangle$. This is similar to the q-Wick theorem given in [14], and our rook placements are equivalent to the Feynman diagram of this reference.

Proposition 6 *For any n, $\langle W|(y\hat{D} + \hat{E})^n|V\rangle$ is equal to the sum of weights of all rook placements of half-perimeter n.*

The enumeration of rook placements leads to an evaluation of $\langle W|(y\hat{D}+\hat{E})^{n-1}|V\rangle$, hence of $\langle W|(yD+E)^{n-1}|V\rangle$ via the inversion formula (4).

3.1 Rook placements and involutions

Given a rook placement R of half-perimeter n, we define an involution $\alpha(R)$ by the following construction: label the north-east boundary of R with integers from 1 to n. This implies that each column or row has a label between 1 and n. If a column, or row, is labelled by i and does not contain a rook, it is a fixed point of $\alpha(R)$. Also, if there is a rook at the intersection of column i and row j, then $\alpha(R)$ sends i to j (and j to i).

Given a rook placement R of half-perimeter n, we also define a Young diagram $\beta(R)$ by the following construction: if we remove all rows and columns of R containing a rook, the remaining cells form a Young diagram, which we denote by $\beta(R)$. We also define $\phi(R) = (\alpha(R), \beta(R))$. See Figure 1 for an example.

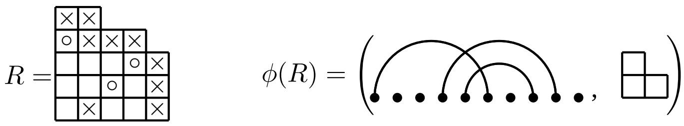

Fig. 1: Example of a rook placement and its image by the map ϕ.

Proposition 7 *The map ϕ is a bijection between rooks placements in Young diagrams of half-perimeter n, and ordered pairs (I, λ) where I is an involution on $\{1, \ldots, n\}$ and λ a Young diagram of half-perimeter $|Fix(I)|$. If $\phi(R) = (I, \lambda)$, the number of crosses in R is the sum of $|\lambda|$ and some parameter $\mu(I)$.*

Proof: This kind of bijection rather classical, see for instance [4, 13]. Note that the pairs (I, λ) may be seen as involutions on $\{1, \ldots, n\}$ with a weight 2 on each fixed point. For the second part of the proposition, we just have to distinguish different kinds of crosses in the rook placement R. For example, the crosses with no rook in the same line and column are enumerated by $|\lambda|$. $\Box$

Corollary 2 *Let $T_{j,k,n}$ be the sum of weights of rook placements of half perimeter n, with k lines and j lines without rooks. Then for any j,k,n, we have:*

$$T_{j,k,n} = \begin{bmatrix} n - 2k + 2j \\ j \end{bmatrix}_q y^j T_{0,k-j,n}. \tag{6}$$

Proof: The previous proposition means that the number of crosses is an additive parameter with respect to the decomposition $R \mapsto (I, \lambda)$. This naturally lead to a factorisation of the generating function. $\Box$

3.2 The recurrence

Proposition 8 *We have the following recurrence relation:*

$$T_{0,k,n} = T_{0,k,n-1} + py[n+1-2k]_q T_{0,k-1,n-1}. \tag{7}$$

Proof: We have the relation $T_{0,k,n} = T_{0,k,n-1} + pT_{1,k,n-1}$. Indeed, we can distinguish two cases, whether a rook placement enumerated by $T_{0,k,n}$ has a rook in its first column or not. These two cases give respectively the two terms of the previous identity. To end the proof we can apply identity (6) to the second term. □

The recurrence (7) is solved by the following formula.

Proposition 9

$$T_{0,k,n} = q^{-2k} \sum_{i=0}^{k} (-1)^i q^{\frac{i(i+1)}{2}} \begin{bmatrix} n-2k+i \\ i \end{bmatrix}_q \left(\binom{n}{k-i} - \binom{n}{k-i-1} \right). \tag{8}$$

It is worth noticing that we can get the Touchard-Riordan formula as a special case when n is even and $k = \frac{n}{2}$. Actually there is also a bijective proof of (8), which generalizes Penaud's bijective proof of the Touchard-Riordan formula [22].

From this proposition, identity (6), and a q-binomial identity, we derive a formula for $T_{j,k,n}$.

Proposition 10

$$\sum_{j=0}^{k} T_{j,k,n} = \sum_{j=0}^{k} \left(\tbinom{n}{j} - \tbinom{n}{j-1} \right) \left(\frac{q^{(k+1-j)(n-k-j)} - q^{(k-j)(n-k-j)} + q^{(k-j)(n+1-k-j)} - q^{(k+1-j)(n+1-k-j)}}{(1-q)q^n} \right).$$

Summing this identity over k gives the following result.

Proposition 11

$$\langle W|(y\hat{D} + \hat{E})|V\rangle = (1+y)G(n) - G(n+1), \tag{9}$$

$$\textit{where} \quad G(n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n}{j} - \binom{n}{j-1} \right) \sum_{i=0}^{n-2j} y^{i+j-1} q^{i(n+1-2j-i)}.$$

This formula is a linear combination of the polynomials $P_k = \sum_{i=0}^{k} y^i q^{i(k+1-i)}$, the coefficients being polynomials in y, just as in Theorem 1. With this result and the inversion formula (4), we can prove Theorem 1: the last step is an elementary binomial simplification.

4 Applications

Among all the objects of the list in Proposition 1, the most studied are probably permutations and the pattern 13-2, see for example [5, 7, 20, 21, 15]. In particular, in [5, 21] we can find methods for obtaining, as a function of n for a given k, the number of permutations of size n with exactly k occurrences of pattern 13-2. By taking the Taylor series of (1), we obtain direct and quick proofs for these results. As an illustration we give the formulae for $k \leq 3$ in the following proposition.

Proposition 12 *The order 3 Taylor series of $\langle W|(D+E)^{n-1}|V\rangle$ is:*

$$\langle W|(D+E)^{n-1}|V\rangle = C_n + \binom{2n}{n-3}q + \frac{n}{2}\binom{2n}{n-4}q^2 + \frac{(n+1)(n+2)}{6}\binom{2n}{n-5}q^3 + O(q^4),$$

where C_n is the nth Catalan number.

More generally, a computer algebra system can provide higher order terms, for example it takes no more than a few seconds to obtain the following closed formula for $[q^{10}]\langle W|(D+E)^{n-1}|V\rangle$:

$$\begin{aligned}\tfrac{(2n)!}{10!(n+12)!(n-8)!}\Big(&n^{13} + 70\,n^{12} + 2093\,n^{11} + 32354\,n^{10} + 228543\,n^9 - 318990\,n^8 \\ &-17493961\,n^7 - 104051458\,n^6 - 6828164\,n^5 + 2022876520\,n^4 \\ &+6310831968\,n^3 + 5832578304\,n^2 + 14397419520\,n + 5748019200\Big),\end{aligned}$$

which is quite an improvement compared to the methods of [21]. In addition to exact formula, we can give asymptotic estimates, for example for the number of permutations with a given number of occurrences of pattern 13-2.

Theorem 2 *For any fixed $m \geq 0$,*

$$[q^m]\langle W|(D+E)^{n-1}|V\rangle \sim \frac{4^n n^{m-\frac{3}{2}}}{\sqrt{\pi}m!} \quad \text{as } n \to \infty.$$

Proof: When $n \to \infty$, the numbers $\binom{2n}{n-k} - \binom{2n}{n-k-2}$ are dominated by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. This implies that in $(1-q)^n\langle W|(D+E)^{n-1}|V\rangle$, each higher order term grows at most as fast as the constant term C_n. On the other side, the coefficient of q^m in $(1-q)^{-n}$ is asymptotically $n^m/m!$. □

Since any occurrence of the pattern 13-2 in a permutation is also an occurrence of the pattern 1-3-2, a permutation with k occurrences of the pattern 1-3-2 has at most k occurrences of the pattern 13-2. So we get the following corollary. This could also be obtained with the methods of [15], which gives an algorithm to obtain the generating functions of permutations with a given number of occurrences of 1-3-2.

Corollary 3 *Let $\psi_k(n)$ be the number of permutations in $\mathfrak{S}_n$ with at most k occurrences of the pattern 1-3-2. For any constant $C > 1$ and $k \geq 0$, we have*

$$\psi_k(n) \leq C\frac{4^n n^{k-\frac{3}{2}}}{\sqrt{\pi}k!}$$

when n is sufficiently large.

So far we have only used Corollary 1. Now we illustrate what can be done with the refined formula given in Theorem 1. For example, when $q = 0$ then the coefficient of y^m is given by the expression $\sum_{k=0}^{n}(-1)^k\left(\binom{n}{m}\binom{n}{m+k} - \binom{n}{m-1}\binom{n}{m+k+1}\right)$. This is equal to the Narayana number $N(n,m) = \frac{1}{n}\binom{n}{m}\binom{n}{m-1}$ (see [29] for a combinatorial proof).

We can also get the coefficients for higher powers of q. For example it is conjectured in [29] that the coefficient of qy^m in $\langle W|y(yD+E)^{n-1}|V\rangle$ is equal to $\binom{n}{m+1}\binom{n}{m-2}$. Applying our results we can prove:

Proposition 13 *The coefficients of qy^m and q^2y^m in $\langle W|y(yD+E)^{n-1}|V\rangle$ are respectively:*

$$\binom{n}{m+1}\binom{n}{m-2} \qquad \textit{and} \qquad \binom{n+1}{m-2}\binom{n+1}{m+2}\frac{nm+m-m^2-4}{2(n+1)}.$$

Proof: A naive expansion of the Taylor series in q gives a lengthy formula, which is simplified easily after noticing that it is the product of $\binom{n}{m}^2$ and a rational fraction of n and m. $\square$

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On wiring and tiling diagrams related to bases of tropical Plücker functions

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We consider the class of bases B of tropical Plücker functions on the Boolean n-cube such that B can be obtained by a series of flips from the basis formed by the intervals of the ordered set of n elements. We show that these bases are representable by special wiring diagrams and by certain arrangements generalizing rhombus tilings on a zonogon.

Keywords: Plücker relations, octahedron recurrence, wiring diagram, rhombus tiling, TP-mutations

1 Introduction

This paper deals with bases of tropical Plücker functions defined on a Boolean (hyper)cube and is devoted to a combinatorial description of a wide class of such bases via a relationship to certain classes of wiring and rhombus tiling diagrams.

For a positive integer n, let $[n]$ denote the ordered set of elements $1, 2, \dots, n$. Consider a real-valued function f on the subsets of $[n]$, or on the Boolean cube $2^{[n]}$. Following [1], f is said to be a *tropical Plücker function*, or a *TP-function* for short, if it satisfies

$$f(Xik) + f(Xj) = \max\{f(Xij) + f(Xk),\ f(Xi) + f(Xjk)\} \tag{1.1}$$

for any triple $i < j < k$ in $[n]$ and any subset $X \subseteq [n] - \{i, j, k\}$. Hereinafter for brevity we write $Xi' \dots j'$ instead of $X \cup \{i'\} \cup \dots \cup \{j'\}$. The set of TP-functions on $2^{[n]}$ is denoted by $\mathcal{TP}_n$.

Definition. A collection $B \subseteq 2^{[n]}$ is called a *TP-basis*, or simply a *basis*, if the restriction map $res : \mathcal{TP}_n \to \mathbb{R}^B$ is a bijection. In other words, each TP-function is determined by its values on B, and the values on B can be chosen arbitrarily.

Such a basis does exist and the simplest instance is the set $\mathcal{I}_n$ of all intervals $\{p, p+1, \dots, q\}$ in $[n]$ (including the empty set); see, e.g., [2]. In particular, the dimension of the polyhedral conic complex $\mathcal{TP}_n$ is equal to $|\mathcal{I}_n| = \binom{n+1}{2} + 1$. The basis $\mathcal{I}_n$ is called *standard*.

(Note that the notion of a TP-function is extended to other domains, of which most popular are an *integer box* $\mathbf{B}^{n,a} := \{x \in \mathbb{Z}^{[n]} : 0 \le x \le a\}$ for $a \in \mathbb{Z}^{[n]}$ and a *hyper-simplex* $\Delta_n^m := \{S \subseteq [n] : |S| = m\}$ for $m \in \mathbb{Z}$ (in the later case, (1.1) should be replaced by a relation on quadruples $i < j < k < \ell$).

Aspects involving TP-bases or related objects are encountered in [1, 5, 8, 9, 10, 11, 12] and some other works. Generalizing some earlier known examples, [2] constructs a TP-basis for a "truncated integer box" $\{x \in \mathbf{B}^{n,a} \colon m \le x_1 + \ldots + x_n \le m'\}$, where $0 \le m \le m' \le a_1 + \ldots + a_n$. The domains different from Boolean cubes are beyond this paper; for some generalizations, see [3].)

Once we are given a basis B, we can produce more bases by making a series of elementary transformations relying on (1.1). More precisely, suppose (X, i, j, k) is a cortege such that the four sets occurring in the right hand side of (1.1) and one set $Y \in \{Xj, Xik\}$ in the left hand side belong to B. Then the replacement in B of Y by the other set Y' in the left hand side results in a basis B' as well (and we can further transform the latter basis in a similar way). The basis B' is said to be obtained from B by the *flip* (or *mutation*) with respect to X, i, j, k. When Xj is replaced by Xik, the flip is called *raising*; otherwise the flip is called *lowering*. The standard basis $\mathcal{I}_n$ does not admit lowering flips, whereas its complementary basis co-$\mathcal{I}_n := \{[n] - I \colon I \in \mathcal{I}_n\}$ does not admit raising flips.

We further distinguish between two sorts of flips, inspiring consideration of two classes of bases.

Definitions. For a TP-basis B and a cortege (X, i, j, k) as above, the corresponding flip is called *strong* if both sets X and $Xijk$ belong to B as well, and *weak* in general. A basis is called *normal* (in terminology of [2]) if it can be obtained by a series of *strong* flips starting from $\mathcal{I}_n$. A basis is called *semi-normal* if it can be obtained by a series of *weak* flips starting from $\mathcal{I}_n$.

Leclerc and Zelevinsky [8] showed that the normal bases (in our terminology) are exactly the collections $C \subseteq 2^{[n]}$ of maximum possible size $|C|$ that possess the strong separation property (defined later). Also the normal bases admit a nice "graphical" representation, even for a natural generalization to the integer boxes (see [2, 4]): such bases correspond to the rhombus tilings on the related *zonogon*.

The purpose of this paper is to characterize the class of semi-normal TP-bases for the Boolean cube $2^{[n]}$, denoted as $\mathcal{B}_n$. (It should be noted that it is still open at present whether there exists a non-semi-normal, or "wild", basis; we conjecture that there is none.) We give two characterizations for $\mathcal{B}_n$: via a bijection to special collections of curves, that we call *proper wirings*, and via a bijection to certain graphical arrangements, that we call *generalized tilings*, or *g-tilings* for short (in fact, these characterizations are interrelated via planar duality). We associate to a proper wiring W (a g-tiling T) a certain collection of subsets of $[n]$ called its *spectrum*. It turns out that proper wirings and g-tilings are rigid objects, in the sense that any of these is determined by its spectrum (see Theorem 3.3).

Roughly speaking, by a *wiring* we mean a set of n directed non-self-intersecting curves $w_1, \ldots, w_n$ in a region R of the plane homeomorphic to a circle, where each w_i begins at a point s_i and ends at a point s'_i, and the points $s_1, \ldots, s_n, s'_1, \ldots, s'_n$ are different and occur in this order in the boundary of R. A *special* wiring W is defined by three axioms (W1)–(W3). Axiom (W1) is standard, it says that W preserves (topologically) under small deformations. (W2) says that the common points of w_i, w_j follow in the opposed orders along these wires. The crucial axiom (W3) says that in the planar graph induced by W, there is a certain bijection between the faces whose boundary is a directed cycle and the regions ("lenses") surrounded by pieces of two wires between their consecutive common points. W is called *proper* if none of "cyclic" faces is a whole lens. The spectrum of W is the collection of subsets $X \subseteq [n]$ associated to the "non-cyclic" faces F, where X consists of the elements i such that F "lies on the left" from w_i. When any two wires intersect exactly once, the dual planar graph is realized by a rhombus tiling, and vice versa (for a more general result of this sort, see [6]). The construction of a g-tiling is more intricate. Axiom (W2) occurs in [9]. Another sort of wirings, related to hyper-simplexes, is studied in [9, 10].

Our main result is the following

Theorem 1.1 *For $B \subseteq 2^{[n]}$, the following statements are equivalent:*
(i) *B is a semi-standard TP-basis;*
(ii) *B is the spectrum of a proper wiring;*
(iii) *B is the spectrum of a generalized tiling.*

The paper is organized as follows. Section 2 gives precise definitions of proper wirings and generalized tilings. Section 3 outlines ideas of the proof of Theorem 1.1. It consists of four subsections, concerning implications (i)→(iii), (iii)→(i), (iii)→(ii), and (ii)→(iii), respectively. (In fact, g-tilings are the central objects of treatment; we take advantages from their nice graphical visualization and structural features, and all implications that we explicitly prove involve just g-tilings. Another advantage of g-tilings is that they admit "local" defining axioms; see the Remark in Section 2.)

Complete proofs of the above-mentioned results are given in the full version [3] of this paper. Moreover, combinatorial methods and technical tools elaborated in those proofs give rise to additional results presented there. Apparently the most important among them is the affirmative answer to a conjecture of Leclerc and Zelevinsky [8] on weakly separated set-systems having maximum possible cardinality.

A commentary: For $A, B \subseteq [n]$, let us write $A \prec B$ if $i < j$ for any $i \in A$ and $j \in B$. Following [8], a pair A, B is called *strongly separated* if $A - B \prec B - A$ or $B - A \prec B - A$, and is called *weakly separated* if, up to renaming (A, B) as (B, A), one holds: $|A| \geq |B|$ and $B - A$ can be partitioned into a disjoint union $B' \sqcup B''$ so that $B' \prec A - B \prec B''$. Accordingly, a collection $\mathcal{C} \subseteq 2^{[n]}$ is called strongly (weakly) separated if any two members of $\mathcal{C}$ are strongly (resp. weakly) separated. It is shown in [8] that: (a) any weakly separated collection in $2^{[n]}$ has cardinality at most $\binom{n+1}{2} + 1$; and (b) the set $\mathbf{C}_n$ of weakly separated collections $\mathcal{C} \subseteq 2^{[n]}$ with $|\mathcal{C}| = \binom{n+1}{2} + 1$ includes $\mathcal{B}_n$. In fact, the above-mentioned conjecture in [8, Conjecture 1.8] is that this inclusion turns into equality: $\mathbf{C}_n = \mathcal{B}_n$.

(As is seen from a discussion in [8], an interest in studying weakly separated collections is inspired, in particular, by the problem of characterizing all families of quasicommuting quantum flag minors, which in turn comes from exploration of Lusztig's canonical bases for certain quantum groups. It is proved in [8] that, in an $n \times n$ generic q-matrix, the flag minors with column sets $I, J \subseteq [n]$ quasicommute if and only if the sets I, J are weakly separated. See also [7].)

2 Wirings and tilings

Wiring and tiling diagrams that we deal with live within a zonogon, which is defined as follows.

In the upper half-plane $\mathbb{R} \times \mathbb{R}_+$, take n non-colinear vectors $\xi_1, \ldots, \xi_n$ so that:

(2.1) (i) $\xi_1, \ldots, \xi_n$ follow in this order clockwise around $(0, 0)$, and
(ii) all integer combinations of these vectors are different.

Then the set

$$Z = Z_n := \{\lambda_1\xi_1 + \ldots + \lambda_n\xi_n \colon \lambda_i \in \mathbb{R},\ 0 \leq \lambda_i \leq 1,\ i = 1, \ldots, n\}$$

is a $2n$-gone. Moreover, Z is a *zonogon*, as it is the sum of n line-segments $\{\lambda\xi_i \colon 1 \leq \lambda \leq 1\}$, $i = 1, \ldots, n$. Also it is the image by a linear projection π of the solid cube $conv(2^{[n]})$ into the plane $\mathbb{R}^2$, defined by $\pi(x) := x_1\xi_1 + \ldots + x_n\xi_n$. The boundary $bd(Z)$ of Z consists of two parts: the *left boundary* $lbd(Z)$ formed by the points (vertices) $p_i := \xi_1 + \ldots + \xi_i$ connected by the line-segments $p_{i-1}p_i := p_{i-1} + \{\lambda\xi_i \colon 0 \leq \lambda \leq 1\}$, and the *right boundary* $rbd(Z)$ formed by the points $p'_i := \xi_{i+1} + \ldots + \xi_n$ connected by the segments $p'_ip'_{i-1}$ $(i = 0, \ldots, n)$. So $p_0 = p'_n$ is the minimal vertex and $p_n = p'_0$ is the maximal vertex of Z. We orient each segment $p_{i-1}p_i$ from p_{i-1} to p_i and orient each segment $p'_ip'_{i-1}$ from p'_i to p'_{i-1}. Let s_i (resp. s'_i) denote the median point in the segment $p_{i-1}p_i$ (resp. $p'_ip'_{i-1}$).

Although the generalized tiling model will be used more extensively later on, we prefer to start with describing the special wiring model, which looks more transparent.

2.1 Wiring diagrams

A *special wiring diagram*, also called a *W-diagram* or a *wiring* for brevity, is a collection W of n wires $w_1, \dots, w_n$ satisfying three axioms below. A *wire* w_i is a continuous injective map of the segment $[0, 1]$ into Z (or the curve in the plane induced by this map) such that $w_i(0) = s_i$, $w_i(1) = s'_i$, and $w_i(\lambda)$ lies in the interior of Z for $0 < \lambda < 1$. We say that w_i begins at s_i and ends at s'_i, and orient w_i from s_i to s'_i. The diagram W is considered up to a homeomorphism of Z stable on $bd(Z)$, and up to parameterizations of the wires. Axioms (W1)–(W3) specify W as follows.

(W1) No three different wires w_i, w_j, w_k have a common point, i.e., there are no $\lambda, \lambda', \lambda''$ such that $w_i(\lambda) = w_j(\lambda') = w_k(\lambda'')$. Any two different wires w_i, w_j intersect at a finite number of points, and at each of their common points v, the wires *cross*, not touch (i.e., when passing v, the wire w_i goes from one connected component of $Z - w_j$ to the other).

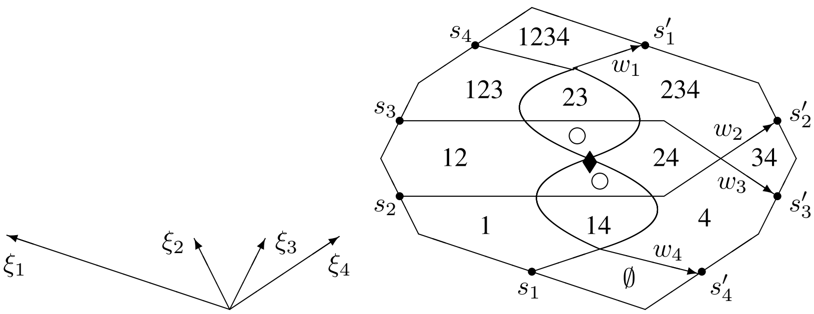
(W2) for $1 \le i < j \le n$, the common points of w_i, w_j follow in opposed orders along these wires, i.e., if $w_i(\lambda_q) = w_j(\lambda'_q)$ for $q = 1, \dots, r$ and if $\lambda_1 < \ldots < \lambda_r$, then $\lambda'_1 > \ldots > \lambda'_r$.

Since the order of s_i, s_j in $\ell bd(Z)$ is different from the order of s'_i, s'_j in $rbd(Z)$, wires w_i, w_j always intersect; moreover, the number $r = r_{ij}$ of their common points is odd. Assuming that $i < j$, we denote these points as $x_{ij}(1), \dots, x_{ij}(r)$ following the direction of w_i. When $r > 1$, the region in the plane surrounded by the pieces of w_i, w_j between $x_{ij}(q)$ and $x_{ij}(q + 1)$ (where $q = 1, \dots, r - 1$) is denoted by $L_{ij}(q)$ and called the q-th *lens* for w_i, w_j. The points $x_{ij}(q)$ and $x_{ij}(q + 1)$ are regarded as the *lower* and *upper* points of $L_{ij}(q)$, respectively. When q is odd (even), we say that $L_{ij}(q)$ is an *odd* (resp. *even*) lens. Note that at each point $x_{ij}(q)$ with q odd the wire with the bigger number, namely, w_j, crosses the wire with the smaller number (w_i) *from left to right* w.r.t. the direction of the latter; we call such a point *white*. In contrast, when q is even, w_j crosses w_i at $x_{ij}(q)$ *from right to left*; in this case, we call $x_{ij}(q)$ *black*, or *orientation-reversing*, and say that this point is the *root* of the lenses $L_{ij}(q - 1)$ and $L_{ij}(q)$. In the simplest case, when any two distinct wires intersect exactly once, there are no lenses at all and all intersection points for W are white. (The adjectives “white” and “black” for intersection points of wires will match terminology that we use for corresponding elements of tilings.)

The wiring W is associated, in a natural way, with a planar directed graph G_W embedded in Z. The vertices of G_W are the points p_i, p'_i, s_i, s'_i and the intersection points of wires. The edges of G_W are the corresponding directed line-segments in $bd(Z)$ and the pieces of wires between neighboring points of intersection with other wires or with the boundary, which are directed according to the direction of wires. We say that an edge contained in a wire w_i has *color* i, or is an *i-edge*. Let $\mathcal{F}_W$ be the set of (inner) faces of G_W. Here each face F is considered as the closure of a maximal connected component in $Z - \cup(w \in W)$. We say that a face F is *cyclic* if its boundary $bd(F)$ is a directed cycle in G_W.

(W3) There is a bijection ϕ between the set $\mathcal{L}(W)$ of lenses in W and the set $\mathcal{F}_W^{cyc}$ of cyclic faces in G_W. Moreover, for each lens L, $\phi(L)$ is the (unique) face lying in L and containing its root.

We say that W is *proper* if none of cyclic faces is a whole lens, i.e., for each lens $L \in \mathcal{L}(W)$, there is at least one wire going across L. An instance of proper wirings for $n = 4$ is illustrated in the picture; here the cyclic faces are marked by circles and the black rhombus indicates the unique black point.



$$B_W = \{\emptyset, 1, 4, 12, 14, 23, 24, 34, 123, 234, 1234\}$$

Now we associate to W a set-system $B_W \subseteq 2^{[n]}$ as follows. For each face F, let $X(F)$ be the set of elements $i \in [n]$ such that F lies *on the left* from the wire w_i, i.e., F and the maximal point p_n lie in the same of the two connected components of $Z - w_i$. We define

$$B_W := \{X \subseteq [n] \colon X = X(F) \text{ for some } F \in \mathcal{F}_W - \mathcal{F}_W^{cyc}\},$$

referring to it as the *effective spectrum*, or simply the *spectrum* of W; this is just the object occurring in (ii) of Theorem 1.1. Sometimes it is also useful to consider the *full spectrum* $\widehat{B}_W$ consisting of all sets $X(F)$, $F \in \mathcal{F}_W$. (One proves that when W is proper, all sets in $\widehat{B}_W$ are different. When W is not proper, there are different faces F, F' with $X(F) = X(F')$. One can turn W into a proper wiring W' by getting rid of lenses forming faces (by making a series of Reidemeister moves of type II: $() \to)($ operations). This preserves the effective spectrum: $B_{W'} = B_W$, whereas the full spectrum may decrease.)

Note that when any two wires intersect at exactly one point (i.e., when no black points exist), B_W is a normal basis, and conversely, any normal basis is obtained in this way (see [2]).

2.2 Generalized tilings

When it is not confusing, we identify a subset $X \subseteq [n]$ with the corresponding vertex of the n-cube and with the point $\sum_{i \in X} \xi_i$ in the zonogon Z. Due to (2.1)(ii), all such points in Z are different.

Assuming that the vectors ξ_i have the same Euclidean norm, a *rhombus tiling diagram* is a subdivision T of Z into rhombi of the form $x + \{\lambda\xi_i + \lambda'\xi_j \colon 0 \le \lambda, \lambda' \le 1\}$ for some $i < j$ and some point x in Z, i.e., the rhombi are pairwise non-overlapping (have no common interior points) and their union is Z. It follows that for i, j, x as above, x represents a subset in $[n] - \{i, j\}$. We associate to T the directed planar graph G_T whose vertices and edges are the vertices and side segments of the rhombi, respectively. An edge connecting X and Xi is directed from the former to the latter. It is shown in [2, 4] that the vertex set of G_T forms a normal basis and that each normal basis is obtained in this way.

In fact, it makes no difference whether we take vectors $\xi_1, \dots, \xi_n$ with equal or arbitrary norms (subject to (2.1)); to simplify technical details and visualization, we further assume that these vectors have *unit height*, i.e., each ξ_i is of the form $(x, 1)$. Then we obtain a subdivision T of Z into parallelograms of height 2, and for convenience refer to T as a *tiling* and to its elements as *tiles*. A tile τ defined by X, i, j (with $i < j$) is called an ij*-tile* at X and denoted by $\tau(X; i, j)$. Its vertices X, Xi, Xj, Xij are called the *bottom, left, right, top* vertices of τ and denoted by $b(\tau)$, $\ell(\tau)$, $r(\tau)$, $t(\tau)$, respectively.

In a *generalized tiling*, or a *g-tiling*, the union of tiles is again Z but some tiles may overlap. It is a collection T of tiles partitioned into two subcollections T^w and T^b, of *white* and *black* tiles (say), respectively, obeying axioms (T1)–(T4) below. When $T^b = \emptyset$, we will obtain a tiling as before, for convenience referring to it as a *pure tiling*. Let V_T and E_T denote the sets of vertices and edges, respectively, occurring in tiles of T, not counting multiplicities. For a vertex $v \in V_T$, the set of edges incident with v is denoted by $E_T(v)$, and the set of tiles having a vertex at v is denoted by $F_T(v)$.

(T1) All tiles are contained in Z. Each boundary edge of Z belongs to exactly one tile. Each edge in E_T not contained in $bd(Z)$ belongs to exactly two tiles. All tiles in T are different (in the sense that no two coincide in the plane).

(T2) Any two white tiles having a common edge do not overlap. If a white tile and a black tile share an edge, then these tiles do overlap. No two black tiles share an edge.

(T3) Let τ be a black tile. None of $b(\tau), t(\tau)$ is a vertex of another black tile. All edges in $E_T(b(\tau))$ *leave* $b(\tau)$ (i.e., are directed from $b(\tau)$). All edges in $E_T(t(\tau))$ *enter* $t(\tau)$ (are directed to $t(\tau)$).

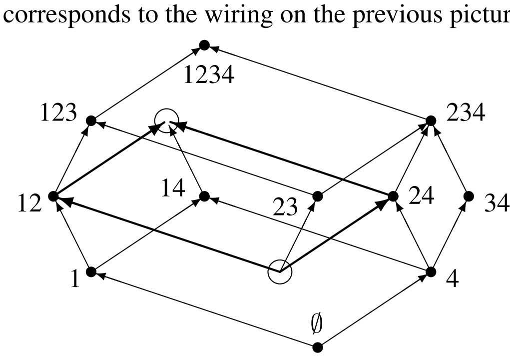
We distinguish between three sorts of vertices by saying that $v \in V_T$ is: (a) a *terminal* vertex if it is the bottom or top vertex of some black tile; (b) an *ordinary* vertex if all tiles in $F_T(v)$ are white; and (c) a *mixed* vertex otherwise (i.e. v is the left or right vertex of some black tile). Note that a mixed vertex may belong, as the left or right vertex, to several black tiles.

Each tile $\tau \in T$ is associated, in a natural way, to a square in the solid cube $conv(2^{[n]})$, denoted by $\sigma(\tau)$: if $\tau = \tau(X; i, j)$ then $\sigma(\tau)$ is the convex hull of the points X, Xi, Xj, Xij in the cube. In view of (T1), the interiors of these squares are disjoint, and $\cup(\sigma(\tau) \colon \tau \in T)$ forms a 2-dimensional surface, denoted by D_T, whose boundary is the preimage by π of the boundary of Z. The last axiom is:

(T4) D_T is a disc (i.e., is homeomorphic to $\{x \in \mathbb{R}^2 \colon x_1^2 + x_2^2 \leq 1\}$).

The *reversed* g-tiling T^{rev} of a g-tiling T is formed by replacing each tile $\tau(X; i, j)$ of T by the tile $\tau([n] - Xij; i, j)$ (or by changing the orientation of all edges in E_T, in particular, in $bd(Z)$). Clearly (T1)–(T4) remain valid for T^{rev}.

The *effective spectrum*, or simply the *spectrum*, of a g-tiling T is the collection B_T of (the subsets of $[n]$ represented by) *non-terminal* vertices in G_T; this is just the object occurring in (iii) of Theorem 1.1. The *full spectrum* $\widehat{B}_T$ is formed by all vertices in G_T. An example of g-tilings for $n = 4$ is drawn in the picture, where the unique black tile is indicated in bold and the terminal vertices are surrounded by circles (this corresponds to the wiring on the previous picture).



$$B_T = \{\emptyset, 1, 4, 12, 14, 23, 24, 34,$$
$$123, 234, 1234\}$$

It turns out that for each semi-normal basis B, there are precisely one proper wiring W and precisely one g-tiling T such that $B_W = B_T = B$ (see Theorem 3.3); this is similar to the one-to-one correspondence between the normal bases and pure tilings.

In the rest of this section we point out some (relatively simple) consequences from axioms (T1)–(T4).

1. For a g-tiling T, an edge e of the graph $G_T = (V_T, E_T)$ is called *black* if there is a black tile containing e (as a side edge); otherwise e is called *white*. The sets of white and black edges incident with a vertex v are denoted by $E^w_T(v)$ and $E^b_T(v)$, respectively. For a vertex v of a tile τ, let $C(\tau, v)$ denote the minimal cone at v containing τ (i.e., generated by the pair of edges of τ incident to v), and let $\alpha(\tau, v)$ denote the angle of this cone taken with sign $+$ if τ is white, and $-$ if τ is black. The *full rotation angle* at v is the sum $\sum(\alpha(\tau, v) \colon \tau \in F_T(v))$, denoted by $\rho(v)$. The terminal vertices behave as follows.

Corollary 2.1 *Let v be a terminal vertex belonging to a black ij-tile τ. Then:*
(i) *v is not connected by edge with another terminal vertex (whence $|E^b_T(v)| = 2$);*
(ii) $|E_T(v)| \geq 3$ *(whence $E^w_T(v) \neq \emptyset$);*
(iii) *each edge $e \in E^w_T(v)$ lies in the cone $C(\tau, v)$ (whence e is a q-edge for some $i < q < j$);*
(iv) $\rho(v) = 0$*;*
(v) *v does not belong to the boundary of Z (whence any tile containing a boundary edge of Z is white).*

2. The rotation angles at non-terminal vertices behave as follows (this is proved by using Euler formula applied to the planar embedding of G_T in the disc D_T).

Lemma 2.2 *Let $v \in V_T$ be a non-terminal vertex.*
(i) *If v is in $bd(Z)$, then $\rho(v)$ is equal to the angle between the boundary edges incident to v.*
(ii) *If v is inner (i.e., not in $bd(Z)$), then $\rho(v) = 2\pi$.*

Remark One shows that if property (ii) in Lemma 2.2 is postulated as axiom (T4') and added to axioms (T1)–(T3), then one can eliminate axiom (T4); in other words, (T4') and (T4) are equivalent subject to (T1)–(T3). Note that each of axioms (T1)–(T3),(T4') is "local"; due to Theorem 1.1, this gives rise to a local characterization for the semi-normal TP-bases.

3. An important fact following immediately from (2.1)(ii) is that for any g-tiling T, the graph G_T is *graded* for each color $i \in [n]$, which means that for any closed path P in G_T, the numbers of forward i-edges and backward i-edges in P are equal.

3 Ideas of proofs

As mentioned in the Introduction, the proof of Theorem 1.1 falls into four stages, each consisting in showing one of the implications involved there. Below we outline ideas of our approach.

3.1 From semi-normal bases to generalized tilings

The first stage is devoted to showing that any semi-normal TP-basis is representable as the spectrum of some g-tiling, yielding (i)→(iii) in Theorem 1.1.

Consider a g-tiling T on the zonogon $Z = Z_n$. By an *M-configuration* in T we mean a quintuple of vertices of the form Xi, Xj, Xk, Xij, Xjk with $i < j < k$ (as it resembles the letter "M"), which is denoted as $CM(X; i, j, k)$. By a *W-configuration* in T we mean a quintuple of vertices Xi, Xk, Xij, Xik, Xjk with $i < j < k$ (as resembling "W"), denoted as $CW(X; i, j, k)$. A configuration is called *feasible* if all five vertices are non-terminal, i.e., they belong to B_T.

Since any normal basis B (in particular, $B = \mathcal{I}_n$) is expressed as B_T for some pure tiling T, it suffices to prove the following assertion saying that the set of g-tilings is closed under transformations analogous to flips for semi-normal bases.

Proposition 3.1 *Let a g-tiling T contain five non-terminal vertices Xi, Xk, Xij, Xjk, Y, where $i < j < k$ and $Y \in \{Xik, Xj\}$. Then there exists a g-tiling T' such that $B_{T'}$ is obtained from B_T by replacing Y by the other member of $\{Xik, Xj\}$.*

To prove this, one may assume that $Y = Xik$, in which case we have a feasible W-configuration $CW(X; i, j, k)$ (since any M-configuration in T turns into a W-configurations in the reversed g-tiling T^{rev}). We rely on the following two facts (whose proofs are nontrivial).

(P1) Any pair of *non-terminal* vertices $X', X'i'$ in G_T is connected by edge.

(Note that vertices $X', X'i'$ need not be connected by edge if some of them is terminal.) So, by (P1), G_T contains the edges (Xi, Xij), (Xi, Xik), (Xk, Xik) and (Xk, Xjk).

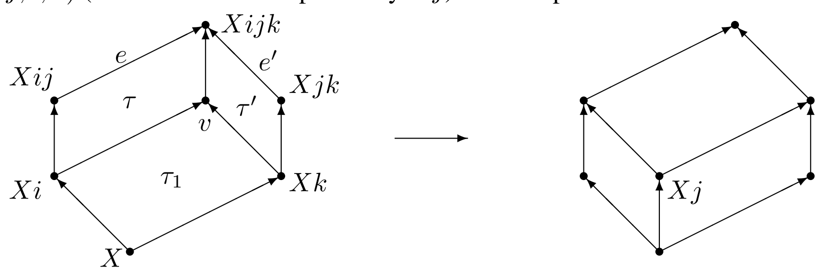
(P2) There exist two white tiles τ, τ' in T such that τ contains the edges (Xi, Xij) and (Xi, Xik), and τ' contains the edges (Xk, Xik) and (Xk, Xjk). (These τ, τ' share the edge $(Xik, Xijk)$.

In addition, one shows (which is not difficult) that the vertex $v := Xik$ is ordinary.

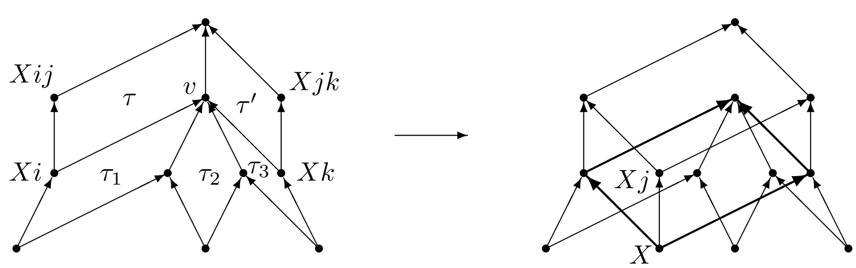
Based on the these facts, the construction of the desired g-tiling T' is as follows. Let $e_0, \ldots, e_q$ be the sequence of edges entering v in the counterclockwise order. Since v is ordinary, $e_0 = (Xi, Xik)$ and $e_q = (Xk, Xik)$, and each pair e_{p-1}, e_p ($p = 1, \ldots, q$) belongs to a white tile τ_p. We consider two possible cases, each case being divided into two subcases.

Case 1: The edges $e := (Xij, Xijk)$ and $e' := (Xjk, Xijk)$ do not belong to the same black tile.

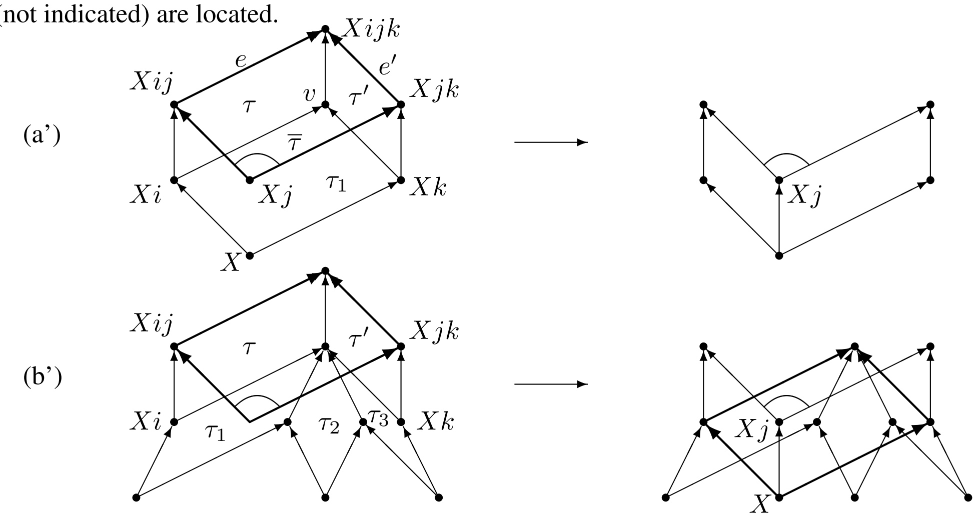
Subcase 1a: $q = 1$. We replace in T the tiles τ, τ', τ_1 by three new white tiles: $\tau(X; i, j)$, $\tau(X; j, k)$ and $\tau(Xj; i, k)$ (so the vertex v is replaced by Xj). See the picture.



Subcase 1b: $q > 1$. We remove the tiles τ, τ' and add four new tiles: the white tiles $\tau(X; i, j)$, $\tau(X; j, k)$, $\tau(Xj; i, k)$ (as before) and the black tile $\tau(X; i, k)$ (so v becomes terminal). See the picture; here $q = 3$ and the added black tile is indicated in bold.



Case 2: Both edges e and e' belong to a black tile $\overline{\tau}$ (which is $\tau(Xj;i,k)$). We act as in Case 1 with the only differences that $\overline{\tau}$ is removed from T and that the white ik-tile at Xj (which is a copy of $\overline{\tau}$) is not added. Then the vertex $Xijk$ vanishes, v either vanishes or becomes terminal, and Xj becomes non-terminal. See the picture. Here (a') and (b') concern *Subcase 2a*: $q=1$, and *Subcase 2b*: $q>1$, respectively, and the arc above the vertex Xj indicates the bottom cone of $\overline{\tau}$ in which some white edges (not indicated) are located.



One proves that in all cases the resulting collection T' of tiles satisfies axioms (T1)–(T4). Also it is seen from the construction that $B_{T'}=(B_T-\{Xik\})\cup\{Xj\}$, as required in Proposition 3.1.

3.2 From generalized tilings to semi-normal bases

The second stage consists in showing that for any g-tiling T, its spectrum B_T is a semi-normal TP-basis, yielding (iii)→(i) in Theorem 1.1.

If T has no black tile, then B_T is a normal basis, and we are done. So assume $T^b\neq\emptyset$. Our aim is to show the existence of a feasible W-configuration $CW(X;i,j,k)$ for T. Then we can transform T into a g-tiling T' as in Proposition 3.1, i.e., with $B_{T'}=(B_T-\{Xik\})\cup\{Xj\}$. Under such a *lowering flip*, the sum of sizes of the sets in $B_\bullet$ decreases. Then the result will follow by induction on $\sum(|X'|\colon X'\in B_T)$.

By the *height* $h(v)$ of a vertex $v\in V_T$ we mean the size of the corresponding subset of $[n]$. The height $h(\tau)$ of a tile $\tau\in T$ is defined to be the height of its left (or right) vertex.

In fact, we present a sharper version of the desired property.

Proposition 3.2 *Let $h\in[n]$. If a g-tiling T has a black tile τ of height h, then there exists a feasible W-configuration $CW(X;i,j,k)$ with $|X|=h-2$. Moreover, such a $CW(X;i,j,k)$ can be chosen so that $Xijk$ is the top vertex of some black tile (of height h).*

To prove this, starting from $\tau_0:=\tau$, choose a vertex u_0 adjacent to $v_0:=t(\tau_0)$ and different from $\ell(\tau_0),r(\tau_0)$ (it exists by (ii) in Corollary 2.1). Then there are white tiles τ',τ'' such that $t(\tau')=t(\tau'')=v_0$ and $r(\tau')=\ell(\tau'')=u_0$. One easily shows that if u_0 is ordinary, then both vertices $b(\tau'),b(\tau'')$ are non-terminal, implying that these vertices together with $u_0,\ell(\tau'),r(\tau'')$ form a feasible W-configuration (as required in the proposition). And if u_0 is mixed, then there is a black tile τ_1 (of the same height h) having u_0 as the left or right vertex. We treat τ_1 in a similar way as τ_0, by choosing a vertex u_1 adjacent to $v_1 := t(\tau_1)$ and different from $\ell(\tau_1), r(\tau_1)$. If u_1 is mixed again, we continue the process. Sooner or later, we obtain a black tile τ_k and a chosen vertex u_k such that either u_k is ordinary, in which case we are done, or τ_k coincides with τ_0. In the latter case, one shows (a key) that there exists $i \in [n]$ such that in the cycle passing $v_0, u_0, v_1, u_1, v_2, \ldots, u_{k-1}, v_k = v_0$, the numbers of forward and backward edges with color i are not equal, contrary to the fact that G_T is graded.

3.3 From generalized tilings to proper wirings

The third stage consists in showing that for any g-tiling T on the zonogon $Z = Z_n$, there exists a proper wiring W on Z such that $B_W = B_T$, yielding (iii)→(ii) in Theorem 1.1.

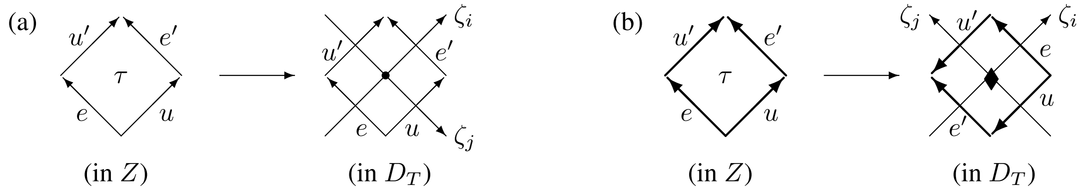
To prove this, we use the notion of an *i-strip* (or a *dual i-path*) for T, where $i \in [n]$. This is a maximal sequence $Q = (e_0, \tau_1, e_1, \ldots, \tau_r, e_r)$ of edges and tiles such that: (a) $\tau_1, \ldots, \tau_r$ are different tiles, each being an ij- or ji-tile for some j, and (b) for $p = 1, \ldots, r$, e_{p-1} and e_p are the opposite i-edges of τ_p (recall that when speaking of an $i'j'$-tile, one assumes $i' < j'$.) Clearly Q is determined uniquely, up to reversing it and shifting cyclically (when $e_0 = e_r$), by any of its edges or tiles. Using the fact that G_T is graded, one shows that Q cannot be cyclic, i.e., the edges e_0 and e_r are different. Then one of e_0, e_r lies on the left boundary, and the other on the right boundary of Z; we may assume that $e_0 \in \ell bd(Z)$.

For convenience we identify the tiles in T with the corresponding squares in the disc D_T (whose interiors are pairwise disjoint). To construct the desired wiring W, each i-strip $Q_i = (e_0, \tau_1, e_1, \ldots, \tau_r, e_r)$ for T is regarded as a sequence of straight-line segments and squares on D_T, and we draw the "median" piece-wise linear curve ζ_i within Q_i. More precisely, for $q = 1, \ldots, r$, draw the line-segment on τ_q connecting the median points of the edges e_{r-1} and e_r. This segment meets the central point of τ_q, denoted by $c(\tau_q)$. The concatenation of these segments is just ζ_i; we direct it according to the direction of Q_i.

Now fix a homeomorphic map $\gamma : D_T \to Z$ that brings the boundary of D_T to $bd(Z)$ in a natural way. This turns the above curves into the wires $w_i := \gamma(\zeta_i)$ on Z, where w_i begins at the median point s_i of $p_{i-1}p_i$ on $\ell bd(Z)$ and ends at the medial point s'_i of $p'_i p'_{i-1}$ on $rbd(Z)$). We assert that the wiring $W = (w_1, \ldots, w_n)$ is as required.

Clearly W satisfies axiom (W1). To verify the other axioms, we first should explain how the planar graphs G_T and $H := \gamma^{-1}(G_W)$ on D_T are related to each other. The vertices of H are the central points $c(\tau)$ of squares τ and the points s_i, s'_i (identifying the boundaries of D_T and Z by γ). Each vertex v of G_T corresponds to the face of H where v is located, denoted by v^*. The edges of color i in H correspond to the i-edges of G_T. More precisely, if an i-edge $e \in E_T$ belongs to squares τ, τ' and if τ, e, τ' occur in this order in the i-strip, then the i-edge e^* of H corresponding to e is the piece of ζ_i between $c(\tau)$ and $c(\tau')$, and this e^* is directed from $c(\tau)$ to $c(\tau')$. Observe that e crosses e^* *from right to left* on the disc. The first (last) piece of ζ_i corresponds to the boundary i-edge $p_{i-1}p_i$ (resp. $p'_i p'_{i-1}$) of G_T.

Consider an ij-tile $\tau \in T$, and let e, e' be its i-edges, and u, u' its j-edges, where e, u leave $b(\tau)$ and e', u' enter $t(\tau)$. One can see that: (a) if τ is white, then e occurs in Q_i before e', while u occurs in Q_j after u', and (b) if τ is black, then e occurs in Q_i after e', while u occurs in Q_j before u'. In the disc D_T, both e, e' cross the wire ζ_i from right to left (w.r.t. the direction of ζ_i), and similarly both u, u' cross ζ_j from right to left. Axioms (T1),(T2) for T imply that when τ is white, the orientations of τ in Z and in D_T are the same, whereas when τ is black, the clockwise orientation of τ in Z turns in the counterclockwise orientation of τ in D_T. It follows that: in case (a), ζ_j crosses ζ_i at $c(\tau)$ from left to right, and in case (b), ζ_j crosses ζ_i at $c(\tau)$ from right to left; see the picture. So the white (black) tiles of T generate the white (resp. black) vertices of G_W.



For a vertex v of G_T and an edge $e \in E_T(v)$, the edge e^* belongs to the boundary of the face v^* of H. Since e crosses e^* from right to left, e^* is directed clockwise around v^* if e leaves v, and counterclockwise if e enters v. Then axiom (T3) for T implies that the terminal vertices of G_T and only them generate cyclic faces of G_W, yielding validity of (W3).

Next, for each $i \in [n]$, removing from D_T the interior of the i-strip Q_i results in two closed regions Ω_1, Ω_2 containing the vertices $\emptyset$ and $[n]$, respectively. Also all edges in Q_i go from Ω_1 to Ω_2, whence each vertex v of G_T occurring in Ω_1 (Ω_2) is a subset of $[n]$ not containing (resp. containing) the element i. So $i \notin X(v^*)$ if and only if $v \in \Omega_1$. This implies the desired equality for spectra: $B_W = B_T$.

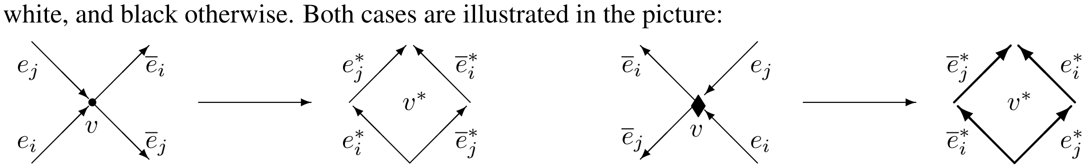
A verification of (W2) for W is less trivial; this is done by using the fact that G_T is graded (we omit the details). Finally, since $|E_T(v)| \geq 3$ for each terminal vertex v in G_T, each cyclic face in G_W is surrounded by at least three edges, and therefore, this face cannot be a lens. So the wiring W is proper.

3.4 From proper wirings to generalized tilings

The final, fourth, stage is devoted to showing that for any proper wiring W on the zonogon $Z = Z_n$, there exists a g-tiling T on Z such that $B_T = B_W$, yielding (ii)→(iii) in Theorem 1.1. The construction of T is converse, in a sense, to that described in the previous subsection; it combines planar duality techniques and geometric arrangements.

We associate to each inner face F of the graph G_W the point (viz. the subset) $X(F)$ in the zonogon, also denoted as F^*. These points are just the vertices of tiles in T. The edges of G_T are defined as follows. Let faces $F, F' \in \mathcal{F}_W$ have a common edge e formed by a piece of a wire w_i, and let F lie on the right from w_i according to the direction of this wire. Then the vertices F^*, F'^* are connected by edge e^* going from F^* to F'^*. Note that in view of the evident relation $X(F') = X(F) \cup \{i\}$, the direction of e^* matches the edge direction for g-tilings.

The tiles in T correspond to the intersection points of wires in W. More precisely, let v be a common point of wires w_i, w_j with $i < j$. Then the vertex v of G_W has four incident edges $e_i, \overline{e}_i, e_j, \overline{e}_j$ such that: $e_i, \overline{e}_i \subset w_i$; $e_j, \overline{e}_j \subset w_j$; e_i, e_j enter v; and $\overline{e}_i, \overline{e}_j$ leave v. One can see that for the four faces F containing v, the subsets $X(F)$ are of the form X, Xi, Xj, Xij for some $X \subset [n]$. The tile surrounded by the edges $e_i^*, \overline{e}_i^*, e_j^*, \overline{e}_j^*$ connecting these subsets (regarded as points) is just the ij-tile in T corresponding to v, denoted as v^*. Observe that the edges $e_i, e_j, \overline{e}_i, \overline{e}_j$ follow in this order clockwise around v if v is white, and counterclockwise if v is black. According to this, the tile v^* is assigned to be white in T if v is white, and black otherwise. Both cases are illustrated in the picture:



A proof that T is indeed a correct g-tiling on Z and that $B_T = B_W$ consists of several verifications of which some are not straightforward; we omit the details.

Properties of g-tilings and proper wirings established during the whole proof of Theorem 1.1 enable us to obtain the following rigidity result.

Theorem 3.3 *For each semi-normal basis B, there are a unique g-tiling T and a unique proper wiring W such that $B = B_T = B_W$.*

Also one can offer an efficient (polynomial-time) algorithm that, given a collection $B \subset 2^{[n]}$, decides whether B is representable as the spectrum of a g-tiling, and if so, explicitly constructs this g-tiling.

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Characters of symmetric groups in terms of free cumulants and Frobenius coordinates

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Free cumulants are nice and useful functionals of the shape of a Young diagram, in particular they give the asymptotics of normalized characters of symmetric groups $\mathfrak{S}(n)$ in the limit $n \to \infty$. We give an explicit combinatorial formula for normalized characters of the symmetric groups in terms of free cumulants. We also express characters in terms of Frobenius coordinates. Our formulas involve counting certain factorizations of a given permutation. The main tool are Stanley polynomials which give values of characters on multirectangular Young diagrams.

Résumé. Les cumulants libres sont des fonctions agréables et utiles sur l'ensemble des diagrammes de Young, en particulier, ils donnent le comportement asymptotiques des caractères normalisés du groupe symétrique $\mathfrak{S}(n)$ dans la limite $n \to \infty$. Nous donnons une formule combinatoire explicite pour les caractères normalisés du groupe symétrique en fonction des cumulants libres. Nous exprimons également les caractères en fonction des coordonnées de Frobenius. Nos formules font intervenir le nombre de certaines factorisations d'une permutation donnée. L'outil principal est la famille de polynômes de Stanley donnant les valeurs des caractères sur les diagrammes de Young multirectangulaires.

Keywords: characters of symmetric groups, free cumulants, Kerov polynomials, Stanley polynomials

1 Introduction

This contribution is an extended abstract of a full version [DFŚ08] which will be published elsewhere.

1.1 Dilations of Young diagrams and normalized characters

For a Young diagram λ and an integer $s \geq 1$ we denote by $s\lambda$ the *dilation* of λ by factor s. This operation can be easily described on a graphical representation of a Young diagram: we just dilate the picture of λ or, alternatively, we replace each box of λ by a grid of $s \times s$ boxes.

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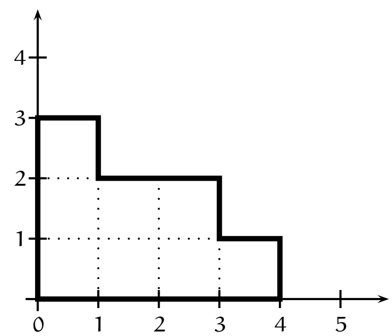

Fig. 1: Young diagram $(4, 3, 1)$ drawn in the French convention

Any permutation $\pi \in \mathfrak{S}(k)$ can be also regarded as an element of $\mathfrak{S}(n)$ if $k \leq n$ (we just declare that $\pi \in \mathfrak{S}(n)$ has additional $n - k$ fixpoints). For any $\pi \in \mathfrak{S}(k)$ and an irreducible representation ρ^λ of the symmetric group $\mathfrak{S}(n)$ corresponding to the Young diagram λ we define the *normalized character*

$$\Sigma^\lambda_\pi = \begin{cases} \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\text{dimension of } \rho^\lambda} & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We shall concentrate our attention on the characters on cycles, therefore we will use the special notation

$$\Sigma^\lambda_k = \Sigma^\lambda_{(1,2,\dots,k)},$$

where we treat the cycle $(1, 2, \dots, k)$ as an element of $\mathfrak{S}(k)$ for any integer $k \geq 1$.

The notion of dilation of a Young diagram is very useful from the viewpoint of the asymptotic representation theory because it allows us to ask the following question:

Problem 1 *What can we say about the characters of the symmetric groups in the limit when the Young diagram λ tends in some sense to infinity in a way that λ preserves its shape?*

This informal problem can be formalized as follows: for fixed λ and k we ask about (asymptotic) properties of the normalized characters $\Sigma^{s\lambda}_k$ in the limit $s \to \infty$. The reason why we decided to study this particular normalization of characters is the following well-known yet surprising result.

Fact 2 *For any Young diagram λ and integer $k \geq 2$ the normalized character on a dilated diagram*

$$\mathbb{N} \ni s \mapsto \Sigma^{s\lambda}_{k-1} \tag{1}$$

is a polynomial function of degree (at most) k.

1.2 Generalized Young diagrams

Any Young diagram drawn in the French convention can be identified with its graph which is equal to the set $\{(x, y) : 0 \leq x, 0 \leq y \leq f(x)\}$ for a suitably chosen function $f : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$, cf. Figure 1. It is therefore natural to define the set of *generalized Young diagrams* $\mathbb{Y}$ *(in the French convention)* as the set of bounded, non-increasing functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ with a compact support; in this way any Young diagram can be regarded as a generalized Young diagram. Notice that with the notion of generalized Young diagrams we may consider dilations $s\lambda$ for any real $s \geq 0$.

1.3 How to describe the shape of a Young diagram?

Our ultimate goal is to explicitly express the polynomials (1) in terms of the shape of λ. However, before we start this task we must ask ourselves: how to describe the shape of λ in the best way? In the folowing we shall present two approaches to this problem.

We define the *fundamental functionals of shape* of a Young diagram λ by an integral over the area of λ

$$S_k^\lambda = (k-1) \iint_{(x,y)\in\lambda} (\text{contents}_{(x,y)})^{k-2}\, dx\, dy$$

for integer $k \geq 2$, and where $\text{contents}_{(x,y)} = x - y$ is the contents of a point of a Young diagram. When it does not lead to confusions we will skip the explicit dependence of the fundamental functionals on Young diagrams and instead of S_k^λ we shall simply write S_k. Clearly, each functional S_k is a homogeneous function of the Young diagram with degree k, i.e. $S_k^{s\lambda} = s^k S_k^\lambda$.

For a Young diagram with Frobenius coordinates $\lambda = \begin{bmatrix} a_1 & \cdots & a_l \\ b_1 & \cdots & b_l \end{bmatrix}$ we define its *shifted Frobenius coordinates* with $A_i = a_i + \frac{1}{2}$ and $B_i = b_i + \frac{1}{2}$. Shifted Frobenius coordinates have a simple interpretation as positions (up to the sign) of the particles and holes in the Dirac sea corresponding to a Young diagram [Oko01]. Functionals S_k^λ can be nicely expressed in terms of (shifted) Frobenius coordinates as follows:

$$S_k^\lambda = \sum_i \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[(A_i + z)^{k-1} - (-B_i - z)^{k-1} \right] dz,$$

$$\frac{S_k^\lambda}{|\lambda|^{k-1}} = \sum_i \left[\alpha_i^{k-1} - (-\beta_i)^{k-1} \right] + O\left(\frac{1}{|\lambda|^2} \right), \qquad \text{where } \alpha_i = \frac{A_i}{|\lambda|} \text{ and } \beta_i = \frac{B_i}{|\lambda|}. \tag{2}$$

Another way of describing the shape of a Young diagram λ is to use its *free cumulants* $R_2^\lambda, R_3^\lambda, \dots$ which are defined as the coefficients of the leading terms of the polynomials (1):

$$R_k^\lambda = [s^k]\Sigma_{k-1}^{s\lambda} = \lim_{s\to\infty} \frac{1}{s^k} \Sigma_{k-1}^{s\lambda} \qquad \text{for integer } k \geq 2.$$

Later on we shall show how to calculate free cumulants directly from the shape of a Young diagram. R_k is a homogeneous function of the Young diagram with degree k, i.e. $R_k^{s\lambda} = s^k R_k^\lambda$.

The importance of homogeneity of S_k^λ and R_k^λ becomes clear when one wants to solve asymptotic problems, such as understanding coefficients of the polynomial (1).

1.4 Character polynomials and their applications

It is not very difficult to show [DFŚ08] that for each integer $k \geq 1$ there exists a polynomial with rational coefficients $J_k(S_2, S_3, \dots)$ with a property that

$$\Sigma_k^\lambda = J_k(S_2^\lambda, S_3^\lambda, \dots)$$

holds true for any Young diagram λ. For example, we have

$$\Sigma_1 = S_2, \qquad \Sigma_2 = S_3, \qquad \Sigma_3 = S_4 - \frac{3}{2}S_2^2 + S_2, \qquad \Sigma_4 = S_5 - 4S_2S_3 + 5S_3,$$

$$\Sigma_5 = S_6 - 5S_2S_4 - \frac{5}{2}S_3^2 + \frac{25}{6}S_2^3 + 15S_4 - \frac{35}{2}S_2^2 + 8S_2.$$

The polynomials J_n are very useful, when one studies the asymptotics of characters in the limit when the parameters $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ converge to some limits and the number of boxes of λ tends to infinity. Equation (2) shows that for such scaling it is convenient to consider a different gradation, in which the degree of S_k is equal to $k-1$. We leave it as an exercise to the Reader to use the results of this paper to show that with respect to this gradation polynomial J_k has the form

$$\Sigma_k = S_{k+1} - \frac{k}{2} \sum_{j_1+j_2=k+1} S_{j_1}S_{j_2} + \text{(terms of smaller degree)}.$$

The dominant part of the right-hand side (the first summand) coincides with the estimate of Wassermann [Was81] and with Thoma character on $\mathfrak{S}(\infty)$ [VK81]. In a similar way it is possible to obtain next terms in the expansion.

One can also show that for each integer $k \geq 1$ there exists a polynomial with integer coefficients $K_k(R_2, R_3, \dots)$, called *Kerov character polynomial* [Ker00, Bia03] with a property that

$$\Sigma_k^\lambda = K_k(R_2^\lambda, R_3^\lambda, \dots)$$

holds true for any Young diagram λ. For example,

$$\Sigma_1 = R_2, \qquad \Sigma_2 = R_3, \qquad \Sigma_3 = R_4 + R_2, \qquad \Sigma_4 = R_5 + 5R_3,$$

$$\Sigma_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2, \qquad \Sigma_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3.$$

The advantage of Kerov polynomials K_k over polynomials J_k comes from the fact that they usually have a much simpler form, involve smaller number of summands and are more suitable for studying asymptotics of characters in the case of *balanced Young diagrams*, i.e. for example in the case of characters $\Sigma_k^{s\lambda}$ of dilated Young diagrams [Bia03].

1.5 The main result: explicit form of character polynomials

For a permutation π we denote by $C(\pi)$ the set of the cycles of π.

Theorem 3 (Dołęga, Féray, Śniady [DFŚ08]) *The coefficients of polynomials J_k are explicitly described as follows:*

$$\frac{\partial}{\partial S_{j_1}} \cdots \frac{\partial}{\partial S_{j_l}} J_k \bigg|_{S_2=S_3=\cdots=0}$$

is equal to $(-1)^{l-1}$ times the number of the number of the triples $(\sigma_1, \sigma_2, \ell)$ where

- *$\sigma_1, \sigma_2 \in \mathfrak{S}(k)$ are such that $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$,*
- *$\ell : C(\sigma_2) \to \{1, \dots, l\}$ is a bijective labeling,*

- *for each $1 \leq i \leq l$ there are exactly $j_i - 1$ cycles of σ_1 which intersect cycle $\ell^{-1}(i)$ and which do not intersect any of the cycles $\ell^{-1}(i+1), \ell^{-1}(i+2), \dots$.*

Theorem 4 (Dołęga, Féray, Śniady [DFŚ08]) *The coefficient of $R_2^{s_2} R_3^{s_3} \cdots$ in the Kerov polynomial K_k is equal to the number of triples (σ_1, σ_2, q) with the following properties:*

(a) *$\sigma_1, \sigma_2 \in \mathfrak{S}(k)$ are such that $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$;*

(b) *the number of cycles of σ_2 is equal to the number of factors in the product $R_2^{s_2} R_3^{s_3} \cdots$; in other words $|C(\sigma_2)| = s_2 + s_3 + \cdots$;*

(c) *the total number of cycles of σ_1 and σ_2 is equal to the degree of the product $R_2^{s_2} R_3^{s_3} \cdots$; in other words $|C(\sigma_1)| + |C(\sigma_2)| = 2s_2 + 3s_3 + 4s_4 + \cdots$;*

(d) *$q : C(\sigma_2) \to \{2, 3, \dots\}$ is a coloring of the cycles of σ_2 with a property that each color $i \in \{2, 3, \dots\}$ is used exactly s_i times (informally, we can think that q is a map which to cycles of $C(\sigma_2)$ associates the factors in the product $R_2^{s_2} R_3^{s_3} \cdots$);*

(e) *for every set $A \subset C(\sigma_2)$ which is nontrivial (i.e., $A \neq \emptyset$ and $A \neq C(\sigma_2)$) there are more than $\sum_{i \in A} \big(q(i) - 1\big)$ cycles of σ_1 which intersect $\bigcup A$.*

Only condition (e) is rather complicated, therefore we will provide two equivalent combinatorial conditions below.

1.6 Marriage and transportation interpretations of condition (e)

Let (σ_1, σ_2, q) be a triple which fulfills conditions (a)–(d) of Theorem 4. We consider the following polyandrous interpretation of Hall marriage theorem. Each cycle of σ_1 will be called a *boy* and each cycle of σ_2 will be called a *girl*. For each girl $j \in C(\sigma_2)$ let $q(j) - 1$ be the *desired number of husbands of j*. We say that a boy $i \in C(\sigma_1)$ is a possible candidate for a husband for a girl $j \in C(\sigma_2)$ if cycles i and j intersect. Hall marriage theorem applied to our setup says that there exists an arrangement of marriages $\mathcal{M} : C(\sigma_1) \to C(\sigma_2)$ which assigns to each boy his wife (so that each girl j has exactly $q(j) - 1$ husbands) if and only if for every set $A \subseteq C(\sigma_2)$ there are at least $\sum_{i \in A} \big(q(i) - 1\big)$ cycles of σ_1 which intersect $\bigcup A$. As one easily see, the above condition is similar but not identical to (e). The following Proposition shows the connection between these two problems.

Proposition 5 *Condition (e) is equivalent to each of the following two conditions:*

(e^2) *for every nontrivial set of girls $A \subset C(\sigma_2)$ (i.e., $A \neq \emptyset$ and $A \neq C(\sigma_2)$) there exist two ways of arranging marriages $\mathcal{M}_p : C(\sigma_1) \to C(\sigma_2)$, $p \in \{1, 2\}$ for which the corresponding sets of husbands of wives from A are different:*

$$\mathcal{M}_1^{-1}(A) \neq \mathcal{M}_2^{-1}(A),$$

(e^3) *there exists a strictly positive solution to the following system of equations:*

Set of variables
$\big\{x_{i,j} :\ i \in C(\sigma_1) \text{ and } j \in C(\sigma_2) \text{ are intersecting cycles}\big\}$

$$\textbf{Equations} \begin{cases} \forall i, \sum_j x_{i,j} = 1 \\ \forall j, \sum_i x_{i,j} = q(j) - 1 \end{cases}$$

Note that the possibility of arranging marriages can be rephrased as existence of a solution to the above system of equations with a requirement that $x_{i,j} \in \{0,1\}$.

The system of equations in condition (e^3) can be interpreted as a transportation problem where each cycle of σ_1 is interpreted as a factory which produces a unit of some ware and each cycle j of σ_2 is interpreted as a consumer with a demand equal to $q(j) - 1$. The value of $x_{i,j}$ is interpreted as amount of ware transported from factory i to the consumer j.

1.7 General conjugacy classes

An analogue of Theorem 3 holds true with some minor modifications also for the analogues of polynomials J giving the values of characters on general permutations, not just cycles.

In case of the analogues of the Kerov polynomials giving the values of characters on more complex permutations π than cycles the situation is slightly more diffcult. Namely, an analogue of Theorem 4 holds true if the character Σ_π is replaced by some quantities which behave like classical cumulants of cycles constituting π and the sum on the right-hand side is taken only over transitive factorizations. Since the expression of characters in terms of classical cumulants of cycles is straightforward, we obtain an expression of characters in terms of free cumulants.

1.8 Applications of the main result

The results of this article (Theorem 4 in particular) can be used to obtain new asymptotic inequalities for characters of the symmetric groups. This vast topic is outside of the scope of this article and will be studied in a subsequent paper.

1.9 Contents of this article

In this article we shall prove Theorem 3. Also, since the proof of Theorem 4 is rather long and technical [DFŚ08], in this overview article we shall highlight just the main ideas and concentrate on the first non-trivial case of quadratic terms of Kerov polynomials.

Due to lack of space we were not able to show the full history of the presented results and to give to everybody the proper credits. For more history and bibliographical references we refer to the full version of this article [DFŚ08].

2 Ingredients of the proof of the main result

2.1 Polynomial functions on the set of Young diagrams

Surprisingly, the normalized characters Σ_π^λ can be extended in a natural way for any generalized Young diagram $\lambda \in \mathbb{Y}$. The algebra they generate will be called *algebra of polynomial functions on (generalized) Young diagrams.* It is well-known that many natural families of functions on Young diagrams generate the same algebra, for example the family of free cumulants (R_k^λ) or the family of fundamental functionals (S_k^λ).

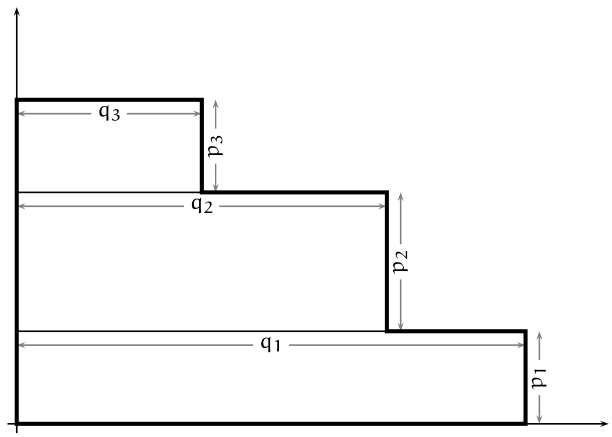

Fig. 2: Generalized Young diagram $\mathbf{p} \times \mathbf{q}$ drawn in the French convention

2.2 Stanley polynomials

For two finite sequences of positive real numbers $\mathbf{p} = (p_1, \dots, p_m)$ and $\mathbf{q} = (q_1, \dots, q_m)$ with $q_1 \geq \cdots \geq q_m$ we consider a multirectangular generalized Young diagram $\mathbf{p} \times \mathbf{q}$, cf Figure 2. In the case when $p_1, \dots, p_m, q_1, \dots, q_m$ are natural numbers $\mathbf{p} \times \mathbf{q}$ is a partition

$$\mathbf{p} \times \mathbf{q} = (\underbrace{q_1, \dots, q_1}_{p_1 \text{ times}}, \underbrace{q_2, \dots, q_2}_{p_2 \text{ times}}, \dots).$$

Proposition 6 *Let $\mathcal{F} : \mathbb{Y} \to \mathbb{R}$ be a polynomial function on the set of generalized Young diagrams. Then $(\mathbf{p}, \mathbf{q}) \mapsto \mathcal{F}(\mathbf{p} \times \mathbf{q})$ is a polynomial in indeterminates $p_1, \dots, p_m, q_1, \dots, q_m$, called* Stanley polynomial*.*

Proof: It is enough to prove this proposition for some family of generators of the algebra of polynomial functions on $\mathbb{Y}$. In the case of functionals $S_2, S_3, \dots$ it is a simple exercise. □

Lemma 7 *If we treat $\mathbf{p}$ as variables and $\mathbf{q}$ as constants then for every $k \geq 2$ and all $i_1 < \cdots < i_s$*

$$[p_{i_1} \cdots p_{i_s}] S_k^{\mathbf{p} \times \mathbf{q}} = \begin{cases} (-1)^{s-1} \, (k-1)_{s-1} \, q_{i_s}^{k-s} & \text{if } 1 \leq s \leq k-1, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Proof: The integral over the Young diagram $\mathbf{p} \times \mathbf{q}$ can be split into several integrals over rectangles constituting $\mathbf{p} \times \mathbf{q}$ therefore

$$S_k^{\mathbf{p}\times\mathbf{q}} = (k-1) \iint_{(x,y)\in\mathbf{p}\times\mathbf{q}} (x-y)^{k-2}\, dx\, dy =$$
$$(k-1)! \sum_{1\le r\le k-1} (-1)^{r-1} \iint_{(x,y)\in\mathbf{p}\times\mathbf{q}} \frac{x^{k-1-r}}{(k-1-r)!} \frac{y^{r-1}}{(r-1)!}\, dx\, dy =$$
$$(k-1)! \sum_{1\le r\le k-1} (-1)^{r-1} \sum_j \frac{q_j^{k-r}}{(k-r)!} \frac{(p_1+\cdots+p_j)^r - (p_1+\cdots+p_{j-1})^r}{r!}.$$

For any $i_1 < \cdots < i_s$

$$\frac{\partial^s}{\partial p_{i_1}\cdots\partial p_{i_s}} \frac{(p_1+\cdots+p_j)^r - (p_1+\cdots+p_{j-1})^r}{r!}\bigg|_{p_1=p_2=\cdots=0} = \begin{cases} 1 & \text{if } s=r \text{ and } i_s=j, \\ 0 & \text{otherwise} \end{cases}$$

which finishes the proof. □

Theorem 8 *Let $\mathcal{F} : \mathbb{Y} \to \mathbb{R}$ be a polynomial function on the set of generalized Young diagrams, we shall view it as a polynomial in $S_2, S_3, \dots$ Then for any $j_1, \dots, j_l \ge 2$*

$$\frac{\partial}{\partial S_{j_1}} \cdots \frac{\partial}{\partial S_{j_l}} \mathcal{F}\bigg|_{S_2=S_3=\cdots=0} = [p_1 q_1^{j_1-1} \cdots p_l q_l^{j_l-1}] \mathcal{F}^{\mathbf{p}\times\mathbf{q}}.$$

Proof: By linearity is enough to consider the case when $\mathcal{F} = S_{m_1} \cdots S_{m_r}$. Clearly, the left hand side is equal to the number of permutations of the sequence $(m_1, \dots, m_r)$ which are equal to the sequence $(j_1, \dots, j_l)$. Lemma 7 shows that the same holds true for the right-hand side. □

Corollary 9 *If $j_1, \dots, j_l \ge 2$ then*

$$[p_1 q_1^{j_1-1} \cdots p_l q_l^{j_l-1}] \mathcal{F}^{\mathbf{p}\times\mathbf{q}}$$

does not depend on the order of the elements of the sequence $(j_1, \dots, j_l)$.

2.3 Stanley polynomials for characters

The following theorem gives explicitly the Stanley polynomial for normalized characters of symmetric groups. It was conjectured by Stanley [Sta06] and proved by Féray [Fér06] and therefore we refer to it as Stanley-Féray character formula. For a more elementary proof we refer to [FŚ07].

Theorem 10 *The value of the normalized character on $\pi \in \mathfrak{S}(k)$ for a multirectangular Young diagram $\mathbf{p} \times \mathbf{q}$ for $\mathbf{p} = (p_1, \dots, p_r)$, $\mathbf{q} = (q_1, \dots, q_r)$ is given by*

$$\Sigma_\pi^{\mathbf{p}\times\mathbf{q}} = \sum_{\substack{\sigma_1,\sigma_2\in\mathfrak{S}(k)\\ \sigma_1\circ\sigma_2=\pi}} \sum_{\phi_2 : C(\sigma_2)\to\{1,\dots,r\}} (-1)^{\sigma_1} \left[\prod_{b\in C(\sigma_1)} q_{\phi_1(b)} \prod_{c\in C(\sigma_2)} p_{\phi_2(c)} \right], \tag{4}$$

where $\phi_1 : C(\sigma_1) \to \{1, \dots, r\}$ *is defined by*

$$\phi_1(c) = \max_{\substack{b \in C(\sigma_2), \\ b \text{ and } c \text{ intersect}}} \phi_2(b).$$

Notice that Theorem 8 and the above Theorem 10 give immediately the proof of Theorem 3.

2.4 Relation between free cumulants and fundamental functionals

Corollary 11 *The value of the* k*-th free cumulant for a multirectangular Young diagram* $\mathbf{p} \times \mathbf{q}$ *for* $\mathbf{p} = (p_1, \dots, p_r)$, $\mathbf{q} = (q_1, \dots, q_r)$ *is given by*

$$R_k^{\mathbf{p}\times\mathbf{q}} = \sum_{\substack{\sigma_1,\sigma_2 \in \mathfrak{S}(k-1) \\ \sigma_1 \circ \sigma_2 = (1,2,\dots,k-1) \\ |C(\sigma_1)|+|C(\sigma_2)|=k}} \sum_{\phi_2 : C(\sigma_2) \to \{1,\dots,r\}} (-1)^{\sigma_1} \left[\prod_{b \in C(\sigma_1)} q_{\phi_1(b)} \prod_{c \in C(\sigma_2)} p_{\phi_2(c)} \right], \tag{5}$$

where $\phi_1 : C(\sigma_1) \to \{1, \dots, r\}$ *is defined as in Theorem 10.*

Proof: It is enough to consider the homogeneous part with degree k of both sides of (4) for $\pi = (1, \dots, k-1) \in \mathfrak{S}(k-1)$. □

Proposition 12 *For any integer* $n \geq 2$

$$R_k = \sum_{l \geq 1} \frac{1}{l!} (-k+1)^{l-1} \sum_{\substack{j_1,\dots,j_l \geq 2 \\ j_1+\dots+j_l=k}} S_{j_1} \cdots S_{j_l}.$$

Before the proof notice that the above formula shows that free cumulants can be explicitly and directly calculated from the shape of a Young diagram.

Proof: For simplicity, we shall proof a weaker form of this result, namely

$$R_k = S_k - \frac{k-1}{2} \sum_{j_1+j_2=k} S_{j_1} S_{j_2} + (\text{terms involving at least three factors } S_j). \tag{6}$$

Theorem 8 shows that the expansion of R_k in terms of (S_j) involves coefficients of Stanley polynomials and the latter are given by Corollary 11. We shall use this idea in the following.

Notice that the condition $|C(\sigma_1)|+|C(\sigma_2)| = k$ appearing in (5) is equivalent to $|\sigma_1|+|\sigma_2| = |\sigma_1 \circ \sigma_2|$ where $|\pi|$ denotes the length of the permutation, i.e. the minimal number of factors necessary to write π as a product of transpositions. In other words, $\pi_1 \circ \pi_2 = (1, \dots, k-1)$ is a *minimal factorization* of a cycle. Such factorizations are in a bijective correspondence with non-crossing partitions of $k-1$-element set [Bia96]. It is therefore enough to enumerate appropriate non-crossing partitions. We present the details of this reasoning below.

The linear term $[S_k]R_k = [p_1 q_1^{k-1}] R_k^{\mathbf{p}\times\mathbf{q}}$ is equal to the number of minimal factorizations such that σ_2 consists of one cycle and σ_1 consists of $k-1$ cycles. Such factorizations corresponds to non-crossing partitions of $k-1$ element set which have exactly one block and clearly there is only one such partition.

Since both free cumulants (R_j) and fundamental functionals of shape are homogeneous, by comparing the degrees we see that $[S_j]R_k = 0$ if $j \neq k$. The same argument shows that $[S_{j_1} S_{j_2}]R_k = 0$ if $j_1 + j_2 \neq k$.

Instead of finding the quadratic terms $[S_{j_1} S_{j_2}]R_k$ is better to find the derivative $\frac{\partial^2}{\partial S_{j_1} \partial S_{j_2}} R_k \Big|_{S_{j_1}=S_{j_2}=0}$ since it better takes care of the symmetric case $j_1 = j_2$. The latter derivative is equal (up to the sign) to the number of minimal factorizations such that σ_2 consists of two labeled cycles c_1, c_2 and σ_1 consists of $k-2$ cycles. Furthermore, we require that there are $j_2 - 1$ cycles of σ_1 which intersect cycle c_2. This is equivalent to counting non-crossing partitions of $k-1$-element set which consist of two labeled blocks b_1, b_2 and we require that the block b_2 consists of $j_2 - 1$ elements. It is easy to see that all such non-crossing partitions can be transformed into each other by a cyclic rotation hence there are $k-1$ of them which finishes the proof.

The general case can be proved by analogous but more technically involved combinatorial considerations. □

2.5 Identities fulfilled by coefficients of Stanley polynomials

The coefficients of Stanley polynomials $[p_1^{s_1} q_1^{r_1} \cdots]\mathcal{F}^{\mathbf{p}\times\mathbf{q}}$ for a polynomial function $\mathcal{F}$ are not linearly independent; in fact they fulfill many identities. In the following we shall show just one of them.

Lemma 13 *For any polynomial function $\mathcal{F} : \mathbb{Y} \to \mathbb{R}$*

$$(j_1 + j_2 - 1) \left[p_1 q_1^{j_1+j_2-1}\right] \mathcal{F}^{\mathbf{p}\times\mathbf{q}} = -\left[p_1 p_2 q_2^{j_1+j_2-2}\right] \mathcal{F}^{\mathbf{p}\times\mathbf{q}}. \tag{7}$$

Proof: It is enough to prove the Lemma if $\mathcal{F} = S_{k_1} \dots S_{k_r}$ is a monomial in fundamental functionals. Lemma 7 shows that the left-hand side of (7) is non-zero only if $\mathcal{F} = S_{j_1+j_2}$ (it is also a consequence of Theorem 8); otherwise every monomial in $\mathbf{p}$ and $\mathbf{q}$ with a nonzero coefficient would be at least quadratic with respect to the variables $\mathbf{p}$. The same argument shows that if the right-hand side is non-zero then either $\mathcal{F} = S_k$ is linear (in this case $k = j_1 + j_2$ by comparing the degrees) or $\mathcal{F} = S_{k_1} S_{k_2}$ is quadratic. In the latter case, an inspection of the coefficient

$$[p_1 p_2] S_{k_1}^{\mathbf{p}\times\mathbf{q}} S_{k_2}^{\mathbf{p}\times\mathbf{q}} = [p_1] S_{k_1}^{\mathbf{p}\times\mathbf{q}} \cdot [p_2] S_{k_2}^{\mathbf{p}\times\mathbf{q}} + [p_1] S_{k_2}^{\mathbf{p}\times\mathbf{q}} \cdot [p_2] S_{k_1}^{\mathbf{p}\times\mathbf{q}} = q_1^{k_1-1} q_2^{k_2-1} + q_1^{k_2-1} q_2^{k_1-1}$$

thanks to (3) leads to a contradiction.

It remains to show that for $\mathcal{F} = S_{j_1+j_2}$ the Lemma holds true, but this is an immediate consequence of Lemma 7. □

3 Toy example: Quadratic terms of Kerov polynomials

We shall prove Theorem 4 in the simplest non-trivial case of the quadratic coefficients $[R_{j_1} R_{j_2}]K_k$. In this case Theorem 4 takes the following equivalent form.

Theorem 14 *For all integers $j_1, j_2 \geq 2$ and $k \geq 1$ the derivative*

$$\frac{\partial^2}{\partial R_{j_1} \partial R_{j_2}} K_k \Big|_{R_2 = R_3 = \cdots = 0}$$

is equal to the number of triples (σ_1, σ_2, q) with the following properties:

(a) σ_1, σ_2 *is a factorization of the cycle; in other words* $\sigma_1, \sigma_2 \in \mathfrak{S}(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \dots, k)$*;*

(b) σ_2 *consists of two cycles;*

(c) σ_1 *consists of* $j_1 + j_2 - 2$ *cycles;*

(d) $\ell : C(\sigma_2) \to \{1, 2\}$ *is a bijective labeling of the two cycles of* σ_2*;*

(e) for each cycle $c \in C(\sigma_2)$ *there are at least* $j_{\ell(c)}$ *cycles of* σ_1 *which intersect nontrivially* c*.*

Proof: Equation (6) shows that for any polynomial function $\mathcal{F}$ on the set of generalized Young diagrams

$$\frac{\partial^2}{\partial R_{j_1} \partial R_{j_2}} \mathcal{F} = \frac{\partial^2}{\partial S_{j_1} \partial S_{j_2}} \mathcal{F} + (j_1 + j_2 - 1) \frac{\partial}{\partial S_{j_1 + j_2}} \mathcal{F},$$

where all derivatives are taken at $R_2 = R_3 = \cdots = S_2 = S_3 = \cdots = 0$. Theorem 8 shows that the right-hand side is equal to

$$\left[p_1 p_2 q_1^{j_1 - 1} q_2^{j_2 - 1} \right] \mathcal{F}^{\mathbf{p} \times \mathbf{q}} + (j_1 + j_2 - 1) \left[p_1 q_1^{j_1 + j_2 - 1} \right] \mathcal{F}^{\mathbf{p} \times \mathbf{q}}.$$

Lemma 13 applied to the second summand shows therefore that

$$\frac{\partial^2}{\partial R_{j_1} \partial R_{j_2}} \mathcal{F} = \left[p_1 p_2 q_1^{j_1 - 1} q_2^{j_2 - 1} \right] \mathcal{F}^{\mathbf{p} \times \mathbf{q}} - \left[p_1 p_2 q_2^{j_1 + j_2 - 2} \right] \mathcal{F}^{\mathbf{p} \times \mathbf{q}}. \tag{8}$$

On the other hand, let us compute the number of the triples $(\sigma_1, \sigma_2, \ell)$ which contribute to the quantity presented in Theorem 14. By inclusion-exclusion principle it is equal to

$$\begin{aligned} &\big(\text{number of triples which fulfill conditions (a)–(d)}\big) + \\ &\quad (-1)\big(\text{number of triples for which the cycle } \ell^{-1}(1) \text{ intersects at most } j_1 - 1 \text{ cycles of } \sigma_1\big) + \\ &\qquad (-1)\big(\text{number of triples for which the cycle } \ell^{-1}(2) \text{ intersects at most } j_2 - 1 \text{ cycles of } \sigma_1\big). \end{aligned} \tag{9}$$

At first sight it might seem that the above formula is not complete since we should also add the number of triples for which the cycle $\ell^{-1}(1)$ intersects at most $j_1 - 1$ cycles of σ_1 and the cycle $\ell^{-1}(2)$ intersects at most $j_2 - 1$ cycles of σ_1, however this situation is not possible since σ_1 consists of $j_1 + j_2 - 2$ cycles and $\langle \sigma_1, \sigma_2 \rangle$ acts transitively.

By Stanley-Féray character formula (4) the first summand of (9) is equal to

$$(-1) \sum_{\substack{a+b = j_1 + j_2 - 2, \\ 1 \le b}} \left[p_1 p_2 q_1^a q_2^b \right] \Sigma_k^{\mathbf{p} \times \mathbf{q}}, \tag{10}$$

the second summand of (9) is equal to

$$\sum_{\substack{a+b = j_1 + j_2 - 2, \\ 1 \le a \le j_1 - 1}} \left[p_1 p_2 q_1^b q_2^a \right] \Sigma_k^{\mathbf{p} \times \mathbf{q}}, \tag{11}$$

and the third summand of (9) is equal to

$$\sum_{\substack{a+b=j_1+j_2-2,\\ 1\leq b\leq j_2-1}} \left[p_1 p_2 q_1^a q_2^b\right] \Sigma_k^{\mathbf{p}\times\mathbf{q}}.$$

We can apply Corollary 9 to the summands of (11); it follows that (11) is equal to

$$\sum_{\substack{a+b=j_1+j_2-2,\\ 1\leq a\leq j_1-1}} \left[p_1 p_2 q_1^a q_2^b\right] \Sigma_k^{\mathbf{p}\times\mathbf{q}}. \tag{12}$$

It remains now to count how many times a pair (a, b) contributes to the sum of (10), (11), (12). It is not difficult to see that the only pairs which contribute are $(0, j_1 + j_2 - 2)$ and $(j_1 - 1, j_2 - 1)$, therefore the number of triples described in the formulation of the Theorem is equal to the right-hand of (8) which finishes the proof. □

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k-distant crossings and nestings of matchings and partitions

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Abstract. We define and consider k-distant crossings and nestings for matchings and set partitions, which are a variation of crossings and nestings in which the distance between vertices is important. By modifying an involution of Kasraoui and Zeng (Electronic J. Combinatorics 2006, research paper 33), we show that the joint distribution of k-distant crossings and nestings is symmetric. We also study the numbers of k-distant noncrossing matchings and partitions for small k, which are counted by well-known sequences, as well as the orthogonal polynomials related to k-distant noncrossing matchings and partitions. We extend Chen et al.'s r-crossings and enhanced r-crossings.

Résumé. Nous définissons les notions de croisements et imbrications k-distants sur les appariements et les partitions d'ensemble, qui sont une variation sur les notions usuelles prenant en compte la distance entre les sommets. En modifiant une involution de Kasraoui et Zeng (Electronic J. Combinatorics 2006, research paper 33), nous montrons que la distribution jointe des croisements et imbrications k-distants est symétrique. Nous étudions le nombre d'involutions et de partitions sans croisement k-distant pour de petites valeurs de k, qui sont des suites d'entiers bien connues, ainsi que les polynômes orthogonaux qui leur sont reliés. Nous étendons les notions de r-croisements et r-croisements amliorés dues à Chen et al.

Keywords: crossings, nestings, set partitions, matchings

1 Introduction

A *(set) partition* of $[n] = \{1, 2, \ldots, n\}$ is a set of disjoint subsets of $[n]$ whose union is $[n]$. Each element of a partition is called a *block*. We will write a partition as a sequence of blocks, for instance, $\{1, 4, 8\}\{2, 5, 9\}\{3\}\{6, 7\}$. Let Π_n denote the set of partitions of $[n]$.

Let π be a partition of $[n]$. A *vertex* of π is an integer $i \in [n]$. An *edge* of π is a pair (i, j) of vertices satisfying either (1) $i < j$, and i and j are in the same block with no vertex between them in that block, or (2) $i = j$ and the block containing i has no other vertex. Thus when we arrange vertices of $\pi = \{1, 5\}\{2, 4, 9\}\{3\}\{6, 12\}\{7, 10, 11\}\{8\}$, in a line in increasing order and draw edges we get Figure 1.

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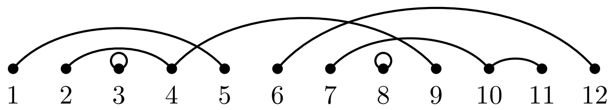

Fig. 1: Diagram for $\{1,5\}\{2,4,9\}\{3\}\{6,12\}\{7,10,11\}\{8\}$.

A vertex v of π is called an *opener* (resp. *closer*) if v is the smallest (resp. largest) element of a block consisting of at least two integers. A vertex v is called a *singleton* if v itself makes a block. A vertex v is called a *transient* if there are two edges connected to v. Let $\mathcal{O}(\pi)$ (resp. $\mathcal{C}(\pi)$, $\mathcal{S}(\pi)$, $\mathcal{T}(\pi)$) be the set of openers (resp. closers, singletons, transients) of π. Let $\mathrm{type}(\pi) = (\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi), \mathcal{T}(\pi))$ and $\mathrm{type}'(\pi) = (\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi) \cup \mathcal{T}(\pi))$. For the partition in Figure 1, the type of π is $\mathrm{type}(\pi) = (\{1,2,6,7\}, \{5,9,11,12\}, \{3,8\}, \{4,10\})$.

A *(complete) matching* is a partition without singletons or transients; this is the same thing as a partition in which all blocks have size 2.

Now we can define the main object of our study.

Definition. Let k be a nonnegative integer. A *k-distant crossing* of π is a pair of edges (i_1, j_1) and (i_2, j_2) of π satisfying $i_1 < i_2 \le j_1 < j_2$ and $j_1 - i_2 \ge k$. A *k-distant nesting* of π is a set of two edges (i_1, j_1) and (i_2, j_2) of π satisfying $i_1 < i_2 \le j_2 < j_1$ and $j_2 - i_2 \ge k$.

Let $\mathrm{dcr}_k(\pi)$ (resp. $\mathrm{dne}_k(\pi)$) denote the number of k-distant crossings (k-distant nestings) in π. Thus $\mathrm{dcr}_1(\pi)$ is the number of usual crossings of π.

For example, in the partition in Figure 1, the edges $(4,9)$ and $(6,12)$ form a 3-distant crossing (as well as an i-distant crossing for $i = 0,1,2$), the edges $(1,5)$ and $(2,4)$ form a 2-distant nesting, the edges $(2,4)$ and $(4,9)$ form a 0-distant crossing, and the edges $(7,10)$ and $(8,8)$ form a 0-distant nesting. That partition has $\mathrm{dcr}_0(\pi) = 5$, $\mathrm{dcr}_2(\pi) = 2$, and $\mathrm{dne}_2(\pi) = 2$.

Kasraoui and Zeng [5] found an involution $\varphi : \Pi_n \to \Pi_n$ such that $\mathrm{type}(\varphi(\pi)) = \mathrm{type}(\pi)$ and $\mathrm{dcr}_1(\varphi(\pi)) = \mathrm{dne}_1(\pi), \mathrm{dne}_1(\varphi(\pi)) = \mathrm{dcr}_1(\pi)$. Modifying this involution, for $k \ge 0$, we find an involution $\varphi_k : \Pi_n \to \Pi_n$ such that $\mathrm{dcr}_k(\varphi_k(\pi)) = \mathrm{dne}_k(\pi)$, $\mathrm{dne}_k(\varphi_k(\pi)) = \mathrm{dcr}_k(\pi)$ and $\mathrm{type}(\varphi_k(\pi)) = \mathrm{type}(\pi)$ if $k \ge 1$; $\mathrm{type}'(\varphi_k(\pi)) = \mathrm{type}'(\pi)$ if $k = 0$.

Noncrossing partitions and matchings are interesting and pervasive objects that arise frequently in diverse areas of mathematics; see [10] and [11] and the references therein for an introduction to noncrossing partitions. A partition π is called *k-distant noncrossing* if π has no k-distant crossing. Let $NCM_k(n)$ denote the number of k-distant noncrossing matchings of $[n]$. Let $NCP_k(n)$ denote the number of k-distant noncrossing partitions of $[n]$.

Table 1 and Table 2 show $NCM_k(n)$ and $NCP_k(n)$ for small values of n and k. We use $k = \infty$ to indicate that i-distant crossing is allowed for any positive integer i, so that $NCM_\infty(n)$ and $NCP_\infty(n)$ equal the total number of matchings of $[2n]$ and partitions of $[n]$, respectively. A matching or partition cannot have a k-distant crossing for $k > n-3$, so for fixed n, $NCM_k(n)$ and $NCP_k(n)$ will "converge" to the number of matchings and number of partitions, respectively; for readability we omit those numbers in the tables. The $n = 0$ column is all 1's for both tables, of course.

It is well known that noncrossing matchings of $[2n]$ and noncrossing partitions of $[n]$ are counted by the Catalan number C_n. Thus $NCM_0(2n) = NCM_1(2n) = NCP_1(n) = C_n$. We will show that $NCM_2(2n) = s_n$ and $NCP_0(n) = M_n$, where s_n and M_n are the little Schröder numbers (A001003 in [12]) and the Motzkin numbers (A001006 in [12]) respectively. We will also find the generating functions

$k \backslash n$	2	4	6	8	10	12	14	16	18	20
1	1	2	5	14	42	132	429	1430	4862	16796
2		3	11	45	197	903	4279	20793	103049	518859
3			14	71	387	2210	13053	79081	488728	3069007
4			15	91	581	3906	27189	194240	1416168	10494328
5				102	753	5752	45636	372360	3101523	26266917
6				105	873	7541	66690	607128	5657520	53631564
7					930	8985	88450	885394	9067611	94719138
8					945	9885	107847	1187376	13233511	150234570
9						10290	122115	1476948	17933348	219754737
10						10395	130515	1715475	22701570	300724081
11							134190	1881495	26969370	386669322
12							135135	1975995	30306045	468680940
13								2016630	32546745	538581120
14								2027025	33794145	591287445
15									34324290	625810185
16									34459425	652702050
17										644729085
18										654729075
∞	1	3	15	105	945	10395	135135	2027025	34459425	654729075

Tab. 1: k-distant noncrossing matchings. The $k = 0$ row is omitted because, as matchings have no transient vertices, the $k = 0$ row is the same as $k = 1$ row; both, of course, are counted by the Catalan numbers (A000108). The $k = 2$ row is the little Schröder numbers (A001003).

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	4	9	21	51	127	323	835	2188	5798	15511
1			5	14	42	132	429	1430	4862	16796	58786	208012
2				15	51	188	731	2950	12235	51822	223191	974427
3					52	201	841	3726	17213	82047	400600	1993377
4						203	872	4037	19796	101437	537691	2926663
5							877	4125	20802	110950	618777	3575688
6								4140	21095	114663	657698	3943294
7									21147	115772	673019	4118232
8										115975	677693	4187838
9											678570	4209457
10												4213597
∞	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597

Tab. 2: k-distant noncrossing partitions. The $k = 0, 1$, and 2 rows are counted by Motzkin numbers (A001006), the Catalan numbers, and A007317, respectively.

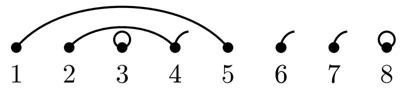

Fig. 2: The 8-th trace $T_8(\pi)$ of π in Figure 1. The vacant vertices are $4, 6$ and 7.

for $NCP_2(n)$ and $NCM_3(2n)$.

Throughout this paper we will frequently refer to sequences in the Online Encyclopedia of Integer Sequences [12] using their "A number"; we will usually omit the citation to [12] and consider it understood that things like "A000108" are a reference to the corresponding sequence in the OEIS.

The rest of this paper is organized as follows. In section 2, we modify Kasraoui and Zeng's involution to prove the joint distribution of k-distant crossings and nestings is symmetric. In section 3, we review a bijection between partitions and Charlier diagrams. In section 4 and section 5, we study the number of k-distant noncrossing matchings and partitions, and, in section 6, we consider the orthogonal polynomials related to these numbers. In section 7, we extend r-crossings and enhanced r-crossings of Chen et al. [1].

2 Modification of the involution of Kasraoui and Zeng

Kasraoui and Zeng [5] found an involution $\varphi : \Pi_n \to \Pi_n$ such that $\mathrm{dcr}_1(\varphi(\pi)) = \mathrm{dne}_1(\pi)$, $\mathrm{dne}_1(\varphi(\pi)) = \mathrm{dcr}_1(\pi)$ and $\mathrm{type}(\varphi(\pi)) = \mathrm{type}(\pi)$. In this section, for fixed $k \geq 0$, we find an involution $\varphi_k : \Pi_n \to \Pi_n$ such that $\mathrm{dcr}_k(\varphi_k(\pi)) = \mathrm{dne}_k(\pi)$ and $\mathrm{dne}_k(\varphi_k(\pi)) = \mathrm{dcr}_k(\pi)$.

We will follow Kasraoui and Zeng's notations. We will identify a partition π to its diagram as shown in Figure 1.

The *i-th trace* $T_i(\pi)$ of π is the diagram obtained from π by removing vertices greater than i. If a vertex $v \leq i$ is connected to $u > i$ in π then make a *half edge* from v in $T_i(\pi)$. Each vertex with a half edge is called *vacant* vertex. For an example, see Figure 2.

Let k be a fixed nonnegative integer. We define $\varphi_k : \Pi_n \to \Pi_n$ as follows.

1. Set $T_0^{(k)} = \emptyset$.

2. For $1 \leq i \leq n$, $T_i^{(k)}$ is obtained as follows.

   (a) Let $T_i^{(k)}$ (resp. $T_i'(\pi)$) be $T_{i-1}^{(k)}$ (resp. $T_{i-1}(\pi)$) with new vertex i.

   (b) If $i \in \mathcal{O}(\pi) \cup \mathcal{S}(\pi) \cup \mathcal{T}(\pi)$, then make a half edge from i both in $T_i^{(k)}$ and $T_i'(\pi)$.

   (c) If $i \in \mathcal{C}(\pi) \cup \mathcal{S}(\pi) \cup \mathcal{T}(\pi)$, let j be the vertex connected to i in π.

      i. If $i - j < k$, then j must be a vacant vertex in $T_i^{(k)}$. Remove the half edge from j and add an edge (i, j) in $T_i^{(k)}$.

      ii. If $i - j \geq k$, then let U (resp. V) be the set of all vacant vertices v in $T_i^{(k)}$ (resp. $T_i'(\pi)$) such that $i - v \geq k$. Let $\gamma_i^{(k)}(\pi)$ denote the integer r such that j is the r-th largest element of V. Let j' be the $\gamma_i^{(k)}(\pi)$-th smallest element of U. Remove the half edge from j' and add an edge (j', i) in $T_i^{(k)}$.

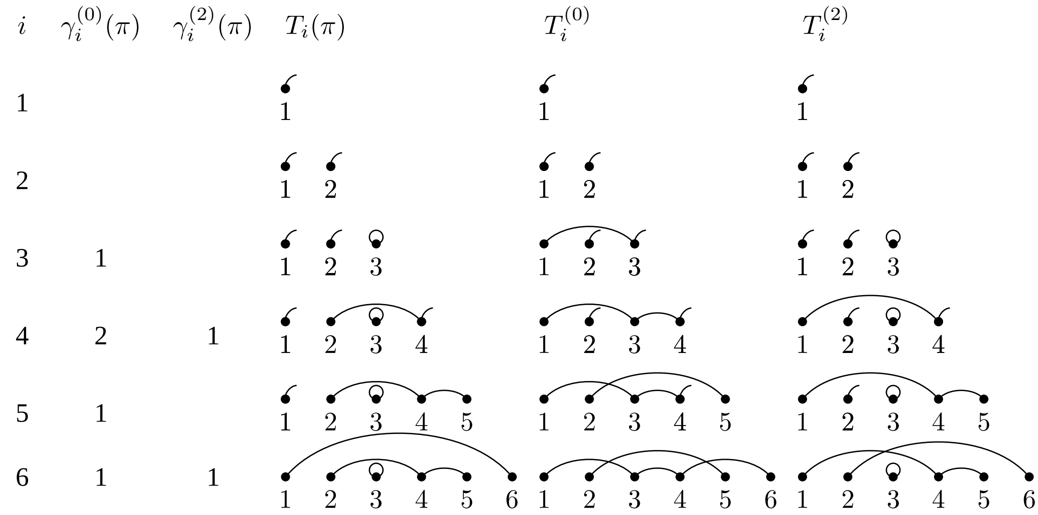

Fig. 3: Construction of $\varphi_0(\pi) = T_6^{(0)}$ and $\varphi_2(\pi) = T_6^{(2)}$ for $\pi = \{1, 6\}\{2, 4, 5\}\{3\}$.

3. Set $\varphi_k(\pi) = T_n^{(k)}$.

For example, see Figure 3. Using the same argument as in [5], we can prove that φ_k is an involution and satisfies $\mathrm{dcr}_k(\varphi_k(\pi)) = \mathrm{dne}_k(\pi)$, $\mathrm{dne}_k(\varphi_k(\pi)) = \mathrm{dcr}_k(\pi)$, $\mathrm{type}(\varphi_k(\pi)) = \mathrm{type}(\pi)$ if $k \geq 1$; $\mathrm{type}'(\varphi_k(\pi)) = \mathrm{type}'(\pi)$ if $k = 0$. Thus we have the following.

Theorem 2.1. *Let k be a nonnegative integer. Then*

$$\sum_{\pi \in \Pi_n} x^{\mathrm{dcr}_k(\pi)} y^{\mathrm{dne}_k(\pi)} = \sum_{\pi \in \Pi_n} x^{\mathrm{dne}_k(\pi)} y^{\mathrm{dcr}_k(\pi)}.$$

3 Motzkin paths and Charlier diagrams

In this section, we recall a bijection between partitions and Charlier diagrams [4, 5].

A *step* is a pair (p, q) of points $p, q \in \mathbb{Z} \times \mathbb{Z}$. The *height* of a step (p, q) is the second component of p, i.e, if $p = (a, b)$ then the height of the step (p, q) is b. A step (p, q) is called an *up* (resp. *down*, *horizontal*) step if the component-wise difference $q - p$ is $(1, 1)$ (resp. $(1, -1)$, $(1, 0)$). A *path* of length n is a sequence $(p_0, p_1, p_2, \ldots, p_n)$ of $n + 1$ points in $\mathbb{Z} \times \mathbb{Z}$. The *$i$-th step* of a path $(p_0, p_1, p_2, \ldots, p_n)$ is (p_{i-1}, p_i). A *nonnegative path* of length n is a path from $(0, 0)$ to $(n, 0)$, which never goes below the x-axis. A *Motzkin path* of length n is a nonnegative path of length n consisting of up steps, down steps and horizontal steps. A *Charlier diagram* of length n is a pair (M, e) where $M = (p_0, p_1, \ldots, p_n)$ is a Motzkin path of length n and $e = (e_1, e_2, \ldots, e_n)$ is a sequence of integers such that

1. if the i-th step is an up step then $e_i = 0$,
2. if the i-th step is a down step of height h then $1 \leq e_i \leq h$,
3. if the i-th step is a horizontal step of height h then $0 \leq e_i \leq h$.

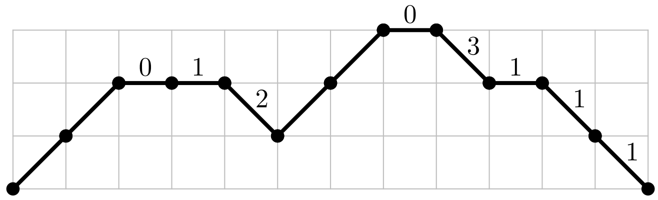

Fig. 4: The Charlier diagram for the partition of Figure 1. The label e_i is written above the horizontal and down steps.

We will identify a Charlier diagram (M, e) with the sequence $(s_1, s_2, \dots, s_n)$ of labeled letters in $\{U, D_1, D_2, \dots, H_0, H_1, H_2, \dots\}$ such that $s_i = U$ (resp. $s_i = D_{e_i}$, $s_i = H_{e_i}$) if the i-th step of M is an up (resp. down, horizontal) step.

Let π be a partition of $[n]$. Recall that in the previous section, if i is a closer or transient, then $\gamma_i^{(1)}(\pi)$ is the integer r such that i is connected to the r-th largest integer in $T_{i-1}^{(1)}(\pi)$.

The corresponding Charlier diagram $\mathrm{Ch}(\pi) = (s_1, s_2, \dots, s_n)$ is defined as follows:

1. if i is an opener in π then $s_i = U$,
2. if i is a closer in π and $\gamma_i^{(1)}(\pi) = r$ then $s_i = D_r$,
3. if i is a singleton in π then $s_i = H_0$,
4. if i is a transient in π and $\gamma_i^{(1)}(\pi) = r$ then $s_i = H_r$.

For example, see Figure 4.

It is easy to see that if there is a step D_ℓ or H_ℓ with $\ell \geq 2$ in $\mathrm{Ch}(\pi)$, than π has an $(\ell - 1)$-distant crossing.

4 k-distant noncrossing matchings

In this section we will find the number of k-distant noncrossing matchings for $k = 0, 1, 2$ and 3. Note that since there is no matching of $[2n+1]$ we have $NCM_k(2n+1) = 0$ for all n and k. Thus we will only consider $NCM_k(2n)$.

4.1 0- and 1-distant noncrossing matchings

Since matchings have no transient vertices, being 0-distant crossing is equivalent to being 1-distant crossing.

We can easily see that a matching π is 1-distant noncrossing if and only if $\mathrm{Ch}(\pi)$ consists of U and D_1. Thus a 1-distant noncrossing matching corresponds to a Dyck path.

Theorem 4.1. *We have*

$$NCM_0(2n) = NCM_1(2n) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

4.2 2-distant noncrossing matchings

Let π be a 2-distant noncrossing matching. Then $\mathrm{Ch}(\pi)$ consists of U, D_1, and D_2. By definition of $\mathrm{Ch}(\pi)$, D_2 is of height at least 2. Moreover, since π has no 2-distant crossing, D_2 must immediately follow U. Thus we can consider $\mathrm{Ch}(\pi)$ as a nonnegative path consisting of the three steps $U = (1, 1)$, $D_1 = (1, -1)$ and $UD_2 = (2, 0)$ such that UD_2 never touches the x-axis. This is exactly the definition of a *little Schröder path*, see [13]. Thus we get the following theorem.

Theorem 4.2. *We have*

$$NCM_2(2n) = s_n,$$

where s_n is the little Schröder number (A001003).

4.3 3-distant noncrossing matchings

Let π be a 3-distant noncrossing matching. One can check that $\mathrm{Ch}(\pi)$ consists of U, D_1, D_2, and D_3 satisfying the following.

1. D_ℓ is of height at least ℓ for $\ell = 1, 2, 3$.
2. D_3 can only occur after two consecutive U, and
3. D_2 can only occur after U or after either D_2 or D_3 which follows U.

Thus we can consider $\mathrm{Ch}(\pi)$ as a nonnegative path consisting of the 6 steps U, D_1, UD_2, UUD_3, UD_2D_2, UUD_3D_2 such that the last four steps must be above the line $y = 1$. Let $g(n)$ be the number of nonnegative paths of length n consisting of $U, D_1, UD_2, UUD_3, UD_2D_2$, and UUD_3D_2. Let $F(x) = \sum_{n\geq 0} NCM_3(2n)x^n$ and $G(x) = \sum_{n\geq 0} g(2n)x^n$.

Decomposing nonnegative paths, we get that $G(x) = 1 + (x + x^2)G(x) + (x^{\frac{1}{2}} + x^{\frac{3}{2}})^2 G(x)^2$ and $F(x) = 1 + xG(x)F(x)$. Thus we get

$$G(x) = \frac{1 - x - x^2 - \sqrt{1 - 6x - 9x^2 - 2x^3 + x^4}}{2x(x + 1)^2}.$$

Now we get the generating function for $NCM_3(2n)$.

Theorem 4.3. *We have*

$$\begin{aligned} \sum_{n\geq 0} NCM_3(2n)x^n &= \frac{2(x + 1)^2}{1 + 5x + 3x^2 + \sqrt{1 - 6x - 9x^2 - 2x^3 + x^4}} \\ &= 1 + x + 3\,x^2 + 14\,x^3 + 71\,x^4 + 387\,x^5 + 2210\,x^6 + 13053\,x^7 + \cdots . \end{aligned}$$

5 k-distant noncrossing partitions

5.1 0-distant noncrossing partitions

Let π be a 0-distant noncrossing partition. Then $\mathrm{Ch}(\pi)$ consists of U, D_1, H_0. Thus $\mathrm{Ch}(\pi)$ is a Motzkin path.

Theorem 5.1. *The number of 0-distant noncrossing partitions of $[n]$ is equal to the number of Motzkin paths of length $[n]$ (A001006).*

5.2 1-distant noncrossing partitions

Let π be a 1-distant noncrossing partition. Then π is a usual noncrossing partition. It is well known that the number of noncrossing partitions of $[n]$ is the Catalan number C_n.

Theorem 5.2. *We have*

$$NCP_1(n) = C_n.$$

5.3 2-distant noncrossing partitions

Let π be a 2-distant noncrossing partition. Then $\mathrm{Ch}(\pi)$ consists of U, D_1, D_2, H_0, H_1, and H_2 and satisfies

1. D_ℓ and H_ℓ are of height at least ℓ,
2. H_2 and D_2 can only occur after U, H_1, or H_2.

Thus we can consider $\mathrm{Ch}(\pi)$ as a nonnegative path with the following steps:

$$UH_2^k, UH_2^kD_2, H_1H_2^k, H_1H_2^kD_2, H_0, \text{and } D_1,$$

where k is a nonnegative integer and H_2^k means k consecutive H_2 steps.

Let $a(n)$ (resp. $b(n)$) denote the number of nonnegative paths of length n consisting of the above steps such that D_ℓ and H_ℓ is of height at least $\ell - 2$ (resp. at least $\ell - 1$). In fact, the height condition is unnecessary for $a(n)$ since every step is of height at least 0. Let $F(x) = \sum_{n\geq 0} NCP_2(n)x^n$, $A(x) = \sum_{n\geq 0} a(n)x^n$, and $B(x) = \sum_{n\geq 0} b(n)x^n$.

Note that the steps which increase the y-coordinate by 1 are

$$UH_2^k, \quad k \geq 0,$$

the steps which do not change the y-coordinate are

$$H_0, H_1H_2^k, UH_2^kD_2, \quad k \geq 0,$$

and the steps which decrease the y-coordinate by 1 are

$$D_1, H_1H_2^kD_2, \quad k \geq 0.$$

Thus, by decomposing nonnegative paths, we get

$$\begin{aligned}
A(x) &= 1 + \left(x + \frac{x}{1-x} + \frac{x^2}{1-x}\right) A(x) + \frac{x}{1-x} \cdot \left(x + \frac{x^2}{1-x}\right) A(x)^2,\\
B(x) &= 1 + \left(2x + \frac{x^2}{1-x}\right) B(x) + \frac{x}{1-x} \cdot \left(x + \frac{x^2}{1-x}\right) A(x)B(x),\\
F(x) &= 1 + xF(x) + x^2B(x)F(x).
\end{aligned}$$

Solving these equations, we get the following theorem.

Theorem 5.3. *We have*

$$\sum_{n\geq 0} NCP_2(n)x^n = \frac{3-3x-\sqrt{1-6x+5x^2}}{2(1-x)} = \frac{3}{2} - \frac{1}{2}\sqrt{\frac{1-5x}{1-x}}$$
$$= 1 + x + 2\,x^2 + 5\,x^3 + 15\,x^4 + 51\,x^5 + 188\,x^6 + 731\,x^7 + 2950\,x^8 + \cdots .$$

This sequence is A007317. Mansour and Severini [9] proved that the generating function for the number of 12312-avoiding partitions is equal to that in 5.3. Thus the number of 2-distant noncrossing partitions of $[n]$ is equal to the number of 12312-avoiding partitions of $[n]$. Yan [16] found a bijection from 12312-avoiding partitions of $[n]$ to UH-free Schröder paths of length $2n-2$. Composing several bijections including Yan's bijection, Kim [7] found a bijection between 2-distant noncrossing partitions and 12312-avoiding partitions.

6 Orthogonal polynomials

Given a sequence $\{\mu_n\}_{n\geq 0}$, one may try to define a sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ that are orthogonal with respect to $\{\mu_n\}$; that is, if we define a measure with $\mu_n = \int x^n \,\mathrm{d}\mu$, then

$$\int P_n(x)P_m(x)\,\mathrm{d}\mu = 0$$

whenever $n \neq m$. These polynomials must satisfy a three-term recurrence relation of the form

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x), \tag{1}$$

with $P_0(x) = 1$ and $P_1(x) = x - b_0$. Viennot showed [14, 15] that for any sequence $\{\mu_n\}$—which are called the *moments*—one can interpret the moment μ_n as the generating function for weighted Motzkin paths of length n in which up steps have weight 1, horizontal steps of height k have weight b_k, and down steps of height k have weight λ_k; then the polynomials in (1) will be orthogonal with respect to $\{\mu_n\}_{n\geq 0}$.

Many classical combinatorial sequences have been interpreted as the moment sequences for a set of orthogonal polynomials, and the corresponding orthogonality relation proved with a sign-reversing involution. In particular, it is known that:

- If $\mu_{2n+1} = 0$ and $\mu_{2n} = C_n$, the Catalan number, then $b_n = 0$ and $\lambda_n = 1$; the corresponding polynomials are Chebyshev polynomials of the second kind [2], which may be defined by

  $$U_{n+1}(x) = xU_n(x) - U_{n-1}(x),$$

  with $U_0(x) = 1$ and $U_1(x) = x$. These moments are $NCM_0(n)$.

- If $\mu_{2n+1} = 0$ and $\mu_{2n} = (2n-1)!!$, then $b_n = 0$ and $\lambda_n = n$; the corresponding polynomials are Hermite polynomials [14]. These moments are $NCM_\infty(n)$.

- If $\mu_n = M_n$, the n-th Motzkin number, then $b_n = 1$, $\lambda_n = 1$; the corresponding polynomials are shifted Chebyshev polynomials of the second kind: $U_n(x-1)$. See [3, section 4.1]. These moments are $NCP_0(n)$.

- If $\mu_n = B_n$, the number of partitions of $[n]$, then $b_n = n+1$ and $\lambda_n = n$; the corresponding polynomials are Charlier polynomials (with $a = 1$) [14]. These moments are $NCP_\infty(n)$.

With these observations, it is natural to try to use, say, $NCM_k(n)$ as a sequence of moments. Letting k go from 0 to infinity would then allow us to interpolate between Chebyshev polynomials and Hermite polynomials; using $NCP_k(n)$ would give the corresponding interpolation between shifted Chebyshev and Charlier polynomials.

If we use $NCM_2(n)$ for the moments, then we have $b_n = 0$, $\lambda_{2n+1} = 1$, and $\lambda_{2n} = 2$. This follows from the work of Kim and Zeng [6]: use $U_n(x,2)$ in their paper. In their paper, they derive formulas for the moments of $U_n(x,2)$ which are the same as known formulas for $NCM_2(n)$, which are the little Schröder numbers.

If we attempt to do the same with $NCM_3(n)$, we get stuck: since $NCM_3(2n+1) = 0$, we know that $b_n = 0$, but the λ_n sequence starts with

$$1, 2, \frac{5}{2}, \frac{3}{10}, \frac{76}{5}, -\frac{680}{57}, -\frac{2311}{7752}, \frac{1246001}{314296}, \frac{114710016}{151553069}, \dots. \tag{2}$$

Not only are some λ_n's fractions, but some are negative, which means prospects for polynomials with nice combinatorics are dim.

Let us try the same line of attack with k-distant noncrossing partitions. Using $NCP_1(n)$—Catalan numbers—for a set of moments, we get a shifted version of Chebyshev polynomials of the second kind: $b_0 = 1$, all other $b_n = 2$, and all $\lambda_n = 1$. These polynomials can be written $U_n(x-2)$, with slightly different initial conditions: $U_0(x) = 1$ and $U_1(x) = x - 1$. The easiest way to see why these recurrence coefficients and initial conditions are orthogonal with respect to the Catalan numbers is with a bijection between Motzkin paths of length n with the above weighting and Dyck paths of length $2n$: take each up step U and make it UU, take each down step D and make it DD, and take each horizontal step H and make it either UD or DU—except for the horizontal step at height zero, which can only be made into UD. This process turns a weighted Motzkin path of length n into a Dyck path of length $2n$ and is easily shown to be a bijection.

When using $NCP_2(n)$ and $NCP_3(n)$ as the moments, we again get some fractional coefficients, but they seem much nicer. We have computed the following with Maple: if $\mu_n = NCP_2(n)$ then

$$\{b_n\}_{n\geq 0} = \left\{1, 3-1, 3-\frac{1}{2}, 3-\frac{1}{10}, 3-\frac{1}{65}, 3-\frac{1}{442}, 3-\frac{1}{3026}, \dots\right\} \text{ and}$$
$$\{\lambda_n\}_{n\geq 1} = \left\{1, 1+1, 1+\frac{1}{4}, 1+\frac{1}{25}, 1+\frac{1}{169}, 1+\frac{1}{1156}, 1+\frac{1}{7921}, \dots\right\};$$

if $\mu_n = NCP_3(n)$ then

$$\{b_n\}_{n\geq 0} = \{1,2,3,3,3,\dots\} \quad \text{and} \quad \{\lambda_n\}_{n\geq 1} = \{1,2,2,2,2,\dots\}.$$

The first case is very interesting. The sequences of denominators of b_n's and λ_n's appear in A064170 and A081068 respectively. Based on the above evidence, we make the following conjectures.

Conjecture 6.1. *If $\mu_n = NCP_2(n)$ then $b_0 = 1, b_1 = 2, \lambda_1 = 1$, and for $n \geq 2$*

$$b_n = 3 - \frac{1}{F_{2n-1}F_{2n-3}} \quad \textit{and} \quad \lambda_n = 1 + \frac{1}{(F_{2n-3})^2},$$

where F_n is the n-th Fibonacci number, i.e., $F_{n+1} = F_n + F_{n-1}$ and $F_1 = F_2 = 1$.

Conjecture 6.2. *If $\mu_n = NCP_3(n)$ then $b_0 = 1, b_1 = 2, b_2 = 3, \lambda_1 = 1, \lambda_2 = 2$, and, for $n \geq 3$, $b_n = 3$ and $\lambda_n = 2$.*

7 k-distant r-crossing

Chen et al. [1] considered a different kind of crossing number. Our definition of k-distant crossing can be applied to their definition.

Let $k \geq 0$ and $r \geq 2$ be integers. A *k-distant r-crossing* is a set of r edges $(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$ such that $i_1 < i_2 < \cdots < i_r \leq j_1 < j_2 < \cdots < j_r$ and $j_1 - i_r \geq k$. Similarly, a *k-distant r-nesting* is a set of r edges $(i_1, j_1), (i_2, j_2), \ldots, (i_r, j_r)$ such that $i_1 < i_2 < \cdots < i_r \leq j_r < j_{r-1} < \cdots < j_1$ and $j_r - i_r \geq k$. In [1], they defined an r-crossing and an enhanced r-crossing, which are a 1-distant r-crossing and a 0-distant r-crossing respectively.

Let $\mathrm{DCR}_k(\pi)$ (resp. $\mathrm{DNE}_k(\pi)$) be the maximal r such that π has a k-distant r-crossing (resp. k-distant r-nesting). Let $f_{n,S,T}(k; i, j)$ denote the number of partitions π of $[n]$ such that $\mathrm{DCR}_k(\pi) = i$, $\mathrm{DNE}_k(\pi) = j$, $\mathcal{O}(\pi) = S$ and $\mathcal{C}(\pi) = T$. Chen et al. [1] proved that $f_{n,S,T}(k; i, j) = f_{n,S,T}(k; j, i)$ for $k = 0, 1$. Krattenthaler [8] extended this result using growth diagrams.

Using Krattenthaler's growth diagram method, we can get the following theorem.

Theorem 7.1. *Let $n \geq 1$ and $k \geq 0$ be integers. Then*

$$f_{n,S,T}(k; i, j) = f_{n,S,T}(k; j, i).$$

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Permutations realized by shifts

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Abstract. A permutation π is realized by the shift on N symbols if there is an infinite word on an N-letter alphabet whose successive left shifts by one position are lexicographically in the same relative order as π. The set of realized permutations is closed under consecutive pattern containment. Permutations that cannot be realized are called forbidden patterns. It was shown in [J.M. Amigó, S. Elizalde and M. Kennel, *J. Combin. Theory Ser. A* 115 (2008), 485–504] that the shortest forbidden patterns of the shift on N symbols have length $N+2$. In this paper we give a characterization of the set of permutations that are realized by the shift on N symbols, and we enumerate them with respect to their length.

Résumé. Une permutation π est réalisée par le *shift* avec N symboles s'il y a un mot infini sur un alphabet de N lettres dont les déplacements successifs d'une position à gauche sont lexicographiquement dans le même ordre relatif que π. Les permutations qui ne sont pas réalisées s'apellent des motifs interdits. On sait [J.M. Amigó, S. Elizalde and M. Kennel, *J. Combin. Theory Ser. A* 115 (2008), 485–504] que les motifs interdits les plus courts du *shift* avec N symboles ont longueur $N+2$. Dans cet article on donne une caractérisation des permutations réalisées par le *shift* avec N symboles, et on les dénombre selon leur longueur.

Keywords: shift, consecutive pattern, forbidden pattern

1 Introduction and definitions

This paper is motivated by an innovative application of pattern-avoiding permutations to dynamical systems (see (1; 2; 4)), which is based on the following idea. Given a piecewise monotone map on a one-dimensional interval, consider the finite sequences (orbits) that are obtained by iterating the map, starting from any point in the interval. It turns out that the relative order of the entries in these sequences cannot be arbitrary. This means that, for any given such map, there will be some order patterns that will never appear in any orbit. The set of such patterns, which we call forbidden patterns, is closed under consecutive pattern containment. These facts can be used to distinguish random from deterministic time series.

A natural question that arises is how to determine, for a given map, what its forbidden patterns are. While this problem is wide open in general, in the present paper we study it for a particular kind of maps, called (one-sided) shift systems. Shift systems are interesting for two reasons. One one hand, they exhibit all important features of low-dimensional chaos. On the other hand, they are natural maps from a combinatorial perspective, and the study of their forbidden patterns can be done in an elegant combinatorial way.

Forbidden patterns in shift systems were first considered in (1; 2). The authors prove that the smallest forbidden pattern of the shift on N symbols has length $N+2$. They also conjecture that, for any N, there are exactly six forbidden patterns of minimal length. In the present paper we give a complete characterization of forbidden patterns of shift systems, and enumerate them with respect to their length.

We will start with some background on consecutive pattern containment, forbidden patterns in maps, and shift systems. In Section 2 we give a formula for the parameter that determines how many symbols are needed in order for a permutation to be realized by a shift. This characterizes allowed and forbidden patterns of shift maps. In Section 3 we give another equivalent characterization involving a transformation on permutations, and we prove that the shift on N symbols has six forbidden patterns of minimal length $N+2$, as conjectured in (1). In Section 4 we give a formula for the number of patterns of a given length that are realized by the binary shift, and then we generalize it to the shift on N symbols, for arbitrary N. Many of the proofs are omitted in this extended abstract, but they can be found in the full version (5).

1.1 Permutations and consecutive patterns

We denote by $\mathcal{S}_n$ the set of permutations of $\{1,2,\dots,n\}$. If $\pi\in\mathcal{S}_n$, we will write its one-line notation as $\pi=[\pi(1),\pi(2),\dots,\pi(n)]$ (or $\pi=\pi(1)\pi(2)\dots\pi(n)$ if it creates no confusion). The use of square brackets is to distinguish it from the cycle notation, where π is written as a product of cycles of the form $(i,\pi(i),\pi^2(i),\dots,\pi^{k-1}(i))$, with $\pi^k(i)=i$. For example, $\pi=[2,5,1,7,3,6,4]=(1,2,5,3)(4,7)(6)$.

Given a permutation $\pi=\pi(1)\pi(2)\dots\pi(n)$, let $D(\pi)$ denote the *descent set* of π, that is, $D(\pi)=\{i\ :\ 1\le i\le n-1,\ \pi(i)>\pi(i+1)\}$. Let $\mathrm{des}(\pi)=|D(\pi)|$ be the number of descents. The Eulerian polynomials are defined by $A_n(x)=\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{des}(\pi)+1}$. Its coefficients are called the Eulerian numbers. The descent set and the number of descents can be defined for any sequence of integers $a=a_1a_2\dots a_n$ by letting $D(a)=\{i\ :\ 1\le i\le n-1,\ a_i>a_{i+1}\}$.

Let X be a totally ordered set, and let $x_1,\dots,x_n\in X$ with $x_1<x_2<\cdots<x_n$. Any permutation of these values can be expressed as $[x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(n)}]$, where $\pi\in\mathcal{S}_n$. We define its *reduction* to be $\rho([x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(n)}])=[\pi(1),\pi(2),\dots,\pi(n)]=\pi$. Note that the reduction is just a relabeling of the entries with the numbers from 1 to n, keeping the order relationships among them. For example $\rho([4,7,1,6.2,\sqrt{2}])=[3,5,1,4,2]$. If the values $y_1,\dots,y_n$ are not all different, then $\rho([y_1,\dots,y_n])$ is not defined.

Given two permutations $\sigma\in\mathcal{S}_m$, $\pi\in\mathcal{S}_n$, with $m\ge n$, we say that σ *contains* π *as a consecutive pattern* is there is some i such that $\rho([\sigma(i),\sigma(i+1),\dots,\sigma(i+n-1)])=\pi$. Otherwise, we say that σ *avoids* π *as a consecutive pattern*. The set of permutations in $\mathcal{S}_n$ that avoid π as a consecutive pattern is denoted by $\mathrm{Av}_n(\pi)$. We let $\mathrm{Av}(\pi)=\bigcup_{n\ge1}\mathrm{Av}_n(\pi)$. Consecutive pattern containment was first studied in (6), where the sets $\mathrm{Av}_n(\pi)$ are enumerated for certain permutations π.

1.2 Allowed and forbidden patterns in maps

Let f be a map $f:X\to X$. Given $x\in X$ and $n\ge1$, we define

$$\mathrm{Pat}(x,f,n)=\rho([x,f(x),f^2(x),\dots,f^{n-1}(x)]),$$

provided that there is no pair $1\le i<j\le n$ such that $f^{i-1}(x)=f^{j-1}(x)$. If there is such a pair, then $\mathrm{Pat}(x,f,n)$ is not defined. When it is defined, we have $\mathrm{Pat}(x,f,n)\in\mathcal{S}_n$. If $\pi\in\mathcal{S}_n$ and there is some $x\in X$ such that $\mathrm{Pat}(x,f,n)=\pi$, we say that π is *realized* by f (at x), or that π is an *allowed pattern* of f. The set of all permutations realized by f is denoted by $\mathrm{Allow}(f)=\bigcup_{n\ge1}\mathrm{Allow}_n(f)$, where

$$\mathrm{Allow}_n(f)=\{\mathrm{Pat}(x,f,n):x\in X\}\subseteq\mathcal{S}_n.$$

The remaining permutations are called *forbidden patterns*, and denoted by $\operatorname{Forb}(f) = \bigcup_{n\geq 1} \operatorname{Forb}_n(f)$, where $\operatorname{Forb}_n(f) = \mathcal{S}_n \setminus \operatorname{Allow}_n(f)$.

We are introducing some variations to the notation and terminology used in (1; 2; 4). The main change is that our permutation $\pi = \operatorname{Pat}(x, f, n)$ is essentially the inverse of the permutation of $\{0, 1, \ldots, n-1\}$ that the authors of (1) refer to as the *order pattern defined by* x. Our convention, aside from simplifying the notation, will be more convenient from a combinatorial point of view. The advantage is that now the set $\operatorname{Allow}(f)$ is closed under consecutive pattern containment, in the standard sense used in the combinatorics literature, and we no longer need to talk about *outgrowth forbidden patterns* like in (1). Indeed, if $\sigma \in \operatorname{Allow}(f)$ and σ contains τ as a consecutive pattern, then $\tau \in \operatorname{Allow}(f)$. An equivalent statement is that if $\pi \in \operatorname{Forb}(f)$, then $\operatorname{Allow}(f) \subseteq \operatorname{Av}(\pi)$. The minimal elements of $\operatorname{Forb}(f)$, i.e., those permutations in $\operatorname{Forb}(f)$ that avoid all other patterns in $\operatorname{Forb}(f)$, will be called *basic forbidden patterns* of f. The set of these patterns will be denoted $\operatorname{BF}(f)$. Note that basic patterns are the inverses of root patterns as defined in (1).

Let us consider now the case in which X is a closed interval in $\mathbb{R}$, with the usual total order on real numbers. An important incentive to study the set of forbidden patterns of a map comes from the following result, which is a consequence of (4).

Proposition 1.1 *If $I \subset \mathbb{R}$ is a closed interval and $f : I \to I$ is piecewise monotone, then* $\operatorname{Forb}(f) \neq \emptyset$.

Recall that piecewise monotone means that there is a finite partition of I into intervals such that f is continuous and strictly monotone on each of those intervals. It fact, it is shown in (4) that for such a map, the number of allowed patterns of f grows at most exponentially, i.e., there is a constant C such that $|\operatorname{Allow}_n(f)| < C^n$ for n large enough. The value of C is related to the *topological entropy* of f (see (4) for details). Since the growth of the total number of permutations of length n is super-exponential, the above proposition follows.

Proposition 1.1, together with the above observation that $\operatorname{Allow}(f)$ is closed under consecutive pattern containment, provides an interesting connection between dynamical systems on one-dimensional interval maps and pattern avoiding permutations. An important application is that forbidden patterns can be used to distinguish random from deterministic time series. Indeed, in a sequence $(x_1, x_2, x_3, \dots)$ where each x_i has been chosen independently at random from some continuous probability distribution, any pattern $\pi \in \mathcal{S}_n$ appears as $\pi = \rho([x_i, x_{i+1}, \ldots, x_{i+n-1}])$ for some i with nonvanishing probability, and this probability approaches one as the length of the sequence increases. On the other hand, if the sequence has been generated by defining $x_{k+1} = f(x_k)$ for $k \geq 1$, where $f : I \to I$ is a piecewise monotone map, then Proposition 1.1 guarantees that some patterns (in fact, most of them) will never appear in the sequence. The practical aspect of these applications has been considered in (3).

A structural property of the set of allowed patterns of a map is that it is closed under consecutive pattern containment. A new and interesting direction of research is to study more properties of the sets $\operatorname{Allow}(f)$. Some natural problems that arise are the following.

1. Understand how $\operatorname{Allow}(f)$ and $\operatorname{BF}(f)$ depend on the map f.
2. Describe and/or enumerate (exactly or asymptotically) $\operatorname{Allow}(f)$ and $\operatorname{BF}(f)$ for a particular f.
3. Among the sets of patterns Σ such that $\operatorname{Av}_n(\Sigma)$ grows at most exponentially in n (this is a necessary condition), characterize those for which there exists a map f such that $\operatorname{BF}(f) = \Sigma$.
4. Given a map f, determine the length of its smallest forbidden pattern.

Most of this paper is devoted to solving problem 2 for a specific family of maps, that we describe next.

1.3 One-sided shifts

We will concentrate on the set of allowed patterns of certain maps called *one-sided shift maps*, or simply *shifts* for short. For a detailed definition of the associated dynamical system, called the *one-sided shift space*, we refer the reader to (1).

The totally ordered set X considered above will now be the set $\mathcal{W}_N = \{0, 1, \dots, N-1\}^{\mathbb{N}}$ of infinite words on N symbols, equipped with the lexicographic order. Define the (one-sided) shift transformation

$$\begin{array}{rccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1 w_2 w_3 \dots & \mapsto & w_2 w_3 w_4 \dots . \end{array}$$

We will use Σ instead of Σ_N when it creates no confusion.

Given $w \in \mathcal{W}_N$, $n \geq 1$, and $\pi \in \mathcal{S}_n$, we have from the above definition that $\mathrm{Pat}(w, \Sigma, n) = \pi$ if, for all indices $1 \leq i, j \leq n$, $\Sigma^{i-1}(w) < \Sigma^{j-1}(w)$ if and only if $\pi(i) < \pi(j)$. For example,

$$\mathrm{Pat}(2102212210\dots, \Sigma, 7) = [4, 2, 1, 7, 5, 3, 6], \tag{1}$$

because the relative order of the successive shifts is

$$\begin{array}{rc} 2102212210\dots & 4 \\ 102212210\dots & 2 \\ 02212210\dots & 1 \\ 2212210\dots & 7 \\ 212210\dots & 5 \\ 12210\dots & 3 \\ 2210\dots & 6, \end{array}$$

regardless of the entries in place of the dots. The case $N = 1$ is trivial, since the only allowed pattern of Σ_1 is the permutation of length 1. In the rest of the paper, we will assume that $N \geq 2$.

If $x \in \{0, 1, \dots, N-1\}$, we will use the notation $x^\infty = xxx\dots$. If $w \in \mathcal{W}_N$, then w_n denotes the n-th letter of w, and we write $w = w_1 w_2 w_3 \dots$. We will also write $w_{[k,\ell]} = w_k w_{k+1} \dots w_\ell$, and $w_{[k,\infty)} = w_k w_{k+1} \dots$. Note that $w_{[k,\infty)} = \Sigma^{k-1}(w)$.

It is shown in (1) that Σ_N has the same set of forbidden patterns as the so-called *sawtooth map* defined by $x \mapsto Nx \mod 1$ for $x \in [0, 1]$. This map is piecewise linear, and therefore has forbidden patterns by Proposition 1.1. Forbidden patterns of shift systems were first studied in (1), where the main result is the following.

Proposition 1.2 ((1)) *Let $N \geq 2$. We have that*

(a) $\mathrm{Forb}_n(\Sigma_N) = \emptyset$ *for every* $n \leq N + 1$,
(b) $\mathrm{Forb}_n(\Sigma_N) \neq \emptyset$ *for every* $n \geq N + 2$.

Example 1. It can be checked that the smallest forbidden patterns of Σ_4 are $615243, 324156, 342516, 162534, 453621, 435261$.

Recall that a word $w \in \{0, 1, \dots, N-1\}^k$ is *primitive* if it cannot be written as a power of any proper subword, i.e., it is not of the form $w = u^m$ for any $m > 1$, where the exponent indicates concatenation of u with itself m times. Let $\psi_N(k)$ denote the number of primitive words of length k over an N-letter alphabet. It is well known that $\psi_N(k) = \sum_{d|k} \mu(d) N^{k/d}$, where μ denotes the Möbius function.

2 The number of symbols needed to realize a pattern

Given a permutation $\pi \in \mathcal{S}_n$, let $N(\pi)$ be the smallest number N such that $\pi \in \mathrm{Allow}(\Sigma_N)$. The value of $N(\pi)$ indicates what is the minimum number of symbols needed in the alphabet in order for π to be realized by a shift. For example, if $\pi = [4, 2, 1, 7, 5, 3, 6]$, then $N(\pi) \leq 3$ because of equation (1), and it is not hard to see that $N(\pi) = 3$. The main result in this section is a formula for $N(\pi)$.

Theorem 2.1 *Let $n \geq 2$. For any $\pi \in \mathcal{S}_n$, $N(\pi)$ is given by*

$$N(\pi) = 1 + |A(\pi)| + \Delta(\pi), \tag{2}$$

where

$$A(\pi) = \{a \,:\, 1 \leq a \leq n-1 \text{ such that if } i = \pi^{-1}(a),\ j = \pi^{-1}(a+1),\ \text{then } i, j < n \text{ and } \pi(i+1) > \pi(j+1)\},$$

and $\Delta(\pi) = 0$ except in the following three cases, in which $\Delta(\pi) = 1$:

(I) $\pi(n) \notin \{1, n\}$, and if $i = \pi^{-1}(\pi(n) - 1)$, $j = \pi^{-1}(\pi(n) + 1)$, then $\pi(i + 1) > \pi(j + 1)$;
(II) $\pi(n) = 1$ and $\pi(n - 1) = 2$; or
(III) $\pi(n) = n$ and $\pi(n - 1) = n - 1$.

Note that $A(\pi)$ is the set of entries a in the one-line notation of π such that the entry following $a + 1$ is smaller than the entry following a. For example, if $\pi = [4, 3, 6, 1, 5, 2]$, then $A(\pi) = \{3, 4, 5\}$, so Theorem 2.1 says that $N(\pi) = 1 + 3 + 0 = 4$. The following lemma, whose proof is omitted here, will be useful in the proof.

Lemma 2.2 *Suppose that* $\mathrm{Pat}(w, \Sigma, n) = \pi$.

1. *If $1 \leq i, j < n$, $\pi(i) < \pi(j)$, and $\pi(i + 1) > \pi(j + 1)$, then $w_i < w_j$.*
2. *If $1 \leq i < k \leq n$ are such that $|\pi(i) - \pi(k)| = 1$, then the word $w_{[i,k-1]}$ is primitive.*

We will prove Theorem 2.1 in two parts. First we show that $N(\pi) \geq 1 + |A(\pi)| + \Delta(\pi)$ by proving that if $w \in \mathcal{W}_N$ is such that $\mathrm{Pat}(w, \Sigma, n) = \pi$, then necessarily $N \geq 1 + |A(\pi)| + \Delta(\pi)$. This fact is a consequence of the following lemma.

Lemma 2.3 *Suppose that* $\mathrm{Pat}(w, \Sigma, n) = \pi$*, and let $b = \pi(n)$. The entries of w satisfy*

$$w_{\pi^{-1}(1)} \leq w_{\pi^{-1}(2)} \leq \cdots \leq w_{\pi^{-1}(n)}, \tag{3}$$

with strict inequalities $w_{\pi^{-1}(a)} < w_{\pi^{-1}(a+1)}$ for each $a \in A(\pi)$. Additionally, if $\Delta(\pi) = 1$, then in each of the three cases from Theorem 2.1 we have, respectively, that

(I) one of the inequalities $w_{\pi^{-1}(b-1)} \leq w_n \leq w_{\pi^{-1}(b+1)}$ is strict;
(II) $\cdots \leq w_{n+2} \leq w_{n+1} \leq w_n \leq w_{n-1}$ and one of these inequalities is strict;
(III) $w_{n-1} \leq w_n \leq w_{n+1} \leq w_{n+2} \leq \cdots$ and one of these inequalities is strict.

In all cases, the entries of w must satisfy $|A(\pi)| + \Delta(\pi)$ strict inequalities.

Proof: The condition $\mathrm{Pat}(w, \Sigma, n) = \pi$ is equivalent to

$$w_{[\pi^{-1}(1),\infty)} < w_{[\pi^{-1}(2),\infty)} < \cdots < w_{[\pi^{-1}(n),\infty)}, \tag{4}$$

which clearly implies equation (3). If we remove the term w_n from it, we get

$$\begin{cases} \text{(a)} \quad w_{\pi^{-1}(1)} \le w_{\pi^{-1}(2)} \le \cdots \le w_{\pi^{-1}(b-1)} \le w_{\pi^{-1}(b+1)} \le \cdots \le w_{\pi^{-1}(n)} & \text{if } b \notin \{1,n\}, \\ \text{(b)} \quad w_{\pi^{-1}(2)} \le w_{\pi^{-1}(3)} \le \cdots \le w_{\pi^{-1}(n)} & \text{if } b = 1, \\ \text{(c)} \quad w_{\pi^{-1}(1)} \le w_{\pi^{-1}(2)} \le \cdots \le w_{\pi^{-1}(n-1)} & \text{if } b = n. \end{cases} \tag{5}$$

For every $a \in A(\pi)$, the inequality $w_{\pi^{-1}(a)} < w_{\pi^{-1}(a+1)}$ in (5) has to be strict, by Lemma 2.2 with $i = \pi^{-1}(a)$ and $j = \pi^{-1}(a+1)$. Let us now see that in the three cases when $\Delta(\pi) = 1$, an additional strict inequality must be satisfied.

Consider first case (I). Let $i = \pi^{-1}(b-1)$ and $j = \pi^{-1}(b+1)$. Since $\pi(i+1) > \pi(j+1)$, Lemma 2.2 implies that $w_i < w_j$, so the inequality $w_{\pi^{-1}(b-1)} < w_{\pi^{-1}(b+1)}$ (equivalently, $w_i < w_j$) in (5a) has to be strict. In case (II), the leftmost inequality in (4) is $w_{[n,\infty)} < w_{[n-1,\infty)}$. For this to hold, we need $\cdots \le w_{n+2} \le w_{n+1} \le w_n \le w_{n-1}$ and at least one of these inequalities must be strict. Similarly, in case (III), the rightmost inequality in (4) is $w_{[n-1,\infty)} < w_{[n,\infty)}$. This forces $w_{n-1} \le w_n \le w_{n+1} \le w_{n+2} \le \cdots$ with at least one strict inequality. □

We will refer to the $|A(\pi)| + \Delta(\pi)$ strict inequalities in Lemma 2.3 as the *required strict inequalities*. Combined with the weak inequalities from the lemma, they force the number of symbols used in w to be at least $1 + |A(\pi)| + \Delta(\pi)$. Examples 2 and 3 illustrate how this lemma is used.

Now we show that $N(\pi) \le 1 + |A(\pi)| + \Delta(\pi)$. We will show how for any given $\pi \in \mathcal{S}_n$ one can construct a word $w \in \mathcal{W}_N$ with $\mathrm{Pat}(w, \Sigma, n) = \pi$, where $N = 1 + |A(\pi)| + \Delta(\pi)$. We need w to satisfy condition (4). Again, let $b = \pi(n)$.

The first important observation is that, if we can only use N different symbols, then the $|A(\pi)|+\Delta(\pi) = N-1$ required strict inequalities from Lemma 2.3 determine the values of the entries $w_1 w_2 \dots w_{n-1}$. This fact is restated as Corollary 2.9. Consequently, we are forced to assign values to these entries as follows:

(a) If $b \notin \{1, n\}$, assign values to the variables in equation (5a) from left to right, starting with $w_{\pi^{-1}(1)} = 0$ and increasing the value by 1 at each required strict inequality.
(b) If $b = 1$, assign values to the variables in equation (5b) from left to right, starting with $w_{\pi^{-1}(2)} = 0$ if $\pi(n-1) \neq 2$, or with $w_{\pi^{-1}(2)} = 1$ if $\pi(n-1) = 2$ (this is needed in order for condition (II) in Lemma 2.3 to hold), and increasing the value by 1 at each required strict inequality.
(c) If $b = n$, assign values to the variables in equation (5c) from left to right, starting with $w_{\pi^{-1}(1)} = 0$ and increasing the value by 1 at each required strict inequality. (Note that when $\Delta(\pi) = 1$, the last assigned value is $w_{\pi^{-1}(n-1)} = w_{n-1} = |A(\pi)| = N - 2$.)

It remains to assign the values to w_m for $m \ge n$. Before we do this, let us prove some facts about the entries $w_1 \dots w_{n-1}$. In the following two lemmas, whose proof can be found in (5), π is any permutation in $\mathcal{S}_n$ with $N(\pi) = N$ and $w_1 \dots w_{n-1}$ are the values in $\{0, 1, \dots, N-1\}$ assigned above in order to satisfy the required strict inequalities.

Lemma 2.4 *Let $i < n$. If $\pi(i) > \pi(i+1)$, then $w_i \ge 1$. If $\pi(i) < \pi(i+1)$, then $w_i \le N - 2$.*

Lemma 2.5 *If $1 \le i, j < n$ are such that $\pi(i) < \pi(j)$ and $\pi(i+1) > \pi(j+1)$, then $w_i < w_j$.*

Once the values $w_1 \dots w_{n-1}$ have been determined, there are several ways to assign values to w_m for $m \ge n$. Two possibilities are the following.

A. Assume that $b \neq n$. Let $k = \pi^{-1}(b+1)$. Let $u = w_1 w_2 \dots w_{k-1}$ and $p = w_k w_{k+1} \dots w_{n-1}$. Let m be any integer satisfying $m \geq 1 + \frac{n-2}{n-k}$ (for definiteness, we can pick $m = n-1$). Let $w_A(\pi) = up^m 0^\infty$.
B. Assume that $b \neq 1$. Let $k = \pi^{-1}(b-1)$. Let $u = w_1 w_2 \dots w_{k-1}$ and $p = w_k w_{k+1} \dots w_{n-1}$. Again, let m be such that $m \geq 1 + \frac{n-2}{n-k}$ (for definiteness, we can pick $m = n-1$). Let $w_B(\pi) = up^m (N{-}1)^\infty$.

Clearly, $w_A(\pi)$ and $w_B(\pi)$ use N different symbols. It remains to prove that if w is any of these two words, $\mathrm{Pat}(w, \Sigma, n) = \pi$, which is equivalent to showing that w satisfies condition (4). Let us now prove that this is the case for $w = w_A(\pi)$, when $b \neq n$.

In the following two lemmas (see the proof in (5)) and in Proposition 2.8, π is any permutation in $\mathcal{S}_n$ with $\pi(n) \neq n$, and $w = w_A(\pi)$. Also, k, u, p and m are as defined in case A above.

Lemma 2.6 *The word $p = w_k w_{k+1} \dots w_{n-1}$ is primitive and has some nonzero entry.*

Lemma 2.7 *We have that $w_{[n,\infty)} < w_{[k,\infty)}$. Moreover, there is no $1 \leq s \leq n$ such that $w_{[n,\infty)} < w_{[s,\infty)} < w_{[k,\infty)}$.*

Proposition 2.8 *If $1 \leq i, j \leq n$ are such that $\pi(i) < \pi(j)$, then $w_{[i,\infty)} < w_{[j,\infty)}$.*

The above proposition proves that $\mathrm{Pat}(w_A(\pi), \Sigma, n) = \pi$. If $b \neq 1$, proving that $\mathrm{Pat}(w_B(\pi), \Sigma, n) = \pi$ is analogous. We can complete the proof of the upper bound on $N(\pi)$ as follows. Let $\pi \in \mathcal{S}_n$ be given, and let $N = 1 + |A(\pi)| + \Delta(\pi)$. If $\pi(n-1) > \pi(n)$, let $w = w_A(\pi)$. If $\pi(n-1) < \pi(n)$, let $w = w_B(\pi)$. Since $\mathrm{Pat}(w, \Sigma, n) = \pi$ and $w \in \mathcal{W}_N$, the theorem is proved.

Example 2. Let $\pi = [4, 3, 6, 1, 5, 2]$. By Theorem 2.1, $N(\pi) = 4$. If $\mathrm{Pat}(w, \Sigma, n) = \pi$, then Lemma 2.3 implies that $w_4 \leq w_6 \leq w_2 < w_1 < w_5 < w_3$, and there are no more required strict inequalities. We assign $w_4 = w_2 = 0$, $w_1 = 1$, $w_5 = 2$, $w_3 = 3$. Since $\pi(5) > \pi(6)$ and $b = \pi(6) = 2$, we can take $w = w_A(\pi)$ (with $m = 2$), so $k = \pi^{-1}(3) = 2$, $u = w_1 = 1$, and $p = w_2 w_3 w_4 w_5 = 0302$. We get $w = up^2 0^\infty = 1030203020^\infty$.

The following consequence of the proof of Theorem 2.1 will be used in Section 4.

Corollary 2.9 *Let $\pi \in \mathcal{S}_n$, $N = N(\pi)$, and let $w \in \mathcal{W}_N$ be such that $\mathrm{Pat}(w, \Sigma, n) = \pi$. Then the entries $w_1 w_2 \dots w_{n-1}$ are uniquely determined by π.*

Note that, however, with the conditions of Corollary 2.9, w_n is not always determined. In the case that $\pi(n) \notin \{1, n\}$ and $\Delta(\pi) = 1$, we have two choices for w_n. In general, there is a lot of flexibility in the choice of w_m for $m \geq n$. The choices $w = w_A(\pi)$ and $w = w_B(\pi)$ in the proof of Theorem 2.1 were made to simplify the proof of Proposition 2.8 for all cases at once.

3 An equivalent characterization

We start this section by giving an expression for $N(\pi)$ that is sometimes more convenient to work with than the one in Theorem 2.1. We denote by $\mathcal{C}_n$ the set of permutations in $\mathcal{S}_n$ whose cycle decomposition consists of a unique cycle of length n. Let $\mathcal{T}_n$ be the set of permutations $\pi \in \mathcal{C}_n$ with one distinguished entry $\pi(i)$, for some $1 \leq i \leq n$. We call the elements of $\mathcal{T}_n$ *marked cycles*. We will use the symbol $\star$ to denote the distinguished entry, both in one-line and in cycle notation. Note that it is not necessary to keep track of its value, since it is determined once we know all the remaining entries. For example, $\mathcal{T}_3 = \{[\star, 3, 1], [2, \star, 1], [2, 3, \star], [\star, 1, 2], [3, \star, 2], [3, 1, \star]\}$. Clearly, $|\mathcal{T}_n| = (n-1)! \cdot n = n!$, since there are n ways to choose the distinguished entry.

Define a map $\theta : \mathcal{S}_n \to \mathcal{T}_n$ sending $\pi \mapsto \hat{\pi}$ as follows. For each $1 \leq i \leq n$ with $i \neq \pi(n)$, let $\hat{\pi}(i)$ be the entry immediately to the right of i in the one-line notation of π. For $i = \pi(n)$, let $\hat{\pi}(i) = \star$ be the distinguished entry.

We can also give the following equivalent definition of $\hat{\pi}$. If $\pi = [\pi(1), \pi(2), \ldots, \pi(n)]$, then $\hat{\pi}$ is the permutation with cycle decomposition $(\pi(1), \pi(2), \ldots, \pi(n))$ with the entry $\pi(1)$ distinguished. We write $\hat{\pi} = (\star, \pi(2), \ldots, \pi(n))$. For example, if $\pi = [8, 9, 2, 3, 6, 4, 1, 5, 7]$, then $\hat{\pi} = (\star, 9, 2, 3, 6, 4, 1, 5, 7)$, or in one-line notation, $\hat{\pi} = [5, 3, 6, 1, 7, 4, \star, 9, 2]$.

The map θ is a bijection between $\mathcal{S}_n$ and $\mathcal{T}_n$, since it is clearly invertible. Indeed, to recover π from $\hat{\pi} \in \mathcal{T}_n$, write $\hat{\pi}$ in cycle notation, replace the $\star$ with the entry in $\{1, \ldots, n\}$ that is missing, and turn the parentheses into brackets, thus recovering the one-line notation of π.

For $\hat{\pi} \in \mathcal{T}_n$, let $\operatorname{des}(\hat{\pi})$ denote the number of descents of the sequence that we get by deleting the $\star$ from the one-line notation of $\hat{\pi}$. That is, if $\hat{\pi} = [a_1, \ldots, a_j, \star, a_{j+1}, \ldots, a_{n-1}]$, then $\operatorname{des}(\hat{\pi}) = |\{i : 1 \leq i \leq n-2,\ a_i > a_{i+1}\}|$. We can now state a simpler formula for $N(\pi)$.

Proposition 3.1 *Let* $\pi \in \mathcal{S}_n$, $\hat{\pi} = \theta(\pi)$. *Then* $N(\pi)$ *is given by*

$$N(\pi) = 1 + \operatorname{des}(\hat{\pi}) + \epsilon(\hat{\pi}),$$

where

$$\epsilon(\hat{\pi}) = \begin{cases} 1 & \text{if } \hat{\pi} = [\star, 1, \ldots] \text{ or } \hat{\pi} = [\ldots, n, \star], \\ 0 & \text{otherwise.} \end{cases}$$

For example, if $\pi = [8, 9, 2, 3, 6, 4, 1, 5, 7]$, then $\hat{\pi} = [5, 3, 6, 1, 7, 4, \star, 9, 2]$ has 4 descents, so $N(\pi) = 1 + 4 + 0 = 5$. If $\pi = [8, 9, 3, 1, 4, 6, 2, 7, 5]$, then $\hat{\pi} = [4, 7, 1, 6, \star, 2, 5, 9, 3]$ has 3 descents, so $N(\pi) = 1 + 3 + 0 = 4$. If $\pi = [3, 4, 2, 1]$, then $\hat{\pi} = [\star, 1, 4, 2]$ has 1 descent, so $N(\pi) = 1 + 1 + 1 = 3$.

If $\pi \in \mathcal{S}_n$, we have by definition that $N(\pi) = \min\{N : \pi \notin \operatorname{Forb}_n(\Sigma_N)\} = \min\{N : \pi \in \operatorname{Allow}_n(\Sigma_N)\}$. As a consequence of Proposition 3.1 we recover Proposition 1.2, which in terms of the statistic $N(\pi)$ can be reformulated as follows.

Corollary 3.2 *Let* $n \geq 3$. *We have that*

(a) for every $\pi \in \mathcal{S}_n$, $N(\pi) \leq n-1$*;*
(b) there is some $\pi \in \mathcal{S}_n$ *such that* $N(\pi) = n-1$.

We define $\mathcal{S}_{n,N} = \{\pi \in \mathcal{S}_n : N(\pi) = N\}$. We are interested in the numbers $a_{n,N} = |\mathcal{S}_{n,N}|$. To avoid the trivial cases, we will assume that $n, N \geq 2$. From the definitions, $\operatorname{Allow}_n(\Sigma_M) = \bigcup_{N=2}^{M} \mathcal{S}_{n,N}$, $\operatorname{Forb}_n(\Sigma_M) = \bigcup_{N=M+1}^{n-1} \mathcal{S}_{n,N}$. Since the sets $\mathcal{S}_{n,N}$ are disjoint, we have that

$$|\operatorname{Allow}_n(\Sigma_M)| = \sum_{N=2}^{M} a_{n,N}, \qquad |\operatorname{Forb}_n(\Sigma_M)| = \sum_{N=M+1}^{n-1} a_{n,N}.$$

The first few values of $a_{n,N}$ are given in Table 1. By symmetry considerations (5) it follows easily that all the $a_{n,N}$ are even.

$n \backslash N$	2	3	4	5	6	7
2	2					
3	6					
4	18	6				
5	48	66	6			
6	126	402	186	6		
7	306	2028	2232	468	6	
8	738	8790	19426	10212	1098	6

Tab. 1: The numbers $a_{n,N} = |\{\pi \in \mathcal{S}_n : N(\pi) = N\}|$ for $n \leq 8$.

The next result shows that, independently of n, there are exactly six permutations of length n that require the maximum number of symbols (i.e., $n-1$) in order to be realized. This settles a conjecture from (1). Given a permutation $\pi \in \mathcal{S}_n$, we will use π^{rc} to denote the permutation such that $\pi^{rc}(i) = n+1-\pi(n+1-i)$ for $1 \leq i \leq n$. If σ is a marked cycle, then σ^{rc} is defined similarly, where if $\sigma(i)$ is the marked entry of π, then $\sigma^{rc}(n+1-i)$ is the marked entry of σ^{rc}. It will be convenient to visualize $\pi \in \mathcal{S}_n$ as an $n \times n$ array with dots in positions $(i, \pi(i))$, for $1 \leq i \leq n$. The first coordinate refers to the row number, which increases from left to right, and the second coordinate is the column number, which increases from bottom to top. Then, the array of π^{rc} is obtained from the array of π by a 180-degree rotation. Of course, the array of π^{-1} is obtained from the one of π by reflecting it along the diagonal $y = x$. Notice also that the cycle structure of π is preserved in π^{-1} and in π^{rc}. A marked cycle can be visualized in the same way, replacing the dot corresponding to the distinguished element with a $\star$.

Proposition 3.3 *For every $n \geq 3$, $a_{n,n-1} = 6$.*

Proof: First we show that $a_{n,n-1} \geq 6$ by giving six permutations in $\mathcal{S}_{n,n-1}$. Let $m = \lceil n/2 \rceil$, and let

$$\sigma = [n, n-1, \ldots, m+1, \star, m, m-1, \ldots, 2], \quad \tau = [\star, 1, n, n-1 \ldots, m+2, m, m-1, \ldots, 2] \in \mathcal{T}_n$$

(see Figure 1). Using Proposition 3.1, it is easy to check that if $\hat{\pi} \in \{\sigma, \sigma^{rc}, \sigma^{-1}, (\sigma^{-1})^{rc}, \tau, \tau^{rc}\}$, then $N(\pi) = n-1$, and that the six permutations in the set are different.

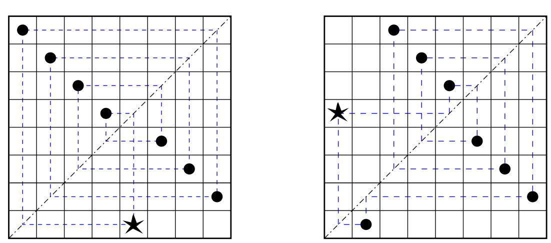

Fig. 1: The arrays of σ and τ for $n = 8$, with dotted lines indicating the cycle structure.

Let us now show that there are no other permutations with $N(\pi) = n-1$. We know by Proposition 3.1 that $N(\pi) = n-1$ can only happen if $\operatorname{des}(\hat{\pi}) = n-2$, or if $\operatorname{des}(\hat{\pi}) = n-3$ and $\epsilon(\hat{\pi}) = 1$.

Case 1: $\mathrm{des}(\hat{\pi}) = n-2$. In this case, all the entries in $\hat{\pi}$ other that the $\star$ must be in decreasing order. If the distinguished entry is neither $\hat{\pi}(1)$ nor $\hat{\pi}(n)$, then the $\star$ must be replacing either 1 or n; otherwise we would have that $\hat{\pi}(1) = n$ and $\hat{\pi}(n) = 1$, so $\hat{\pi}$ would not be an n-cycle. It follows that in the array of $\hat{\pi}$, the entry corresponding to the $\star$ is either in the top or bottom row, or in the leftmost or rightmost column.

If the $\star$ is replacing 1 (i.e, it is is the bottom row of the array), we claim that the only possible n-cycle in which the other entries are in decreasing order is $\hat{\pi} = \sigma$. Indeed, if we consider the cycle structure of $\hat{\pi} = (1, \hat{\pi}(1), \hat{\pi}^2(1), \dots, \hat{\pi}^{n-1}(1))$, we see that $\hat{\pi}(1) = n$ and $\hat{\pi}^2(1) = \hat{\pi}(n) = 2$. Now, $\hat{\pi}^i(1) \neq 1$ for $3 \leq i \leq n-1$, so the decreasing condition on the remaining entries forces $\hat{\pi}^3(1) = \hat{\pi}(2) = n-1$, $\hat{\pi}^4(1) = \hat{\pi}(n-1) = 3$, and so on. A similar argument, considering that rotating the array 180 degrees preserves the cycle structure, shows that if the $\star$ is replacing n (i.e, it is in the top row of the array), then necessarily $\hat{\pi} = \sigma^{rc}$.

If the distinguished entry is $\hat{\pi}(1)$ (i.e, it is in the leftmost column of the array), then a symmetric argument, reflecting the array along $y = x$), shows that $\hat{\pi} = \sigma^{-1}$. Similarly, if the distinguished entry is $\hat{\pi}(n)$ (i.e, it is is the rightmost column of the array), then necessarily $\hat{\pi} = (\sigma^{-1})^{rc}$.

Case 2: $\mathrm{des}(\hat{\pi}) = n-3$ *and* $\epsilon(\hat{\pi}) = 1$. The second condition forces $\hat{\pi} = [\star, 1, \dots]$ or $\hat{\pi} = [\dots, n, \star]$. Let us restrict to the first case (the second one can be argued in a similar way if we rotate the array 180 degrees). We must have $\hat{\pi}(3) > \hat{\pi}(4) > \cdots > \hat{\pi}(n)$. We claim that the only such $\hat{\pi}$ that is also an n-cycle is $\hat{\pi} = \tau$. Indeed, looking at the cycle structure $\hat{\pi} = (\hat{\pi}^{-(n-1)}(1), \dots, \hat{\pi}^{-1}(1), 1)$, we see that $\hat{\pi}^{-1}(1) = 2$. Now, $\hat{\pi}^{-i}(1) \neq 1$ for $2 \leq i \leq n-1$, so the decreasing condition on the remaining entries forces $\hat{\pi}^{-2}(1) = \hat{\pi}^{-1}(2) = n$, $\hat{\pi}^{-3}(1) = \hat{\pi}^{-1}(n) = 3$, $\hat{\pi}^{-4}(1) = \hat{\pi}^{-1}(3) = n-1$, and so on. $\square$

4 The number of allowed patterns of a shift

In the rest of the paper, we will assume for simplicity that $w_A(\pi)$ and $w_B(\pi)$ are defined taking $m = n-1$, so they are of the form $up^{n-1}x^{\infty}$, with $x = 0$ or $x = N-1$ respectively. The following variation of Lemma 2.7 will be useful later.

Lemma 4.1 *Let* $w = up^{n-1}0^{\infty} \in \mathcal{W}_N$*, where* $|u| = k-1$ *and* $|p| = n-k$ *for some* $1 \leq k \leq n-1$*, and* p *is primitive. If* $\pi = \mathrm{Pat}(w, \Sigma, n)$ *is defined, then* $\pi(n) = \pi(k) - 1$.

For $n \geq 2$, the set of patterns of length n that are realized by the shift on two symbols is $\mathrm{Allow}_n(\Sigma_2) = \mathcal{S}_{n,2}$. The next result gives the number of these permutations. Recall that $a_{n,2} = |\mathcal{S}_{n,2}|$ and that $\psi_2(t)$ is the number of primitive binary words of length t.

Theorem 4.2 *For* $n \geq 2$*,*

$$a_{n,2} = \sum_{t=1}^{n-1} \psi_2(t) 2^{n-t-1}.$$

Proof: Fix $n \geq 2$. We will construct a set $W \subset \mathcal{W}_2$ with the following four properties:

(i) for all $w \in W$, $\mathrm{Pat}(w, \Sigma_2, n)$ is defined,
(ii) for all $w, w' \in W$ with $w \neq w'$, we have that $\mathrm{Pat}(w, \Sigma_2, n) \neq \mathrm{Pat}(w', \Sigma_2, n)$,
(iii) for all $\pi \in \mathrm{Allow}_n(\Sigma_2)$, there is a word $w \in W$ such that $\mathrm{Pat}(w, \Sigma_2, n) = \pi$,
(iv) $|W| = \sum_{t=1}^{n-1} \psi_2(t) 2^{n-t-1}$.

Properties (i)-(iii) imply that the map from W to $\mathcal{S}_{n,2}$ sending w to $\mathrm{Pat}(w, \Sigma_2, n)$ is a bijection. Thus, $a_{n,2} = |W|$ and the result will follow from property (iv).

Let

$$W = \bigcup_{t=1}^{n-1} \{up^{n-1}x^\infty : u \in \{0,1\}^{n-t-1}, p \in \{0,1\}^t \text{ is a primitive word, and } x = \bar{p_t}\},$$

where we use the notation $\bar{0} = 1$, $\bar{1} = 0$. Given binary words u, p of lengths $n-t-1$ and t respectively, where p is primitive, and $x = \bar{p_t}$, we will denote $v(u,p) = up^{n-1}x^\infty$.

To see that W satisfies (i), we have to show that for any $w \in W$ and any $1 \le i < j \le n$, we have $w_{[i,\infty)} \ne w_{[j,\infty)}$. This is clear because if $x = 0$ (resp. $x = 1$) both $w_{[i,\infty)}$ and $w_{[j,\infty)}$ end with 10^∞ (resp. 01^∞), with the last 1 (resp. 0) being in different positions in $w_{[i,\infty)}$ and $w_{[j,\infty)}$.

Now we prove that W satisfies (ii). Let u, u' be binary words of lengths $n-t-1$, $n-t'-1$, respectively, and let p, p' be primitive binary words of lengths t, t', respectively. Let $w = v(u,p)$ and $w' = v(u',p')$, and let $\pi = \mathrm{Pat}(w, \Sigma_2, n)$, $\pi' = \mathrm{Pat}(w', \Sigma_2, n)$. We assume that $w \ne w'$, and want to show that $\pi \ne \pi'$. From $w \ne w'$ it follows that $u \ne u'$ or $p \ne p'$.

Corollary 2.9 for $N = 2$ implies that if $w_1 w_2 \dots w_{n-1} \ne w'_1 w'_2 \dots w'_{n-1}$, then $\mathrm{Pat}(w, \Sigma_2, n) \ne \mathrm{Pat}(w', \Sigma_2, n)$. In particular, if $t = t'$, then $up \ne u'p'$, so $\pi \ne \pi'$.

We are left with the case that $t \ne t'$and $up = u'p' = w_1 w_2 \dots w_{n-1}$. Let us first assume that $w_{n-1} = 1$ (and so $p_t = p'_{t'} = 1$). By Lemma 4.1 with $k = n-t$, we have that $\pi(n) = \pi(n-t) - 1$, and similarly $\pi'(n) = \pi'(n-t') - 1$. If we had that $\pi = \pi'$, then $\pi(n) = \pi'(n)$ and so $\pi(n-t) = \pi'(n-t') = \pi(n-t')$. But $t \ne t'$, so this is a contradiction. In the case $w_{n-1} = 0$, an analogous argument to the proof of Lemma 4.1 implies that $w_{[n-t,\infty)} = p^{n-1}1^\infty < p^{n-2}1^\infty = w_{[n,\infty)}$ and there is no s such that $w_{[s,\infty)}$ is strictly in between the two. Thus, $\pi(n) = \pi(n-t) + 1$, and similarly $\pi'(n) = \pi'(n-t') + 1$, so again $\pi \ne \pi'$.

To see that W satisfies (iii) we use the construction from the proof of the upper bound in Theorem 2.1. Let $\pi \in \mathrm{Allow}_n(\Sigma_2)$. If $\pi(n-1) > \pi(n)$, let $w = w_A(\pi) = up^{n-1}0^\infty$. By Lemma 2.4, $w_{n-1} = 1$, so $w \in W$. Similarly, if $\pi(n-1) < \pi(n)$, let $w = w_B(\pi) = up^{n-1}1^\infty$. Then $w_{n-1} = 0$, so $w \in W$. In both cases, $\mathrm{Pat}(w, \Sigma_2, n) = \pi$, so this construction is the inverse of the map $w \mapsto \mathrm{Pat}(w, \Sigma_2, n)$.

To prove (iv), observe that the union in the definition of W is a disjoint union. This is because the value of t determines the position of the last entry in w that is not equal to x. For fixed t, there are 2^{n-t-1} choices for u and $\psi_2(t)$ choices for t, so the formula follows. $\square$

Example 3. For $n = 4$, we have

$$\begin{aligned} W = \quad & \{\underline{00}\,0\,0\,0\,1^\infty, \underline{00}\,1\,1\,1\,0^\infty, \underline{01}\,0\,0\,0\,1^\infty, \underline{01}\,1\,1\,1\,0^\infty, \underline{10}\,0\,0\,0\,1^\infty, \underline{10}\,1\,1\,1\,0^\infty, \underline{11}\,0\,0\,0\,1^\infty, \underline{11}\,1\,1\,1\,0^\infty, \\ & \underline{0}\,01\,01\,01\,0^\infty, \underline{0}\,10\,10\,10\,1^\infty, \underline{1}\,01\,01\,01\,0^\infty, \underline{1}\,10\,10\,10\,1^\infty, \\ & 001\,001\,001\,0^\infty, 010\,010\,010\,1^\infty, 011\,011\,011\,0^\infty, 100\,100\,100\,1^\infty, 101\,101\,101\,0^\infty, 110\,110\,110\,1^\infty\}, \end{aligned}$$

where each word is written as $w = \underline{u}\,p\,p\,p\,x^\infty$. The permutations corresponding to these words are

$$\begin{aligned} \mathrm{Allow}_4(\Sigma_2) = \quad & \{1234, 1243, 3412, 1432, 4123, 2143, 4312, 4321, \\ & 1342, 1324, 4231, 4213, \\ & 2341, 2413, 2431, 3124, 3142, 3214\}. \end{aligned}$$

Theorem 4.2 can be generalized to find a formula for the numbers $a_{n,N}$, which count permutations that can be realized by the shift on N symbols but not by the shift on $N-1$ symbols. The proof of the next result is more involved and is omitted here due to lack of space, but it can be found in (5).

Theorem 4.3 *For any $n, N \geq 2$,*

$$a_{n,N} = \sum_{i=0}^{N-2} (-1)^i \binom{n}{i} \left((N-i-2)(N-i)^{n-2} + \sum_{t=1}^{n-1} \psi_{N-i}(t)(N-i)^{n-t-1} \right). \tag{6}$$

We finish with two curious conjectures that came up while studying forbidden patterns of shift systems. They are derived from experimental evidence, and it would be interesting to find combinatorial proofs.

For the first one, let $\mathcal{T}_n^0$ be the set of n-cycles where one entry has been replaced with 0. The set $\mathcal{T}_n^0$ is essentially the same as $\mathcal{T}_n$, with the only difference that the $\star$ symbol in each element is replaced with a 0, so that it produces a descent if there is an entry to its left. We have checked this conjecture by computer for n up to 9.

Conjecture 4.4 *For any n and any subset $D \subseteq \{1, 2, \ldots, n-1\}$,*

$$|\{\sigma \in \mathcal{T}_n^0 \,:\, D(\sigma) = D\}| = |\{\pi \in \mathcal{S}_n \,:\, D(\pi) = D\}|.$$

In particular, the statistic des *has the same distribution in $\mathcal{T}_n^0$ as in $\mathcal{S}_n$, i.e,*

$$\sum_{\sigma \in \mathcal{T}_n^0} x^{\operatorname{des}(\sigma)+1} = A_n(x).$$

Our last conjecture concerns a divisibility property of the numbers $a_{n,N}$ which is not apparent from Theorem 4.3.

Conjecture 4.5 *For every $n, N \geq 3$, $a_{n,N}$ is divisible by 6.*

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Median clouds and a fast transposition median solver

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Abstract. The median problem seeks a permutation whose total distance to a given set of permutations (the base set) is minimal. This is an important problem in comparative genomics and has been studied for several distance measures such as reversals. The transposition distance is less relevant biologically, but it has been shown that it behaves similarly to the most important biological distances, and can thus give important information on their properties. We have derived an algorithm which solves the transposition median problem, giving all transposition medians (the median cloud). We show that our algorithm can be modified to accept median clouds as elements in the base set and briefly discuss the new concept of median iterates (medians of medians) and limit medians, that is the limit of this iterate.

Résumé. Le problème de la médiane est de trouver une permutation dont la distance totale à un ensemble donné de permutations (l´ensemble de base) est minimale. C'est un problème important en génomique comparative et il a été étudié pour certaines mesures de distance. La distance de transposition n'est pas directement liée à la biologie, mais il a été démontré que son comportement est similaire à celui des distances biologiques essentielles, et elle peut donc donner des indications sur leurs propriétés. Nous construisons un algorithme qui résoud le problème de la médiane pour la transposition, et donne toutes les transpositions médianes (le nuage des médianes). Nous démontrons que notre algorithme peut être modifié pour admettre des nuages de médianes comme éléments de l´ensemble de base et introduisons le concept de médianes itérées (médianes de médianes) et de médianes limites, c-à-d de limites de ces itérations.

Keywords: median, transposition, reversal, DCJ, median cloud

1 Introduction

The median problem in comparative genomics calls for a permutation such that the total distance to a given set S of permutations is minimised. Using the permutations in S as models for some species' genomes, by regarding the genome as a permutation of the genes therein, the median permutation is an approximation of the gene order of these species' closest ancestor. Using median computations, biologists can infer phylogenetic trees, which show how different species are related (8; 2; 7; 1).

The gene order typically changes in a species by **reversals**, where a segment is taken out and inserted backwards at the same place (changing for instance 1234567 to 1543267), **block transpositions**, where a segment is taken out and inserted, possibly backwards, at another place (changing 1234567 to 1456237, for instance), or **Double Cut and Join** (DCJ), which generalise reversals to genomes with several chromosomes, noting that a reversal can be seen as cutting the genome in two places and then putting it together again). Usually one also attaches a sign (+/-) to each gene, changing the sign of every gene in a reversed segment to indicate that the reading directions of these genes have been flipped. Distances are measured in the number of operations (reversals, block transposition, DCJ or combinations) needed to transform one permutation into another. There are also simpler distances, such as the number of elements in one permutation which are followed by different elements in the two permutations under comparison. Such positions are called **breakpoints**.

Depending on which distance measure we use, the median problem may be easy or hard. However, for all these distances, including the simple breakpoint distance, the median problem is NP-hard, see (4; 9; 11; 10) and references in the latter. Thus, variations on this problem which could shed some light on how to simplify it are most welcome.

In this paper, we consider the median problem under the usual transposition distance (exchanging positions of any two elements). While this operation has no relevance in genomic development, the distance function behaves very similarly to the reversal and the DCJ distances for signed genomes (6), which both take the number of genes, subtract the number of cycles and then add some more terms which for most permutations are zero (3). Studying the transposition median would therefore be regarded as a somewhat simpler version of the reversal median and the DCJ median.

We give a branch and bound algorithm which computes the transposition median. This algorithm resembles algorithms for the reversal median (4) and the DCJ median (12; 11) and has a comparable running time. We conjecture that the transposition median problem is NP-hard as well and expect that this can be proved by using the same techniques as Caprara, but it does not seem trivial to change his proof for undirected graphs into a similar proof for directed graphs.

Interestingly, this algorithm gives all transposition medians. Previous studies of the transposition median have explained why transposition medians, and medians in general, are not unique when the base set is fairly separated (5). We now consider the entire set of medians, here called the **median cloud**, and try to extract more information from it than we would get from any single median. We also revise the algorithm to accept median clouds in the base set, instead of only permutations.

First, we consider what happens when we compute the median of a median cloud. There are reasons to believe that the second median would be closer to the true ancestor, and this is also the case, even though the difference is not large. Iterating *ad infinitum*, we obtain the limit median, in case of convergence. We give some results on the appearance of the limit median.

Second, we give an example on how median clouds can be used to enhance computations of inner nodes in a given phylogeny. We show that methods based on median cloud in general outperform methods based on a single genome.

There are good reasons to believe that median solvers for other distances (breakpoints, reversals, DCJ) can be extended to compute median clouds and accepting them in their base sets. We are thus confident that our results will improve on biologically relevant median computations.

2 Background and definitions

Let $S = \{\pi_1, \pi_2, \ldots, \pi_k\}$, $\pi_i \in \mathfrak{S}_n$, be a set of permutations called the **base set**. We will use both one line notation (for example $\pi = 3412$) and cycle notation ($\pi = (13)(24)$). Unless otherwise stated, k is the number of elements in S and n is the length of the permutations. Given any distance function $d(\cdot, \cdot)$

between two permutations, the distance between a permutation $\pi \in \mathfrak{S}_n$ and S is defined to be

$$d(\pi, S) = \sum_i d(\pi, \pi_i)$$

and a **median** is any $\mu \in \mathfrak{S}_n$ which minimises $d(\mu, S)$. The set of medians is denoted $M(S)$ and we let $d(S) = d(\mu, S)$ for $\mu \in M(S)$. The choice of distance measure $d(\cdot, \cdot)$ gives rise to several interesting median problems; in this article we focus on the **transposition median problem** (TMP).

It is well known that the following bounds for $d(S)$ hold under any metric distance.

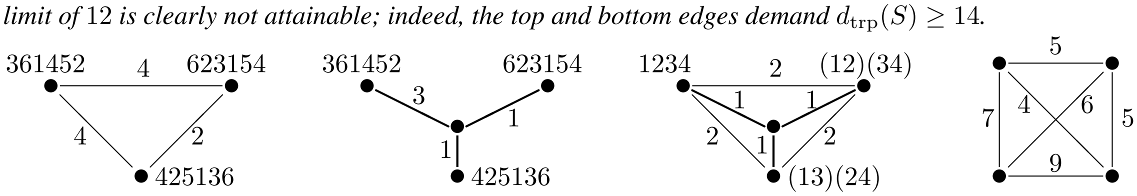
Lemma 2.1 *For any distance measure $d(\cdot, \cdot)$, the median distance $d(S)$ for $S = \{\pi_1, \ldots, \pi_k\}$ is bounded by*

$$\frac{\sum_{i<j} d(\pi_i, \pi_j)}{k-1} \leq d(S) \leq \min_i \sum_j d(\pi_i, \pi_j).$$

Proof: For the lower bound, we note that by the triangle inequality, $d(\pi_i, \pi_j) \leq d(\mu, \pi_i) + d(\mu, \pi_j)$, and hence $\sum_{i<j} d(\pi_i, \pi_j) \leq (k-1) \sum_i d(\mu, \pi_i)$. The upper bound is the minimum of $d(\pi_i, S)$. $\square$

We note that the upper bound gives a $(2 - 2/k)$-approximation of $d(S)$, and hence the median problem is trivial for $k \leq 2$, as expected. In addition, since the transposition distance changes parity for every transposition applied, we can always assign edge lengths in the tree with three (or less) genomes as leaves and a single inner node, which attains the lower limit without breaking the triangle inequality. However, we can not always find a median which attains the lower bound. For $k \geq 4$, the lower limit is only rarely realisable in the tree without breaking the triangle inequality.

Example 2.1 *Consider the three permutations in (a) with given transposition distances. The tree which attains the lower limit can be found in (b). In this case, the unique median which attains the lower limit is $\mu = 423156$. On the other hand, the base set S in (c) has $d_{\mathrm{trp}}(S)$ strictly larger than the lower limit; in fact, $M(S) = S$, giving $d_{\mathrm{trp}}(S) = 4$, while the lower limit is 3. In (d), with the distances given, the lower limit of 12 is clearly not attainable; indeed, the top and bottom edges demand $d_{\mathrm{trp}}(S) \geq 14$.*



(a) (b) (c) (d)

In the following, the term **graph** refers to edge coloured directed graphs $G = (V(G), E(G))$, unless otherwise stated. An edge from v_1 to v_2 of colour j is denoted $(v_1 \xrightarrow{j} v_2)$. By $\deg_{\mathrm{in}}(G, v, j) = |\{u : (u \xrightarrow{j} v)\}|$ and $\deg_{\mathrm{out}}(G, v, j) = |\{u : (v \xrightarrow{j} u)\}|$ we denote the in/out-degree of colour j at vertex v. The number of edges from u to v in G is denoted $|(u \longrightarrow v)|_G$, suppressing G if no confusion can arise. An **alternating path** with colours c_1 and c_2 in a graph G is a sequence of vertices $v_1, v_2, \ldots, v_{2m}$ such that G contains edges $(v_{2i-1} \xrightarrow{c_1} v_{2i})$ coloured c_1 for $1 \leq i \leq m$ and edges $(v_{2i} \xrightarrow{c_2} v_{2i+1})$ coloured c_2

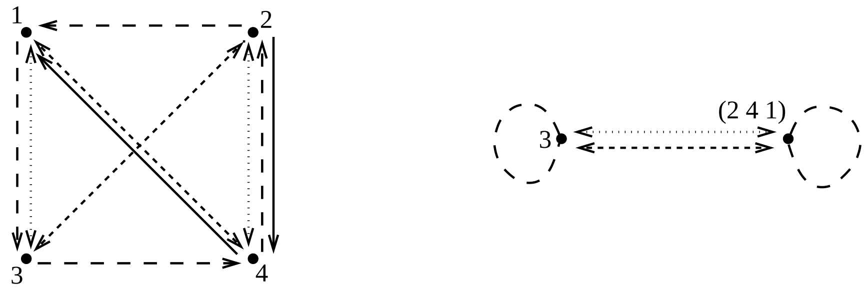


Fig. 1: A cycle graph $G(S, \mu_2)$ with $S = \{3142, 3412, 4321\}$ and $\mu_2 = \cdot 4 \cdot 1$, where a dot at position i indicates that $i \notin A_2$, and its reduced graph $\mathrm{red}(G)$.

for $1 \leq i \leq m-1$, and $j = i + 2a \Rightarrow v_i \neq v_j$ for $a > 0$. A **maximal alternating path** is an alternating path which cannot be elongated, and an **alternating cycle** is an alternating path with $v_{2m-1} = v_1$.

Let μ_b be a partial permutation with b values, that is μ_b is an injective map from $A_b \subseteq [n]$ to $[n]$ with $|A_b| = b$. Our algorithm will give a sequence $\{\mu_1, \mu_2, \dots, \mu_n\}$ such that $A_1 \subset A_2 \subset \dots \subset A_n$ and $\mu_c(j) = \mu_b(j)$ for $c \leq b$ and $j \in A_c$. We thus tacitly assume that if $\mu_b(u) = v$, then $\mu_j(u) = v$ for all $j \geq b$. Any μ_n fulfilling this criterion for a given μ_b is called a **completion** of μ_b.

The **cycle graph** of S, $G = G(S, \mu_b)$, is a graph on n vertices labelled $1, \dots, n$, with $(v_1 \xrightarrow{j} v_2)$ if $\pi_j(v_1) = v_2$. It corresponds to the breakpoint graph which is often considered when studying reversal distance problems, but has directed edges instead of undirected. The cycle graph also contains b edges of colour $k+1$, from here on called **black**, which indicate the inverse of the partial permutation μ_b: we have $(v_1 \xrightarrow{k+1} v_2)$ if $\mu_b(v_2) = v_1$. We may conclude that for all $v \in V(G)$ we have $\deg_{\text{in}}(G, v, j) = \deg_{\text{out}}(G, v, j) = 1$ for $1 \leq j \leq k$, and also $\deg_{\text{in}}(G, v, k+1) \leq 1$ and $\deg_{\text{out}}(G, v, k+1) \leq 1$.

Given a cycle graph $G = G(S, \mu_b)$ with b black edges, the **reduced cycle graph** $G' = \mathrm{red}(G)$ is a graph defined as follows. For each maximal path $(v_1 \xrightarrow{k+1} v_2 \xrightarrow{k+1} \cdots \xrightarrow{k+1} v_m)$ in G, we get the vertex $(v_1 v_2 \dots v_m)$ in G'. For each maximal alternating path $(v_1 \xrightarrow{j} v_2 \xrightarrow{k+1} \cdots \xrightarrow{j} v_{2m})$ in G, we add the edge $(v_1 \dots) \xrightarrow{j} (\dots v_{2m})$ in G'. We note that since the alternating path is maximal, there is no black edge going to v_1; hence v_1 is the first vertex in the black path giving the vertex $(v_1 \dots) \in V(G')$, and similarly v_{2m} is the last vertex in its black path. We thus observe that the reduced cycle graph $G' = \mathrm{red}(G)$ is a cycle graph on $n - b$ vertices.

Example 2.2 *Consider the cycle graph G to the left in Figure 1, with $k = 3$ and two black edges. With black edges $(2 \xrightarrow{4} 4 \xrightarrow{4} 1)$, we get the vertex $(2\ 4\ 1)$ in $\mathrm{red}(G)$ to the right in the figure. With the long dashes as colour 1, we get the maximal alternating paths $(3 \xrightarrow{1} 4 \xrightarrow{4} 1 \xrightarrow{1} 3)$ and $(2 \xrightarrow{1} 1)$ in G, giving edges $(3 \xrightarrow{1} 3)$ and $((2\ 4\ 1) \xrightarrow{1} (2\ 4\ 1))$ in $\mathrm{red}(G)$.*

3 Efficient bounds on $d_{\mathrm{trp}}(\mu, S)$ and optimal assignments

The transposition distance between any two permutations $\sigma, \tau \in \mathfrak{S}_n$ is easy to compute using this classical theorem.

Theorem 3.1 *The transposition distance between $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_n$ is given by*

$$d_{\mathrm{trp}}(\sigma, \tau) = n - c(\sigma^{-1}\tau),$$

where $c(\pi)$ is the number of cycles in π.

The standard proof uses the fact that any transposition $(a\ b)$ will either merge the two cycles in $\sigma^{-1}\tau$ containing a and b, respectively, or split the cycle containing both a and b. In addition, $\sigma^{-1}\tau$ has n cycles if and only if $\sigma = \tau$.

The distance between π_i and π_j in S can of course be computed directly from $G(S, \mu_0)$. Each cycle in $\pi_i^{-1}\pi_j$ corresponds to an alternating cycle with colours i and j, provided that all edges $(v_1 \xrightarrow{i} v_2)$ are flipped into $(v_2 \xrightarrow{i} v_1)$. This alternating cycle may also be written $(v_1 \xleftarrow{i} v_2 \xrightarrow{j} v_3 \xleftarrow{i} \cdots \xrightarrow{j} v_1)$. We thus have $c(\pi_i^{-1}\pi_j) = c(G, i, j)$, where $c(G, i, j)$ is the number of alternating cycles with colours i and j in G, provided that the edges coloured i are flipped.

Similarly, the distance between the black coloured median $\mu = \mu_n$ and any base permutation π_i is given by the number of alternating cycles with colours $k+1$ and i, not flipping any edges since the black edges give μ^{-1}. We use $c(G, i)$ to denote this quantity. But we can also say something about this distance given only μ_b. To this end, we let $p(G, i)$ be the number of maximal alternating paths and cycles with colours $k+1$ and i in G.

Lemma 3.2 *Given a cycle graph $G = G(S, \mu_b)$, the transposition distance between any completion μ of μ_b and $\pi_i \in S$ satisfies*

$$d_{\mathrm{trp}}(\mu, \pi_i) \geq n - p(G, i).$$

Proof: The lemma is clearly true for $b = 0$, since $p(G(S, \mu_0), i) = n$, and for $b = n$, since $p(G(S, \mu_n), i) = c(G(S, \mu_n), i)$. But it is also clear that $p(G(S, \mu_{b-1}), i) - p(G(S, \mu_b), i) \in \{0, 1\}$, since adding a black edge will either turn a path into a cycle or unite two paths. $\square$

Combining the previous lemma with the upper and lower bounds of Lemma 2.1, we get strong bounds on $d_{\mathrm{trp}}(\mu, S)$, given μ_b.

Lemma 3.3 *Let $S = \{\pi_1, \pi_2, \ldots, \pi_k\}$, and let $G = G(S, \mu_b)$ and $G' = \mathrm{red}(G)$. For any completion μ of μ_b, we have*

$$\frac{\sum_{i<j}((n-b) - c(G', i, j))}{k-1} \leq d_{\mathrm{trp}}(\mu, S) - \sum_i (n - p(G, i)) \leq \min_i \sum_j ((n-b) - c(G', i, j)).$$

Proof: It follows from Lemma 3.2 that $d_{\mathrm{trp}}(\mu, S) - \sum(n - p(G, i)) \geq 0$. This quantity is obtained by adding black edges to $G(S, \mu_b)$, or equivalently to $G' = \mathrm{red}(G(S, \mu_b))$. Since G' is a cycle graph, we can invoke Lemma 2.1, and this lemma follows. $\square$

Example 3.1 *Returning to Figure 1, the transposition median distance of $G(S, \mu_0)$ is bounded by $(3 + 1 + 2)/2 \leq d_{\mathrm{trp}}(\mu, S) \leq (1 + 2)$, that is $3 \leq d_{\mathrm{trp}}(\mu, S) \leq 3$, and thus one permutation in the base set (the one marked with dots) actually gives a median. For $G(S, \mu_2)$, we have $(1 + 1 + 0)/2 \leq d_{\mathrm{trp}}(\mu, S) - (1 + 1 + 1) \leq 1 + 0$. Thus, any completion of the given μ_2 gives $d_{\mathrm{trp}}(\mu, S) \geq 4$. We have made at least one bad choice among the black edges.*

We can now make a couple of observations of the influence an added black edge has on the lower limit of $d_{\text{trp}}(\mu, S)$.

Lemma 3.4 *Assume we set $\mu_{b+1}(v_2) = v_1$, that is we add the black edge $(v_1 \xrightarrow{k+1} v_2)$ to $G(S, \mu_b)$, obtaining $G(S, \mu_{b+1})$. Then,*

$$\sum_i (p(G(S, \mu_b), i) - p(G(S, \mu_{b+1}), i)) = k - |((v_2 \ldots) \longrightarrow (\ldots v_1))|_{\text{red}(G(S,\mu_b))}.$$

Proof: If there is an edge $((v_2 \ldots) \xrightarrow{i} (\ldots v_1))$ in $\text{red}(G(S, \mu_b))$, adding the edge will close an alternating path in $G(S, \mu_b)$ into a cycle in $G(S, \mu_{b+1})$, thus not changing p. Otherwise, two alternating paths in $G(S, \mu_b)$ will be united, reducing p by one. $\square$

Lemma 3.5 *If the edges $((v_2 \ldots) \xrightarrow{c_1} (\ldots v_1))$ and $((v_2 \ldots) \xrightarrow{c_2} (\ldots v_1))$ both belong to $E(\text{red}(G(S, \mu_b)))$, then letting $\mu_{b+1}(v_2) = v_1$ gives $c(\text{red}(G(S, \mu_b)), c_1, c_2) - c(\text{red}(G(S, \mu_{b+1})), c_1, c_2) = 1$.*

Proof: The alternating cycle $((v_2 \ldots) \xrightarrow{c_1} (\ldots v_1) \xleftarrow{c_2} (v_2 \ldots))$ in $\text{red}(G(S, \mu_b))$ will have disappeared in $\text{red}(G(S, \mu_{b+1}))$ and no other alternating cycles with colours c_1 and c_2 are affected. $\square$

Lemma 3.6 *If $((v_2 \ldots) \xrightarrow{c_1} (\ldots v_1)) \in E(\text{red}(G(S, \mu_b)))$, but $((v_2 \ldots) \xrightarrow{c_2} (\ldots v_1)) \notin E(\text{red}(G(S, \mu_b)))$, then letting $\mu_{b+1}(v_2) = v_1$ does not change the number of cycles, that is $c(\text{red}(G(S, \mu_b)), c_1, c_2) - c(\text{red}(G(S, \mu_{b+1})), c_1, c_2) = 0$.*

Proof: With $u_1 = (\ldots v_1)$ and $u_2 = (v_2 \ldots)$, the alternating cycle $(u_0 \xrightarrow{c_2} u_1 \xleftarrow{c_1} u_2 \xrightarrow{c_2} u_3 \xleftarrow{c_1} \ldots \xleftarrow{c_1} u_0)$ will be reduced to $(u_0 \xrightarrow{c_2} u_3 \xleftarrow{c_1} \ldots \xleftarrow{c_1} u_0)$. All other alternating cycles are untouched. $\square$

Lemma 3.7 *Let $u_1 = (\ldots v_1)$ and $u_2 = (v_2 \ldots)$. Assume that $(u_0 \xrightarrow{c_1} u_1)$ and $(u_2 \xrightarrow{c_2} u_3)$, where $u_0 \neq u_2, u_1 \neq u_3$, belong to $E(\text{red}(G(S, \mu_b)))$, which gives that neither $(u_2 \xrightarrow{c_1} u_1)$ nor $(u_2 \xrightarrow{c_2} u_1)$ belong to $E(\text{red}(G(S, \mu_b)))$. Then, letting $\mu_{b+1}(v_2) = v_1$ implies $c(\text{red}(G(S, \mu_b)), c_1, c_2) - c(\text{red}(G(S, \mu_{b+1})), c_1, c_2) = -1$ if $(u_0 \xrightarrow{c_1} u_1 \xleftarrow{c_2} u_4 \xrightarrow{c_1} \cdots \xrightarrow{c_1} u_3 \xleftarrow{c_2} u_2 \xrightarrow{c_1} u_5 \xleftarrow{c_2} \cdots \xleftarrow{c_2} u_0)$ is an alternating cycle of $\text{red}(G(S, \mu_b))$ and 1 otherwise.*

Proof: If the alternating cycle $(u_0 \xrightarrow{c_1} u_1 \xleftarrow{c_2} u_4 \xrightarrow{c_1} \cdots \xrightarrow{c_1} u_3 \xleftarrow{c_2} u_2 \xrightarrow{c_1} u_5 \xleftarrow{c_2} \cdots \xleftarrow{c_2} u_0)$ exists, it will be split in two, namely $(u_0 \xrightarrow{c_1} u_5 \xleftarrow{c_2} \cdots \xleftarrow{c_2} u_0)$ and $(u_4 \xrightarrow{c_2} u_3 \xleftarrow{c_1} \cdots \xleftarrow{c_1} u_4)$. Otherwise, we have the two alternating cycles $(u_0 \xrightarrow{c_1} u_1 \xleftarrow{c_2} u_4 \xrightarrow{c_1} \cdots \xleftarrow{c_2} u_0)$ and $(u_5 \xleftarrow{c_1} u_2 \xrightarrow{c_2} u_3 \xleftarrow{c_1} \cdots \xrightarrow{c_2} u_3)$, which unite into $(u_0 \xrightarrow{c_1} u_5 \xleftarrow{c_2} \cdots \xrightarrow{c_1} u_3 \xleftarrow{c_2} u_4 \xrightarrow{c_1} \cdots \xleftarrow{c_2} u_0)$. Remaining alternating cycles are untouched. $\square$

We are now way on our way to find $M(S)$. Using the above lemmata, we can control the lower limit of $d_{\text{trp}}(S)$ as we add edges to μ_b.

Theorem 3.8 *Let $G = G(S, \mu_b)$ and $G' = \mathrm{red}(G)$. For $u_1 = (\ldots v_1)$ and $u_2 = (v_2 \ldots)$, assume that $j = |(u_2 \longrightarrow u_1)|_{G'}$ and that there are m alternating cycles in colours $1 \leq c_1 < c_2 \leq k$ with an odd number of edges between u_1 and u_2. Then, letting $\mu_{b+1}(v_2) = v_1$ will increase the lower limit of $d_{\mathrm{trp}}(S)$,*

$$\frac{\sum_{c_1<c_2}((n-b) - c(G', c_1, c_2))}{k-1} + \sum_{c_1}(n - p(G, c_1))$$

by

$$\delta(v_1, v_2) = \frac{2}{k-1}\left(\binom{k-j}{2} - m\right).$$

In particular, for $k = 3$, the integral lower limit stays unchanged for $j \geq 2$, increases by at most 1 for $j = 1$ and at most 3 for $j = 0$.

Proof: It is clear from Lemma 3.4 that $\sum(n - p(G, c_1))$ increases with $k - j$. Next, consider colour pairs $1 \leq c_1 < c_2 \leq k$. If $(u_2 \xrightarrow{c_1} u_1)$ and $(u_2 \xrightarrow{c_2} u_1)$ are both present in $E(G')$, Lemma 3.5 gives that $((n-b) - c(G', c_1, c_2))$ does not change. If only one of these edges is present, $((n-b) - c(G', c_1, c_2))$ decreases by 1 (Lemma 3.6). Finally, Lemma 3.7 says that if none of the edges are present, $((n-b) - c(G', c_1, c_2))$ decreases by 2 if the cycle passes both u_1 and u_2 with an odd number of edges in between, and stays unchanged otherwise.

Summing up, we get that the bound increases with

$$\delta(v_1, v_2) = (k-j) - \frac{(k-j)j + 2m}{k-1} = \frac{(k-j)^2 - (k-j) - 2m}{k-1} = \frac{2}{k-1}\left(\binom{k-j}{2} - m\right).$$

□

It is not obvious that adding an edge which does not increase the lower bound is optimal. In fact, it is not even true. However, there are some black edges which are guaranteed to be optimal.

Theorem 3.9 *Assume that μ_b can be completed to all medians in $M(S)$. If $|((v_2 \ldots) \longrightarrow (\ldots v_1))|_{\mathrm{red}(G(S,\mu_b))} > k/2$, then $\mu(v_2) = v_1$ for all $\mu \in M(S)$. If $|((v_2 \ldots) \longrightarrow (\ldots v_1))|_{\mathrm{red}(G(S,\mu_b))} = k/2$, then $\mu(v_2) = v_1$ for some $\mu \in M(S)$.*

Proof: Assume that a median μ has $\mu^{-1}(v_1) = v_3 \neq v_2$. If $((v_2 \ldots) \xrightarrow{c_1} (\ldots v_1)) \in E(\mathrm{red}(G(S, \mu)))$, then the alternating cycle $(v_2 \xrightarrow{c_1} \ldots \xrightarrow{c_1} v_1 \xrightarrow{k+1} v_3 \xleftarrow{c_1} \ldots \xrightarrow{k+1} v_2)$ will split if v_1 is redirected to v_2. Hence, $c(G(S, \mu \circ (v_2\, v_3))) - c(G(S, \mu), c_1) = 1$, and summing over all colours we get $d_{\mathrm{trp}}(\mu \circ (v_2\, v_3), S) < d_{\mathrm{trp}}(\mu, S)$, contradicting the fact that μ is a median. Similarly, if $|((v_2 \ldots) \longrightarrow (\ldots v_1))| = k/2$, we obtain a median $\mu \circ (v_2\, v_3)$ which satisfies $(\mu \circ (v_2\, v_3))(v_2) = \mu(v_3) = v_1$. □

4 A median solver

Based on the theorems in the previous section, we have devised a transposition median solver which gives all medians, that is $M(S)$, for any S. We start with $\mu_0 = 0$ and then make a depth first search through the space of μ_b. At any node μ_b in the search tree, if $|((v_2 \ldots) \longrightarrow (\ldots v_1))|_{\mathrm{red}(G(S,\mu_b))} > k/2$ then $\mu_{b+1}(v_2) = v_1$ is optimal. Otherwise, we search all subtrees of μ_b, stopping as soon as the lower bound on that subtree rises above the lowest value on $d_{\mathrm{trp}}(\mu, S)$ found so far. A formal description of the algorithm Median can be found in Algorithm 1 (last page).

Algorithm 1 can of course be improved upon. For instance, to achieve a more effective pruning, we compute the increase of $d_{\mathrm{trp}}(\mu, S)$ for each assignment $\mu_{b+1}(v_2) = v_1$, where $(\dots v_1), (v_2 \dots) \in \mathrm{red}(G(S, \mu))$. Keeping v_2 fixed, it is clear that $d_{\mathrm{trp}}(\mu, S) \geq d_{\mathrm{trp}}(\mu_b, S) + \min_{v_1} \delta(v_1, v_2)$, since we are free to assign $\mu_{b+1}(v_2)$ at any stage. Similarly, keeping v_1 fixed, we have $d_{\mathrm{trp}}(\mu, S) \geq d_{\mathrm{trp}}(\mu_b, S) + \min_{v_2} \delta(v_1, v_2)$. This leads to more effective pruning. Our implementation in Matlab is available upon request. The speed of this implementation is comparable to the DCJ median solver by Xu (11).

5 Median clouds

Given the set of medians $M(S)$, there are some parts which are common to all medians and some parts which vary more or less between the medians. If a median is chosen at random from this set, the choice of the parts which vary between medians will be impossible to distinguish from the parts which are common between all medians, and they will probably effect later computations using this median. To minimise this effect, we would like to keep as much information as possible about $M(S)$ instead of just choosing a single median.

Since our median solver gives the complete median cloud $M(S)$, we would like to keep this cloud and use it in further calculation. In those calculations, this median cloud should play the role of a single permutation in a base set. How can we revise the median solver to accept such base sets, that is to compute the median of a base set of sets, $S = (S_1, S_2, \dots, S_k)$?

One method which seems tempting is to take the permutation matrices A_j of all permutations in each set S_i and compute their arithmetical mean, $\sum A_j / |S_i|$. However, since the algorithm requires not only the extent of which a set S_i maps v_1 to v_2, but also the alternating cycle structure, we lose too much information in this process. Instead, we need to consider each pair $\pi_1 \in S_i$ and $\pi_2 \in S_j$ separately.

To be more precise, we give each permutation $\pi \in S_i$ weight $w(\pi)$ such that $\sum_{\pi \in S_i} w(\pi) = 1$. Usually, $w(\pi) = |S_i|^{-1}$ will do. Then, it is easy to see that if we define

$$d^w(\mu, S) = \sum_i \sum_{\pi \in S_i} d(\mu, \pi) w(\pi),$$

a lower transposition median distance limit of any completion of μ_b is given by

$$\frac{\sum\limits_{1 \leq i < j \leq k} \sum\limits_{\pi \in S_i, \sigma \in S_j} ((n - b) - c(G', c_1, c_2)) w(\pi) w(\sigma)}{k - 1} + \sum_i \sum_{\pi \in S_i} (n - p(G, c_1)) w(\pi).$$

We can thus use Algorithm 1 almost unchanged. We note, however, that the running time is proportional to $\max_i |S_i|^2$ in the worst case and median sets $M(S)$ grow fast when we scatter the base set. However, pruning may be more effective when median distances are given rational numbers instead of integers.

6 Limit medians

Medians and median clouds are often used to estimate the ancestor of three or more contemporary species. Medians are approximations of the ancestor and should "surround" the ancestor. This leads us to compute the median of a median cloud, which could improve on the estimate of the ancestor, although not on the distance $d(\mu, S)$.

Definition 6.1 *Given a base set S, the kth* **median iterate** *of S is $M^k(S)$, where $M^k(S) = M(M^{k-1}(S))$. If the limit*

$$M^{\infty}(S) = \lim_{k\to\infty} M^k(S)$$

exists, we say that $M^{\infty}(S)$ is a **limit median**.

It is obvious that $M^{\infty}(S) = M(S)$ if $M(S)$ is a singleton. But what can be said if $|M(S)| > 1$?

Proposition 6.1 *If $S = \{\mathrm{id}, (1\ 2\ \dots\ m)\}$, then $M(S)$ contains all permutations $\pi \in \mathfrak{S}_n$ such that $d_{\mathrm{trp}}(\pi, \mathrm{id}) + d_{\mathrm{trp}}(\pi, (1\ 2\ \dots\ m)) = m - 1$.*

Proof: The assertion is given directly by the triangle inequality, since the set contains all permutation on a shortest path from id to $(1\ 2\ \dots\ m)$. ☐

We conjecture based on extensive calculations that for $S = \{\mathrm{id}, (1\ 2\ \dots\ m)\}$, $M^2(S) = S$. If this holds, $M^k(S)$ is periodic with period 2.

Proposition 6.2 *With $S = \{\mathrm{id}, (1\ 2\ \dots\ n)\}$, we have*

$$|\{\pi \in M(S) : d_{\mathrm{trp}}(\pi, \mathrm{id}) = k\}| = N(n, k+1) = \frac{\binom{n}{k}\binom{n-1}{k}}{k+1} = \frac{\binom{n}{k}\binom{n}{k+1}}{n},$$

where $N(n, k)$ are the Narayana numbers. Hence, $|M(S)| = C_n$, the nth Catalan number.

Proof: The Narayana numbers $N(n, k+1)$ count the number of Dyck paths of length n with $n - k$ peaks. Extend each peak into a mountain, that is continue the steps $(1, 1)$ and $(1, -1)$ which constitute the peak until they cut the x-axis. Draw left parenthesis at positions where the left mountain sides cut the line $y = 1/2$ and right parenthesis where the right mountain sides cut the same line. Then, inserting the numbers $j \in [n]$ at positions $2j - 1$ gives the permutations in $M(S)$ with $n - k$ cycles, given that we recursively interpret the expression $(a\ \dots\ b\ (c\ \dots\ d)\ f \dots\ g)$ as $(a\ \dots\ b\ f\ \dots\ g)(c\ \dots\ d)$. ☐

The following proposition follows directly from independence of disjoint cycles.

Proposition 6.3 *If $S = \{\mathrm{id}, \pi\}$ and the cycles in π are given by $\pi = c_1 c_2 \dots c_m$, each $\tau \in M(S)$ can be written as a product of permutations $\tau_1 \tau_2 \dots \tau_m$ such that $\tau_j \in M(\{\mathrm{id}, c_j\})$.*

For all base sets S we have looked at, the sequence $M^k(S)$ has either had a limit or been eventually periodic with period 2. In fact, we have yet to discover a base set S such that $M^4(S) \neq M^2(S)$.

7 Computing ancestral permutations

Median clouds can be used to facilitate median computations in a given phylogeny in two different ways. First, previously computed inner nodes are used to compute the remaining inner nodes, and these computations may be improved on by using median clouds instead of just a single median. Second, if the inner node we seek to approximate with a median has three edges leading to several leaves in each direction, we can take the leaves of each direction and merge them into a cloud, instead of choosing one of these leaves at random. To test these approaches, we have made simulations and compared different methods for approximating the inner nodes of a known tree from the leaves.

Consider the phylogenetic tree in Figure 2. Given edge lengths, we simulate leaf permutations using transpositions chosen randomly and independently with uniform distribution. We then use five different median methods to estimate the inner nodes as closely as possible. Thus, we get indications on the quality of the methods. In particular, we wish to examine if median clouds can be used to enhance our abilities to find the inner nodes.

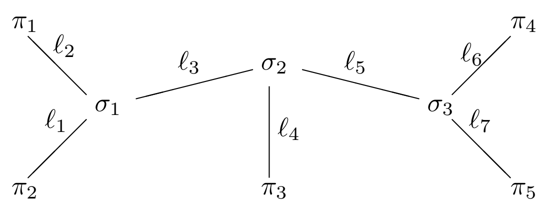


Fig. 2: The phylogenetic tree of $\pi_1, \ldots, \pi_5$.

The five methods used are the following: First, we compute the median of three leaves. This can be done in three ways for σ_1 and σ_3, and four ways for σ_2. Second, we use medians computed with the first method; for instance, we approximate σ_1 with $\tau \in M(\pi_1, \pi_2, \mu)$, where $\mu \in M(\pi_1, \pi_3, \pi_4)$). Third, we compute medians using all leaves on each side of the median. For instance, we approximate σ_1 with $\tau \in M(\pi_1, \pi_2, \{\pi_3, \pi_4, \pi_5\})$. The fourth method is similar to the second, except that we use the median clouds from the third method instead of a single median from the first. Fifth, for comparison we use the inner nodes, that is σ_1 is approximated by $M(\pi_1, \pi_2, \sigma_2)$. This gives a lower limit on the error we can achieve using information only on the leaves.

Tab. 1: Comparing five methods for estimating permutations at inner nodes in the phylogenetic tree in Figure 2. Edge lengths are as below and $n = 40$. In the table, mean distances to the correct inner node are given, summing both over 500 simulations and over all ways to compute the inner node using the respective methods. We note that the results improve significantly as we refine the methods. We should add that the first two methods can be improved upon in the case where edge lengths are as different as in the second row by always choosing the closest leaf on each side, but the fourth method is still somewhat better.

Edge lengths	First	Second	Third	Fourth	Fifth
$(7, 7, 7, 14, 7, 7, 7)$	4.2	2.7	2.1	1.9	0.8
$(15, 3, 4, 15, 4, 4, 12)$	4.0	2.3	1.6	1.4	0.5

The five methods are compared in Table 1. We find that the third and fourth methods constitute a significant improvement over the first two, both with similar edge length and a mixture of long and short edges. In the second case, choosing closely related permutations improves on the mean results, but the third and fourth methods are still better even in this extreme case.

8 Open problems

Our results leave several open problems. Which sets are medians clouds for some base set? Which sets are limit clouds for some base set? Are there base sets whose median sequence $M^k(S)$ is not periodic, or has a longer period than 2? What kind of regularities and symmetries can we expect to find in a limit cloud? All these questions are also interesting under other distances, for example reversals.

We are also anxious to see if median clouds can be incorporated into median computations under other distances, such as breakpoints, reversals and DCJ. In addition, a proof that the transposition median problem is, as conjectured, NP-complete (or even better, a polynomial time solver) would of course be welcomed.

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Algorithm 1: Median: A simplified branch-and-bound algorithm for finding $M(S)$ under the transposition distance. It is called with $\mu = (0, 0, \dots, 0)$ and $B = \infty$. The ApplyOptimal algorithm iteratively applies all majority rule assignments according to Theorem 3.9.

Data: S, μ, B
Result: $M(S)$, B
$\mu \leftarrow$ ApplyOptimal(S, μ);
if $\mu \in \mathfrak{S}_n$ **then**
    if $d(\mu, S) < B$ **then**
        $M(S) \leftarrow \{\mu\}$;
        $B \leftarrow d(\mu, S)$;
    else if $d(\mu, S) = B$ **then**
        $M(S) \leftarrow M(S) \cup \{\mu\}$;
    end
else
    $e \leftarrow \min\{j : j \notin \mu\}$;
    foreach i *such that* $\mu(i) = 0$ **do**
        $\mu(i) \leftarrow e$;
        if $d(\mu, S) \leq B$ **then**
            $(M(S), B) \leftarrow$ Median(S, μ, B);
        end
    end
end

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Enumeration of derangements with descents in prescribed positions

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We enumerate derangements with descents in prescribed positions. A generating function was given by Guo-Niu Han and Guoce Xin in 2007. We give a combinatorial proof of this result, and derive several explicit formulas. To this end, we consider fixed point λ-coloured permutations, which are easily enumerated. Several formulae regarding these numbers are given, as well as a generalisation of Euler's difference tables. We also prove that except in a trivial special case, if a permutation π is chosen uniformly among all permutations on n elements, the events that π has descents in a set S of positions, and that π is a derangement, are positively correlated.

Keywords: Permutation statistic, fixed point, descent

1 Introduction

In a permutation $\pi \in \mathfrak{S}_n$, a *descent* is a position i such that $\pi_i > \pi_{i+1}$, and an *ascent* is a position where $\pi_i < \pi_{i+1}$. A *fixed point* is a position i where $\pi_i = i$. If $\pi_i > i$, then i is called an *excedance*, while if $\pi_i < i$, i is a *deficiency*. Richard Stanley (11) conjectured that permutations in $\mathfrak{S}_{2n}$ with descents at and only at odd positions (commonly known as *alternating permutations*) and n fixed points are equinumerous with permutations in $\mathfrak{S}_n$ without fixed points, commonly known as *derangements*.

The conjecture was given a bijective proof by Chapman and Williams in 2007 (1). The solution is quite straightforward: Assume $\pi \in \mathfrak{S}_{2n}$ is alternating and $F \subseteq [2n]$ is the set of fixed points, $|F| = n$. Then removing the fixed points gives a permutation τ in $\mathfrak{S}_{[2n]\setminus F}$ without fixed points, and π can be easily reconstructed from τ.

For instance, removing the fixed points in $\pi = 326451$ gives $\tau = 361$ or $\tau = 231$ if we reduce it to $\mathfrak{S}_3$. To recover π, we note that the fixed points in the first two descending blocks must be at the respective second positions, 2 and 4, since both τ_1 and τ_2 are excedances, that is above the fixed point diagonal $\tau_i = i$. On the other hand, since $\tau_3 < 3$, the fixed point in the third descending block comes in its first position, 5. With this information, we immediately recover π.

Alternating permutations are permutations which fall in and only in blocks of length two. A natural generalisation comes by considering permutations which fall in blocks of lengths

$a_1, a_2, \ldots, a_k$ and have k fixed points (this is obviously the maximum number of fixed points, since each descending block can have at most one). These permutations are in bijection with derangements which descend in blocks of length $a_1 - 1, a_2 - 1, \ldots, a_k - 1$, and possibly also between them, a fact which was proved by Guo-Niu Han and Guoce Xin (9).

In this article we compute the number of derangements which have descents in prescribed blocks and possibly also between them. A generating function was given by Han and Xin using a representation theory argument. We start by computing the generating function using simple combinatorial arguments (Section 3), and then proceed to extract a closed formula in Section 4.

Interestingly, this formula, which is a combination of factorials, can also be written as the same combination of an infinite family of other numbers, including the derangement numbers. We give a combinatorial interpretation of these families as the number of *fixed point λ-coloured* permutations.

For a uniformly chosen permutation, the events that it is a derangement and that its descent set is included in a given set are not independent. We prove that except for the permutations of odd length with no ascents, these events are positively correlated. In fact, we prove that the number of permutations which are derangements when sorted decreasingly in each block is larger when there are few and large blocks, compared to many small blocks. The precise statement is found in Section 7.

Finally, in Section 8, we generalise some results concerning Euler's difference triangles from (10) to fixed point λ-coloured permutations, using a new combinatorial interpretation. This interpretation is in line with the rest of this article, counting permutations having an initial descending segment and λ-coloured fixed points to the right of the initial segment. In addition, we also derive a relation between difference triangles with different values of λ.

There are many papers devoted to counting permutations with prescribed descent sets and fixed points, see for instance (6; 8) and references therein. More recent related papers include (4), where Corteel et al. considered the distribution of descents and major index over permutations without descents on the last i positions, and (2), where Chow considers the problem of enumerating the involutions with prescribed descent set.

This paper is an extended abstract, with some proofs missing. These can be found in the full paper (7).

2 Definitions and examples

Let $[i, j] = \{i, i+1, \ldots, j\}$ and $[n] = [1, n]$. We think of $[n]$ as being decomposed into blocks of lengths $a_1, \ldots, a_k$, and we will consider permutations that decrease within these blocks. The permutations are allowed to decrease or increase in the breaks between the blocks.

Consider a sequence $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)$ of nonnegative integers, with $\sum_i a_i = n$, and let $c_j = \sum_{i=1}^{j} a_i$. We denote by A_j the j:th block of $\boldsymbol{a}$, that is the set $A_j = [c_{j-1} + 1, c_j] \subseteq [n]$. Throughout the paper, k will denote the number of blocks in a given composition. We let $\mathfrak{S}_{\boldsymbol{a}} \subseteq \mathfrak{S}_n$ be the set of permutations that have descents at every place within the blocks, and may or may not have descents in the positions c_j. In particular we have $\mathfrak{S}_n = \mathfrak{S}_{(1,1,\ldots,1)}$.

Example 2.1 *If $n = 6$ and $\boldsymbol{a} = (4,2)$, then we consider permutations that are decreasing in* [1,4] *and in* [5,6]*. Such a permutation is uniquely determined by the partition of the numbers 1–6 into* *these blocks, so the total number of such permutations is*

$$\binom{6}{4,2} = 15.$$

Of these 15 permutations, those that are derangements are

$$\begin{array}{c} 6543|21 \\ 6542|31 \\ 6541|32 \\ 6521|43 \\ 5421|63 \\ 5321|64 \\ 4321|65 \end{array}$$

We define $D(\boldsymbol{a})$ to be the subset of $\mathfrak{S}_{\boldsymbol{a}}$ consisting of derangements, and our objective is to enumerate this set. For simplicity, we also define $D_n = D(1, \ldots, 1)$.

For every composition $\boldsymbol{a}$ of n, there is a natural map $\Phi_{\boldsymbol{a}} : \mathfrak{S}_n \to \mathfrak{S}_{\boldsymbol{a}}$, given by simply sorting the entries in each block in decreasing order. For example, if $\sigma = 25134$, we have $\Phi_{(3,2)}(\sigma) = 52143$. Clearly each fiber of this map has $a_1! \ldots a_k!$ elements.

The following maps on permutations will be used frequently in the paper.

Definition 2.1 *For $\sigma \in \mathfrak{S}_n$, let $\phi_{j,k}(\sigma) = \tau_1 \ldots \tau_{j-1} k \tau_j \ldots \tau_n$, where*

$$\tau_i = \begin{cases} \sigma_i & \text{if } \sigma_i < k \\ \sigma_i + 1 & \text{if } \sigma_i \geq k \end{cases}$$

Similarly, let $\psi_j(\sigma) = \tau_1 \ldots \tau_{j-1} \tau_{j+1} \ldots \tau_n$ where

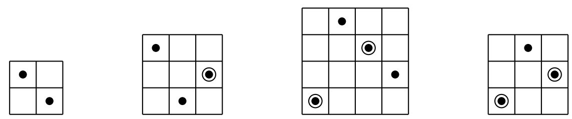
$$\tau_i = \begin{cases} \sigma_i & \text{if } \sigma_i < \sigma_j; \\ \sigma_i - 1 & \text{if } \sigma_i > \sigma_j. \end{cases}$$

Thus, $\phi_{j,k}$ inserts the element k at position j, increasing elements larger than k by one and shifting elements to the right of position j one step further to the right. The map ψ_j removes the element at position j, decreasing larger elements by one and shifting those to its right one step left.

We will often use the map $\phi_j = \phi_{j,j}$ which inserts a fixed point at position j. The generalisations to a set F of fixed points to be inserted or removed are denoted $\phi_F(\sigma)$ and $\psi_F(\sigma)$, inserting elements in increasing order and removing them in decreasing order.

The maps ϕ and ψ are perhaps most obvious in terms of permutation matrices. For a permutation $\sigma \in \mathfrak{S}_n$, we get $\phi_{j,k}(\sigma)$ by adding a new row below the k:th one, a new column before the j:th one, and an entry at their intersection. Similarly, $\psi_j(\sigma)$ is obtained by deleting the j:th column and the σ_j:th row.

Example 2.2 *We illustrate by showing some permutation matrices. For $\pi = 21$ and $F = \{1,3\}$, we get*



π $\quad\phi_{3,2}(\pi)\quad$ $\phi_F(\pi)\quad$ $\psi_4 \circ \phi_F(\pi)$

where inserted points are labeled with an extra circle.

3 A generating function

Guo-Niu Han and Guoce Xin gave a generating function for $D(a)$ ((9), Theorem 9). In fact they proved this generating function for another set of permutations, equinumerous to $D(\boldsymbol{a})$ by ((9), Theorem 1). What they proved was the following:

Theorem 3.1 *The number $|D(\boldsymbol{a})|$ is the coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in the expansion of*

$$\frac{1}{(1+x_1)\cdots(1+x_k)(1-x_1-\cdots-x_k)}.$$

The proof uses scalar products of symmetric functions. We give a more direct proof, with a combinatorial flavour. The proof uses the following definition, and the bijective result of Lemma 3.2.

Definition 3.1 *We denote by $D_j(\boldsymbol{a})$ the set of permutations in $\mathfrak{S}_{\boldsymbol{a}}$ that have no fixed points in blocks $A_1, \ldots, A_j$. Thus, $D(\boldsymbol{a}) = D_k(\boldsymbol{a})$.*

Moreover, let $D_j^*(\boldsymbol{a})$ be the set of permutations in $\mathfrak{S}_{\boldsymbol{a}}$ that have no fixed points in the first $j-1$ blocks, but have a fixed point in A_j.

Lemma 3.2 *There is a bijection between $D_j(a_1, \ldots, a_k)$ and $D_j^*(a_1, \ldots, a_{j-1}, a_j+1, a_{j+1}, \ldots, a_k)$.*

Proof: Let $\sigma = \sigma_1 \ldots \sigma_n$ be a permutation in $D_j(a_1, \ldots, a_k)$, and consider the block $A_j = \{p, p+1, \ldots, q\}$. Then there is an index r such that $\sigma_p \ldots \sigma_{r-1}$ are excedances, and $\sigma_r \ldots \sigma_q$ are deficiencies.

Now $\phi_r(\sigma)$ is a permutation of $[n+1]$. It is easy to see that

$$\phi_r(\sigma) \in \mathfrak{S}_{(a_1,\ldots,a_{j-1},a_j+1,a_{j+1},\ldots,a_k)}.$$

All the fixed points of σ are shifted one step to the right, and one new is added in the j:th block, so

$$\phi(\sigma) \in D_j^*(a_1, \ldots, a_{j-1}, a_j+1, a_{j+1}, \ldots, a_k).$$

We see that $\psi_r(\phi_r(\sigma)) = \sigma$, so the map $\sigma \mapsto \phi_r(\sigma)$ is injective.

Similarily, for a permutation $\tau \in D_j^*(a_1, \ldots, a_{j-1}, a_j+1, a_{j+1}, \ldots, a_k)$, let r be the fixed point in A_j. Then $\psi_r(\tau) \in D_j(a_1, \ldots, a_k)$ and $\phi_r(\psi_r(\sigma)) = \sigma$. Thus, $\sigma \mapsto \phi_r(\sigma)$ is a bijection. $\square$

We now obtain a generating function for $|D(\boldsymbol{a})|$, with a purely combinatorial proof. In fact, we even strengthen the result to give generating functions for $|D_j(\boldsymbol{a})|$, $j = 0, \ldots, k$. Theorem 3.1 then follows by letting $j = k$.

Theorem 3.3 *The number $|D_j(\boldsymbol{a})|$ is the coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in the expansion of*

$$\frac{1}{(1+x_1)\cdots(1+x_j)(1-x_1-\cdots-x_k)}. \tag{1}$$

Proof: Let $F_j(\boldsymbol{x})$ be the generating function for $|D_j(\boldsymbol{a})|$, so that $|D_j(a_1, \ldots, a_k)|$ is the coefficient for $x_1^{a_1} \cdots x_k^{a_k}$ in $F_j(\boldsymbol{x})$. We want to show that $F_j(x)$ is given by (1).

By definition, $|D_0(\boldsymbol{a})| = |\mathfrak{S}_{\boldsymbol{a}}|$. But a permutation in $\mathfrak{S}_{\boldsymbol{a}}$ is uniquely determined by the set of a_1 numbers in the first block, the set of a_2 numbers in the second, etc. So $|D_0|$ is the multinomial coefficient $\binom{n}{a_1,a_2,\ldots,a_k}$. This is also the coefficient of $x_1^{a_1} \cdots x_k^{a_k}$ in the expansion of $1 + (\sum x_i) + (\sum x_i)^2 + \cdots$, since any such term must come from the $(\sum x_i)^n$-term. Thus,

$$F_0(\boldsymbol{x}) = 1 + \left(\sum x_i\right) + \left(\sum x_i\right)^2 + \cdots = \frac{1}{(1-x_1-\cdots-x_k)}. \tag{2}$$

Note that for any j, $D_{j-1}(\boldsymbol{a}) = D_j(\boldsymbol{a}) \cup D_j^*(\boldsymbol{a})$, and the two latter sets are disjoint. Indeed, a permutation in D_{j-1} either does or does not have a fixed point in the j:th block. Hence by Lemma 3.2, we have the identity

$$|D_{j-1}(\boldsymbol{a})| = |D_j(\boldsymbol{a})| + |D_j(a_1, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots, a_k)|. \tag{3}$$

This holds also if $a_j = 0$, if the last term is interpreted as 0 in that case.

In terms of generating functions, this gives the recursion $F_{j-1}(\boldsymbol{x}) = (1 + x_j)F_j(\boldsymbol{x})$. Hence $F_0(\boldsymbol{x}) = F_j(\boldsymbol{x}) \prod_{i \le j}(1 + x_i)$. Thus,

$$F_j(\boldsymbol{x}) = \frac{F_0(\boldsymbol{x})}{(1+x_1)\cdots(1+x_j)} = \frac{1 + (\sum x_i) + (\sum x_i)^2 + \cdots}{(1+x_1)\cdots(1+x_j)}, \tag{4}$$

and $|D_j(\boldsymbol{a})|$ is the coefficient for $x_1^{a_1} \cdots x_k^{a_k}$ in the expansion of F_j. □

Proof of Theorem 3.1: The set of derangements in $\mathfrak{S}_{\boldsymbol{a}}$ is just $D(\boldsymbol{a}) = D_k(\boldsymbol{a})$. Letting $j = k$ in Theorem 3.3 gives the generating function for $|D(\boldsymbol{a})|$. □

4 An explicit enumeration

It is not hard to explicitly calculate the numbers $|D(\boldsymbol{a})|$ from here. We will use $\boldsymbol{x}^{\boldsymbol{a}}$ as shorthand for $\prod_i x_i^{a_i}$.

Every term $\boldsymbol{x}^{\boldsymbol{a}}$ in the expansion of $F(\boldsymbol{x})$ is obtained by choosing $x_i^{b_i}$ from the factor

$$\frac{1}{1+x_i} = \sum_{j \ge 0} (-x_i)^j,$$

for some $0 \leq b_i \leq a_i$. This gives us a coefficient of $(-1)^{\sum b_i}$. For each choice of $b_1, \dots, b_k$ we should multiply by $\boldsymbol{x}^{\boldsymbol{a}-\boldsymbol{b}}$ from the factor

$$\frac{1}{(1-x_1-\cdots-x_k)} = 1 + \left(\sum x_i\right) + \left(\sum x_i\right)^2 + \cdots .$$

But every occurence of $\boldsymbol{x}^{\boldsymbol{a}-\boldsymbol{b}}$ in this expression comes from the term $(\sum x_i)^{n-\sum b_j}$. Thus the coefficient of $\boldsymbol{x}^{\boldsymbol{a}-\boldsymbol{b}}$ is the multinomial coefficient

$$\binom{n-\sum b_j}{a_1-b_1,\dots,a_k-b_k} = \frac{(n-\sum b_j)!}{(a_1-b_1)!\cdots(a_k-b_k)!}.$$

Now since $|D(\boldsymbol{a})|$ is the coefficient of $\boldsymbol{x}^{\boldsymbol{a}}$ in $F_k(\boldsymbol{x})$, we conclude that

$$\begin{aligned} |D(\boldsymbol{a})| &= \sum_{\boldsymbol{0}\leq\boldsymbol{b}\leq\boldsymbol{a}} (-1)^{\sum b_j} \frac{(n-\sum b_j)!}{(a_1-b_1)!\cdots(a_k-b_k)!} \\ &= \frac{1}{\prod_i a_i!} \sum_{\boldsymbol{0}\leq\boldsymbol{b}\leq\boldsymbol{a}} (-1)^{\sum b_j} \left(n-\sum b_j\right)! \prod_i \binom{a_i}{b_i} b_i!. \end{aligned} \tag{5}$$

While the last expression in Equation (5) seems a bit more involved than necessary, it turns out to generalise in a nice way.

5 Fixed point coloured permutations

A *fixed point coloured permutation* in λ colours, or a fixed point λ-coloured permutation, is a permutation where we require each fixed point to take one of λ colours. More formally it is a pair (π, C) with $\pi \in \mathfrak{S}_n$ and $C : F_\pi \to [\lambda]$, where F_π is the set of fixed points of π. When there can be no confusion, we denote the coloured permutation (π, C) by π. Thus, fixed point 1-coloured permutatations are simply ordinary permutations and fixed point 0-coloured permutations are derangements. The set of fixed point λ-coloured permutations on n elements is denoted $\mathfrak{S}_n^\lambda$.

For the number of λ-fixed point coloured permutations on n elements, we use the notation $|\mathfrak{S}_n^\lambda| = f_\lambda(n)$, the λ-*factorial* of n. Of course, we have $f_0(n) = D_n$ and $f_1(n) = n!$. Clearly,

$$f_\lambda(n) = \sum_{\pi\in\mathfrak{S}_n} \lambda^{\mathrm{fix}(\pi)},$$

where $\mathrm{fix}(\pi)$ is the number of fixed points in π, and we use this formula as the definition of $f_\lambda(n)$ for $\lambda \notin \mathbb{N}$.

Lemma 5.1 *For $\nu, \lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, we have*

$$f_\nu(n) = \sum_j \binom{n}{j} f_\lambda(n-j) \cdot (\nu-\lambda)^j.$$

Proof: It suffices to show this for $\nu, \lambda, n \in \mathbb{N}$, since the identity is polynomial in ν and λ, so if it holds on $\mathbb{N} \times \mathbb{N}$ it must hold on all of $\mathbb{C} \times \mathbb{C}$.

We divide the proof into three parts. First, assume $\nu = \lambda$. Then all terms in the sum vanish except for $j = 0$, when we get $f_\nu(n) = f_\lambda(n)$.

Secondly, assuming $\nu > \lambda$, we let j denote the number of fixed points in $\pi \in \mathfrak{S}_n^\nu$ which are coloured with colours from $[\lambda + 1, \nu]$. These fixed points can be chosen in $\binom{n}{j}$ ways, there are $f_\lambda(n-j)$ ways to permute and colour the remaning elements, and the colours of the high coloured fixed points can be chosen in $(\nu - \lambda)^j$ ways. Thus, the equality holds.

Finally, assuming $\nu < \lambda$, we prescribe j fixed points in $\pi \in \mathfrak{S}_n^\lambda$ which only get to choose their colours from $[\nu + 1, \lambda]$. These fixed points can be chosen in $\binom{n}{j}$ ways, the remaining elements can be permuted in $f_\lambda(n - j)$ ways and the chosen fixed points can be coloured in $(\lambda - \nu)^j$ ways, so by the principle of inclusion-exclusion, the equality holds. $\square$

With $\lambda = 1$ and replacing ν by λ, we find that

$$f_\lambda(n) = n! \left(1 + \frac{(\lambda - 1)}{1!} + \frac{(\lambda - 1)^2}{2!} + \cdots + \frac{(\lambda - 1)^n}{n!}\right) = n! \exp_n(\lambda - 1). \tag{6}$$

Here we use $\exp_n$ to denote the truncated series expansion of the exponential function. In fact, $\lim_{n\to\infty} n!\mathrm{e}^{(\lambda-1)} - f_\lambda(n) = 0$ for all $\lambda \in [-1,1]$, although we cannot in general approximate $f_\lambda(n)$ by the nearest integer of $n!\mathrm{e}^{\lambda-1}$ as for derangements.

The formula (6) also shows that

$$f_\lambda(n) = n f_\lambda(n - 1) + (\lambda - 1)^n, \qquad f_\lambda(0) = 1 \tag{7}$$

which generalises the well known recursions $|D_n| = n|D_{n-1}| + (-1)^n$ and $n! = n(n - 1)!$.

6 Enumerating $D(\boldsymbol{a})$ using fixed point coloured permutations

Another consequence of Equation (6) is that the λ-factorial satisfies the following rule for differentiation, which is similar to the rule for differentiating powers of λ:

$$\frac{d}{d\lambda} f_\lambda\left(n\right) = n \cdot f_\lambda\left(n - 1\right). \tag{8}$$

Regarding n as the cardinality of a set X, the differentiation rule (8) translates to

$$\frac{d}{d\lambda} f_\lambda\left(|X|\right) = \sum_{x \in X} f_\lambda\left(|X \setminus \{x\}|\right). \tag{9}$$

Products of λ-factorials can of course be differentiated by the product formula. This implies that if $X_1, \ldots X_k$ are disjoint sets, then

$$\frac{d}{d\lambda} \prod_i f_\lambda\left(|X_i|\right) = \sum_{x \in \cup X_j} \prod_i f_\lambda\left(|X_i \setminus \{x\}|\right).$$

Now consider the expression

$$\sum_{B\subseteq[n]} (-1)^{|B|} f_\lambda\left(|[n]\smallsetminus B|\right) \prod_{i}^{k} f_\lambda\left(|A_i\cap B|\right). \tag{10}$$

This is obtained from the right-hand side of Equation (5) by deleting the factor $1/\prod_i a_i!$ and replacing the other factorials by λ-factorials. For $\lambda = 1$, the expression (10) is therefore $\left|\Phi_{\boldsymbol{a}}^{-1}(D(\boldsymbol{a}))\right|$, the number of permutations that, when sorted in decreasing order within the blocks, have no fixed points. We want to show that (10) is independent of λ. The derivative of (10) is, by the rule (9) of differentiation,

$$\sum_{B\subseteq[n]} (-1)^{|B|} \sum_{x=1}^{n} f_\lambda\left(|[n]\smallsetminus B\smallsetminus\{x\}|\right) \prod_{i=1}^{k} f_\lambda\left(|(A_i\cap B)\smallsetminus\{x\}|\right). \tag{11}$$

Here each product of λ-factorials occurs once with $x \in B$ and once with $x \notin B$. Because of the sign $(-1)^{|B|}$, these terms cancel. Therefore (11) is identically zero, which means that (10) is independent of λ. Hence we have proven the following theorem:

Theorem 6.1 *For any $\lambda \in \mathbb{C}$, the identity*

$$\left|\Phi_{\boldsymbol{a}}^{-1}(D(\boldsymbol{a}))\right| = \sum_{0\le\boldsymbol{b}\le\boldsymbol{a}} (-1)^{\sum b_j}\cdot f_\lambda\left(n-\sum b_j\right)\prod_i \binom{a_i}{b_i}\cdot f_\lambda\left(b_i\right) \tag{12}$$

holds.

A particularly interesting special case is when we put $\lambda = 0$. In this case, $f_0(n) = D_n$, so

$$|D(\boldsymbol{a})| = \frac{1}{\prod_i a_i!}\sum_{\boldsymbol{0}\le\boldsymbol{b}\le\boldsymbol{a}} (-1)^{\sum b_j} D_{n-\sum b_j}\prod_i \binom{a_i}{b_i} D_{b_i}. \tag{13}$$

This equation has some advantages over Equation (5). It has a clear main term, the one with $\boldsymbol{b} = \boldsymbol{0}$. Moreover, since $D_1 = 0$, the number of terms does not increase if blocks of length 1 are added.

It should be pointed out that Theorem 6.1 can be proven directly for all λ in a recursive manner, using neither the differentiation rule 8, nor the generating function in Theorem 3.3. This alternative proof is rather lengthy, and can be found in (7).

We also note that Theorem 6.1 can be used to enumerate permutations in $\mathfrak{S}_{\boldsymbol{a}}$ with μ allowed fixed point colours, and even μ_i fixed point colours in block A_i.

Corollary 6.2 *For any $\lambda \in \mathbb{C}$ and natural numbers $\mu_i, 1 \le i \le k$, the number of permutations (π, C) where $\pi \in \mathfrak{S}_{\boldsymbol{a}}$ and $(j \in A_i, \pi(j) = j) \Rightarrow C(j) = [\mu_i]$ is given by*

$$\sum_{\boldsymbol{0}\le\boldsymbol{c}\le\boldsymbol{1}}\ \sum_{\boldsymbol{0}\le\boldsymbol{b}\le\boldsymbol{a}-\boldsymbol{c}} (-1)^{\sum b_j} f_\lambda\left(\sum a_j-\sum c_j-\sum b_j\right)\prod_i \binom{a_i-c_i}{b_i} f_\lambda(b_i)\cdot\mu_i^{c_i}.$$

Proof: The numbers c_i are one if A_i contains a fixed point and zero otherwise. We may remove these fixed points and consider a fixed point free permutation, enumerated above. We then reinsert the fixed points and colour them in every allowed combination. $\square$

7 A correlation result

Taking a permutation at uniformly random in $\mathfrak{S}_n$, the chances are about 1/e that it is a derangement, since there are $n!\exp_n(-1)$ derangements in $\mathfrak{S}_n$. Moreover, there are $n!/\boldsymbol{a}!$ permutations in $\mathfrak{S}_{\boldsymbol{a}}$.

If belonging to $\mathfrak{S}_{\boldsymbol{a}}$ and being fixed point free were two independent events, we would have $n!\exp_n(-1)$ permutations in $\Phi^{-1}(D(\boldsymbol{a}))$. This is not the case, although the main term in Equation (13) is this very number. The special case $\boldsymbol{b} = (1,1,\ldots,1)$ of Theorem 7.1 says, that belonging to $\mathfrak{S}_{\boldsymbol{a}}$ and being a derangement are almost always positively correlated events. The sole exception is when $\boldsymbol{a}$ is a single block of odd length, in which case every permutation gets a fixed point when sorted.

For two compositions $\boldsymbol{a}$ and $\boldsymbol{b}$ of n, we say that $\boldsymbol{a} \geq \boldsymbol{b}$ if, when sorted decreasingly, $\sum_{i\leq j} a_i \geq \sum_{i\leq j} b_i$ for all j. Then we get the following monotonicity theorem.

Theorem 7.1 *If $\boldsymbol{a} \geq \boldsymbol{b}$ and $\boldsymbol{a}$ is not a single block of odd size, then*

$$|\Phi_{\boldsymbol{a}}^{-1}(D(\boldsymbol{a}))| \geq |\Phi_{\boldsymbol{b}}^{-1}(D(\boldsymbol{b}))|.$$

The theorem follows from a series of lemmata, all included in (7). The main point is proving that shifting any position from a smaller block to a larger one almost never decreases the number $|\Phi_{\boldsymbol{a}}^{-1}(D(\boldsymbol{a}))|$. Equivalently, for fixed $a_3, \ldots a_k$, and $a = a_1 + a_2$ fixed, the function $|\Phi_{\boldsymbol{a}}^{-1}(D(\boldsymbol{a}))|$ is unimodal in a_1 (with the trivial exception).

A weaker, but perhaps more natural version of the correlation result is the following.

Corollary 7.2 *Let $n \geq 2$ and let $\pi \in \mathfrak{S}_n$ be chosen uniformly at random. Then the number of descents in π is positively correlated with the event that π is a derangement.*

Proof: For any i let χ_i be the indicator variable of the event that i is a descent of π. Let $\mathrm{Der}(\pi)$ be the event that π is a derangement. With $\boldsymbol{a} = (1,\ldots,1,2,1,\ldots,1)$ and $\boldsymbol{b} = (1,\ldots,1)$ in Theorem 7.1, we see that each χ_i is positively correlated with $\mathrm{Der}(\pi)$.

The number of descents in π is just $\sum_i \chi_i$, and thus also has positive correlation with $\mathrm{Der}(\pi)$. $\square$

8 Euler's difference tables fixed point coloured

Leonard Euler introduced the integer table $(e_n^k)_{0\leq k\leq n}$ by defining $e_n^n = n!$ and $e_n^{k-1} = e_n^k - e_{n-1}^{k-1}$ for $1 \leq k \leq n$. Apparently, he never gave a combinatorial interpretation, but a simple one was given by Dumont and Randrianarivony in (5). Indeed, e_n^k gives the number of permutations $\pi \in \mathfrak{S}_n$ such that there are no fixed points on the last $n-k$ positions. Thus, $e_n^0 = D_n$. A q-analogue of the same result was given in (3).

It is clear from the recurrence that $k!$ divides e_n^k. Thus, we can define the integers $d_n^k = e_n^k/k!$. These have recently been studied by Fanja Rakotondrajao (10), and the combinatorial interpretation of d_n^k given there was that they count the number of permutations $\pi \in \mathfrak{S}_n$ such that there are no fixed points on the last $n-k$ positions and such that the first k elements are all in different cycles.

We will now generalise these integer tables to any number λ of fixed point colours, and give a combinatorial interpretation that is more in line with the context of this article. The relations in (10) generalise nicely to the case with general λ.

Let $e_n^k(\lambda)$ be defined by $e_n^n(\lambda) = n!$ and $e_n^{k-1}(\lambda) = e_n^k(\lambda) + (\lambda-1)e_{n-1}^{k-1}(\lambda)$. Then, a natural combinatorial interpretation for non-negative integer λ is that $e_n^k(\lambda)$ count the number of permutations $\pi \in \mathfrak{S}_n$ such that fixed points on the last $n-k$ positions may be coloured in any one of λ colours.

Similarly, we can define $d_n^k(\lambda) = e_n^k(\lambda)/k!$ and interpret these numbers as counting the number of permutations $\pi \in \mathfrak{S}_{(k,1,1,\dots,1)} \subseteq \mathfrak{S}_n$ such that fixed points on the last $n-k$ positions may be coloured in any one of λ colours. The set of these permutations is denoted $D_n^k(\lambda)$. Thus, our intepretation for $\lambda = 0$ states that apart from forbidding fixed points at the end, we also demand that the first k elements are in descending order. Equivalently, we could have considered permutations ending with $k-1$ ascents and having λ fixed point colours in the first $n-k$ positions, to be closer to the setting in (4).

There are a couple of relations that can proven bijectively with this interpretation, generalising the results with $\lambda = 0$ from (10). The following propositions are all given bijective proofs in (7).

Proposition 8.1 *For integers $1 \le k \le n$ and $\lambda \in \mathbb{C}$ we have*

$$d_n^{k-1}(\lambda) = kd_n^k(\lambda) + (\lambda-1)d_{n-1}^{k-1}(\lambda).$$

Proposition 8.2 *For integers $0 \le k \le n-1$ and $\lambda \in \mathbb{C}$ we have*

$$d_n^k(\lambda) = nd_{n-1}^k(\lambda) + (\lambda-1)d_{n-2}^{k-1}(\lambda).$$

Proposition 8.3 *For integers $0 \le k \le n-1$ and $\lambda \in \mathbb{C}$ we have*

$$d_n^k(\lambda) = (n + (\lambda-1))\, d_{n-1}^k(\lambda) - (\lambda-1)\,(n-k-1)\, d_{n-2}^k(\lambda).$$

These formulae allow us to once again deduce the recursion for the λ-factorials. Using Proposition 8.3 extended to $k=-1$ and $d_{-1}^{-1}(\lambda) = 1$, we get by induction $d_n^{-1} = (\lambda-1)d_{n-1}^{-1}$ and hence $d_n^{-1} = (\lambda-1)^{n+1}$. Thus, by Proposition 8.2 we have $f_\lambda(n) = d_n^0 = nd_{n-1}^0 + (\lambda-1)^n$. We can also use Proposition 8.3 to obtain, using (7) and $f_\lambda(n) = d_n^0$, that

$$\begin{aligned} f_\lambda(n) &= (n+\lambda-1)f_\lambda(n-1) - (\lambda-1)(n-1)f_\lambda(n-2) \\ &= (n-1)\left(f_\lambda(n-1) + f_\lambda(n-2)\right) + \lambda\left(f_\lambda(n-1) - (n-1)f_\lambda(n-2)\right) \\ &= (n-1)\left(f_\lambda(n-1) + f_\lambda(n-2)\right) + \lambda(\lambda-1)^{n-1}, \end{aligned}$$

which specialises to the well-known

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

and $n! = (n-1)((n-1)! + (n-2)!)$.

We close this section by noting that Lemma 5.1 can be generalised to $d_n^k(\lambda)$ as follows. The proof is completely analogous.

Proposition 8.4 *For $\nu, \lambda \in \mathbb{C}$ and $0 \le k \le n \in \mathbb{N}$, we have*

$$d_n^k(\nu) = \sum_j \binom{n-k}{j} d_{n-j}^k(\lambda)(\nu-\lambda)^j.$$

9 Open problems

While many of our results have been shown bijectively, there are a few that still seek their combinatorial explanation. The most obvious are these.

Problem 9.1 *Give a combinatorial proof, using the principle of inclusion-exclusion, of Theorem 6.1.*

Problem 9.2 *Give a bijection $f : \mathfrak{S}_n \to \mathfrak{S}_n$ such that $\pi \in D(a_1, a_2, \ldots a_k) \Rightarrow f(\pi) \in D(a_1 + 1, a_2 - 1, a_3, \ldots, a_k)$ whenever $a_1 \ge a_2$ and $\boldsymbol{a} \ne (2m, 1)$.*

We would also like the rearrangement of blocks in $D(\boldsymbol{a})$ to get a simple description.

Problem 9.3 *For any $(a_1, \ldots, a_k)$ and any $\sigma \in \mathfrak{S}_k$, give a simple bijection $f : D(a_1, \ldots, a_k) \to D(a_{\sigma_1}, \ldots, a_{\sigma_k})$.*

Instead of specifying descents, we could specify spots where the permutation must not descend. This would add some new features to the problem, as ascending blocks can contain several fixed points, whereas descending blocks can only contain one.

Problem 9.4 *Given a composition $\boldsymbol{a}$, find the number of derangements that ascend within the blocks.*

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Bijections between noncrossing and nonnesting partitions for classical reflection groups

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We present *type preserving* bijections between noncrossing and nonnesting partitions for all classical reflection groups, answering a question of Athanasiadis and Reiner. The bijections for the abstract Coxeter types B, C and D are new in the literature. To find them we define, for every type, sets of statistics that are in bijection with noncrossing and nonnesting partitions, and this correspondence is established by means of elementary methods in all cases. The statistics can be then seen to be counted by the generalized Catalan numbers $\mathrm{Cat}(W)$ when W is a classical reflection group. In particular, the statistics of type A appear as a new explicit example of objects that are counted by the classical Catalan numbers.

Keywords: noncrossing partition, nonnesting partition, reflection group, catalan number, coxeter group

1 Introduction and background

The Coxeter-Catalan combinatorics is an active field of study in the theory of Coxeter groups, having at its core the numerological concurrences according to which several independently motivated sets of objects to do with a Coxeter group W have the cardinality $\prod_{i=1}^{n}(h+d_i)/d_i$, where h is the Coxeter number of W and $d_1,\ldots,d_n$ its degrees. Two of these sets of objects are

- the *noncrossing partitions* $NC(W)$, which in their classical (type A) avatar are a long-studied combinatorial object harking back at least to Kreweras [6], and in their generalisation to arbitrary Coxeter groups are due to Bessis and Brady and Watt [4, 5]; and
- the *nonnesting partitions* $NN(W)$, introduced by Postnikov [9] for all the finite crystallographic reflection groups simultaneously.

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Athanasiadis [2], and Athanasiadis and Reiner [3] proved in a case-by-case fashion that $|NN(W)| = |NC(W)|$ for all finite Weyl groups W, and asked for a bijective proof. Their work also proved equidistribution by type, cited as Theorem 1.17 below. Our contribution has been to provide a family of bijective proofs, one for each type of classical reflection group, that also address equidistribution by type. The bijections in types B, C, and D have not appeared before in the literature. The ultimate goal in connecting $NN(W)$ and $NC(W)$ from this perspective, a proof both uniform and bijective, remains open.

In the remainder of this section we lay out briefly the definitions of the objects involved: in §1.1, the uniform definitions of nonnesting and noncrossing partitions; in §1.2, a mode of extracting actual partitions from these definitions which our bijections rely upon; in §1.3, the resulting notions for classical reflection groups. In section 2 we present a family of type-preserving bijections between noncrossing and nonnesting partitions for all the classical reflection groups, one type at a time.

Two other papers presenting combinatorial bijections between noncrossing and nonnesting partitions independent of ours, one by Stump [12] and by Mamede [8], appeared essentially simultaneously to it. Both of these limit themselves to types A and B, whereas we also treat type D; our approach is also distinct in its type preservation and in providing additional statistics characterising the new bijections.

1.1 Uniform noncrossing and nonnesting partitions

For noncrossing partitions we follow Armstrong [1, §2.4–6]. The treatment of nonnesting partitions is due to Postnikov [9].

Let W be a finite Coxeter group and consider the dual Coxeter system (W, T).

The set $NC(W)$ of (uniform) noncrossing partitions of W is defined as an interval of the absolute order.

Definition 1.1 *The* absolute order $\mathrm{A}bs(W)$ *of W is the partial order on W such that for $w, x \in W$, $w \leq x$ if and only if*

$$l_T(x) = l_T(w) + l_T(w^{-1}x),$$

where $l_T(w)$ is the minimum length of any expression for w as a product of elements of T. A word for w in T of length $l_T(w)$ will be called a reduced T-word *for w.*

Definition 1.2 *A* standard Coxeter element *of (W, S) is any element of the form $c = s_{\sigma(1)}s_{\sigma(2)}\ldots s_{\sigma(n)}$, where σ is a permutation of the set $[n]$ and $S = \{s_1, s_2, \ldots, s_n\}$, the set of simple generators of W. A* Coxeter element *is any conjugate of a standard Coxeter element in W.*

Definition 1.3 *Relative to any Coxeter element c, the poset of (*uniform*)* noncrossing partitions *is the interval $NC(W, c) = [1, c]$ in the absolute order.*

This definition does not depend on the choice of Coxeter element c, see Armstrong [1]. We use the notation $NC(W)$ for the poset of noncrossing partitions of W with respect to any c.

Now assume W is crystallographic. The set $NN(W)$ of nonnesting partitions is defined in terms of the usual root poset of W.

Definition 1.4 *A (*uniform*)* nonnesting partition *for W is an antichain in the root poset of W. We denote the set of nonnesting partitions of W by $NN(W)$.*

To each root α we have an orthogonal hyperplane $\alpha^\perp$ with respect to $\langle \cdot, \cdot \rangle$, and these define a hyperplane arrangement and a poset of intersections.

Definition 1.5 *The* partition lattice $\Pi(W)$ *of* W *is the intersection poset of reflecting hyperplanes*

$$\{\bigcap_{\alpha \in S} \alpha^\perp : S \subseteq \Phi^+\}.$$

1.2 Classical partitions

Our drawings of partitions are taken from Athanasiadis and Reiner [3]. We have reversed the orderings of the ground sets from Athanasiadis's presentation.

Let W be a classical reflection group.

Definition 1.6 *A* classical partition *for* W *is a partition* $\mathrm{Part}(L)$ *of the set*

$$\{\pm e_i : i = 1, \ldots, n\} \cup \{0\},$$

induced by the partition into fibers of the orthogonal projection to L, for some $L \in \Pi(W)$ *in the standard choice of coordinates.*

We will streamline the notation of classical partitions by writing $\pm i$ for $\pm e_i$. Thus, a classical partition for W is a partition of $\pm[n] = \{1, \ldots, n, -1, \ldots, -n, 0\}$ for some n, symmetric under negation. A classical partition always contains exactly one part fixed by negation, which contains the element 0, namely the fiber over $0 \in L$. Since the position of 0 is predictable given the other elements, in many circumstances we will omit it altogether. If the block containing 0 contains other elements as well, we shall call it a *zero block*. A detailed discussion of what these partitions are for each specific type is found in [2] and [3].

Finally, we introduce the type of a partition.

Definition 1.7 *Let* $\pi = \mathrm{Part}(L)$ *be a classical partition for a classical reflection group* W. *The* type $\mathrm{type}(\pi)$ *of* π *is the conjugacy class of L under the action of* W *on* $\Pi(W)$.

1.3 Classical noncrossing and nonnesting partitions

Definitions of the classes of noncrossing and nonnesting classical partitions are most intuitively presented in terms of a diagrammatic representation, motivating the names "noncrossing" and "nonnesting". After Armstrong [1, §5.1] we call these *bump diagrams*.

Let P be a partition of a totally ordered ground set $(S, <)$.

Definition 1.8 *Let* $G(P)$ *be the graph with vertex set* S *and edge set*

$$\{(s, s') : s <_P s' \text{ and } \nexists s'' \in S \text{ s.t. } s <_P s'' <_P s'\}$$

where $s <_P s'$ *iff* $s < s'$ *and* s *and* s' *are in the same block of* P.

A bump diagram *of* P *is a drawing of* $G(P)$ *in the plane in which the elements of* S *are arrayed along a horizontal line in their given order, all edges lie above this line, and no two edges intersect more than once.*

Definition 1.9 *P is* noncrossing *if its bump diagram contains no two crossing edges, equivalently if $G(P)$ contains no two edges of form (a, c), (b, d) with $a < b < c < d$.*

Definition 1.10 *P is* nonnesting *if its bump diagram contains no two nested edges, equivalently if $G(P)$ contains no two edges of form (a, d), (b, c) with $a < b < c < d$.*

We will abuse the terminology slightly and refer to the bump diagram of P as noncrossing, resp. nonnesting, if P is. We will denote the set of classical noncrossing and nonnesting partitions for W by $NC^{\mathrm{cl}}(W)$, resp. $NN^{\mathrm{cl}}(W)$. To define these sets it remains only to specify the ordered ground set.

Definition 1.11 *A* classical nonnesting partition *for a classical reflection group W is a classical partition for W nonnesting with respect to the ground set*

$$\begin{array}{cl} 1 < \cdots < n+1 & \textit{if } W = A_n; \\ -n < \cdots < -1 < 0 < 1 < \cdots < n & \textit{if } W = B_n; \\ -n < \cdots < -1 < 1 < \cdots < n & \textit{if } W = C_n; \\ -n < \cdots < -1, 1 < \cdots < n & \textit{if } W = D_n. \end{array}$$

Classical nonnesting partitions for B_n differ from those for C_n, reflecting the different root posets. We have specified that 0 is part of the ordered ground set for B_n. Despite that, 0 cannot occur in a classical partition. It is easily seen that its presence is necessary when drawing bump diagrams: the dot 0 "ties down" a problematic edge of the zero block in the middle, preventing it from nesting with the others.

Definition 1.12 *A* classical noncrossing partition *for a classical reflection group W not of type D is a classical partition for W noncrossing with respect to the ground set*

$$\begin{array}{cl} 1 < \cdots < n+1 & \textit{if } W = A_n; \\ -1 < \cdots < -n < 1 < \cdots < n & \textit{if } W = B_n; \\ -1 < \cdots < -n < 1 < \cdots < n & \textit{if } W = C_n. \end{array}$$

Observe that the order $<$ in these ground sets differs from those for nonnesting partitions.

We will draw these circularly. Arrange dots labelled $-2, \ldots, -n$, $2, \ldots, n$ in a circle and place 1 and -1 in the middle. We let 1 and -1 be drawn coincidently, after [3], although it would be better to use two circles as in [7], with a smaller one in the center on which only 1 and -1 lie. There is a standard way to represent partitions of D_n in this setting. The edges we will supply in our diagrams are those delimiting the convex hulls of the blocks.

Definition 1.13 *A* classical noncrossing partition *π for D_n is a classical partition for D_n such that no two blocks have intersecting convex hulls in the circular diagram representing π, except possibly two blocks $\pm B$ meeting only at the middle point.*

See the figures in Section 2 for examples of bump diagrams of every type.

We state the relations between these classical noncrossing and nonnesting partitions and the uniform ones. For $w \in W$, let the *fixed space* $\mathrm{Fix}(w)$ of w be the subspace of $V(W)$ consisting of vectors fixed by w.

Proposition 1.14 *The map $f_{NC} : w \mapsto \mathrm{Part}(\mathrm{Fix}(w))$ is a bijection between $NC(W,c)$ and $NC^{\mathrm{cl}}(W)$, where c is the usual choice of standard Coxeter element. Furthermore, it is an isomorphism of posets, where $NC(W,c)$ is ordered by the absolute order and $NC^{\mathrm{cl}}(W)$ by the refinement order.*

Proposition 1.15 *The map $f_{NN} : S \mapsto \mathrm{Part}(\bigcap_{\alpha\in S}\alpha^{\perp})$ is a bijection between $NN(W)$ and $NN^{\mathrm{cl}}(W)$.*

The distribution of classical noncrossing and nonnesting partitions with respect to type is well-behaved. In the noncrossing case, the images of the conjugacy classes of the group W itself are the same as these conjugacy classes of the action of W on $\Pi(W)$.

One can check that

Proposition 1.16 *Two subspaces $L, L' \in \Pi(W)$ are conjugate if and only if both of the following hold:*

- *the multisets of block sizes $\{|C| : C \in \mathrm{Part}(L)\}$ and $\{|C| : C \in \mathrm{Part}(L')\}$ are equal;*
- *if either $\mathrm{Part}(L)$ or $\mathrm{Part}(L')$ has a zero block, then both do, and these zero blocks have equal size.*

We close this section with the statement of the uniform equidistribution result of Athanasiadis and Reiner.

Theorem 1.17 *Let W be a Weyl group. Let f_{NC} and f_{NN} be the functions of Propositions 1.14 and 1.15. For any type λ we have*

$$|(\mathrm{type}\circ f_{NC})^{-1}(\lambda)| = |(\mathrm{type}\circ f_{NN})^{-1}(\lambda)|.$$

2 Type-preserving classical bijections

Throughout, partitions will be drawn and considered drawn with the greatest elements of their ground sets to the left.

Given any partition, define the order $<_1$ on those of its blocks containing positive elements so that $B <_1 B'$ if and only if the least positive element of B is less than the least positive element of B'.

2.1 Type A

The bijection in type A, which forms the foundation of the ones for the other types, is due to Athanasiadis [2, §3]. We include it here to make this foundation explicit and to have bijections for all the classical groups in one place.

Let π be a classical partition for A_n. Let $M_1 <_1 \cdots <_1 M_m$ be the blocks of π, and a_i the least element of M_i, so that $a_1 < \cdots < a_m$. Let μ_i be the cardinality of M_i. Define the two statistics $a(\pi) = (a_1, \ldots, a_m)$ and $\mu(\pi) = (\mu_1, \ldots, \mu_m)$.

We will say that a list of partition statistics S establishes a bijection for a classical reflection group W if, given either a classical noncrossing partition π^{NC} or a classical nonnesting partition π^{NN} for W, the other one exists uniquely such that $s(\pi^{\mathrm{NC}}) = s(\pi^{\mathrm{NN}})$ for all $s \in S$. We will say it establishes a type-preserving bijection if furthermore π^{NC} and π^{NN} always have the same type.

Theorem 2.1 *The statistics (a, μ) establish a type-preserving bijection for A_n.*

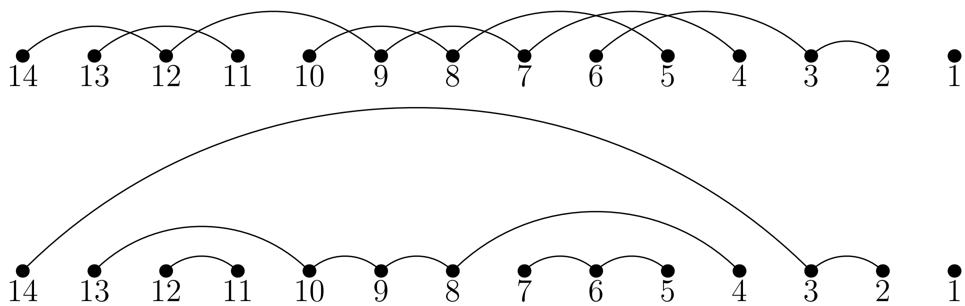

Fig. 1: The type A classical nonnesting (top) and noncrossing (bottom) partitions corresponding to $a = (1, 2, 4, 5, 11)$, $\mu = (1, 3, 5, 3, 2)$.

Figure 1 illustrates this bijection.

The type-preserving assertion is obtained by μ preservation. As for the bijection itself, we describe a process for converting back and forth between classical noncrossing and nonnesting partitions with the same tuples a, μ. Routine verifications are left out of it.

Proof: View $M_1, \dots, M_m$ as *chains*, *i.e.* connected components, in the bump diagram for π, each corresponding to a block. By virtue of μ we know the length of each chain, so we can view the chains as abstract unlabeled graphs in the plane and our task as that of interposing the vertices of these chains in such a way that the result is nonnesting or noncrossing, as desired.

Suppose we start with π^{NN}. To build the noncrossing diagram of π^{NC}, we will inductively place the chains $M_1, \dots, M_m$, in that order. Suppose that, for some $j \leq n$, we have placed M_i for all $i < j$. To place M_j, we insert its rightmost vertex so as to become the a_jth vertex counting from right to left, relative to the chains $M_{j-1}, \dots, M_1$ already placed. We then insert the remaining vertices of M_j in the unique possible way so that no pair of crossing edges are formed. (In this instance, this means that all the vertices of M_j should be placed consecutively, in immediate succession.)

Now, suppose we start with π^{NC} and want π^{NN}. We again build the bump diagram chain by chain, at each step placing the rightmost vertex in exactly the same way and placing the remaining vertices in the unique way so that no pair of nesting edges are formed. (This can be achieved if every edge is drawn the same size, with its vertices at constant Euclidean distance 1.)

Note that, in both directions, all the choices we made were unique, so the resulting partitions are unique. $\square$

A careful study of this proof provides a useful characterisation of the pairs of tuples a, μ that are the statistics of a classical nonnesting or noncrossing partition of type A.

Corollary 2.2 *Suppose we are given a pair of tuples of positive integers $a = (a_1, ..., a_{m_1})$, $\mu = (\mu_1, ..., \mu_{m_2})$ and let $n > 0$. Define $a_0 = 0$ and $\mu_0 = 1$. Then, a and μ represent a classical noncrossing or nonnesting partition for A_n if and only if*

1. *$m_1 = m_2 = m$;*
2. *$n + 1 = \sum_{k=1}^{m} \mu_k$; and*

3. $a_{i-1} < a_i \leq \sum_{k=0}^{i-1} \mu_k$ *for* $i = 1, 2, ..., m$.

2.2 Type C

In the classical reflection groups other than A_n, the negative elements of the ground set must be treated, and so it will be useful to have some terminology to deal with these.

Definition 2.3 *A* positive block *of a classical partition* π *is a block of* π *that contains some positive integer; similarly a* negative block *contains a negative integer. A* switching block *of* π *is a block of* π *that contains both positive and nonpositive elements, and a* nonswitching block *is one that contains positive elements but not negative ones, or the reverse.*

A single edge of the bump diagram is positive *or* negative *or* switching *or* nonswitching *if it would have those properties as a block of size 2.*

Note that in later sections, where there will be elements of the ground set that are neither positive nor negative, there may be blocks that consist only of these and so are neither switching nor nonswitching.

Let π be a classical partition for C_n. Given π, let $M_1 <_1 \cdots <_1 M_m$ be the nonswitching positive blocks of π, and a_i the least element of M_i. Let μ_i be the cardinality of M_i. These two tuples are reminiscent of type A. Let $P_1 <_1 \cdots <_1 P_k$ be the switching blocks of π, let p_i be the least positive element of P_i, and let ν_i be the number of positive elements of P_i. Define the three statistics $a(\pi) = (a_1, \ldots, a_m)$, $\mu(\pi) = (\mu_1, \ldots, \mu_m)$, $\nu(\pi) = (\nu_1, \ldots, \nu_k)$. We have

$$n = \sum_{i=1}^{m} \mu_i + \sum_{j=1}^{k} \nu_j. \tag{1}$$

Theorem 2.4 *The statistics* (a, μ, ν) *establish a type-preserving bijection for* C_n.

Figure 2 illustrates the bijection.

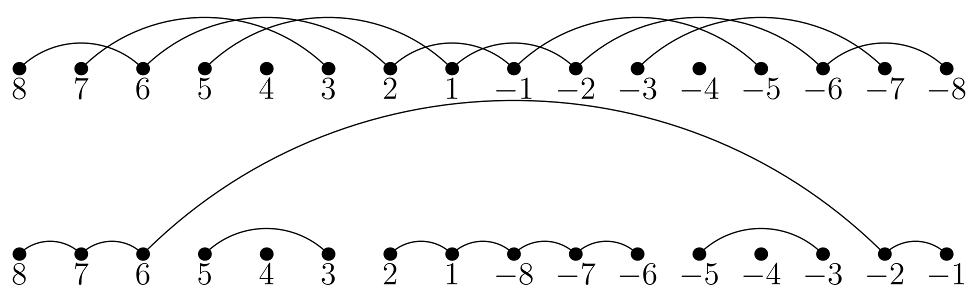

Fig. 2: The type C nonnesting (top) and noncrossing (bottom) partitions corresponding to $a = (3, 4)$, $\mu = (2, 1)$, $\nu = (2, 3)$.

Proof: We state a procedure for converting back and forth between classical noncrossing and nonnesting partitions that preserve the values a, μ, and ν. Suppose we start with a partition π, be it noncrossing π^{NN} or nonnesting π^{NC}, and we want to find the partition π', which respectively would precisely be π^{NC} or

π^{NN}. To do this, we construct the positive side of π' inductively from these tuples, which will determine π' completely.

In the bump diagram of π, consider the labeled connected component representing P_i, which we call the *chain* P_i. Let the *partial chain* P_i' be the abstract unlabeled connected graph obtained from the chain P_i by removing its negative nonswitching edges and negative vertices, leaving the unique switching edge incomplete, *i.e.* partially drawed so that it becomes a clear half-edge in our new abstract graph, and losing the labeling. Notice how the tuple ν allows us to draw these partial chains. Whatever procedure we followed for type A will generalise to this case, treating the positive parts of the switching edges first.

We want to obtain the bump diagram for π', so we begin by using ν to partially draw the chains representing its switching blocks: we draw *only* the positive edges (switching and nonswitching) of every chain, leaving the unique switching edge incomplete. This is done by reading ν from back to front and inserting each partial switching chain P_i' in turn with its rightmost dot placed to the right of all existing chains. In the noncrossing case, we end up with every vertex of P_i' being strictly to the right of every vertex of P_j' for $i < j$. In the nonnesting case, the vertices of the switching edges will be exactly the k first positions from right to left among all the vertices of $P_1', \ldots, P_k'$. It remains to place the nonswitching chains $M_1, M_2, \ldots, M_m$, and this we do as in the type A bijection, except that at each step, we place the rightmost vertex of M_j so as to become the a_jth vertex, counting from right to left, relative to the chains $M_{j-1}, \ldots, M_1$ and the partial chains $P_1', P_2', \ldots, P_k'$ already placed.

Now we have the positive side of π'. We copy these blocks down again with all parts negated, and end up with a set of incomplete switching blocks $P_1*, \ldots, P_k*$ on the positive side and another equinumerous set $-P_1*, \ldots, -P_k*$ on the negative side that we need to pair up and connect with edges in the bump diagram.

There is a unique way to connect these incomplete blocks to get the partition π', be it π^{NC} or π^{NN}. In every case P_i* gets connected with $-P_{k+1-i}*$, and in particular symmetry under negation is attained. If there is a zero block it arises from $P_{(k+1)/2}*$.

Finally, π and π' have the same type. Since the P_i* are paired up the same way in each, including any zero block, μ and ν determine the multiset of block sizes of π and π' and the size of any zero block, in identical fashion in either case. Then this is Proposition 1.16. ☐

Again, a careful look at the preceding proof gives the characterization of the tuples that describe classical noncrossing and nonnesting partitions for type C.

Corollary 2.5 *Suppose we are given some tuples of positive integers $a = (a_1, ..., a_{m_1})$, $\mu = (\mu_1, ..., \mu_{m_2})$, $\nu = (\nu_1, ..., \nu_k)$ and let $n > 0$. Define $a_0 = 0$ and $\mu_0 = 1$. Then, a, μ and ν represent a classical noncrossing or nonnesting partition for C_n if and only if*

1. $m_1 = m_2 = m$;
2. $n = \sum_{i=1}^{m} \mu_i + \sum_{j=1}^{k} \nu_j$;
3. $a_{i-1} < a_i \leq \sum_{k=0}^{i-1} \mu_k + \sum_{j=1}^{k} \nu_j$ *for* $i = 1, 2, ..., m$.

2.3 Type B

We will readily be able to modify our type C bijection to handle type B. Indeed, if it were not for our concern about type, we would already possess a bijection for type B, differing from the type C bijection only in pairing up the incomplete switching blocks in a way respecting the presence of the element 0. Our task is thus to adjust that bijection to recover the type-preservation.

If π is a classical partition for B_n, we define the tuples $a(\pi)$, $\mu(\pi)$ and $\nu(\pi)$ as in type C.

Notice that classical noncrossing partitions for B_n and for C_n are identical, and that the strictly positive part of any classical nonnesting partition for B_n is also the strictly positive part of some nonnesting C_n-partition, though not necessarily one of the same type. Thus Corollary 2.5 characterises the classical noncrossing or nonnesting partitions for B_n just as well as for C_n.

Suppose π is a classical nonnesting partition for B_n. In two circumstances its tuples $a(\pi)$, $\mu(\pi)$, $\nu(\pi)$ also describe a unique nonnesting partition for C_n of the same type: to be explicit, this is when π does not contain a zero block, and when the unique switching chain in π is the one representing the zero block. If $P_1 <_1 \cdots <_1 P_k$ are the switching blocks of π, then π contains a zero block and more than one switching chain if and only if k is odd and $k > 1$. We notice that P_k must be the zero block. On the other hand, if π^{C} is a classical nonnesting partition for C_n, the zero block must be $P_{(k+1)/2}$. Reflecting this, our bijection will be forced to reorder ν to achieve type preservation.

Generalizing our prior language, we will say that two lists $S^{\mathrm{NC}} = \{s_1^{\mathrm{NC}}, \dots, s_a^{\mathrm{NC}}\}$ and $S^{\mathrm{NN}} = \{s_1^{\mathrm{NN}}, \dots, s_a^{\mathrm{NN}}\}$ of partition statistics, in that order, and a list $\Sigma = \{\sigma_1, \dots, \sigma_a\}$ of bijections establish a (type-preserving) bijection for a classical reflection group W if, given either a classical noncrossing partition π^{NC} or a classical nonnesting partition π^{NN} for W, the other one exists uniquely such that $\sigma_i(s_i^{\mathrm{NC}}(\pi^{\mathrm{NC}})) = s_i^{\mathrm{NN}}(\pi^{\mathrm{NN}})$ for all $1 \le i \le a$ (and furthermore π^{NC} and π^{NN} have the same type).

Suppose we have a tuple $\nu = (\nu_1, \dots, \nu_k)$ with k odd. Define the reordering

$$\sigma_B(\nu) = (\nu_1, \dots, \nu_{(k-1)/2}, \nu_{(k+3)/2}, \dots, \nu_k, \nu_{(k+1)/2}).$$

If k is not odd then let $\sigma_B(\nu) = \nu$. Clearly σ_B is bijective. For explicitness, we write for k odd

$$\sigma_B^{-1}(\nu) = (\nu_1, \dots, \nu_{(k-1)/2}, \nu_k, \nu_{(k+1)/2}, \dots, \nu_{k-1})$$

and for k even $\sigma_B^{-1}(\nu) = \nu$.

Theorem 2.6 *The lists of statistics (a, μ, ν) and (a, μ, ν) establish a type-preserving bijection for B_n via the bijections $(\mathrm{id}, \mathrm{id}, \sigma_B)$.*

Proof (Sketch): We use the same procedures as in type C to convert back and forth between classical nonnesting and noncrossing partitions, except that we rearrange ν as appropriate. Notice ν can be reordered in any way, so we order it to have type preservation. □

Figure 3 illustrates the resulting bijection.

2.4 Type D

The handling of type D partitions is a further modification of our treatment of the foregoing types, especially type B.

In classical partitions for D_n, the elements ± 1 will play much the same role as the element 0 of classical nonnesting partitions for B_n. So when applying the order $<_1$ and the terminology of Definition 2.3 in type D we will regard ± 1 as being neither positive nor negative.

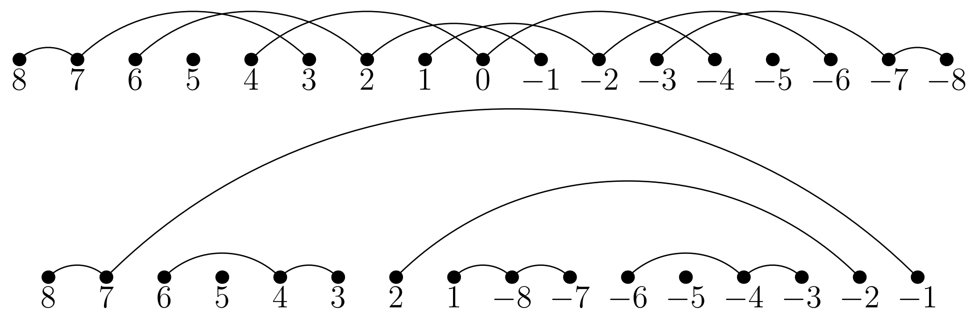

Fig. 3: The type B nonnesting (top) and noncrossing (bottom) partitions corresponding to $a = (3, 5)$, $\mu = (3, 1)$, and respectively $\nu = (1, 2, 1)$ and $\nu = (1, 1, 2)$. Note that $\sigma_B((1, 1, 2)) = (1, 2, 1)$. These correspond under the bijection of Theorem 2.6.

Given $\pi \in NN^{\mathrm{cl}}(D_n)$, define the statistics $a(\pi)$, $\mu(\pi)$ and $\nu(\pi)$ as in type B. Let $R_1 <_1 \cdots <_1 R_l$ be the blocks of π which contain both a positive element and either 1 and -1. It is clear that $l \leq 2$. Define the statistic $c(\pi) = (c_1, \ldots, c_l)$ by $c_i = R_i \cap \{1, -1\}$. To streamline the notation we shall usualy write c_i as one of the symbols $+$, $-$, $\pm$. Observe that π contains a zero block if and only if $c(\pi) = (\pm)$.

To get a handle on type D classical noncrossing partitions, we will transform them into type B ones. Let $NC^{\mathrm{cl}}_{\mathrm{r}}(B_{n-1})$ be a relabelled set of classical noncrossing partitions for B_{n-1}, in which the parts $1, \ldots, (n-1)$ and $-1, \ldots, -n-1$ are changed respectively to $2, \ldots, n$ and $-2, \ldots, -n$. Define a map $CM : NC^{\mathrm{cl}}(D_n) \to NC^{\mathrm{cl}}_{\mathrm{r}}(B_{n-1})$, which we will call *central merging*, such that for $\pi \in NN^{\mathrm{cl}}(D_n)$, $CM(\pi)$ is the classical noncrossing B_{n-1}-partition obtained by first merging the blocks containing ± 1 (which we have drawn at the center of the circular diagram) into a single part, and then discarding these elements ± 1. Define the statistics a, μ and ν for π to be equal to those for $CM(\pi)$, where the entries of a should acknowledge the relabelling and thus be chosen from $\{2, \ldots, n\}$.

These statistics do not uniquely characterise π, so we define additional statistics $c(\pi)$ and $\xi(\pi)$. The definition of $c(\pi)$ is analogous to the nonnesting case: let $R_1 <_1 \cdots <_1 R_l$ be the blocks of π which intersect $\{1, -1\}$, and define $c(\pi) = (c_1, \ldots, c_l)$ where $c_i = R_i \cap \{1, -1\}$. Also define $\zeta(\pi) = (\zeta_1, \ldots, \zeta_l)$ where $\zeta_l = \#(R_l \cap \{2, \ldots, n\})$ is the number of positive parts of R_l.

Observe that $CM(\pi)$ lacks a zero block if and only if $c(\pi) = ()$, the case that 1 and -1 both belong to singleton blocks of π. In this case $CM(\pi)$ is just π with the blocks $\{1\}$ and $\{-1\}$ removed, so that π is uniquely recoverable given $CM(\pi)$. Otherwise, $CM(\pi)$ has a zero block. If $c(\pi) = (\pm)$ this zero block came from a zero block of π, and π is restored by resupplying ± 1 to this zero block. Otherwise two blocks of π are merged in the zero block of $CM(\pi)$. Suppose the zero block of $CM(\pi)$ is $\{c_1, \ldots, c_j, -c_1, \ldots, -c_j\}$, with $0 < c_1 < \cdots < c_j$, so that $j = \sum_{i=1}^{l} \zeta_l$. By the noncrossing and symmetry properties of π, one of the blocks of π which was merged into this block has the form $\{-m_{i+1}, \ldots, -m_j, m_1, \ldots, m_i, s\}$ where $1 \leq i \leq j$ and $s \in \{1, -1\}$. Then, by definition, $c(\pi) = (s, -s)$ and $\xi(\pi) = (i, j - i)$, except that if $j - i = 0$ the latter component of each of these must be dropped.

Let a *tagged* noncrossing partition for B_{n-1} be an element $\pi \in NC^{\mathrm{cl}}_{\mathrm{r}}(B_{n-1})$ together with tuples $c(\pi)$ of nonempty subsets of $\{1, -1\}$ and $\zeta(\pi)$ of positive integers such that:

1. the entries of $c(\pi)$ are pairwise disjoint;
2. $c(\pi)$ and $\zeta(\pi)$ have equal length;
3. the sum of all entries of $\zeta(\pi)$ is the number of positive elements in the zero block of π.

We have the following important lemmas whose proof we omit in this abstract.

Lemma 2.7 *Central merging gives a bijection between classical noncrossing partitions for D_n and tagged noncrossing partitions for B_{n-1}.*

Lemma 2.8 *A classical nonnesting partition π for D_n is uniquely determined by the values of $a(\pi)$, $\mu(\pi)$, $\nu(\pi)$, and $c(\pi)$.*

All that remains to obtain a bijection is to describe the modifications to ν that are needed for correct handling of the zero block and its components (rather as in type B). For a classical nonnesting partition π for D_n, find the tuples $a(\pi)$, $\mu(\pi)$, $\nu(\pi) = (\nu_1, \ldots, \nu_k)$, and $c(\pi)$. Let $\xi(\pi)$ be the tuple of the last l entries of $\nu(\pi)$, where l is the length of $c(\pi)$. Define

$$\hat{\nu}(\pi) = \begin{cases} (\nu_1, \ldots, \nu_{k/2-1}, \nu_{k-1} + \nu_k, \nu_{k/2} \ldots, \nu_{k-2}) & \text{if } l = 2 \\ (\nu_1, \ldots, \nu_{(k-1)/2}, \nu_k, \nu_{(k+1)/2} \ldots, \nu_{k-1}) & \text{if } l = 1 \\ \nu(\pi) & \text{if } l = 0 \end{cases}$$

Define a bijection σ_D by $\sigma_D(\nu(\pi)) = (\hat{\nu}(\pi), \xi(\pi))$. This gives us all the data for a tagged noncrossing partition $CM(\pi')$ for B_{n-1}, which corresponds via central merging with a noncrossing partition π' for D_n. Going backwards, from a noncrossing partition π' we recover a nonnesting partition π by applying central merging, finding the list of statistics $(a(\pi), \mu(\pi), \nu(\pi), c(\pi))$ via the equality

$$\nu(\pi) = \sigma_D^{-1}(\nu(\pi'), \xi(\pi'))$$

(the other statistics remain equal) and using these statistics to make a nonnesting partition as usual. Type preservation may be seen to be implied within these modifications of the statistics.

All in all, we have just proved the following theorem.

Theorem 2.9 *The lists of statistics $(a, \mu, (\nu, \xi), c)$ and (a, μ, ν, c) establish a type-preserving bijection for D_n via the bijections* $(\mathrm{id}, \mathrm{id}, (\sigma_D)^{-1}, \mathrm{id})$.

Figures 4 and 5 illustrate this bijection.

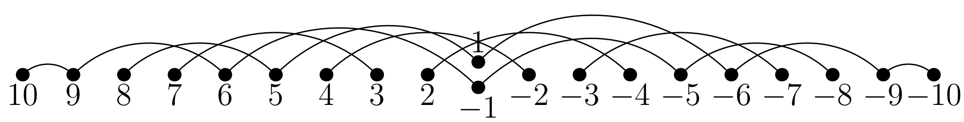

Fig. 4: The D_{10} nonnesting partition corresponding to $a = (3)$, $\mu = (2)$, $\nu = (1, 1, 2, 3)$, $c = (+, -)$ (so $\hat{\nu} = (1, 5, 1)$).

We omit the characterization of the tuples representing noncrossing and nonnesting partitions of type D.

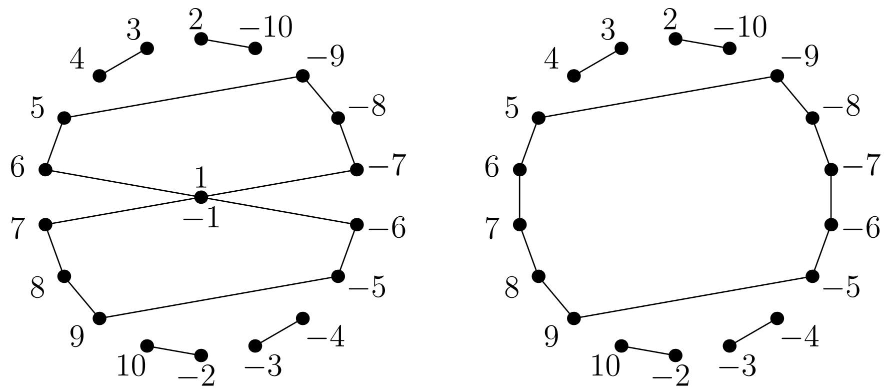

Fig. 5: (left) The D_{10} noncrossing partition corresponding to $a = (3)$, $\mu = (2)$, $\nu = (1, 5, 1)$, $\xi = (2, 3)$, $c = (+, -)$. (right) The relabelled type B noncrossing partition obtained via central merging.

3 Acknowledgments

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New Hopf Structures on Binary Trees

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Abstract. The multiplihedra $\mathcal{M}_\bullet = (\mathcal{M}_n)_{n\geq 1}$ form a family of polytopes originating in the study of higher categories and homotopy theory. While the multiplihedra may be unfamiliar to the algebraic combinatorics community, it is nestled between two families of polytopes that certainly are not: the permutahedra $\mathfrak{S}_\bullet$ and associahedra $\mathcal{Y}_\bullet$. The maps $\mathfrak{S}_\bullet \twoheadrightarrow \mathcal{M}_\bullet \twoheadrightarrow \mathcal{Y}_\bullet$ reveal several new Hopf structures on tree-like objects nestled between the Hopf algebras $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$. We begin their study here, showing that $\mathcal{M}Sym$ is a module over $\mathfrak{S}Sym$ and a Hopf module over $\mathcal{Y}Sym$. An elegant description of the coinvariants for $\mathcal{M}Sym$ over $\mathcal{Y}Sym$ is uncovered via a change of basis—using Möbius inversion in posets built on the 1-skeleta of $\mathcal{M}_\bullet$. Our analysis uses the notion of an *interval retract* that should be of independent interest in poset combinatorics. It also reveals new families of polytopes, and even a new factorization of a known projection from the associahedra to hypercubes.

Résumé. Les multiplièdres $\mathcal{M}_\bullet = (\mathcal{M}_n)_{n\geq 1}$ forment une famille de polytopes en provenant de l'étude des catègories supérieures et de la théorie de l'homotopie. Tandis que les multiplihèdres sont peu connus dans la communauté de la combinatoire algébrique, ils sont nichés entre deux familles des polytopes qui sont bien connus: les permutahèdres $\mathfrak{S}_\bullet$ et les associahèdres $\mathcal{Y}_\bullet$. Les morphismes $\mathfrak{S}_\bullet \twoheadrightarrow \mathcal{M}_\bullet \twoheadrightarrow \mathcal{Y}_\bullet$ dévoilent plusieurs nouvelles structures de Hopf sur les arbres binaires entre les algèbres de Hopf $\mathfrak{S}Sym$ et $\mathcal{Y}Sym$. Nous commençons son étude ici, en démontrant que $\mathcal{M}Sym$ est un module sur $\mathfrak{S}Sym$ et un module de Hopf sur $\mathcal{Y}Sym$. Une description élégante des coinvariants de $\mathcal{M}Sym$ sur $\mathcal{Y}Sym$ est trouvée par moyen d'une change de base—en utilisant une inversion de Möbius dans certains posets construits sur le 1-squelette de $\mathcal{M}_\bullet$. Notre analyse utilise la notion d'*interval retract*, ce devrait être intéressante par soi-même dans la théorie des ensembles partiellement ordonés. Notre analyse donne lieu également a des nouvelles familles des polytopes, et même une nouvelle factorisation d'une projection connue des associahèdres aux hypercubes.

Keywords: multiplihedron, permutations, permutahedron, associahedron, binary trees, Hopf algebras

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Introduction

In the past 30 years, there has been an explosion of interest in combinatorial Hopf algebras related to the classical ring of symmetric functions. This is due in part to their applications in combinatorics and representation theory, but also in part to a viewpoint expressed in the elegant commuting diagram

$$\begin{tikzcd} NSym \arrow[r, hook] \arrow[d, two heads] & \mathfrak{S}Sym \arrow[d, two heads] \\ Sym \arrow[r, hook] & \mathcal{Q}Sym \end{tikzcd} .$$

Namely, much information about an object may be gained by studying how it interacts with its surroundings. From this picture, we focus on the right edge, $\mathfrak{S}Sym \twoheadrightarrow \mathcal{Q}Sym$. We factor this map through finer and finer structures (some well-known and some new) until this edge is replaced by a veritable zoo of Hopf structures. A surprising feature of our results is that each of these factorizations may be given geometric meaning—they correspond to successive polytope quotients from permutahedra to hypercubes.

The (known) cast of characters

Let us reacquaint ourselves with some of the characters who have already appeared on stage.

$\mathfrak{S}Sym$ – the Hopf algebra introduced by Malvenuto and Reutenauer [13] to explain the isomorphism $\mathcal{Q}Sym \simeq (NSym)^*$. A graded, noncommutative, noncocommutative, self-dual Hopf algebra, with basis indexed by permutations, it offers a natural setting to practice noncommutative character theory [4].

$\mathcal{Y}Sym$ – the (dual of the) Hopf algebra of trees introduced by Loday and Ronco [11]. A graded, noncommutative, noncocommutative Hopf algebra with basis indexed by planar binary trees, it is important for its connections to the Connes-Kreimer renormalization procedure.

$\mathcal{Q}Sym$ – The Hopf algebra of quasisymmetric functions introduced by Gessel [9] in his study of P-partitions. A graded, commutative, noncocommutative Hopf algebra with basis indexed by compositions, it holds a special place in the world of combinatorial Hopf algebras [1].

The new players

In this extended abstract, we study in detail a family of planar binary trees that we call *bi-leveled trees*, which possess two types of internal nodes (circled or not, subject to certain rules). These objects are the vertices of Stasheff's multiplihedra [18], originating from his study of A_∞ categories. The multiplihedra were given the structure of CW-complexes by Iwase and Mimura [10] and realized as polytopes later [8]. They persist as important objects of study, among other reasons, because they catalog all possible ways to multiply objects in the domain and range of a function f, when both have nonassociative multiplication rules. More recently, they have appeared as moduli spaces of "stable quilted discs" [14].

In Section 2, we define a vector space $\mathcal{M}Sym$ with basis indexed by these bi-leveled trees. We give $\mathcal{M}Sym$ a module structure for $\mathfrak{S}Sym$ by virtue of the factorization

$$\mathfrak{S}Sym \xrightarrow{\boldsymbol{\beta}} \mathcal{M}Sym \xrightarrow{\boldsymbol{\phi}} \mathcal{Y}Sym$$

(evident on the level of planar binary trees) and a splitting $\mathcal{M}Sym \hookrightarrow \mathfrak{S}Sym$. We also show that $\mathcal{M}Sym$ is a Hopf module for $\mathcal{Y}Sym$ and we give an explicit realization of the fundamental theorem of Hopf modules. That is, we find the coinvariants for this action. Our proof, sketched in Section 3, rests on a result about poset maps of independent interest.

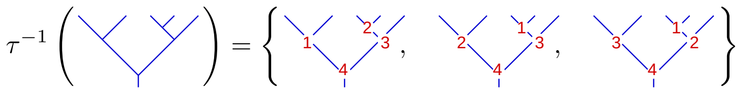
We conclude in Section 4 with a massive commuting diagram—containing several new families of planar binary trees—that further factors the map from $\mathfrak{S}Sym$ to $\mathcal{Q}Sym$. The remarkable feature of this diagram is that it comes from polytopes (some of them even new) and successive polytope quotients. Careful study of the interplay between the algebra and geometry will be carried out in future work.

1 Basic combinatorial data

1.1 Ordered and planar binary trees

We recall a map τ from permutations $\mathfrak{S}_\bullet = \bigcup_n \mathfrak{S}_n$ to planar binary trees $\mathcal{Y}_\bullet = \bigcup_n \mathcal{Y}_n$ that has proven useful in many contexts [19, 12]. Its behavior is best described in the reverse direction as follows. Fix a tree $t \in \mathcal{Y}_n$. The n internal nodes of t are equipped with a partial order, viewing the root node as maximal. An *ordered tree* is a planar binary tree, together with a linear extension of the poset of its nodes. These are in bijection with permutations, as the nodes are naturally indexed left-to-right by the numbers $1, \ldots, n$. The map τ takes an ordered tree (permutation) to the unique tree whose partial order it extends.

Example 1 The permutations 1423, 2413, and 3412 share a common image under τ:



There are two right inverses to τ that will be useful later. Let $\mathsf{min}(t)$ (respectively, $\mathsf{max}(t)$) denote the unique 231-avoiding (132-avoiding) permutation mapping to t under τ. Loday and Ronco show that $\tau^{-1}(t)$ is the interval $[\mathsf{min}(t), \mathsf{max}(t)]$ in the weak Bruhat order on the symmetric group [12, Thm. 2.5], and that min and max are both order-preserving with respect to the Tamari order on $\mathcal{Y}_n$.

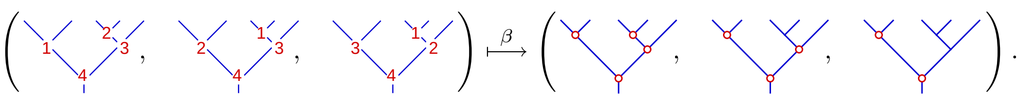
1.2 Bi-leveled trees and the multiplihedra

We next describe a family of *bi-leveled trees* intermediate between the ordered and unordered ones. These trees arrange themselves as vertices of the *multiplihedra* $\mathcal{M}_\bullet = \bigcup_n \mathcal{M}_n$, a family of polytopes introduced by Stasheff in 1970 [18] (though only proven to be polytopes much later [8]). Stasheff introduced this family to represent the fundamental structure of a weak map f between weak structures, such as weak n-categories or A_n spaces. The vertices of $\mathcal{M}_n$ correspond to associations of n objects, pre- and post-application of f, e.g., $(f(a)f(b))f(c)$ and $f(a)f(bc)$. This leads to a natural description of $\mathcal{M}_n$ in terms of "painted binary trees" [5], but we use here the description of Saneblidze and Umble [16].

A *bi-leveled tree* is a pair (t, C) with $t \in \mathcal{Y}_n$ and $C \subseteq [n]$ designating some nodes of t as lower than the others (indexing the nodes from left-to-right by $1, \ldots, n$). Viewing t as a poset with root node maximal, C is an increasing order ideal in t where the leftmost node is a minimal element. Graphically, C indexes a collection of nodes of t circled according to the rules: (i) the leftmost node is circled and has no circled children; (ii) if a node is circled, then its parent node is circled.

Define a map β from $\mathfrak{S}_n$ to $\mathcal{M}_n$ as follows. Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, first represent σ as an ordered tree. Next, forget the ordering on the nodes, save for circling all nodes σ_i with $\sigma_i \geq \sigma_1$.

Example 2 Consider again the permutations 1423, 2413, and 3412 of Example 1. Viewed as ordered trees, their images under β are distinct:



Denote by ϕ the map from bi-leveled trees to trees that forgets which nodes are circled. The map ϕ helps define a partial order on bi-leveled trees that extends the Tamari lattice on planar binary trees: say that the bi-leveled tree s precedes the bi-leveled tree t in the partial order if $\phi(s) \leq \phi(t)$ and the circled nodes satisfy $C_t \subseteq C_s$. We call this the weak order on bi-leveled trees. See Figure 1 below for an example.

The equality $\phi \circ \beta = \tau$ is evident. Remarkably, this factorization

$$\mathfrak{S}_\bullet \xrightarrow{\beta} \mathcal{M}_\bullet \xrightarrow{\phi} \mathcal{Y}_\bullet$$

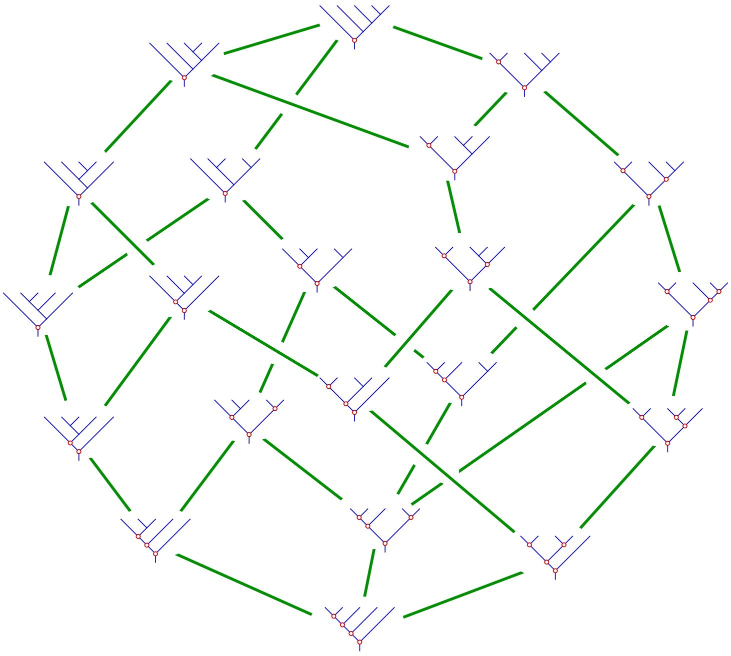


FIG. 1: The weak order on $\mathcal{M}_4$, the bi-leveled trees on 4 nodes.

extends to the level of face maps between polytopes (see Figure 2). Our point of departure was the observation that it is also a factorization as poset maps.

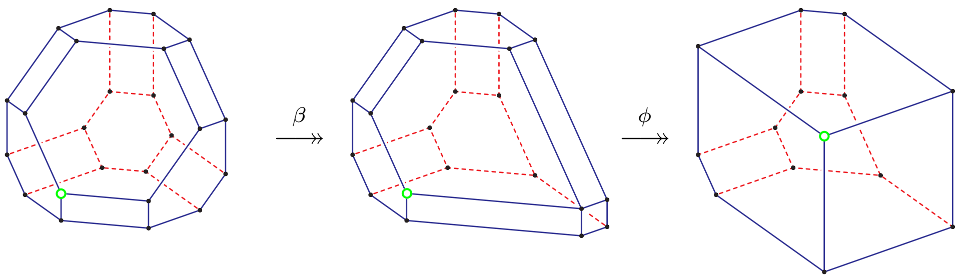


FIG. 2: β and ϕ extend to face (and poset) maps from the permutahedra to the associahedra. The distinguished vertices 1234, $\beta(1234)$, and $\phi(\beta(1234))$ are indicated.

1.3 Dimension enumeration

Fix a field k of characteristic zero and let $\mathfrak{S}Sym = \bigoplus_{n\geq 0} \mathfrak{S}Sym_n$ denote the graded vector space whose n^{th} graded piece has the "fundamental" basis $\{F_\sigma \mid \sigma$ an ordered tree in $\mathfrak{S}_n\}$. Define $\mathcal{M}Sym$ and $\mathcal{Y}Sym$ similarly, replacing $\mathfrak{S}_n$ by $\mathcal{M}_n$ and $\mathcal{Y}_n$, respectively. We follow convention and say that $\mathfrak{S}Sym_0$ and $\mathcal{Y}Sym_0$ are 1-dimensional. By contrast, we agree that $\mathcal{M}Sym_0 = \{0\}$. (See [7] for categorical rationale; briefly, Stasheff's $\mathcal{M}_1$ is already 0-dimensional, so $\mathcal{M}_0$ has no clear significance.)

In Section 2, we give these three vector spaces a variety of algebraic structures. Here we record some information about the dimensions of the graded pieces for later reference.

$$\mathrm{Hilb}_q(\mathfrak{S}Sym) = \sum_{n\geq 0} n!\, q^n = 1 + q + 2q^2 + 6q^3 + 24q^4 + 120q^5 \cdots \tag{1}$$

$$\mathrm{Hilb}_q(\mathcal{M}Sym) = \sum_{n\geq 1} A_n q^n = \qquad q + 2q^2 + 6q^3 + 21q^4 + 80q^5 + \cdots \tag{2}$$

$$\mathrm{Hilb}_q(\mathcal{Y}Sym) = \sum_{n\geq 0} C_n q^n = 1 + q + 2q^2 + 5q^3 + 14q^4 + 42q^5 + \cdots \tag{3}$$

Of course, C_n is the n^{th} Catalan number. The enumeration of bi-leveled trees is less familiar: the n^{th} term satisfies $A_n = C_{n-1} + \sum_{k=1}^{n-1} A_i\, A_{n-i}$ [17, A121988]. A little generating function arithmetic can show that the quotient of (2) by (3) expands as a power series with nonnegative coefficients,

$$\frac{\mathrm{Hilb}_q(\mathcal{M}Sym)}{\mathrm{Hilb}_q(\mathcal{Y}Sym)} = q + q^2 + 3q^3 + 11q^4 + 44q^5 + \cdots . \tag{4}$$

We will recover this with a little algebra in Section 2.3. The positivity of the quotient of (1) by (3) is established by [3, Theorem 7.2].

2 The Hopf module $\mathcal{M}Sym$

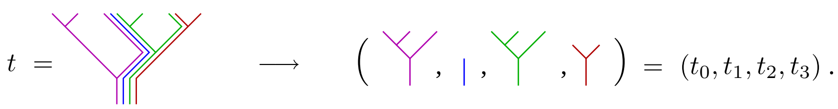
Let $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, and $\boldsymbol{\phi}$ be the maps between the vector spaces $\mathfrak{S}Sym$, $\mathcal{M}Sym$, and $\mathcal{Y}Sym$ induced by τ, β, and ϕ on the fundamental bases. That is, for permutations σ and bi-leveled trees t, we take

$$\boldsymbol{\tau}(F_\sigma) = F_{\tau(\sigma)} \qquad \boldsymbol{\beta}(F_\sigma) = F_{\beta(\sigma)} \qquad \boldsymbol{\phi}(F_t) = F_{\phi(t)}$$

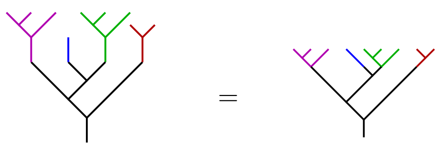
Below, we recall the product and coproduct structures on the Hopf algebras $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$. In [13] and [11], these were defined in terms of the fundamental bases. Departing from these definitions, rich structural information was deduced about $\mathfrak{S}Sym$, $\mathcal{Y}Sym$, and the Hopf algebra map $\boldsymbol{\tau}$ between them in [2, 3]. This information was revealed via a change of basis—from fundamental to "monomial"—using Möbius inversion. We take the same tack below with $\mathcal{M}Sym$ and meet with similar success.

2.1 The Hopf algebras $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$

Following [3], we define the product and coproduct structures on $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ in terms of p-*splittings* and *graftings* of trees. A *p-splitting* of a tree t with n nodes is a forest (sequence) of $p+1$ trees with n nodes in total. This sequence is obtained by choosing p leaves of t and splitting them (and all parent branchings) right down to the root. By way of example, consider the 3-splitting below (where the third leaf is chosen twice and the fifth leaf is chosen once).



Denote a p-splitting of t by $t \xrightarrow{\curlyvee} (t_0, \dots, t_p)$. The *grafting* of a forest $(t_0, t_1, \dots, t_p)$ onto a tree with p nodes is also best described in pictures; for the forest above and $s = \tau(213)$, the tree



is the grafting of $(t_0, t_1, \ldots, t_p)$ onto s, denoted $(t_0, t_1, t_2, t_3)/s$. Splittings and graftings of ordered trees are similarly defined. One remembers the labels originally assigned to the nodes of t in a p-splitting, and if t has q nodes, then one increments the labels of s by q in a grafting $(t_0, t_1, t_2, t_3)/s$. See [3] for details.

Definition 3 Fix two ordered or ordinary trees s and t with p and q internal nodes, respectively. We define the product and coproduct by

$$F_t \cdot F_s \;=\; \sum_{t \xrightarrow{\curlyvee} (t_0, t_1, \ldots, t_p)} F_{(t_0, t_1, \ldots, t_p)/s} \qquad \text{and} \qquad \Delta(F_t) \;=\; \sum_{t \xrightarrow{\curlyvee} (t_0, t_1)} F_{t_0} \otimes F_{t_1}\,. \tag{5}$$

(In the coproduct for ordered trees, the labels in t_0 and t_1 are reduced to be permutations of $|t_0|$ and $|t_1|$.)

2.2 Module and comodule structures

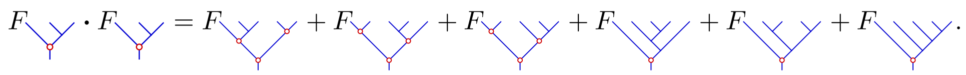
We next modify the structure maps in (5) to give $\mathcal{M}Sym$ the structure of (left) $\mathfrak{S}Sym$-module and (right) $\mathcal{Y}Sym$–Hopf module. Given a bi-leveled tree b, let $b \xrightarrow{\curlyvee} (b_0, \ldots, b_p)$ represent any p-splitting of the underlying tree, together with a circling of all nodes in each b_i that were originally circled in b.

Definition 4 **(***action* of $\mathfrak{S}Sym$ on $\mathcal{M}Sym$**)** For $w \in \mathfrak{S}_{\boldsymbol{\cdot}}$ and $s \in \mathcal{M}_p$, write $b = \beta(w)$ and set

$$F_w \cdot F_s \;=\; \sum_{b \xrightarrow{\curlyvee} (b_0, b_1, \ldots, b_p)} F_{(b_0, b_1, \ldots, b_p)/s} \tag{6}$$

where the circling rules in $(b_0, b_1, \ldots, b_p)/s$ are as follows: every node originating in s is circled whenever $|b_0| > 0$, otherwise, every node originating in $b = \beta(w)$ is uncircled.

This action may be combined with any section of β to define a product on $\mathcal{M}Sym$. For example,



Theorem 5 *The action $\mathfrak{S}Sym \otimes \mathcal{M}Sym \to \mathcal{M}Sym$ and the product $\mathcal{M}Sym \otimes \mathcal{M}Sym \to \mathcal{M}Sym$ are associative. Moreover, putting $\mathcal{M}Sym_0 := k$, they make $\boldsymbol{\beta}$ into an algebra map that factors $\boldsymbol{\tau}$.*

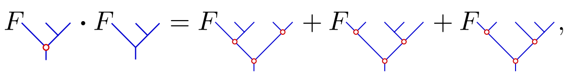
Unfortunately, no natural coalgebra structure exists on $\mathcal{M}Sym$ that makes $\boldsymbol{\beta}$ into a Hopf algebra map.

Definition 6 **(***action* and *coaction* of $\mathcal{Y}Sym$ on $\mathcal{M}Sym$**)** Given $b \in \mathcal{M}_{\boldsymbol{\cdot}}$, let $b \xrightarrow{\curlyvee_+} (b_0, \ldots, b_p)$ denote a p-splitting satisfying $|b_0| > 0$. For $s \in \mathcal{Y}_p$, set

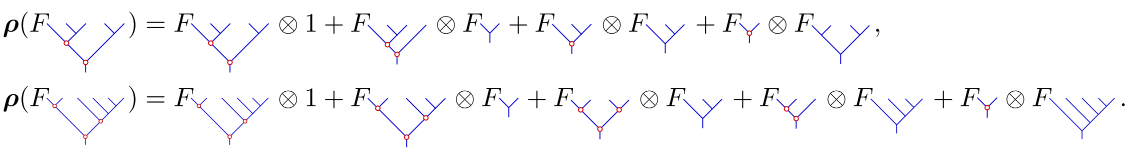
$$F_b \cdot F_s \;=\; \sum_{b \xrightarrow{\curlyvee_+} (b_0, b_1, \ldots, b_p)} F_{(b_0, b_1, \ldots, b_p)/s} \qquad \text{and} \qquad \boldsymbol{\rho}(F_b) \;=\; \sum_{b \xrightarrow{\curlyvee_+} (b_0, b_1)} F_{b_0} \otimes F_{\phi(b_1)}\,, \tag{7}$$

where in $(b_0, b_1, \ldots, b_p)/s$ every node originating in s is circled, and in $\phi(b_1)$ all circles are forgotten.

Example 7 In the fundamental bases of $\mathcal{M}Sym$ and $\mathcal{Y}Sym$, the action looks like



while the coaction looks like



The significance of our definition of $\boldsymbol{\rho}$ will be seen in Corollary 11. Our next result requires only slight modifications to the original proof that $\mathcal{Y}Sym$ is a Hopf algebra (due to the restricted p-splittings).

Theorem 8 *The maps* $\boldsymbol{\cdot} : \mathcal{M}Sym \otimes \mathcal{Y}Sym \to \mathcal{M}Sym$ *and* $\boldsymbol{\rho} : \mathcal{M}Sym \to \mathcal{M}Sym \otimes \mathcal{Y}Sym$ *are associative and coassociative, respectively. They give* $\mathcal{M}Sym$ *the structure of* $\mathcal{Y}Sym$*–Hopf module. That is,* $\boldsymbol{\rho}(F_b \boldsymbol{\cdot} F_s) = \boldsymbol{\rho}(F_b) \boldsymbol{\cdot} \Delta(F_s)$.

2.3 Main results

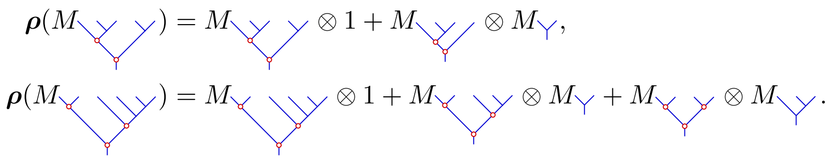
We next introduce "monomial bases" for $\mathfrak{S}Sym$, $\mathcal{M}Sym$, and $\mathcal{Y}Sym$. Given $t \in \mathcal{M}_n$, define

$$M_t = \sum_{t \leq t'} \mu(t, t') F_{t'},$$

where $\mu(\,\cdot\,,\,\cdot\,)$ is the Möbius function on the poset $\mathcal{M}_n$. Define the monomial bases of $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ similarly (see (13) and (17) in [3]). The coaction $\boldsymbol{\rho}$ in this basis is particularly nice, but we need a bit more notation to describe it. Given $t \in \mathcal{M}_p$ and $s \in \mathcal{Y}_q$, let $t\backslash s$ denote the bi-leveled tree on $p + q$ internal nodes formed by grafting the root of s onto the rightmost leaf of t.

Theorem 9 *Given a bi-leveled tree* t*, the coaction* $\boldsymbol{\rho}$ *on* M_t *is given by* $\boldsymbol{\rho}(M_t) = \sum_{t = t' \backslash s} M_{t'} \otimes M_s$.

Example 10 Revisiting the trees in the previous example, the coaction in the monomial bases looks like



Recall that the *coinvariants* of a Hopf module M over a Hopf algebra H are defined by $M^{\mathrm{co}} = \{m \in M \mid \boldsymbol{\rho}(m) = m \otimes 1\}$. The fundamental theorem of Hopf modules provides that $M \simeq M^{\mathrm{co}} \otimes H$. The monomial basis of $\mathcal{M}Sym$ demonstrates this isomorphism explicitly.

Corollary 11 *A basis for the coinvariants in the Hopf module* $\mathcal{M}Sym$ *is given by* $\{M_t\}_{t \in \mathcal{T}}$*, where* $\mathcal{T}$ *comprises the bi-leveled trees with no uncircled nodes on their right branches.*

This result explains the phenomenon observed in (4). It also parallels Corollary 5.3 of [3] to an astonishing degree. There, the right-grafting idea above is defined for pairs of planar binary trees and used to describe the coproduct structure of $\mathcal{Y}Sym$ in its monomial basis.

3 Towards a proof of the main result

We follow the proof of [3, Theorem 5.1], which uses properties of the monomial basis of $\mathfrak{S}Sym$ developed in [2] to do the heavy lifting. In [3], the section $\mathcal{Y}_{\boldsymbol{\cdot}} \xrightarrow{\mathsf{max}} \mathfrak{S}_{\boldsymbol{\cdot}}$ of τ is shown to satisfy $\boldsymbol{\tau}(M_{\mathsf{max}(t)}) = M_t$ and $\boldsymbol{\tau}(M_\sigma) = 0$ if σ is not 132-avoiding. This was proven using the following result about Galois connections.

Theorem 12 ([15, Thm. 1]) *Suppose* P *and* Q *are two posets related by a* Galois connection*, i.e., a pair of order-preserving maps* $\varphi : P \to Q$ *and* $\gamma : Q \to P$ *such that for any* $v \in P$ *and* $t \in Q$*,* $\varphi(v) \leq t \iff v \leq \gamma(t)$*. Then the Möbius functions* μ_P *and* μ_Q *are related by*

$$\forall v \in P \text{ and } t \in Q, \qquad \sum_{\substack{w \in \varphi^{-1}(t),\\ v \leq w}} \mu_P(v, w) \;=\; \sum_{\substack{s \in \gamma^{-1}(v),\\ s \leq t}} \mu_Q(s, t).$$

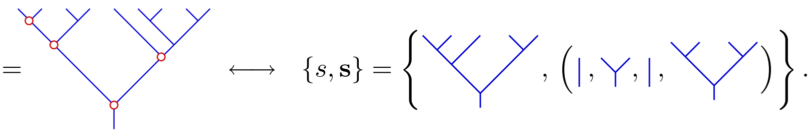
There is a twist in our present situation. Specifically, no Galois connection exists between $\mathfrak{S}_n$ and $\mathcal{M}_n$. On the other hand, we find that no order-preserving map $\iota : \mathcal{M}_{\bullet} \hookrightarrow \mathfrak{S}_{\bullet}$ satisfies $\boldsymbol{\beta}(M_{\iota(t)}) = M_t$. Rather,

$$\boldsymbol{\beta}\Bigg(\sum_{\sigma\in\beta^{-1}(t)} M_\sigma\Bigg) = M_t. \tag{8}$$

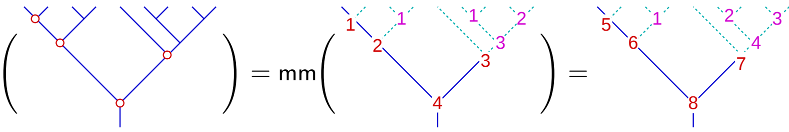
This fact is the key ingredient in our proof of Theorem 11. Its verification required modification of the notion of Galois connection—a relationship between posets that we call an *interval retract* (Section 3.2).

3.1 *Sections of the map* $\beta : \mathfrak{S}_{\bullet} \to \mathcal{M}_{\bullet}$

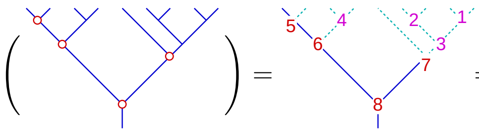
Bi-leveled trees t are in bijection with pairs $\{s, \mathbf{s}\}$, where s is a planar binary tree, with p nodes say, and $\mathbf{s} = (s_1, \ldots, s_p)$ is a forest (sequence) of planar binary trees. In the bijection, s comprises the circled nodes of t and s_i is the binary tree (of uncircled nodes) sitting above the i^{th} leaf of s. For example,



A natural choice for a section $\iota : \mathcal{M}_n \to \mathfrak{S}_n$ would be to, say, build $\mathsf{min}(s)$ and $\mathsf{min}(s_i)$ for each i and splice these permutations together in some way to build a word on the letters $\{1, 2, \ldots n\}$. Let $\mathsf{mm}(t)$ denote the choice giving s_1 smaller letters than s_2, s_2 smaller letters than s_3, ..., s_{p-1} smaller letters than s_p, and s_p smaller letters than s:



This choice does not induce a poset map. The similarly defined MM also fails (chosing maximal permutations representing $\boldsymbol{s}$ and s), but Mm has the properties we need:



We define this map carefully. Given $t \in \mathcal{Y}_n$ and any subset $S \subseteq \mathbb{N}$ of cardinality n, write $\mathsf{min}_S(t)$ for the image of $\mathsf{min}(t)$ under the unique order-preserving map from $[n]$ to S; define $\mathsf{max}_S(t)$ similarly.

Definition 13 [The section Mm] Let $t \leftrightarrow \{s, \mathbf{s}\}$ be a bi-leveled tree on n nodes with p circled nodes. Write $u = u_1 \cdots u_p = \mathsf{min}_{[a,b]}(s)$ for $[a, b] = \{n-p+1, \ldots, n\}$ and write $v^i = \mathsf{max}_{[a_i,b_i]}(s_i)$ $(1 \le i \le p)$, where the intervals $[a_i, b_i]$ are defined recursively as follows:

$$a_p = 1 \quad \text{and} \quad b_p = a_p + |s_p| - 1,$$
$$a_i = 1 + \max \bigcup_{j>i} S_j \quad \text{and} \quad b_i = a_i + |s_i| - 1.$$

Finally, define $\mathsf{Mm}(t)$ by the concatenation $\mathsf{Mm}(t) = u_1 v^1 u_2 v^2 \cdots u_p v^p$.

Remark 14 Alternatively, $\mathsf{Mm}(t)$ is the unique $w \in \beta^{-1}(t)$ avoiding the pinned patterns $\underline{0}231$, $\underline{3}021$, and $\underline{2}031$, where the underlined letter is the first letter in w. The first two patterns fix the embeddings of s_i $(0 \le i \le p)$, the last one makes the letters in s_i larger than those in s_{i+1} $(1 \le i < p)$.

The important properties of $\mathsf{Mm}(t)$ are as follows.

Proposition 15 *The section* $\iota : \mathcal{M}_n \to \mathfrak{S}_n$ *given by* $\iota(t) = \mathsf{Mm}(t)$ *is an embedding of posets. The map* $\beta : \mathfrak{S}_n \to \mathcal{M}_n$ *satisfies* $\beta(\iota(t)) = t$ *for all* $t \in \mathcal{M}_n$ *and* $\beta^{-1}(t) \subseteq \mathfrak{S}_n$ *is the interval* $[\mathsf{mm}(t), \mathsf{MM}(t)]$.

3.2 Interval retracts

Let $\varphi : P \to Q$ and $\gamma : Q \to P$ be two order-preserving maps between posets P and Q. If

$$\forall t \in Q \qquad \varphi(\gamma(t)) = t \quad \text{and} \quad \varphi^{-1}(t) \;\text{ is an interval,}$$

then we say that φ and γ demonstrate P as an *interval retract* of Q.

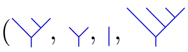
Theorem 16 *If* P *and* Q *are two posets related by an interval retract* (φ, γ)*, then the Möbius functions* μ_P *and* μ_Q *are related by*

$$\forall s < t \in Q \qquad \sum_{\substack{v \in \varphi^{-1}(s) \\ w \in \varphi^{-1}(t)}} \mu_P(v, w) \;=\; \mu_Q(s, t).$$

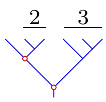
The proof of Theorem 16 exploits Hall's formula for Möbius functions. An immediate consequence is a version of (8) for any P and Q related by an interval retract. Verifying that (β, Mm) is an interval retract between $\mathfrak{S}_n$ and $\mathcal{M}_n$ (Proposition 15) amounts to basic combinatorics of the weak order on $\mathfrak{S}_n$.

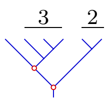
4 More families of binary trees and their polytopes

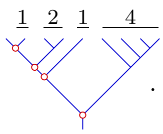
We have so far ignored the algebra $\mathcal{Q}Sym$ of quasisymmetric functions advertised in the introduction. A basis for its n^{th} graded piece is naturally indexed by compositions of n, but may also be indexed by trees as follows. To a composition $(a_1, a_2, \ldots)$, say $(3, 2, 1, 4)$, we associate a sequence of *left-combs*

(),

i.e., trees with a_i leaves and all internal leaves rooted to the rightmost branch and left-pointing. These may be hung on another tree, a right-comb with right-pointing leaves, to establish a bijection between compositions of n and "combs of combs" with n internal nodes:

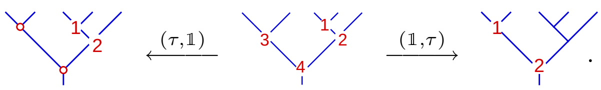
$(23) \leftrightarrow$ 

$(32) \leftrightarrow$ 

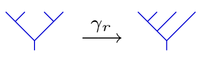
$(1214) \leftrightarrow$  .

To see how $\mathcal{Q}Sym$ and the hypercubes fit into the picture, we briefly revisit the map β of Section 1.2.

We identified bi-leveled trees with pairs $\{s, \mathbf{s}\}$, where s is the tree of circled nodes and $\mathbf{s} = (s_0, \ldots, s_p)$ is a forest of trees (the uncircled nodes). Under this identification, β may be viewed as a pair of maps (τ, τ)—with the first factor τ making a (planar binary) trees out of the nodes greater than or equal to σ_1, and the second factor τ making trees out of the smaller nodes:



See also Figure 3. Two more fundamental maps are γ_r and γ_l, taking trees to (right- or left-) combs, e.g.,



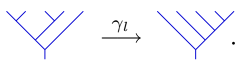
and
 .

Figure 3 displays several combinations of the maps τ, γ_r, and γ_l. The algebra $\mathcal{Q}Sym$ corresponds to the terminal object there—the set denoted $\{\frac{\cdots\text{ combs }\cdots}{\text{comb}}\}$.

The new binary tree–like structures appearing in the factorization of $\mathfrak{S}Sym \twoheadrightarrow \mathcal{Q}Sym$ (i.e., those trees not appearing on the central, vertical axis of Figure 3) will be studied in upcoming papers. It is no surprise that $\mathfrak{S}Sym \twoheadrightarrow \mathcal{Q}Sym$ factors through so many intermediate structures. What is remarkable, and what our binary tree point-of-view reveals, is that each family of trees in Figure 3 can be arranged into a family of polytopes. See Figure 4. The (Hopf) algebraic and geometric implications of this phenomenon will also be addressed in future work.

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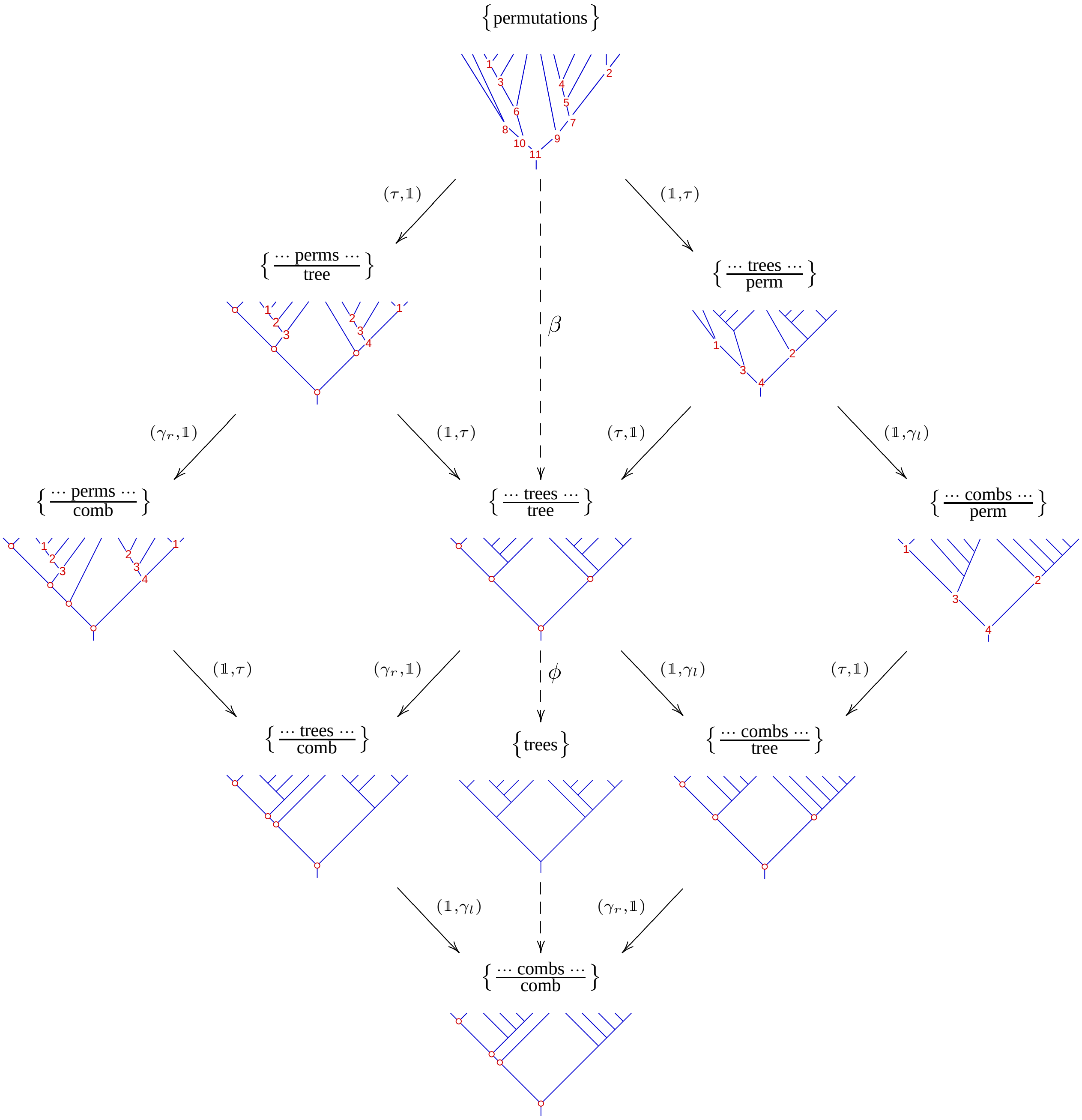

FIG. 3: A commuting diagram of tree-like objects. The spaces $\mathfrak{S}Sym$, $\mathcal{M}Sym$, $\mathcal{Y}Sym$ and $\mathcal{Q}Sym$ appear, top to bottom, along the center. The unlabeled dashed line represents the usual map from $\mathcal{Y}Sym$ to $\mathcal{Q}Sym$ (see [11], Section 4.4). It is incompatible with the given map $(\gamma_r, \gamma_l) : \mathcal{M}Sym \to \mathcal{Q}Sym$.

We explore the Hopf module structures of objects mapping to $\mathcal{Y}Sym$ and $\mathcal{Q}Sym$ in future work. At least some of these will be full-fledged Hopf algebras (e.g., note that there is a bijection of sets between $\left\{\frac{\cdots\text{ trees }\cdots}{\text{comb}}\right\}$ and $\{\text{trees}\}$, the latter indexing the Hopf algebra $\mathcal{Y}Sym$).

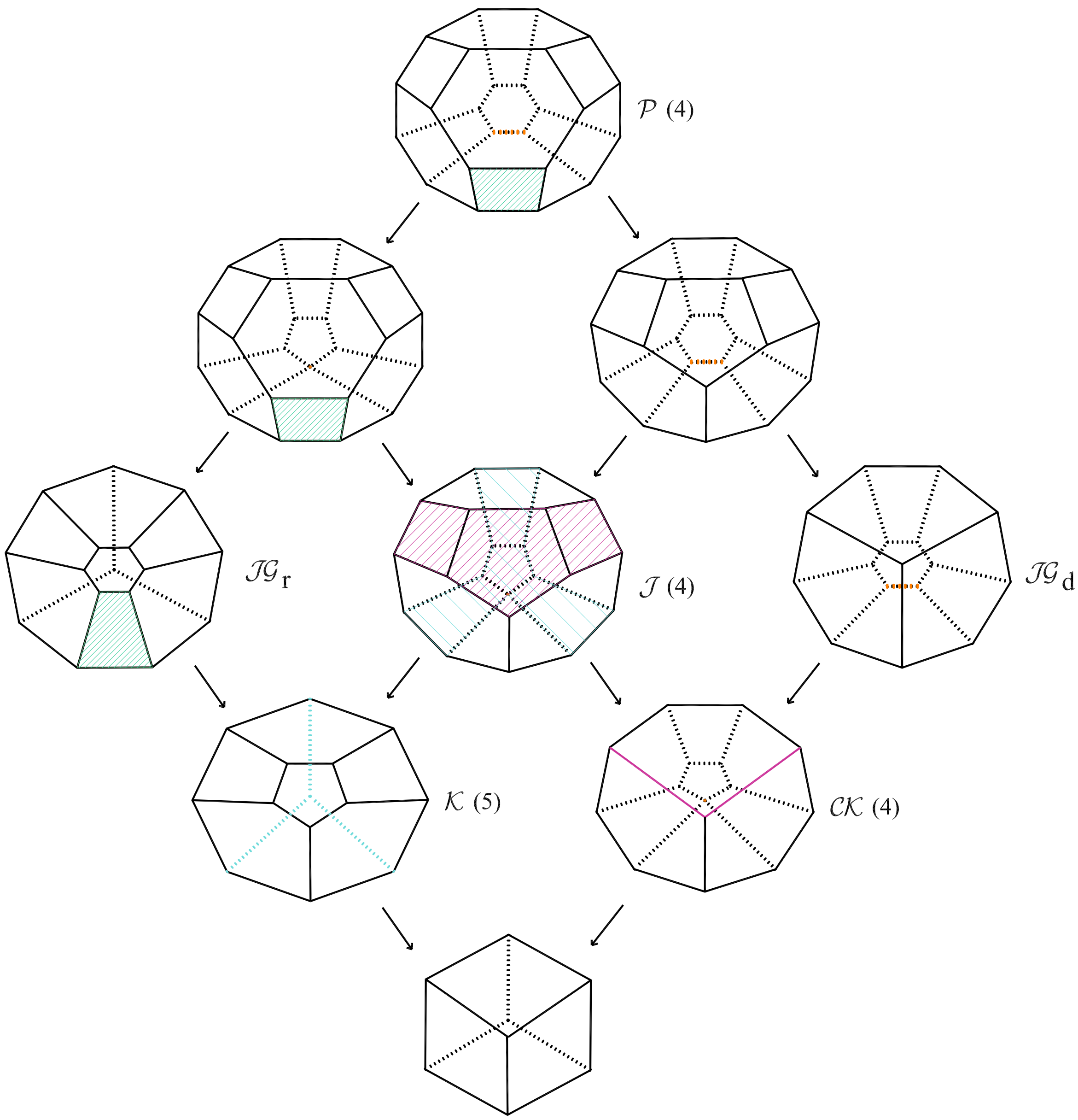

FIG. 4: A commuting diagram of polytopes based on tree-like objects with 4 nodes, corresponding position-wise to Figure 3 (image of ϕ is suppressed). Notation is taken from [6]: $\mathcal{P}(4)$ is the permutohedron, $\mathcal{J}(4)$ is the multiplihedron, $\mathcal{K}(5)$ is the associahedron, and $\mathcal{CK}(4)$ is the composihedron. $\mathcal{JG}_d$ is the domain quotient of the permutohedron and $\mathcal{JG}_r$ is its range quotient.

The cellular projections shown include neither the Tonks projection nor the Loday Ronco projection from the associahedron to the hypercube. However, the map from the multiplihedron to the cube passing through the associahedron appears in [5].

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Perfectness of Kirillov–Reshetikhin crystals for nonexceptional types

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Abstract. For nonexceptional types, we prove a conjecture of Hatayama et al. about the prefectness of Kirillov–Reshetikhin crystals.

Résumé. Pour les types non-exceptionnels, on démontre une conjecture de Hatayama et al. concernant la perfection des cristaux de Kirillov–Reshetikhin.

Keywords: Crystal bases, combinatorial models for Kirillov–Reshetikhin crystals, perfectness

1 Introduction

Kirillov–Reshetikhin (KR) crystals $B^{r,s}$ are crystals corresponding to finite-dimensional $U_q'(\mathfrak{g})$-modules [3, 4], where $\mathfrak{g}$ is an affine Kac–Moody algebra. Recently, a lot of progress has been made regarding long outstanding problems concerning these crystals which appear in mathematical physics and the path realization of affine highest weight crystals [13]. In [20, 21] the existence of KR crystals was shown. In [5] a major step in understanding these crystals was provided by giving explicit combinatorial realizations for all nonexceptional types. This abstract is based on [5, 6]. We prove a conjecture of Hatayama, Kuniba, Okado, Takagu, and Tsuboi [8, Conjecture 2.1] about the perfectness of these KR crystals.

Conjecture 1.1 *[8, Conjecture 2.1] The Kirillov-Reshetikhin crystal $B^{r,s}$ is perfect if and only if $\frac{s}{c_r}$ is an integer with c_r as in Table 1. If $B^{r,s}$ is perfect, its level is $\frac{s}{c_r}$.*

In [14], this conjecture was proven for all $B^{r,s}$ for type $A_n^{(1)}$, for $B^{1,s}$ for nonexceptional types (except for type $C_n^{(1)}$), for $B^{n-1,s}$, $B^{n,s}$ of type $D_n^{(1)}$, and $B^{n,s}$ for types $C_n^{(1)}$ and $D_{n+1}^{(2)}$. When the highest weight is given by the highest root, level-1 perfect crystals were constructed in [1]. For $1 \le r \le n-2$

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	$(c_1,\ldots,c_n)$
$A_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$	$(1,\ldots,1)$
$B_n^{(1)}$	$(1,\ldots,1,2)$
$C_n^{(1)}$	$(2,\ldots,2,1)$

Tab. 1: List of c_r

for type $D_n^{(1)}$, $1 \le r \le n-1$ for type $B_n^{(1)}$, and $1 \le r \le n$ for type $A_{2n-1}^{(2)}$, the conjecture was proved in [22]. The case $G_2^{(1)}$ and $r=1$ was treated in [24] and the case $D_4^{(3)}$ and $r=1$ was treated in [16]. Naito and Sagaki [18] showed that the conjecture holds for twisted algebras, if it is true for the untwisted simply-laced cases.

In this paper we prove Conjecture 1.1 in general for nonexceptional types.

Theorem 1.2 *If $\mathfrak{g}$ is of nonexceptional type, Conjecture 1.1 is true.*

The paper is organized as follows. In Section 2 we give basic notation and the definition of perfectness in Definition 2.1. In Section 3 we review the realizations of the KR crystals of nonexceptional types as recently provided in [5]. Section 4 is reserved for the proof of Theorem 1.2 and an explicit description of the minimal elements $B^{r,c_rs}_{\min}$ of the perfect crystals. A long version of this article containing further details and examples is available at [6].

2 Definitions and perfectness

We follow the notation of [12, 5]. Let $\mathcal{B}$ be a $U_q'(\mathfrak{g})$-crystal [15]. Denote by α_i and Λ_i for $i \in I$ the simple roots and fundamental weights and by c the canoncial central element associated to $\mathfrak{g}$, where I is the index set of the Dynkin diagram of $\mathfrak{g}$ (see Table 2). Let $P = \oplus_{i\in I}\mathbb{Z}\Lambda_i$ be the weight lattice of $\mathfrak{g}$ and P^+ the set of dominant weights. For a positive integer ℓ, the set of level-ℓ weights is

$$P_\ell^+ = \{\Lambda \in P^+ \mid \mathrm{lev}(\Lambda) = \ell\}.$$

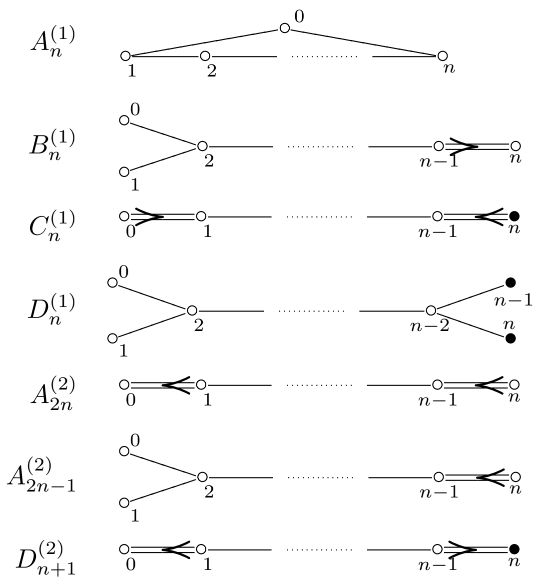
where $\mathrm{lev}(\Lambda) := \Lambda(c)$. The set of level-0 weights is denoted by P_0. We identify dominant weights with partitions; each Λ_i yields a column of height i (except for spin nodes). For more details, please consult [11].

We denote by $f_i, e_i : \mathcal{B} \to \mathcal{B} \cup \{\emptyset\}$ for $i \in I$ the Kashiwara operators and by $\mathrm{wt} : \mathcal{B} \to P$ the weight function on the crystal. For $b \in \mathcal{B}$ we define $\varepsilon_i(b) = \max\{k \mid e_i^k(b) \neq \emptyset\}$, $\varphi_i(b) = \max\{k \mid f_i^k(b) \neq \emptyset\}$, and

$$\varepsilon(b) = \sum_{i\in I} \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i.$$

Next we define perfect crystals, see for example [11].

Definition 2.1 *For a positive integer $\ell > 0$, a crystal $\mathcal{B}$ is called perfect crystal of level ℓ, if the following conditions are satisfied:*



Tab. 2: Dynkin diagrams

1. *$\mathcal{B}$ is isomorphic to the crystal graph of a finite-dimensional $U_q^{'}(\mathfrak{g})$-module.*
2. *$\mathcal{B} \otimes \mathcal{B}$ is connected.*
3. *There exists a $\lambda \in P_0$, such that* $\mathrm{wt}\,(\mathcal{B}) \subset \lambda + \sum_{i \in I \setminus \{0\}} \mathbb{Z}_{\leq 0} \alpha_i$ *and there is a unique element in $\mathcal{B}$ of classical weight λ.*
4. *$\forall\, b \in \mathcal{B}, \;\; \mathrm{lev}(\varepsilon(b)) \geq \ell$.*
5. *$\forall\, \Lambda \in P_\ell^+$, there exist unique elements $b_\Lambda, b^\Lambda \in \mathcal{B}$, such that*

$$\varepsilon(b_\Lambda) = \Lambda = \varphi(b^\Lambda).$$

We denote by $\mathcal{B}_{\min}$ the set of minimal elements in $\mathcal{B}$, namely

$$\mathcal{B}_{\min} = \{b \in \mathcal{B} \mid \mathrm{lev}(\varepsilon(b)) = \ell\}.$$

Note that condition (5) of Definition 2.1 ensures that $\varepsilon, \varphi : \mathcal{B}_{\min} \to P_\ell^+$ are bijections. They induce an automorphism $\tau = \varepsilon \circ \varphi^{-1}$ on P_ℓ^+.

In [22, 5] $\pm$-diagrams were introduced, which describe the branching $X_n \to X_{n-1}$ where $X_n = B_n, C_n, D_n$. A $\pm$-diagram P of shape Λ/λ is a sequence of partitions $\lambda \subset \mu \subset \Lambda$ such that Λ/μ and μ/λ are horizontal strips (i.e. every column contains at most one box). We depict this $\pm$-diagram by the skew tableau of shape Λ/λ in which the cells of μ/λ are filled with the symbol $+$ and those of Λ/μ are filled with the symbol $-$. There are further type specific rules which can be found in [5, Section 3.2]. There exists a bijection Φ between $\pm$-diagrams and the X_{n-1}-highest weight vectors inside the X_n crystal of highest weight Λ.

3 Realization of KR-crystals

Throughout the paper we use the realization of $B^{r,s}$ as given in [5, 21, 22]. In this section we briefly recall the main constructions.

3.1 KR crystals of type $A_n^{(1)}$

Let $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_n\Lambda_n$ be a dominant weight. Then the level is given by

$$\mathrm{lev}(\Lambda) = \ell_0 + \cdots + \ell_n.$$

A combinatorial description of $B^{r,s}$ of type $A_n^{(1)}$ was provided by Shimozono [23]. As a $\{1, 2, \ldots, n\}$-crystal

$$B^{r,s} \cong B(s\Lambda_r).$$

The Dynkin diagram of $A_n^{(1)}$ has a cyclic automorphism $\sigma(i) = i + 1 \pmod{n+1}$ which extends to the crystal in form of the promotion operator. The action of the affine crystal operators f_0 and e_0 is given by

$$f_0 = \sigma^{-1} \circ f_1 \circ \sigma \qquad \text{and} \qquad e_0 = \sigma^{-1} \circ e_1 \circ \sigma.$$

3.2 KR crystals of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

Let $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_n\Lambda_n$ be a dominant weight. Then the level is given by

$$\begin{aligned}
\mathrm{lev}(\Lambda) &= \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \cdots + 2\ell_{n-2} + \ell_{n-1} + \ell_n && \text{for type } D_n^{(1)} \\
\mathrm{lev}(\Lambda) &= \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + \ell_n && \text{for type } B_n^{(1)} \\
\mathrm{lev}(\Lambda) &= \ell_0 + \ell_1 + 2\ell_2 + 2\ell_3 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + 2\ell_n && \text{for type } A_{2n-1}^{(2)}.
\end{aligned} \tag{3.1}$$

We have the following realization of $B^{r,s}$. Let $X_n = D_n, B_n, C_n$ be the classical subalgebra for $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$, respectively.

Definition 3.1 *Let $1 \le r \le n-2$ for type $D_n^{(1)}$, $1 \le r \le n-1$ for type $B_n^{(1)}$, and $1 \le r \le n$ for type $A_{2n-1}^{(2)}$. Then $B^{r,s}$ is defined as follows. As an X_n-crystal*

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda), \tag{3.2}$$

where the sum runs over all dominant weights Λ that can be obtained from $s\Lambda_r$ by the removal of vertical dominoes. The affine crystal operators e_0 and f_0 are defined as

$$f_0 = \sigma^{-1} \circ f_1 \circ \sigma \quad \text{and} \quad e_0 = \sigma^{-1} \circ e_1 \circ \sigma, \tag{3.3}$$

where σ is the crystal automorphism defined in [22, Definition 4.2].

Definition 3.2 *Let $B^{n,s}_{A^{(2)}_{2n-1}}$ be the $A^{(2)}_{2n-1}$-KR crystal. Then $B^{n,s}$ of type $B^{(1)}_n$ is defined through the unique injective map $S : B^{n,s} \to B^{n,s}_{A^{(2)}_{2n-1}}$ such that*

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \quad \text{for } i \in I,$$

where $(m_i)_{0\le i\le n} = (2, 2, \dots, 2, 1)$.

In addition, the $\pm$-diagrams of $A^{(2)}_{2n-1}$ that occur in the image are precisely those which can be obtained by doubling a $\pm$-diagram of $B^{n,s}$ (see [5, Lemma 3.5]). S induces an embedding of dominant weights of $B^{(1)}_n$ into dominant weights of $A^{(2)}_{2n-1}$, namely $S(\Lambda_i) = m_i\Lambda_i$. It is easy to see that for any $\Lambda \in P^+$ we have $\mathrm{lev}(S(\Lambda)) = 2\,\mathrm{lev}(\Lambda)$ using (3.1).

For the definition of $B^{n,s}$ and $B^{n-1,s}$ of type $D^{(1)}_n$, see for example [5, Section 6.2].

3.3 KR crystal of type $C^{(1)}_n$

The level of a dominant $C^{(1)}_n$ weight $\Lambda = \ell_0\Lambda_0 + \cdots + \ell_n\Lambda_n$ is given by

$$\mathrm{lev}(\Lambda) = \ell_0 + \cdots + \ell_n.$$

We use the realization of $B^{r,s}$ as the fixed point set of the automorphism σ [22, Definition 4.2] (see Definition 3.1) inside $B^{r,s}_{A^{(2)}_{2n+1}}$ of [5, Theorem 5.7].

Definition 3.3 *For $1 \le r < n$, the KR crystal $B^{r,s}$ of type $C^{(1)}_n$ is defined to be the fixed point set under σ inside $B^{r,s}_{A^{(2)}_{2n+1}}$ with the operators*

$$e_i = \begin{cases} e_0 e_1 & \text{for } i = 0, \\ e_{i+1} & \text{for } 1 \le i \le n, \end{cases}$$

where the Kashiwara operators on the right act in $B^{r,s}_{A^{(2)}_{2n+1}}$. Under the crystal embedding $S : B^{r,s} \to B^{r,s}_{A^{(2)}_{2n+1}}$ we have

$$\Lambda_i \mapsto \begin{cases} \Lambda_0 + \Lambda_1 & \text{for } i = 0, \\ \Lambda_{i+1} & \text{for } 1 \le i \le n. \end{cases}$$

Under the embedding S, the level of $\Lambda \in P^+$ doubles, that is $\mathrm{lev}(S(\Lambda)) = 2\,\mathrm{lev}(\Lambda)$.

For $B^{n,s}$ of type $C^{(1)}_n$ we refer to [5, Section 6.1].

3.4 KR crystals of type $A^{(2)}_{2n}$, $D^{(2)}_{n+1}$

Let $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_n\Lambda_n$ be a dominant weight. The level is given by

$$\mathrm{lev}(\Lambda) = \ell_0 + 2\ell_1 + 2\ell_2 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + 2\ell_n \qquad \text{for type } A^{(2)}_{2n}$$
$$\mathrm{lev}(\Lambda) = \ell_0 + 2\ell_1 + 2\ell_2 + \cdots + 2\ell_{n-2} + 2\ell_{n-1} + \ell_n \qquad \text{for type } D^{(2)}_{n+1}.$$

Define positive integers m_i for $i \in I$ as follows:

$$(m_0, m_1, \dots, m_{n-1}, m_n) = \begin{cases} (1,2,\dots,2,2) & \text{for } A_{2n}^{(2)}, \\ (1,2,\dots,2,1) & \text{for } D_{n+1}^{(2)}. \end{cases} \tag{3.4}$$

Then $B^{r,s}$ can be realized as follows.

Definition 3.4 *For $1 \le r \le n$ for $\mathfrak{g} = A_{2n}^{(2)}$, $1 \le r < n$ for $\mathfrak{g} = D_{n+1}^{(2)}$ and $s \ge 1$, there exists a unique injective map $S : B_{\mathfrak{g}}^{r,s} \longrightarrow B_{C_n^{(1)}}^{r,2s}$ such that*

$$S(e_i b) = e_i^{m_i} S(b), \quad S(f_i b) = f_i^{m_i} S(b) \qquad \text{for } i \in I.$$

The $\pm$-diagrams of $C_n^{(1)}$ that occur in the image of S are precisely those which can be obtained by doubling a $\pm$-diagram of $B^{r,s}$ (see [5, Lemma 3.5]). S induces an embedding of dominant weights for $A_{2n}^{(2)}, D_{n+1}^{(2)}$ into dominant weights of type $C_n^{(1)}$, with $S(\Lambda_i) = m_i \Lambda_i$. This map preserves the level of a weight, that is $\mathrm{lev}(S(\Lambda)) = \mathrm{lev}(\Lambda)$.

For the case $r = n$ of type $D_{n+1}^{(2)}$ we refer to [5, Definition 6.2].

4 Proof of Theorem 1.2

For type $A_n^{(1)}$, perfectness of $B^{r,s}$ was proven in [14]. For all other types, in the case that $\frac{s}{c_r}$ is an integer, we need to show that the 5 defining conditions in Definition 2.1 are satisfied:

1. This was recently shown in [21].

2. This follows from [7, Corollary 6.1] under [7, Assumption 1]. Assumption 1 is satisfied except for type $A_{2n}^{(2)}$: The regularity of $B^{r,s}$ is ensured by (1), the existence of an automorphism σ was proven in [5, Section 7], and the unique element $u \in B^{r,s}$ such that $\varepsilon(u) = s\Lambda_0$ and $\varphi(u) = s\Lambda_\nu$ (where $\nu = 1$ for r odd for types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, $\nu = r$ for $A_n^{(1)}$, and $\nu = 0$ otherwise) is given by the classically highest weight element in the component $B(0)$ for $\nu = 0$, $B(s\Lambda_1)$ for $\nu = 1$, and $B(s\Lambda_r)$ for $\nu = r$. Note that $\Lambda_0 = \tau(\Lambda_\nu)$, where $\tau = \varepsilon \circ \varphi^{-1}$. For type $A_{2n}^{(2)}$, perfectness follows from [18].

3. The statement is true for $\lambda = s(\Lambda_r - \Lambda_r(c)\Lambda_0)$, which follows from the decomposition formulas [2, 9, 10, 19].

Conditions (4) and (5) will be shown in the following subsections using case by case considerations: Section 4.1 for type $A_n^{(1)}$, Sections 4.2, 4.3, and 4.4 for types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$, Sections 4.5 and 4.6 for type $C_n^{(1)}$, Section 4.7 for type $A_{2n}^{(2)}$, and Sections 4.8 and 4.9 for type $D_{n+1}^{(2)}$.

When $\frac{s}{c_r}$ is not an integer, we show in the subsequent sections that the minimum of the level of $\varepsilon(b)$ is the smallest integer exceeding $\frac{s}{c_r}$, and provide examples that contradict condition (5) of Definition 2.1 for each crystal, thereby proving that $B^{r,s}$ is not perfect. In the case that $\frac{s}{c_r}$ is an integer, we provide an explicit construction of the minimal elements of $B^{r,s}$.

4.1 Type $A_n^{(1)}$

It was already proven in [14] that $B^{r,s}$ is perfect. We give below its associated automorphism τ and minimal elements. τ on P is defined by

$$\tau(\sum_{i=0}^{n} k_i\Lambda_i) = \sum_{i=0}^{n} k_i\Lambda_{i-r \bmod n+1}.$$

Recall that $B^{r,s}$ is identified with the set of semistandard tableaux of $r \times s$ rectangular shape over the alphabet $\{1, 2, \ldots, n+1\}$. For $b \in B^{r,s}$ let $x_{ij} = x_{ij}(b)$ denote the number of letters j in the i-th row of b for $1 \le i \le r, 1 \le j \le n+1$. Set $r' = n+1-r$, then

$$x_{ij} = 0 \quad \text{unless} \quad i \le j \le i + r'.$$

Let $\Lambda = \sum_{i=0}^{n} \ell_i\Lambda_i$ be in P_s^+, that is, $\ell_0, \ell_1, \ldots, \ell_n \in \mathbb{Z}_{\ge 0}, \sum_{i=0}^{n} \ell_i = s$. Then $x_{ij}(b)$ of the minimal element b such that $\varepsilon(b) = \Lambda$ is given by

$$\begin{aligned} x_{ii} &= \ell_0 + \sum_{\alpha=i}^{r-1} \ell_{\alpha+r'}, \\ x_{ij} &= \ell_{j-i} \quad (i < j < i + r'), \\ x_{i,i+r'} &= \sum_{\alpha=0}^{i-1} \ell_{\alpha+r'} \end{aligned} \tag{4.1}$$

for $1 \le i \le r$.

4.2 Types $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n-1}^{(2)}$

Conditions (4) and (5) of Definition 2.1 for $1 \le r \le n-2$ for type $D_n^{(1)}$, $1 \le r \le n-1$ for type $B_n^{(1)}$, and $1 \le r \le n$ for type $A_{2n-1}^{(2)}$ were shown in [22, Section 6]. To a given fundamental weight Λ_k a $\pm$-diagram $\mathrm{diagram}(\Lambda_k)$ was associated. This map can be extended to any dominant weight $\Lambda = \ell_0\Lambda_0 + \cdots + \ell_n\Lambda_n$ by concatenating the columns of the $\pm$-diagrams of each piece. To every fundamental weight Λ_k a string of operators $f(\Lambda_k)$ can be associated as in [22, Section 6].

The minimal element b in $B^{r,s}$ that satisfies $\varepsilon(b) = \Lambda$ can now be constructed as follows

$$b = f(\Lambda_n)^{\ell_n} \cdots f(\Lambda_2)^{\ell_2}\ \Phi(\mathrm{diagram}(\Lambda)).$$

For $\Lambda = \sum_{i=0}^{n} \ell_i\Lambda_i \in P_s^+$, we have

$$\tau(\Lambda) = \begin{cases} \Lambda & \text{if } r \text{ is even,} \\ \ell_0\Lambda_1 + \ell_1\Lambda_0 + \sum_{i=2}^{n} \ell_i\Lambda_i & \text{if } r \text{ is odd,} \\ & \quad \text{types } B_n^{(1)}, A_{2n-1}^{(2)}, \\ \ell_0\Lambda_1 + \ell_1\Lambda_0 + \sum_{i=2}^{n-2} \ell_i\Lambda_i + \ell_{n-1}\Lambda_n + \ell_n\Lambda_{n-1} & \text{if } r \text{ is odd, type } D_n^{(1)}. \end{cases}$$

4.3 Type $D_n^{(1)}$ for $r = n-1, n$

The cases when $r = n, n-1$ for type $D_n^{(1)}$ were treated in [14]. We refer to [14] or [6, Section 4.3] for an explicit description of the minimal elements.

The automorphism τ is given by

$$\tau\big(\sum_{i=0}^{n} \ell_i \Lambda_i\big) = \ell_0 \Lambda_{n-1} + \ell_1 \Lambda_n + \sum_{i=2}^{n-2} \ell_i \Lambda_{n-i} + \begin{cases} \ell_{n-1}\Lambda_0 + \ell_n \Lambda_1 & n \text{ even,} \\ \ell_{n-1}\Lambda_1 + \ell_n \Lambda_0 & n \text{ odd.} \end{cases}$$

4.4 Type $B_n^{(1)}$ for $r = n$

In this section we consider the perfectness of $B^{n,s}$ of type $B_n^{(1)}$.

Proposition 4.1 *We have*

$$\min\{\mathrm{lev}(\varepsilon(b)) \mid b \in B^{n,2s+1}\} \geq s+1,$$
$$\min\{\mathrm{lev}(\varepsilon(b)) \mid b \in B^{n,2s}\} \geq s.$$

Proof: Suppose, there exists an element $b \in B^{n,2s+1}$ with $\mathrm{lev}(\varepsilon(b)) = p < s+1$. Since $B^{n,2s+1}$ is embedded into $B^{n,2s+1}_{A^{(2)}_{2n-1}}$ by Definition 3.2, this would yield an element $\tilde{b} \in B^{n,2s+1}_{A^{(2)}_{2n-1}}$ with $\mathrm{lev}(\tilde{b}) < 2s+1$. But this is not possible, since $B^{n,2s+1}_{A^{(2)}_{2n-1}}$ is a perfect crystal of level $2s+1$.

Suppose there exists an element $b \in B^{n,2s}$ with $\mathrm{lev}(\varepsilon(b)) = p < s$. By the same argument one obtains a contradiction to the level of $B^{n,2s}_{A^{(2)}_{2n-1}}$. □

Hence to show that $B^{n,2s+1}$ is not perfect, it is enough to provide two elements $b_1, b_2 \in B^{n,2s+1}_{A^{(2)}_{2n-1}}$ which are in the realization of $B^{r,s}$ under S and satisfy $\varepsilon(b_1) = \varepsilon(b_2) = \Lambda$, where $\mathrm{lev}(\Lambda) = 2s+2$. We use the notation $f_{\vec{a}} = f_{a_1}^{m_1} \cdots f_{a_k}^{m_k}$ for $\vec{a} = (a_1^{m_1}, \ldots, a_k^{m_k})$.

Proposition 4.2 *Define the following elements $b_1, b_2 \in B^{n,2s+1}_{A^{(2)}_{2n-1}}$: For n odd, let P_1 be the $\pm$-diagram corresponding to one column of height n containing one $+$, and $2s$ columns of height 1 each containing a $-$ sign, and P_2 the analogous $\pm$-diagram but with a $-$ in the column of height n. Set $\vec{a} = (n, (n-1)^2, n, (n-2)^2, (n-1)^2, n, \ldots, 2^2, \ldots, (n-1)^2, n)$ and*

$$b_1 = f_{\vec{a}}(\Phi(P_1)) \quad \textit{and} \quad b_2 = f_{\vec{a}}(\Phi(P_2)).$$

For n even, replace the columns of height 1 with columns of height 2 and fill them with $\pm$-pairs. Then $b_1, b_2 \in S(B^{n,2s+1})$ and $\varepsilon(b_1) = \varepsilon(b_2) = 2s\Lambda_1 + \Lambda_n$, which is of level $2s+2$.

Proof: It is clear from the construction that the $\pm$-diagrams corresponding to b_1 and b_2 can be obtained by doubling a $B_n^{(1)}$ $\pm$-diagram (see [5, Lemma 3.5]). Hence $\Phi(P_1), \Phi(P_2) \in S(B^{n,2s+1})$. The sequence $\vec{a}$ can be obtained by doubling a type $B_n^{(1)}$ sequence using $(m_1, m_2, \ldots, m_n) = (2, \ldots, 2, 1)$, so by Definition 3.2 b_1 and b_2 are in the image of the embedding S that realizes $B^{n,2s+1}$. The claim that $\varepsilon(b_1) = \varepsilon(b_2) = 2s\Lambda_1 + \Lambda_n$ can be checked explicitly. □

Corollary 4.3 *The KR crystal $B^{n,2s+1}$ of type $B_n^{(1)}$ is not perfect.*

Proof: This follows directly from Proposition 4.2 using the embedding S of Definition 3.2. □

Proposition 4.4 *There exists a bijection, induced by ε, from $B^{n,2s}_{\min}$ to P_s^+. Hence $B^{n,2s}$ is perfect of level s.*

Proof: Let S be the embedding from Definition 3.2. Then we have an induced embedding of dominant weights Λ of $B_n^{(1)}$ into dominant weights of $A_{2n-1}^{(2)}$ via the map S, that sends $\Lambda_i \mapsto m_i\Lambda_i$.

In [22, Section 6] (see Section 4.2) the minimal elements for $A_{2n-1}^{(2)}$ were constructed by giving a $\pm$-diagram and a sequence from the $\{2,\dots,n\}$-highest weight to the minimal element. Since $(m_0,\dots,m_n) = (2,\dots,2,1)$ and columns of height n for type $A_{2n-1}^{(2)}$ are doubled, it is clear from the construction that the $\pm$-diagrams corresponding to weights $S(\Lambda)$ are in the image of S of $\pm$-diagrams for $B_n^{(1)}$ (see [5, Lemma 3.5]). Also, since under S all weights Λ_i for $1 \le i < n$ are doubled, it follows that the sequences are "doubled" using the m_i. Hence a minimal element of $B^{n,2s}$ of level s is in one-to-one correspondence with those minimal elements in $B^{n,2s}_{A_{2n-1}^{(2)}}$ that can be obtained from doubling a $\pm$-diagram of $B^{n,2s}$. This implies that ε defines a bijection between $B^{n,2s}_{\min}$ and P_s^+. □

The automorphism τ of the perfect KR crystal $B^{n,2s}$ is given by

$$\tau\Big(\sum_{i=0}^{n} \ell_i\Lambda_i\Big) = \begin{cases} \sum_{i=0}^{n} \ell_i\Lambda_i & \text{if } n \text{ is even,}\\ \ell_0\Lambda_1 + \ell_1\Lambda_0 + \sum_{i=2}^{n} \ell_i\Lambda_i & \text{if } n \text{ is odd.}\end{cases}$$

4.5 Type $C_n^{(1)}$

In this section we consider $B^{r,s}$ of type $C_n^{(1)}$ for $r < n$.

Proposition 4.5 *Let $r < n$. Then*

$$\min\{\mathrm{lev}(\varepsilon(b)) \mid b \in B^{r,2s+1}\} \ge s+1,$$
$$\min\{\mathrm{lev}(\varepsilon(b)) \mid b \in B^{r,2s}\} \ge s.$$

Proof: By Definition 3.3, the crystal $B^{r,s}$ is realized inside $B^{r,s}_{A_{2n+1}^{(2)}}$. The proof is similar to the proof of Proposition 4.1 for type $B_n^{(1)}$. □

Hence to show that $B^{r,2s+1}$ is not perfect, it is suffices to give two elements $b_1, b_2 \in B^{r,2s+1}_{A_{2n+1}^{(2)}}$ that are fixed points under σ with $\varepsilon(b_1) = \varepsilon(b_2) = \Lambda$, where $\mathrm{lev}(\Lambda) = 2s+2$.

Proposition 4.6 *Let $b_1, b_2 \in B^{r,2s+1}_{A_{2n+1}^{(2)}}$, where b_1 consists of s columns of the form read from bottom to top $(1,2,\dots,r)$, s columns of the form $(\overline{r},\overline{r-1},\dots,\overline{1})$, and a column $(\overline{r+1},\dots,\overline{2})$. In b_2 the last column is replaced by $(r+2,\dots,2r+2)$ if $2r+2 \le n$ and $(r+2,\dots,n,\overline{n},\dots,\overline{k})$ of height n otherwise. Then*

$$\varepsilon(b_1) = \varepsilon(b_2) = \begin{cases} s\Lambda_r + \Lambda_{r+1} & \text{if } r > 1,\\ s(\Lambda_0+\Lambda_1) + \Lambda_2 & \text{if } r = 1,\end{cases}$$

which is of level $2s+2$.

Proof: The claim is easy to check explicitly. □

Corollary 4.7 *The KR crystal $B^{n,2s+1}$ of type $C_n^{(1)}$ is not perfect.*

Proof: The $\{2,\dots,n\}$-highest weight elements in the same component as b_1 and b_2 of Proposition 4.6 correspond to $\pm$-diagrams that are invariant under σ. Hence, by Definition 3.3, b_1 and b_2 are fixed points under σ. Combining this result with Proposition 4.5 proves that $B^{r,2s+1}$ is not perfect. □

Proposition 4.8 *There exists a bijection, induced by ε, from $B^{r,2s}_{\min}$ to P_s^+. Hence $B^{r,2s}$ is perfect of level s.*

Proof: By Definition 3.3, $B^{r,s}$ of type $C_n^{(1)}$ is realized inside $B^{r,s}_{A^{(2)}_{2n+1}}$ as the fixed points under σ. Under the embedding S, it is clear that a dominant weight $\Lambda = \ell_0\Lambda_0 + \ell_1\Lambda_1 + \cdots + \ell_{n+1}\Lambda_{n+1}$ of type $A^{(2)}_{2n+1}$ is in the image if and only if $\ell_0 = \ell_1$. Hence it is clear from the construction of the minimal elements for $A^{(2)}_{2n+1}$ as described in Section 4.2 that the minimal elements corresponding to Λ with $\ell_0 = \ell_1$ are invariant under σ. By [22, Theorem 6.1] there is a bijection between all dominant weights Λ of type $A^{(2)}_{2n+1}$ with $\ell_0 = \ell_1$ and $\mathrm{lev}(\Lambda) = 2s$ and minimal elements in $B^{r,2s}_{A^{(2)}_{2n+1}}$ that are invariant under σ. Hence using S, there is a bijection between dominant weights in P_s^+ of type $C_n^{(1)}$ and $B^{r,2s}_{\min}$. □

The automorphism τ of the perfect KR crystal $B^{r,2s}$ is given by the identity.

4.6 Type $C_n^{(1)}$ for $r = n$

This case is treated in [14]. For the minimal elements, we follow the construction in Section 4.2. To every fundamental weight Λ_k we associate a column tableau $T(\Lambda_k)$ of height n whose entries are $k+1, k+2, \dots, n, \overline{n}, \dots, \overline{n-k+1}$ $(1, 2, \dots, n$ for $k = 0)$ reading from bottom to top. Let $f(\Lambda_k)$ be defined such that $T(\Lambda_k) = f(\Lambda_k)b_1$, where b_k is the highest weight tableau in $B(k\Lambda_n)$. Then the minimal element b in $B^{n,s}$ such that $\varepsilon(b) = \Lambda = \sum_{i=0}^n \ell_i\Lambda_i \in P_s^+$ is constructed as

$$b = f(\Lambda_n)^{\ell_n} \cdots f(\Lambda_1)^{\ell_1} b_s.$$

The automorphism τ is given by

$$\tau(\sum_{i=0}^n \ell_i\Lambda_i) = \sum_{i=0}^n \ell_i\Lambda_{n-i}.$$

4.7 Type $A^{(2)}_{2n}$

For type $A^{(2)}_{2n}$ one may use the result of Naito and Sagaki [18, Theorem 2.4.1] which states that under their [18, Assumption 2.3.1] (which requires that $B^{r,s}$ for $A^{(1)}_{2n}$ is perfect) all $B^{r,s}$ for $A^{(2)}_{2n}$ are perfect. Here we provide a description of the minimal elements via the emebdding S into $B^{r,2s}_{C_n^{(1)}}$.

Proposition 4.9 *The minimal elements of $B^{r,s}$ of level s are precisely those that corresponding to doubled $\pm$-diagrams in $B^{r,2s}_{C_n^{(1)}}$.*

Proof: In Proposition 4.8 a description of the minimal elements of $B^{r,2s}_{C_n^{(1)}}$ is given. We have the realization of $B^{r,s}$ via the map S from Definition 3.4. In the same way as in the proof of Proposition 4.4 one can show, that the minimal elements of $B^{r,2s}_{C_n^{(1)}}$ that correspond to doubled dominant weights are precisely those in the realization of $B^{r,s}$, hence ε defines a bijection between $B^{r,s}_{\min}$ and P_s^+. □

The automorphism τ is given by the identity.

4.8 Type $D_{n+1}^{(2)}$ for $r < n$

Proposition 4.10 *Let $r < n$. There exists a bijection $B^{r,s}_{\min}$ to P_s^+, defined by ε. Hence $B^{r,s}$ is perfect.*

Proof: This proof is analogous to the proof of Proposition 4.9. □

The automorphism τ is given by the identity.

4.9 Type $D_{n+1}^{(2)}$ for $r = n$

This case is already treated in [14], which we summarize below. As a B_n-crystal it is isomorphic to $B(s\Lambda_n)$. There is a description of its elements in terms of semistandard tableaux of $n \times s$ rectangular shape with letters from the alphabet $\mathcal{A} = \{1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$. Moreover, each column does not contain both k and $\overline{k}$. Let c_i be the ith column, then the action of e_i, f_i $(i = 1, \dots, n)$ is calculated through that of $c_s \otimes \cdots \otimes c_1$ of $B(\Lambda_n)^{\otimes s}$. With this realization the minimal element b_Λ such that $\varepsilon(b_\Lambda) = \Lambda = \sum_{i=0}^n \ell_i \Lambda_i \in P_s^+$ is given as follows. Let x_{ij} $(1 \le i \le n, j \in \mathcal{A})$ be the number of j in the ith row. Note that $x_{ij} = 0$ unless $i \le j \le \overline{n-i+1}$. The table (x_{ij}) of b_Λ is then given by $x_{ii} = \ell_0 + \cdots + \ell_{n-i}$ $(1 \le i \le n)$, $x_{ij} = \ell_{j-i}$ $(i+1 \le j \le n)$, $x_{i\overline{j}} = \ell_j + \cdots + \ell_n$ $(n-i+1 \le j \le n)$. The automorphism τ is given by

$$\tau(\sum_{i=0}^n \ell_i \Lambda_i) = \sum_{i=0}^n \ell_i \Lambda_{n-i}.$$

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Refinements of the Littlewood-Richardson rule

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Abstract.

We refine the classical Littlewood-Richardson rule in several different settings. We begin with a combinatorial rule for the product of a Demazure atom and a Schur function. Building on this, we also describe the product of a quasisymmetric Schur function and a Schur function as a positive sum of quasisymmetric Schur functions. Finally, we provide a combinatorial formula for the product of a Demazure character and a Schur function as a positive sum of Demazure characters. This last rule implies the classical Littlewood-Richardson rule for the multiplication of two Schur functions.

Résumé.

Nous décrivons trois nouvelles règles de Littlewood-Richardson, et chaque nouvelle règle partage la vielle règle de Littlewood-Richardson. La première règle multiplie un atome de Demazure et un fonction de Schur. Le deuxième multiplie un fonction de quasisymmetric-Schur et un fonction de Schur. Le troisième multiplie un caractère de Demazure et un fonction de Schur. Cette dernière règle est une description de la vielle règle de Littlewood-Richardson.

Keywords: key polynomials, nonsymmetric Macdonald polynomials, Littlewood-Richardson rule, quasisymmetric functions, Schur functions

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1 Introduction

In (5), Haglund, Haiman, and Loehr obtained a new combinatorial formula for the type A nonsymmetric Macdonald polynomial $E_\alpha(x_1,\ldots,x_n;q,t)$ first introduced by Macdonald (8), where α is a (weak) composition into n nonnegative parts. This formula involves *inversion triples*, a combinatorial construct introduced in the study of symmetric Macdonald polynomials (3), (4). By letting $q=t=0$, we obtain a new combinatorial formula for the $E_\alpha(X;0;0)$, which are known (13) to be certain B-module characters studied by Demazure, now commonly referred to as Demazure characters. Marshall (9) has shown that many of the nice analytic properties of type A symmetric Macdonald polynomials, such as their occurrence in a generalization of Selberg's Integral, are shared by type A nonsymmetric Macdonald polynomials as well. In his work he used a modified version obtained from the E_α by replacing q,t by $1/q,1/t$, reversing the order of the x_i-variables, and reversing the order of the parts of α. The combinatorics of the case $q=t=0$ of these polynomials, i.e. $E_{\alpha_n,\ldots,\alpha_1}(x_n,\ldots,x_1;\infty,\infty)$, was investigated in (10), (11), including a direct combinatorial proof that they are in fact the same as polynomials introduced by Lascoux and Schützenberger (7) in connection with the study of Schubert polynomials, which they called standard bases, and which equal the characters of quotients of Demazure modules. The Demazure characters are sometimes called key polynomials (12) and in prior work as well as in this article the standard bases of Lascoux and Schützenberger are referred to as Demazure atoms. It is known that the Demazure character is a positive sum of Demazure atoms, and that the Schubert polynomial is a positive sum of Demazure characters.

Schur functions are special cases of both Demazure characters and Schubert polynomials, and the decomposition of a Schur function as a positive sum of Demazure atoms was proved directly in (10) using an extension of the RSK algorithm. It is well-known (2) that the Schur function $s_\lambda(x_1,\ldots,x_n)$ is a sum of Gessel's fundamental quasisymmetric functions, one for each standard Young tableau of shape λ. It is natural to investigate how this decomposition correlates with the expansion of s_λ into Demazure atoms. In the predecessor to this article (6), the authors introduced a new basis for the ring of quasisymmetric functions called "quasisymmetric Schur functions" denoted $\mathcal{S}_\beta(x_1,\ldots,x_n)$, where β is a (strong) composition. They defined $\mathcal{S}_\beta$ as a sum of certain Demazure atoms, and it follows immediately from the results in (10) and the decomposition of Schur functions into atoms that $s_\lambda=\sum_\beta \mathcal{S}_\beta$, where the sum is over all multiset permutations β of the parts of λ. Note that if the Ferrers shape of λ is a rectangle, then the sum has only one term and $s_\lambda=\mathcal{S}_\lambda$. The authors showed that if you multiply a quasisymmetrc Schur function by an elementary symmetric function $e_k(=s_{1^k})$ or a complete homogeneous symmetric function $h_k(=s_k)$ then this result can be expressed in a simple combinatorial way as a positive sum of quasisymmetric Schur functions. From this rule the classical Pieri rule for multiplying a Schur function by an e_k or h_k can be easily derived.

In this article we generalize the result by showing that the product of a Schur function with a quasisymmetric Schur function (respectively Demazure character, Demazure atom) expands positively into quasisymmetric Schur functions (respectively Demazure characters, atoms), and we give a simple combinatorial rule for the coefficients in this expansion. The description of the rule contains many elements in common with the classical Littlewood-Richardson (LR) rule for the multiplication of two Schur functions, and the authors' proof is essentially a refinement of the proof of the LR-rule in Fulton's book on Young Tableaux (1) involving the combinatorial constructs (such as inversion triples) occurring in the new combinatorial formulas for quasisymmetric Schur functions, Demazure characters, and Demazure atoms. One can obtain the classical LR-rule from the rule for Demazure atoms by careful bookkeeping combined with the decomposition of Schur functions into atoms.

1.1 Sequences and words

A *strong (resp. weak) composition* is a finite sequence of positive (resp. nonnegative) integers. A *partition* is a multiset of nonnegative integers, which we usually present as a weakly decreasing sequence. By $\widetilde{\gamma}$ we denote the underlying partition of γ, and by γ^+ the strong composition underlying γ i.e. γ with its zero parts removed. When necessary, any of these may be considered to be an infinite sequence of integers in which all but a finite number of entries is zero. By γ^* we denote the sequence that contains the parts of γ in reverse order.

For any finite sequence $\alpha=(\alpha_1,\ldots,\alpha_r)$ we denote $\ell(\alpha):=r$ and $|\alpha|:=\sum_i\alpha_i$. Given two (possibly weak) compositions $\alpha=(\alpha_1,\ldots,\alpha_r)$ and $\beta=(\beta_1,\ldots,\beta_r)$ of the same length, we say that α is *contained* in β, denoted $\alpha\subseteq\beta$, if $\alpha_i\le\beta_i$ for all $1\le i\le r$.

A finite sequence w of positive integers is called a *lattice word* if for every positive integer i, every prefix of w contains at least as many i's as $(i+1)$'s. A finite sequence w of positive integers is called a *reverse lattice word* (or *Yamanouchi word*) if for every positive integer i, every suffix of w contains at least as many i's as $(i+1)$'s. The *weight* of a word w is the sequence $\lambda=(\lambda_1,\ldots,\lambda_r)$ such that for every $1\le i\le r$, w contains exactly λ_i elements of value i. Note that if w is a lattice or reverse lattice word, then its weight will be a partition. Define the function $\phi_n : \mathbb{Z} \to \mathbb{Z}$ by $\phi_n(k) = n + 1 - k$. Let $w = (w_1, \ldots, w_t)$ be a sequence of integers with largest element r. Define $\Phi(w) = (\phi_r(w_1), \ldots, \phi_r(w_t))$. Then we will say that a word w is a *contre-lattice* word if $\Phi(w)$ is a lattice word and the weight of $\Phi(w)$ is a partition of length r, i.e. the weight of w is λ^* for some partition λ. Similarly, we will say that w is a *reverse contre-lattice* (or *contre-Yamanouchi*) word if w^* is a contre-lattice word.

1.2 Diagrams and tableaux

Given any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, we have a *partition diagram*, denoted $dg(\lambda)$, consisting of the usual left-justified arrangement of rows of cells (sometimes called squares or boxes), one row for each part of λ, the part λ_i giving the number of cells in row i. We use the English convention for our diagrams, so that the longest row (of length λ_1) at the top of the diagram. We index the cells of a diagram by $(row, column)$ pairs of positive integers.

A *semistandard Young tableau* (SSYT) is a partition diagram filled in such a way that the entries within each row increase weakly left-to-right and the entries within each column increase strictly top-to-bottom. A *standard* Young tableau (SYT) is a SSYT in which the set of entries is exactly $[n] = \{1, \ldots, n\}$, where the diagram has n cells altogether. We use the word *tableau* without modifiers to refer to an SSYT unless otherwise indicated. Given partitions $\mu \subseteq \lambda$, the diagram of *skew shape* λ/μ consists of those cells of $dg(\lambda)$ that are not in $dg(\mu)$. Skew tableaux, both standard and semistandard, are defined analogously in the obvious way.

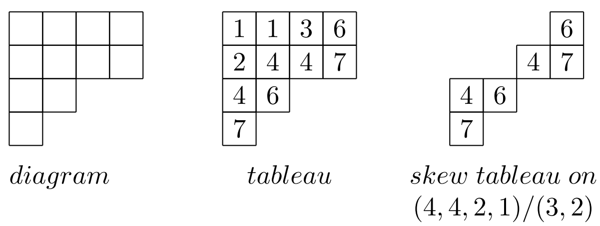


Fig. 1: Diagram and tableau examples

The *content* of a tableau T is the weak composition γ for which γ_i is the number of entries of T with value i, for all positive i. For example, in Figure 1, the first tableau has content $(2, 1, 1, 3, 0, 2, 2)$. Given a set of variables $X = \{x_1, x_2, x_3, \ldots\}$ indexed by positive integers, the *(monomial) weight* of T, denoted $\mathbf{x}^T$, is the monomial for which the exponent of x_i is γ_i, the number of entries of T with value i. For the aforementioned example, the monomial weight is $x_1^2 x_2 x_3 x_4^3 x_6^2 x_7^2$.

The *row reading order* is a total ordering of the cells of a (possibly skew) diagram where $(i, j) <_{row} (i', j')$ if either $i > i'$ or ($i = i'$ and $j < j'$). That is, the row reading order reads the cells from left-to-right in each row, starting with the bottommost row and proceeding upwards to the top row. The *row reading word* of a tableau T, denoted $w_{row}(T)$ is the sequence of integers formed by the entries of T taken in row reading order. A *Littlewood-Richardson skew tableau* is a skew tableau whose row reading word is a reverse lattice word.

We will also make use of a slightly different reading order on diagrams, which we will refer to as the *column reading order*. In the column reading order, we have $(i, j) <_{col} (i', j')$ if either $j > j'$ or ($j = j'$ and $i < i'$). That is, the column reading order reads the cells from top-to-bottom within each column, starting with the rightmost column and working leftwards. The *column reading word* of a tableau T, denoted $w_{col}(T)$, is the sequence of integers formed by the entries of T taken in column reading order.

1.3 The Littlewood-Richardson rule

The graded algebra of symmetric functions

$$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots \subseteq \mathbb{Z}[[x_1, x_2, x_3, \ldots]]$$

has each graded piece given by

$$\Lambda_n = \text{ span } \{s_\lambda | \lambda \text{ is a partition of } n\}$$

where $\Lambda_0 = \{1\}$. The *Schur function* s_λ can in turn be defined as

$$s_\lambda = \sum_{\substack{T \in SSYT \\ shape(T)=\lambda}} \mathbf{x}^T.$$

The product of two Schur functions expands as a sum of Schur functions whose coefficients are nonnegative. The coefficients are called Littlewood-Richardson coefficients and also arise in representation theory and algebraic geometry. The rule for the computing the coefficients is called the Littlewood-Richardson rule, and can be described as follows.

Theorem 1 (Littlewood-Richardson rule) *Let λ, μ, ν be partitions. In the expansion*

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu,$$

the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is the number of Littlewood-Richardson skew tableaux of shape ν/λ with content μ.

1.4 Skyline and composition diagrams

Earlier papers (5; 10; 11) introduce *column diagrams*, or *skyline diagrams*, similar to partition diagrams, whose shapes are indexed by weak compositions. The parts of the composition specify the number of cells in the respective *columns* of the diagram. These diagrams are usually augmented by a *basement*, an extra row on the bottom (row 0) whose entries contain positive integers. For example, for the augmented diagram indexed by the weak composition $(2, 0, 3, 1, 2)$ with increasing basement (left-to-right) is shown in the leftmost diagram of Figure 2.

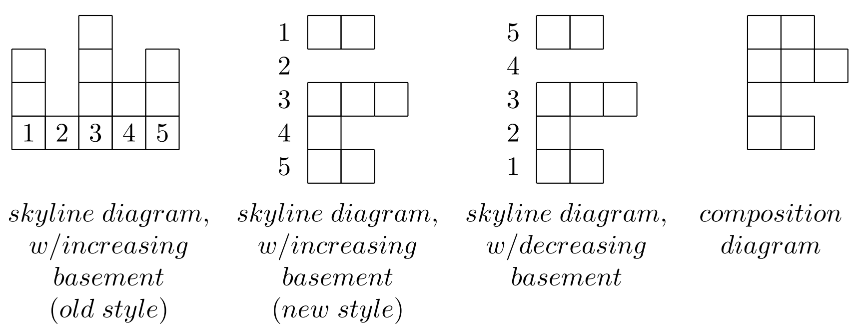
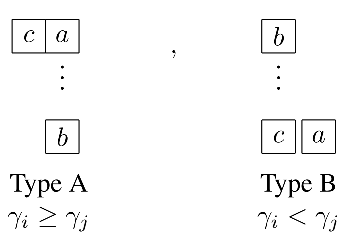


Fig. 2: Skyline and composition diagram examples

However, for consistency with other diagrams, including tableaux, in this paper we will draw our skyline diagrams in sideways fashion, the columns then becoming rows. As with partition diagrams, we number the rows in the English style with row 1 at the top. We will also introduce the related notion of *composition diagrams*, which are indexed by strong compositions, and for which we typically do not indicate a basement. In Figure 2, the skyline diagrams are indexed by the weak composition $(2, 0, 3, 1, 2)$, while the composition diagram is indexed by the composition $(2, 3, 1, 2)$.

Just as we fill partition diagrams according to certain restrictions to obtain tableaux, we define fillings for composition diagrams subject to certain rules. To state the rules, we first define the notion of a *triple* of cells, of which there are two types.



A *type A triple* of a diagram of shape γ is a set of three cells a, b, c of the form $(i, k), (j, k), (i, k-1)$ for some pair of rows $i < j$ of the diagram and some column $k > 0$, where row i is at least as long as row j, i.e. $\gamma_i \geq \gamma_j$. A *type B triple* is a set of three cells a, b, c of the form $(j, k+1), (i, k), (j, k)$ for some pair of rows $i < j$ of the diagram and some column $k \geq 0$, where row i is strictly shorter than row j, i.e. $\gamma_i < \gamma_j$. Note that basement cells can be elements of triples. We say that a triple of either type is an *inversion* triple if the relative order of the entries is either $b < a \leq c$ or $a \leq c < b$.

We say that a skyline diagram filling is *semistandard* if

(i) each row is weakly decreasing left-to-right (including the basement), and

(ii) all triples (including triples with cells in the basement) are inversion triples.

We refer to such a filled skyline diagram as a *semistandard augmented filling* (SSAF), or simply as a *skyline*.

We say that a composition diagram filling is *semistandard* if

(i) the first column is strictly increasing, top-to-bottom,

(ii) each row is weakly decreasing left-to-right, and

(iii) all triples are inversion triples.

We refer to such a filled composition diagram as a *semistandard composition tableau* (SSCT), or simply as a *composition tableau*.

1.5 Combinatorial formulas for formal power series

We have already mentioned the well-known combinatorial formula for Schur functions, which restricts to *Schur polynomials* over the variables $X = \{x_1, \ldots, x_n\}$:

$$s_\lambda(X) = \sum_{\substack{T \in SSYT(n), \\ shape(T)=\lambda}} \mathbf{x}^T, \tag{1.1}$$

where $SSYT(n)$ is the set of all semistandard tableaux with entries in $[n]$. The formula for the Schur function over the infinite variable set $X = \{x_1, x_2, x_3, \ldots\}$ is the same except that the tableau entries are taken over $\mathbb{Z}_{>0}$ instead of $[n]$. Earlier works (10; 11) have similarly provided combinatorial formulas for *Demazure atoms* and *Demazure characters*, respectively, in terms of skylines:

$$\mathcal{A}_\gamma(X) = \sum_{\substack{Y \in SSAFI(n), \\ shape(Y)=\gamma}} \mathbf{x}^Y, \tag{1.2}$$

$$\kappa_\gamma(X) = \sum_{\substack{Y \in SSAFD(n), \\ shape(Y)=\gamma^*}} \mathbf{x}^Y, \tag{1.3}$$

where $SSAFI(n)$ is the set of all semistandard augmented fillings with increasing basement with entries in $[n]$, and $SSAFD(n)$ is the set of all semistandard augmented fillings with decreasing basement with entries in $[n]$. The analogous formula for *quasisymmetric Schur functions* over the variable set $X = \{x_1, x_2, x_3, \ldots\}$ is given by

$$\mathcal{S}_\alpha = \sum_{\substack{T \in SSCT, \\ shape(T)=\alpha}} \mathbf{x}^T, \tag{1.4}$$

where $SSCT$ is the set of all semistandard composition tableaux. We are now ready to provide Littlewood-Richardson rules for these latter three formal power series.

2 Littlewood-Richardson rule for Demazure atoms

Given a weak composition γ with $\ell(\gamma) = n$, we say that an *extended basement* of shape γ is a skyline in which the basement entries (in the 0-th column) are distinct integers, each of which is strictly greater than n, and the entries in each row of the diagram are equal to the column 0 entry for the respective row. The *standard decreasing (respectively increasing) extended basement* of shape γ is the extended basement of shape γ skyline in which the basement entries (in the 0-th column) are decreasing (respectively increasing) , beginning with $2n$ in the first row and ending with $n+1$ in the last row (respectively $n+1$ to $2n$).

Given weak compositions $\gamma \subset \delta$, where we assume $\ell(\gamma) = \ell(\delta)$, the *skew skyline diagram* (or simply *skew diagram*) of shape δ/γ is the set of those cells of the skyline diagram of shape δ that are not in the skyline diagram of shape γ. For this section, we consider the skew diagram to be "resting on" a standard decreasing extended basement of shape γ. Finally, a *Littlewood-Richardson skew skyline* (LRS) of shape δ/γ will be a filling $\sigma : S \to [n]$ of the empty cells of a skew diagram S resting on a standard decreasing extended basement of shape γ, where $n = \ell(\delta)$, that satisfies the rules for skyline diagram fillings and for which the column reading word (excluding extended basement entries) is a contre-lattice word. Figure 3 shows an example of a standard decreasing extended basement of shape $\gamma = (2, 0, 3, 1, 2)$, a skew diagram of shape δ/γ where $\delta = (3, 1, 4, 2, 5)$, and an LRS of the same skew shape with column reading word 3231321, which is contre-lattice of weight $(3, 2, 2)^*$. We can now state our Littlewood-Richardson rule for the product of Schur polynomials and Demazure atoms.

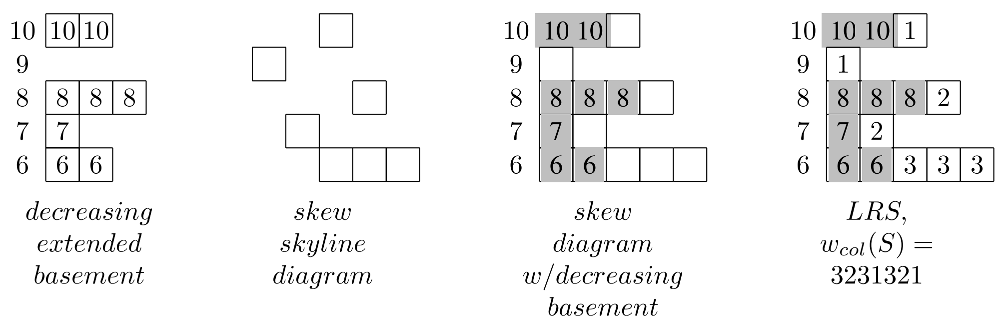


Fig. 3: Skyline extended basement and LRS

Theorem 2 *Let λ be a partition and γ, δ be weak compositions. In the expansion*

$$\mathcal{A}_\gamma \cdot s_\lambda = \sum_\delta a^\delta_{\gamma\lambda} \mathcal{A}_\delta, \tag{2.1}$$

the coefficient $a^\delta_{\gamma\lambda}$ is the number of LRS of shape δ/γ with content λ^*.

3 Littlewood-Richardson rule for quasisymmetric Schur functions

The method of proof for Theorem 2 leads easily to a corresponding Littlewood-Richardson rule for the product of a Schur function and a quasisymmetric Schur function. To state it we define the analogue of an LRS for composition diagrams.

Given a strong composition β and a weak composition γ such that $\ell(\gamma) = \ell(\beta)$ and $\gamma \subset \beta$, we say that the *skew composition diagram of shape β/γ* is the set of cells of the composition diagram of β that are not in the diagram of γ. We naturally associate to this skew composition diagram a skew skyline diagram with decreasing extended basement, where all of the of the entries in the basement are larger than the number of empty cells in the skew shape β/γ. Now to every LRS filling of this skew skyline diagram we associate the corresponding *Littlewood-Richardson skew composition tableau* (LRC), which is simply the filling of the cells of the skew composition diagram with the corresponding (nonbasement) entries from the LRS. An example is given in Figure 4. We can now state our Littlewood-Richardson rule for the product of Schur functions and quasisymmetric Schur functions.

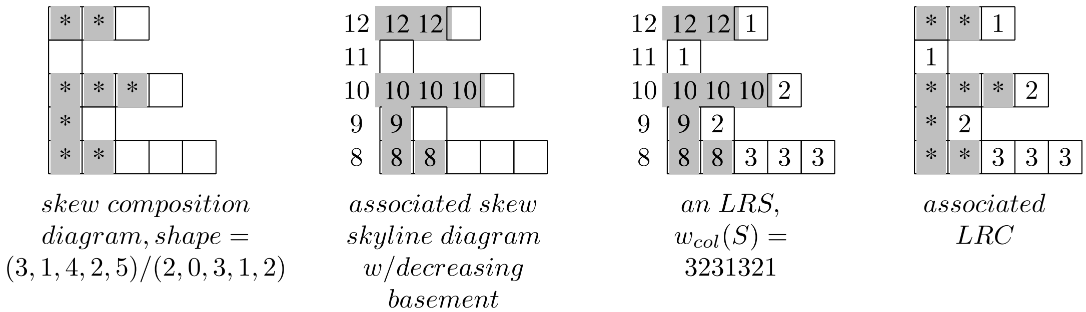


Fig. 4: Composition skew diagram and LRC

Theorem 3 *Let λ be a partition and α, β be strong compositions. In the expansion*

$$\mathcal{S}_\alpha \cdot s_\lambda = \sum_\beta C^\beta_{\alpha\lambda} \mathcal{S}_\beta, \tag{3.1}$$

the coefficient $C^\beta_{\alpha\lambda}$ is the number of LRC of shape β/γ with content λ^* and $\gamma^+ = \alpha$.

4 Littlewood-Richardson rule for Demazure characters

Given a standard increasing extended basement of shape α, with $\ell(\alpha) = n$, any weak composition β such that $\ell(\beta) = n$ and $\alpha \subset \beta$, we define a *Littlewood-Richardson key skyline* (LRK) of shape β/α to be a filling $\sigma : S \to [n]$ of the empty cells of a skew diagram S that satisfies the skyline diagram filling rules and for which the column reading word (excluding extended basement entries) is a contre-lattice word. Figure 5 provides an example of a standard increasing extended basement of shape $\alpha = (2, 0, 1, 2, 3)$, a skew diagram of shape β/α where $\beta = (5, 1, 3, 2, 4)$, and an LRK of the same shape with column reading word 3323121, which is contre-lattice of weight $(3, 2, 2)^*$. We are now ready to state our Littlewood-Richardson rule for the product of Schur polynomials and Demazure characters.

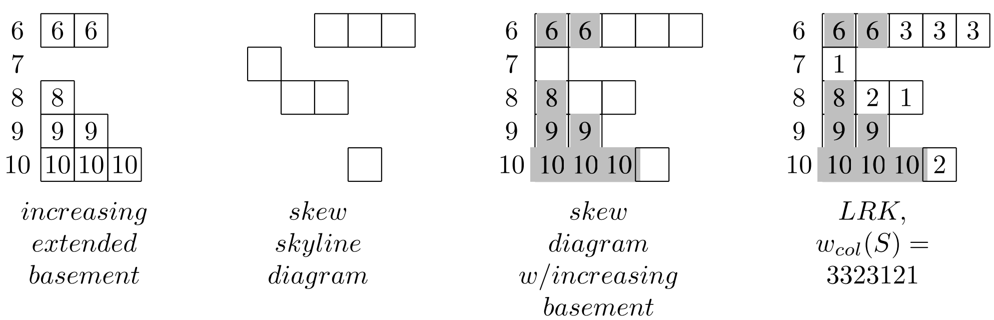


Fig. 5: Skyline extended basement and LRK

Theorem 4 *Let λ be a partition and γ, δ be weak compositions. In the expansion*

$$\kappa_\gamma \cdot s_\lambda = \sum_\delta b^\delta_{\gamma\lambda} \kappa_\delta, \tag{4.1}$$

the coefficient $b^\delta_{\gamma\lambda}$ is the number of LRK of shape δ^*/γ^* with content λ^*.

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On the Monotone Column Permanent conjecture

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Abstract. We discuss some recent progress on the Monotone Column Permanent (MCP) conjecture. We use a general method for proving that a univariate polynomial has real roots only, namely by showing that a corresponding multivariate polynomial is stable. Recent connections between stability of polynomials and the strong Rayleigh property revealed by Brändén allows for a computationally feasible check of stability for multi-affine polynomials. Using this method we obtain a simpler proof for the $n = 3$ case of the MCP conjecture, and a new proof for the $n = 4$ case. We also show a multivariate version of the stability of Eulerian polynomials for $n \leq 5$ which arises as a special case of the multivariate MCP conjecture.

Résumé. Nous présentons des développements récents concernant la conjecture Monotone Column Permanent (MCP). Nous utilisons une méthode générale pour prouver qu'un polynôme univarié a uniquement des racines réelles, c'est-à-dire que nous prouvons qu'un polynôme correspondant a plusieurs variables est stable. Les nouveaux liens, établis par Brändén, entre la stabilité des polynômes et la propriété forte de Rayleigh, permettent de vérifier facilement la stabilité de polynômes multi-affines. En utilisant cette méthode nous obtenons une preuve plus simple pour la conjecture MCP pour le cas $n = 3$, et la première preuve pour le cas $n = 4$. Nous présentons également une version multivarée de stabilité des polynômes d'Euler pour le cas $n \leq 5$, qui apparaît comme un cas spécial de la conjecture MCP multivarée.

Keywords: Monotone column, permanent, polynomials with real roots only, stability, strong Rayleigh property

1 Introduction

We discuss some recent progress on the Monotone Column Permanent (MCP) conjecture of Haglund, Ono and Wagner (HOW99; Hag00).

The Monotone Column Permanent (MCP) conjecture Let A be an $n \times n$ matrix with real entries weakly increasing down columns, *i.e.*, $a_{i,j} \leq a_{i+1,j}$ for $i = 1, \dots, n-1, j = 1, \dots, n$. Then, the polynomial $p(z) = \operatorname{per}(B)$, the permanent of matrix B with $b_{i,j} = a_{i,j} + z$ has only real zeros.

The conjecture was proven for some special cases in (HOW99), but the general case was left open for $n > 3$ and the proof for the $n = 3$ case was rather lengthy (May). In this paper we give a new proof for the $n = 3$ case and prove the $n = 4$ case. We also prove a special case for $n = 5$ and conjecture a multivariate version of the stability of Eulerian polynomials.

1.1 Real rootedness and stability

To prove real rootedness of a polynomial f, *i.e.*, that all roots of f are all real, we will use a method of showing that a multivariate generalization of f is stable. Similar ideas have been applied before in different contexts, *e.g.*, the multivariate Heilmann–Lieb and Lee–Young theorems (HL72; Sok05), and recently remarkable results were proved concerning reality of roots, using stable polynomials in (BB08).

We start our discussion with some necessary definitions first.

Stability We call a polynomial $f \in \mathbb{R}[z_1, \dots, z_n]$ stable if

$$(\forall i : \Im(z_i) > 0) \Rightarrow f(z_1, \dots, z_n) \neq 0.$$

For a univariate polynomial $f(z) \in \mathbb{R}[z]$ stability is equivalent to the fact that $f(z)$ has only real roots. Observe that if $f(z_1, \dots, z_n)$ is a stable multivariate polynomial then $g(z) = f(z, \dots, z)$ is also stable. By this observation it is clear that the following conjecture would imply the MCP conjecture:

Multivariate MCP conjecture Let A be an $n \times n$ matrix with real entries weakly increasing down columns, *i.e.*, $a_{i,j} \leq a_{i+1,j}$ for $i = 1, \dots, n-1$, $j = 1, \dots, n$. Then, the multivariate multi-affine polynomial $f(z_1, \dots, z_n) = \operatorname{per}(B)$ with $b_{i,j} = a_{i,j} + z_j$ is stable.

In fact, the authors conjecture a stronger version of the above conjecture.

Multivariate k-permanent MCP conjecture Let A be an $n \times m$ matrix with real entries weakly increasing down columns, *i.e.*, $a_{i,j} \leq a_{i+1,j}$ for $i = 1, \dots, n-1$, $j = 1, \dots, m$. Let B be the $n \times m$ matrix with entries $b_{i,j} = a_{i,j} + z_j$. Let I, J be index sets of size $k \leq \min(n, m)$ and denote by $[B]_{I,J}$ the $k \times k$ submatrix of B containing the rows $I = \{i_1, \dots, i_k\}$ and columns $J = \{j_1, \dots, j_k\}$. Then, the multivariate multi-affine polynomial $f(z_1, \dots, z_m) = \sum_{I,J} \operatorname{per}([B]_{I,J})$ is stable. The sum goes over all possible I and J, k-subsets of $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively.

1.2 Brändén's criterion

In our discussion we will deal with a restricted class of multivariate polynomials:

Multi-affine polynomial A multivariate polynomial is *multi-affine* if it has degree at most one in each variable.

Recently, in (Brä07) the following useful characterization of multi-affine stable polynomials was shown.

Theorem 1.1 (Brändén) *A multivariate multi-affine polynomial with real coefficients $f \in \mathbb{R}[z_1, \ldots, z_n]$ is stable, if and only if for all $\xi \in \mathbb{R}^n$ and $1 \le i < j \le n$*

$$\Delta_{i,j} f := \frac{\partial f}{\partial z_i}(\xi) \cdot \frac{\partial f}{\partial z_j}(\xi) - \frac{\partial^2 f}{\partial z_i \partial z_j}(\xi) \cdot f(\xi) \ge 0 \tag{1}$$

This equivalent condition is often referred to as the *strong Rayleigh property*.

The k-permanent conjecture is trivial for $k = 1$. Consider the $n = k = 2$ case. Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ denote a monotone column matrix. Then for

$$f(z_1, z_2) = \operatorname{per} \begin{pmatrix} z_1 + a & z_2 + c \\ z_1 + b & z_2 + d \end{pmatrix} = (z_1 + a)(z_2 + d) + (z_1 + b)(z_2 + c)$$

using Brändén's criterion we need to show that

$$\Delta_{1,2} f = (2z_1 + a + b)(2z_2 + c + d) - 2\left((z_1 + a)(z_2 + d) + (z_1 + b)(z_2 + c)\right) =$$

$$= (a + c)(b + d) - 2(ad + bc) = (a - b)(c - d) \ge 0,$$

which is a consequence of the monotone column property of A.

Note that it is possible to apply this method straightforwardly to the $n = k = 3$ case and perhaps larger matrices, however the computations become soon intractable. In the following section we present observations that allow us to restrict ourselves to $0 - 1$ matrices.

2 Reducing the conjecture to 0-1 matrices

Lemma 2.1 *If there is a counterexample to the k-permanent MCP conjecture, then there is a counterexample A such that there are only two different entries in each column of A.*

Proof: If there are no counterexamples to the conjecture the lemma is true. Otherwise, let k denote the smallest number for which the k-permanent MCP conjecture is false. Clearly, $k > 1$. Let A be a minimal size counterexample for this k with n rows and m columns, and assume that A has a column with at least three different values in it. *W.l.o.g.*, we can assume that this is the first column, *i.e.*, $\alpha = a_{k-1,1} < a_{k,1} = \beta = a_{\ell,1} < a_{\ell+1,1} = \gamma$ for some $1 < k \le l < n$. We will show that by changing all occurrences of β to α or γ we obtain a matrix which is also a counterexample. Clearly, the first column of the matrix will have one less different values (and all other columns remain unchanged). Hence, by repeating this procedure in each column we will arrive at a counterexample matrix that has only two different entries per column.

Using the notation of the conjecture, denote the matrix with entries $b_{i,j} = a_{i,j} + z_j$ by B and the multivariate multi-affine polynomial obtained by summing over all k-permanental minors of B by f. Since A is a counterexample there are complex numbers $\xi_i, i = 1 \ldots, m$ with positive imaginary part such that $f(\xi_1, \ldots, \xi_m) = 0$. On the other hand, by expanding the permanent along the first column we get that

$$f(z_1, \xi_2, \ldots, \xi_m) = z_1 p(\xi_2, \ldots, \xi_m) + \beta q(\xi_2, \ldots, \xi_m) + r(\xi_2, \ldots, \xi_m)$$

where the polynomials $p(z_2, \ldots, z_m)$, $q(z_2, \ldots, z_m)$ and $r(z_2, \ldots, z_m)$ do not depend on z_1 nor on β. Note that p is the polynomial obtained by summing over all $k-1$-permanental polynomial of a monotone matrix (obtained from A by deleting the first column). And since $\Im(\xi_i) > 0$ for $i = 2 \ldots m$, by the minimality of A and k we get that $p(\xi_2, \ldots, \xi_m) \neq 0$.

We need to show that if we change all occurrences of β to α (or all occurrences of β to γ) in the column then the modified matrix is also a counterexample, *i.e.*, $z_1 p + \alpha q + r = 0$ (or $z_1 p + \gamma q + r = 0$) for some z_1 with positive imaginary part. Let $w_1 = -\frac{\alpha q + r}{p}$ and $w_2 = -\frac{\gamma q + r}{p}$. Since z_1 is a linear function of β in $z_1 p + \beta q + r = 0$, and $\Im(\xi_1) > 0$ where $\xi_1 = -\frac{\beta q + r}{p}$, it must be the case that $\Im(w_1) > 0$ or $\Im(w_2) > 0$. $\square$

Lemma 2.2 *If there is a counterexample to the k-permanent MCP conjecture, then there is a counterexample A with entries* 0 *and* 1 *only.*

Proof: By the previous lemma we can assume that each column j in A has at most two different entries. Consider the case when there are two different values in each column, namely $c_j < d_j$. Since multiplying a complex number z by a positive real number and adding a real number to z does not change the sign of its imaginary part, $\Im(z)$, it is easy to see that $f(z_1, \ldots, z_m) = \mathrm{per}(a_{ij} + z_j)$ is stable if and only if $g(z_1, \ldots, z_m) = f\left((d_1 - c_1)z_1 - c_1, \ldots, (d_n - c_n)z_m - c_m\right)$ is stable. To conclude, note that $g(z_1, \ldots, z_m) = \mathrm{per}(\tilde{a}_{ij} + z_j)$ where $\tilde{a}_{ij} \in \{0, 1\}$ for all i, j. For the case when $c_i = d_i$ for some i the proof is similar. $\square$

3 Results for the MCP conjecture

Note that in Lemma 2.1 the same proof goes through if we only consider matrices A of size $n \times m$ with $m \leq n$, and restrict ourselves to the $k = m$ case. Then the $p \neq 0$ assumption for the coefficient of z_1 still holds, because p is in fact the sum of the $(m-1)$-permanents of a matrix of size $n \times (m-1)$.

3.1 A new proof for the 3×3 case

By Lemmas 2.1 and 2.2, in the 3×3 case we only need to verify the monotone column 0-1 matrices. Due to symmetry considerations we can restrict ourselves to $\binom{6}{3} = 20$ matrices, the number of Ferrers boards fitting in a 3×3 square. Furthermore, 16 out of these matrices have an all 0 or all 1 column, which means that we can factor z_j or $z_j + 1$, respectively, from the permanent and reduce the question to a 3×2 matrix. Let us check the conjecture for these matrices first.

In the 3×2 case if we have an all 0 (or all 1) column the problem is trivial since we can factor the polynomial, hence it is stable. There are 3 matrices which do not have an all 0 or all 1 column:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Denote by $f_{(c_1, \ldots, c_m)}$ the polynomial obtained by taking the permanent of matrix B with $b_{ij} = a_{ij} + z_j$, where the corresponding matrix A is an $n \times m$ Ferrers matrix with c_j ones in column j for all j.

The permanents

$$\begin{aligned}
f_{(1,1)} &= z_1(2z_2 + 1) + z_1(2z_2 + 1) + (z_1 + 1)2z_2 = 6z_1z_2 + 2z_1 + 2z_2, \\
f_{(1,2)} &= z_1(2z_2 + 2) + z_1(2z_2 + 1) + (z_1 + 1)(2z_2 + 1) = 6z_1z_2 + 4z_1 + 2z_2 + 1, \\
f_{(2,2)} &= z_1(2z_2 + 2) + (z_1 + 1)(2z_2 + 1) + (z_1 + 1)(2z_2 + 1) = 6z_1z_2 + 4z_1 + 4z_2 + 2
\end{aligned}$$

all satisfy (1), *i.e.*, they all have the strong Rayleigh property:

$$\begin{array}{rcl} \Delta_{1,2}f_{(1,1)} & = & (6z_2+2)(6z_1+2)-6(6z_1z_2+2z_1+2z_2)=4\geq 0, \\ \Delta_{1,2}f_{(1,2)} & = & (6z_2+4)(6z_1+2)-6(6z_1z_2+4z_1+2z_2+1)=2\geq 0, \\ \Delta_{1,2}f_{(2,2)} & = & (6z_2+4)(6z_1+4)-6(6z_1z_2+4z_1+4z_2+2)=4\geq 0. \end{array}$$

Now we only need to check the following 3×3 matrices:

$$\begin{pmatrix} 0&0&0\\0&0&0\\1&1&1 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0\\0&0&1\\1&1&1 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0\\0&1&1\\1&1&1 \end{pmatrix}, \quad \begin{pmatrix} 0&0&0\\1&1&1\\1&1&1 \end{pmatrix}.$$

Following the notation from above, the permanents

$$\begin{array}{rcl} f_{(1,1,1)} & = & 6z_1z_2z_3+2z_1z_2+2z_1z_3+2z_2z_3, \\ f_{(1,1,2)} & = & 6z_1z_2z_3+4z_1z_2+2z_1z_3+2z_2z_3+z_1+z_2, \\ f_{(1,2,2)} & = & 6z_1z_2z_3+4z_1z_2+4z_1z_3+2z_2z_3+2z_1+z_2+z_3, \\ f_{(2,2,2)} & = & 6z_1z_2z_3+4z_1z_2+4z_1z_3+4z_2z_3+2z_1+2z_2+2z_3 \end{array}$$

all have the strong Rayleigh property

$$\begin{array}{rcl} \Delta_{1,2}f_{(1,1,1)} & = & 4z_3^2\geq 0, \\ \Delta_{1,2}f_{(1,1,2)} & = & (2z_3+1)^2\geq 0, \\ \Delta_{1,3}f_{(1,1,2)} & = & 2z_3^2\geq 0, \\ \Delta_{1,2}f_{(1,2,2)} & = & 2(z_3+1)^2\geq 0, \\ \Delta_{2,3}f_{(1,2,2)} & = & (2z_1+1)^2\geq 0, \\ \Delta_{1,2}f_{(2,2,2)} & = & 4(z_3+1)^2\geq 0. \end{array}$$

Note we did not check $\Delta_{i,j}f\geq 0$ for all possible i,j pairs, the remaining cases follow by symmetry.

As a consequence we obtain that the MCP conjecture holds for 3×3 matrices.

3.2 A proof for the 4×4 case

The above computation reveals one weakness of this method, namely that we have to check the strong Rayleigh property for all pairs, *i.e.*, we need to verify $\Delta_{i,j}f\geq 0$ for all $i<j$. In (WW09) a new criterion was introduced to reduce the computation required.

Theorem 3.1 (Wagner and Wei) *Let $f(z_1,\dots,z_n)$ be a multi-affine polynomial with positive coefficients. Then f has the strong Rayleigh property if and only if*

$$\frac{\partial f}{\partial z_\ell} \text{ and } f_{|z_\ell=0}=f(z_1,\dots,z_{\ell-1},0,z_{\ell+1},\dots,z_n)$$

have the strong Rayleigh property for all ℓ, and (1) holds for some pair $i<j$.

This theorem is helpful because it is sufficient to check now $\Delta_{i,j} f \geq 0$ for only one index pair. We were checking the stability of $\partial f / \partial z_\ell$ already. Since, if A is an $n \times n$ matrix then $\partial f / \partial z_\ell$ is the permanent corresponding to the $n \times (n-1)$ matrix obtained by removing column ℓ from A (we were checking this, in case column ℓ had only zeros or only ones). Checking the stability of $f_{|z_\ell=0}$ is an extra overhead but overall saves more time than if we had to check $\Delta_{i,j} f \geq 0$ for all index pairs.

Let us verify the conjecture for 4×4 matrices. Again, to verify the conjecture for matrices with an all 0 or all 1 columns we reduce the problem to 4×3 and 4×2 matrices consequently. The permanents of 4×1 matrices are trivially stable. Here are the 6 matrices of size 4×2 with no all 0 or all 1 columns:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We need to verify the strong Rayleigh property for the corresponding polynomials:

$$\begin{aligned}
f_{(1,1)} &= 12z_1z_2 + 3z_1 + 3z_2 \\
f_{(1,2)} &= 12z_1z_2 + 6z_1 + 3z_2 + 1 \\
f_{(1,3)} &= 12z_1z_2 + 9z_1 + 3z_2 + 2 \\
f_{(2,2)} &= 12z_1z_2 + 6z_1 + 6z_2 + 2 \\
f_{(2,3)} &= 12z_1z_2 + 9z_1 + 6z_2 + 4 \\
f_{(3,3)} &= 12z_1z_2 + 9z_1 + 9z_2 + 6
\end{aligned}$$

They are all stable since,

$$\begin{aligned}
\Delta_{1,2} f_{(1,1)} = (12z_1+3)(12z_2+3) - 12(12z_1z_2 + 3z_1 + 3z_2) &= 9 \geq 0 \\
\Delta_{1,2} f_{(1,2)} = (12z_1+3)(12z_2+6) - 12(12z_1z_2 + 6z_1 + 3z_2 + 1) &= 6 \geq 0 \\
\Delta_{1,2} f_{(1,3)} = (12z_1+3)(12z_2+9) - 12(12z_1z_2 + 9z_1 + 3z_2 + 2) &= 3 \geq 0 \\
\Delta_{1,2} f_{(2,2)} = (12z_1+6)(12z_2+6) - 12(12z_1z_2 + 6z_1 + 6z_2 + 2) &= 12 \geq 0 \\
\Delta_{1,2} f_{(2,3)} = (12z_1+6)(12z_2+9) - 12(12z_1z_2 + 9z_1 + 6z_2 + 4) &= 6 \geq 0 \\
\Delta_{1,2} f_{(3,3)} = (12z_1+9)(12z_2+9) - 12(12z_1z_2 + 9z_1 + 9z_2 + 6) &= 9 \geq 0.
\end{aligned}$$

Now, we need to verify the stability of the 4×3 matrices (with no all 0 or all 1 columns):

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The polynomials and their Rayleigh differences are:

$$\begin{aligned}
f_{(1,1,1)} &= 24z_1z_2z_3 + 6z_1z_2 + 6z_1z_3 + 6z_2z_3\\
f_{(1,1,2)} &= 24z_1z_2z_3 + 12z_1z_2 + 6z_1z_3 + 6z_2z_3 + 2z_1 + 2z_2\\
f_{(1,1,3)} &= 24z_1z_2z_3 + 18z_1z_2 + 6z_1z_3 + 6z_2z_3 + 4z_1 + 4z_2\\
f_{(1,2,2)} &= 24z_1z_2z_3 + 12z_1z_3 + 12z_1z_2 + 6z_2z_3 + 4z_1 + 2z_2 + 2z_3\\
f_{(1,2,3)} &= 24z_1z_2z_3 + 18z_1z_2 + 12z_1z_3 + 6z_2z_3 + 8z_1 + 4z_2 + 2z_3 + 1\\
f_{(1,3,3)} &= 24z_1z_2z_3 + 18z_1z_2 + 18z_1z_3 + 6z_2z_3 + 12z_1 + 4z_2 + 4z_3 + 2\\
f_{(2,2,2)} &= 24z_1z_2z_3 + 12z_1z_2 + 12z_1z_3 + 12z_2z_3 + 4z_1 + 4z_2 + 4z_3\\
f_{(2,2,3)} &= 24z_1z_2z_3 + 18z_1z_2 + 12z_1z_3 + 12z_2z_3 + 8z_1 + 8z_2 + 4z_3 + 2\\
f_{(2,3,3)} &= 24z_1z_2z_3 + 18z_1z_2 + 18z_1z_3 + 12z_2z_3 + 12z_1 + 8z_2 + 8z_3 + 4\\
f_{(3,3,3)} &= 24z_1z_2z_3 + 18z_1z_2 + 18z_1z_3 + 18z_2z_3 + 12z_1 + 12z_2 + 12z_3 + 6
\end{aligned}$$

$$\begin{aligned}
\Delta_{1,2}f_{(1,1,1)} &= 36z_3^2\\
\Delta_{1,2}f_{(1,1,2)} &= 4(3z_3+1)^2\\
\Delta_{1,3}f_{(1,1,2)} &= 24z_2^2\\
\Delta_{1,2}f_{(1,1,3)} &= 4(3z_3+2)^2\\
\Delta_{1,3}f_{(1,1,3)} &= 12z_2^2\\
\Delta_{1,2}f_{(1,2,2)} &= 8(3z_3^2+3z_3+1)\\
\Delta_{2,3}f_{(1,2,2)} &= 4(12z_1^2+6z_1+1)\\
\Delta_{1,2}f_{(1,2,3)} &= 2(12z_3^2+18z_3+7)\\
\Delta_{1,3}f_{(1,2,3)} &= 4(3z_2^2+3z_2+1)\\
\Delta_{2,3}f_{(1,2,3)} &= 2(12z_1^2+6z_1+1)\\
\Delta_{1,2}f_{(1,3,3)} &= 12(z_3+1)^2\\
\Delta_{2,3}f_{(1,3,3)} &= 4(3z_1+1)^2\\
\Delta_{1,2}f_{(2,2,2)} &= 16(3z_3^2+3z_3+1)\\
\Delta_{1,2}f_{(2,2,3)} &= 4(12z_3^2+18z_3+7)\\
\Delta_{1,3}f_{(2,2,3)} &= 8(3z_2^2+3z_2+1)\\
\Delta_{1,2}f_{(2,3,3)} &= 24(z_3+1)^2\\
\Delta_{2,3}f_{(2,3,3)} &= 4(3z_1+2)^2\\
\Delta_{1,2}f_{(3,3,3)} &= 36(z_3+1)^2
\end{aligned}$$

Finally, we have to show the stability of 4×4 matrices. Instead of computing the Rayleigh differences potentially $\binom{4}{2} = 6$ times for each matrix, we employ Theorem 3.1 and compute only one Rayleigh difference per matrix. We already have that the partial derivatives are stable, since these are exactly the polynomials which are the permanents of 4×3 matrices. Now we need to show the stability of $f_{|z_\ell=0}$.

These polynomials have one of the three following forms. They can be a permanent of a 3×3 matrix obtained by removing the last row of the 4×3 matrix, or the sum of two permanents (one obtained by removing the third row and another obtained by removing the last row of the original matrix), or three permanents of 3×3 matrices (obtained by the respective submatrices of the given 4×3 matrix by removing the second, third and last row).

The 3×3 case we have already solved in Section 3.1, the sum of two permanents can be reduced to a single 3×3 permanent case, by expanding the permanent along the last row:

$$
\begin{aligned}
\operatorname{per}\begin{pmatrix} z_1+a_1 & z_2+a_2 & z_3+a_3 & 0 \\ z_1+b_1 & z_2+b_2 & z_3+b_3 & 0 \\ z_1+c_1 & z_2+c_2 & z_3+c_3 & 1 \\ z_1+d_1 & z_2+d_2 & z_3+d_3 & 1 \end{pmatrix} &= \operatorname{per}\begin{pmatrix} z_1+a_1 & z_2+a_2 & z_3+a_3 \\ z_1+b_1 & z_2+b_2 & z_3+b_3 \\ z_1+c_1 & z_2+c_2 & z_3+c_3 \end{pmatrix} + \\
&\quad \operatorname{per}\begin{pmatrix} z_1+a_1 & z_2+a_2 & z_3+a_3 \\ z_1+b_1 & z_2+b_2 & z_3+b_3 \\ z_1+d_1 & z_2+d_2 & z_3+d_3 \end{pmatrix} \\
&= 2\operatorname{per}\begin{pmatrix} z_1+a_1 & z_2+a_2 & z_3+a_3 \\ z_1+b_1 & z_2+b_2 & z_3+b_3 \\ z_1+\frac{c_1+d_1}{2} & z_2+\frac{c_2+d_2}{2} & z_3+\frac{c_3+d_3}{2} \end{pmatrix}
\end{aligned}
$$

Note that in this case the other rows are identical, and since the entries in the last rows are the largest ones their average also preserves the monotone column property.

For the sum of three permanents we can also argue similarly. Note that if there are two columns with the same number of ones, then two of the summands are identical and we can sum the permanents by expanding along the row in which they differ. Here factoring out 3 and placing the average in the row will preserve the monotone column property. The only case when all columns have different number of ones is when

$$
A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
$$

The corresponding polynomial $f = 18z_1z_2z_3 + 12z_1z_2 + 8z_1z_3 + 4z_2z_3 + 4z_1 + 2z_2 + z_3$ is stable, because the

$$
\begin{aligned}
\Delta_{1,2}f &= 2(7z_3^2 + 10z_3 + 4) \\
\Delta_{1,3}f &= 4(3z_2^2 + 3z_2 + 1) \\
\Delta_{2,3}f &= 2(12z_1^2 + 6z_1 + 1)
\end{aligned}
$$

differences are always positive.

We check the strong Rayleigh property for all the 15 matrices of size 4×4 with no all 0 or all 1 columns:

$$\begin{aligned}
\Delta_{1,2}f_{(1,1,1,1)} &= 36z_3^2z_4^2\\
\Delta_{1,2}f_{(1,1,1,2)} &= 24z_3^2z_4^2\\
\Delta_{1,2}f_{(1,1,1,3)} &= 8z_3z_4+48z_3^2z_4^2+4z_3^2\ +4z_4^2+24z_3^2z_4+24z_3z_4^2\\
\Delta_{1,2}f_{(1,1,2,2)} &= 8z_3z_4+48z_3^2z_4^2+4z_3^2\ +16z_4^2+24z_3^2z_4+48z_3z_4^2\\
\Delta_{1,2}f_{(1,1,2,3)} &= 16z_3z_4+48z_3^2z_4^2+16z_3^2\ +16z_4^2+48z_3^2z_4+48z_3z_4^2\\
\Delta_{1,2}f_{(1,1,3,3)} &= 12z_3^2z_4^2\\
\Delta_{1,2}f_{(1,2,2,2)} &= 4z_3z_4+24z_3^2z_4^2+2z_3^2\ +2z_4^2+12z_3^2z_4+12z_3z_4^2\\
\Delta_{1,2}f_{(1,2,2,3)} &= 4z_3z_4+24z_3^2z_4^2+2z_3^2\ +8z_4^2+12z_3^2z_4+24z_3z_4^2\\
\Delta_{1,2}f_{(1,2,3,3)} &= 8z_3z_4+24z_3^2z_4^2+8z_3^2\ +8z_4^2+24z_3^2z_4+24z_3z_4^2\\
\Delta_{1,2}f_{(1,3,3,3)} &= 4(3z_3z_4+z_4+z_3)^2\\
\Delta_{1,2}f_{(2,2,2,2)} &= (6z_3z_4+4z_4+2z_3+1)^2\\
\Delta_{1,2}f_{(2,2,2,3)} &= 4(3z_3z_4+2z_4+2z_3+1)^2\\
\Delta_{1,2}f_{(2,2,3,3)} &= 4(3z_4+1)^2(z_3+1)^2\\
\Delta_{1,2}f_{(2,3,3,3)} &= 4(3z_4+2)^2(z_3+1)^2\\
\Delta_{1,2}f_{(3,3,3,3)} &= 36(z_4+1)^2(z_3+1)^2
\end{aligned}$$

The 6 differences which are not complete squares are also non-negative, since they have the following non-positive discriminants (when they are considered as a polynomial in z_3):

$$-192z_4^4,\ \ -192(2z_4+1)^2z_4^2,\ \ -768(z_4+1)^2z_4^2,\ \ -48z_4^4,\ \ -48(2z_4+1)^2z_4^2,\ \ -192(z_4+1)^2z_4^2,$$

and positive leading coefficients:

$$4(12z_4^2{+}6z_4{+}1),\ \ 4(12z_4^2{+}6z_4{+}1),\ \ 16(3z_4^2{+}3z_4{+}1),\ \ 2(12z_4^2{+}6z_4{+}1),\ \ 2(12z_4^2{+}6z_4{+}1),\ \ 8(3z_4^2{+}3z_4{+}1).$$

3.3 Proof for a 5×5 matrix and the Eulerian polynomials

There is an interesting connection between the Eulerian polynomial and the multivariate MCP conjecture. Let A be the $n\times n$ matrix with all zeros above and on the diagonal and all ones below. Denote the permanent of B where $b_{ij}=a_{ij}+z_j$ by $f(z_1,\ldots,z_n)$. The univariate polynomial obtained by setting $z_i=z$ for all i is the Eulerian polynomial modified by a rational change of variables:

$$f(z,\ldots,z)=(z+1)^nA_n\left(\frac{z}{1+z}\right)=\sum_{k=1}^{n}A(n,k)z^k(1+z)^{n-k}, \tag{2}$$

where $A_n(z)$ is the generating polynomial of the Eulerian numbers, $A(n,k)$, *e.g.*, the number of permutations of n letters with k weak excedances. (A number i is a weak excedance in permutation $\pi=\pi_1\pi_2\cdots\pi_n$ if $\pi_i\geq i$.) To show (2) we can identify permutations of n letters with placements of n rooks on an $n\times n$ board, with a rook on (i,j) interpreted as $\pi_i=j$ in the permutation $\pi=\pi_1\pi_2\cdots\pi_n$. The

$z^k(1+z)^{n-k}$ terms in the expansion of the permanent $f(z,\ldots,z)$ correspond to rook placements with k rooks on or above the diagonal (each rook contributes a a factor of z to the term). Hence the coefficient of $z^k(1+z)^{n-k}$ is exactly $A(n,k)$.

Therefore, this multivariate generalization of the Eulerian polynomials is a natural candidate to be proven stable, which would imply the well-known fact that Eulerian polynomials have only real zeros.

We already proved that for $n \leq 4$ these polynomials are stable. Now we show that for $n = 5$ this special case of the multivariate MCP conjecture also holds.

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let $f = f_{(0,1,2,3,4)}$. We check the strong Rayleigh property for all $1 \leq i < j \leq 5$. Since we can factor out z_1 from f note that $\Delta_{1,2}f = \Delta_{1,3}f = \Delta_{1,4}f = \Delta_{1,5}f = 0$.

$$\begin{aligned}
\Delta_{2,3}f &= 2z_1^2(216z_4^2z_5^2 + 336z_4^2z_5 + 240z_4z_5^2 + 354z_4z_5 + 132z_4^2 + 132z_4 + 72z_5^2 + 102z_5 + 37) \\
\Delta_{2,4}f &= 4z_1^2(75z_3z_5 + 120z_3^2z_5 + 48z_3z_5^2 + 72z_3^2z_5^2 + 12z_5^2 + 18z_5 + 7 + 51z_3^2 + 30z_3) \\
\Delta_{2,5}f &= 4z_1^2(27z_3z_4 + 48z_3^2z_4 + 24z_3z_4^2 + 36z_3^2z_4^2 + 6z_4^2 + 6z_4 + 2 + 18z_3^2 + 9z_3) \\
\Delta_{3,4}f &= 2z_1^2(150z_2z_5 + 96z_5^2z_2 + 288z_2^2z_5^2 + 480z_2^2z_5 + 12z_5^2 + 18z_5 + 7 + 60z_2 + 204z_2^2) \\
\Delta_{3,5}f &= 4z_1^2(27z_2z_4 + 72z_2^2z_4^2 + 24z_4^2z_2 + 3z_4^2 + 3z_4 + 1 + 96z_2^2z_4 + 9z_2 + 36z_2^2) \\
\Delta_{4,5}f &= 2z_1^2(216z_2^2z_3^2 + 66z_2z_3 + 96z_3^2z_2 + 1 + 192z_2^2z_3 + 12z_2 + 12z_3^2 + 6z_3 + 48z_2^2)
\end{aligned}$$

We verified by a computer that these are always non-negative by the following procedure. For example, consider $g(z_2,z_3) = \Delta_{4,5}f/2z_1^2$. The coefficient of z_3^2 in $g(z_2,z_3)$ is $12(18z^2 + 8z_2 + 1) > 0$ and the discriminant of $g(z_2,z_3)$ viewed as a polynomial in z_3 is $-48(384z_2^4 + 288z_2^3 + 93z_2^2 + 14z_2 + 1) < 0$, since this quartic polynomial has negative leading coefficient and has no real roots.

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Counting Quiver Representations over Finite Fields Via Graph Enumeration

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Abstract. Let Γ be a quiver on n vertices $v_1, v_2, \ldots, v_n$ with g_{ij} edges between v_i and v_j, and let $\boldsymbol{\alpha} \in \mathbb{N}^n$. Hua gave a formula for $A_\Gamma(\boldsymbol{\alpha}, q)$, the number of isomorphism classes of absolutely indecomposable representations of Γ over the finite field $\mathbb{F}_q$ with dimension vector $\boldsymbol{\alpha}$. We use Hua's formula to show that the derivatives of $A_\Gamma(\boldsymbol{\alpha}, q)$ with respect to q, when evaluated at $q = 1$, are polynomials in the variables g_{ij}, and we can compute the highest degree terms in these polynomials. The formulas for these coefficients depend on the enumeration of certain families of connected graphs. This note simply gives an overview of these results; a complete account of this research is available on the arXiv and has been suboldsymbolitted for publication.

Résumé. Soit Γ un carquois sur n sommets $v_1, v_2, \ldots, v_n$ avec g_{ij} arêtes entre v_i et v_j, et soit $\boldsymbol{\alpha} \in \mathbb{N}^n$. Hua a donné une formule pour $A_\Gamma(\boldsymbol{\alpha}, q)$, le nombre de classes d'isomorphisme absolument indécomposables de représentations de Γ sur le corps fini $\mathbb{F}_q$ avec vecteur de dimension $\boldsymbol{\alpha}$. Nous utilisons la formule de Hua pour montrer que les dérivées de $A_\Gamma(\boldsymbol{\alpha}, q)$ par rapport à q, alors évaluée à $q = 1$, sont des polynômes dans les variables g_{ij}, et on peut calculer les termes de plus haut degré de ces polynômes. Les formules pour ces coefficients dépendent de l'énumération de certaines familles de graphes connectés. Cette note donne simplement un aperçu de ces résultats, un compte rendu complet de cette recherche est disponible sur arXiv et a été soumis pour publication.

Keywords: quiver representation, finite field, graph enumeration, absolutely indecomposable representation

1 Introduction

Let Γ be a quiver on n vertices $v_1, v_2, \ldots, v_n$ with g_{ij} edges between vertices v_i and v_j for $1 \leq i \leq j \leq n$. All of the following results are independent of the orientation of these edges. Let $\mathbf{0} \neq \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ (throughout the paper, vectors will be represented by boldface symbols, and $\mathbb{N}$ denotes the set of non-negative integers). We are interested in $A_\Gamma(\boldsymbol{\alpha}, q)$, the number of isomorphism classes of absolutely indecomposable representations of Γ over the finite field $\mathbb{F}_q$ with dimension vector $\boldsymbol{\alpha}$. Kac [6] proved that $A_\Gamma(\boldsymbol{\alpha}, q)$ is a polynomial in q with integer coefficients and that it is independent of the orientation of Γ. He conjectured that the coefficients of $A_\Gamma(\boldsymbol{\alpha}, q)$ are non-negative and that if Γ has no loops, then the constant term of $A_\Gamma(\boldsymbol{\alpha}, q)$ is equal to the multiplicity of $\boldsymbol{\alpha}$ in the Kac-Moody algebra

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defined by Γ. Both conjectures are true for quivers of finite and tame type and remain open for quivers of wild type (see Crawley-Boevey and Van den Bergh [1]). A proof of the multiplicity statement in Kac's conjectures for general quivers was recently announced by Hausel [3].

Our goal is to understand $A_\Gamma(\boldsymbol{\alpha}, 1)$, and more generally $\left(\frac{d^s}{dq^s}\, A_\Gamma(\boldsymbol{\alpha}, q)\right)\Big|_{q=1}$, as a function of the variables g_{ij}. The paper [4] offers complete descriptions and proofs of our results; this note is an extended abstract of that paper, content with stating the main theorems. Our primary impetus for studying $A_\Gamma(\boldsymbol{\alpha}, 1)$ comes from the work of Hausel and Rodriguez-Villegas [2]. They show that when Γ is the quiver S_g consisting of one vertex v with g self-loops, $A_{S_g}(\alpha, 1)$ (where $\boldsymbol{\alpha} = \alpha \in \mathbb{N}$) is (conjecturally) the dimension of the middle cohomology group of a character variety parameterizing certain representations of the fundamental group of a closed genus-g Riemann surface to $\mathrm{GL}_n(\mathbb{C})$.

One can imagine that specializing to $q = 1$ will relate $A_\Gamma(\boldsymbol{\alpha}, q)$ to counting representations of Γ in the category of finite sets; this hope follows a well-known philosophy about the significance of letting $q \to 1$ in formulas that depend on a finite field $\mathbb{F}_q$, although it seems hard to make this philosophy precise. In this paper we show in Theorems 4.3 and 5.1 that $\left(\frac{d^s}{dq^s}\, A_\Gamma(\boldsymbol{\alpha}, q)\right)\Big|_{q=1}$ is a polynomial in the variables g_{ij}, and we give a formula for its leading coefficients. This formula relies on the number of connected graphs in a family determined by Γ and on Stirling numbers of the second kind, which arise from derivatives of q-binomial coefficients. The description of the graphs in question is given prior to Theorem 3.1. Unfortunately, our proofs of Theorems 4.3 and 5.1 do not give any conceptual indication as to why our results should involve the enumeration of connected graphs.

To illustrate the type of result found in this paper, consider $\Gamma = S_g$. Using a formula of Hua [5, Theorem 4.6] for $A_\Gamma(\boldsymbol{\alpha}, q)$, which we will present in Section 2 and which is the starting point for our results, we can compute the polynomial $A_{S_g}(\alpha, q)$ for small α and g. These computations are displayed in the following table:

$A_{S_g}(\alpha, q)$	$g=1$	$g=2$	$g=3$	$g=4$
$\alpha=1$	q	q^2	q^3	q^4
$\alpha=2$	q	q^5+q^3	$q^9+q^7+q^5$	$q^{13}+q^{11}+\cdots$
$\alpha=3$	q	$q^{10}+q^8+\cdots$	$q^{19}+q^{17}+\cdots$	$q^{28}+q^{26}+\cdots$
$\alpha=4$	q	$q^{17}+q^{15}+\cdots$	$q^{33}+q^{31}+\cdots$	$q^{49}+q^{47}+\cdots$

Evaluating each polynomial at $q = 1$ gives the following values for $A_{S_g}(\alpha, 1)$:

$A_{S_g}(\alpha, 1)$	$g=1$	$g=2$	$g=3$	$g=4$	$g=5$	$g=6$
$\alpha=1$	1	1	1	1	1	1
$\alpha=2$	1	2	3	4	5	6
$\alpha=3$	1	6	15	28	45	66
$\alpha=4$	1	22	95	252	525	946

Fitting each row of the above table to a polynomial gives empirical evidence that the next table is correct:

	$A_{S_g}(\alpha, 1)$
$\alpha = 1$	1
$\alpha = 2$	$\binom{g}{1}$
$\alpha = 3$	$4\binom{g}{2} + \binom{g}{1}$
$\alpha = 4$	$32\binom{g}{3} + 20\binom{g}{2} + \binom{g}{1}$

This suggests that $A_{S_g}(\alpha, 1)$ is a polynomial in g of degree $\alpha - 1$ with leading coefficient $2^{\alpha-1}\alpha^{\alpha-2}/\alpha!$. We prove this and a generalization to all quivers in Theorem 4.3 below. Theorem 5.1 offers a similar result for any derivative (with respect to q) of $A_\Gamma(\boldsymbol{\alpha}, q)$ evaluated at $q = 1$.

The fact that the leading coefficient of $A_{S_g}(\alpha, 1)$ equals $2^{\alpha-1}\alpha^{\alpha-2}/\alpha!$ was mentioned (without proof) in [2, Remark 4.4.6]. As mentioned above, in the context of that paper, S_g corresponds to a closed Riemann surface of genus g and it seems more appropriate to use its Euler characteristic $2g - 2$ instead of g as a variable. Then $A_{S_g}(\alpha, 1)$ is a polynomial in $2g - 2$ of degree $\alpha - 1$ with leading coefficient $\alpha^{\alpha-2}/\alpha!$, and the disappearance of the factor $2^{\alpha-1}$ suggests that $2g - 2$ may be the "right" variable to use, though we do not know of a similar approach for the general case. Finally, we note that $\alpha^{\alpha-2}$ appears in the formula for the leading coefficient of $A_{S_g}(\alpha, 1)$ because $\alpha^{\alpha-2}$ is the number of trees on α labeled vertices by Cayley's Theorem. As indicated above, for other quivers, the leading coefficient formula involves the enumeration of other families of graphs.

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2 Hua's Formula

We begin with a presentation of Hua's formula for $A_\Gamma(\boldsymbol{\alpha}, q)$. Let $\boldsymbol{T} = (T_1, T_2, \ldots, T_n)$ be a vector of indeterminates. Let $\mathcal{P}$ denote the set of all integer partitions, including the unique partition of 0. If λ and μ are partitions with transposes λ' and μ' respectively, let

$$\langle \lambda, \mu \rangle := \sum_{1 \le i} \lambda'_i \mu'_i.$$

Also, let

$$b_\lambda(q) := \prod_{1 \le i} \prod_{1 \le j \le n_i} (1 - q^j),$$

where λ has n_i parts of size i for each i. As a notational convenience, we will write monomials as a vector with a vector exponent, as in $\boldsymbol{T}^{\boldsymbol{\alpha}} = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$. If λ is a vector (say a partition or a weak composition), let $|\lambda|$ denote the sum of the parts of λ.

Finally, define the function $P_\Gamma(\boldsymbol{T}, q)$ by

$$P_\Gamma(\boldsymbol{T}, q) := \sum_{\lambda^1, \ldots, \lambda^n \in \mathcal{P}} \frac{\prod_{1 \le i \le j \le n} q^{g_{ij} \langle \lambda^i, \lambda^j \rangle}}{\prod_{1 \le i \le n} q^{\langle \lambda^i, \lambda^i \rangle} b_{\lambda^i}(q^{-1})} \, T_1^{|\lambda^1|} \cdots T_n^{|\lambda^n|} \tag{1}$$

and the function $H_\Gamma(\boldsymbol{\alpha}, q)$ implicitly by

$$\log P_\Gamma(\boldsymbol{T}, q) = \sum_{\boldsymbol{0} \neq \boldsymbol{\alpha} \in \mathbb{N}^n} \frac{H_\Gamma(\boldsymbol{\alpha}, q)}{\overline{\boldsymbol{\alpha}}} \boldsymbol{T}^{\boldsymbol{\alpha}}, \tag{2}$$

where $\overline{\boldsymbol{\alpha}} = \gcd(\alpha_1, \ldots, \alpha_n)$ (since $\boldsymbol{\alpha} \neq \boldsymbol{0}$, this is well-defined if we consider every integer to be a divisor of 0). Hua expresses $A_\Gamma(\boldsymbol{\alpha}, q)$ in terms of $H_\Gamma(\boldsymbol{\alpha}, q)$.

Theorem 2.1 (Hua [5, Theorem 4.6]).

$$A_\Gamma(\boldsymbol{\alpha}, q) = \frac{q-1}{\overline{\boldsymbol{\alpha}}} \sum_{d | \overline{\boldsymbol{\alpha}}} \mu(d) H_\Gamma(\boldsymbol{\alpha}/d, q^d). \tag{3}$$

Although we want to understand $A_\Gamma(\boldsymbol{\alpha}, 1)$, we cannot use Equations (1), (2), and (3) directly, since the summands in $P_\Gamma(\boldsymbol{T}, q)$ have poles at $q = 1$. We proceed instead by introducing extra variables, computing certain limits as q approaches 1, and then specializing the results. The remainder of this section analyzes $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$, a generalization of $A_\Gamma(\boldsymbol{\alpha}, q)$, while Sections 4 and 5 apply the results to $A_\Gamma(\boldsymbol{\alpha}, q)$.

In what follows, vectors $\boldsymbol{u} \in \mathbb{N}^{n(n+1)/2}$ will have components u_{ij} for $1 \leq i \leq j \leq n$, and for $\boldsymbol{\ell} \in \mathbb{N}^n$ we let $\boldsymbol{u}^{\boldsymbol{\ell}} := \prod_{1 \leq i \leq j \leq n} u_{ij}^{\ell_i \ell_j}$. Let $\boldsymbol{u} \in \mathbb{N}^{n(n+1)/2}$. Define functions $P_\Gamma(\boldsymbol{T}, \boldsymbol{u}, q)$, $H_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$, and $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ by the formulas

$$P_\Gamma(\boldsymbol{T}, \boldsymbol{u}, q) \quad := \quad \sum_{\lambda^1, \ldots, \lambda^n \in \mathcal{P}} \frac{\prod_{1 \leq i \leq j \leq n} u_{ij}^{\langle \lambda^i, \lambda^j \rangle}}{\prod_{1 \leq i \leq n} q^{\langle \lambda^i, \lambda^i \rangle} b_{\lambda^i}(q^{-1})} T_1^{|\lambda^1|} \cdots T_n^{|\lambda^n|}, \tag{4}$$

$$\log P_\Gamma(\boldsymbol{T}, \boldsymbol{u}, q) \quad := \quad \sum_{\boldsymbol{0} \neq \boldsymbol{\alpha} \in \mathbb{N}^n} \frac{H_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)}{\overline{\boldsymbol{\alpha}}} \boldsymbol{T}^{\boldsymbol{\alpha}}, \text{ and} \tag{5}$$

$$A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q) \quad := \quad \frac{q-1}{\overline{\boldsymbol{\alpha}}} \sum_{d | \overline{\boldsymbol{\alpha}}} \mu(d) H_\Gamma(\boldsymbol{\alpha}/d, \boldsymbol{u}^d, q^d). \tag{6}$$

Observe that $P_\Gamma(\boldsymbol{T}, \boldsymbol{u}, q)$, $H_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$, and $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ specialize to $P_\Gamma(\boldsymbol{T}, q)$, $H_\Gamma(\boldsymbol{\alpha}, q)$, and $A_\Gamma(\boldsymbol{\alpha}, q)$ respectively when $u_{ij} = q^{g_{ij}}$ for $1 \leq i \leq j \leq n$. However, $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ typically is not a polynomial in q even though $A_\Gamma(\boldsymbol{\alpha}, q)$ is. For $\boldsymbol{\ell} \in \mathbb{N}^n$ let $\boldsymbol{\ell}! := \ell_1! \cdots \ell_n!$ and for $\boldsymbol{u} \in \mathbb{N}^{n(n+1)/2}$ let $\boldsymbol{u}! := u_{11}! \cdots u_{ij}! \cdots u_{nn}!$. Our first result computes a limit involving $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$.

Proposition 2.2.

$$\lim_{q \to 1} (q-1)^{|\boldsymbol{\alpha}|-1} A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q) \quad = \quad [\textit{the coefficient of } \boldsymbol{T}^{\boldsymbol{\alpha}} \textit{ in}\,] \log \sum_{\boldsymbol{\ell} \in \mathbb{N}^n} \boldsymbol{u}^{\boldsymbol{\ell}} \frac{\boldsymbol{T}^{\boldsymbol{\ell}}}{\boldsymbol{\ell}!}. \tag{7}$$

3 $A_\Gamma(\boldsymbol{\alpha}, \boldsymbol{u}, q)$ and Connected Graphs

The limit in Proposition 2.2, namely Equation (7), can be rewritten using a multivariate version of the Exponential Formula applied to the enumeration of graphs. To describe this enumerative result, we must introduce some more notation. If $\boldsymbol{\ell} \in \mathbb{N}^n$, let $\mathcal{G}^{\boldsymbol{\ell}}$ be the set of graphs on the vertices $v_1, v_2, \ldots, v_{|\boldsymbol{\ell}|}$. Let

$V_1 := \{v_1, \ldots, v_{\ell_1}\}, V_2 := \{v_{\ell_1+1}, \ldots, v_{\ell_2}\}, \ldots, V_n := \{v_{|\boldsymbol{\ell}|-\ell_n+1}, \ldots, v_{|\boldsymbol{\ell}|}\}$. If $\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}$, then let $\mathcal{G}_{\boldsymbol{k}}^{\boldsymbol{\ell}}$ be the set of graphs in $\mathcal{G}^{\boldsymbol{\ell}}$ that have k_{ij} edges between V_i and V_j for $1 \le i \le j \le n$ and let $G_{\boldsymbol{k}}^{\boldsymbol{\ell}}$ be the number of connected graphs in $\mathcal{G}_{\boldsymbol{k}}^{\boldsymbol{\ell}}$. Now let $\boldsymbol{x} = (x_{11}, \ldots, x_{ij}, \ldots, x_{nn})$ be a vector of $n(n+1)/2$ indeterminates, where $1 \le i \le j \le n$, and define the weight of $G \in \mathcal{G}_{\boldsymbol{k}}^{\boldsymbol{\ell}}$ to be $\boldsymbol{x}^{\boldsymbol{k}} := \prod_{1 \le i \le j \le n} x_{ij}^{k_{ij}}$.

Theorem 3.1.

$$\log\left(\sum_{\boldsymbol{\ell}\in\mathbb{N}^n}\left(\prod_{1\le i<j\le n}(1+x_{ij})^{\ell_i\ell_j}\right)\left(\prod_{1\le i\le n}(1+x_{ii})^{\binom{\ell_i}{2}}\right)\frac{\boldsymbol{X}^{\boldsymbol{\ell}}}{\boldsymbol{\ell}!}\right) \tag{8}$$

$$= \sum_{0\ne\boldsymbol{\alpha}\in\mathbb{N}^n}\ \sum_{\boldsymbol{k}\in\mathbb{N}^{n(n+1)/2}} G_{\boldsymbol{k}}^{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{k}}\frac{\boldsymbol{X}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!},$$

where $G_{\boldsymbol{k}}^{\boldsymbol{\alpha}}$ is the number of connected graphs in $\mathcal{G}_{\boldsymbol{k}}^{\boldsymbol{\alpha}}$.

Perhaps Theorem 3.1 is known, but we do not know of any reference. Presumbly there is no explicit formula for $G_{\boldsymbol{k}}^{\boldsymbol{\alpha}}$ in general. However, when $|\boldsymbol{k}| = |\boldsymbol{\alpha}| - 1$ (that is, when the connected graphs in $\mathcal{G}_{\boldsymbol{k}}^{\boldsymbol{\alpha}}$ are trees), certain sums of the numbers $G_{\boldsymbol{k}}^{\boldsymbol{\alpha}}$ can be computed by the methods in Knuth [7].

To understand Equation (7) better, we obtain a corollary of Theorem 3.1 by rewriting Equation (8) with the substitutions $1 + x_{ij} = u_{ij}$ $(1 \le i < j \le n)$, $1 + x_{ii} = u_{ii}^2$ $(1 \le i \le n)$, and $X_i = u_{ii}T_i$ $(1 \le i \le n)$. This allows us to rewrite the result of Proposition 2.2. For each $\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}$, let

$$S_{\boldsymbol{k}} = \left\{\boldsymbol{p} \in \mathbb{N}^{n(n+1)/2} \ :\ \begin{array}{l} k_{ii} \ge p_{ii} \text{ for } 1 \le i \le n \\ k_{ij} = p_{ij} \text{ for } 1 \le i < j \le n \end{array}\right\}. \tag{9}$$

Proposition 3.2. *For each $\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}$,*

$$\big[\textit{the coefficient of } (\boldsymbol{u}-\boldsymbol{1})^{\boldsymbol{k}} \textit{ in}\,\big] \lim_{q\to 1}\,(q-1)^{|\boldsymbol{\alpha}|-1}A_\Gamma(\boldsymbol{\alpha},\boldsymbol{u},q) \tag{10}$$

$$= \frac{1}{\boldsymbol{\alpha}!}\sum_{\boldsymbol{p}\in S_{\boldsymbol{k}}} c_{\boldsymbol{kp}}^{\boldsymbol{\alpha}}\, G_{\boldsymbol{p}}^{\boldsymbol{\alpha}},$$

where

$$c_{\boldsymbol{kp}}^{\boldsymbol{\alpha}} := \prod_{1\le i\le n}\left(\sum_{j=0}^{\infty}\binom{p_{ii}}{j}\binom{\alpha_i}{k_{ii}-p_{ii}-j}2^{p_{ii}-j}\right)$$

and

$$(\boldsymbol{u}-\boldsymbol{1})^{\boldsymbol{k}} := \prod_{1\le i\le j\le n}(u_{ij}-1)^{k_{ij}}.$$

In particular, if $|\boldsymbol{k}| = |\boldsymbol{\alpha}| - 1$, then

$$\big[\textit{the coefficient of } (\boldsymbol{u}-\boldsymbol{1})^{\boldsymbol{k}} \textit{ in}\,\big] \lim_{q\to 1}\,(q-1)^{|\boldsymbol{\alpha}|-1}A_\Gamma(\boldsymbol{\alpha},\boldsymbol{u},q) \tag{11}$$

$$= \frac{1}{\boldsymbol{\alpha}!}2^{t(\boldsymbol{k})}\, G_{\boldsymbol{k}}^{\boldsymbol{\alpha}},$$

where

$$t(\boldsymbol{k}) := \sum_{1\le i\le n} k_{ii}.$$

Observe that the left- and right-hand sides of Equation (10) are nonzero for finitely many $\boldsymbol{k}$ and that the sum over j is actually finite by the definition of binomial coefficients. Also, the sum over j can be expressed in terms of a hypergeometric series as

$$\begin{aligned}&\sum_{j=0}^{\infty}\binom{p_{ii}}{j}\binom{\alpha_i}{k_{ii}-p_{ii}-j}2^{p_{ii}-j}\\ =\;& 2^{p_{ii}}\binom{\alpha_i}{k_{ii}-p_{ii}}\,{}_2F_1(-p_{ii},-k_{ii}+p_{ii};a_i-k_{ii}+p_{ii}+1;1/2),\end{aligned}$$

if desired.

4 A Mahler-type Expansion for $A_\Gamma(\boldsymbol{\alpha},\boldsymbol{u},q)$

We can use Proposition 3.2 to understand $A_\Gamma(\boldsymbol{\alpha},q)$ if we rewrite $A_\Gamma(\boldsymbol{\alpha},\boldsymbol{u},q)$ using the Mahler-type expansion given in the following theorem.

Theorem 4.1. *If $f\in\mathbb{Q}(q)[x_1,\dots,x_r]$ and $f(q^{b_1},\dots,q^{b_r})\in\mathbb{Z}[q]$ for all non-negative integers $b_1,\dots,b_r$, then there are polynomials $\{c_{\boldsymbol{\ell}}(q)\in\mathbb{Z}[q]\ :\ \boldsymbol{\ell}\in\mathbb{N}^r\}$ such that*

$$f \;=\; \sum_{\boldsymbol{\ell}\in\mathbb{N}^r} c_{\boldsymbol{\ell}}(q)\prod_{1\le i\le r}\left\langle {x_i \atop \ell_i}\right\rangle_q, \tag{12}$$

where

$$\left\langle {x \atop \ell}\right\rangle_q \;:=\; \prod_{1\le i'\le \ell}\frac{(x/q^{i'-1}-1)}{(q^{i'}-1)} \tag{13}$$

and $c_{\boldsymbol{\ell}}(q)=0$ for all but finitely many $\boldsymbol{\ell}$.

The proof of this theorem was communicated to us by Keith Conrad. Note that $\left\langle {x \atop \ell}\right\rangle_q=\left[{b \atop \ell}\right]_q$ when $x=q^b$ and that

$$\lim_{q\to1}(q-1)^{\ell}\left\langle {x \atop \ell}\right\rangle_q \;=\; \frac{(x-1)^{\ell}}{\ell!}. \tag{14}$$

Here $\left[{b \atop \ell}\right]_q$ is a q-binomial coefficient. By Theorem 4.1 and the fact that $A_\Gamma(\boldsymbol{\alpha},q)\in\mathbb{Z}[q]$, we can write

$$A_\Gamma(\boldsymbol{\alpha},\boldsymbol{u},q) \;=\; \sum_{\boldsymbol{k}\in\mathbb{N}^{n(n+1)/2}} a_\Gamma(\boldsymbol{\alpha},\boldsymbol{k},q)\left\langle {\boldsymbol{u} \atop \boldsymbol{k}}\right\rangle_q, \tag{15}$$

for some $a_\Gamma(\boldsymbol{\alpha},\boldsymbol{k},q)\in\mathbb{Z}[q]$, where

$$\left\langle {\boldsymbol{u} \atop \boldsymbol{k}}\right\rangle_q := \prod_{1\le i\le j\le n}\left\langle {u_{ij} \atop k_{ij}}\right\rangle_q.$$

Hence

$$A_\Gamma(\boldsymbol{\alpha}, q) \quad = \sum_{\boldsymbol{k}\in\mathbb{N}^{n(n+1)/2}} a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q) \begin{bmatrix}\boldsymbol{g}\\ \boldsymbol{k}\end{bmatrix}_q, \tag{16}$$

where

$$\begin{bmatrix}\boldsymbol{g}\\ \boldsymbol{k}\end{bmatrix}_q := \prod_{1\le i\le j\le n} \begin{bmatrix}g_{ij}\\ k_{ij}\end{bmatrix}_q.$$

It turns out that Proposition 3.2 leads to a formula for the derivatives of $a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q)$ evaluated at $q = 1$, given in Proposition 4.2, which in turn produces a formula for the derivatives of $A_\Gamma(\boldsymbol{\alpha}, q)$ evaluated at $q = 1$ (see Theorems 4.3 and 5.1).

Proposition 4.2. *For $\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2}$ such that $|\boldsymbol{k}| > |\boldsymbol{\alpha}|$ we have*

$$\left.\frac{a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q)}{(q-1)^{|\boldsymbol{k}|-|\boldsymbol{\alpha}|+1}}\right|_{q=1} \quad = \quad \frac{\boldsymbol{k}!}{\boldsymbol{\alpha}!} \sum_{\boldsymbol{p}\in S_{\boldsymbol{k}}} c^{\boldsymbol{\alpha}}_{\boldsymbol{k}\boldsymbol{p}}\, G^{\boldsymbol{\alpha}}_{\boldsymbol{p}}. \tag{17}$$

Note that if $|\boldsymbol{k}| \le |\boldsymbol{\alpha}|$, Proposition 4.2 says nothing about $a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, q)$ at $q = 1$. This is why Theorems 4.3 and 5.1 below only give information about leading coefficients. The first consequence of Proposition 4.2 appears when we evaluate $A_\Gamma(\boldsymbol{\alpha}, 1)$. By Equation (16),

$$A_\Gamma(\boldsymbol{\alpha}, 1) \quad = \sum_{\boldsymbol{k}\in\mathbb{N}^{n(n+1)/2}} a_\Gamma(\boldsymbol{\alpha}, \boldsymbol{k}, 1) \binom{\boldsymbol{g}}{\boldsymbol{k}}, \tag{18}$$

where

$$\binom{\boldsymbol{g}}{\boldsymbol{k}} := \prod_{1\le i\le j\le n} \binom{g_{ij}}{k_{ij}}.$$

Theorem 4.3. *The quantity $A_\Gamma(\boldsymbol{\alpha}, 1)$ is a polynomial in the variables g_{ij} whose homogeneous component of highest degree $A^*_\Gamma(\boldsymbol{\alpha}, 1)$ has total degree $|\boldsymbol{\alpha}| - 1$ and has the form*

$$A^*_\Gamma(\boldsymbol{\alpha}, 1) = \frac{1}{\boldsymbol{\alpha}!} \sum_{|\boldsymbol{k}|=|\boldsymbol{\alpha}|-1} C^{\boldsymbol{\alpha}}_{\Gamma,\boldsymbol{k}}\, \boldsymbol{g}^{\boldsymbol{k}}, \tag{19}$$

where

$$C^{\boldsymbol{\alpha}}_{\Gamma,\boldsymbol{k}} := 2^{t(\boldsymbol{k})}\, G^{\boldsymbol{\alpha}}_{\boldsymbol{k}} \quad \text{and} \quad t(\boldsymbol{k}) := \sum_{1\le i\le n} k_{ii}.$$

As a special case of this theorem, we can consider the quiver S_g from Section 1, which has a single vertex (so $n = 1$) and g loops, and $\boldsymbol{\alpha} = \alpha$. In this case, $A_\Gamma(\alpha, 1)$ is a polynomial in g of degree $\alpha - 1$ and leading coefficient $2^{\alpha-1}G^{\alpha}_{\alpha-1}/\alpha!$ But $G^{\alpha}_{\alpha-1}$ is just the number of (spanning) trees on α labeled vertices, which is $\alpha^{\alpha-2}$ by Cayley's Theorem. So the leading coefficient is $2^{\alpha-1}\alpha^{\alpha-2}/\alpha!$ as claimed in the introduction.

5 The derivatives $\frac{d^s}{dq^s} A_\Gamma(\boldsymbol{\alpha}, q)$ at $q = 1$

We can proceed further by differentiating Equation (16) to obtain information about the highest order terms of the s-th derivative of $A_\Gamma(\boldsymbol{\alpha}, q)$ evaluated at $q = 1$. If $\boldsymbol{k}, \boldsymbol{\ell} \in \mathbb{N}^{n(n+1)/2}$, we write $\boldsymbol{k} \leq \boldsymbol{\ell}$ if $k_{ij} \leq \ell_{ij}$ for all $1 \leq i \leq j \leq n$. To simplify the notation we let

$$A_{\Gamma,s}(\boldsymbol{\alpha}, q) := \frac{d^s}{dq^s} A_\Gamma(\boldsymbol{\alpha}, q).$$

Theorem 5.1. *The quantity $A_{\Gamma,s}(\boldsymbol{\alpha}, 1)$ is a polynomial in the variables g_{ij} whose homogeneous component of highest degree $A^*_{\Gamma,s}(\boldsymbol{\alpha}, 1)$ has total degree $s + |\boldsymbol{\alpha}| - 1$ and is given by*

$$A^*_{\Gamma,s}(\boldsymbol{\alpha}, 1) = \frac{1}{\boldsymbol{\alpha}!} \sum_{|\boldsymbol{\ell}|=s+|\boldsymbol{\alpha}|-1} C^{\boldsymbol{\alpha}}_{\Gamma,s,\boldsymbol{\ell}} \, \boldsymbol{g}^{\boldsymbol{\ell}}, \tag{20}$$

where

$$C^{\boldsymbol{\alpha}}_{\Gamma,s,\boldsymbol{\ell}} := \frac{s!}{\boldsymbol{\ell}!} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}^{n(n+1)/2} \\ \boldsymbol{k} \leq \boldsymbol{\ell}}} S(\boldsymbol{k}, \boldsymbol{\ell}) \, \boldsymbol{k}! \sum_{\boldsymbol{p} \in S_{\boldsymbol{k}}} c^{\alpha}_{\boldsymbol{kp}} \, G^{\boldsymbol{\alpha}}_{\boldsymbol{p}},$$

$$S(\boldsymbol{\ell}, \boldsymbol{k}) := \prod_{1 \leq i \leq j \leq n} S(\ell_{ij}, k_{ij}),$$

and $S(\ell, k)$ is the Stirling number of the second kind.

Incidentally, one ingredient in the proof of Theorem 5.1 is an auxiliary theorem which shows that for each $t \geq 0$, the quantity $\left(\frac{d^t}{dq^t} \begin{bmatrix} b \\ k \end{bmatrix}_q \right) \Big|_{q=1}$ is a polynomial in b of degree $k + t$ with leading coefficient $\frac{t!}{(k+t)!} \cdot S(k + t, k)$.

As a special case of this theorem, we can consider the quiver S_g from Section 1. In this case, $A_{\Gamma,s}(\boldsymbol{\alpha}, 1)$ is a polynomial in g of degree $s + \alpha - 1$ and leading coefficient

$$\frac{s!}{\alpha!(s + \alpha - 1)!} \sum_{k=\alpha-1}^{s+\alpha-1} S(s + \alpha - 1, k) \, k! \sum_{p=\alpha-1}^{k} G^\alpha_p \sum_{j=0}^{\infty} \binom{p}{j} \binom{\alpha}{k - p - j} 2^{p-j}.$$

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Abstract. Algebraic complexes whose "faces" are indexed by partitions and plane partitions are introduced, and their homology is proven to be concentrated in even dimensions with homology basis indexed by fixed points of an involution, thereby explaining topologically two quite important instances of Stembridge's $q = -1$ phenomenon. A more general framework of invariant and coinvariant complexes with coefficients taken mod 2 is developed, and as a part of this story an analogous topological result for necklaces is conjectured.

Résumé Complexes algébriques dont les "faces" sont indexées par des partitions et des partitions planes sont introduits. Il est démontré que leur homologie est concentrée en dimensions paires, avec base de homologie indexée par des points fixes d'une involution. Ce résultat explique d'une manière topologique deux instances du phénomène $q = -1$ du a Stembridge. De plus, un cadre plus général des complexes invariants et coinvariants dont les coefficients sont pris modulo 2 est développé. Comme part de cette histoire, nous conjecturons un résultat analogue pour des colliers.

Keywords: plane partitions, discrete Morse theory, $q = -1$ phenomenon, homology basis, down operator

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The $q = -1$ phenomenon for bounded (plane) partitions via homology concentration

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1 Introduction

There is a rich history surrounding the enumeration of partitions in a rectangle or higher dimensional box, as well as the enumeration of classes of partitions possessing various symmetries (see e.g. (1), (10)). One reason for so much interest comes from connections to physics, while another is the important role they play in representation theory, specifically in the theory of canonical bases (see e.g. (11) and (12)). Richard Stanley used the Littlewood-Richardson rule in (10) to prove a recursive formula for the number of complementary plane partitions of bounded value. John Stembridge proved that semistandard domino tableaux are counted by this same formula, by showing that their enumeration formula satisfies the same recurrence. This proved that the set of fixed points in a fundamental involution of Lusztig on a type A canonical basis also has this same cardinality, by virtue of a bijection due to Berenstein and Zelevinsky between the elements of a canonical basis and semistandard Young tableaux (actually Gelfand-Tsetlin patterns) such that this bijection sends Lusztig's involution to evacuation. Stembridge examined this connection between self-complementary partitions and canonical bases more closely, unveiling in the process a phenomenon he dubbed the "$q = -1$ phenomenon".

In (12), Stembridge defines the $q = -1$ phenomenon as the following situation. One has a set of combinatorial objects B (such as tableaux), together with a generating function $X(q)$ that enumerates the objects in B according to some weight depending on q. The $q = -1$ phenomenon occurs when there is a "natural" involution on B such that $X(-1)$ is the number of fixed points of the involution. Stembridge established various instances of this phenomenon by interpreting $X(-1)$ as the trace of a matrix which is conjugate to a permutation matrix for the involution.

It is natural to ask if the $q = -1$ phenomenon can be explained by an Euler characteristic computation. To this end, we define a complex whose ranks of chain groups are the coefficients in the polynomial $X(q)$ so that its Euler characteristic is $X(-1)$. On the other hand, the Euler characteristic is the alternating sum of ranks of homology groups, so whenever homology is concentrated in even dimensions with homology

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basis indexed by the fixed points of the involution, this would imply the phenomenon. Moreover, the homology generating function then offers a natural q-analogue of the integer which is the Euler characteristic. We carry out this plan in two quite central cases: the partitions in a rectangle and the plane partitions of bounded value in a rectangle, i.e., the partitions in a three dimensional box.

In §2 we associate to any action of a finite group G regarded as a subgroup of S_n four algebraic complexes over a field with 2 elements, and prove in Proposition 2.3 that their Euler characteristic is the number of fixed points of the complementation involution in the action of G on the subsets of $[n]$. The aforementioned case of partitions in a rectangle arises as the special case where G is a wreath product of symmetric groups. We prove acyclicity of these complexes for any G in which n is odd and also somewhat more generally. In §3 we give an algebraic Morse matching lemma, which is then applied in §4 and §5 to partitions in a rectangle and in a three dimensional box, respectively, establishing homology concentration in even dimensions and explicit homology bases. The results in §4 and §5 may be regarded as sign-reversing involutions with some extra topological structure. In §6, we conclude with an example showing that not all G give rise to complexes with homology concentrated in even dimensions, namely Example 6.1, and finally we propose in Conjecture 6.5 that homology concentration in even ranks nonetheless does hold for necklaces, i.e. the case where G is a cyclic group.

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2 The algebraic complexes

In this section we define the algebraic complexes, which are quotient complexes of the Boolean algebra over a field of two elements. The boundary map is closely related to the down operator introduced in (9). Let G be a subgroup of $\mathfrak{S}_n$, so G acts on $[n] := \{1, 2, \ldots, n\}$. Then G also permutes the elements of the Boolean algebra $2^{[n]}$ of all subsets of $[n]$, and permutes the subsets $\binom{[n]}{i}$ of a given cardinality i. Consider the following generating function that counts such G-orbits according to their cardinality, letting S be an element in the orbit $\overline{S}$, we have

$$\begin{aligned} X(G,q) &:= \sum_{G-\mathrm{orbits}\ \bar{\mathrm{S}}\ \mathrm{in}\ 2^{[\mathrm{n}]}/\mathrm{G}} q^{|S|} \\ &= \sum_{i=0}^{n} \left| \binom{[n]}{i} / G \right| q^i \end{aligned}$$

It has been well-studied historically via algebraic means, perhaps starting with Redfield and Polya.

Theorem 2.1 *((9; 5)) The polynomial $X(G, q)$ has symmetric, unimodal coefficients.*

The idea is to show that the coefficients in $X(G, q)$ are the ranks of the weight spaces in a representation of $sl_2(\mathbb{C})$ by showing that the three operators $D, U, DU - UD$ satisfy the appropriate relations.

See (6, Corollary 6.2) for the next result, which is due to de Bruijn.

Theorem 2.2 *(de Bruijn) $X(G, -1)$ is the number of G-orbits $\bar{S}$ of subsets of $[n]$ which are self-complementary in the sense that $\bar{S} = \overline{[n]\backslash S}$.*

This was proven by finding two conjugate matrices, one having $X(G, -1)$ as its trace and the other of which is a permutation matrix acting on the G-orbits of subsets of $[n]$ by complementation. As an example, if $G = \langle (12), (34) \rangle$, then $\overline{S} = \{\{1, 3\}, \{2, 4\}\}$ is a self-complementary orbit. Theorem 2.2 may be applied to our first main example, namely the case of partitions in a rectangle, but does not seem to apply to our second example of plane partitions of bounded value in a rectangle.

One algebraic approach introduces, for any field $\mathbb{F}$, the graded vector space $\mathcal{C}(\mathbb{F}) := \bigoplus_{i=0}^{n} \mathcal{C}_i(\mathbb{F})$ in which $\mathcal{C}_i(\mathbb{F})$ has an $\mathbb{F}$-basis $\{e_S : S \in \binom{[n]}{i}\}$. It is useful to identify $\mathcal{C} \cong V^{\otimes n}$, where $V \cong \mathbb{F}^2$ has $\mathbb{F}$-basis $\{e_0, e_1\}$. Under this identification, the $\mathbb{F}$-basis element e_S in $\mathcal{C}(\mathbb{F})$ corresponds to the decomposable tensor $e_{i_1} \otimes \cdots \otimes e_{i_n}$ in which $i_j = 1$ for $i_j \in S$ and $i_j = 0$ otherwise.

Define the *up* and *down* maps $U, D : \mathcal{C}(\mathbb{F}) \to \mathcal{C}(\mathbb{F})$ by

$$\begin{aligned} U(e_S) &= \sum_{i \in [n] \setminus S} e_{S \cup \{i\}} \\ D(e_S) &= \sum_{j \in S} e_{S \setminus \{j\}}. \end{aligned}$$

Note that U, D both commute with the G-action. As a consequence, they give well-defined maps on the graded vector spaces of G*-invariants* $\mathcal{C}(\mathbb{F})^G$ and G*-coinvariants*[(i)] $\mathcal{C}(\mathbb{F})_G$. Note that both $\mathcal{C}(\mathbb{F})^G, \mathcal{C}(\mathbb{F})_G$ will have $\mathbb{F}$-bases indexed by G-orbits $\bar{S}$ of subsets of $[n]$: for $\mathcal{C}(\mathbb{F})^G$, a typical basis element $e_{\bar{S}}$ is a sum of e_S as S varies over the elements of the orbit, while for $\mathcal{C}(\mathbb{F})_G$, a typical basis element is the image $\overline{e_S}$ of e_S in the quotient for any $S \in \overline{S}$.

We let $\mathbb{F} = \mathbb{F}_2$ henceforth, in order to obtain a complex.

Proposition 2.3 *The map D induced on $\mathcal{C}(\mathbb{F}_2)^G$ or on $\mathcal{C}(\mathbb{F}_2)_G$ make them algebraic chain complexes of $\mathbb{F}_2$-vector spaces, i.e., $D^2 = 0$ in each case. Likewise the map U makes them into cochain complexes, i.e., $U^2 = 0$.*

Each of these four complexes has Euler characteristic, i.e. alternating sum of the ranks of its chain groups, equalling $X(-1)$, or in other words the number of self-complementary G-orbits. In particular, each Euler characteristic is nonnegative.

Proof: Working over $\mathbb{F}_2$, these maps D, U on $\mathcal{C}(\mathbb{F}_2)$ coincide with the boundary and coboundary maps ∂_i and ∂^i in the usual simplicial chain complex for a simplex having vertex set $[n]$. Hence $D^2 = U^2 = 0$, and the same holds after taking G-invariants or G-coinvariants.

For the second assertion, note that

$$\dim_{\mathbb{F}_2} \mathcal{C}(\mathbb{F}_2)_i^G = \dim_{\mathbb{F}_2} \mathcal{C}(\mathbb{F}_2)_G^i = \left| \binom{[n]}{i} / G \right|.$$

and now apply Theorem 2.2. □

Given a G-orbit $\bar{S}$, let $e_{\bar{S}} := \sum_{S' \in \bar{S}} e_{S'}$ denote the basis element of $\mathcal{C}^G(\mathbb{F})$ corresponding to $\bar{S}$; let $\overline{e_S}$ denote the basis element of $\mathcal{C}(\mathbb{F})_G$ corresponding to $\bar{S}$

Proposition 2.4 *The map sending $e_{\bar{S}} \mapsto \overline{e_S}$ induces isomorphisms of cochain complexes*

$$\begin{aligned} (\mathcal{C}(\mathbb{F}_2)^G, U) &\cong \mathrm{Hom}_{\mathbb{F}_2}((\mathcal{C}(\mathbb{F}_2)_G, D), \mathbb{F}_2) \\ (\mathcal{C}(\mathbb{F}_2)^G, D) &\cong \mathrm{Hom}_{\mathbb{F}_2}((\mathcal{C}(\mathbb{F}_2)_G, U), \mathbb{F}_2). \end{aligned}$$

(i) Recall that the *coinvariant space* U_G for an $\mathbb{F}$-vector space U with a linear G-action is the quotient space $U/\mathbb{F}\{u - g(u) : u \in U, g \in G\}$.

The map sending $e_{\bar{S}} \mapsto \overline{e_{[n]\setminus S}}$ induces isomorphisms of chain complexes

$$\begin{aligned} (\mathcal{C}(\mathbb{F}_2)^G, D) &\cong (\mathcal{C}(\mathbb{F}_2)^G, U)^{\mathrm{op}} \\ (\mathcal{C}(\mathbb{F}_2)_G, D) &\cong (\mathcal{C}(\mathbb{F}_2)_G, U)^{\mathrm{op}}, \end{aligned}$$

where here $\mathcal{C}^{\mathrm{op}}$ for a cochain complex $\mathcal{C}$ denotes the opposite chain complex that one obtains by reindexing in the opposite order and reversing all the arrows.

Proof: Let $\bar{S}, \bar{T}$ be G-orbits of subsets with $|T| = |S| + 1$. Then the boundary map coefficient $D_{\bar{S},\bar{T}}$ in $\mathcal{C}(\mathbb{F}_2)^G$ is the number of elements T' in the orbit $\bar{T}$ which *contain* the fixed set S in $\bar{S}$. Meanwhile the boundary coefficient $D_{\bar{S},\bar{T}}$ in $\mathcal{C}(\mathbb{F}_2)_G$ is the number of elements S' in the orbit $\bar{S}$ which are *contained in* the fixed set T in $\bar{T}$. There are similar formulae for the coefficients $U_{\bar{S},\bar{T}}$ in the two complexes. The isomorphisms are not hard to verify, using the fact that set-complementation is an inclusion-reversing bijection. $\square$

In light of the previous proposition, one may consider any one of the four complexes, as its homology determines the homology of the others (either by turning it around in homological degree, or by taking dual $\mathbb{F}_2$-vector spaces, or both).

Proposition 2.5 *The complex $(\mathcal{C}^G, D)$ is acyclic when n is odd. More generally, it is acyclic whenever G has at least one orbit in its action on $[n]$ of odd cardinality. Whenever $(\mathcal{C}^G, D)$ is not acyclic, it must have $H_0 = \mathbb{F}_2$ and $H_1 = 0$.*

Proof: Let $S \subseteq [n]$ be G-stable (although not necessarily pointwise fixed by G). Then one forms S-*masked* versions of the up and down maps in $\mathcal{C}$ as follows:

$$\begin{aligned} U^{(S)}(e_T) &:= \sum_{i \in S \setminus T} e_{T \cup \{i\}} \\ D^{(S)}(e_T) &:= \sum_{j \in S \cap T} e_{T - \{j\}}. \end{aligned}$$

The G-stability of S implies that these S-masked up and down maps commute with the G-action: the crucial point in all these calculations is that $g(S) = S$, so that, for example,

$$\begin{aligned} S \setminus g(T) &= g(S \setminus T) \\ S \cap g(T) &= g(S \cap T), \end{aligned}$$

Hence one gets induced S-masked up and down maps $U^{(S)}, D^{(S)}$ on the G-invariant and G-coinvariant complexes also.

An easy calculation, generalizing the commutator calculation $DU - UD = (n - 2i)I$ on $\mathcal{C}_i$ is the following:

$$\begin{aligned} (D \cdot U^{(S)} - U^{(S)} \cdot D)(e_T) &= (|S \setminus T| - |S \cap T|) \cdot e_T \\ &= (|S| - 2|S \cap T|) \cdot e_T. \end{aligned}$$

When working with $\mathbb{F}_2$ coefficients, as in $\mathcal{C}(\mathbb{F}_2), \mathcal{C}(\mathbb{F}_2)^G, \mathcal{C}(\mathbb{F}_2)_G$, this gives

$$D \cdot U^{(S)} + U^{(S)} \cdot D = |S| \cdot I.$$

Thus when $|S|$ is odd, the S-masked up map $U^{(S)}$ gives an algebraic chain-contraction, showing that the complex with D as boundary map is acyclic. We now analyze the consequences of this for the first few boundary maps in $(\mathcal{C}(\mathbb{F}_2)^G, D)$.

The boundary map D out of $\mathcal{C}_0(\mathbb{F}_2)^G = \mathbb{F}_2$ is *always* the zero map, regardless of G. If all G-orbits have even cardinality, the boundary map D out of $\mathcal{C}_1(\mathbb{F}_2)^G$ will also be the zero map, so the assertion about H_0 follows.

It remains to show that when all G-orbits have even cardinality, the map D out of $\mathcal{C}_2(\mathbb{F}_2)^G$ is surjective. But for this we can work within each G-orbit X on $[n]$. That is, it suffices to show that there is some G-orbit $Y = \overline{\{i,j\}}$ of pairs with $i, j \in X$ for which $D(e_Y)$ has coefficient 1 on e_X, not zero. However, fixing X, one can see that the sum of all of such boundary map coefficients incident to X and *coming from G-orbits of pairs contained in X* will be $|X| - 1$, an odd number. Thus one of them must be non-zero in $\mathbb{F}_2$, as desired. $\square$

Remark 2.6 *Examples like $G = \langle (123)(456) \rangle$ in $\mathfrak{S}_6$, where G has orbits on $[n]$ of odd size but n is even and G is not a product $G_1 \times G_2$ show that the Künneth formula doesn't suffice to prove the previous proposition (or at least it's not obvious how to deduce it from Künneth).*

In light of Proposition 2.3 and the calculations of H_0, H_1 in Proposition 2.5, one might be tempted to make the conjecture (true up through $n = 5$) that the homology is always concentrated in even dimension. However, Section 6 gives a counterexample to this. Section 4 does confirm this behavior for wreath products of symmetric groups, and we conjecture it for cyclic groups in Section 6 as well.

3 An algebraic Morse matching lemma

In this section we give a general result, Lemma 3.2, on matchings in partially ordered sets, called Morse matchings. If the complex of §2 is supported by a partially ordered set, and the Morse matching is good enough, in a sense that will be made precise, it may be used to give a homology basis and prove homology concentration in even dimensions for the complex. In this case the basis is indexed by fixed points of the Morse matching, which must be equinumerous with $X(G, -1)$.

Definition 3.1 *Say that a graded poset $P = \bigsqcup_{i=0}^{n} P_i$ supports an algebraic complex $(\mathcal{C}, d)$ of $\mathbb{F}$-vector spaces if $\mathcal{C}_i$ has an $\mathbb{F}$-basis $\{e_p\}$ indexed by $p \in P_i$, and the differential d_i has $(d_i)_{p,q} = 0$ unless $q <_P p$.*

As usual, say that a partial matching M of the Hasse diagram of a poset P is an acyclic matching, or Morse matching, if the digraph D, obtained by starting with the Hasse diagram directed downward and then reversing all the directions of the edges in M, is acyclic.

Given an acyclic matching M on P, let the subsets

$$P^{\mathrm{M}}, P^{\mathrm{unM}}, P_i^{\mathrm{M}}, P_i^{\mathrm{unM}}$$

respectively denote the M-matched and M-unmatched elements in P, and the same sets restricted to rank i. The elements of P^{unM} are called critical elements. Let $q = M(p)$ denote that q is matched by M to p. Let $<_D$ be the partial order on P that results from taking the transitive closure of D.

Lemma 3.2 *Let P be a graded poset supporting an algebraic complex $(\mathcal{C}, d)$ and assume P has a Morse matching M such that for all $q = M(p)$ with $q < p$, one has $d_{p,q} \in \mathbb{F}^{\times}$. Let Q_i be the set of poset elements of rank i which are matched with elements of rank $i-1$. Then*

(i) $\dim H_i(\mathcal{C}, d) \leq |P_i^{\mathrm{unM}}|$.

(ii) if $|Q_i| = rank(d_i)$ for every i, then $\dim H_i = |P_i^{\mathrm{unM}}|$. (For example, this condition is met whenever all unmatched poset elements live in ranks of the same parity.)

(iii) if $d_{q,p} = d_{p,r} = 0$ for all $p \in P^{\mathrm{unM}}$ and all $q, r \in P$, then the homology $H(\mathcal{C}, d)$ has $\mathbb{F}$-basis $\{e_p : p \in P^{\mathrm{unM}}\}$.

Proof: To prove (i), we want to show that the boundary maps d_i have sufficiently large ranks, which we'll do by showing that they have some large, nonsingular square submatrices. We make the following claim: consider the subset Q_i of P_i^{M} consisting of those elements matched *below* them into P_{i-1}^{M}. Then ordering Q_i as $q_1, \ldots, q_r$ by any linear extension of the partial order $<_D$, the square submatrix of d_i having columns indexed $q_1, , \ldots, q_r$ and rows indexed $M(q_1), \ldots, M(q_r)$ will be invertible upper-triangular.

To prove the claim, note that the hypothesis $d_{p,q} \in \mathbb{F}^{\times}$ for $q < p$ implies that the diagonal entries of this square matrix are all in $\mathbb{F}^{\times}$, since $q = M(p)$ implies $q < p$ or $p < q$. Hence one only needs to verify upper-triangularity. So assume that the boundary map d_i has $(d_i)_{q_j, M(q_k)} \neq 0$ for some $k \neq j$. Since the complex $\mathcal{C}$ was supported on P, this implies that $q_j >_P M(q_k)$ and hence D has an edge directed $q_j \rightarrow M(q_k)$. There is also the matching edge in D, directed upward as $M(q_k) \rightarrow q_k$, and thus by transitivity, $q_j <_D q_k$. Hence $j < k$, yielding the claim.

The claim implies that $\mathrm{rank}(d_i) \geq |Q_i|$ for all i. As usual letting $Z_i = \ker d_i$ and $B_i = \mathrm{im} d_{i+1}$, note that

$$\begin{aligned}
\dim H_i &= \dim Z_i - \dim B_i \\
&= |P_i| - (\mathrm{rank}(d_i) + \mathrm{rank}(d_{i+1})) \\
&\leq |P_i| - (|Q_i| + |Q_{i+1}|) \\
&= |P_i| - |P_i^{\mathrm{M}}| \\
&= |P_i^{\mathrm{unM}}|
\end{aligned}$$

as desired.

Now the hypothesis in (ii) makes the weak inequality into an equality in the above string of equalities and weak inequalities, implying the desired equality in (ii). To prove (iii), note that the hypothesis $d_{p,r} = 0$ for all $r < p$ ensures that $\{e_p | p \in P_i^{\mathrm{unM}}\}$ spans a subspace of $H_i(\mathcal{C}, d)$ for each i, while $d_{q,p} = 0$ for all $q > p$ implies linear independence of the set. Since $|P_i^{\mathrm{unM}}| \geq \dim H_i(\mathcal{C}, d)$, this set also must span $H_i(\mathcal{C}, d)$, hence is a homology basis. $\square$

Lemma 3.2 is closely related to results of Jöllenbeck-Welker, of Sköldberg, and of Kozlov (see (3), (8), and (4)) developing algebraic versions of discrete Morse theory.

4 Application to partitions in a rectangle

Consider the subgroup $G = \mathfrak{S}_\ell \wr \mathfrak{S}_k$ inside $\mathfrak{S}_{k\ell}$, acting on the cells of a rectangle with k rows and ℓ columns, by permuting arbitrarily within each row, but also allowing wholesale swaps of one row for another. In this section, we will show how all three parts of Lemma 3.2 apply to the complex $(\mathcal{C}(\mathbb{F}_2)_G, D)$ for $G = \mathfrak{S}_\ell \wr \mathfrak{S}_k$, yielding homology concentration in even dimensions. The G-orbits are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ with

$$\ell \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0.$$

The number λ_i indicates the number of boxes in row i not belonging to the set S serving as orbit representative. Thus, the complex $\mathcal{C}(\mathbb{F}_2)_G$ is supported on the Gaussian poset $P(k, \ell)$ of all such λ ordered by reverse inclusion of their Ferrers diagrams. One may check directly that the entries in the boundary maps d take the following form: if μ is obtained from λ by reducing the part λ_i to $\lambda_i - 1$, then

$$d_{\lambda,\mu} = (\ell - \lambda_i + 1)(\mathrm{mult}_\lambda(\lambda_{i-1}) + 1) \tag{1}$$

where $\mathrm{mult}_\lambda(s)$ is the multiplicity of the part s in λ and μ covers λ in the poset supporting the complex. In fact, d is derived from the down operator D which acts on faces of a simplex, by using the fact that D commutes with the group action. D deletes a box from the orbit representative S in all possible ways, each of which has the impact of lengthening some part of λ by one.

Here is the matching M we will use. Given a partition λ, find the *smallest* part λ_i which is either

- non-zero and of the same parity as ℓ, in which case you should subtract 1 from it in order to obtain $M(\lambda)$, or
- possibly zero and of opposite parity to ℓ, but with odd multiplicity, in which case you should add 1 to it in order to obtain $M(\lambda)$.

It is not hard to check that this is indeed a well-defined matching.

Remark 4.1 *The unmatched partitions also correspond to the lattice paths in a $k \times l$ rectangle delineating the shape λ, specifically those lattice paths from (l, k) to $(0, 0)$ comprised of steps $(-2, 0)$ and $(0, -2)$ until either $(i, 0)$ or $(i, 1)$ is reached, after which there is a step $(0, -1)$ in the latter case.*

To see that these are equinumerous with self-complementary partitions in a $k \times l$ rectangle, notice that there is a bijection sending an unmatched path to a path fixed under 180 degree rotation by replacing each step of length 2 with a step in the same direction of length 1 to obtain the first half of the path with 180 degree rotational symmetry.

Theorem 4.2 *The above matching on the Gaussian poset $P(k, \ell)$ is acyclic, with the partitions λ in P^{unM} being those described in Remark 4.1. Moreover, the homology is concentrated in even dimensions.*

Proof: It is easy to verify hypotheses (i), (ii), and (iii) of Lemma 3.2 and the description of P^{unM} directly from these descriptions and (1). Let us check *acyclicity* of M. Suppose one had a directed cycle C. If λ_i is the smallest even value ever incremented in C, then there must be some downward step in C decrementing the value $\lambda_i + 1$. However, since no smaller values ever change, this downward step is across a matching edge, a contradiction. $\square$

Question 4.3 *Is there some connection between the Poincaré series for the homology here and T. Eisenkölbl's recent (-1)-enumerations of self-complementary plane partitions which appears in (2)? In particular, what happens in her situation when the $k \times \ell \times m$ box in which the plane partitions live has $m = 1$?*

5 Application to plane partitions of bounded value in a rectangle

In this section, we construct a mod 2 complex $(\mathcal{C}(c,r,t),d)$ whose i-dimensional cells are indexed by plane partitions of volume i in a $c \times r \times t$ box. We prove homology concentration in Theorem 5.6 and also give a homology basis. Since we are not aware of a way to regard partitions in a $c \times r \times t$ box as orbits of a group action permuting cells, we devised a different, though related construction.

It will be more convenient to use another indexing set of equal cardinality, namely the semistandard Young tableaux (SSYT) of shape $\lambda = (c)^r$ in which all entries have value between 0 and $r + t - 1$. The bijection comes from taking a Young tableaux filling with entries that weakly increase in rows and columns to a column strict one by adding $i - 1$ to each entry in row i.

We now define the complex $(\mathcal{C}(c,r,t),d)$, with coefficients taken mod 2, by letting the chain group generators be the SSYT of $c \times r$ rectangular shape with entries between 0 and $t+r-1$. Let us call an odd value $2i+1$ in row R decrementable if it does not have the value $2i$ immediately above it in row $R-1$. Define the boundary map d as follows. For T an SSYT, dT is a sum over SSYT obtained by subtracting one from the leftmost copy in some row R some odd value $2k+1$ having the following property: there are an odd number of decrementable copies of $2k+1$ in row R. In other words,

$$dT = \sum_{T' \in d(T)} T'$$

for $d(T)$ the set of SSYT obtained by deleting one from a decrementable odd entry $\lambda_{i,j}$ of T located at position (i,j). One may verify:

Proposition 5.1 *$(\mathcal{C}(c,r,t),d)$ is a chain complex, i.e., $d^2 = 0$.*

Next we give an acyclic matching M on the set of such SSYT. M will have the property that for each pair S, T of matched tableaux with $|S| < |T|$, S appears with nonzero coefficient in dT.

Matching M: Let T be a SSYT satisfying our requirements on its entries. Consider the earliest row R in T having at least one of the following items:

1. an odd value i such that there are an odd number of decrementable copies of i in row R
2. an even value i in row R not having the value $i+1$ immediately below it such that the number of decrementable copies of $i+1$ in row R is even (possibly zero)

Match T to $M(T)$ by choosing the smallest value i in the chosen row R meeting one of these conditions; now subtract one from the leftmost such copy of i in row R if i is odd, or add one to the rightmost such copy of i in row R if i is even.

Proposition 5.2 *M is a matching and is acyclic, hence a Morse matching.*

The proof is quite similar to the one discussed in the last section. One may also check that M meets the requirements of Lemma 3.2. Now we describe P^{unM}.

Lemma 5.3 *In any element of P^{unM}, every even value $2i$ with $2i < t + r - 1$ has the odd value $2i+1$ just below it. For each odd value $2i+1$ with $2i+1 < t+r-1$, the number of decrementable copies of $2i+1$ in a given row is even.*

Proof: Start with the top row, and proceed downward from row R to row $R+1$ by induction as follows. In row 1, notice that each odd value $2i+1$ must occur with even multiplicity, since otherwise we could match by decrementing by one the leftmost copy of $2i+1$. Thus, each even value $2i$ in row 1 will have an even number of copies (possibly 0) of $2i+1$ just to its right; therefore we could match by increasing the rightmost copy of $2i$ to $2i+1$ unless there were a $2i+1$ just below it. Since our fillings are semistandard, this implies we must have $2i+1$ just below all the other copies of $2i$ in that row, putting each $2i$ in row one in a vertical domino and each $2i+1$ in a horizontal domino.

The same argument works at row $R+1$ once the claim has been proven through row R: we may have some odd values in row $R+1$ which already belong to dominoes shared with row R, but all remaining spots to be filled in row $R+1$ will have odd values just above them. This ensures that each odd value $2i+1$ in row $R+1$ must occur with even multiplicity (not counting those in dominoes shared with row R) to avoid matching by decrementing $2i+1$. And again any even value $2i$ in row $R+1$ will have an even number of decrementable copies of $2i+1$ to its right, allowing matching by incrementing the rightmost $2i$ unless either it has a copy of $2i+1$ just below it or it is the absolute largest allowable value. $\square$

Corollary 5.4 *The first row has even sum. If row lengths are odd, then row sums alternate in parity. Otherwise, all row sums have even parity. Therefore the critical cells are concentrated in dimensions all of the same parity.*

Proposition 5.5 *The elements of P^{unM} are all in ranks of the same parity. They are also in bijection with the semistandard domino tableaux of $c \times r$ rectangular shape comprised of odd values between 0 and $t+r-1$.*

The idea is to show that our description of P^{unM} amounts to saying the shapes are tiled by two types of dominos: (1) horizontal dominos in which both entries of the domino have odd value $2i+1$, and (2) vertical dominos in which the top entry is $2i$ and the bottom entry is $2i+1$, along with perhaps some monominoes in the bottom row of maximal allowed value.

Theorem 5.6 *The complex $(\mathcal{C}(c,r,t),d)$. has homology concentration in ranks all of the same parity. Moreover, a basis is given by Lemma 5.3.*

Now we will use the following result from (12):

Theorem 5.7 (Stembridge, Theorem 3.1) *The following quantities are equal.*

- *(a) The number of self-evacuating tableaux in S_λ*
- *(b) $(-1)^{k(\lambda)}s_\lambda((-1)^0,(-1)^1,(-1)^2,\ldots,(-1)^{(n-1)})$ for $k(\lambda)=\lambda_2+\lambda_4+\cdots$*

Moreover, if n is even, or more generally if we allow monominos of maximal value in a domino tableau, then these quantities also equal

- *(c) The number of semistandard domino tableaux with entries $\leq n/2$ and shape λ*

Since the map sending a plane partition in a rectangle to a SSYT by adding $i-1$ to each entry of row i has the effect of sending the self-complementary partitions exactly to the self-evacuating tableaux of the same rectangular shape, we may use Stembridge's result to deduce that our homology basis has the same cardinality as the self-complementary plane partitions with entries of value at most t.

Our matching, together with a straightforward verification that quantity (b) has positive sign, also gives a combinatorial proof that (b) equals (c) by a sign-reversing involution on the SSYT counted by (b) which has as its fixed points a set that is in trivial bijection with the objects counted by (c).

6 A counterexample and a conjecture

Example 6.1 *Embed $G = PSL_2(\mathbb{F}_5)$ in $\mathfrak{S}_6$ via its action on the 6 points of the projective line over $\mathbb{F}_5$. To be explicit, one can start with these generators for $SL_2(\mathbb{F}_5)$*

$$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and then if one numbers the points of the projective line over $\mathbb{F}_5$ as $1,2,3,4,5,6$ according their slopes $0,1,2,3,4,\infty$, the images of a,b,c in $PSL_2(\mathbb{F}_5)$ permute $[6]$ as follows:

$$\begin{aligned} a &= (16)(34) \\ b &= (25)(34) \\ c &= (12345). \end{aligned}$$

One can check that this subgroup $G = \langle a,b,c \rangle$ of $\mathfrak{S}_6$ acts transitively on $\binom{[6]}{i}$ for $i = 0,1,2,4,5,6$, and has these two orbits on $\binom{[6]}{3}$:

$$\{123, 234, 345, 145, 125, 136, 246, 356, 146, 256\},$$
$$\{124, 235, 134, 245, 135, 126, 236, 346, 456, 156\}.$$

An easy computation then shows that $\mathcal{C}(\mathbb{F}_2)^G$ has $H_0 = H_3 = \mathbb{F}_2$ and no other nonvanishing homology groups.

Proposition 6.2 *$G \leq \mathfrak{S}_n$ has a self-complementary orbit if and only if a Sylow 2-subgroup of G contains a derangement.*

Proof: Note first that a Sylow 2-subgroup of G contains a derangement if and only if some element of g contains no cycle of odd length in its cycle decomposition. Indeed, a derangement of 2-power order is such an element, while if g has no cycle of odd length and order $2^k d$ with d odd then g^d is a derangement of order 2^k. Now if g has no cycle of odd length then it is easy to construct $S \subseteq [n]$ with $Sg = [n] \setminus S$. On the other hand, say $Sg = [n] \setminus S$ and $(i_1 \dots i_k)$ is a cycle in g. We may assume that $i_1 \in S$. Then $i_j \in S$ if and only if j is odd, and since $i_k g = i_1$, we must have $i_k \notin S$, so k is even. □

Corollary 6.3 *If G is a transitive subgroup of $\mathfrak{S}_n$ with n even, and G contains no derangement in a Sylow 2-subgroup, then the homology of $\mathcal{C}(\mathbb{F}_2)^G$ is not concentrated in even dimensions.*

Proof: By Theorem 2.2 and Proposition 6.2, the Euler characteristic of $\mathcal{C}(\mathbb{F}_2)^G$ is zero. On the other hand, by Proposition 2.5, we have $H_0(\mathcal{C}(\mathbb{F}_2)^G) \neq 0$. □

Although the homology of our mod 2 complexes is not always concentrated in even dimensions, we are still interested in interesting families of groups G for which this concentration does occur. In this situation, the Poincaré polynomial for the homology of, say $\mathcal{C}(\mathbb{F}_2)^G$, can be interpreted as giving a grading on the set of self-complementary G-orbits.

We conclude with evidence that this holds for G a cyclic group C_n generated by an n-cycle in $\mathfrak{S}_n$. In this case the orbits are necklaces, as in (7). Here are homology calculations from `Mathematica` on C_n for n even; the case of n odd already follows from Proposition 2.5.

n	homologyranks
2	$1,0,0$
4	$1,0,1,0,0$
6	$1,0,0,0,1,0,0$
8	$1,0,1,0,1,0,1,0,0$
10	$1,0,0,0,2,0,0,0,1,0,0$
12	$1,0,1,0,2,0,2,0,1,0,1,0,0$
14	$1,0,0,0,3,0,2,0,3,0,0,0,1,0,0$
16	$1,0,1,0,3,0,5,0,5,0,3,0,1,0,1,0,0$
18	$1,0,0,0,4,0,6,0,8,0,6,0,4,0,0,0,1,0,0$

The following calculation of $X(C_n,q), X(C_n,-1)$ may be done using Polya-Redfield theory or Burnside's lemma.

Proposition 6.4 *For $n>2$,*

$$
\begin{aligned}
X(C_n,q) &= \frac{1}{n}\sum_{d:d|n}\varphi(d)(1+q^d)^{\frac{n}{d}}\\
X(C_n,-1) &= \frac{1}{n}\sum_{\substack{d:d|n\\ d \text{even}}}\varphi(d)2^{\frac{n}{d}}
\end{aligned}
$$

Note that the above homology data suggests that

- $H_i(\mathcal{C}^{C_{2n}})=0$ for i odd, and for $i=2n-1,2n$, and
- $H_{2j}(\mathcal{C}^{C_{2n}})=H_{2n-2-2j}(\mathcal{C}^{C_{2n}})$.

But one can be much more precise. Note that using the formula in Proposition 6.4 for $X(C_n,-1)$, one can easily check that it satisfies the recursion

$$
X(C_{2n},-1) = \frac{X(C_n,-1)+X(C_n,1)}{2}
$$

This recursion may be modified to a recursion predicting the homology Poincaré series. For notational convenience, define

$$
A_n(q) := \sum_{i=1}^{n}\dim H_i(\mathcal{C}(\mathbb{F}_2)^{C_n})q^i \qquad \text{and} \qquad \mathrm{X_n(q) := X(C_n,q)}.
$$

Conjecture 6.5 *For any positive integer n, one has $H_i(\mathcal{C}^{C_{2n}})=0$ if i is odd, so that $A_{2n}(q)$ is a polynomial in q^2, and one has the recursion*

$$
A_{2n}(q^{1/2}) = \frac{qA_n(q)+X_n(q)}{1+q}
$$

Conjecture 6.5 has some strong evidence. It is correct at $q = 1$. It holds through $n = 18$, as shown earlier. We have proven it for n odd, although we do not include a proof here. In addition, it is correct with regard to its prediction about $H_2(\mathcal{C}^{C_{2n}})$.

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Colored Tutte polynomials and composite knots

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Abstract. Surveying the results of three recent papers and some currently ongoing research, we show how a generalization of Brylawski's tensor product formula to colored graphs may be used to compute the Jones polynomial of some fairly complicated knots and, in the future, even virtual knots.

Résumé. En faisant une revue de trois articles récents et de la recherche en cours, nous montrons comment une généralisation aux graphes colorés de la formule de Brylawski sur le produit tensoriel peut être utilisée à calculer le polynôme de Jones de quelques nœuds et, dans la future, même de quelques nœuds virtuels, bien compliqués.

Keywords: knots, Jones polynomial, Tutte polynomial, signed graphs, tensor product of matroids

Introduction

Tutte polynomials are known to be useful in the computation of the Jones polynomials of alternating knots. One of the most famous applications may be found in the work of Jaeger, Vertigan, and Welsh (11), where they use Brylawski's (3) tensor product formula to show that computing the Jones polynomial of even an alternating knot is NP-complete. The technique used also indicates the existence of large knots whose composite structure (their face graph being the tensor product of "manageable" graphs) allows the computation of their Jones polynomials using Brylawski's formula.

This presentation will take through the main results of three recent papers by the presenting authors which show how Brylawski's formula may be generalized to colored graphs, using a notion of a Tutte polynomial whose existence follows from the work of Bollobás and Riordan (1). The proof of the resulting formula indicates a new, Tutte-style proof of Brylawski's original result, and has many potential applications outside knot theory. As an example we indicate the application to the random-cluster model introduced by Fortuin and Kasteleyn (10).

The result should be generalizable to calculating the Jones polynomial of virtual knots (16) arising as a tensor product, however, for the moment, we only believe to have a suitable notion of a relative Tutte polynomial that allows to compute the Jones polynomial. Our research indicates that the theory of Tutte polynomials of morphisms of matroids, developed by Las Vergnas (17) is worth revisiting from the point of view of colored generalizations.

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1 The Tutte polynomial of a colored graph

In this section we review our implementation of the colored Tutte polynomial, introduced by Bollobás and Riordan (1).

Definition 1.1 *Let G be a connected graph with n edges whose edges are labeled $1, 2, \ldots, n$, and let T be a spanning tree of G. An edge e of T is internally active if for any edge $f \neq e$ in G such that $(T \setminus e) \cup f$ is a spanning tree of G, the label of e is less than the label of f. Otherwise e is internally inactive. An edge f of $G \setminus T$ is externally active if f has the smallest label among the edges in the unique cycle contained in $T \cup f$. Otherwise, f is externally inactive.*

Bollobás and Riordan (1) use Tutte's notion of activities but generalize Tutte's variable assignments as follows. Let G be a colored and connected graph and T a spanning tree of G. For each edge e in G with color λ, we assign one of the variables X_λ, Y_λ, x_λ and y_λ to it according to the activities of e as shown below (with respect to the tree T):

internally active	X_λ	externally active	Y_λ
internally inactive	x_λ	externally inactive	y_λ

Tab. 1: The variable assignment of an edge with respect to a spanning tree T.

Definition 1.2 *Let G be a connected colored graph (whose edges are labeled as in Definition 1.1). For a spanning tree T of G, let $C(T)$ be the product of the variable contributions from each edge of G according to the variable assignment above. The Tutte polynomial $T(G)$ is defined as the sum of $C(T)$ over all spanning trees T of G.*

Tutte's original variable assignment may be recovered by setting all $X_\lambda = x$, $Y_\lambda = y$, $x_\lambda = 1$ and $y_\lambda = 1$ for $\lambda \in \Lambda$. It is Tutte's main result (18) is that the total contribution of all spanning trees is labeling independent in the non-colored case. In the colored case, labeling independence is preserved only if the polynomial ring $\mathbb{Z}[\Lambda] := \mathbb{Z}[X_\lambda, Y_\lambda, x_\lambda, y_\lambda : \lambda \in \Lambda]$ is factored with an appropriate ideal I. The following theorem by Bollobás and Riordan (1) gives the exact description of all such ideals.

Theorem 1.3 (Bollobás-Riordan) *Assume I is an ideal of $\mathbb{Z}[\Lambda]$. Then the homomorphic image of $T(G)$ in $\mathbb{Z}[\Lambda]/I$ is independent of the labeling of the edges of G if and only if for all $\lambda, \mu, \nu \in \Lambda$ the polynomials*
$\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix}$, $Y_\nu\left(\det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix}\right)$,
and $X_\nu\left(\det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix}\right)$ *belong to I.*

Let I_0 be the ideal generated by the polynomials listed in Theorem 1.3. The image of $T(G)$ in $\mathbb{Z}[\Lambda]/I_0$ is the most general colored Tutte polynomial whose definition is independent of the labeling. Many important polynomials may be obtained from this polynomial by substitution, and most such substitutions, including all those we want to consider, map $\mathbb{Z}[\Lambda]/I_0$ into an integral domain in such a way that the images of the variables $x_\lambda, X_\lambda, y_\lambda$ and Y_λ are nonzero. All such maps factor through the the canonical map $\mathbb{Z}[\Lambda]/I_0 \to \mathbb{Z}[\Lambda]/I_1$, where $I_1 \supset I_0$ is the ideal generated by all polynomials of the forms

$\det\begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix}$ and $\det\begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix} - \det\begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix}$. This role of I_1 is implicitly noted in (1, Corollary 3). Thus we will consider the colored Tutte polynomial to be an element of $\mathbb{Z}[\Lambda]/I_1$. Let us highlight the following algebraic observation, making (1, Corollary 3) truly useful.

Lemma 1.4 (Diao-Hetyei-Hinson) *The ideal I_1 is a prime ideal. More generally, given any integral domain R, the ideal I_1 in $R[\Lambda]$ is prime.*

The proof of this lemma uses (2, Theorem (2.10)) stating that given any integral domain R and a matrix X of variables, the ideal generated by the maximal minors of X in the polynomial ring $R[X]$ is prime.

As noted in (1, Remark 3), our definitions and statements may be generalized to *matroids* by modifying the definitions of activities using “matroid basis” instead of “spanning tree” and interpreting a “cycle” as a “minimal dependent set”. By replacing the phrase “spanning tree” with “spanning forest”, one can easily generalize our Tutte polynomial to *disconnected graphs*. Given a disconnected graph G with connected components $G_1, \ldots, G_k$, the Tutte polynomial $T(G)$ obtained via this generalization is simply the product of the Tutte polynomials of its components: $T(G) = \prod_{i=1}^{k} T(G_i)$. On the other hand, Bollobás and Riordan (1) introduced a different generalized form of a Tutte polynomial by multiplying a variable $\alpha_{k(G)}$ (that only depends on the number $k(G)$ of connected components in G) with $T(G)$. This allows one to keep track of the number of connected components. For details we refer the reader to (1, Corollary 4). Here we only highlight the following consequence.

Corollary 1.5 *Let $\mathbb{Z}_{\Lambda,\alpha}$ be the polynomial ring $\mathbb{Z}_\Lambda[\alpha_i : i = 1, 2 \ldots]$, then the polynomial $\alpha_{k(G)}T(G)$, considered as an element of $\mathbb{Z}_{\Lambda,\alpha}/I_1$ is labeling independent.*

To avoid confusion, we will refer to $T(G)$ as the *ordinary* Tutte polynomial of the colored graph, and to $\alpha_{k(G)}T(G)$ as the *enriched* Tutte polynomial of G. The ordinary Tutte polynomial may be obtained from the enriched Tutte polynomial by sending all variables α_i to 1. Using the same I_1 to denote the ideal generated by the same elements in different rings will not cause confusion.

2 The signed Tutte polynomial and the Jones polynomial

The special case when the color set has two colors, referred to as *signs*, the resulting *signed Tutte polynomial* may be used to compute the Kauffman bracket and the Jones polynomial of a knot. Here we only outline this well-known construction, the details may be found in (1), (14), and (15). Kauffman's uses different letters to denote his variables, inactive edges correspond to variables A_+, A_-, B_+, B_-, active edges correspond to x_+, x_-, y_+, and y_-. A full concordance between Kauffman's (14), (15) and the Bollobás-Riordan notation (1) is given in Table 2.

Kauffman	ε	ε_i	x_ε	y_ε	A_ε	B_ε
Bollobás-Riordan	λ	λ_i	X_λ	Y_λ	x_λ	y_λ

Tab. 2: Concordance between the Kauffman and the Bollobás-Riordan notations

Consider a regular projection D of a knot K. We shade the regions of D either “white” or “dark” in a checkerboard fashion, so that no two regions of the same color are adjacent. We usually consider the infinite region surrounding the knot projection to be white. Next we construct a dual graph of D by converting the dark regions in D into vertices in a graph G and converting the crossings in D between two dark regions into edges incident to the corresponding vertices in G. We look at each crossing in the knot projection. If, after the upper strand passes over the lower, the dark region is to the left of the upper strand, then we denote this as a positive crossing, otherwise we denote it as a negative crossing. Then our signed graph is obtained by marking each edge of G with the same sign as the crossing of K to which it corresponds. See Figure 1 below and Figure 2 in Section 4.

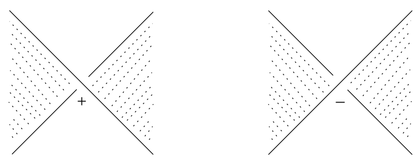

Fig. 1: The assignment of signs at a crossing (vertex) for the graph G.

The following theorem is due to Kauffman (14; 15).

Theorem 2.1 *Let G be the (signed) dual graph of a regular knot projection D of K as described above, then $T(G)$ equals the Kauffman bracket polynomial $\langle K\rangle$ under the following variable substitutions:*

$$x_+ \mapsto -A^{-3},\ x_- \mapsto -A^3,\ y_+ \mapsto -A^3,\ y_- \mapsto -A^{-3}$$
$$A_+ \mapsto A,\ A_- \mapsto A^{-1},\ B_+ \mapsto A^{-1},\ B_- \mapsto A.$$

Furthermore, the Jones polynomial $V_K(t)$ of K can be obtained from

$$V_K(t) = (-A^{-3})^{w(K)}\langle K\rangle \tag{1}$$

by setting $A = t^{-\frac{1}{4}}$, where $w(K)$ is the writhe of the projection D.

A regular projection D of a knot is *alternating* if all edges of the corresponding graph G have the same sign (w.l.o.g positive). A regular knot projection is alternating if and only if each "overcorssing" is followed by an "undercrossing" and vice versa, motivating the choice of the name. As a consequence of Theorem 2.1, the Jones polynomial of an alternating knot may be computed from the (original) Tutte polynomial of a (non-colored) graph.

3 Brylawski's tensor product formula and its signed generalization

The tensor product of colored graphs we introduce in this section is a colored generalization of the tensor product operation introduced by Brylawski (3) for (non-colored) matroids. The matroid generalization of the definition below is obvious.

Definition 3.1 *Let M and N be two graphs colored with the set Λ, $\lambda \in \Lambda$ a fixed color, and e a distinguished edge of N that is neither a loop nor a bridge. The λ-tensor product of M and N, denoted by $M \otimes_\lambda N$ is the colored graph obtained by replacing each edge in M of color λ with a copy of $N \setminus e$, where the distinguished edge e is to be identified with the replaced edge of M.*

We may recover Brylawski's original definition for $M \otimes N$ by setting $|\Lambda| = 1$. Brylawski's formula (3) was rephrased in the famous paper of Jaeger, Vertigan and Welsh (11) as follows.

Theorem 3.2 (Brylawski) *The Tutte polynomial of $M \otimes N$ is given by*

$$T(M \otimes N; x, y) = T_C(N; x, y)^{|S|-\mathrm{rank}(S)} T_L(N; x, y)^{\mathrm{rank}(S)} T(M; X, Y) \quad \textit{where}$$

$$X = \frac{(x-1)T_C(N; x, y) + T_L(N; x, y)}{T_L(N; x, y)}, \quad Y = \frac{T_C(N; x, y) + (y-1)T_L(N; x, y)}{T_C(N; x, y)},$$

and the polynomials T_C and T_L are defined by the equations

$$\begin{array}{rcl} (x-1)T_C(N; x, y) + T_L(N; x, y) & = & T(N \setminus e; x, y), \\ T_C(N; x, y) + (y-1)T_L(N; x, y) & = & T(N/e; x, y). \end{array} \tag{2}$$

Here, and from now on, $N \setminus e$ resp. N/e stands for the graph or matroid obtained after the *deletion* resp. *contraction* of e.

Brylawski's result was used by Jaeger, Vertigan and Welsh (11) to show that the computation of the Tutte polynomial and of the Jones polynomial of an alternating knot is NP-complete. Finding a generalization of the tensor product formula stated in Theorem 3.2 allows to compute the Jones polynomials of large *non-alternating* knots that have a regular projection whose associated graph arises as a tensor product of signed graphs of "manageable size".

For this purpose we first define a colored generalization of the polynomials T_C and T_L that appear in Theorem 3.2.

Definition 3.3 *Let N be a colored connected graph with a distinguished edge e that is neither a loop nor a bridge. Then $T_L(N, e)$ is the polynomial defined by the same rule that defines the ordinary colored Tutte polynomial $T(N \setminus e)$ except that internally active edges on a cycle closed by e will be considered as internally inactive instead. Similarly, $T_C(N, e)$ is the polynomial defined by the same rule that defines the ordinary colored Tutte polynomial $T(N/e)$ except that externally active edges that would close a cycle containing e will be considered as externally inactive instead.*

To interpret the second part of the definition, note that the spanning trees of N/e are identifiable with those spanning trees of N which contain e. We say that an external edge closes a cycle containing e with a spanning tree T of N/e, if the same holds for the corresponding spanning tree $T \cup \{e\}$ of N.

Definition 3.3 appears to depend on the labeling of the edges, but it is not, because of the following result (7):

Theorem 3.4 (Diao-Hetyei-Hinson) *Let N be a colored graph with a distinguished edge e that is neither a loop nor a bridge and let $\lambda \in \Lambda$ be any color. Then the following two equalities hold:*

$$x_\lambda(T(N/e) - T_C(N, e)) = (Y_\lambda - y_\lambda)T_L(N, e), \tag{3}$$

$$y_\lambda(T(N \setminus e) - T_L(N, e)) = (X_\lambda - x_\lambda)T_C(N, e). \tag{4}$$

As a consequence of Theorem 3.4 the pair of polynomials $z_C = T_C(N, e)$ and $z_L = T_L(N, e)$ provides a solution of the linear system of equations

$$\begin{array}{rcl} (Y_\lambda - y_\lambda)z_L + x_\lambda z_C & = & x_\lambda T(N/e) \\ y_\lambda z_L + (X_\lambda - x_\lambda)z_C & = & y_\lambda T(N \setminus e) \end{array} \tag{5}$$

for the unknowns z_L and z_C. The givens belong to $\mathbb{Z}[\Lambda]/I_1$, an integral domain by Lemma 1.4. Cramer's rule is applicable in the quotient field of $\mathbb{Z}[\Lambda]/I_1$, and we have

$$\det\begin{pmatrix} Y_\lambda - y_\lambda & x_\lambda \\ y_\lambda & X_\lambda - x_\lambda \end{pmatrix} = X_\lambda Y_\lambda - X_\lambda y_\lambda - y_\lambda X_\lambda$$

which is a nonzero element of $\mathbb{Z}[\Lambda]/I_1$, independent of the choice of λ, since each element of I_1 is a $\mathbb{Z}$-linear combination of monomials involving at least two colors from Λ. Thus the solution of (5) is unique and definition of $T_C(N,e)$ and $T_L(N,e)$ is in deed independent of the labeling. The proof of Theorem 3.4 greatly depends on the following algebraic formula (7):

Lemma 3.5 *The following identities hold in the ring $\mathbb{Z}[\Lambda]/I_1$ for all $k \geq 1$ and all $\lambda, \lambda_1, \ldots, \lambda_k \in \Lambda$ (all empty products are treated as 1):*

$$x_\lambda \left(\prod_{i=1}^{k} Y_{\lambda_i} - \prod_{i=1}^{k} y_{\lambda_i} \right) = (Y_\lambda - y_\lambda) \sum_{i=1}^{k} x_{\lambda_i} \prod_{j=1}^{i-1} Y_{\lambda_j} \prod_{t=i+1}^{k} y_{\lambda_t} \tag{6}$$

$$y_\lambda \left(\prod_{i=1}^{k} X_{\lambda_i} - \prod_{i=1}^{k} x_{\lambda_i} \right) = (X_\lambda - x_\lambda) \sum_{i=1}^{k} y_{\lambda_i} \prod_{j=1}^{i-1} X_{\lambda_j} \prod_{t=i+1}^{k} x_{\lambda_t}. \tag{7}$$

Note that the second equation in Lemma 3.5 follows from the first by exchanging each x_λ with y_λ and each X_λ with Y_λ. The equations in Theorem 3.4 may be rephrased as follows.

$$\det\begin{pmatrix} T_L(N,e) & T_C(N,e) \\ x_\lambda & y_\lambda \end{pmatrix} = \det\begin{pmatrix} T_L(N,e) & T(N/e) \\ x_\lambda & Y_\lambda \end{pmatrix} \tag{8}$$

and

$$\det\begin{pmatrix} T_L(N,e) & T_C(N,e) \\ x_\lambda & y_\lambda \end{pmatrix} = \det\begin{pmatrix} T(N \setminus e) & T_C(N,e) \\ X_\lambda & y_\lambda \end{pmatrix} \tag{9}$$

Using this determinantal form it is easy to show the following (7):

Lemma 3.6 (Diao-Hetyei-Hinson) *The endomorphism of $\mathbb{Z}[\Lambda]$, given by $X_\lambda \mapsto T(N \setminus e)$, $x_\lambda \mapsto T_L(N,e)$, $Y_\lambda \mapsto T(N/e)$, $y_\lambda \mapsto T_C(N,e)$ and by $X_\mu \mapsto X_\mu$, $x_\mu \mapsto x_\mu$, $Y_\mu \mapsto Y_\mu$, $y_\mu \mapsto y_\mu$ for all $\mu \neq \lambda$, sends I_1 into itself.*

This lemma allows us to state our main result on the tensor product of colored graphs (and matroids) as follows (7):

Theorem 3.7 *Let M be a colored graph and N a colored graph with a distinguished edge e that is neither a loop nor a bridge. Then the ordinary Tutte polynomial $T(M \otimes_\lambda N)$ can be computed from $T(M)$ by keeping all variables of color $\mu \neq \lambda$ unchanged, and using the substitutions $X_\lambda \mapsto T(N \setminus e)$, $x_\lambda \mapsto T_L(N,e)$, $Y_\lambda \mapsto T(N/e)$, $y_\lambda \mapsto T_C(N,e)$.*

Remark 3.8 *For non-colored graphs and matroids our reasoning may be substantially simplified. Thus, in (9), we obtained a new , "Tutte-style" proof of Brylawski's original result (3).*

4 Applications and non-connected generalizations of the colored tensor product formula

The first important specialization of Theorem 3.7 is the case of signed graphs. Using Kauffman's notation, in this special case Theorem 3.7 may be restated as follows (6):

Theorem 4.1 *Let M be a signed graph and N a signed graph with a distinguished edge e. Then $T(M \otimes_+ N)$ can be computed from $T(M)$ by keeping the negative variables unchanged and using the substitutions*

$$x_+ \mapsto T(N\backslash e) \quad A_+ \mapsto T_L(N,e) \quad y_+ \mapsto T(N/e) \quad B_+ \mapsto T_C(N,e).$$

Similarly, $T(M \otimes_- N)$ can be computed from $T(M)$ by keeping the positive variables unchanged and using the substitutions

$$x_- \mapsto T(N\backslash e) \quad A_- \mapsto T_L(N,e) \quad y_- \mapsto T(N/e) \quad B_- \mapsto T_C(N,e).$$

Theorem 4.1 allowed us to compute the Jones polynomial of the non-alternating knot shown in Figure 2 in (6). "Historically," the computation of this Jones polynomial was the challenge that motivated all our results presented here.

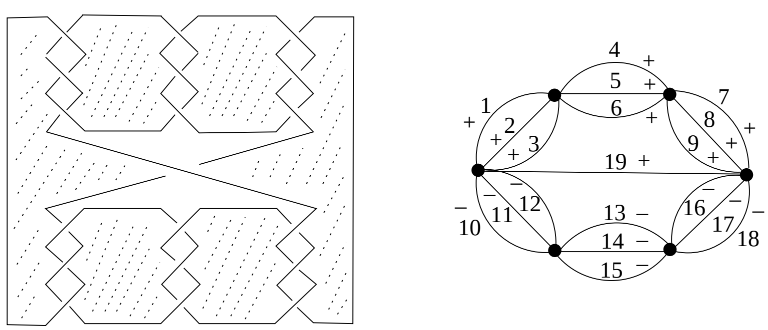

Fig. 2: A 19 crossing knot diagram and its corresponding signed graph.

The generalization of Theorem 3.7 to enriched Tutte polynomials of disconnected graphs is discussed in (7). The number $k(M \otimes_\lambda N)$ of connected components of $M \otimes_\lambda N$ satisfies the equation

$$k(M \otimes_\lambda N) = k(M) + |E_\lambda(M)| \cdot (k(N \setminus e) - 1). \tag{10}$$

Here $E_\lambda(M)$ is the set of edges of color λ in M. The enriched Tutte polynomial of $M \otimes_\lambda N$ is thus equal to

$$\alpha_{k(M)+|E_\lambda(M)|\cdot(k(N\setminus e)-1)} \cdot T(M \otimes_\lambda N),$$

where $T(M \otimes_\lambda N)$ is the ordinary colored Tutte polynomial, to which Theorem 3.7 is applicable. In particular, substituting $k(N \setminus e) = 1$ into (10) yields $k(M \otimes_\lambda N) = k(M)$ thus we have the following consequence.

Corollary 4.2 *Let M be a colored graph and N a colored graph with a distinguished edge e that is neither a loop nor a bridge. Assume that $N \setminus e$ is connected. Then the enriched Tutte polynomial of $M \otimes_\lambda N$ can be computed from the enriched Tutte polynomial of M by keeping all variables of color $\mu \neq \lambda$ and the variables α_n ($n \geq 1$) unchanged, and using the substitutions*

$$X_\lambda \mapsto T(N \setminus e) \quad x_\lambda \mapsto T_L(N, e) \quad Y_\lambda \mapsto T(N/e) \quad y_\lambda \mapsto T_C(N, e).$$

Corollary 4.2 may be applied to the *random-cluster model* introduced by Fortuin and Kasteleyn (10) in 1972 as a generalization of various models such as the percolation model, the two-state Ising model and the Potts model. This model can be thought of as a graph $G(V, E)$ that is associated with a function $p : E \longrightarrow [0, 1]$. We may think of p_e as the probability that the edge $e \in E$ "survives" an accident, and $q_e = 1 - p_e$ as the probability that the edge e "breaks" in an accident. Fortuin and Kasteleyn (10) introduced the following polynomial of the variable κ as a cluster-generating function $Z(G; p, \kappa)$:

$$Z(G; p, \kappa) = \sum_{C \subseteq E} p^C q^{E \setminus C} \kappa^{k(C)}. \tag{11}$$

Here p^C is a shorthand for the product $\prod_{e \in C} p_e$, $q^{E \setminus C}$ is a shorthand for the product $\prod_{e \in E \setminus C} q_e$, and $k(C)$ is the number of connected components in the subgraph consisting of the edges of C and the vertices incident to these edges. Bollobás and Riordan (1) have shown that the polynomial $Z(G; p, \kappa)$ can be computed from the enriched Tutte polynomial of a colored graph. We may think of the colors as different "materials" and assume that edges of the same color have the same probability to "break" in an accident. After computing the Tutte polynomial of the colored graph, we have to make the following substitutions (as directed in (1)):

$$x_\lambda \mapsto p_\lambda \quad y_\lambda \mapsto q_\lambda \quad X_\lambda \mapsto p_\lambda + \kappa q_\lambda \quad Y_\lambda \mapsto 1. \tag{12}$$

Corollary 4.2 becomes applicable when our network is a "network of networks" $M \otimes_\lambda N$, i.e. when the edges of color λ of a network M are networks themselves, associated to the same colored graph N with a distinguished edge e indicating how $N \setminus e$ should be inserted as a substitute of $e \in M$.

Definition 4.3 *Let N be a colored graph with a distinguished edge e that is neither a loop nor a bridge, such that each color λ has an associated probability p_λ. We define the* pointed random-cluster-generating functions *$Z_C(N, e; p, \kappa)$ and $Z_L(N, e; p, \kappa)$ as the homomorphic images of $T_C(N, e)$ and $T_L(N, e)$ respectively under the homomorphism induced by the substitutions given in (12).*

As a consequence of Corollary 4.2 and the substitution rule (12) we have the following result.

Theorem 4.4 *Let M and N be colored graphs and p a function associating to each color a probability. Assume N has a distinguished edge e that is neither a loop nor a bridge, and that $N \setminus e$ is connected. Let λ be a fixed color. Then the cluster-generating function $Z(M \otimes_\lambda N; p, \kappa)$ may be obtained from the enriched colored Tutte polynomial $\alpha_{k(M)} T(M)$ by sending α_n into κ^n and making the following substitutions:*

$$X_\lambda \mapsto Z(N \setminus e; p, \kappa) \quad x_\lambda \mapsto Z_L(N, e; p, \kappa) \quad Y_\lambda \mapsto Z(N/e; p, \kappa) \quad y_\lambda \mapsto Z_C(N, e; p, \kappa).$$

For all other colors μ we apply the substitutions given in (12).

It was shown in (7) that the pointed random cluster generating functions $Z_C(N, e; p, \kappa)$ and $Z_L(N, e; p, \kappa)$ may be equivalently given by

$$Z_C(N, e; p, \kappa) = \frac{Z(N \setminus e; p, \kappa) - Z(N/e; p, \kappa)}{\kappa - 1} \quad \text{and} \tag{13}$$

$$Z_L(N,e;p,\kappa) = \frac{\kappa Z(N/e;p,\kappa) - Z(N \setminus e;p,\kappa)}{\kappa - 1}. \tag{14}$$

5 Relative Tutte polynomials and virtual knots

Kauffman (16) has generalized the Jones polynomial to *virtual knot diagrams* which may be thought of as knot diagrams drawn on nontrivial surfaces. These drawings my be represented in the plane by allowing a few "virtual" crossings: crossings that do not exist on the surface only in the plane, due to two distinct points of the surface being represented by the same point in the plane. We may then apply the process of associating a signed graph as outlined in Section 2 with the additional rule that "virtual crossings" correspond to zero colored edges. An example is shown in Figure 3. If we wanted to extend Theorem 3.7

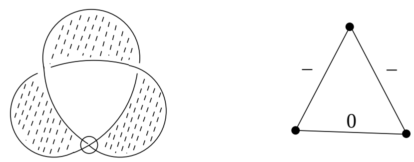

Fig. 3: The virtual trefoil knot (which is not checkerboard colorable) and its face graph.

to virtual knots, we would first need some computational rule expressing the Jones polynomial of a virtual knot (as defined by Kauffman (16)) in terms of the Tutte polynomial of the 3-colored graph associated to the virtual knot diagram. Unfortunately, the"zero" edges do not abide to the same deletion-contraction rules as the other edges. There have been some efforts to overcome this difficulty. These efforts so far seem to concentrate on changing the underlying graph to a "ribbon graph" so that the "zero edges" would go away (5; 12; 13). This approach only applies to those virtual link diagrams that are "checkerboard colorable". Figure 3 shows the virtual trefoil, which is not checkerboard colorable.

To overcome the difficulty, a theory of *relative Tutte polynomials* was developed in (8).

Definition 5.1 *Let G be a connected graph and $\mathcal{H}$ a subset of its edge set $E(G)$. A subset $\mathcal{C}$ of the edge set $E(G)\setminus\mathcal{H}$ is called a* contracting set *of G with respect to $\mathcal{H}$ if $\mathcal{C}$ contains no cycles and $\mathcal{D} := E(G)\setminus(\mathcal{C}\cup\mathcal{H})$ contains no cocycles (and $\mathcal{D}$ is called a* deleting set*).*

Lemma 5.2 (Diao-Hetyei) *In the above definition, if $\mathcal{H} = \emptyset$, then $\mathcal{C} \subseteq E(G)$ is a contracting set if and only if the subgraph $\mathcal{C}$ is a spanning tree of $E(G)$.*

For the sake of matroid-theoretic generalizations, the following observation is useful.

Lemma 5.3 (Diao-Hetyei) *$\mathcal{C}$ is a contracting set with respect to $\mathcal{H}$ if and only if there is a basis $B \subset \mathcal{C} \cup \mathcal{H}$ that contains $\mathcal{C}$.*

Lemma 5.4 (Diao-Hetyei) *Let G be a connected graph and $\mathcal{H}$ a subset of $E(G)$. Let $\mathcal{C}$ be a contracting set of G with respect to $\mathcal{H}$, $\mathcal{D}$ be the corresponding deleting set and $e \in \mathcal{C}$ be any edge in $\mathcal{C}$. Then for any $f \in \mathcal{D}$, $\mathcal{C}' = \{f\} \cup (\mathcal{C} \setminus \{e\})$ is also a contracting set with respect to $\mathcal{H}$ if the triplet $(\mathcal{C}, e, f)$ has either of the following properties:*

(i) *$\mathcal{C} \cup \{f\}$ contains a cycle containing $\{e\}$.*

(ii) $\mathcal{D} \cup \{e\}$ *contains a cocycle containing* $\{f\}$.

Moreover, if the triplet $(\mathcal{C}, e, f)$ *satisfies (i) or (ii) then the triplet* $(\mathcal{C}', f, e)$ *has the same properties.*

Definition 5.5 Let G be a connected graph and $\mathcal{H}$ be a subset of $E(G)$. Let us assume that a labeling of G is given in such a way that all edges in $\mathcal{H}$ are labeled with number 0 and all other edges are labeled with distinct positive integers. Such a labeling is called a *proper labeling* or a *relative labeling* (with respect to $\mathcal{H}$). In other words, a proper labeling of the edges of G with respect to $\mathcal{H}$ is a mapping $\phi: \ E(G) \longrightarrow \mathbb{Z}$ such that $\phi(e) = 0$ for any $e \in \mathcal{H}$ and ϕ is an injective map from $E(G) \setminus \mathcal{H}$ to $\mathbb{Z}^+$. We say that e_1 is larger than e_2 if $\phi(e_1) > \phi(e_2)$. Let $\mathcal{C}$ be a contracting set of G with respect to $\mathcal{H}$, then

a) an edge $e \in \mathcal{C}$ is called *internally active* if $\mathcal{D} \cup \{e\}$ contains a cocycle D_0 in which e is the smallest edge, otherwise it is *internally inactive*.

b) an edge $f \in \mathcal{D}$ is called *externally active* if $\mathcal{C} \cup \{f\}$ contains a cycle C_0 in which f is the smallest edge, otherwise it is *externally inactive*.

Definition 5.6 *Let* ψ *be a mapping defined on the isomorphism classes of finite connected graphs with values in a ring* $\mathcal{R}$*. We say that* ψ *is a* block invariant *if for all positive integer* n *there is a function* $f_n : \mathcal{R}^n \to \mathcal{R}$ *that is symmetric under permuting its input variables such that for any connected graph* G *having* n *blocks* $G_1, \ldots, G_n$ *we have*

$$\psi(G) = f_n(\psi(G_1), \ldots, \psi(G_n)).$$

In other words, we require the ability to compute $\psi(G)$ from the value of ψ on the blocks of G, and this computation should not depend on the order in which the blocks are listed.

Lemma 5.7 (Diao-Hetyei) *Let* G *be a connected graph and* $\mathcal{H}$ *be a subset of* $E(G)$*. Assume that* $\mathcal{C}$ *is a contracting set with respect to* $\mathcal{H}$ *and that the triplet* $(\mathcal{C}, e, f)$ *has at least one of the properties listed in Lemma 5.4. Let* $\mathcal{C}' := (\mathcal{C} \cup \{f\}) \setminus \{e\}$*. Then the multiset of blocks of* $\mathcal{H}_{\mathcal{C}}$ *is the same as the multiset of blocks of* $\mathcal{H}_{\mathcal{C}'}$*.*

Similarly, for matroids we have:

Lemma 5.8 (Diao-Hetyei) *Let* M *be a matroid and* $\mathcal{H}$ *a subset of its elements. Assume that* $\mathcal{C}$ *is a contracting set with respect to* $\mathcal{H}$ *and that the triplet* $(\mathcal{C}, e, f)$ *has at least one of the properties listed in Lemma 5.4. Let* $\mathcal{C}' := (\mathcal{C} \cup \{f\}) \setminus \{e\}$*. Then the cycle matroid of the graph* $\mathcal{H}_{\mathcal{C}}$ *is the same as the cycle matroid of the graph* $\mathcal{H}_{\mathcal{C}'}$*.*

Let G be a connected graph and $\mathcal{H} \subseteq E(G)$. Assume we are given a mapping c from $E(G) \setminus \mathcal{H}$ to a color set Λ. Assume further that ψ is a block invariant associating an element of a fixed integral domain $\mathcal{R}$ to each connected graph. For any contracting set $\mathcal{C}$ of G with respect to $\mathcal{H}$, let $\mathcal{H}_{\mathcal{C}}$ be the graph obtained by deleting all edges in $\mathcal{D}$ and contracting all edges in $\mathcal{C}$ (so that the only edges left in $\mathcal{H}_{\mathcal{C}}$ are the zero edges). Finally, we will assign a proper labeling to the edges of G. We now define the relative Tutte polynomial of G with respect to $\mathcal{H}$ and ψ as

$$T^{\psi}_{\mathcal{H}}(G) = \sum_{\mathcal{C}} \Big(\prod_{e \in G \setminus H} w(G, c, \phi, \mathcal{C}, e) \Big) \psi(\mathcal{H}_{\mathcal{C}}) \in \mathcal{R}[\Lambda], \tag{15}$$

where the summation is taken over all contracting sets $\mathcal{C}$ and $w(G, c, \phi, \mathcal{C}, e)$ is the *weight* of the edge e with respect to the contracting set $\mathcal{C}$, which is defined as (assume that e has color λ):

$$w(G, c, \phi, \mathcal{C}, e) = \begin{cases} X_\lambda & \text{if } e \text{ is internally active;} \\ Y_\lambda & \text{if } e \text{ is externally active;} \\ x_\lambda & \text{if } e \text{ is internally inactive;} \\ y_\lambda & \text{if } e \text{ is externally inactive.} \end{cases} \tag{16}$$

To simplify the notation somewhat, we will be using $T_{\mathcal{H}}(G)$ for $T^{\psi}_{\mathcal{H}}(G)$, with the understanding that some ψ has been chosen, unless there is a need to stress what ψ really is. Following (1), we then write

$$W(G, c, \phi, \mathcal{C}) = \prod_{e \in G \setminus \mathcal{H}} w(G, c, \phi, \mathcal{C}, e)$$

so that

$$T_{\mathcal{H}}(G, \phi) = \sum_{\mathcal{C}} W(G, c, \phi, \mathcal{C}) \psi(\mathcal{H}_{\mathcal{C}}). \tag{17}$$

We may now extend Theorem 1.3 of Bollobás and Riordan (1) to $T_{\mathcal{H}}$ as follows.

Theorem 5.9 (Diao-Hetyei) *Assume I is an ideal of $\mathcal{R}[\Lambda]$. Then the homomorphic image of $T_{\mathcal{H}}(G, \phi)$ in $\mathcal{R}[\Lambda]/I$ is independent of ϕ (for any G and ψ) if and only if for all $\lambda, \mu \in \Lambda$ the polynomials* $\det \begin{pmatrix} X_\lambda & y_\lambda \\ X_\mu & y_\mu \end{pmatrix} - \det \begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix}$ *and* $\det \begin{pmatrix} x_\lambda & Y_\lambda \\ x_\mu & Y_\mu \end{pmatrix} - \det \begin{pmatrix} x_\lambda & y_\lambda \\ x_\mu & y_\mu \end{pmatrix}$ *belong to I.*

Using Theorem 5.9 it is possible to define a relative Tutte polynomial that may be used to compute the Jones polynomial of a virtual knot. For details see (8, Section 5).

6 Concluding remarks

The immediate goal prompted by Theorem 5.9 is to extend the tensor product formula for colored graphs to relative Tutte polynomials and we are currently working on such an extension. It is worth observing that for graphs, our relative Tutte polynomial generalizes the *set-pointed* Tutte polynomial of matroids introduced and discussed in (17). Looking into results on that invariant with the purpose of further signed generalizations seems a worthwhile project. The ideal I_1 introduced in Section 1 is a *determinantal ideal*, and Gröbner basis theory is known to be useful in the theory of such ideals. Thus it is conceivable that Gröbner basis theory may be used to develop new or improve on existing algorithms to compute the Jones polynomial.

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polymake and Lattice Polytopes

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The `polymake` software system deals with convex polytopes and related objects from geometric combinatorics. This note reports on a new implementation of a subclass for lattice polytopes. The features displayed are enabled by recent changes to the `polymake` core, which will be discussed briefly.

Keywords: polymake system, lattice polytope, Hilbert basis, toric geometry

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1 Introduction

polymake is a software system designed for analyzing convex polytopes, finite simplicial complexes, graphs, and other objects. While the system exists for more than a decade [14] it was continuously developed and expanded. The most recent version fundamentally changes the way to interact with the system. It now offers an interface which looks similar to many computer algebra systems. However, on the technical level polymake differs from most mathematical software systems: rule based computations and an extendible dual Perl/C++ interface are the most important characteristics.

polymake can now also handle hierarchies of objects where each level may come with additional sets of rules. polymake handles casts between subclasses based on property requests. We will explain this feature by means of a new subclass LatticePolytope that is derived from the existing class Polytope<Rational>. However, some of the new functions can also be applied to any rational polytope. A *lattice polytope* is a polytope whose vertices are contained in a lattice $\Lambda \subset \mathbb{R}^n$ [6, 2]. polymake always assumes $\Lambda = \mathbb{Z}^n$. This subclass reflects a new use of polymake in toric geometry, where lattice polytopes encode properties of toric varieties and toric ideals [21, 12]. String theorists have been interested in special lattice polytopes, as they led to the construction of mirror pairs of Calabi-Yau varieties [4]. Gröbner bases of toric ideals have been applied to optimization problems [27].

We will explain all relevant concepts for our exposition on the way. Lattice polytopes have also become an important subject in other areas of mathematics. Enumerating non-negative solutions of Diophantine equations can be interpreted as counting lattice points in a polyhedron [25]. Contingency tables in statistics can be modeled by lattice polytopes [11]. Sampling then corresponds to finding integral points in the polytope.

The paper is organized as follows. First we will review the recent changes and the new polymake interface. Then we will report on our new implementation of a subclass for lattice polytopes. In particular, this comprises interfaces to 4ti2 [1], Latte macchiato [10, 16], and normaliz2 [8]. We will show how the user can benefit from the common interface to these systems via polymake and how one can extend their functionality by combining with polymake's features. Rather than discussing implementation details we will explain the functions available with one easy running example. This note then concludes with a final section analyzing a specific 6-dimensional polyhedral cone which was found to be a counter-example to a conjecture of Sebő [23] by Bruns et al. [7].

2 polymake – the Next Generation

The general ideas which lead to the design and the implementation of the polymake system more than ten years ago are still valid. The key goals are the following.

- ▷ The system should be scalable with the user's ability to write programs. This means that basic usage should not require any programming skills, while it should be powerful enough not to restrain the programming expert.
- ▷ The system should not try to "re-invent the wheel". There is a multitude of valuable pieces of software for individual tasks; so they should be suitably interfaced rather than their functionality be duplicated.
- ▷ The system should be really easy to extend. It should be possible to model new mathematical objects and to integrate them into the existing framework.

These "golden rules" are most natural, and most users of mathematical software systems would probably agree that all of these are very desirable. For instance, the SAGE system is following a similar strategy albeit on a somewhat larger scale [26]. In polymake we are focusing on convex polytopes and related objects from the realm of geometric combinatorics. The "golden rules" already have a number of implications, some obvious and some less obvious. The most important design decisions which can be derived are: The system requires both a compiled and an interpreted programming language (we settled for C++ and Perl), and the system must be an Open Source project (we settled for the GNU Public License). By far the most difficult to accomplish is the third rule. And, in fact, a large part of polymake's code evolution over the last decade can be seen as an attempt to re-interpret this rule again and again with an increasing level of abstraction.

A word of warning to the experienced polymake user. On a technical level the new version of polymake is very different from previous versions. From the point of view of the working mathematician this results in a number of benefits. In particular, the overall usability is improved, while we gained additional flexibility and speed. The unavoidable drawback is that the interface had to be changed in a substantial way.

Using polymake now means to start a program named "polymake" from the command line, and then to work in a shell-type environment typical for most computer algebra systems. The language for interacting with the system is Perl, but we added a few features in order to easy the usability. We give a very brief overview of how to get started with the new system.

```
Welcome to polymake version 2.9.6, rev. 9033
Copyright (c) 1997-2009
Ewgenij Gawrilow (TU Berlin), Michael Joswig (TU Darmstadt)
http://www.math.tu-berlin.de/polymake,  mailto:polymake@math.tu-berlin.de

This is free software licensed under GPL; see the source for copying conditions.
There is NO warranty; not even for MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE.

Type 'help;' for basic instructions.

Application polytope uses following third-party software (for details: help 'credits';)
4ti2, azove, cddlib, lrslib, nauty, normaliz2, porta, qhull, splitstree, topcom, vinci
polytope >
```

By now there are several different *applications* known to polymake. Each application comprises a main *object* type, *properties* which describe an object of this type, and a set of *rules*. By default the first application to start is the one dealing with convex polytopes, and this is made visible by showing the command line prompt "polytope >". The last line before the prompt lists the programs whose interfaces are loaded. Since everything (the application, the objects, the properties, the rules, the interfaces, and the defaults) can be modified or extended by the user, what shows up exactly very much depends on the local installation. The main purpose of this note is to explain how a new sub-type for lattice polytopes is organized within the object hierarchy for general polytopes.

The following simple example can explain the polymake concept in a nutshell. The first command produces a 3-dimensional cube with ± 1-coordinates (and assigns it to the variable $P), while the second one (separated by ";") prints its f-vector, that is, the number of faces per dimension. Clearly we have eight vertices, 12 edges, and six facets.

```
polytope > $P=cube(3); print $P->F_VECTOR;
8 12 6
```

The function `cube` returns a polytope object of type `Polytope<Rational>`, and `F_VECTOR` is a property of this class, which models polytopes with rational coordinates. Notice that there are polytopes whose combinatorial type does not admit any rational representation [28, §6.5]. `polymake` reduces computing the f-vector of this cube to finding a suitable sequence of rules and to execute them one after another. These rules can be shown as follows. To this end we restart from scratch.

```
polytope > $P=cube(3); print join(", ", $P->list_properties);
AMBIENT_DIM, DIM, FACETS, VERTICES_IN_FACETS, BOUNDED
polytope > print $P->type->full_name;
Polytope<Rational>
```

Our cube `$P` is "born" as an object with the five initial properties `AMBIENT_DIM`, `DIM`, `FACETS`, `VERTICES_IN_FACETS`, `BOUNDED`, all of which are redundant except for `FACETS`, which gives a description of the cube as the intersection of six affine halfspaces.

```
polytope > print $P->FACETS;
1 1 0 0
1 0 0 1
1 0 1 0
1 0 -1 0
1 -1 0 0
1 0 0 -1
```

Each line is a vector $(\alpha_0, \alpha_1, \ldots, \alpha_d)$ representing the linear inequality $\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_d x_d \geq 0$. The property `AMBIENT_DIM`, for instance, is the dimension of the space where our polytope lives in, that is the number d, which can easily be derived from each facet by counting the number of columns. Now, asking for the f-vector means that it has to be computed from the data given somehow. We can look at the schedule of rules necessary to accomplish this task.

```
polytope > $schedule=$P->get_schedule("F_VECTOR");
polytope > print join("\n", $schedule->list);
HASSE_DIAGRAM : VERTICES_IN_FACETS
F_VECTOR, F2_VECTOR : HASSE_DIAGRAM
```

Each line is one rule. Each rule has its *targets* to the left of the "`:`" and its *sources* to the right. The first line says: "I can produce the Hasse diagram (of the face lattice) if I know which vertex is incident with which facet". This is clear since it follows from the facet that the face lattice is co-atomic, that is, each face is the intersection of facets [28, §2.2]. The second line says: "I know how to compute the f-vector (and something else that we do not care to discuss now) from the Hasse diagram. Each rule comes with a piece of (Perl) code which actually implements what the rule heads shown promise. The schedule is an object of its own right, and it can be applied to the cube, which means that the corresponding Perl code is executed in the order of the schedule.

```
polytope > $schedule->apply($P);
polytope > print join(", ", $P->list_properties);
AMBIENT_DIM, DIM, FACETS, VERTICES_IN_FACETS, BOUNDED, HASSE_DIAGRAM, F_VECTOR,
F2_VECTOR
```

We see that the list of properties known about our cube changed. Three new properties have been added, and these correspond to the total of three targets of the two rules above. If we now ask for the f-vector this information is already stored with the cube object, and it is read from memory rather than re-computed.

As far as technology is concerned, the function `cube` which was called to produce the cube is written in C++. On top of the standard Perl-interface to C we built a shared memory mechanism to access an object from the C++ and the Perl side. This is also fully extendible, which means that the user is welcome to add new functions to produce other polytopes, new properties of polytopes, or new rules to compute existing properties in a different way. The integration of new functions, properties, and rules is seamless, that is, they cannot be distinguished from the built-in ones.

There are many more things to be said about this concept both from the logical and the technical point of view, but for the details we refer the reader to [14] and to further documentation at `http://www.opt.tu-darmstadt.de/polymake`.

3 Lattice Polytopes as a Subclass

The new version of `polymake` can now handle derived classes of objects specified by some *preconditions* that inherit all properties and rules from their base class but may provide additional rules that are specific for their class. A user may, but doesn't have to, specify, that the object he defines falls in this class. `polymake` decides upon what properties a user asks for, whether the object should be *cast* into this subclass. Of course, before performing the cast, `polymake` checks whether the object meets the requirements for the subclass. The first occurrence of this new mechanism is in the class `LatticePolytope` derived from `Polytope<Rational>`. In `polymake`, a lattice polytope is a bounded rational polytope whose vertices are in the integer lattice $\mathbb{Z}^n$. The new rules in this object class concern properties of such polytopes in connection with toric algebra and algebraic geometry.

The main focus of our implementation concerning lattice polytopes is toric geometry, so we explain this connection here. Let P be a lattice polytope. The *normal fan* $\mathcal{N}_P$ defines a projective *toric variety* X_P [12, 27]. The defining ideal $\mathcal{I}_P$ of X_p is a homogeneous *toric ideal*. Many properties of the variety are reflected in the corresponding polytope. We will see some entries in this "dictionary" which translates back and forth below.

There are several software packages available which proved to be useful in applications in this area. `normaliz2` by Bruns and Ichim [8] computes Hilbert bases and h^*-polynomials. `Latte macchiato` by Köppe [16] builds on previous work by De Loera et al. [10], and its key application is to count lattice points and to compute Ehrhart polynomials. `4ti2` by Hemmecke et. al. [1] solves integral equations over $\mathbb{Z}$ and it computes convex hulls as well as Hilbert bases. `polymake` now provides a unified access to these programs. Additionally, we implemented various rules to compute further important properties of lattice polytopes which can be derived. We will browse through the main features by using the 3-cube from above as our running example. As already mentioned, our cube is the convex hull of all ± 1-vectors, so the vertices do lie in the $\mathbb{Z}^3$-lattice. We can let `polymake` check this for us.

```
polytope > print $P->LATTICE;
1
```

Here the output "1" represents the boolean value "true". For instance, we can ask for the number of lattice points contained in the cube, that is, for the number $|[-1,1]^3 \cap \mathbb{Z}^3|$. In our case, we should obtain "27" as the answer, there is exactly one lattice point contained in the relative interior of each non-empty face.

```
polytope > print $P->N_LATTICE_POINTS;
polymake: used package latte
 LattE macchiato is an improved version of LattE, a free software dedicated
```

```
 to the problems of counting and detecting lattice points inside convex polytopes,
 and the solution of integer programs.
 Copyright by Matthias Koeppe, Jesus A. De Loera and others.
 http://www.math.ucdavis.edu/~mkoeppe/latte/

27
```

As shown `Latte macchiato` was called for the computation. It uses an enhanced version of Barvinok's algorithm[17, 3]. By default `polymake` gives credit to a program when it calls it for the first time. The corresponding output is omitted in some of the computations below; but we will explain which package was called in each case.

```
polytope > print $P->N_INTERIOR_LATTICE_POINTS;
1
```

Sometimes it is important to know how many of the lattice points in a polytope are contained in the interior. While the Barvinok algorithm avoids to enumerate the points, the user can force the complete enumeration. This will be computed by `4ti2`.

```
print $P->INTERIOR_LATTICE_POINTS;
1 0 0 0
```

Up to this point, none of the rules used was specific to lattice polytopes. To the contrary, all this makes perfect sense for any rational polytope.

Let us now switch to some properties that are only defined for lattice polytopes. A lattice polytope is *reflexive*, if the origin is in the interior of the polytope and all facets have integral distance one from the origin. Equivalently, a lattice polytope is reflexive, if also its polar is a lattice polytope. In algebraic geometry, these polytopes correspond to Gorenstein toric Fano varieties. These polytopes were introduced by Batyrev [4] to construct mirror pairs of *Calabi-Yau* varieties in the context of string theory. A necessary condition for a polytope to be reflexive is, that the origin is the unique interior lattice point, so the cube is a candidate.

```
polytope > print $P->REFLEXIVE;
1
```

Of course, this is not a surprise, as the polar dual of the cube (with ± 1-coordinates) is the regular octahedron, the convex hull of the standard basis vectors and their negatives. Reflexivity is a property that is only defined for lattice polytopes, and so at this point, `polymake` has internally cast the cube to the subclass `LatticePolytope`.

```
polytope > print $P->type->full_name;
LatticePolytope
```

Notice the difference to the first call of the same command at the very beginning. Many further properties of Fano varieties can be checked via the corresponding polytope. Reflexive polyhedra have been classified in dimensions up to 4, see [18]. In dimension 3, there are 124 of these, of which 18 correspond to smooth Fano varieties. The toric variety X_P is *smooth* (or *non-singular*) if every cone in the normal fan $\mathcal{N}_P$ is unimodular. A cone is *unimodular*, if its minimal integral generators can be extended to a basis of $\mathbb{Z}^n$.

```
polytope > print $P->SMOOTH;
1
```

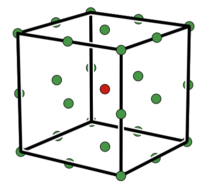

Fig. 1: The 3-dimensional cube with its lattice points. The interior lattice points are drawn in a different color.

We will now explore a different aspect of lattice polytopes. Stanley showed [24] that for any d-dimensional polytope P there is a polynomial $h^* \in \mathbb{Z}[t]$ of degree at most d with non-negative coefficients such that

$$\sum_{k \geq 0} |kP \cap \mathbb{Z}^d| t^k \;=\; \frac{h^*(t)}{(1-t)^{d+1}} \,.$$

The polynomial $h^*(t) = \sum_{k=0}^{d} h_k^* t^k$ is the h^*-*polynomial* of P. It is closely related to the Ehrhart polynomial. Some of the coefficients have a combinatorial meaning. For instance, h_d^* counts the number of interior lattice points, while the sum of all coefficients is the normalized volume of the polytope. The *normalized volume* of a d-dimensional polytope is $d!$ times the d-dimensional Euclidean volume. We can compute the coefficients for the cube (starting with the constant coefficient).

```
polytope > print $P->H_STAR_VECTOR;
1 23 23 1
polytope > print $P->LATTICE_VOLUME;
48
```

`polymake` calls `normaliz2` to compute this. The *degree* δ of P is defined as the degree of the h^*-polynomial.

```
polytope > print $P->LATTICE_DEGREE;
3
```

The value $d + 1 - \delta$ is the smallest factor by which we have to dilate P so that it has an interior lattice point. This is the *co-degree* of the polytope. `polymake` computes it with the command `LATTICE_CODEGREE`. In our case, this gives 1, as the cube contains the origin. Recent results suggest that the degree is a more relevant invariant of a lattice polytope than the dimension. For instance, it is known that for given degree d and normalized volume V or linear coefficient h_1^* there is a constant c such that any lattice polytope of dimension $d \geq c$ is a lattice pyramid (a pyramid where the apex is a lattice point with height 1 over the base) [5, 20]. The h^*-polynomials of a lattice polytope P and the lattice pyramid with base P coincide.

Finally, we can use `polymake` to draw our cube, together with its lattice points. By default the command

```
polytope > $P->VISUAL->LATTICE_COLORED;
```

triggers the visualization with `JavaView` [22]. See Figure 1.

4 Analyzing an Example

Let us look at the cone $C \subset \mathbb{R}^6$ positively spanned by the rows of the 10×6-matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 & 2 & 0 \end{pmatrix} . \quad (1)$$

The cone C is *pointed*, that is, it does not contain any line. Equivalently, C is projectively equivalent to a polytope $\bar{C}$. The rows of M are precisely the rays (or *generators*) of C, that is, they correspond to the vertices of $\bar{C}$. The key fact about C is the following.

Theorem 1 (Bruns et al. [7]) *The vector $(9, 13, 13, 13, 13, 13)$ lies in C, but it cannot be written as a non-negative integral linear combination of six generators of C.*

This says that C does not satisfy the *integral Carathéodory property*, and thus it is a counter-example to a conjecture of Sebő [23]. We will sketch how this can be verified using `polymake`. Moreover, we will reveal the combinatorial structure. There is an integral transformation which maps C to a cone with $0/1$-coordinates [7], and there is also a realization of $\bar{C}$ as a lattice polytope. Both other representations could be used in the sequel with the same results. The following command creates a new matrix object representing the matrix M above. Here the user types in the coefficient directly; alternatively, they could also be read from a file.

```
polytope > $M=new Matrix<Rational>(<<".");
polytope (2)> 0 1 0 0 0 0
polytope (3)> 0 0 1 0 0 0
polytope (4)> 0 0 0 1 0 0
polytope (5)> 0 0 0 0 1 0
polytope (6)> 0 0 0 0 0 1
polytope (7)> 1 0 2 1 1 2
polytope (8)> 1 2 0 2 1 1
polytope (9)> 1 1 2 0 2 1
polytope (10)> 1 1 1 2 0 2
polytope (11)> 1 2 1 1 2 0
polytope (12)> .
```

`polymake` can work with pointed polyhedral cones right away, so it is legal to write

```
polytope > $C=new Polytope<Rational>(POINTS=>$M);
```

C in terms of the (rows of the) matrix M. The first step is to verify that the generators actually form the Hilbert basis of C. Each integral cone admits a unique minimal family of vectors such that any integral point inside can be written as a non-negative linear combination of these. Moreover, this family is finite, and this is the *Hilbert basis* of the cone. `polymake` cannot compute Hilbert bases directly, but instead it relies of `normaliz2` [8], which uses an algorithm of Bruns and Koch [9]. The alternative implementation in `4ti2` [1] uses a lift and project approach described in [15].

```
polytope > print $C->HILBERT_BASIS;
0 0 0 0 0 1
0 0 0 0 1 0
0 0 0 1 0 0
0 0 1 0 0 0
1 0 2 1 1 2
0 1 0 0 0 0
1 1 1 2 0 2
1 1 2 0 2 1
1 2 0 2 1 1
1 2 1 1 2 0
```

The output coincides with our first input, and this says that the generators of C do form a Hilbert basis. One can show that it suffices to check if the vector $x = (9, 13, 13, 13, 13, 13)$ can be written as a non-negative integral linear combination of six *linearly independent* generators. The following `polymake` code enumerates all possibilities.

```
$x=new Vector<Rational>([9,13,13,13,13,13]);
foreach (all_subsets_of_k(6,0..9)) {
  $B=$M->minor($_,All);
  if (det($B)) {
    print lin_solve(transpose($B),$x), "\n";
  }
}
```

For each non-vanishing maximal minor B we solve the linear system of equations $yB = x$, and we print the unique solution to the screen. The resulting 185 lines of output can be checked by hand: All coefficients are integral, and each solution has at least one negative coefficient. Clearly, adding one or two more lines of code would also leave this final check to `polymake`.

In the remainder of this section we want to exploit `polymake`'s features to further investigate the cone C or rather the projectively equivalent polytope $\bar{C}$ from the combinatorial point of view. The first thing is to look at the facets (which had been computed by `cddlib` [13] before). There are 27 of them. Instead of printing them all we only look at two, and instead of printing the coordinates we list the numbers of the generators incident.

```
polytope > print $C->VERTICES_IN_FACETS->[8];
{0 1 2 3 4}
polytope > print $C->VERTICES_IN_FACETS->[22];
{5 6 7 8 9}
```

This shows that $\bar{C}$ has two disjoint facets of five vertices each. Since $\dim \bar{C} = 5$ each facet is a 4-polytope, and this shows that both facets must be simplices. The numbers of the facets depend on the sequence of the output of `cddlib`, but the numbers of the vertices correspond to the matrix M as defined above. `polymake` uses the first coordinate to homogenize. By looking at (1) we see that the first five points have a leading zero coordinate, and hence the facet numbered 8 is the face at infinity of C. There is another popular 5-polytope which happens to be the joint convex hull of two disjoint 4-dimensional simplices, and this is the 5-dimensional cross polytope.

```
polytope > $cross5 = cross(5);
polytope > print isomorphic($C->GRAPH->ADJACENCY,$cross5->GRAPH->ADJACENCY);
1
```

The vertex-edge graph of $\bar{C}$ turns out to be isomorphic (as an abstract graph) to the graph of the cross polytope. This has been verified by `polymake`'s interface to `nauty` [19]. In fact, one can show that both polytopes even share the same 2-skeleton. If we compare the f-vectors we see that the cross polytope has five more facets and five more *ridges* (faces of codimension 2) than $\bar{C}$.

```
polytope > print $cross5->F_VECTOR - $C->F_VECTOR;
0 0 0 5 5
```

This leads to a natural conjecture: What if, combinatorially, $\bar{C}$ can be constructed from the cross polytope by picking five pairs of adjacent facets and "straightening" them? Equivalently, the dual graph of $\bar{C}$ would result from the dual graph of the cross polytope by contracting a partial matching of five edges. This can be verified as follows. First let us look at two more facets or rather the set of generators incident with them.

```
polytope > print $C->VERTICES_IN_FACETS->[12];
{0 2 5 7 8}
polytope > print $C->VERTICES_IN_FACETS->[13];
{1 2 5 7 8}
```

The facets 12 and 13 with the vertices $\{0,2,5,7,8\}$ and $\{1,2,5,7,8\}$, respectively, are adjacent in the dual graph (via the common ridge with vertex set $\{2,5,7,8\}$). This edge and four others can be contracted in a copy of the dual graph of `$cross5`. Taking a copy first is necessary since `polymake`'s objects are immutable.

```
polytope > $g=new props::Graph($cross5->DUAL_GRAPH->ADJACENCY);
polytope > $g->contract_edge(12,13);
polytope > $g->contract_edge(24,26);
polytope > $g->contract_edge(17,21);
polytope > $g->contract_edge(3,11);
polytope > $g->contract_edge(6,22);
polytope > $g->squeeze;
```

It is only the last command which turns `$g` into a valid graph again. The reason for this is that `polymake`'s graphs necessarily have their nodes consecutively numbered. Contracting an edge means to destroy one node (actually the second one). Squeezing renumbers the remaining vertices properly.

```
polytope > print isomorphic($C->DUAL_GRAPH->ADJACENCY,$g);
1
```

This final computation by `nauty` explains the combinatorial structure of the cone C in Theorem 1, or the projectively equivalent polytope $\bar{C}$, completely.

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The absolute order on the hyperoctahedral group†

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Abstract. The absolute order on the hyperoctahedral group B_n is investigated. It is shown that every closed interval in this order is shellable, those closed intervals which are lattices are characterized and their zeta polynomials are computed. Moreover, using the notion of strong constructibility, it is proved that the order ideal generated by the Coxeter elements of B_n is homotopy Cohen-Macaulay and the Euler characteristic of the order complex of the proper part of this ideal is computed. Finally, an example of a non Cohen-Macaulay closed interval in the absolute order on the group D_4 is given and the closed intervals of D_n which are lattices are characterized.

Résumé. Nous étudions l'ordre absolu sur le groupe hyperoctahédral B_n. Nous montrons que chaque intervalle fermé de cet ordre est shellable, caractérisons les treillis parmi ces intervalles et calculons les polynômes zêta de ces derniers. De plus, en utilisant la notion de constructibilité forte, nous prouvons que l'idéal engendré par les éléments de Coxeter de B_n est Cohen-Macaulay pour l'homotopie, et nous calculons la caractéristique d'Euler du complexe associé à cet idéal. Pour finir, nous exhibons un exemple d'intervalle fermé non Cohen-Macaulay dans l'ordre absolu du groupe D_4, et caractérisons les intervalles fermés de D_n qui sont des treillis.

Keywords: Coxeter group, hyperoctaherdal group, absolute order, Cohen-Macaulay poset, shellability

1 Introduction and results

Coxeter groups are fundamental combinatorial structures which appear in several areas of mathematics. Partial orders on Coxeter groups often provide an important tool for understanding the questions of interest. Examples of such partial orders are the Bruhat order and the weak order. We refer the reader to [7, 10, 15] for background on Coxeter groups and their orderings.

In this work we study the absolute order. Let W be a finite Coxeter group with respect to the set $\mathcal{T}$ of *all* reflections in W. The absolute order on W is denoted by $\mathrm{Abs}(W)$ and defined as the partial order on W whose Hasse diagram is obtained from the Cayley graph of W with respect to $\mathcal{T}$ by directing its edges away from the identity (see Section 2.1 for a precise definition). The poset $\mathrm{Abs}(W)$ is locally self-dual

†The present research will be part of the author's Doctoral Dissertation at the University of Athens

and graded. It has a minimum element, the identity $e \in W$, but will typically not have a maximum, since every Coxeter element of W is a maximal element of Abs(W). Its rank function is called the absolute length and is denoted by $\ell_{\mathcal{T}}$. The absolute length and order arise naturally in combinatorics [2], group theory [5, 11], statistics [13] and invariant theory [15]. For instance, $\ell_{\mathcal{T}}(w)$ can also be defined as the codimension of the fixed space of w, when W acts faithfully as a group generated by orthogonal reflections on a vector space V by its standard geometric representation. In this case, the rank generating polynomial of Abs(W) satisfies

$$\sum_{w \in W} t^{\ell_{\mathcal{T}}(w)} = \prod_{i=1}^{\ell} (1 + e_i t),$$

where $e_1, e_2, \ldots, e_\ell$ are the exponents [15, Section 3.20] of W and ℓ is its rank. We refer to [2, Section 2.4] and [4, Section 1] for further discussion of the importance of the absolute order and related historical remarks.

In this paper we will be interested in the combinatorics and topology of Abs(W). These have been studied extensively for the interval $[e, c] := NC(W, c)$ of Abs(W), known as the poset of noncrossing partitions associated to W, where $c \in W$ denotes a Coxeter element. For instance, it was shown in [3] that $NC(W, c)$ is shellable for every finite Coxeter group W. In particular, $NC(W, c)$ is homotopy Cohen-Macaulay and the order complex of $NC(W, c) \setminus \{e, c\}$ has the homotopy type of a wedge of spheres. The problem to determine the topology of the poset Abs$(W) \setminus \{e\}$ and to decide whether Abs(W) is Cohen-Macaulay or shellable, was naturally posed by Athanasiadis (unpublished) and Reiner [1, Problem 3.1], see also [19, Problem 3.3.7]. Computer calculations carried out by Reiner showed that the absolute order is not Cohen-Macaulay for the group D_4. In the case of the symmetric group, it is still not known whether Abs(S_n) is shellable. However, the following result was obtained in [4].

[4, Theorem 1.1] *The poset Abs(S_n) is homotopy Cohen-Macaulay for all $n \geq 1$. In particular, the order complex of Abs$(S_n) \setminus \{e\}$ is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres and Cohen-Macaulay over $\mathbb{Z}$.*

Here we focus on the hyperoctahedral group B_n. Contrary to the case of the symmetric group, not every maximal element of the absolute order on B_n is a Coxeter element. The maximal intervals in Abs(B_n) include the posets $NC^B(n)$ of noncrossing partitions of type B [17] and $NC^B(p, q)$ of annular noncrossing partitions, introduced and studied recently by Nica and Oancea [16]. Our main results are as follows. In Section 3 we prove that every interval of B_n is shellable and present an example of a maximal element x of Abs(D_4) for which the interval $[e, x]$ is not Cohen-Macaulay over any field (Example 3.3). In Section 4 we comment on the proof a B_n-analogue of [4, Theorem 1.1], stating that the order ideal $\mathcal{J}_n$ of Abs(B_n) generated by the set of Coxeter elements of B_n is homotopy Cohen-Macaulay for all $n \geq 2$ (see Theorem 4.1). In particular, the order complex of $\mathcal{J}_n \setminus \{e\}$ is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres and Cohen-Macaulay over $\mathbb{Z}$. The number of such spheres is also computed (see Theorem 4.8). We conjecture that the poset Abs(B_n) is Cohen-Macaulay for every $n \geq 2$ [(i)]. Finally, in Section 5 we characterize the maximal intervals of Abs(B_n) and Abs(D_n) which are lattices and compute some of their enumerative invariants. We refer the reader to [18, Chapter 3] and [9, 19] for background on partially ordered sets and the topology of simplicial complexes, respectively.

[(i)] This conjecture has now been proved by the author

2 Preliminaries

2.1 The absolute length and absolute order

Let W be a finite Coxeter group with set of all reflections $\mathcal{T}$. Given $w \in W$, let $\ell_{\mathcal{T}}(w)$ denote the smallest integer k such that w can be written as a product of k reflections in $\mathcal{T}$. The *absolute order*, or *reflection length order*, is the partial order on W denoted by $\preceq$ and defined by letting

$$u \preceq v \quad \text{if and only if} \quad \ell_{\mathcal{T}}(u) + \ell_{\mathcal{T}}(u^{-1}v) = \ell_{\mathcal{T}}(v)$$

for $u, v \in W$. Equivalently, $\preceq$ is the partial order on W with covering relations $w \to wt$, where $w \in W$ and $t \in \mathcal{T}$ are such that $\ell_{\mathcal{T}}(w) < \ell_{\mathcal{T}}(wt)$. In that case we write $w \xrightarrow{t} wt$. The poset Abs(W) is graded with rank function $\ell_{\mathcal{T}}$.

2.2 The posets Abs(B_n) and Abs(D_n)

We view the hyperoctahedral group B_n as the group of permutations u of the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $u(-i) = -u(i)$ for every $1 \leq i \leq n$. Following [11], the permutation which has cycle form $(a_1\, a_2\, \cdots\, a_k)(-a_1\, -a_2\, \cdots\, -a_k)$ is denoted by $(\!(a_1, a_2, \ldots, a_k)\!)$ and is called a *paired* k-cycle, while the cycle $(a_1\, a_2\, \cdots\, a_k\, -a_1\, -a_2\, \cdots\, -a_k)$ is denoted by $[a_1, a_2, \ldots, a_k]$ and is called a *balanced* k-cycle. Every element $u \in B_n$ can be written (uniquely) as a product of disjoint paired or balanced cycles, called cycles of u. With this notation, the set $\mathcal{T}$ of reflections of B_n is equal to the union

$$\{[i] : 1 \leq i \leq n\} \cup \{(\!(i, j)\!), (\!(i, -j)\!) : 1 \leq i < j \leq n\}. \tag{1}$$

The length $\ell_{\mathcal{T}}(u)$ of $u \in B_n$ is equal to $n - \gamma(u)$, where $\gamma(u)$ denotes the number of paired cycles in the cycle decomposition of u. An element $u \in B_n$ is maximal in Abs(B_n) if and only if it can be written as a product of disjoint balanced cycles whose lengths sum to n. The Coxeter elements of B_n are precisely the balanced n-cycles. To simplify the notation, we will denote by ℓ the absolute length $\ell_{\mathcal{T}}$. The covering relations $w \xrightarrow{t} wt$ of Abs(B_n), when w and t are non-disjoint cycles, can be described as follows: for $1 \leq i < j \leq m \leq n$ we have:

(a) $(\!(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)\!) \xrightarrow{(\!(a_{i-1}, a_i)\!)} (\!(a_1, \ldots, a_m)\!)$

(b) $(\!(a_1, \ldots, a_m)\!) \xrightarrow{[a_i]} [a_1, \ldots, a_{i-1}, a_i, -a_{i+1}, \ldots, -a_m]$

(c) $(\!(a_1, \ldots, a_m)\!) \xrightarrow{(\!(a_i, -a_j)\!)} [a_1, \ldots, a_i, -a_{j+1}, \ldots, -a_m][a_{i+1}, \ldots, a_j]$

(d) $[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m] \xrightarrow{(\!(a_{i-1}, a_i)\!)} [a_1, \ldots, a_m]$

(e) $[a_1, \ldots, a_j](\!(a_{j+1}, \ldots, a_m)\!) \xrightarrow{(\!(a_j, a_m)\!)} [a_1, \ldots, a_m]$

where $a_1, \ldots, a_m$ are elements of the set $\{\pm 1, \ldots, \pm n\}$ with pairwise distinct absolute values.

The Coxeter group D_n is the subgroup of index two of the group B_n generated by the set of reflections

$$\{(\!(i, j)\!), (\!(i, -j)\!) : 1 \leq i < j \leq n\}. \tag{2}$$

(these are all reflections in D_n). The absolute length on D_n is the restriction of absolute length of B_n on the set D_n. The number of balanced cycles of any element $u \in D_n$ is even and every Coxeter element of D_n has the form $[a_1, a_2, \ldots, a_{n-1}][a_n]$, where $|a_i| \in \{1, 2, \ldots, n\}$ and $|a_i| \neq |a_j|$ for all $i \neq j$.

3 Shellability

In this section we prove the following theorem.

Theorem 3.1 *Every interval of Abs*(B_n) *is shellable.*

Proof: (sketch) We show that every closed interval of Abs(B_n) admits an EL-labeling. The result then follows, since EL-shellability implies shellability (we refer to [8] for the definition of EL-labeling and EL-shellability).

Let $C(B_n)$ be the set of covering relations of Abs(B_n) and $(a, b) \in C(B_n)$. Then $a^{-1}b$ is a reflection of B_n, thus either $a^{-1}b = [i]$ for some $i \in \{1, 2, \ldots, n\}$, or there exist $i, j \in \{1, 2, \ldots, n\}$, with $i < j$, such that $a^{-1}b = ((i, j))$ or $a^{-1}b = ((i, -j))$. We define a map $\lambda : C(B_n) \to \{1, 2, \ldots, n\}$ as follows:

$$\lambda(a, b) = \begin{cases} i & \text{if } a^{-1}b = [i], \\ j & \text{if } a^{-1}b = ((i, j)) \text{ or } ((i, -j)). \end{cases}$$

A similar labeling was used by Biane [6] in order to study the maximal chains of the poset $NC^B(n)$ of noncrossing B_n-partitions. Figure 1 illustrates the Hasse diagram of the interval $[e, x]$, for $n = 4$ and $x = [3, -4]((1, 2))$, together with the corresponding labels.

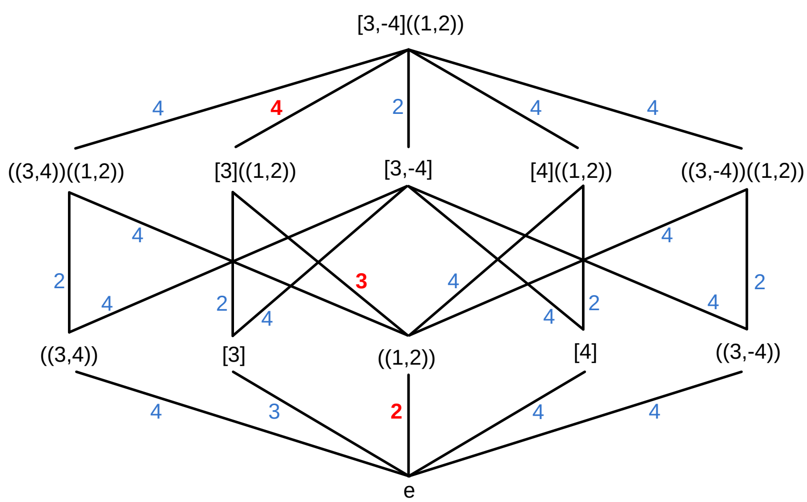

Fig. 1: The interval $[e, x]$ for $x = [3, -4]((1, 2))$

The restricion of the map λ to the interval $[x, y]$ is an EL-labeling for all $x, y \in B_n$ with $x \preceq y$. To prove that, it suffices to show that for every $u \in B_n$ the map $\lambda|_{[e,u]}$ is an EL-labeling. Indeed, let

$x, y \in B_n$ with $x \preceq y$ and define the map $\phi : [x, y] \to [e, x^{-1}y]$ by $\phi(t) = x^{-1}t$. Clearly, ϕ is a poset isomorphism. Moreover, if $(a, b) \in C([x, y])$, then $\phi(a)^{-1}\phi(b) = (x^{-1}a)^{-1}x^{-1}b = a^{-1}xx^{-1}b = a^{-1}b$, which implies that $\lambda(a, b) = \lambda(\phi(a), \phi(b))$.

Let $u = b_1 b_2 \cdots b_k\, p_1 p_2 \cdots p_l$ be written as a product of disjoint cycles, where $b_i = [b_i^1, \ldots, b_i^{k_i}]$ for $i \leq k$ and $p_j = (\!(p_j^1, \ldots, p_j^{l_j})\!)$ with $p_j^1 = \min\{|p_j^m| : 1 \leq m \leq l_j\}$ for $j \leq l$. We consider the sequence of positive integers obtained by placing the numbers $|b_i^h|$ and $|p_j^m|$, for $i, j, h \geq 1$ and $m > 1$, in increasing order. There are $r = \ell(u)$ such integers. To simplify the notation, we denote by $c(u) = (c_1, c_2, \ldots, c_r)$ this sequence and say that c_μ ($\mu = 1, 2, \ldots, r$) belongs to a balanced (respectively paired) cycle if it is equal to some $|b_i^h|$ (respectively $|p_j^m|$). Clearly we have $c_1 < c_2 < \cdots < c_r$ and $\lambda(a, b) \in \{c_1, c_2, \ldots, c_r\}$ for every pair $a, b \in [e, u]$, with $a \to b$. To the sequence $(c_1, c_2, \ldots, c_r)$ corresponds a unique maximal chain

$$\mathcal{C}_u : \; u_0 = e \xrightarrow{c_1} u_1 \xrightarrow{c_2} u_2 \xrightarrow{c_3} \cdots \xrightarrow{c_r} u_r = u,$$

which can be constructed inductively as follows (here, the integer κ in $a \xrightarrow{\kappa} b$ denotes the label $\lambda(a, b)$). If c_1 belongs to a balanced cycle, then $u_1 = [c_1]$. Otherwise, if c_1 belongs to some p_i, say p_1, then we set u_1 to be either $(\!(p_1^1, c_1)\!)$ or $(\!(p_1^1, -c_1)\!)$, so that $u_1 \preceq p_1$ holds. In both cases $\lambda(e, u_1) = c_1$ and $\lambda(e, u_1) < \lambda(e, x)$ for any other atom $x \in [e, u]$. Suppose now that we have uniquely defined the elements $u_1, u_2, \ldots, u_j$, so that for every $i = 1, 2, \ldots, j$ we have $u_{i-1} \to u_i$ with $\lambda(u_{i-1}, u_i) = c_i$ and $\lambda(u_{i-1}, u_i) < \lambda(u_{i-1}, x)$ for every $x \in [e, u]$ such that $x \neq u_i$ and $u_{i-1} \to x$. We consider the number c_{j+1} and distinguish two cases.

Case 1: c_{j+1} belongs to a cycle whose elements have not been used. In this case, if c_{j+1} belongs to a balanced cycle, then we set $u_{j+1} = u_j[c_{j+1}]$, while if c_{j+1} belongs to p_s for some $s \in \{1, 2, \ldots, l\}$, then we set u_{j+1} to be either $u_j\,(\!(p_s^1, c_{j+1})\!)$ or $u_j\,(\!(p_s^1, -c_{j+1})\!)$, so that $u_j^{-1}u_{j+1} \preceq p_s$ holds.

Case 2: c_{j+1} belongs to a cycle some element of which has been used. Then there exist an $i < j+1$ such that c_i belongs to the same cycle as c_{j+1}. If c_i, c_{j+1} belong to some b_s, then there is a balanced cycle of u_j, say a, that contains c_i. In this case we set u_{j+1} to be the permutation that we obtain from u_j if we add the number c_{j+1} in the cycle a in the same order and with the same sign that it appears in b_s. We proceed similarly if c_i, c_{j+1} belong the same paired cycle.

Using the relations written in Section 2.2, one can show that $\mathcal{C}_u$ is lexicographically first and the unique strictly increasing chain in $[e, u]$. Thus Theorem 3.1 is proved. □

Example 3.2 (i) Let $n = 7$ and $u = [1, -7][3](\!(2, -6, -5)\!)(\!(4)\!) \in B_7$. Then $c(u) = (1, 3, 5, 6, 7)$ and $\mathcal{C}_u : e \xrightarrow{1} [1] \xrightarrow{3} [1][3] \xrightarrow{5} [1][3](\!(2, -5)\!) \xrightarrow{6} [1][3](\!(2, -6, -5)\!) \xrightarrow{7} u$.

(ii) Let $n = 4$ and $v = [3, -4](\!(1, 2)\!)$. Then $c(v) = (2, 3, 4)$ and $\mathcal{C}_v : e \xrightarrow{2} (\!(1, 2)\!) \xrightarrow{3} (\!(1, 2)\!)[3] \xrightarrow{4} v$.

Example 3.3 Figure 2 illustrates the Hasse diagram of the interval $I = [e, x]$ of Abs(D_4), where $x = [1][2][3][4]$. Note that the Hasse diagram of the open interval (e, x) is disconnected and, therefore, I is not Cohen-Macaulay. It follows that Abs(D_n) is neither Cohen-Macaulay nor shellable for $n \geq 4$ [19, Corollary 3.1.9]. This is in accordance with Reiner's computations showing that Abs(D_4) is not Cohen-Macaulay and answers in the negative a question raised by Athanasiadis (personal communication), asking whether all intervals of the absolute order on a Coxeter group are shellable.

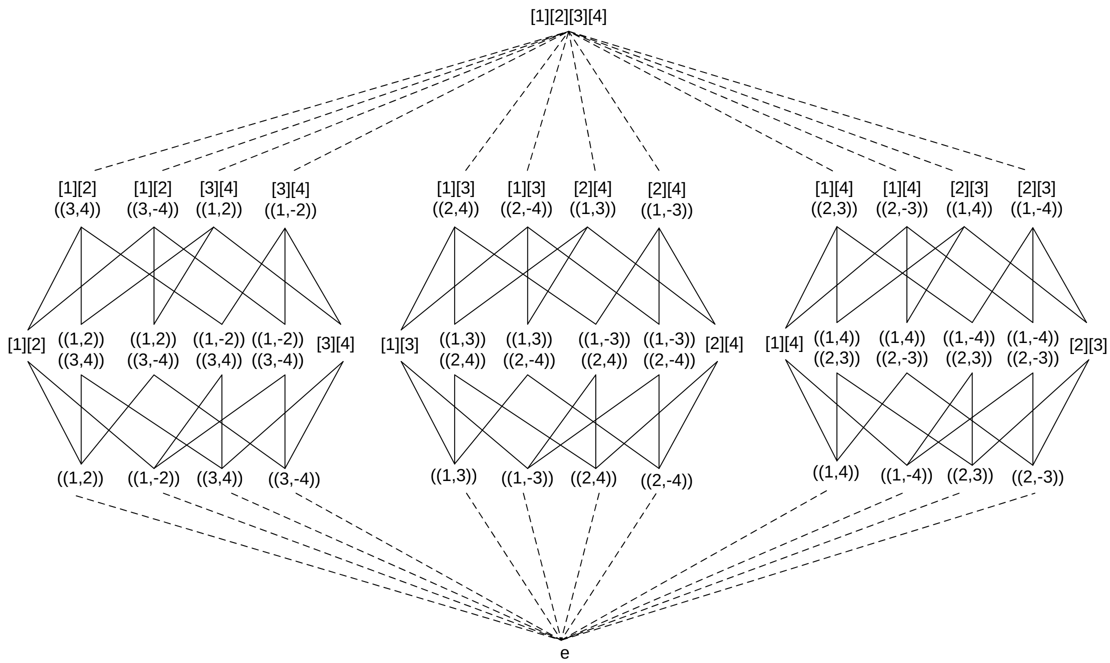

Fig. 2: The interval $[e, [1][2][3][4]]$ in D_4

4 The ideal of Coxeter elements

Recall that the Coxeter elements of B_n are precisely the balanced n-cycles.

Theorem 4.1 *The order ideal $\mathcal{J}_n$ of Abs(B_n) generated by the set of Coxeter elements of B_n is homotopy Cohen-Macaulay for all $n \geq 2$. In particular, the order complex of $\mathcal{J}_n \setminus \{e\}$ is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres and Cohen-Macaulay over $\mathbb{Z}$.*

Since the set of maximal elements of Abs(S_n) coincides with the set of Coxeter elements of S_n, Theorem 4 can be considered as a B_n-analogue of [4, Theorem 1.1]. It is not known whether the order ideal generated by the Coxeter elements is Cohen-Macaulay for every Coxeter group W. To prove Theorem 4.1 we will use the notion of strong constructibility, introduced in [4]. We first review some definitions and results given in [4].

Definition 4.2 *A d-dimensional simplicial complex Δ is constructible if it is a simplex or it can be written as $\Delta = \Delta_1 \cup \Delta_2$, where Δ_1, Δ_2 are d-dimensional constructible simplicial complexes such that $\Delta_1 \cap \Delta_2$ is constructible of dimension at least $d-1$.*

We do not know whether this notion of constructibility coincides with the classical notion, which differs in that the dimension of the intersection $\Delta_1 \cap \Delta_2$ has to equal to $d-1$. However, it is proved in [4] that every constructible simplicial complex, in the sense of Definition 4.2, is homotopy Cohen-Macaulay. Figure 3 illustrates two 2-dimensional strongly constructible complexes, Δ_1 and Δ_2, the intersection of which is the 2-dimensional simplex F_3. Thus, the union $\Delta_1 \cup \Delta_2$ is strongly constructible as well.

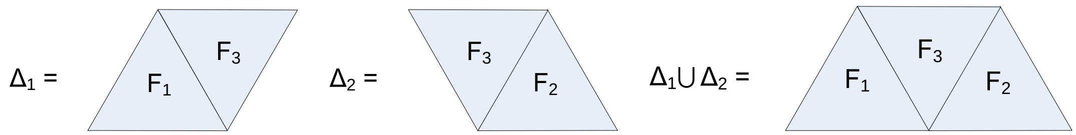

Fig. 3: A 2-dimensional strongly constructible simplicial complex

Definition 4.3 *A finite poset P of rank d with a minimum element is strongly constructible if it is bounded and pure shellable or it can be written as a union $P = I_1 \cup I_2$ of two strongly constructible proper ideals I_1, I_2 of rank d, such that $I_1 \cap I_2$ is strongly constructible of rank at least $d-1$.*

Proposition 4.4 *The order complex of any strongly constructible poset is constructible.*

Remark 4.5 Every strongly constructible poset is homotopy Cohen-Macaulay.

Proposition 4.6 *The poset Abs(S_n) is strongly constructible for every $n \geq 1$.*

The main idea to prove Proposition 4.6 is to partition the set of maximal elements (n-cycles) of Abs(S_n) by placing x and y in the same part of the partition if $x(1) = y(1)$. This is the partition of S_n into the left cosets of the subgroup which consists of the permutations of the set $\{2, 3, \ldots, n\}$. Then we show that the order ideal generated by each part is strongly constructible and that so is the intersection of two or more of these ideals. We extend this construction to the case of $\mathcal{J}_n \subset B_n$ by defining the following equivalence relation on the set of cycles of B_n.

Definition 4.7 *Given cycles u, v of B_n, we write $u \sim v$ if*

- *u, v are either both paired or both balanced cycles and*
- *$u(i) = \pm v(i)$ for every $i = 1, 2, \ldots, n$.*

We denote by $\bar{u}$ the equivalence class of $u \in B_n$. If $u_1, u_2, \ldots, u_k$ are disjoint cycles of B_n, we set $\bar{u}_1 \bar{u}_2 \cdots \bar{u}_k = \{v_1 v_2 \cdots v_k : v_i \in \bar{u}_i, i = 1, 2, \ldots, k\} \subset B_n$. For example,

$$\overline{(\!(1,2)\!)}\,\overline{[3,4]} = \{(\!(1,2)\!)[3,4], (\!(1,-2)\!)[3,4], (\!(1,2)\!)[3,-4], (\!(1,-2)\!)[3,-4]\}.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a sequence of distinct positive integers, with $\alpha_i \leq n$ for every $i = 1, 2, \ldots, n$. To the sequence α we associate the permutations $[\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_k]$ and $(\!(\alpha)\!) = (\!(\alpha_1, \alpha_2, \ldots, \alpha_k)\!)$ of B_n and the cycle $(\alpha) = (\alpha_1\, \alpha_2 \cdots \alpha_k)$ of S_n. Let A be a subset of S_n consisting of permutations that have length equal to $n-k$ (i.e. permutations of S_n that have exactly k cycles in their decomposition). To the order ideal $\langle A \rangle$ of Abs(S_n), which has rank $n-k$, we associate the order ideal $\overline{\langle A \rangle}$ of Abs(B_n), which has rank $n-k+1$ and is defined as:

$$\overline{\langle A \rangle} := \langle x \in \overline{[\alpha_1]}\,\overline{(\!(\alpha_2)\!)}\,\overline{(\!(\alpha_3)\!)} \cdots \overline{(\!(\alpha_k)\!)} : (\alpha_1)(\alpha_2)(\alpha_3)\cdots(\alpha_k) \in A \rangle.$$

Let A_1, A_2 be subsets of S_n as above. Then $\overline{\langle A_1 \rangle} \cup \overline{\langle A_2 \rangle} = \overline{\langle A_1 \cup A_2 \rangle}$ and $\overline{\langle A_1 \rangle} \cap \overline{\langle A_2 \rangle} = \overline{\langle \langle A_1 \rangle \cap \langle A_2 \rangle \rangle}$. This connection allows us to compute intersections of ideals generated by certain equivalence classes, using the intersections of ideals generated by the corresponding cycles in S_n and to adapt the proof of [4, Proposition 4.2].

The next result is a B_n-analogue of [4, Theorem 1.2].

Theorem 4.8 *Let $\mathcal{J}_n$ denote the order ideal of Abs(B_n) generated by the Coxeter elements of B_n and $\bar{\mathcal{J}}_n = \mathcal{J}_n \setminus \{\hat{0}\}$. The reduced Euler characteristic of the order complex $\Delta(\bar{\mathcal{J}}_n)$ satisfies*

$$\sum_{n\geq 2}(-1)^n \tilde{\chi}(\Delta(\bar{\mathcal{J}}_n))\frac{t^n}{n!} = 1 - \sqrt{C(2t)}\exp\{-2tC(2t)\}\left(1+\sum_{n\geq 1}2^{n-1}\binom{2n-1}{n}\frac{t^n}{n}\right),$$

where $C(t) = \frac{1}{2t}\left(1-\sqrt{1-4t}\right)$ is the ordinary generating function for the Catalan numbers.

5 Combinatorics of intervals

5.1 Intervals with the lattice property

In this section we characterize the maximal intervals in Abs(B_n) and Abs(D_n) which are lattices. It is known that the interval $[e, c]$ of Abs(W) is a lattice for every finite Coxeter group W and Coxeter element c of W (see [5, Fact 2.3.1], [11, Section 4], [12]). Moreover, it was shown in [16, Theorem 1.6] that $[e, x]$ is a lattice for every maximal element x of Abs(B_n) that is a product of exactly two Coxeter elements, one of which is a reflection.

Theorem 5.1 *Let x be a maximal element of Abs(B_n). The interval $[e, x]$ of Abs(B_n) is a lattice if and only if x has the form*

$$x = [a_1, a_2, \ldots, a_k][a_{k+1}][a_{k+2}]\cdots[a_n]$$

where $k \in \{0, 1, \ldots, n\}$ and the $a_i \in \{\pm 1, \pm 2, \ldots, \pm n\}$ have pairwise disjoint absolute values.

We now consider the absolute order on the group D_n and prove the following theorem.

Theorem 5.2 *Let x be a maximal element of Abs(D_n). The interval $[e, x]$ of Abs(D_n) is a lattice if and only if x is a Coxeter element or $n = 4$ and $x = [1][2][3][4]$.*

Proof: As previously mentioned, the interval $[e, x]$ of Abs(D_n), where x ia a Coxeter element of D_n, is known to be a lattice. Let x be a maximal non-Coxeter element of Abs(D_n) such that the interval $[e, x]$ of Abs(D_n) is a lattice. One can show that in this case at most one cycle of x is not a reflection. Thus we may write $x = [a_1, a_2, \ldots, a_m][b_2]\cdots[b_k]$, where $k > 2$ and $m + k - 1 = n$. Suppose that $m \geq 2$. Then $u = [a_1, a_2][b_2]$ and $v = [a_1, a_2][b_3]$ are elements of $[e, x]$. However, the intersection $[e, u] \cap [e, v] \subset$ Abs(D_n) has two maximal elements, namely the paired reflections $(\!(a_1, a_2)\!)$ and $(\!(a_1, -a_2)\!)$. This implies that the elements u and v do not have a meet in $[e, x]$ and, therefore, the interval $[e, x]$ is not a lattice. Thus we must have $m = 1$, so $k = n$ and $x = [1][2]\cdots[n]$. Suppose that $n \geq 5$. We consider the elements $u = [1][2][3][4]$ and $v = [1][2][3][5]$ of $[e, x]$ and note that the intersection $[e, u] \cap [e, v]$ has three maximal elements, namely $[1][2]$, $[1][3]$ and $[2][3]$. This implies that the interval $[e, x]$ is not a lattice, contradicting our assumption. Thus $n = 4$ and $x = [1][2][3][4]$. Figure 2 shows that the interval $[e, [1][2][3][4]]$ is indeed a lattice and Theorem 5.2 is proved. □

Remark 5.3 For the remainder of this paper we denote by $L(k, r)$ the order ideal of Abs(B_n) generated by the element $[1, 2, \ldots, k][k + 1]\cdots[k + r]$, where k, r are nonnegative integers such that $k + r \leq n$. Moreover, we set $L_n := L(0, n) = [e, [1][2]\cdots[n]]$.

5.2 The lattice L_n

We compute some of the basic enumerative invariants of the lattice L_n, as follows.

Proposition 5.4 For the lattice L_n the following hold:

(i) The number of elements of L_n is equal to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{n-k} (2k-1)!!.$$

(ii) The number of elements of L_n of rank r is equal to

$$\sum_{k=0}^{\min\{r,n-r\}} \frac{n!}{k!(r-k)!(n-r-k)!}.$$

(iii) The zeta polynomial of L_n is given by the formula

$$Z_n(m) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} m^{n-k} (m-1)^k (2k-1)!!.$$

(iv) The number of maximal chains of L_n is equal to

$$n! \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!.$$

(v) For the Möbius function of L_n we have

$$\mu_n(\hat{0}, \hat{1}) = (-1)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^k (2k-1)!!,$$

where $\hat{0}$ and $\hat{1}$ denotes the minimum and the maximum element of L_n, respectively.

Remark 5.5 By the proof of Theorem 3.1, the lattice L_n is EL-shellable. We describe two more EL-labelings for L_n.

- Let $\Lambda = \{[i] : i = 1, 2, \ldots, n\} \cup \{(\!(i,j)\!) : 1 \leq i < j \leq n\}$. We linearly order the elements of Λ in the following way. We first order the balanced reflections so that $[i] <_\Lambda [j]$ if and only if $i < j$. Then we order the paired reflections lexicographically. Finally, we set $[n] <_\Lambda (\!(1,2)\!)$. The map $\lambda_1 : C(B_n) \to \Lambda$ defined as

$$\lambda_1(a,b) = \begin{cases} [i] & \text{if } a^{-1}b = [i], \\ (\!(i,j)\!) & \text{if } a^{-1}b = (\!(i,j)\!) \text{ or } (\!(i,-j)\!) \end{cases}$$

  is an EL-labeling for L_n.

- Let $\mathcal{T}$ be the set of reflections of B_n. We define a total order $<_{\mathcal{T}}$ on $\mathcal{T}$ which extends the order $<_{\Lambda}$, by ordering the reflections $(\!(i,-j)\!)$, for $1 \leq i < j \leq n$, lexicographically and letting $(\!(n-1,n)\!) <_{\mathcal{T}} (\!(1,-2)\!)$. Let t_i be the i-th reflection in the order above. We define a map $\lambda_2 : C(B_n) \to \{1,2,\ldots,n^2\}$ as

$$\lambda_2(a,b) = \min_{1 \leq i \leq n^2} \{i : \, t_i \vee a = b\},$$

where $t_i \vee a$ denotes the join of t_i and a in the lattice L_n. The map λ_2 is an EL-labeling for L_n.

See Figure 4 for an example of these two EL-labelings when $n = 2$.

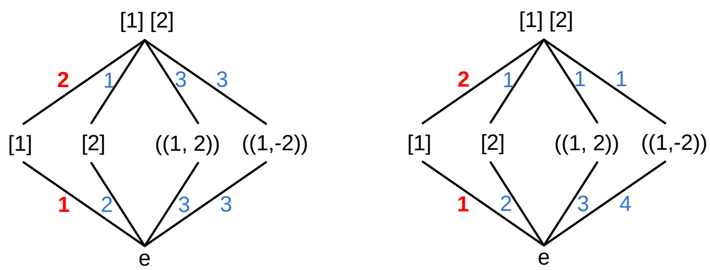

Fig. 4: EL-labelings for L_2

5.3 Enumerative combinatorics of $L(k,r)$

In this section we compute the cardinality, zeta polynomial and Möbius function of $L(k,r)$, where k,r are nonnegative integers with $k+r=n$. The case $k=n-1$ was treated by Goulden, Nica and Oancea [14]. We will use their results, as well as the formulas for the cardinality and zeta polynomial for $NC^B(n)$ and Proposition 5.4, to find the corresponding formulas for $L(k,r)$.

Proposition 5.6 *Let $\alpha_r = |L_r|$, $\beta_r(m) = Z(L_r,m)$ and $\mu_r = \mu_r(L_r)$, where $\alpha_r = \beta_r(m) = \mu_r = 1$ for $r \in \{0,1\}$. For fixed nonnegative integers k,r such that $k+r=n$, the cardinality, zeta polynomial and Möbius function of the lattice*

$$L(k,r) = [e, [1,2,\ldots,k][k+1]\cdots[k+r]]$$

are given by:

- $\#L(k,r) = \binom{2k}{k}\left(\dfrac{2\,r\,k}{k+1}\,\alpha_{r-1} + a_r\right),$

- $Z(L(k,r),m) = \binom{mk}{k}\left(\dfrac{2\,r\,k}{k+1}(m-1)\,\beta_{r-1}(m) + \beta_r(m)\right),$

- $\mu(L(k,r)) = (-1)^n\binom{2k-1}{k}\left(\dfrac{4\,r\,k}{k+1}\,|\mu_{r-1}| + |\mu_r|\right).$

Proof: (sketch) We denote by A the subset of $L(k,r)$ which consists of elements x with the following property: every cycle of x that contains at least one of $\pm 1, \pm 2, \ldots, \pm k$ is less than or equal to the element $[1,2,\ldots,k]$ in Abs(B_n). Let $x = x_1 x_2 \cdots x_\nu \in A$, written as a product of disjoint cycles. Without loss of generality, we may assume that there is a $t \in \{0,1,\ldots,\nu\}$ such that $x_1 x_2 \cdots x_t \preceq [1,2,\ldots,k]$ and $x_{t+1} x_{t+2} \cdots x_\nu \preceq [k+1][k+2]\cdots[k+r]$. Clearly, there exist a poset isomorphism

$$\begin{aligned} f : A \to NC^B(k) \ &\times \ [e, [k+1]\cdots[k+r]] \\ x \mapsto (x_1 \cdots x_t \quad &, \quad x_{t+1}\cdots x_\nu), \end{aligned}$$

so that

$$A \cong NC^B(k) \times L_r. \tag{3}$$

Let $C := L(k,r) \setminus A$ and $x = x_1 x_2 \cdots x_\nu \in C$, written as a product of disjoint cycles. Then there exists a paired cycle of x, say x_1, and a reflection $((i,j))$ with $|j| \in \{1,2,\ldots,k\}$, $j \in \{k+1, k+2, \ldots, k+r\}$, such that $((i,j)) \preceq x_1$. Note that the cycle x_1 and the reflection $((i,j))$ are unique with this property. For every $j \in \{k+1, k+2, \ldots, k+r\}$ denote by C_j the set of permutations $x \in L(k,r)$ which have a cycle, say x_1, such that $((i,j)) \preceq x_1$ for some $i \in \{\pm 1, \pm 2, \ldots, \pm k\}$. Thus, for every $x \in C$ there exists an ordering $x_1, x_2, \ldots, x_\nu$ of the cycles of x and a unique index $t \in \{1,2,\ldots,\nu\}$ such that $x_1 x_2 \cdots x_t \preceq [1,2,\ldots,k][j]$ and $x_{t+1} x_{t+2} \cdots x_\nu \preceq [k+1][k+2]\cdots[j-1][j+1]\cdots[k+r]$. Let

$$E_j = \{x \in C : x \preceq [1,2,\ldots,k][j]\}.$$

Clearly, there exist a poset isomorphism

$$\begin{aligned} g_j : C_j \to \quad E_j \quad &\times \ [e, [k+1]\cdots[j-1][j+1]\cdots[k+r]] \\ x \mapsto (x_1 \cdots x_t \quad &, \quad x_{t+1}\cdots x_\nu), \end{aligned}$$

so that

$$C_j \cong E_l \times L(0, r-1). \tag{4}$$

The results follow by using the poset isomorphisms (3) and (4) and [14, Section 5]. □

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Rationality, irrationality, and Wilf equivalence in generalized factor order

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Abstract. Let P be a partially ordered set and consider the free monoid P^* of all words over P. If $w, w' \in P^*$ then w' is a factor of w if there are words u, v with $w = uw'v$. Define generalized factor order on P^* by letting $u \leq w$ if there is a factor w' of w having the same length as u such that $u \leq w'$, where the comparison of u and w' is done componentwise using the partial order in P. One obtains ordinary factor order by insisting that $u = w'$ or, equivalently, by taking P to be an antichain.

Given $u \in P^*$, we prove that the language $\mathcal{F}(u) = \{w \ : \ w \geq u\}$ is accepted by a finite state automaton. If P is finite then it follows that the generating function $F(u) = \sum_{w \geq u} w$ is rational. This is an analogue of a theorem of Björner and Sagan for generalized subword order.

We also consider $P = \mathbb{P}$, the positive integers with the usual total order, so that $\mathbb{P}^*$ is the set of compositions. In this case one obtains a weight generating function $F(u; t, x)$ by substituting tx^n each time $n \in \mathbb{P}$ appears in $F(u)$. We show that this generating function is also rational by using the transfer-matrix method. Words u, v are said to be Wilf equivalent if $F(u; t, x) = F(v; t, x)$ and we can prove various Wilf equivalences combinatorially.

Björner found a recursive formula for the Möbius function of ordinary factor order on P^*. It follows that one always has $\mu(u, w) = 0, \pm 1$. Using the Pumping Lemma we show that the generating function $M(u) = \sum_{w \geq u} |\mu(u, w)| w$ can be irrational.

Résumé. Soit P un ensemble partiellement ordoné. Nous considérons le monoïde libre P^* de tous les mots utilisant P comme alphabet. Si $w, w' \in P^*$, on dit que w' est un facteur de w s'il y a des mots u, v avec $w = uw'v$. Nous definissons l'ordre facteur généralisé sur P^* par: $u \leq w$ s'il y a un facteur w' de w ayant la même longueur que u tel que $u \leq w'$, où la comparison de u avec w' est faite lettre par lettre utilisant l'ordre en P. On obtient l'ordre facteur usuel si on insiste que $u = w'$ ou, ce qui est la même chose, en prenant P comme antichaîne.

Pour n'importe quel $u \in P^*$, nous démontrons que le langage $\mathcal{F}(u) = \{w \ : \ w \geq u\}$ est accepté par un automaton avec un nombre fini d'états. Si P est fini, ca implique que la fonction génératrice $F(u) = \sum_{w \geq u} w$ est rationnelle. Björner et Sagan ont démontré le théorème analogue pour l'ordre où, en la définition au-dessus, w' est un sous-mot de w.

Nous considérons aussi le cas $P = \mathbb{P}$, les entiers positifs avec l'ordre usuel, donc P^* est l'ensemble des compositions. En ce cas on obtient une fonction génératrice pondéré $F(u; t, x)$ en remplaçant tx^n chaque fois on trouve $n \in \mathbb{P}$

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en $F(u)$. Nous démontrons que cette fonction génératrice est aussi rationnelle en utilisant la Méthode Matrice de Tranfert. On dit que let mots u, v sont Wilf-équivalents si $F(u;t,x) = F(v;t,x)$. Nous pouvons démontré quelques équivalances dans une manière combinatorie.

Björner a trouvé une formule recursive pour la fonction Möbius de l'ordre facteur usuel sur P^*. Cette formule implique qu'on a toujours $\mu(u,w) = 0, \pm 1$. En utilisant le Lemme de Pompage, nous démontrons que la fonction génératrice $M(u) = \sum_{w \geq u} |\mu(u,w)|w$ peut être irrationnelle.

Keywords: composition, factor order, finite state automaton, partially ordered set, rational generating function, Wilf equivalence

1 Introduction and definitions

Let P be a set and consider the corresponding *free monoid* or *Kleene closure* of all words over P:

$$P^* = \{w = w_1 w_2 \ldots w_\ell \ : \ \ell \geq 0 \text{ and } w_i \in P \text{ for all } i\}.$$

Let ϵ be the empty word and for any $w \in P^*$ we denote its cardinality or *length* by $|w|$. Given $w, w' \in P^*$, we say that w' is a *factor* of w if there are words u, v with $w = uw'v$, where adjacency denotes concatenation. For example, $w' = 322$ is a factor of $w = 12213221$ starting with the fifth element of w. *Factor order* on P^* is the partial order obtained by letting $u \leq_{\text{fo}} w$ if and only if there is a factor w' of w with $u = w'$.

Now suppose that we have a poset $(P, \leq)$. We define *generalized factor order* on P^* by letting $u \leq_{\text{gfo}} w$ if there is a factor w' of w such that

(a) $|u| = |w'|$, and

(b) $u_i \leq w'_i$ for $1 \leq i \leq |u|$.

We call w' an *embedding* of u into w, and if the first element of w' is the jth element of w, we call j an *embedding index* of u into w. We also say that in this embedding u_i is in *position* $j + i - 1$. To illustrate, suppose $P = \mathbb{P}$, the positive integers with the usual order relation. If $u = 322$ and $w = 12213431$ then $u \leq_{\text{gfo}} w$ because of the embedding factor $w' = 343$ which has embedding index 5, and the two 2's of u are in positions 6 and 7. Note that we obtain ordinary factor order by taking P to be an antichain. Also, we will henceforth drop the subscript gfo since context will make it clear what order relation is meant. Generalized factor order is the focus of this extended abstract.

Returning to the case where P is an arbitrary set, let $\mathbb{Z}\langle\langle P \rangle\rangle$ be the algebra of formal power series with integer coefficients and having the elements of P as noncommuting variables. In other words,

$$\mathbb{Z}\langle\langle P \rangle\rangle = \left\{ f = \sum_{w \in P^*} c_w w \ : \ c_w \in \mathbb{Z} \text{ for all } w \right\}.$$

If $f \in \mathbb{Z}\langle\langle P \rangle\rangle$ has no constant term, i.e., $c_\epsilon = 0$, then define

$$f^* = \epsilon + f + f^2 + f^3 + \cdots = (\epsilon - f)^{-1}.$$

(We need the restriction on f to make sure that the sums are well defined as formal power series.) We say that f is *rational* if it can be constructed from the elements of P using only a finite number of applications of the algebra operations and the star operation.

A *language* is any $\mathcal{L} \subseteq P^*$. It has an associated generating function

$$f_{\mathcal{L}} = \sum_{w \in \mathcal{L}} w.$$

The language $\mathcal{L}$ is *regular* if $f_{\mathcal{L}}$ is rational.

Consider generalized factor order on P^* and fix a word $u \in P^*$. There is a corresponding language and generating function

$$\mathcal{F}(u) = \{w \, : \, w \geq u\} \quad \text{and} \quad F(u) = \sum_{w \geq u} w.$$

Our first result is as follows.

Theorem 1.1 *If P is a finite poset and $u \in P^*$ then $F(u)$ is rational.*

Theorem 1.1 is an analogue of a result of Björner and Sagan [**?**] for generalized subword order on P^*. *Generalized subword order* is defined exactly like generalized factor order except that w' is only required to be a subword of w, i.e., the elements of w' need not be consecutive in w. For related results, also see Goyt [**?**].

Given any set, P, a *nondeterministic finite automaton* or *NFA* over P is a digraph (directed graph) Δ with vertices V and arcs $\vec{E}$ having the following properties.

1. The elements of V are called *states* and $|V|$ is finite.
2. There is a designated *initial state* α and a set Ω of *final states*.
3. Each arc of $\vec{E}$ is labeled with an element of P.

Given a (directed) path in Δ starting at α, we construct a word in P^* by concatenating the elements on the arcs on the path in the order in which they are encountered. The *language accepted by* Δ is the set of all such words which are associated with paths ending in a final state. It is a well-known theorem that, for $|P|$ finite, a language $\mathcal{L} \subseteq P^*$ is regular if and only if there is a NFA accepting $\mathcal{L}$. (See, for example, the text of Hopcroft and Ullman [**?**, Chapter 2].)

We will demonstrate Theorem 1.1 by constructing a NFA accepting the language for $F(u)$. This will be done in the next section. In fact, the NFA still exists even if P is infinite, suggesting that more can be said about the generating function in this case.

We are particularly interested in the case of $P = \mathbb{P}$ with the usual order relation. So $\mathbb{P}^*$ is just the set of *compositions* (ordered integer partitions). Given $w = w_1 w_2 \ldots w_\ell \in \mathbb{P}^*$, we define its *norm* to be

$$\Sigma(w) = w_1 + w_2 + \cdots + w_\ell.$$

Let t, x be commuting variables. Replacing each $n \in w$ by tx^n, we get an associated monomial called the *weight* of w

$$wt(w) = t^{|w|} x^{\Sigma(w)}.$$

For example, if $w = 213221$ then

$$wt(w) = tx^2 \cdot tx \cdot tx^3 \cdot tx^2 \cdot tx^2 \cdot tx = t^6 x^{11}.$$

We also have the associated *weight generating function*

$$F(u; t, x) = \sum_{w \geq u} wt(w).$$

Our NFA will demonstrate, via the transfer-matrix method, that this is also a rational function of t and x. The details will be given in Section 3.

Call $u, w \in \mathbb{P}^*$ *Wilf equivalent* if $F(u; t, x) = F(v; t, x)$. This definition is modelled on the one used in the theory of pattern avoidance. See the survey article of Wilf [**?**] for more information about this subject. Section 4 is devoted to stating various Wilf equivalences all of which can be proved combinatorially.

Björner [**?**] gave a recursive formula for the Möbius function of (ordinary) factor order. It follows from his theorem that $\mu(u, w) = 0, \pm 1$ for all $u, w \in P^*$. Using the Pumping Lemma [**?**, Lemma 3.1] we show that there are finite sets P and $u \in P^*$ such that the language

$$\mathcal{M}(u) = \{w \ : \ \mu(u, w) \neq 0\}$$

is not regular. This is done in Section 5. The final section is devoted to comments and open questions.

2 Construction of automata

We will now introduce another language which is related to $\mathcal{F}(u)$ and which will be useful in proving Theorem 1.1. We say that u is a *suffix* (respectively, *prefix*) of w if $w = vu$ (respectively, $w = uv$) for some word v. Let $\mathcal{S}(u)$ be all the $w \in \mathcal{F}(u)$ such that, in the definition of generalized factor order, the only possible choice for w' is a suffix of w. Let $S(u)$ be the corresponding generating function.

The next result follows easily from the definitions and so we omit the proof. In it, we will use the notation Q to stand both for a subset of P and for the generating function $Q = \sum_{a \in Q} a$. Context will make it clear which is meant.

Lemma 2.1 *Let P be any poset and let $u \in P^*$. Then we have the following relationships*

$$\mathcal{F}(u) = \mathcal{S}(u) P^* \quad \textit{and} \quad F(u) = S(u)(\epsilon - P)^{-1}$$

between the languages and between the generating functions. □

We will now prove that the two languages we have defined are accepted by NFAs. An example follows the proof so the reader may want to read it in parallel.

Theorem 2.2 *Let P be any poset and let $u \in P^*$. Then there are NFAs accepting $\mathcal{F}(u)$ and $\mathcal{S}(u)$.*

Proof: We first construct an NFA, Δ, for $\mathcal{S}(u)$. Let $\ell = |u|$. The states of Δ will be all subsets T of $\{1, \ldots, \ell\}$. The initial state is $\emptyset$. Let $w = w_1 \ldots w_m$ be the word corresponding to a path from $\emptyset$ to T. The NFA will be constructed so that if the path is continued, the only possible embedding indices are those in the set $\{m - t + 1 \ : \ t \in T\}$. In other words, for each $t \in T$ we have

$$u_1 u_2 \ldots u_t \leq w_{m-t+1} w_{m-t+2} \ldots w_m, \tag{1}$$

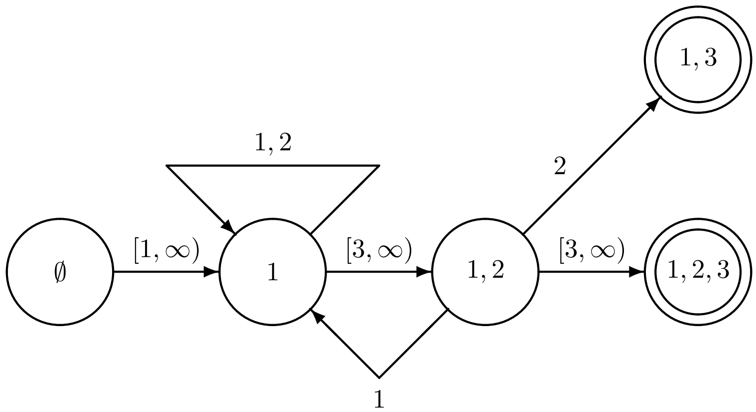

Fig. 1: A NFA accepting $\mathcal{S}(132)$

for each $t \in \{1,\ldots,\ell\} - T$ this inequality does not hold, and $u \not\leq w'$ for any factor w' of w starting at an index smaller then $m - \ell + 1$. From this description it is clear that the final states should be those containing ℓ.

The definition of the arcs of Δ is forced by the interpretation of the states. There will be no arcs out of a final state. If T is a nonfinal state and $a \in P$ then there will be an arc from T to

$$T' = \{t + 1 \ : \ t \in T \cup \{0\} \text{ and } u_{t+1} \leq a\}.$$

It is easy to see that (1) continues to hold for all $t' \in T'$ once we append a to w. This finishes the construction of the NFA for $\mathcal{S}(u)$. To obtain an automaton for $\mathcal{F}(u)$, just add loops to the final states of Δ, one for each $a \in P$. □

As an example, consider $P = \mathbb{P}$ and $u = 132$. We will do several things to simplify writing down the automaton. First of all, certain states may not be reachable by a path starting at the initial state. So we will not display such states. For example, we can not reach the state $\{2, 3\}$ since $u_1 = 1 \leq w_i$ for any i and so 1 will be in any state reachable from ϕ. Also, given states T and U there may be many arcs from T to U, each having a different label. So we will replace them by one arc bearing the set of labels of all such arcs. Finally, set braces will be dropped for readability. The resulting digraph is displayed in Figure 1.

Consider what happens as we build a word w starting from the initial state $\emptyset$. Since $u_1 = 1$, any element of $\mathbb{P}$ could be the first element of an embedding of u into w. That is why every element of the interval $[1, \infty) = \mathbb{P}$ produces an arrow from the initial state to the state $\{1\}$. Now if $w_2 \leq 2$, then an embedding of u could no longer start at w_1 and so these elements give loops at the state $\{1\}$. But if $w_2 \geq 3$ then an embedding could start at either w_1 or at w_2 and so the corresponding arcs all go to the state $\{1, 2\}$. The rest of the automaton is explained similarly.

As an immediate consequence of the previous theorem we get the following result which includes Theorem 1.1.

Theorem 2.3 *Let P be a finite poset and let $u \in P^*$. Then the generating functions $F(u)$ and $S(u)$ are rational.* □

3 The positive integers

If $P = \mathbb{P}$ then Theorem 2.3 no longer applies to the generating functions $F(u)$ and $S(u)$. However, we can still show rationality of the weight generating function $F(u; t, x)$ as defined in the introduction. Similarly, we will see that the series $S(u; t, x) = \sum_{w \in \mathcal{S}(u)} wt(w)$ is rational.

Note first that Lemma 2.1 still holds for $\mathbb{P}$ and can be made more explicit in this case. Extend the function wt to all of $\mathbb{Z}\langle\langle\mathbb{P}\rangle\rangle$ by letting it act linearly. Then

$$wt(\epsilon - \mathbb{P})^{-1} = \frac{1}{1 - \sum_{n \geq 1} tx^n} = \frac{1}{1 - tx/(1-x)} = \frac{1-x}{1-x-tx}.$$

We now plug this into the lemma just cited.

Corollary 3.1 *We have $F(u; t, x) = (1-x)S(u; t, x)/(1-x-tx)$.* □

It follows that if one of these three series is rational then the other one is as well.

We will now use the NFA, Δ, constructed in Theorem 2.2 to show that $S(u; t, x)$ is rational. This is essentially an application of the transfer-matrix method. See the text of Stanley [**?**, Section 4.7] for more information about this technique. The *transfer matrix* M for Δ has rows and columns indexed by the states with

$$M_{T,U} = \sum_n wt(n)$$

where the sum is over all n which appear as labels on the arcs from T to U. For example, consider the case where $w = 132$ as done at the end of the previous section. If we list the states in the order

$$\emptyset,\ \{1\},\ \{1,2\},\ \{1,3\},\ \{1,2,3\}$$

then the transfer matrix is

$$M = \begin{bmatrix} 0 & \dfrac{tx}{1-x} & 0 & 0 & 0 \\ 0 & t(x+x^2) & \dfrac{tx^3}{1-x} & 0 & 0 \\ 0 & tx & 0 & tx^2 & \dfrac{tx^3}{1-x} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now M^k has entries $M^k_{T,U} = \sum_w wt(w)$ where the sum is over all words w corresponding to a directed walk of length k from T to U. So to get the weight generating function for walks of all lengths one considers $\sum_{k \geq 0} M^k$. Note that this sum converges in the algebra of matrices over the formal power series algebra $\mathbb{Z}[[t, x]]$ because none of the entries of M has a constant term. It follows that

$$L := \sum_{k \geq 0} M^k = (I - M)^{-1} = \frac{\operatorname{adj}(I - M)}{\det(I - M)} \tag{2}$$

where adj denotes the adjoint.

Now

$$S(u;t,x) = \sum_T L_{\emptyset,T}$$

where the sum is over all final states of Δ. So it suffices to show that each entry of L is rational. From equation (2), this reduces to showing that each entry of M is rational. So consider two given states T, U. If T is final then we are done since the Tth row of M is all zeros. If T is not final, then consider

$$T' = \{t+1 \ : \ t \in T \cup \{0\}\}. \tag{3}$$

If $U = T'$ then there will be an $N \in \mathbb{P}$ such that all the arcs out of T with labels $n \geq N$ go to T'. So $M_{T,T'}$ will contain $\sum_{n \geq N} tx^n = tx^N/(1-x)$ plus a finite number of other terms of the form tx^m. Thus this entry is rational. If $U \neq T'$, then there will only be a finite number of arcs from T to U and so $M_{T,U}$ will actually be a polynomial. This shows that every entry of M is rational and we have proved, with the aid of the remark following Corollary 3.1, the following result.

Theorem 3.2 *If $u \in \mathbb{P}^*$ then $F(u;t,x)$ and $S(u;t,x)$ are rational.* □

4 Wilf equivalence

Recall that $u, v \in \mathbb{P}^*$ are *Wilf equivalent*, written $u \sim v$, if $F(u;t,x) = F(v;t,x)$. By Corollary 3.1, this is equivalent to $S(u;t,x) = S(v;t,x)$. It follows that to prove Wilf equivalence, it suffices to find a weight-preserving bijection $f : \mathcal{L}(u) \to \mathcal{L}(v)$ where $\mathcal{L} = \mathcal{F}$, or $\mathcal{S}$. Since $\sim$ is an equivalence relation, we can talk about the *Wilf equivalence class* of u which is $\{w \ : \ w \sim u\}$. It is worth noting that the automata for the words in a Wilf equivalence class need not bear a resemblance to each other. Part of the motivation for this section is to try to explain as many Wilf equivalences as possible between permutations.

First of all, we consider three operations on words in $\mathbb{P}^*$. The *reversal* of $u = u_1 \ldots u_\ell$ is $u^r = u_\ell \ldots u_1$. It will also be of interest to consider $1u$, the word gotten by prepending one to u. Finally, we will look at u^+ which is gotten by increasing each element of u by one. It is not hard to give combinatorial proofs for the three facts in the next theorem, but due to space limitations we will only do so for the second.

Theorem 4.1 *We have the following Wilf equivalences.*

(a) $u \sim u^r$,

(b) if $u \sim v$ *then* $1u \sim 1v$,

(c) if $u \sim v$ *then* $u^+ \sim v^+$.

Proof: (b) We can assume we are given a weight-preserving bijection $f : \mathcal{S}(u) \to \mathcal{S}(v)$. Since 1 is the minimal element of $\mathbb{P}$,

$$\mathcal{S}(1u) = \{w \in \mathbb{P}^* \ : \ w_2 w_3 \ldots w_{|w|} \in S(u)\}.$$

So f induces a weight-preserving bijection $g : \mathcal{S}(1u) \to \mathcal{S}(1v)$ defined by

$$g(w_1 w_2 \ldots w_n) = w_1 f(w_2 \ldots w_n)$$

and we are done. □

Applying the previous result, we can obtain all the Wilf equivalences in the symmetric groups $\mathfrak{S}_2$ and $\mathfrak{S}_3$. In $\mathfrak{S}_2$ we have $12 \sim 21$ by (a). So $23 \sim 32$ by (c) and $123 \sim 132$ by (b). Continuing in this way, we obtain

$$123 \sim 321 \sim 132 \sim 231 \quad \text{and} \quad 213 \sim 312.$$

These two groups are indeed in different equivalence classes as one can use equation (2) to compute that

$$S(123;t,x) = \frac{t^3x^6}{(1-x)^2(1-x-tx+tx^3-t^2x^4)}$$

while

$$S(213;t,x) = \frac{t^3x^6(1+tx^3)}{(1-x)(1-x+t^2x^4)(1-x-tx+tx^3-t^2x^4)}.$$

We will need a new result to explain some of the equivalences in $\mathfrak{S}_4$ such as $2134 \sim 2143$. This is done by the next result which, in conjunction with Theorem 4.1, can be used to derive all of the equivalences in $\mathfrak{S}_4$. We omit the proof due to space limitations.

Theorem 4.2 *Let $x, y, z \in \{1, \ldots, m\}^*$ and suppose $n > m$. Then*

$$xmynz \sim xnymz.$$

5 The Möbius function

We will now show that the language for the Möbius function of ordinary factor order is not regular. This is somewhat surprising because Björner and Reutenauer [**?**] showed that this language is regular if one considers ordinary subword order, and then Björner and Sagan [**?**] extended this result to generalized subword order. We will begin by reviewing some basic facts about Möbius functions. The reader wishing more details can consult [**?**, Chapter 3].

For any poset P, the *incidence algebra* of P over the integers is

$$I(P) = \{\alpha : P \times P \to \mathbb{Z} \ : \ \alpha(a,b) = 0 \text{ if } a \not\leq b\}.$$

This set is an algebra whose multiplication is given by *convolution* $(\alpha * \beta)(a,b) = \sum_{c\in P} \alpha(a,c)\beta(c,b)$. It is easy to see that the identity for this operation is the *Kronecker delta*

$$\delta(a,b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{else.} \end{cases}$$

So it is possible for incidence algebra elements to have multiplicative inverses.

One of the simplest elements of $I(P)$ is the *zeta function*

$$\zeta(a,b) = \begin{cases} 1 & \text{if } a \geq b, \\ 0 & \text{else.} \end{cases}$$

Note that $F(u)$ can be rewritten as $F(u) = \sum_{w\in P^*} \zeta(u,w)w$. It turns out that ζ has a convolutional inverse μ in $I(P)$. This function is important in enumerative and algebraic combinatorics. Björner [**?**]

has given a formula for μ in ordinary factor order which we will need. To describe this result, we must make some definitions. The *dominant outer factor* of w, denoted $o(w)$, is the longest word other than w which is both a prefix and a suffix of w. Note that we may have $o(w) = \epsilon$. The *dominant inner factor* of $w = w_1 \ldots w_\ell$, written $i(w)$, is $w_2 \ldots w_{\ell-1}$. Finally, a word is *flat* if all its elements are equal. For example, $w = abbaabb$ has $o(w) = abb$ and $i(w) = bbaab$.

Theorem 5.1 (Björner) *In (ordinary) factor order, if $u \leq w$ then*

$$\mu(u,w) = \begin{cases} \mu(u, o(w)) & \text{if } |w| - |u| > 2 \text{ and } u \leq o(w) \not\leq i(w), \\ 1 & \text{if } |w| - |u| = 2,\ w \text{ is not flat, and } u = o(w) \text{ or } i(w), \\ (-1)^{|w|-|u|} & \text{if } |w| - |u| < 2, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

Continuing the example

$$\mu(b, abbaabb) = \mu(b, abb) = 1.$$

Note that this description is inductive. It also implies that $\mu(u, w)$ is ± 1 or 0 for all u, w in factor order.

We will show that the language $\mathcal{M}(u) = \{w \ : \ \mu(u, w) \neq 0\}$ need not be regular. To do this, we will need the Pumping Lemma which we now state. A proof can be found in [**?**, pp. 55–56].

Lemma 5.2 (Pumping Lemma) *Let $\mathcal{L}$ be a regular language. Then there is a constant $n \geq 1$ such that any $z \in \mathcal{L}$ can be written as $z = uvw$ satisfying*

1. $|uv| \leq n$ *and* $|v| \geq 1$,
2. $uv^i w \in \mathcal{L}$ *for all* $i \geq 0$. $\square$

Roughly speaking, any word in a regular language has a prefix of bounded length such that pumping up the end of the prefix keeps one in the language.

Theorem 5.3 *Consider (ordinary) factor order where $P = \{a, b\}$. Then $\mathcal{M}(a)$ is not regular.*

Proof: Suppose, to the contrary, that $\mathcal{M}(a)$ is regular and let n be the constant guaranteed by the pumping lemma. We will derive a contradiction by letting $z = ab^n ab^n a$ where, as usual, b^n represents the letter b repeated n times.

First we show that $z \in \mathcal{M}(a)$. Indeed, $o(z) = ab^n a$ and $i(z) = b^n ab^n$ which implies that $a \leq o(z) \not\leq i(z)$. So we are in the first case of Björner's formula and $\mu(a, z) = \mu(a, ab^n a)$. Repeating this analysis with $ab^n a$ in place of z gives $\mu(a, z) = \mu(a, a) = 1$. Hence $z \in \mathcal{M}(a)$ as promised.

Now pick any prefix uv of z as in the Pumping Lemma. There are two cases. The first is if $u \neq \epsilon$. So $v = b^j$ for some j with $1 \leq j < n$. Picking $i = 2$, we conclude that $z' = uv^2 w = ab^{n+j} ab^n a$ is in $\mathcal{M}(a)$. But $o(z') = a$ and $i(z') = b^{n+j} ab^n$. Thus $|z'| - |a| > 2$ and $a \leq o(z') \leq i(z')$, so z' does not fall into any of the first three cases of Björner's formula. This implies that $\mu(a, z') = 0$ and hence $z' \notin \mathcal{M}(a)$, which is a contradiction in this case.

The second possibility is that $u = \epsilon$ and $v = ab^j$ for some $0 \leq j < n$. Similar considerations to those in the previous paragraph show that if we take $z' = uv^2 w$ then $\mu(a, z') = 0$ again. So we have a contradiction as before and the theorem is proved. $\square$

6 Comments, conjectures, and open questions

6.1 Mixing factors and subwords

It is possible to create languages using combinations of factors and subwords. This is an idea that was first studied by Babson and Steingrímsson [**?**] in the context of pattern avoidance in permutations. Many of the results we have proved can be generalized in this way. We will indicate how this can be done for Theorem 2.2.

A *pattern* p over P is a word in P^* where certain pairs of adjacent elements have been overlined (barred). For example, in the pattern $p = 1\overline{13}\overline{33}24\overline{61}$ the pairs 13, 33, and 61 have been overlined. If $w \in P^*$ we will write $\overline{w}$ for the pattern where every pair of adjacent elements in w is overlined. So every pattern has a unique factorization of the form $p = \overline{y_1}\,\overline{y_2}\ \ldots\ \overline{y_k}$. In the preceding example, the factors are $y_1 = 1$, $y_2 = 133$, $y_3 = 2$, $y_4 = 4$, and $y_5 = 61$.

If $p = \overline{y_1}\,\overline{y_2}\ \ldots\ \overline{y_k}$ is a pattern and $w \in P^*$ then p *embeds* into w, written $p \to w$, if there is a subword $w' = z_1 z_2 \ldots z_k$ of w where, for all i,

1. z_i is a factor of w with $|z_i| = |y_i|$, and
2. $y_i \le z_i$ in generalized factor order.

For example $\overline{32}4 \to 14235$ and there is only one embedding, namely 425. For any pattern p, define the language

$$\mathcal{F}(p) = \{w \in P^* \ :\ p \to w\}$$

and similarly for $\mathcal{S}(p)$. The next result generalizes Theorem 2.2 to an arbitrary pattern. It is proved by pasting together automata like those constructed in that theorem.

Theorem 6.1 *Let P be any poset and let p be a pattern over P. Then there are NFAs accepting $\mathcal{F}(p)$ and $\mathcal{S}(p)$.* □

6.2 Rationality for infinite posets

It would be nice to have a criterion that would imply rationality even for some infinite posets P. To this end, let $\mathbf{x} = \{x_1, \ldots, x_m\}$ be a set of commuting variables and consider the formal power series algebra $\mathbb{Z}[[\mathbf{x}]]$. Suppose we are given a function

$$wt : P \to \mathbb{Z}[[\mathbf{x}]]$$

which then defines a weighting of words $w = w_1 \ldots w_\ell \in P^*$ by

$$wt(w) = \prod_{i=1}^{m} wt(w_i).$$

To make sure our summations will be defined in $\mathbb{Z}[[\mathbf{x}]]$, we assume that there are only finitely many w of any given weight and call such a weight function *regular*.

For $u \in P^*$, let

$$F(u; \mathbf{x}) = \sum_{w \ge u} wt(w)$$

and similarly for $S(u;\mathbf{x})$. Suppose we want to make sure that $S(u;\mathbf{x})$ is rational. As done in Section 3, we can consider a transfer matrix with entries

$$M_{T,U} = \sum_{a} wt(a)$$

where the sum is over all $a \in P$ occurring on arcs from T to U. Equation (2) remains the same, so it suffices to make sure that $M_{T,U}$ is always rational.

If there is an arc labeled a from T to U then we must have $U \subseteq T'$ where T' is given in equation (3). Recalling the definition of Δ from the proof of Theorem 2.2, we see that the a's appearing in the previous sum are exactly those satisfying

1. $a \geq u_{t+1}$ for $t+1 \in U$, and
2. $a \not\geq u_{t+1}$ for $t+1 \in T' - U$.

To state these criteria succinctly, for any subword y of u we write $a \succeq y$ (respectively, $a \not\succeq y$) if $a \geq b$ (respectively, $a \not\geq b$) for all $b \in y$. Finally, note that, from the proof of Theorem 2.2, similar transfer matrices can be constructed for $F(u;\mathbf{x})$ and $A(u;\mathbf{x})$. We have proved the following result which generalizes Theorem 3.2.

Theorem 6.2 *Let P be a poset with a regular weight function $wt : P^* \to \mathbb{Z}[[\mathbf{x}]]$, and let $u \in P^*$. Suppose that for any two subwords y and z of u we have*

$$\sum_{\substack{a \succeq y \\ a \not\succeq z}} wt(a)$$

is a rational function. Then so are $F(u;\mathbf{x})$ and $S(u;\mathbf{x})$. □

6.3 Irrationality for infinite posets

When P is countably infinite it is possible for the generating functions we have considered to be irrational. As an example, fix a distinguished element $a \in P$. For each $A \subseteq P$ with $a \in A$, we define an order $\leq_A$ by insisting that the elements of $P - \{a\}$ form an antichain, and that $a \leq_A b$ if and only if $b \in A$. Consider the corresponding suffix language $\mathcal{S}_A(a)$. Clearly $\mathcal{S}_A(a) = (P - A)^* A$ and so no two of these languages are equal. It follows that the mapping $A \to \mathcal{S}_A(a)$ is injective. So one of the $\mathcal{S}_A(a)$ must be irrational since there are uncountably many possible A but only countably many rational functions in $\mathbb{Z}\langle\langle P \rangle\rangle$.

6.4 Wilf equivalence and strong equivalence

There are a number of open problems and questions raised by our work on Wilf equivalence.

(1) **If $u \sim v$, then must v be a rearrangement of u?** This is the case for all the Wilf equivalences we have proved.

(2) **What about Wilf equivalence in $[m]^*$ where $[m] = \{1, 2, \ldots, m\}$?** Given a positive integer m, one can define Wilf equivalence of words $u, v \in [m]^*$ in the same way that we did for $\mathbb{P}^*$. We write $u \sim_m v$ for this relation. Is it true that $u \sim_m v$ if and only if $u \sim v$?

(3) **If $u^+ \sim v^+$ then is $u \sim v$?** In other words, does the converse of Theorem 4.1 (c) hold? It is not hard to see that the converse of (b) is true.

(4) **Find a theorem which, together with the results already proved, explains all the Wilf equivalences in $\mathfrak{S}_5$.** We have a conjecture that would be helpful in this regard.

Conjecture 6.3 *For any $a, b, c \in [2, \infty)$ we have*

$$a1b2c \sim a2b1c.$$

(5) **Is it always the case that the number of elements of $\mathfrak{S}_n$ Wilf equivalent to a given permutation is a power of 2?** Our computations show that this is always true for $n \leq 5$.

6.5 The language $\mathcal{M}(u)$

We have shown that $\mathcal{M}(u)$ is not always regular and so the corresponding generating function $M(u)$ is not always rational. But this leaves open whether $\mathcal{M}(u)$ might fall into a more general class of languages such as context free grammars (CFGs). There is a Pumping Lemma for CFGs, see [**?**, Section 6.1]. So it is tempting to try and modify the proof of Theorem 5.3 to show that $\mathcal{M}(u)$ is not even a CFG. However, all our attempts in that direction have failed. Is $\mathcal{M}(u)$ a CFG or not?

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Record statistics in integer compositions

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A *composition* $\sigma = a_1a_2 \ldots a_m$ of n is an ordered collection of positive integers whose sum is n. An element a_i in σ is a strong (weak) *record* if $a_i > a_j$ ($a_i \geq a_j$) for all $j = 1, 2, \ldots, i-1$. Furthermore, the position of this record is i. We derive generating functions for the total number of strong (weak) records in all compositions of n, as well as for the sum of the positions of the records in all compositions of n, where the parts a_i belong to a fixed subset A of the natural numbers. In particular when $A = \mathbb{N}$, we find the asymptotic mean values for the number, and for the sum of positions, of records in compositions of n.

Keywords: Composition, Record, Left-to-right maxima, Generating function, Mellin transforms, Asymptotic estimates

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Introduction

Let $\pi = a_1a_2 \cdots a_n$ be any permutation of length n, an element a_i in π is a *record* if $a_i > a_j$ for all $j = 1, 2, \ldots, i-1$. Furthermore, the position of this record is i. The number of records was first studied by Rényi [13], compare also [7]. A survey of results on this topic can be found in [2]. In the literature records are also referred to as a *left–to–right maxima* or *outstanding elements*. In particular the study of records has applications to observations of extreme weather problems, test of randomness, determination of minimal failure, and stresses of electronic components. The recent paper by Kortchemski [8] defines a new statistic $srec$, where $srec(\pi)$ is the sum over the positions of all records in π. For instance, the permutation $\pi = 451632$ has 3 records $4, 5, 6$ and $srec(\pi) = 1 + 2 + 4 = 7$.

A word over an alphabet A, a set of positive integers, is defined as any ordered sequence of possibly repeated elements of A. Recently, Prodinger [12] studied the statistic $srec$ for words over the alphabet $\mathbb{N} = \{1, 2, 3, \ldots\}$, equipped with geometric probabilities $p, pq, pq^2, \ldots$, with $p + q = 1$. In the case of words there two versions: A *strong record* in a word $a = a_1 \cdots a_n$ is an element a_i such that $a_i > a_j$ for all $j = 1, 2, \ldots, i-1$ (that is, must be strictly larger than elements to the left) and *weak record* is an element $a_i \geq a_j$ for all $j = 1, 2, \ldots, i-1$ (must be only larger or equal to elements to the left). Furthermore, the position i is called the position of the strong record (weak record). We denote the sum of

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the positions of all strong (respectively, weak) records in a word a by $ssrec$ (respectively, $wsrec$). In [12], Prodinger found the expected value of the sum of the positions of strong records, in random geometrically distributed words of length n. Previously, Prodinger [10] also studied the number of strong and weak records, in samples of geometrically distributed random variables. He also studied further properties of such records in papers such as [11] and references therein.

A *composition* $\sigma = \sigma_1\sigma_2 \dots \sigma_m$ of n is an ordered collection of positive integers whose sum is n. Thus a composition σ of n with parts in A is a restricted word over the alphabet A. We denote the set of all compositions of n with m parts in A by $C_A(n, m)$. It is well known that the number of compositions of $n \geq 1$ with m parts in $\mathbb{N}$ is given by $\binom{n-1}{m-1}$ and that the total number of compositions of n is 2^{n-1}.

In this paper we find generating functions for these parameters, number of strong records, number of weak records, sum of positions of strong records, and sum positions of weak records in a random composition of n with parts in $A = [d] := \{1, 2, \dots, d\}$ or $A = \mathbb{N}$. We also study the mean values of these parameters as $n \to \infty$ in the case $A = \mathbb{N}$ by means of rational function asymptotics and Mellin transforms. Details of some of the lengthier proofs will be left to the full version of the paper. We remark that in [5], an asymptotic correspondence is established between compositions of n and samples of geometric variable of parameter $p = 1/2$ and length $n/2$. By exploiting this correspondence, and using the already established results of Prodinger for samples of geometric random variables, alternative derivations of our asymptotic results can be obtained.

1 Number strong records and weak records

Let $NSR_A(z, y, q)$ and $NWR_A(z, y, q)$ be the generating function for the number of compositions of n with m parts in A according to the number of strong and weak records, respectively, that is,

$$NSR_A(z, y, q) = \sum_{n,m \geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{nsr(\sigma)},$$
$$NWR_A(z, y, q) = \sum_{n,m \geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{nwr(\sigma)},$$

where $nsr(\sigma)$ and $nwr(\sigma)$ is the number of strong and weak records in composition σ, respectively. In this section we find an explicit formulas for those generating functions.

Theorem 1.1 *The generating function $NSR_{[d]}(z, y, q)$ is given by*

$$NSR_{[d]}(z, y, q) = \prod_{j=1}^{d} \left(1 + \frac{z^j yq}{1 - y \sum_{i=1}^{j} z^i} \right).$$

Proof: We denote the number of occurrences of the part d in the composition $\sigma \in C_{[d]}(n, m)$ by $\ell(\sigma)$. Now let us write equation for the generating function $NSR_{[d]}(z, y, q)$. The contribution of the case $\ell(\sigma) = 0$ is given by $NSR_{[d-1]}(z, y, q)$. Assume $\ell(\sigma) > 0$, then σ can be decomposed as $\sigma' d \sigma''$, where σ' is a composition with parts in $[d-1]$ and σ'' is a composition with parts in $[d]$. Thus, the contribution of the case $\ell(\sigma) > 0$ equals $z^d yq NSR_{[d-1]}(z, y, q) NSR[d](z, y, 1)$. Therefore,

$$NSR_{[d]}(z, y, q) = NSR_{[d-1]}(z, y, q) + z^d yq NSR_{[d-1]}(z, y, q) NSR_{[d]}(z, y, 1).$$

For $q = 1$ and by induction we have that

$$NSR_{[d]}(z, y, 1) = \frac{1}{1 - y\sum_{j=1}^{d} z^j}.$$

Hence,

$$NSR_{[d]}(z, y, q) = \prod_{j=1}^{d}\left(1 + \frac{z^j yq}{1 - y\sum_{i=1}^{j} z^i}\right),$$

as claimed. □ Theorem 1.1 with $q = 1$ gives that the generating function for the number of compositions of n with m parts in $[d]$ is given by $NSR_{[d]}(z, y, 1) = \frac{1}{1-y\sum_{i=1}^{d} z^j}$, see for example [4].

Also from Theorem 1.1 we get that

$$\begin{aligned}\frac{\partial}{\partial q} NSR_{[d]}(z, y, 1) &= \prod_{j=1}^{d}\left(1 + \frac{z^j y}{1 - y\sum_{i=1}^{j} z^i}\right)\left(\sum_{j=1}^{d} \frac{z^j y}{1 - y\sum_{i=1}^{j-1} z^i}\right)\\ &= \frac{1}{1 - y\sum_{i=1}^{d} z^i}\left(\sum_{j=1}^{d} \frac{z^j y}{1 - y\sum_{i=1}^{j-1} z^i}\right).\end{aligned}$$

Hence, the generating function for the number strong records in all compositions of n with parts in $\mathbb{N}$ is given by

$$f(z) := \frac{1}{1 - \sum_{i\geq 1} z^i}\sum_{j\geq 1}\frac{z^j}{1 - \sum_{i=1}^{j-1} z^i} = \frac{1 - z}{1 - 2z}\sum_{j\geq 1}\frac{z^j}{1 - \sum_{i=1}^{j-1} z^i}.$$

Theorem 1.2 *The average number E_n^s of strong left-to-right maxima in the context of compositions of n has the asymptotic expansion*

$$E_n^s = \frac{1}{2}\left[\log_2 n - \frac{1}{2} + \frac{\gamma}{L} - \delta\left(\log_2 n\right)\right] + o(1).$$

Here and in the rest of the paper, $L = \log 2$; γ is Euler's constant and $\delta(x)$ is a periodic function of period 1 and mean 0 and small amplitude, which is given by the Fourier series

$$\delta(x) = \frac{1}{L}\sum_{k\neq 0}\Gamma\left(-\chi_k\right)e^{2k\pi ix}.$$

The complex numbers χ_k are given by $\chi_k = 2k\pi i/L$.

Proof: Firstly by summing the finite geometric series and using partial fraction decomposition,

$$f(z) = \frac{z - z^2}{1 - 2z} + (1 - z)^2\sum_{k\geq 2}\left[\frac{1}{1 - 2z} - \frac{1}{1 - 2z + z^k}\right].$$

Hence the average number E_n^s of strong left-to-right maxima in compositions of n satisfies

$$E_n^s = \frac{1}{2^{n-1}}[z^n]f(z) = \frac{1}{2} + \frac{1}{2^{n-1}}[z^n](1-z)^2 \sum_{k=2}^{n} \left[\frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right].$$

Let ρ_k be the smallest positive root of the denominator polynomial $1-2z+z^k$ that lies between $1/2$ and 1. An application of the principle of the argument or Rouche's Theorem shows such a root to exist with all other roots of modulus greater than $3/4$. By dominant pole analysis,

$$q_{n,k} := [z^n]\frac{(1-z)^2}{1-2z+z^k} = c_k \rho_k^{-n} + O\left(\left(\frac{4}{3}\right)^n\right) \quad \text{with} \quad c_k = \frac{(1-\rho_k)^2}{\rho_k(2-k\rho_k^{k-1})},$$

for large n but fixed k. The denominator polynomial $1-2z+z^k$ behaves like a perturbation of $1-2z$ near $z=1/2$. By "bootstrapping" we find that

$$\rho_k = \frac{1}{2} + 2^{-k-1} + O(k2^{-2k})$$

and hence $c_k = \frac{1}{4} + O(k2^{-k})$. The use of this approximation can be justified for a wide range of values of k and n (see for example [3] or [6]).

Let us now restrict our attention to those k for which $n^{-3} \le 2^{-k} \le \frac{\log n}{n}$. For such k we can show that

$$q_{n,k} = 2^{n-2}\left(\exp\left(-\frac{n}{2^k}\right) + O\left(\frac{\log^3 n}{n}\right)\right). \tag{1}$$

Turning next to smaller values of $k \ge 2$, that is, k such that $2^{-k} > \frac{\log n}{n}$, we find that now the coefficients $q_{n,k}$ are relatively small, since for such k, $q_{n,k} = O(\frac{2^n}{n})$ as $n \to \infty$. Finally we must consider larger values of $k \le n$ that is, k for which $n^{-3} > 2^{-k}$, or equivalently, $k \ge 3\log_2 n$. In this range we find that

$$q_{n,k} = 2^{n-2}\left(\exp\left(-\frac{n}{2^k}\right) + O\left(\frac{1}{n^2}\right)\right). \tag{2}$$

Then combining the estimates for $q_{n,k}$ over the range $2 \le k \le n$ above,

$$\begin{aligned} E_n^s - \frac{1}{2} &= \frac{1}{2}\sum_{k=2}^{n}\left(1 - \frac{q_{n,k}}{2^{n-2}}\right) \\ &\sim \frac{1}{2}\sum_{k\ge 0}\left(1 - \exp\left(-\frac{n}{2^k}\right)\right) - 1, \end{aligned}$$

as the additional tail sum $\sum_{k>n}\left(1-\exp\left(-\frac{n}{2^k}\right)\right)$ is exponentially small. It remains to estimate

$$h(n) := \sum_{k\ge 0}\left(1 - \exp\left(-\frac{n}{2^k}\right)\right),$$

as $n \to \infty$. For this we use Mellin transforms and find (see [1, Appendix B.7, equation (48)])

$$h(n) = \log_2 n + \frac{1}{2} + \frac{\gamma}{L} - \delta\left(\log_2 n\right) + O(1/n). \tag{3}$$

The asymptotic estimate for E_n^s follows. $\square$

Remarks Asymptotically we find that the expected number of strict left-to-right maxima is half the expected size of the *largest part* in a random composition of n (see [9]). Also, as mentioned in the introduction, the asymptotic correspondence established in [5] would allow one to use the results of Prodinger [10] in the case $p = 1/2$, to give an alternative proof of Theorem 1.2.

A similar approach to that of Theorem 1.3 leads to

Theorem 1.3 *The generating function $NWR_{[d]}(z, y, q)$ is given by*

$$NWR_{[d]}(z, y, q) = \prod_{j=1}^{d} \frac{1}{1 - \frac{z^j yq}{1-y\sum_{i=1}^{j-1} z^i}}.$$

The generating function for the total number of weak records in compositions over $\mathbb{N}$ is then

$$g(z) := \frac{\partial NWR_{[\mathbb{N}]}(z, 1, q)}{\partial q}\bigg|_{q=1} = \frac{1-z}{1-2z} \sum_{k\geq 1} \frac{z^k}{1 - \sum_{i=1}^{k} z^i}$$
$$= \frac{(1-z)^2}{z} \sum_{k\geq 2} \left[\frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right].$$

Theorem 1.4 *The average number E_n^w of weak left-to-right maxima in the context of compositions of n has the asymptotic expansion*

$$E_n^w = \log_2 n - \frac{3}{2} + \frac{\gamma}{L} - \delta\left(\log_2 n\right) + o(1).$$

Proof: The average number E_n^w of weak left-to-right maxima in compositions of n satisfies

$$E_n^w = \frac{1}{2^{n-1}}[z^n]g(z) = \frac{1}{2^{n-1}}[z^{n+1}](1-z)^2 \sum_{k=2}^{n} \left[\frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right].$$

Then using the $q_{n,k}$ notation in the proof of Theorem 1.2

$$E_n^w = \sum_{k=2}^{n+1} \left(1 - \frac{q_{n+1,k}}{2^{n-1}}\right) = 2E_{n+1}^s - 1.$$

The asymptotic estimate then follows from that of Theorem 1.2. $\square$

2 The statistics $ssrec$ and $wsrec$ on the set of compositions

Let $NSR_A(z, y, q)$ and $NWR_A(z, y, q)$ be the generating function for the number of compositions of n with m parts in A according to the statistic $ssrec$ and $wsrec$, respectively, that is,

$$PSR_A(z, y, q) = \sum_{n,m\geq 0} \sum_{\sigma\in C_A(n,m)} z^n y^m q^{ssrec(\sigma)},$$
$$PWR_A(z, y, q) = \sum_{n,m\geq 0} \sum_{\sigma\in C_A(n,m)} z^n y^m q^{wsrec(\sigma)}.$$

Theorem 2.1 *The generating function $PSR_{[d]}(z,y,q)$ is given by*

$$1+\sum_{k=1}^{d} q^k \left(\sum_{d\geq j_1>j_2>\cdots>j_k\geq 1} \prod_{i=1}^{k} \frac{z^{j_i} y q^{i-1}}{1-yq^{i-1}\sum_{\ell=1}^{j_i} z^\ell}\right).$$

Proof: We denote the number of occurrences of the part d in the composition $\sigma \in C_{[d]}(n,m)$ by $\ell(\sigma)$. Decomposing according to $\ell(\sigma)=0$ and $\ell(\sigma)>0$ leads to

$$PSR_{[d]}(z,y,q) = PSR_{[d-1]}(z,y,q) + z^d yq PSR_{[d-1]}(z,qy,q) PSR_{[d]}(z,y,1). \tag{4}$$

For $q=1$, $PSR_{[d]}(z,y,1) = \frac{1}{1-y\sum_{j=1}^{d} z^j}$. Hence,

$$\begin{aligned} PSR_{[d]}(z,y,q) &= PSR_{[d-1]}(z,y,q) + \frac{z^d yq}{1-y\sum_{i=1}^{d} z^i} PSR_{[d-1]}(z,qy,q) \\ &= PSR_{[d-2]}(z,y,q) + \sum_{j=d-1}^{d} \frac{z^j yq}{1-y\sum_{i=1}^{j} z^i} PSR_{[j-1]}(z,qy,q) \\ &\vdots \\ &= 1+\sum_{j=1}^{d} \frac{z^j yq}{1-y\sum_{i=1}^{j} z^i} PSR_{[j-1]}(z,qy,q). \end{aligned}$$

Iterating the above recurrence relation d times we get the desired result. □ From this we derive

Corollary 2.2 *The generating function $v_d(z) = \frac{\partial}{\partial q} PSR_{[d]}(z,1,q) \mid_{q=1}$ is given by*

$$\frac{z}{1-\sum_{j=1}^{d} z^j} \sum_{j=0}^{d-1} \frac{z^j}{\left(1-\sum_{i=1}^{j} z^i\right)^2}.$$

The above corollary gives that the generating function for the number of compositions of n according to the total of the statistic $ssrec$ is given by

$$v(z) := \frac{z(1-z)}{1-2z} \sum_{j\geq 0} \frac{z^j}{\left(1-\sum_{i=1}^{j} z^i\right)^2}.$$

The rather lengthy proof of the asymptotic behaviour of the coefficients of $v(z)$ will be left for the journal version of the paper. We obtain

Theorem 2.3 *The average sum of the positions of the strong records e_n^s in compositions of n has the asymptotic expansion*

$$e_n^s = \frac{n}{4\log 2}\big(1+\delta_2\left(\log_2 n\right)\big) + o(n),$$

where $\delta_2(x)$ is a periodic function of period 1, mean zero and small amplitude, which is given by the Fourier series

$$\delta_2(x) = \sum_{k \neq 0} \chi_k \Gamma\left(-1 - \chi_k\right) e^{2k\pi i x}.$$

With reference again to [5], Theorem 2.3 is seen to correspond to the $p = 1/2$ case of the results of Prodinger [12].

The corresponding results for $PWR_{[d]}(z, y, q)$ are as follows.

Theorem 2.4 *The generating function $PWR_{[d]}(z, y, q)$ satisfies the following recurrence relation*

$$PWR_{[d]}(z, y, q) = PWR_{[d-1]}(z, y, q) + \frac{z^d yq}{1 - y\sum_{i=1}^{d-1} z^i} PWR_{[d]}(z, qy, q).$$

Corollary 2.5 *The generating function $w_d(z) = \frac{\partial}{\partial q} PWR_{[d]}(z, 1, q) \mid_{q=1}$ is given by*

$$w_d(z) = \frac{1}{1 - \sum_{j=1}^{d} z^j} \sum_{j \geq 1} \frac{z^j}{\left(1 - \sum_{i=1}^{j} z^i\right)^2}.$$

The above corollary gives that the generating function for the number of compositions of n according to the total of the statistic $swrec$ is given by

$$w(z) := \frac{1 - z}{1 - 2z} \sum_{j \geq 1} \frac{z^j}{\left(1 - \sum_{i=1}^{j} z^i\right)^2}.$$

Theorem 2.6 *The average sum of the positions of the weak records e_n^w in compositions of n has the asymptotic expansion*

$$e_n^w = \frac{n}{2\log 2}\left(1 + \delta_2\left(\log_2 n\right)\right) + o(n),$$

where $\delta_2(x)$ is the same periodic function that occured in Theorem 2.3.

Proof: The generating functions $v(z)$ and $w(z)$ are related as follows,

$$v(z) = \frac{z(1-z)}{1-2z} + zw(z).$$

From this we see that

$$[z^{n+1}]v(z) = 2^{n-1} + [z^n]w(z).$$

So that $e_n^w = 2e_{n+1}^s - 1$. The result then follows from Theorem 2.3. □

Now, our aim is to present a combinatorial explanation for the fact that the number (sum) of the positions of weak records in all compositions of n plus 2^{n-1} equals the number (sum) of the positions of strong records in all compositions of $n+1$, for $n \geq 1$. In order to do that we need the following notations.

Let $sw_{n,r}$ (respectively, $ss_{n,r}$) be the sum of r-th power of the positions of weak (respectively, strong) records in all the compositions of n, namely,

$$\begin{aligned}
sw_{n,r} &= \sum_{\sigma\in C_n} \sum_{\sigma_i \text{ is a weak record of } \sigma} i^r, \\
ss_{n,r} &= \sum_{\sigma\in C_n} \sum_{\sigma_i \text{ is a strong record of } \sigma} i^r, \\
sw'_{n,r} &= \sum_{\sigma\in C_n(A)} \sum_{\sigma_i \text{ is a weak record of } \sigma, i>1} i^r, \\
ss'_{n,r} &= \sum_{\sigma\in C_n(A)} \sum_{\sigma_i \text{ is a strong record of } \sigma, i>1} i^r,
\end{aligned}$$

where $C_n = \cup_{m=1}^{n} C_{n,m}$ is the set of all compositions of n. From the definitions, each first letter is a weak (strong) record. Therefore,

$$sw_{n,r} = |C_n| + sw'_{n,r} \text{ and } ss_{n,r} = |C_n| + ss'_{n,r}, \tag{5}$$

where $|C_n| = 2^{n-1}$ is the number of compositions of n.

Theorem 2.7 *For all $n \geq 1$,*

$$ss_{n+1,r} = sw_{n,r} + 2^{n-1}.$$

Proof: It is not hard to see that $\sigma_1 \cdots \sigma_m$ is a composition of n and σ_i, $i > 1$, is a weak record if and only if $\sigma_1 \cdots \sigma_{i-1}(\sigma_i + 1)\sigma_{i+1} \cdots \sigma_m$ is a composition of n and $\sigma_i + 1$, $i > 1$, is a strong record. Therefore, the multiset of all positions i, $i > 1$, of the weak records in all compositions of n is the same multiset as all positions i, $i > 1$, of the strong records in all compositions on $n+1$. In other words, $ss'_{n+1,r} = sw'_{n,r}$ for all n and r.

Hence, by (5) we have

$$ss_{n+1,r} = 2^n + ss'_{n+1,r} = 2^n + sw'_{n,r} = 2^{n-1} + 2^{n-1} + sw'_{n,r} = 2^{n-1} + sw_{n,r},$$

as requested. □

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Geometry and complexity of O'Hara's algorithm

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In this paper we analyze O'Hara's partition bijection. We present three type of results. First, we see that O'Hara's bijection can be viewed geometrically as a certain scissor congruence type result. Second, we present a number of new complexity bounds, proving that O'Hara's bijection is efficient in most cases and mildly exponential in general. Finally, we see that for identities with finite support, the map of the O'Hara's bijection can be computed in polynomial time, i.e. much more efficiently than by O'Hara's construction.

Keywords: partitions, O'Hara's algorithm, complexity

1 Introduction

Some combinatorial results have an easy proof via generating functions and a more elusive, but also more interesting and important, bijective proof. It would be difficult to think of a better example of this than the generalization of Euler's classical distinct/odd theorem due to George Andrews (Theorem 2.1). The proof via generating functions is a trivial one-line calculation. On the other hand, the simplest bijective proof of this result, O'Hara's algorithm, is distinctly non-trivial and has numerous fascinating properties.

Note that a quest to find bijective proofs of partition identities goes back all to way to the pioneer work of Sylvester and his school. Despite remarkable successes in the last century (see [P06]) and some recent work of both positive and negative nature (see e.g. [P04b, P]), the problem remains ambiguous and largely unresolved. Much of this stems from the lack of clarity as to what exactly constitutes a bijective proof. Depending on whether one accentuates simplicity, ability to generalize, the time complexity, geometric structure, or asymptotic stability, different answers tend to emerge.

In one direction, the subject of partition bijections was revolutionized by Garsia and Milne with their *involution principle* [GM81a, GM81b]. This is a combinatorial construction which allows to use a few basic bijections and involutions to build more involved combinatorial maps. As a consequence, one can start with a reasonable analytic proof of a partition identity and trace every step to obtain a (possibly extremely complicated) bijective construction. Garsia and Milne used this route to obtain a long sought bijection proving the Rogers-Ramanujan identities, resolving an old problem in this sense [GM81b]. Unfortunately, this bijection is too complex to be analyzed and has yet to lead to new Rogers-Ramanujan type partition identities.

After Garsia-Milne paper, there has been a flurry of activity to obtain synthetic bijections for large classes of partition identities. Most of these bijections did not seem to lead anywhere with one notable exception. Remmel and Gordon found (rather involved) bijective proofs of the above-mentioned partition identity due to Andrews [R82, G83]. O'Hara's streamlined proof is in fact a direct generalization of Glaisher's classical bijection proving Euler's theorem. Moreover, in her thesis [O84], O'Hara showed that her bijection is computationally efficient in certain special cases. Until now, the reason why O'Hara's bijection has a number of nice properties distinguishing it from the other "involution principle bijections" remained mysterious.

In this extended abstract, we present results of both positive and negative type. First, we analyze the complexity of O'Hara's bijection, which we view as a discrete algorithm. Theorem 3.2 gives an *exact* formula for the number of steps of the algorithm in certain cases. From here it follows that O'Hara's bijection is computationally efficient in many special cases. On the other hand, perhaps surprisingly, the number of steps can be (mildly) exponential in the worst case (Theorem 3.7 part (3)). This is the first negative result of this kind, proving the analogue of a conjecture that remains open for the Garsia-Milne's "Rogers-Ramanujan bijection" (see Subsection 4.1).

Second, we show that O'Hara's bijection has a rich underlying geometry. In a manner similar to that in [P04a, PV05], we view this bijection as a map between integer points in polytopes which preserves certain linear functionals. We present an advanced generalization of Andrews's result and of O'Hara's bijection in this geometric setting. In a special case, the working of the map corresponds to the Euclid algorithm and, more generally, to terms in the continuing fractions. Thus one can also think of our generalization as a version of multidimensional continuing fractions.

Finally, by combining the geometric and complexity ideas we see that in the finite dimensional case the map defined by O'Hara's bijection is a solution of an integer linear programming problem. This implies that the map defined by the bijection can be computed in polynomial time, i.e. much more efficiently than by O'Hara's bijection.

The extended abstract is structured as follows. We start with definitions and notations in Section 2. In Section 3, we describe the main results on both geometry and complexity. We conclude with final remarks in Section 4.

Due to space constraints, we present almost no proofs. An interested reader is invited to find the proofs and some other results in the paper [KP], on which this abstract is based.

2 Definitions and background

2.1 Andrews's theorem

A *partition* λ is an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$, where the integers λ_i are called the *parts* of the partition. The sum $n = \sum_{i=1}^{\ell} \lambda_i$ is called the *size* of λ, denoted $|\lambda|$; in this case we say that λ is a partition of n, and write $\lambda \vdash n$. We can also write $\lambda = 1^{m_1} 2^{m_2} \cdots$, where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i. The *support* of $\lambda = 1^{m_1} 2^{m_2} \cdots$ is the set $\{i \colon m_i > 0\}$. The set of all positive integers will be denoted by $\mathbb{P}$.

Denote the set of all partitions by $\mathcal{P}$ and the set of all partitions of n by $\mathcal{P}_n$. The number of partitions of n is given by Euler's formula

$$\sum_{\lambda \in \mathcal{P}} t^{|\lambda|} = \sum_{n=0}^{\infty} |\mathcal{P}_n| t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^i}.$$

For a sequence $\overline{a} = (a_1, a_2, \ldots)$ with $a_i \in \mathbb{P} \cup \{\infty\}$, define $\mathcal{A}$ to be the set of partitions λ with $m_i(\lambda) < a_i$ for all i; write $\mathcal{A}_n = \mathcal{A} \cap \mathcal{P}_n$. Denote by $\mathrm{supp}(\overline{a}) = \{i \colon a_i < \infty\}$ the *support* of the sequence $\overline{a}$.

Let $\overline{a} = (a_1, a_2, \ldots)$ and $\overline{b} = (b_1, b_2, \ldots)$. We say that $\overline{a}$ and $\overline{b}$ are *φ-equivalent*, $\overline{a} \sim_\varphi \overline{b}$, if φ is a bijection $\mathrm{supp}(\overline{a}) \to \mathrm{supp}(\overline{b})$ such that $ia_i = \varphi(i) b_{\varphi(i)}$ for all i. If $\overline{a} \sim_\varphi \overline{b}$ for some φ, we say that $\overline{a}$ and $\overline{b}$ are equivalent, and write $\overline{a} \sim \overline{b}$.

Theorem 2.1 (Andrews) *If $\overline{a} \sim \overline{b}$, then $|\mathcal{A}_n| = |\mathcal{B}_n|$ for all n.*

Proof: We use the notation $t^\infty = 0$. Clearly,

$$\sum_{n=0}^{\infty} |\mathcal{A}_n| t^n = \prod_{i=1}^{\infty} \frac{1-t^{ia_i}}{1-t^i} = \prod_{j=1}^{\infty} \frac{1-t^{jb_j}}{1-t^j} = \sum_{n=0}^{\infty} |\mathcal{B}_n| t^n,$$

which means that $|\mathcal{A}_n| = |\mathcal{B}_n|$. $\square$

Consider the classical Euler's theorem on partitions into distinct and odd parts. For $\overline{a} = (2, 2, \ldots)$ and $\overline{b} = (\infty, 1, \infty, 1, \ldots)$, $\mathcal{A}_n$ is the set of all partitions of n into distinct parts, and $\mathcal{B}_n$ is the set of partitions of n into odd parts. The bijection $i \mapsto 2i$ between $\mathrm{supp}(\overline{a}) = \mathbb{P}$ and $\mathrm{supp}(\overline{b}) = 2\mathbb{P}$ satisfies $ia_i = \varphi(i) b_{\varphi(i)}$, so $\overline{a} \sim_\varphi \overline{b}$ and $|\mathcal{A}_n| = |\mathcal{B}_n|$. We refer to this example as the *distinct/odd case*.

2.2 O'Hara's algorithm

The analytic proof of Andrews's theorem shown above does not give an explicit bijection $\mathcal{A}_n \to \mathcal{B}_n$. Such a bijection is, by Theorem 2.3, given by the following algorithm.

Algorithm 2.2 (O'Hara's algorithm on partitions)

`Fix:` sequences $\overline{a} \sim_\varphi \overline{b}$

`Input:` $\lambda \in \mathcal{A}$

`Set:` $\mu \leftarrow \lambda$

`While:` μ contains more than b_j copies of j for some j

  `Do:` remove b_j copies of j from μ, add a_i copies of i to μ, where $\varphi(i) = j$

`Output:` $\psi(\lambda) \leftarrow \mu$

Theorem 2.3 (O'Hara) *Algorithm 2.2 stops after a finite number of steps. The resulting partition $\psi(\lambda) \in \mathcal{B}$ is independent of the order of the parts removed and defines a size-preserving bijection $\mathcal{A} \to \mathcal{B}$.*

Denote by $L_\varphi(\lambda)$ the number of steps O'Hara's algorithm takes to compute $\psi(\lambda)$, and by $\mathcal{L}_\varphi(n)$ the maximum value of $L_\varphi(\lambda)$ over all $\lambda \vdash n$.

Example 2.4 *In the distinct/odd case, O'Hara's algorithm gives the inverse of Glaisher's bijection, which maps $\lambda = 1^{m_1}3^{m_3}\cdots \in \mathcal{B}$ to the partition $\mu \in \mathcal{A}$ which contains $i2^j$ if and only if m_i has a 1 in the j-th position when written in binary.* □

Example 2.5 *Let $\overline{a} = (1,1,4,5,3,1,1,\ldots)$, $\overline{b} = (1,1,5,3,4,1,1,\ldots)$ and $\varphi(3) = 4$, $\varphi(4) = 5$, $\varphi(5) = 3$, $\varphi(i) = i$ for $i \neq 3,4,5$; observe that $\overline{a} \sim_\varphi \overline{b}$. Then O'Hara's algorithm on $\lambda = 3^3 4^4 5^2$ runs as follows:*

$$\begin{array}{cccccccccc} & \mathbf{3^3 4^4 5^2} & \to & 3^7 4^1 5^2 & \to & 3^2 4^1 5^5 & \to & 3^2 4^6 5^1 & \to & 3^6 4^3 5^1 \\ \to & 3^{10} 4^0 5^1 & \to & 3^5 4^0 5^4 & \to & 3^0 4^0 5^7 & \to & 3^0 4^5 5^3 & \to & \mathbf{3^4 4^2 5^3} \end{array}$$

We have $L_\varphi(\lambda) = \mathcal{L}_\varphi(35) = 9$. □

Example 2.6 *Take $\overline{a} = (2,2,1,2,2,1,\ldots)$ and $\overline{b} = (3,1,3,1,\ldots)$. Here $\mathcal{A}$ is the set of partitions into distinct parts $\equiv \pm 1$ mod 3, and $\mathcal{B}$ is the set of partitions into odd parts, none appearing more than twice. Define $\varphi\colon \mathbb{P} \to \mathbb{P}$ as follows:*

$$\varphi(i) = \begin{cases} i & \text{if } i \text{ is divisible by 6} \\ i/3 & \text{if } i \text{ is divisible by 3, but not by 2} \\ 2i & \text{if } i \text{ is not divisible by 3} \end{cases}. \tag{1}$$

Clearly, $\overline{a} \sim_\varphi \overline{b}$. O'Hara's algorithm on $1^1 2^1 8^1 10^1 14^1 20^1$ runs as follows:

$$\begin{array}{cccccccc} & \mathbf{1^1 2^1 8^1 10^1 14^1 20^1} & \to & 1^1 2^1 8^1 10^3 14^1 & \to & 1^1 2^1 7^2 8^1 10^3 & \to & 1^1 2^1 5^2 7^2 8^1 10^2 \\ \to & 1^1 2^1 5^4 7^2 8^1 10^1 & \to & 1^1 2^1 5^6 7^2 8^1 & \to & 1^1 2^1 4^2 5^6 7^2 & \to & 1^1 2^3 4^1 5^6 7^2 \\ \to & 1^1 2^5 5^6 7^2 & \to & 1^3 2^4 5^6 7^2 & \to & 1^5 2^3 5^6 7^2 & \to & 1^7 2^2 5^6 7^2 \\ \to & 1^9 2^1 5^6 7^2 & \to & 1^{11} 5^6 7^2 & \to & 1^{11} 5^3 7^2 15^1 & \to & 1^{11} 7^2 15^2 \\ \to & 1^8 3^1 7^2 15^2 & \to & 1^5 3^2 7^2 15^2 & \to & 1^2 3^3 7^2 15^2 & \to & \mathbf{1^2 7^2 9^1 15^2} \end{array}$$

The bijection ψ is similar in spirit to Glaisher's bijection: given $\lambda = 1^{m_1}2^{m_2}4^{m_4}5^{m_5}\cdots \in \mathcal{A}$ and $j \in \mathbb{P}$, the number of copies of part $2j-1$ in $\psi(\lambda)$ is equal to the k-th digit in the ternary expansion of l, where k is the highest power of 3 dividing $2j-1$, $2j-1 = 3^k r$, and $l = \sum_i 2^i m_{r2^i}$. □

2.3 Equivalent sequences and graphs

Choose equivalent sequences $\overline{a}$, $\overline{b}$. Define a directed graph G_φ on $\mathrm{supp}(\overline{a}) \cup \mathrm{supp}(\overline{b})$ by drawing an edge from i to j if $\varphi(j) = i$; an arrow from i to j therefore means that O'Hara's algorithm simultaneously removes copies of i and adds copies of j. Each vertex v has $\mathrm{indeg}\, v \leq 1$, $\mathrm{outdeg}\, v \leq 1$ and $\mathrm{indeg}\, v + \mathrm{outdeg}\, v \geq 1$. The graph splits into connected components of the following five types:

i. cycles of length $m \geq 1$;

ii. paths of length $m \geq 2$;

iii. infinite paths with an ending point, but without a starting point;

iv. infinite paths with a starting point, but without an ending point;

v. infinite paths without a starting point or an ending point.

Example 2.7 *Figure 1 shows portions of graphs G_φ for certain φ:*

1. $\overline{a} = (1, 1, 4, 5, 3, 1, 1, \ldots)$, $\overline{b} = (1, 1, 5, 3, 4, 1, 1, \ldots)$, $\varphi(3) = 4$, $\varphi(4) = 5$, $\varphi(5) = 3$, $\varphi(i) = i$ *for* $i \neq 3, 4, 5$*; components of* G_φ *are of type* (i)*;*
2. $\overline{a} = (\infty, 1, 2, 3, \infty, \infty, \infty, \ldots)$, $\overline{b} = (2, 3, 4, \infty, \infty, \infty, \infty, \ldots)$, $\varphi(2) = 1$, $\varphi(3) = 2$, $\varphi(4) = 3$*;* G_φ *is of type* (ii)*;*
3. *the distinct/odd case:* $\overline{a} = (2, 2, \ldots)$, $\overline{b} = (\infty, 1, \infty, 1, \ldots)$, $\varphi(i) = 2i$*; components of* G_φ *are of type* (iii)*;*
4. *the odd/distinct case:* $\overline{a} = (\infty, 1, \infty, 1, \ldots)$, $\overline{b} = (2, 2, \ldots)$, $\varphi(i) = i/2$*; components of* G_φ *are of type* (iv)*;*
5. $\overline{a} = (2, 2, 1, 2, 2, 1, \ldots)$ *and* $\overline{b} = (3, 1, 3, 1, \ldots)$*,* φ *given by* (1)*; components of* G_φ *are of types* (i) *and* (v). □

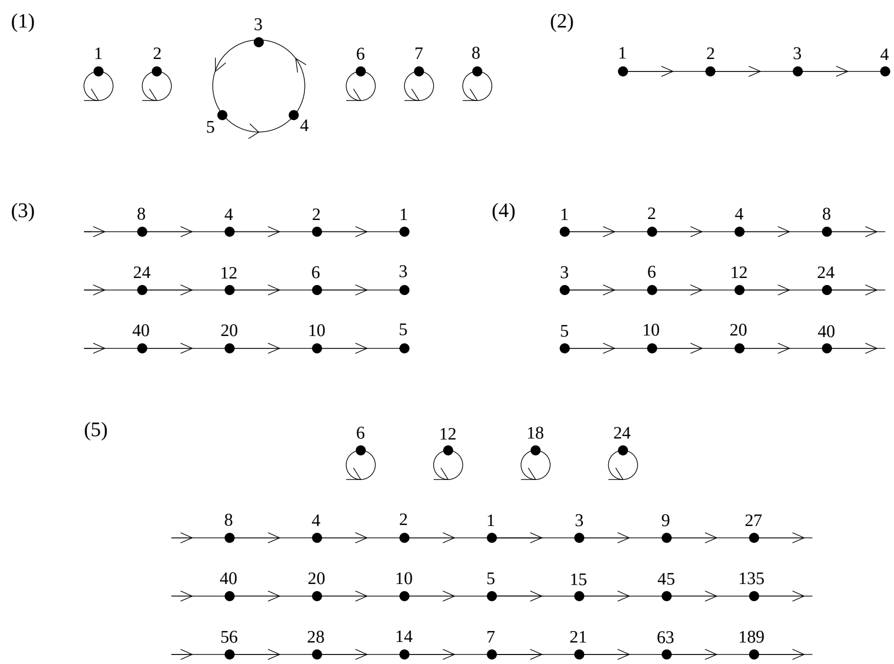


Fig. 1: Examples of graphs G_φ.

2.4 Scissor-congruence and Π-congruence

We say that convex polytopes A, B in $\mathbb{R}^m$ are *congruent*, write $A \simeq B$, if B can be obtained from A by rotation and translation. For convex polytopes $P, Q \subset \mathbb{R}^m$, we say that they are *scissor-congruent* if P can be cut into finitely many polytopes which can be rearranged and assembled into Q, i.e. if P and Q are the disjoint union of congruent polytopes: $P = \cup_{i=1}^n P_i$, $Q = \cup_{i=1}^n Q_i$, $P_i \simeq Q_i$.

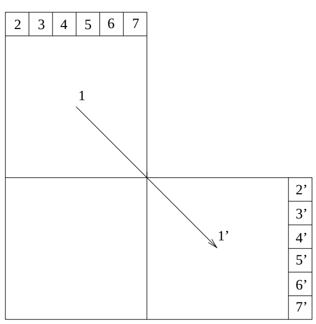
Let π be a linear functional on $\mathbb{R}^m$. If Q_i can be obtained from P_i by a translation by a vector in the hyperplane $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^m : \pi(\mathbf{x}) = 0\}$, we say that P and Q are π-*congruent*. If P and Q are π-congruent for some linear functional π, we say that they are Π-*congruent*.

If P can be cut into countably many polytopes which can be translated by a vector in the hyperplane $\mathcal{H} = \{\mathbf{x} \in \mathbb{R}^m : \pi(\mathbf{x}) = 0\}$ and assembled into Q, we say that P and Q are *approximately* π-*congruent*. We say that they are *approximately* Π-*congruent* if they are approximately π-congruent for some linear functional π. If P and Q are approximately π-congruent, there exist, for every $\varepsilon > 0$, π-congruent polytopes $P_\varepsilon \subseteq P$ and $Q_\varepsilon \subseteq Q$, such that $\mathrm{vol}(P \setminus P_\varepsilon) < \varepsilon$ and $\mathrm{vol}(Q \setminus Q_\varepsilon) < \varepsilon$.

Finally, let $\mathbf{R}(a_1, \ldots, a_m) = [0, a_1) \times \cdots \times [0, a_m)$ be a box in $\mathbb{R}^m$, and let $R(a_1, \ldots, a_m) = \mathbf{R}(a_1, \ldots, a_m) \cap \mathbb{Z}^m$ be the set of its integer points.

Example 2.8 *Let $d = 2$ and $\pi(x, y) = x + y$. Euclid's algorithm on (a, b) yields a π-congruence between $\mathbf{R}(a, b)$ and $\mathbf{R}(b, a)$: if $b = r_1 a + s_1$ with $0 \le s_1 < a$, divide $[0, a) \times [0, r_1 a)$ into r_1 squares with side a, and translate the square $[0, a) \times [ia, (i+1)a)$ by the vector $(ia, -ia)$ to $[ia, (i+1)a) \times [0, a)$. Then write $a = r_2 s_1 + s_2$ with $0 \le s_2 < s_1$, divide $[0, a) \times [r_1 a, b)$ into r_2 squares with side s_1, and translate the square $[is_1, (i+1)s_1) \times [r_1 a, b)$ by the vector $(r_1 a - is_1, is_1 - r_1 a)$ to $[r_1 a, b) \times [is_1, (i+1)s_1)$. Continue until the remainder s_i is equal to 0. The first drawing of Figure 2 gives an example.*

The second drawing shows that boxes $\mathbf{R}(12, 8)$ and $\mathbf{R}(32, 3)$ are π-congruent for $\pi(x, y) = x + 4y$. Finally, in Figure 3 we give a π-congruence between $\mathbf{R}(4, 5, 3)$ and $\mathbf{R}(5, 3, 4)$ for $\pi(x, y, z) = 3x + 4y + 5z$. $\square$



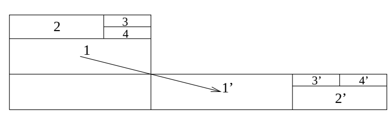


Fig. 2: Two Π-congruences.

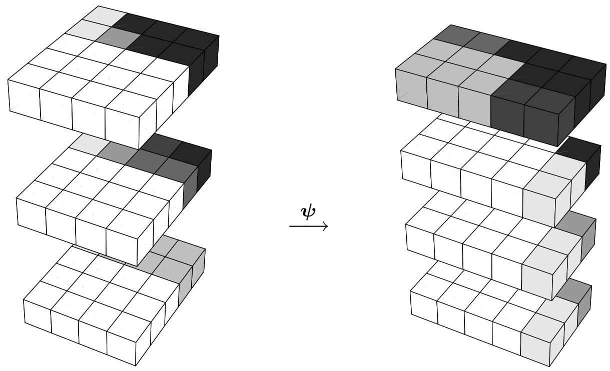


Fig. 3: π-congruence between $\mathbf{R}(4,5,3)$ and $\mathbf{R}(5,3,4)$.

3 Main results

3.1 Continuous O'Hara's algorithm and Π-congruences

Take the case when G_φ is a cycle $i_1 \to i_m \to i_{m-1} \to \ldots \to i_1$. In this case, $\varphi(i_1) = i_2, \varphi(i_2) = i_3$, etc. Throughout this section, identify a partition $i_1^{t_1} \cdots i_m^{t_m}$ with the vector $\mathbf{t} = (t_1, \ldots, t_m)$. By Theorem 2.3, O'Hara's algorithm defines a bijection $\psi\colon R(a_1, \ldots, a_m) \to R(b_1, \ldots, b_m)$, where $i_j a_j = i_{j+1} b_{j+1}$ for all j, where the indices are taken cyclically. The following algorithm (see also Theorem 3.2) generalizes ψ to the continuous setting. It gives a bijection $\boldsymbol{\psi}\colon \mathbf{R}(a_1, \ldots, a_m) \to \mathbf{R}(b_1, \ldots, b_m)$, which is defined also for non-integer a_j, b_j. When a_j, b_j are integers, it is an extension of $\psi\colon R(a_1, \ldots, a_m) \to R(b_1, \ldots, b_m)$. As an immediate corollary, we prove that two boxes with rational coordinates and with equal volume are Π-congruent. We can use Theorem 3.2 to give an alternative proof of Theorem 2.3.

Algorithm 3.1 (continuous O'Hara's algorithm)

`Fix:` $\mathbf{i} = (i_1, \ldots, i_m) \in \mathbb{R}_+^m$

$\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}_+^m$, $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbb{R}_+^m$ with $i_j a_j = i_{j+1} b_{j+1}$

`Input:` $\mathbf{t} \in \mathbf{R}(a_1, \ldots, a_m)$

`Set:` $\mathbf{s} \leftarrow \mathbf{t}$

`While:` $\mathbf{s}$ contains a coordinate $s_j \geq b_j$

`Do:` $s_j \leftarrow s_j - b_j$, $s_{j-1} \leftarrow s_{j-1} + a_{j-1}$

`Output:` $\boldsymbol{\psi}(\mathbf{t}) \leftarrow \mathbf{s}$

It is clear that the algorithm starts with an element of $P = \mathbf{R}(a_1, \ldots, a_m)$ and, if the while loop terminates, outputs an element of $Q = \mathbf{R}(b_1, \ldots, b_m)$. It is not obvious, however, that the loop terminates in every case, or that the output $\boldsymbol{\psi}(\mathbf{t})$ and the number of steps $\mathbf{L}_\varphi(\mathbf{t})$ depend only on $\mathbf{t}$, not on the choices made in the while loop.

Theorem 3.2 *Algorithm 3.1 has the following properties.*

1. *The algorithm stops after a finite number of steps, and the resulting vector $\psi(\mathbf{t})$ and the number of steps $\mathbf{L}_\varphi(\mathbf{t})$ are independent of the choices made during the execution of the algorithm.*

2. *The algorithm defines a bijection $\psi\colon P \to Q$ which satisfies $\psi(\mathbf{t}) - \mathbf{t} \in \mathcal{H}$, where $\mathcal{H}$ is the hyperplane defined by $i_1x_1 + \ldots + i_mx_m = 0$.*

3. *We have*
$$\mathbf{L}_\varphi(\mathbf{t}+\mathbf{t}') \geq \mathbf{L}_\varphi(\mathbf{t}) + \mathbf{L}_\varphi(\mathbf{t}') \textit{ for every } \mathbf{t}, \mathbf{t}', \mathbf{t}+\mathbf{t}' \in P.$$
In particular, $\mathbf{L}_\varphi(\mathbf{t}') \leq \mathbf{L}_\varphi(\mathbf{t})$ if $\mathbf{t}' \leq \mathbf{t}$.

4. *Let $\mathbf{t}, \mathbf{t}' \in P$, $\mathbf{s} = \psi(\mathbf{t})$, with $t_j \leq t'_j < t_j + \varepsilon_j$, where $\varepsilon_j = b_j - s_j$. Then*
$$\psi(\mathbf{t}') - \mathbf{t}' = \psi(\mathbf{t}) - \mathbf{t} \quad \textit{and} \quad \mathbf{L}_\varphi(\mathbf{t}') = \mathbf{L}_\varphi(\mathbf{t}).$$

5. *For all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^m$, we have*
$$\max_{\mathbf{t}\in P} \mathbf{L}_\varphi(\mathbf{t}) \; = \; \operatorname{lcm}(c_1, \ldots, c_m) \cdot \left(\frac{1}{c_1} + \ldots + \frac{1}{c_m}\right) - m,$$
where $c_j = a_1 \cdots a_{j-1} b_j \cdots b_{m-1}$.

We call boxes $P = \mathbf{R}(a_1, \ldots, a_m)$, $Q = \mathbf{R}(b_1, \ldots, b_m)$ *relatively rational* if there exists λ, $\lambda \neq 0$, such that $\lambda a_j \in \mathbb{Z}$, $\lambda b_j \in \mathbb{Z}$. Clearly, two boxes P and Q with rational side-lengths are relatively rational.

Corollary 3.3 *Boxes $P = \mathbf{R}(a_1, \ldots, a_m)$, $Q = \mathbf{R}(b_1, \ldots, b_m)$ with equal volume are approximately Π-congruent. Moreover, when P and Q are relatively rational and have equal volume, they are Π-congruent.*

Proof: For $j = 1, \ldots, m$, take $i_j = a_1 \cdots a_{j-1} b_{j+1} \cdots b_m$. Clearly $i_j a_j = i_{j+1} b_{j+1}$ for $j = 1, \ldots, m-1$, and $a_1 \cdots a_m = b_1 \cdots b_m$ implies $i_m a_m = i_1 b_1$. Therefore, the numbers i_j, a_j, b_j satisfy the conditions of Algorithm 3.1. By Theorem 3.2 part (2), the algorithm defines a bijection $\psi\colon P \to Q$. Parts (4) and (2) of Theorem 3.2 imply that we can cut P into (countably many) smaller boxes, each of which is translated by a vector in the plane $i_1x_1 + \ldots + i_mx_m = 0$.

If P and Q are relatively rational, we can assume without loss of generality that all a_j, b_j are integers. For any integer vector $\mathbf{t}$, we have $\psi(\mathbf{t}') - \mathbf{t}' = \psi(\mathbf{t}) - \mathbf{t}$ and $\mathbf{L}_\varphi(\mathbf{t}') = \mathbf{L}_\varphi(\mathbf{t})$ whenever $t_j \leq t'_j < t_j + 1$, so P and Q are divided into a finite number (at most $a_1 \cdots a_m$) of boxes. □

Example 3.4 *Even in the 3-dimensional case the Π-congruence defined by the algorithm can be quite complex, as the next figure suggests. Here the same shading is used for parallel translations by the same vector.* □

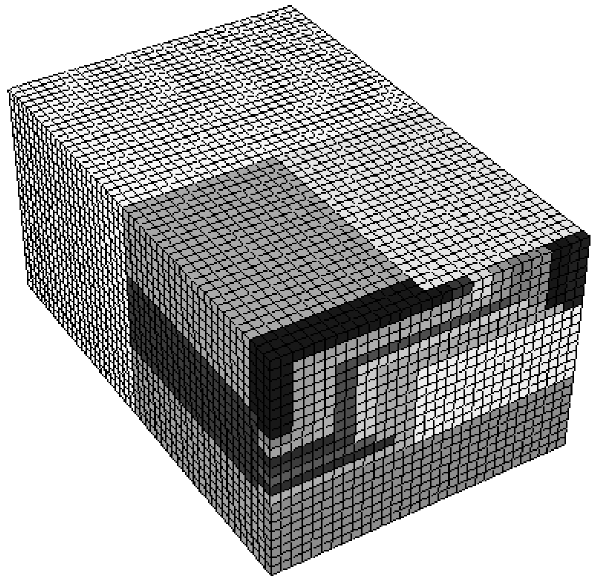


Fig. 4: The decomposition of the box $\mathbf{R}(31, 47, 23)$ given by O'Hara's algorithm (only the top, right, and back sides are shown) .

3.2 Complexity of O'Hara's algorithm

The complexity of O'Hara's algorithm has been an open problem, with the exception of the elementary distinct/odd case (see [O84]).

It turns out that the complexity depends heavily on the type of the graph G_φ defined in Subsection 2.3. Part (5) of Theorem 3.2 gives the maximum number of steps that O'Hara's algorithm takes when G_φ is a cycle. The following lemma gives an estimate for $\mathcal{L}_\varphi(n)$ when G_φ is a path.

Lemma 3.5 *Let G_φ be a finite or infinite path on $\mathcal{I} \subseteq \mathbb{P}$. Then $\mathcal{L}_\varphi(n) \le n(\log n + 1)$. Moreover, if*

$$D = \sum_{i \in \mathcal{I}} \frac{1}{i a_i} = \sum_{j \in \mathcal{I}} \frac{1}{j b_j} < \infty,$$

then $\mathcal{L}_\varphi(n) \le Dn$. Here, by $\log n$ we mean the natural logarithm of n.

Theorem 3.6 *Let $\overline{a}, \overline{b}$ be φ-equivalent sequences.*

1. *If G_φ has only a finite number of cycles of length > 2, then $\mathcal{L}_\varphi(n) = O(n \log n)$, and the constants implied by the O-notation are universal.*

2. *If G_φ has only a finite number of cycles of length $> m$ for some $m > 2$, then $\mathcal{L}_\varphi(n) = O(n^{m-1})$, and the constants implied by the O-notation depend only on m.*

The following theorem gives the corresponding lower bound on the worst case complexity. It shows that the estimates of Theorem 3.6 are close to being sharp.

Theorem 3.7 *There exist φ-equivalent sequences $\overline{a}$ and $\overline{b}$, such that:*

1. *G_φ is a path and $\mathcal{L}_\varphi(n) = \Omega(n \log\log n)$;*
2. *G_φ contains only cycles of length $\le m$ and $\mathcal{L}_\varphi(n) = \Omega(n^{m-1-\varepsilon})$ for every $\varepsilon > 0$;*
3. *$\mathcal{L}_\varphi(n) = \exp\Omega(\sqrt[3]{n})$.*

In other words, depending on the type of the graph, we have nearly matching upper and lower bounds on $\mathcal{L}_\varphi(n)$. For example, for an m-cycle, Theorem 3.6 shows that $\mathcal{L}_\varphi(n)$ is $O(n^{m-1})$, while Theorem 3.7 shows that it is $\Omega(n^{m-1-\varepsilon})$ for every $\varepsilon > 0$. Similarly, part (3) shows that O'Hara's algorithm can be very slow in general since the total number of partitions of n is asymptotically $\exp\Theta(\sqrt{n})$.

3.3 O'Hara's algorithm as an integer linear programming problem

Let us now give a new description of O'Hara's algorithm.

Proposition 3.8 *Let $\mathbf{i}, \mathbf{a}, \mathbf{b} \in$ be as above such that $i_j a_j = i_{j+1} b_{j+1}$ for $j = 1, \ldots, m$. Fix a vector $\mathbf{t} \in \mathbf{R}(a_1, \ldots, a_m)$. Then $\mathbf{s} = \boldsymbol{\psi}(\mathbf{t})$ satisfies the following:*

$$\mathbf{s} = \mathbf{t} + A\mathbf{k},$$

where

$$A = \begin{pmatrix} -b_1 & a_1 & 0 & \cdots & 0 \\ 0 & -b_2 & a_2 & \cdots & 0 \\ 0 & 0 & -b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m & 0 & 0 & \cdots & -b_m \end{pmatrix}$$

and $\mathbf{k} = (k_1, \ldots, k_m)$ is the unique vector minimizing

$$k_1 + \ldots + k_m$$

with constraints

$$\mathbf{k} \in \mathbb{Z}^m, \qquad \mathbf{k} \ge \mathbf{0}, \qquad A\mathbf{k} \ge -\mathbf{t}, \qquad A\mathbf{k} \le \mathbf{b} - \mathbf{1} - \mathbf{t}.$$

Proposition 3.8 can be used to obtain a significant speed-up of (the usual) O'Hara's algorithm, in the case when G_φ contains only cycles of bounded length. Namely, we obtain the following result.

Theorem 3.9 *Let $\overline{a} \sim_\varphi \overline{b}$. If the lengths of cycles of G_φ are bounded, there exists a deterministic algorithm which computes $\psi(\lambda)$ in $O(n \log n)$ steps for $\lambda \in \mathcal{A}_n$.*

Proof: Without loss of generality, the support of $\lambda \in \mathcal{A}_n$ is contained in one of the connected components of G_φ. If this connected component is a path, O'Hara's algorithm takes $O(n \log n)$ steps by Lemma 3.5. If it is a cycle of length m, we can use the algorithm described in, say, [S86, Corollary 18.7b] to compute $\psi(\lambda)$ in $O(\log^c n)$ steps for some c. Obviously the $O(n \log n)$ term dominates. □

Remark 3.10 *Let us note that the inner workings of the algorithm in Theorem 3.9 have a geometric rather than combinatorial nature, and are very different from those of O'Hara's algorithm. However, both kinds of algorithms produce the same partition bijection.*

4 Final remarks

4.1

The polynomial time algorithm in the proof of Theorem 3.9 is given implicitly, by using the general results in integer linear programming. It is saying that the function $\psi : \mathcal{A}_n \to \mathcal{B}_n$ can be computed much faster, by circumventing the elegant construction of O'Hara's algorithm. It would be interesting to give an explicit construction of such an algorithm.

In a different direction, it might prove useful to restate other involution principle bijections in the language of linear programming, such as the Rogers-Ramanujan bijection in [GM81b] or in [BP06]. If this works, this might lead to a new type of a bijection between these two classes of partitions. Alternatively, this might resolve the conjecture by the second author on the mildly exponential complexity of Garsia-Milne's Rogers-Ramanujan bijection, see [P06, Conjecture 8.5].

4.2

Note the gap between the number $\exp \Theta(\sqrt{n})$ of partitions of n and the lower bound $\mathcal{L}_\varphi(n) = \exp \Omega(\sqrt[3]{n})$ in Theorem 3.7. It would be interesting to decide which of the two worst complexity bounds on the number of steps of O'Hara's algorithm is closer to the truth.

Note that we applied our linear programming approach only in the bounded cycle case. We do not know if there is a way to apply the same technique to the general case. However, we believe that there are number theoretic obstacles preventing that and in fact, computing O'Hara's bijection as a function on partitions may be hard in the formal complexity sense.

4.3

Recently, variations on the O'Hara's bijection and applications of rewrite systems were found in [SSM04] and [K04, K07]. It would be interesting to see connections between our analysis and this work.

4.4

Recall also that the 2-dimensional case can be viewed as the Euclid algorithm which in turn corresponds to the usual continued fractions (see Example 2.8). Thus the geometry of ψ can be viewed as a delicate multidimensional extension of continued fractions. Given the wide variety of (different) multidimensional continued fractions available in the literature, it would be interesting to see if there is a connection to at least one of these notions.

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A breakthrough in the theory of (type A) Macdonald polynomials is due to Haglund, Haiman and Loehr, who exhibited a combinatorial formula for these polynomials in terms of fillings of Young diagrams. Recently, Ram and Yip gave a formula for the Macdonald polynomials of arbitrary type in terms of the corresponding affine Weyl group. In this paper, we show that a Haglund-Haiman-Loehr type formula follows naturally from the more general Ram-Yip formula, via compression. Then we extend this approach to the Hall-Littlewood polynomials of type C, which are specializations of the corresponding Macdonald polynomials at $q = 0$. We note that no analog of the Haglund-Haiman-Loehr formula exists beyond type A, so our work is a first step towards finding such a formula.

Keywords: Macdonald polynomials, Hall-Littlewood polynomials, Haglund-Haiman-Loehr formula, alcove walks, Ram-Yip formula, Schwer's formula.

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Combinatorial Formulas for Macdonald and Hall-Littlewood Polynomials

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1 Introduction

Macdonald [14, 15] defined a remarkable family of symmetric orthogonal polynomials depending on parameters q, t, which bear his name. These polynomials generalize several other symmetric polynomials related to representation theory. For instance, at $q = 0$, the Macdonald polynomials specialize to the Hall-Littlewood polynomials (or spherical functions on p-adic groups), and they further specialize to the Weyl characters (upon setting $t = 0$ as well). There has been considerable interest recently in the combinatorics of Macdonald polynomials. This stems in part from a combinatorial formula for the ones corresponding to type A, which is due to Haglund, Haiman, and Loehr [4], and which is in terms of fillings of Young diagrams. This formula uses two statistics on the mentioned fillings, called "inv" and "maj". The Haglund-Haiman-Loehr formula already found important applications, such as new proofs of the Schur positivity for Macdonald polynomials [1, 3]. Let us also note that there is a version of the Haglund-Haiman-Loehr formula for the non-symmetric Macdonald polynomials [5], as well as a different formula for these polynomials due to Lascoux [7].

Schwer [18] gave a formula for the Hall-Littlewood polynomials of arbitrary type. Throughout this paper, we use the version of Schwer's formula that was derived by Ram [16]. Schwer's formula is in terms of so-called alcove walks, which originate in the work of Gaussent-Littelmann [2] and of the author with Postnikov [10, 11] on discrete counterparts to the Littelmann path model [12, 13]. Schwer's formula was recently generalized by Ram and Yip to a similar formula for the Macdonald polynomials [17]. The generalization consists in the fact that the latter formula is in terms of alcove walks with both "positive" and "negative" foldings, whereas in the former only "positive" foldings appear.

In this paper, we relate the Ram-Yip formula to the Haglund-Haiman-Loehr formula. More precisely, we show that we can group the terms in the type A version of the Ram-Yip formula into equivalence classes, such that the sum in each class is a term in a new formula, which is similar to the Haglund-Haiman-Loehr one but contains considerably fewer terms, see [9]. An equivalence class consists of all the terms corresponding to alcove walks that produce the same filling of a Young diagram λ (indexing the Macdonald polynomial) via a simple construction. In fact, in this paper we require that the partition λ is a regular weight; the general case will be considered elsewhere.

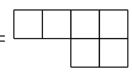
Our approach has the advantage of deriving the Haglund-Haiman-Loehr statistics "inv" and "maj" on fillings of Young diagrams in a natural way, from more general concepts. It also has the advantage of being applicable to other root systems, where no analog of the Haglund-Haiman-Loehr formula exists. As a first step in this direction, we derive here a formula in terms of fillings of Young diagrams for the Hall-Littlewood polynomials of type C indexed by a regular weight; we proceed by compressing the type C version of Schwer's formula. A completely similar formula exists in type B, while type D is slightly more complex.

The structure of this extended abstract is as follows. In Section 2 we present our formula of Haglund-Haiman-Loehr type for the Macdonald polynomials of type A. In Section 3 we present our new formula for the Hall-Littlewood polynomials of type C in terms of fillings of Young diagrams. In Section 4 we give background information on root systems, alcove walks, the Ram-Yip formula, and Schwer's formula. In Section 5 we specialize the Ram-Yip formula to type A and explain how it compresses to our formula for the corresponding Macdonald polynomials. In Section 6 we specialize Schwer's formula to type C and explain how it compresses to our formula for the corresponding Hall-Littlewood polynomials. The full length versions of sections 5 and 6 are [9] and [8], respectively.

2 A new formula of Haglund-Haiman-Loehr type

In this section we present a new formula for the Macdonald polynomials of type A that is similar to the Haglund-Haiman-Loehr one [4]. This formula will be derived by compressing the Ram-Yip formula [17]. It also turns out that the new formula has considerably fewer terms even than the Haglund-Haiman-Loehr formula.

Let us consider a partition with $n-1$ distinct parts $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_{n-1} > 0)$ for a fixed n (this corresponds to a dominant regular weight for the root system of type A_{n-1}). Using standard notation, one defines $n(\lambda) := \sum_i (i-1)\lambda_i$. We identify λ with its Young (or Ferrers) diagram, as usual, and denote by (i,j) the cell in row i and column j, where $1 \leq j \leq \lambda_i$. We draw this diagram in "Japanese style", that is, we embed it in the third quadrant, as shown below:

$\lambda = (4, 2) =$  .

For any cell $u = (i,j)$ of λ with $j \neq 1$, denote the cell $v = (i, j-1)$ directly to the right of u by $\mathrm{r}(u)$.

Two cells $u, v \in \lambda$ are said to *attack* each other if either

(i) they are in the same column: $u = (i,j)$, $v = (k,j)$; or

(ii) they are in consecutive columns, with the cell in the left column strictly above the one in the right column: $u = (i,j)$, $v = (k, j-1)$, where $i < k$.

Remark 2.1 The main difference in our approach compared to the Haglund-Haiman-Loehr one is in the definition of attacking cells; note that in [4] these cells are defined similarly, except that $u = (i,j)$ and $v = (k, j-1)$ with $i > k$ attack each other.

A *filling* is a function $\sigma : \lambda \to [n] := \{1, \ldots, n\}$ for some n, that is, an assignment of values in $[n]$ to the cells of λ. As usual, we define the content of a filling σ as $\mathrm{content}(\sigma) := (c_1, \ldots, c_n)$, where c_i is the number of entries i in the filling, i.e., $c_i := |\sigma^{-1}(i)|$. The monomial $x^{\mathrm{content}(\sigma)}$ in the variables $x_1, \ldots, x_n$ is then given by $x^{\mathrm{content}(\sigma)} := x_1^{c_1} \ldots, x_n^{c_n}$.

Definition 2.2 *A filling $\sigma : \lambda \to [n]$ is called* non-attacking *if $\sigma(u) \neq \sigma(v)$ whenever u and v attack each other. Let $\mathcal{T}(\lambda, n)$ denote the set of non-attacking fillings.*

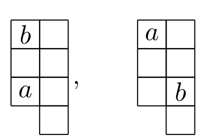
Definition 2.3 *Given a filling σ of λ, let*

$$\mathrm{Des}(\sigma) := \{(i,j) \in \lambda \,:\, (i, j+1) \in \lambda\,,\ \ \sigma(i,j) > \sigma(i,j+1)\}\,,$$
$$\mathrm{Diff}(\sigma) := \{(i,j) \in \lambda \,:\, (i, j+1) \in \lambda\,,\ \ \sigma(i,j) \neq \sigma(i,j+1)\}\,.$$

We define a reading order on the cells of λ as the total order given by considering the columns from right to left (largest to smallest), and by reading each column from top to bottom. Note that this is a different reading order than the usual (French or Japanese) ones.

Definition 2.4 *An* inversion *of σ is a pair (u, v) of attacking cells, where u precedes v in the considered reading order and $\sigma(u) > \sigma(v)$. Let $\mathrm{Inv}(\sigma)$ denote the set of inversions of σ.*

Here are two examples of inversions, where $a < b$:



The *arm* of a cell $u \in \lambda$ is the number of cells strictly to the left of u in the same row; similarly, the *leg* of u is the number of cells strictly below u in the same column.

Definition 2.5 *The* maj *and* inv *statistics on fillings σ are defined by*

$$\mathrm{maj}(\sigma) := \sum_{u \in \mathrm{Des}(\sigma)} \mathrm{arm}(u)\,, \quad \mathrm{inv}(\sigma) := |\mathrm{Inv}(\sigma)| - \sum_{u \in \mathrm{Des}(\sigma)} \mathrm{leg}(u)\,.$$

We are now ready to state a new combinatorial formula for the Macdonald P-polynomials in the variables $X = (x_1, \ldots, x_n)$.

Theorem 2.6 *Given a partition λ with $n-1$ distinct parts, we have*

$$P_\lambda(X;q,t) = \sum_{\sigma \in \mathcal{T}(\lambda,n)} t^{n(\lambda)-\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \left(\prod_{u \in \mathrm{Diff}(\sigma)} \frac{1-t}{1-q^{\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}} \right) x^{\mathrm{content}(\sigma)}\,. \tag{1}$$

3 Hall-Littlewood polynomials of type C_n

In this section we present a new formula for the Hall-Littlewood polynomials of type C in terms of fillings of Young diagrams. This formula will be derived by compressing Schwer's formula [18] (cf. also [16]).

Let $\lambda = (\lambda_1 > \ldots > \lambda_n > 0)$ be a partition with n distinct parts for a fixed $n \geq 2$ (this corresponds to a dominant regular weight for the root system of type C_n). Consider the shape $\widehat{\lambda}$ obtained from λ by replacing each column of height k with k or $2k-1$ (adjacent) copies of it, depending on the given column being the first one or not. We are representing a filling σ of $\widehat{\lambda}$ as a concatenation of columns C_{ij} and C'_{ik}, where $i = 1, \ldots, \lambda_1$, while for a given i we have $j = 1, \ldots, \lambda'_i$ if $i > 1$, $j = 1$ if $i = 1$, and $k = 2, \ldots, \lambda'_i$; the columns C_{ij} and C'_{ik} have height λ'_i. The diagram $\widehat{\lambda}$ is represented in "Japanese style", like in the previous section, i.e., the heights of columns increase from left to right; more precisely, we let

$$\sigma = \mathcal{C}^{\lambda_1} \ldots \mathcal{C}^1\,, \quad \text{where } \mathcal{C}^i := \begin{cases} C'_{i2} \ldots C'_{i,\lambda'_i} C_{i1} \ldots C_{i,\lambda'_i} & \text{if } i > 1 \\ C'_{i2} \ldots C'_{i,\lambda'_i} C_{i1} & \text{if } i = 1\,. \end{cases}$$

Note that the leftmost column is $C_{\lambda_1,1}$, and the rightmost column is C_{11}. For an example, we refer to Section 6.

Essentially, the above description says that the column to the right of C_{ij} is $C_{i,j+1}$, whereas the column to the right of C'_{ik} is $C'_{i,k+1}$. Here we are assuming that the mentioned columns exist, up to the following conventions:

$$C_{i,\lambda'_i+1} = \begin{cases} C'_{i-1,2} & \text{if } i > 1 \text{ and } \lambda'_{i-1} > 1 \\ C_{i-1,1} & \text{if } i > 1 \text{ and } \lambda'_{i-1} = 1\,, \end{cases} \qquad C'_{i,\lambda'_i+1} = C_{i1}\,. \tag{2}$$

We consider the alphabet $[\overline{n}] := \{1 < \ldots < n < \overline{n} < \overline{n-1} < \ldots < \overline{1}\}$, where the barred entries are viewed as negatives, so that $-\overline{\imath} = i$. Next, we consider the set $\mathcal{T}(\widehat{\lambda}, \overline{n})$ of fillings of $\widehat{\lambda}$ with entries in $[\overline{n}]$ which satisfy the following conditions:

1. the rows are weakly decreasing from left to right;
2. no column contains two entries a, b with $a = \pm b$;
3. any two adjacent columns are related as indicated below (essentially, they differ by a "signed cycle").

In order to explain the mentioned relation between adjacent columns, we consider right actions of type C reflections on columns (see Section 6). For instance, $C(a, \overline{b})$ is the column obtained from C by transposing the entries in positions a, b and by changing their signs. Let us first explain the passage from some column C_{ij} to $C_{i,j+1}$. There exist positions $1 \leq r_1 < \ldots < r_p < j$ (possibly $p = 0$) such that $C_{i,j+1}$ differs from $D = C_{ij}(r_1, \overline{\jmath}) \ldots (r_p, \overline{\jmath})$ only in position j, while $C_{i,j+1}(j) \notin \{\pm D(r) \,:\, r \in [\lambda'_i] \setminus \{j\}\}$ and $C_{i,j+1}(j) \leq D(j)$. To include the case $j = \lambda'_i$ in this description, just replace $C_{i,j+1}$ everywhere by $C_{i,j+1}[1, \lambda'_i]$ and use the conventions (2). Let us now explain the passage from some column C'_{ik} to $C'_{i,k+1}$. There exist positions $1 \leq r_1 < \ldots < r_p < k$ (possibly $p = 0$) such that $C'_{i,k+1} = C'_{ik}(r_1, \overline{k}) \ldots (r_p, \overline{k})$. This description includes the case $k = \lambda'_i$, based on the conventions (2).

Let us now define the content of a filling. For this purpose, we first associate with a filling σ a compressed version of it, namely the filling $\overline{\sigma}$ of the partition 2λ. This is defined as follows:

$$\overline{\sigma} = \overline{\mathcal{C}}^{\lambda_1} \ldots \overline{\mathcal{C}}^{1}\,, \quad \text{where } \overline{\mathcal{C}}^{i} := C'_{i2} C_{i1}\,, \tag{3}$$

where the conventions (2) are used again. Now define $\mathrm{content}(\sigma) = (m_1, \ldots, m_n)$, where m_i is half the difference between the number of occurences of the entries i and $\overline{\imath}$ in $\overline{\sigma}$.

We now define two statistics on fillings that will be used in our compressed formula for Hall-Littlewood polynomials. Intervals refer to the discrete set $[\overline{n}]$. Let

$$\sigma_{ab} := \begin{cases} 1 & \text{if } a, b \geq \overline{n} \\ 0 & \text{otherwise}\,. \end{cases}$$

Given a sequence of integers w, we write $w[i, j]$ for the subsequence $w(i)w(i+1) \ldots w(j)$. We use the notation $N_{ab}(w)$ for the number of entries $w(i)$ in the interval (a, b).

Given two columns D, C of the same height d such that $D \geq C$ in the componentwise order, we will define two statistics $N(D, C)$ and $\mathrm{des}(D, C)$ in some special cases, as specified below.

Case 0. If $D = C$, then $N(D, C) := 0$ and $\mathrm{des}(D, C) := 0$.

Case 1. Assume that $C = D(r, \overline{\jmath})$ with $r < j$. Let $a := D(r)$ and $b := D(j)$. In this case, we set

$$N(D, C) := N_{\overline{b}a}(D[r+1, j-1]) + |(\overline{b}, a) \setminus \{\pm D(i) \,:\, i = 1, \ldots, j\}| + \sigma_{ab}\,, \quad \mathrm{des}(D, C) := 1\,.$$

Case 2. Assume that $C = D(r_1, \overline{\jmath}) \ldots (r_p, \overline{\jmath})$ where $1 \leq r_1 < \ldots < r_p < j$. Let $D_i := D(r_1, \overline{\jmath}) \ldots (r_i, \overline{\jmath})$ for $i = 0, \ldots, p$, so that $D_0 = D$ and $D_p = C$. We define

$$N(D, C) := \sum_{i=1}^{p} N(D_{i-1}, D_i)\,, \quad \mathrm{des}(D, C) := p\,.$$

Case 3. Assume that C differs from $D' := D(r_1, \overline{\jmath}) \ldots (r_p, \overline{\jmath})$ with $1 \leq r_1 < \ldots < r_p < j$ (possibly $p = 0$) only in position j, while $C(j) \notin \{\pm D'(r) \,:\, r \in [d] \setminus \{j\}\}$ and $C(j) \leq D'(j)$. We define

$$N(D, C) := N(D, D') + N_{C(j), D'(j)}(D[j+1, d])\,, \quad \mathrm{des}(D, C) := p + 1\,.$$

If the height of C is larger than the height d of D (necessarily by 1), and $N(D, C[1, d])$ can be computed as above, we let $N(D, C) := N(D, C[1, d])$ and $\mathrm{des}(D, C) := \mathrm{des}(D, C[1, d])$. Given a filling σ in $\mathcal{T}(\widehat{\lambda}, \overline{n})$ with columns $C_m, \ldots, C_1$, we set

$$N(\sigma) := \sum_{i=1}^{m-1} N(C_{i+1}, C_i) + \mathrm{inv}(C_1)\,;$$

here $\mathrm{inv}(C_1)$ denotes the number of (ordinary) inversions in C_1, that is, the number of pairs $i < j$ of positions in C_1 with $C_1(i) > C_1(j)$. Furthermore, in the mentioned case, we also set

$$\mathrm{des}(\sigma) := \sum_{i=1}^{m-1} \mathrm{des}(C_{i+1}, C_i)\,.$$

We can now state our new formula for the Hall-Littlewood polynomials of type C. We refer to Remarks 6.6 for more comments on this formula.

Theorem 3.1 *Given a partition λ with n distinct parts, the Hall-Littlewood polynomial $P_\lambda(X;t)$ is given by*

$$P_\lambda(X;t) = \sum_{\sigma \in \mathcal{T}(\widehat{\lambda}, \overline{n})} t^{N(\sigma)}\,(1-t)^{\mathrm{des}(\sigma)}\,x^{\mathrm{content}(\sigma)}\,. \tag{4}$$

4 Alcove walks and Macdonald polynomials

4.1 Root systems

We recall some background information on finite root systems and affine Weyl groups. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra, whose rank is r. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{h}^*$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. Let $\rho := \frac{1}{2}(\sum_{\alpha \in \Phi^+} \alpha)$. Let $\alpha_1, \ldots, \alpha_r \in \Phi^+$ be the corresponding *simple roots*. We denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the non-degenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, we consider the corresponding *coroot* $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ and reflection s_α.

Let W be the corresponding *Weyl group*, whose Coxeter generators are denoted, as usual, by $s_i := s_{\alpha_i}$. The length function on W is denoted by $\ell(\,\cdot\,)$. The *Bruhat order* on W is given by its covers $w \lessdot ws_\beta$, where $\beta \in \Phi^+$, and $\ell(ws_\beta) = \ell(w) + 1$.

The *weight lattice* Λ is given by $\Lambda := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \;:\; \langle\lambda,\alpha^\vee\rangle \in \mathbb{Z}\}$ for any $\alpha \in \Phi$. The weight lattice Λ is generated by the *fundamental weights* $\omega_1, \ldots, \omega_r$, which form the dual basis to the basis of simple coroots, i.e., $\langle\omega_i, \alpha_j^\vee\rangle = \delta_{ij}$. The set Λ^+ of *dominant weights* is given by $\Lambda^+ := \{\lambda \in \Lambda \;:\; \langle\lambda,\alpha^\vee\rangle \geq 0\}$ for any $\alpha \in \Phi^+$. Let $\mathbb{Z}[\Lambda]$ be the group algebra of the weight lattice Λ, which has a $\mathbb{Z}$-basis of formal exponents $\{x^\lambda \;:\; \lambda \in \Lambda\}$ with multiplication $x^\lambda \cdot x^\mu := x^{\lambda+\mu}$.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane

$$H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \;:\; \langle\lambda,\alpha^\vee\rangle = k\}. \tag{5}$$

These reflections generate the *affine Weyl group* W_{aff} for the *dual root system* $\Phi^\vee := \{\alpha^\vee \;:\; \alpha \in \Phi\}$. The hyperplanes $H_{\alpha,k}$ divide the real vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves*. The *fundamental alcove* A° is given by

$$A^\circ := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \;:\; 0 < \langle\lambda,\alpha^\vee\rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

4.2 Alcove walks

We say that two alcoves A and B are *adjacent* if they are distinct and have a common wall. Given a pair of adjacent alcoves $A \neq B$ (i.e., having a common wall), we write $A \xrightarrow{\beta} B$ if the common wall is of the form $H_{\beta,k}$ and the root $\beta \in \Phi$ points in the direction from A to B.

Definition 4.1 [10] *An* alcove path *is a sequence of alcoves such that any two consecutive ones are adjacent. We say that an alcove path $(A_0, A_1, \ldots, A_m)$ is* reduced *if m is the minimal length of all alcove paths from A_0 to A_m.*

We need the following generalization of alcove paths.

Definition 4.2 *An* alcove walk *is a sequence $\Omega = (A_0, F_1, A_1, F_2, \ldots, F_m, A_m, F_\infty)$ such that $A_0, \ldots, A_m$ are alcoves; F_i is a codimension one common face of the alcoves A_{i-1} and A_i, for $i = 1, \ldots, m$; and F_∞ is a vertex of the last alcove A_m. The weight F_∞ is called the* weight *of the alcove walk, and is denoted by $\mu(\Omega)$.*

The *folding operator* ϕ_i is the operator which acts on an alcove walk by leaving its initial segment from A_0 to A_{i-1} intact and by reflecting the remaining tail in the affine hyperplane containing the face F_i. In other words, we define

$$\phi_i(\Omega) := (A_0, F_1, A_1, \ldots, A_{i-1}, F'_i = F_i, A'_i, F'_{i+1}, A'_{i+1}, \ldots, A'_m, F'_\infty)\,,$$

where $A'_j := \rho_i(A_j)$ for $j \in \{i, \ldots, m\}$, $F'_j := \rho_i(F_j)$ for $j \in \{i, \ldots, m\} \cup \{\infty\}$, and ρ_i is the affine reflection in the hyperplane containing F_i. Note that any two folding operators commute. An index j such that $A_{j-1} = A_j$ is called a *folding position* of Ω. Let $\mathrm{fp}(\Omega) := \{j_1 < \ldots < j_s\}$ be the set of folding positions of Ω. If this set is empty, Ω is called *unfolded*. Given this data, we define the operator "unfold", producing an unfolded alcove walk, by

$$\mathrm{unfold}(\Omega) = \phi_{j_1} \ldots \phi_{j_s}(\Omega) .$$

Definition 4.3 *A folding position j of the alcove walk $\Omega = (A_0, F_1, A_1, F_2, \ldots, F_m, A_m, F_\infty)$ is called a* positive folding *if the alcove $A_{j-1} = A_j$ lies on the positive side of the affine hyperplane containing the face F_j. Otherwise, the folding position is called a* negative folding.

Let $\tau_\lambda \in W_{\mathrm{aff}}$ denote the translation by λ. Recall the bijection $A \mapsto v_A$ between alcoves and affine Weyl group elements given by $v_A(A^\circ) = A$. We now fix a dominant weight λ and a reduced alcove path $\Pi := (A_0, A_1, \ldots, A_m)$ from $A^\circ = A_0$ to the alcove A_m corresponding to the minimal element in the coset $\tau_\lambda W$ under the mentioned bijection. Assume that we have

$$A_0 \xrightarrow{\beta_1} A_1 \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_m} A_m , \tag{6}$$

where $\Gamma := (\beta_1, \ldots, \beta_m)$ is a sequence of positive roots. This sequence, which determines the alcove path, is called a λ-*chain* (of roots).

We also let $r_i := s_{\beta_i}$, and let $\widehat{r}_i$ be the affine reflection in the common wall of A_{i-1} and A_i, for $i = 1, \ldots, m$; in other words, $\widehat{r}_i := s_{\beta_i, l_i}$, where $l_i := |\{j \le i \ : \ \beta_j = \beta_i\}|$ is the cardinality of the corresponding set. Given $J = \{j_1 < \ldots < j_s\} \subseteq [m] := \{1, \ldots, m\}$, we define the Weyl group element $\phi(J)$ and the weight $\mu(J)$ by

$$\phi(J) := r_{j_1} \ldots r_{j_s} , \qquad \mu(J) := \widehat{r}_{j_1} \ldots \widehat{r}_{j_s}(\lambda) . \tag{7}$$

4.3 The Ram-Yip formula for Macdonald polynomials

Given $w \in W$ and the alcove path Π considered above, we define the alcove path

$$w(\Pi) := (w(A_0), w(A_1), \ldots, w(A_m)) .$$

Consider the set of alcove paths $\mathcal{P}(\Gamma) := \{w(\Pi) \ : \ w \in W\}$. We identify any $w(\Pi)$ with the obvious unfolded alcove walk of weight $\mu(w(\Pi)) := w(\lambda)$. Let us now consider the set of alcove walks

$$\mathcal{F}(\Gamma) := \{ \text{alcove walks } \Omega \ : \ \mathrm{unfold}(\Omega) \in \mathcal{P}(\Gamma)\} .$$

We can encode an alcove walk Ω in $\mathcal{F}(\Gamma)$ by the pair (w, J) in $W \times 2^{[m]}$, where

$$\mathrm{fp}(\Omega) = J \quad \text{and} \quad \mathrm{unfold}(\Omega) = w(\Pi) .$$

Clearly, we can recover Ω from (w, J) with $J = \{j_1 < \ldots < j_s\}$ by $\Omega = \phi_{j_1} \ldots \phi_{j_s}(w(\Pi))$. We call a pair (w, J) a *folding pair*, and, for simplicity, we denote the set $W \times 2^{[m]}$ of such pairs by $\mathcal{F}(\Gamma)$ as well. Given a folding pair (w, J), the corresponding positive and negative foldings (viewed as a partition of J) are denoted by J^+ and J^-.

Proposition 4.4 (1) *Consider a folding pair (w, J) with $J = \{j_1 < \ldots < j_s\}$. We have $j_i \in J^+$ if and only if $w r_{j_1} \ldots r_{j_{i-1}} > w r_{j_1} \ldots r_{j_{i-1}} r_{j_i}$.* (2) *If $\Omega \mapsto (w, J)$, then $\mu(\Omega) = w(\mu(J))$.*

We call the sequence $w,\, w r_{j_1},\, \ldots,\, w r_{j_1} \ldots r_{j_s} = w\phi(J)$ the Bruhat chain associated to (w, J).

We now restate the Ram-Yip formula [17] for the Macdonald polynomials $P_\lambda(X; q, t)$ in terms of folding pairs. From now on we assume that the weight λ is regular (and dominant), i.e., $\langle \lambda, \alpha^\vee \rangle > 0$ for all positive roots α.

Theorem 4.5 [17] *Given a dominant regular weight λ, we have (based on the notation in Section* 4.2*)*

$$P_\lambda(X; q, t) = \tag{8}$$

$$= \sum_{(w,J) \in \mathcal{F}(\Gamma)} t^{\frac{1}{2}(\ell(w) - \ell(w\phi(J)) - |J|)} \, (1-t)^{|J|} \left(\prod_{j \in J^+} \frac{1}{1 - q^{l_j} t^{\langle \rho, \beta_j^\vee \rangle}} \right) \left(\prod_{j \in J^-} \frac{q^{l_j} t^{\langle \rho, \beta_j^\vee \rangle}}{1 - q^{l_j} t^{\langle \rho, \beta_j^\vee \rangle}} \right) x^{w(\mu(J))} .$$

4.4 Schwer's formula for Hall-Littlewood polynomials

Let us now consider a reduced alcove path from A° to $A^\circ + \lambda$. The associated chain of roots Γ, defined as in (6), will be called an *extended* λ*-chain*. All the previous definitions can be adapted to this setup. Let $\mathcal{F}_+(\Gamma)$ consist of the folding pairs (w, J) with $J_- = \emptyset$, which will be called *positive folding pairs*.

Theorem 4.6 [16, 18] *Given a dominant regular weight* λ*, the Hall-Littlewood polynomial* $P_\lambda(X;t)$ *is given by*

$$P_\lambda(X;t) = \sum_{(w,J)\in\mathcal{F}_+(\Gamma)} t^{\frac{1}{2}(\ell(w)+\ell(w\phi(J))-|J|)}\,(1-t)^{|J|}\,x^{w(\mu(J))}\,. \tag{9}$$

5 Compressing the Ram-Yip formula in type A_{n-1}

We now restrict ourselves to the root system of type A_{n-1}, fow which the Weyl group W is the symmetric group S_n. Permutations $w \in S_n$ are written in one-line notation $w = w(1)\dots w(n)$. We can identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with the quotient space $V := \mathbb{R}^n/\mathbb{R}(1,\dots,1)$, where $\mathbb{R}(1,\dots,1)$ denotes the subspace in $\mathbb{R}^n$ spanned by the vector $(1,\dots,1)$. The action of the symmetric group S_n on V is obtained from the (left) S_n-action on $\mathbb{R}^n$ by permutation of coordinates. Let $\varepsilon_1,\dots,\varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The root system Φ can be represented as $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j \;:\; i \neq j,\ 1 \leq i,j \leq n\}$. The simple roots are $\alpha_i = \alpha_{i,i+1}$, for $i = 1,\dots,n-1$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1,\dots,n-1$. The weight lattice is $\Lambda = \mathbb{Z}^n/\mathbb{Z}(1,\dots,1)$. A dominant weight $\lambda = \lambda_1\varepsilon_1 + \ldots + \lambda_{n-1}\varepsilon_{n-1}$ is identified with the partition $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$ of length at most $n-1$. We fix such a partition λ for the remainder of this section.

For simplicity, we use the same notation (i,j) with $i < j$ for the root α_{ij} and the reflection $s_{\alpha_{ij}}$, which is the transposition of i and j. Consider the following chain of roots, denoted by $\Gamma(k)$:

$$\begin{array}{llll} (\;(k,n), & (k,n-1), & \ldots, & (k,k+1)\,, \\ \;\;(k-1,n), & (k-1,n-1), & \ldots, & (k-1,k+1)\,, \\ & & \ldots & \\ \;\;(1,n), & (1,n-1), & \ldots, & (1,k+1)\;)\,. \end{array} \tag{10}$$

Denote by $\Gamma'(k)$ the chain of roots obtained by removing the root $(i,k+1)$ at the end of each row. Now define a chain Γ as a concatenation $\Gamma := \Gamma_{\lambda_1}\dots\Gamma_2$, where

$$\Gamma_j := \begin{cases} \Gamma'(\lambda'_j) & \text{if } j = \min\{i \;:\; \lambda'_i = \lambda'_j\} \\ \Gamma(\lambda'_j) & \text{otherwise}\,. \end{cases}$$

It is not hard to verify that Γ is a λ-chain in the sense discussed in Section 4.2. The λ-chain Γ is fixed for the remainder of this section. Thus, we can replace the notation $\mathcal{F}(\Gamma)$ with $\mathcal{F}(\lambda)$.

Example 5.1 Consider $n = 4$ and $\lambda = (4,3,1,0)$, for which we have the following λ-chain (the underlined pairs are only relevant in Example 5.2 below):

$$\Gamma = \Gamma_4\Gamma_3\Gamma_2 = (\underline{(1,4)},(1,3)\mid(2,4),\underline{(2,3)},(1,4),\underline{(1,3)}\mid\underline{(2,4)},(1,4))\,. \tag{11}$$

Given the λ-chain Γ above, in Section 4.2 we considered subsets $J = \{j_1 < \ldots < j_s\}$ of $[m]$, where m is the length of the λ-chain. Instead of J, it is now convenient to use the subsequence of Γ indexed by the positions in J. This is viewed as a concatenation with distinguished factors $T = T_{\lambda_1}\dots T_2$ induced by the factorization of Γ as $\Gamma_{\lambda_1}\dots\Gamma_2$. The partition $J = J^+ \sqcup J^-$ induces partitions $T = T^+ \sqcup T^-$ and $T_j = T_j^+ \sqcup T_j^-$. All the notions defined in terms of J are now redefined in terms of T. As such, from now on we will write $\phi(T)$, $\mu(T)$, and $|T|$, the latter being the size of T. If (w,J) is a folding pair, we will use the same name for the corresponding pair (w,T). We will use the notation $\mathcal{F}(\Gamma)$ and $\mathcal{F}(\lambda)$ accordingly. We denote by $wT_{\lambda_1}\dots T_j$ the permutation obtained from w via right multiplication by the transpositions in $T_{\lambda_1},\dots,T_j$, considered from left to right. This agrees with the above convention of using pairs to denote both roots and the corresponding reflections. As such, $\phi(J)$ in (7) can now be written simply T.

Example 5.2 We continue Example 5.1, by picking the folding pair (w, J) with $w = 2341 \in S_4$ and $J = \{1, 4, 6, 7\}$ (see the underlined positions in (11)). Thus, we have

$$T = T_4 T_3 T_2 = ((1,4) \mid (2,3), (1,3) \mid (2,4)) \,.$$

Note that $J^+ = \{1, 7\}$ and $J^- = \{4, 6\}$. Indeed, we have the following Bruhat chain associated to (w, T), where the transposed entries are shown in bold (we represent permutations as broken columns):

$$\begin{array}{cccccccccccccccccc}
 & \boxed{\mathbf{2}} & & \boxed{1} & & \boxed{1} & & \boxed{\mathbf{1}} & & \boxed{3} & & \boxed{3} & & \boxed{3} & & \boxed{3} & \\
 & & & & & \boxed{\mathbf{3}} & & \boxed{4} & & \boxed{4} & & \boxed{\mathbf{4}} & & \boxed{2} & & \boxed{2} & \\
w = & \boxed{3} & > & \boxed{3} & \mid & & < & & < & & \mid & & > & & \mid & \boxed{1} & . \\
 & \boxed{4} & & \boxed{4} & & \boxed{\mathbf{4}} & & \boxed{\mathbf{3}} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & & \\
 & \boxed{\mathbf{1}} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{2} & & \boxed{\mathbf{2}} & & \boxed{4} & & \boxed{4} &
\end{array}$$

Given a folding pair (w, T), we consider the permutations

$$\pi_j = \pi_j(w, T) := w T_{\lambda_1} T_{\lambda_1 - 1} \dots T_{j+1} \,,$$

for $j = 1, \dots, \lambda_1$. In particular, $\pi_{\lambda_1} = w$.

Definition 5.3 *The* filling map *is the map f from folding pairs (w, T) to fillings $\sigma = f(w, T)$ of the shape λ, defined by $\sigma(i, j) := \pi_j(i)$.*

Example 5.4 Given (w, T) as in Example 5.2, we have

$$f(w, T) = \begin{array}{|c|c|c|c|} \hline 2 & 1 & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 3 & 4 & 2 \\ \cline{2-4} \multicolumn{3}{c|}{} & 1 \\ \cline{4-4} \end{array} \,.$$

From now on, we assume that the partition λ corresponds to a regular weight, i.e., $(\lambda_1 > \ldots > \lambda_{n-1} > 0)$. We will now describe the way in which the formula (1) of Haglund-Haiman-Loehr type can be obtained by compressing the Ram-Yip formula (8). Thus, Theorem 2.6 becomes a corollary of Theorem 5.5 below. We start by rewriting the Ram-Yip formula (8) in the type A setup, as follows:

$$P_\lambda(X; q, t) = \sum_{(w,T) \in \mathcal{F}(\Gamma)} t^{\frac{1}{2}(\ell(w) - \ell(wT) - |T|)} \, (1-t)^{|T|} \left(\prod_{j,(i,k) \in T_j^+} \frac{1}{1 - q^{\mathrm{arm}(-j+1,-i)} t^{k-i}} \right) \times$$

$$\times \left(\prod_{j,(i,k) \in T_j^-} \frac{q^{\mathrm{arm}(-j+1,-i)} t^{k-i}}{1 - q^{\mathrm{arm}(-j+1,-i)} t^{k-i}} \right) x^{w(\mu(T))} \,.$$

Theorem 5.5 *We have $f(\mathcal{F}(\lambda)) = \mathcal{T}(\lambda, n)$. Given any $\sigma \in \mathcal{T}(\lambda, n)$ and any $(w, T) \in f^{-1}(\sigma)$, we have* $\mathrm{content}(f(w, T)) = w(\mu(T))$. *Furthermore, the following compression formula holds for any $\sigma \in \mathcal{T}(\lambda, n)$:*

$$\sum_{(w,T) \in f^{-1}(\sigma)} t^{\frac{1}{2}(\ell(w) - \ell(wT) - |T|)} \, (1-t)^{|T|} \left(\prod_{j,(i,k) \in T_j^+} \frac{1}{1 - q^{\mathrm{arm}(i,j-1)} t^{k-i}} \right) \times$$

$$\times \left(\prod_{j,(i,k) \in T_j^-} \frac{q^{\mathrm{arm}(i,j-1)} t^{k-i}}{1 - q^{\mathrm{arm}(i,j-1)} t^{k-i}} \right) = t^{n(\lambda) - \mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)} \left(\prod_{u \in \mathrm{Diff}(\sigma)} \frac{1-t}{1 - q^{\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}} \right) \,.$$

6 Compressing Schwer's formula in type C_n

We now restrict ourselves to the root system of type C_n. We can identify the space $\mathfrak{h}_{\mathbb{R}}^*$ with $V := \mathbb{R}^n$, the coordinate vectors being $\varepsilon_1, \ldots, \varepsilon_n$. The root system Φ can be represented as $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \,:\, 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i \,:\, 1 \le i \le n\}$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $i = 1, \ldots, n-1$ and $\alpha_n = 2\varepsilon_n$. The fundamental weights are $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, for $i = 1, \ldots, n$. The weight lattice is $\Lambda = \mathbb{Z}^n$. A dominant weight $\lambda = \lambda_1\varepsilon_1 + \ldots + \lambda_n\varepsilon_n$ is identified with the partition $(\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n \ge 0)$ of length at most n. We fix such a partition λ for the remainder of this section.

The corresponding Weyl group W is the group of signed permutations B_n. Such permutations are bijections w from $[\overline{n}] := \{1 < \ldots < n < \overline{n} < \overline{n-1} < \ldots < \overline{1}\}$ to $[\overline{n}]$ satisfying $w(\bar{\imath}) = \overline{w(i)}$. We use the window notation $w = w(1) \ldots w(n)$. The group B_n acts on V as usual, by permuting the coordinate vectors and by changing their signs.

For simplicity, we use the same notation (i,j) with $1 \le i < j \le n$ for the positive root $\varepsilon_i - \varepsilon_j$ and the corresponding reflection, which, in the window notation, is the transposition of entries in positions i and j. Similarly, we denote by $(i,\overline{\jmath})$, again for $i < j$, the positive root $\varepsilon_i + \varepsilon_j$ and the corresponding reflection; in the window notation, the latter is the transposition of entries in positions i and j followed by the sign change of those entries. Finally, we denote by $(i,\bar{\imath})$ the positive root $2\varepsilon_i$ and the corresponding reflection, which is the sign change in position i.

Let

$$\Gamma(k) := \Gamma_2' \ldots \Gamma_k' \Gamma_1(k) \ldots \Gamma_k(k)\,,$$

where

$$\Gamma_j' := ((1,\overline{\jmath}), (2,\overline{\jmath}), \ldots, (j-1,\overline{\jmath}))\,,$$

$$\begin{array}{llllll}\Gamma_j(k) := (& (1,\overline{\jmath}), & (2,\overline{\jmath}), & \ldots, & (j-1,\overline{\jmath}), \\ & (j,\overline{k+1}), & (j,\overline{k+2}), & \ldots, & (j,\overline{n}), \\ & (j,\overline{\jmath}), \\ & (j,n), & (j,n-1), & \ldots, & (j,k+1)\,)\,. \end{array}$$

It is not hard to see that $\Gamma(k)$ is an extended ω_k-chain, in the sense discussed in Section 4.4. Hence, we can construct an extended λ-chain as a concatenation $\Gamma := \Gamma^{\lambda_1} \ldots \Gamma^1$, where

$$\Gamma^i = \Gamma(\lambda_i') = \Gamma_{i2}' \ldots \Gamma_{i,\lambda_i'}' \Gamma_{i1} \ldots \Gamma_{i,\lambda_i'}\,, \quad \text{and } \Gamma_{ij} = \Gamma_j(\lambda_i')\,,\ \Gamma_{ij}' = \Gamma_j'\,. \tag{12}$$

This extended λ-chain is fixed for the remainder of this section. Thus, we can replace the notation $\mathcal{F}_+(\Gamma)$ with $\mathcal{F}_+(\lambda)$.

Example 6.1 Consider $n = 3$ and $\lambda = (3,2,1)$, for which we have the extended λ-chain below. The factorization of Γ into subchains is indicated by vertical bars, while the double vertical bars separate the subchains corresponding to different columns. The underlined pairs are only relevant in Example 6.2 below.

$$\begin{aligned} &\Gamma = \Gamma_{31} \,||\, \Gamma_{22}'\Gamma_{21}\Gamma_{22} \,||\, \Gamma_{12}'\Gamma_{13}'\Gamma_{11}\Gamma_{12}\Gamma_{13} = \\ &= ((1,\overline{2}), \underline{(1,\overline{3})}, (1,\overline{1}), (1,3), (1,2) \,||\, \underline{(1,\overline{2})} \,|\, (1,\overline{3}), (1,\overline{1}), (1,3) \,|\, (1,\overline{2}), (2,\overline{3}), \underline{(2,\overline{2})}, \underline{(2,3)} \,|| \\ &(1,\overline{2}) \,|\, (1,\overline{3}), (2,\overline{3}) \,|\, (1,\overline{1}) \,|\, (1,\overline{2}), (2,\overline{2}) \,|\, (1,\overline{3}), (2,\overline{3}), (3,\overline{3}))\,. \end{aligned} \tag{13}$$

Given the extended λ-chain Γ above, in Section 4.2 we considered subsets $J = \{j_1 < \ldots < j_s\}$ of $[m]$, where m is the length of Γ. Instead of J, it is now convenient to use the subsequence of Γ indexed by the positions in J. This is viewed as a concatenation with distinguished factors T_{ij} and T_{ik}' induced by the factorization (12) of Γ. All the notions defined in terms of J are now redefined in terms of T. As such, from now on we will write $\phi(T)$, $\mu(T)$, and $|T|$, the latter being the size of T. If (w,J) is positive folding pair, we will use the same name for the corresponding pair (w,T). We denote by $wT_{\lambda_1,1} \ldots T_{ij}$ and $wT_{\lambda_1,1} \ldots T_{ik}'$ the permutations obtained from w via right multiplication by the reflections in $T_{\lambda_1,1}, \ldots, T_{ij}$ and $T_{\lambda_1,1}, \ldots, T_{ik}'$, considered from left to right. This agrees with the above convention of using pairs to denote both roots and the corresponding reflections. As such, $\phi(J)$ in (7) can now be written simply T.

Example 6.2 We continue Example 6.1, by picking the positive folding pair (w, J) with $w = \overline{1}\,\overline{2}\,\overline{3} \in B_3$ and $J = \{2, 6, 12, 13\}$ (see the underlined positions in (13)). Thus, we have

$$T = T_{31} \,||\, T'_{22}T_{21}T_{22} \,||\, T'_{12}T'_{13}T_{11}T_{12}T_{13} = ((1,\overline{3}) \,||\, (1,\overline{2}) \,|\quad|\, (2,\overline{2}), (2,3) \,||\quad|\quad|\quad|\quad|\quad)\,.$$

We have the following decreasing Bruhat chain associated to (w, T), where the modified entries are shown in bold (we represent signed permutations as broken columns, as in Example 5.2, and we display the splitting of the chain into subchains induced by the above splitting of T):

$$w = \begin{array}{c} \boxed{\overline{\mathbf{1}}} \\ \\ \boxed{\overline{2}} \\ \boxed{\overline{\mathbf{3}}} \end{array} > \begin{array}{c} \boxed{3} \\ \\ \boxed{\overline{2}} \\ \boxed{1} \end{array} \;||\; \begin{array}{c} \boxed{\mathbf{3}} \\ \boxed{\overline{\mathbf{2}}} \\ \\ \boxed{1} \end{array} > \begin{array}{c} \boxed{2} \\ \boxed{\overline{3}} \\ \\ \boxed{1} \end{array} \;|\; \begin{array}{c} \boxed{2} \\ \boxed{\overline{3}} \\ \\ \boxed{1} \end{array} \;|\; \begin{array}{c} \boxed{2} \\ \boxed{\overline{\mathbf{3}}} \\ \\ \boxed{1} \end{array} > \begin{array}{c} \boxed{2} \\ \boxed{\mathbf{3}} \\ \\ \boxed{\mathbf{1}} \end{array} > \begin{array}{c} \boxed{2} \\ \boxed{1} \\ \\ \boxed{3} \end{array} \;||\; \begin{array}{c} \boxed{2} \\ \boxed{1} \\ \boxed{3} \\ \\ \end{array} \;|\; \begin{array}{c} \boxed{2} \\ \boxed{1} \\ \boxed{3} \\ \\ \end{array} \;|\; \begin{array}{c} \boxed{2} \\ \boxed{1} \\ \boxed{3} \\ \\ \end{array} \;|\; \begin{array}{c} \boxed{2} \\ \boxed{1} \\ \boxed{3} \\ \\ \end{array} \;|\; \begin{array}{c} \boxed{2} \\ \boxed{1} \\ \boxed{3} \\ \\ \end{array}\,.$$

Given a positive folding pair (w, T), with T split into factors T_{ij} and T'_{ik} as above, we consider the signed permutations

$$\pi_{ij} = \pi_{ij}(w, T) := wT_{\lambda_1,1} \ldots T_{i,j-1}\,, \qquad \pi'_{ik} = \pi'_{ik}(w, T) := wT_{\lambda_1,1} \ldots T'_{i,k-1}\,;$$

when undefined, $T_{i,j-1}$ and $T'_{i,k-1}$ are given by conventions similar to (2), based on the corresponding factorization (12) of the extended λ-chain Γ. In particular, $\pi_{\lambda_1,1} = w$.

Let us now recall the notation in Section 3.

Definition 6.3 *The* filling map *is the map $\widehat{f}$ from positive folding pairs (w, T) to fillings $\sigma = \widehat{f}(w, T)$ of the shape $\widehat{\lambda}$, defined by $C_{ij} = \pi_{ij}[1, \lambda'_i]$ and $C'_{ik} = \pi'_{ik}[1, \lambda'_i]$.*

Example 6.4 Given (w, T) as in Example 6.2, we have

$$\widehat{f}(w, T) = \begin{array}{ccccccc} \boxed{\overline{1}} & \boxed{3} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{2} \\ & \boxed{\overline{2}} & \boxed{\overline{3}} & \boxed{\overline{3}} & \boxed{1} & \boxed{1} & \boxed{1} \\ & & & & \boxed{3} & \boxed{3} & \boxed{3} \end{array}\,.$$

From now on, we assume that the partition λ corresponds to a regular weight, i.e., $(\lambda_1 > \ldots > \lambda_n > 0)$. We will now describe the way in which the formula (4) can be obtained by compressing Schwer's formula (9). Thus, Theorem 3.1 becomes a corollary of the theorem below.

Theorem 6.5 *We have $\widehat{f}(\mathcal{F}_+(\lambda)) = \mathcal{T}(\widehat{\lambda}, \overline{n})$. Given any $\sigma \in \mathcal{T}(\widehat{\lambda}, \overline{n})$ and $(w, T) \in \widehat{f}^{-1}(\sigma)$, we have $w(\mu(T)) =$ $\mathrm{content}(\widehat{f}(w, T))$. Furthermore, the following compression formula holds for any $\sigma \in \mathcal{T}(\widehat{\lambda}, \overline{n})$:*

$$\sum_{(w,T)\in \widehat{f}^{-1}(\sigma)} t^{\frac{1}{2}(\ell(w)+\ell(w\phi(T))-|T|)}\,(1-t)^{|T|} = t^{N(\sigma)}\,(1-t)^{\mathrm{des}(\sigma)}\,. \tag{14}$$

Remarks 6.6 (1) The Kashiwara-Nakashima tableaux [6] of shape λ index the basis elements of the irreducible representation of $\mathfrak{sp}_{2n}$ of highest weight λ. These tableaux correspond precisely to the surviving fillings in our formula (4) when we set $t = 0$. More precisely, the map $\sigma \mapsto \overline{\sigma}$ (see (3)) is a bijection between the fillings σ in $\mathcal{T}(\widehat{\lambda}, \overline{n})$ with $N(\sigma) = 0$ and the "doubled" versions of the type C Kashiwara-Nakashima tableaux of shape λ.

(2) In some special cases, the statistic $N(\sigma)$ is essentially the Haglund-Haiman-Loehr "inv" statistic (extended naturally to our signed fillings), as explained below. Let σ in $\mathcal{T}(\widehat{\lambda}, \overline{n})$ be a filling satisfying the following properties: (1) $C'_{i,j+1} = C'_{i,j}$ for all i and $j = 1, \ldots, \lambda'_i$; (2) $C_{i,j+1}$ only differs from C_{ij} in position j. Let $\widetilde{\sigma}$ be the filling of λ given by $\widetilde{\sigma} := C_{\lambda_1,1}C_{\lambda_1-1,1} \ldots C_{11}$. Then $N(\sigma) = n(\lambda) - \mathrm{inv}(\widetilde{\sigma})$, where $n(\lambda) := \sum_i (i-1)\lambda_i$ and $\mathrm{inv}(\sigma)$ is the Haglund-Haiman-Loehr "inv" statistic, cf. Remark 2.1 and Definition 2.5.

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On the 2-adic order of Stirling numbers of the second kind and their differences

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Let n and k be positive integers, $d(k)$ and $\nu_2(k)$ denote the number of ones in the binary representation of k and the highest power of two dividing k, respectively. De Wannemacker recently proved for the Stirling numbers of the second kind that $\nu_2(S(2^n,k)) = d(k)-1, 1 \le k \le 2^n$. Here we prove that $\nu_2(S(c2^n,k)) = d(k)-1, 1 \le k \le 2^n$, for any positive integer c. We improve and extend this statement in some special cases. For the difference, we obtain lower bounds on $\nu_2\big(S(c2^{n+1}+u,k) - S(c2^n+u,k)\big)$ for any nonnegative integer u, make a conjecture on the exact order and, for $u = 0$, prove part of it when $k \le 6$, or $k \ge 5$ and $d(k) \le 2$.

The proofs rely on congruential identities for power series and polynomials related to the Stirling numbers and Bell polynomials, and some divisibility properties.

Keywords: Stirling number of the second kind, congruences for power series and polynomials, divisibility

1 Introduction

The study of p-adic properties of Stirling numbers of the second kind is full with challenging problems. Lengyel (1994) proved that

$$\nu_2(S(2^n,k)) = d(k) - 1 \tag{1}$$

for all sufficiently large n, and in fact, $n \ge k-2$ suffices and conjectured that $\nu_2(S(2^n,k)) = d(k)-1$ for all values of $k : 1 \le k \le 2^n$. The conjecture was eventually proved by De Wannemacker.

Theorem 1 (De Wannemacker (2005)) *Let $n, k \in \mathbb{N}$ and $1 \le k \le 2^n$. Then we have*

$$\nu_2(S(2^n,k)) = d(k) - 1. \tag{2}$$

Related results for $k \le 5$ can be found in Amdeberhan et al. (2008). We generalize De Wannemacker's proof in Section 2. We obtain related results in Section 3. For example, we prove that the 2-adic order of $S(a2^n, b2^n)$ becomes constant as $n \to \infty$ for any positive integers $a \ge b$. As a new direction of investigation, we study the differences of Stirling numbers in Section 4. Lower bounds on $\nu_2\left(S(c2^{n+1}+u,k) - S(c2^n+u,k)\right)$ for any nonnegative integer u and a conjecture on the exact order are presented. For $u = 0$, we prove the conjecture provided that $k \le 6$, or $k \ge 5$ and $d(k) \le 2$.

The proofs rely on the use of identity (7) by De Wannemacker (2005), the inclusion-exclusion principle based calculation (20) of the Stirling numbers, their generating function (10) and a family of congruential identities for Bell polynomials (23) by Junod (2002). Section 5 utilizes (23) to improve previous results. Section 6 shows that some of the results can be extended to primes other than two.

We note that **?**, and **?** also use formal power series or umbral calculus based techniques to prove divisibility properties.

Exact 2-adic orders are determined in Theorems 2-5, 7, and 12-13. As a summary, we note that the 2-adic order $\nu_2(S(a2^n+u, b2^n+v))$ is discussed with the particular triplet (u, v, b) of parameters. In general, exact values are obtained (except in Remark 2 in which we determine lower bounds on the 2-adic orders). For instance, $(0, 2^m-1, 0)$ (or $(1, 2^m, 0)$), $2 \le m < \log_2(a2^n+1)$, in Theorem 4 (or in Remark 3); $(0, 2^m, 0), 2 \le m \le n$, in Theorem 4; $(u, u, b), 0 \le u < 2^n$, in Theorem 5; and $(2^m, 0, 1), 0 \le m \le n-1$, in Theorem 6; potentially with some other extra assumptions.

In this paper, we include the proofs of the theorems or their sketches if they use generating function or power series based arguments but omit some other proofs.

We note that generating functions (Section 3) and related formal power series (Section 5) based techniques outlined in this paper might lead to improved congruential identities, p-adic results, or their alternative proofs involving other combinatorial quantities, their lacunary series, and their differences, often proved by other methods.

2 A generalization

Theorem 2 *Let $n, k, c \in \mathbb{N}$ and $1 \le k \le 2^n$, then*

$$\nu_2(S(c2^n, k)) = d(k) - 1. \tag{3}$$

Remark 1 *In other words, for any fixed $k \ge 1$, we have that $\nu_2(S(c2^n, k)) = d(k) - 1$ if $n \ge \lceil \log_2 k \rceil$. Without loss of generality, we may assume that c is an odd integer (otherwise, we can factor c into a power of two and an odd integer). Note that we obtain*

$$\nu_2(S(4c, 5)) \ge 2 > 1 = d(5) - 1 \tag{4}$$

for $c \ge 1$ odd by (Amdeberhan et al., 2008, formula (3.1))

$$S(n, 5) = \frac{1}{24}(5^{n-1} - 4^n + 2 \cdot 3^n - 2^{n+1} + 1), \quad n \ge 1. \tag{5}$$

For the generalization of (4) see Remark 2. In a similar fashion,

$$S(n, 4) = \frac{1}{6}(4^{n-1} - 3^n + 3 \cdot 2^{n-1} - 1), \quad n \ge 1, \tag{6}$$

proves that $S(c,4)$ is even if c is odd (Amdeberhan et al., 2008, identity (2.14)) while $d(4)-1=0$. Also, $S(c2^n, c2^n-1)=\binom{c2^n}{2}$ and therefore,

$$\nu_2(S(c2^n, c2^n-1)) = n-1 < n \le d(c2^n-1)-1 = n+d(c)-2,$$

for $n \ge 1, c > 1$ odd. More involved cases of a different type are covered by Theorems 6 and 7. Thus, we cannot expect to extend Theorems 1 and 2 beyond the range $1 \le k \le 2^n$, i.e., $\lceil \log_2 k \rceil \le n$. On the other hand, we mention some extensions in Remark 2.

Proof of Theorem 2: The proof is by induction on $d(c)$. The initial case is with $d(c)=1$, i.e., when $c2^n$ is a power of two, and it is taken care of by Theorem 1.

For $d(c) \ge 2$, we use the identity from (De Wannemacker, 2005, Theorem 2)

$$S(n+m,k) = \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\frac{(k-i)!}{(k-j)!}S(n,k-i)S(m,j) \tag{7}$$

which plays a crucial role in the proof of Theorem 1 in De Wannemacker (2005). Assume that (3) holds for all $c \ge 1$ with $d(c) \le d-1$ for some $d \ge 2$. We prove that it holds for all c with $d(c)=d$. In fact, let $c'2^n$ be the highest power of two contained in $c2^n$. Then we can write $c2^n$ as the sum $c'2^n+(c-c')2^n$, and by (7) we get that

$$S(c'2^n+(c-c')2^n,k) = \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\frac{(k-i)!}{(k-j)!}S(c'2^n,k-i)S((c-c')2^n,j)$$

since $d(c')=1, d(c-c')=d(c)-1 \le d-1$, and $k-i, j \le 2^n$. By the induction hypothesis

$$\nu_2(S(c'2^n,k-i)S((c-c')2^n,j)) = d(k-i)+d(j)-2,$$

and the proof proceeds exactly the same way as in (De Wannemacker, 2005, Section 3). $\square$

Remark 2 *We can generalize inequality (4) and find that in general, if a is an integer such that $1 \le a \le 2^n-2$ then $\nu_2(S(c2^n, 2^n+a)) \ge d(a)+1 > d(a) = d(2^n+a)-1$ for $c \ge 3$ odd. (On the other hand, $\nu_2(S(c2^n, 2^n+a)) = d(a)$ for $a = 2^n-1$, $n \ge 1$ and $c \ge 2$ as we will see in (9) of Theorem 4.) We leave the proof to the reader but note that it is similar to that of Theorems 1 and 2. In fact, after expanding $S(c2^n, 2^n+a) = S((c-1)2^n+2^n, 2^n+a)$ by identity (7) and focusing on the terms $\binom{j}{i}\frac{(2^n+a-i)!}{(2^n+a-j)!}S((c-1)2^n, 2^n+a-i)S(2^n,j)$, $0 \le i \le j \le 2^n$, the 2-adic order of the terms can now be easily calculated by Theorem 2. It is $\nu_2(\binom{j}{i})+\nu_2((2^n+a-i)!)-\nu_2((2^n+a-j)!)+\nu_2(S((c-1)2^n, 2^n+a-i))+\nu_2(S(2^n,j)) \ge 2^n+a-i-d(2^n+a-i)-(2^n+a-j)+d(2^n+a-j)+d(2^n+a-i)-1+d(j)-1 \ge j-i+d(2^n+a)-2 = d(a)-1+j-i$. (Here we used the fact that $d(2^n+a-j)+d(j) \ge d(2^n+a)$.) Now we can combine the terms with 2-adic orders $d(a)-1$ and $d(a)$ to yield the result. By a similar technique, we can also prove that $\nu_2(S(c2^n+b, 2^n+a)) \ge d(a)-2$ for integers $c \ge 3$ odd and $1 \le b < a < 2^n$. Note that the case with $a=b$ is treated by Theorem 5.*

Note that if c is even then $\nu_2(S(c2^n, 2^n+a)) = d(a)$ for $1 \le a \le 2^n-1$ by Theorem 2. We can further explore the subtle differences between the cases with c odd and even. Numerical experience suggests the following somewhat surprising conjecture.

Conjecture 1 *We have $\nu_2(S((2^r+1)2^n, 2^n+a)) = d(a)+r$ for integers $r \ge 1$, $1 \le a \le 2^{n-1}$, and sufficiently large n.*

We also state the following simplified and limited version of the conjecture. It assumes that the 2-adic order $\nu_2(a)$ of a and thus, n are large. We present its proof after that of Theorem 5.

Theorem 3 *We have $\nu_2(S(c2^n, 2^n+a)) = d(a)+\nu_2(c-1)$ for $c \ge 3$ odd, $1 \le a < 2^n$, if $\nu_2(a)-d(a) > \nu_2(c-1)+1$.*

3 Other properties

Numerical experimentations reveal other interesting properties of the Stirling numbers of the second kind $S(c2^n, k)$. For example, we can slightly improve Theorem 2 for two special values of k.

Theorem 4 *Let $n, c \in \mathbb{N}$ and m be an integer, $2 \le m \le n$, then*

$$S(c2^n, 2^m) \equiv 1 \bmod 4 \tag{8}$$

and for $2 \le m$ with $c2^n > 2^m-1$,

$$S(c2^n, 2^m-1) \equiv 3 \cdot 2^{m-1} \bmod 2^{m+1}. \tag{9}$$

Proof of Theorem 4: For $c = 1$ (or any power of two), the proof of (8) is based on that of Theorem 1. For other values of c, the proof is similar to that of Theorem 2.

The proof of congruence (9), however, is rather different. We leave some details to the reader. The cases with $m = 2$ and 3 are easy. For $m \ge 4$, we use the generating function (cf. Comtet (1974))

$$f_k(x) = \sum_{n=0}^{\infty} S(n+k,k)x^n = \frac{1}{(1-x)(1-2x)\cdots(1-kx)} \tag{10}$$

with $k = 2^m-1$. The proof is based on the formal power series expansion of $f_k(x) \bmod 2^{m+1}$. We note that the coefficient of $x^{c2^n-2^m+1}$ is $S(c2^n, 2^m-1)$. We make two groups of the factors in the denominator. It can be proven that for $m \ge 3$

$$\prod_{i=1}^{2^{m-1}} (1-(2i-1)x) \equiv (1+3x^2)^{2^{m-2}} \bmod 2^{m+1}, \tag{11}$$

and for $m \ge 4$

$$\prod_{i=1}^{2^{m-1}-1} (1-2ix) \equiv 1+2^{m-1}x+2^{m-1}x^2+2^m x^4 \bmod 2^{m+1},$$

and thus,

$$\frac{1}{\prod_{i=1}^{2^{m-1}-1}(1-2ix)} \equiv 1+3\cdot 2^{m-1}x+3\cdot 2^{m-1}x^2+2^m x^4 \bmod 2^{m+1}. \tag{12}$$

For example, to prove (11), we set $g_{m+1}(x) = \prod_{i=1}^{2^m}(1-(2i-1)x)$. Clearly, $g_3(x) \equiv 1+6x^2+9x^4 \equiv (1+3x^2)^2 \bmod 16$, $g_4(x) \equiv 1+12x^2+22x^4+12x^6+17x^8 \equiv (1+3x^2)^4 \bmod 32$, and note that in general, for $m \geq 2$, $g_{m+1}(x) = \prod_{i=1}^{2^m}(1-(2i-1)x) = g_m(x)\prod_{i=2^{m-1}+1}^{2^m}(1-(2i-1)x) = g_m(x)\prod_{i=1}^{2^{m-1}}(1-(2i-1+2^m)x) \equiv g_m(x)(g_m(x)-h_m(x)) \equiv ((1+3x^2)^{2^{m-2}}+c_1 2^{m+1})((1+3x^2)^{2^{m-2}}+c_1 2^{m+1}-h_m(x)) \equiv (1+3x^2)^{2^{m-1}} \bmod 2^{m+2}$ with some integer c_1 and $h_m(x) = 2^m x g_m(x)\left(\frac{1}{1-x}+\frac{1}{1-3x}+\cdots+\frac{1}{1-(2^m-1)x}\right)$, by induction on m.

Here, we also relied on the fact that, for the power sum $S_j = 1^j+3^j+\cdots+(2^m-1)^j$ we have $\nu_2(S_j) \geq m-1 \geq 2$ for $m \geq 3$, which can be easily proven by induction on m (cf. Lengyel (2007)).

Recall that we need the coefficient of $x^{c2^n-2^m+1}$ in $f_{2^m-1}(x) \bmod 2^{m+1}$. When combined, congruences (11) and (12) give $A \equiv 3\cdot 2^{m-1}(-3)^i\binom{2^{m-2}+i-1}{i} \bmod 2^{m+1}$ with $i=(c2^n-2^m)/2$, making i a multiple of 2^{m-1}. Noting that $(-3)^i \equiv 1 \bmod 2^{m+1}$ and $\binom{2^{m-2}+i-1}{i} \equiv 1 \bmod 4$, this implies that $A \equiv 3\cdot 2^{m-1} \bmod 2^{m+1}$, i.e., the congruence (9). □

Remark 3 *We note that the congruence (9) does not require that the exponent n be at least as large as m but that $c2^n > 2^m-1$, and the proof makes no use of Theorem 2. This congruence allows us to prove that*

$$\nu_2(S(c2^n+1,2^m)) = m-1. \tag{13}$$

In fact, by the usual recurrence $S(c2^n+1,2^m) = 2^m S(c2^n,2^m)+S(c2^n,2^m-1)$ and $\nu_2(S(c2^n,2^m-1)) = m-1$, thus (13) follows.

The above proof of congruence (9) can be modified to yield the following

Theorem 5 *Let $a, b,$ and $n \in \mathbb{N}$, $b \leq a$, and n be sufficiently large (in terms of a and b). Then the 2-adic order of $S(a2^n,b2^n)$ becomes constant as $n\to\infty$. In fact, with $g(a,b) = \nu_2\left(\binom{(2a-b)2^{n-2}-1}{(a-b)2^{n-1}}\right) = d((a-b)2^{n-1})+d(b2^{n-2}-1)-d((2a-b)2^{n-2}-1) = d(a-b)+d(b-1)-d(2a-b-1)$, for any $n > \max\{2, g(a,b)+1\}$ we get that*

$$\nu_2\left(S(a2^n,b2^n)\right) = g(a,b), \tag{14}$$

and in general,

$$\nu_2\left(S(a2^n+u,b2^n+u)\right) = g(a+1,b+1),$$

independently of u, for any integer u: $1 \leq u < 2^n$ as long as $\nu_2(u) > \max\{2, g(a+1,b+1)+1\}$. The periodicity of $g(a,b)$ yields that $\nu_2\left(S((a+2^t)2^n,b2^n)\right) = \nu_2\left(S(a2^n,b2^n)\right)$ if $t \geq \lceil\log_2(2a-b)\rceil$ is a nonnegative integer.

Proof of Theorem 5: We need the coefficient of $x^{(a-b)2^n}$ in $f_{b2^n}(x) \equiv (1+3x^2)^{-b2^{n-2}} \bmod 2^{n-1}$ with $n \geq 3$, since here it is sufficient to combine congruences (11) and (12) $\bmod 2^{n-1}$ rather than $\bmod 2^{n+1}$ for $n \geq 4$. Also note that $\prod_{i=1}^{3}(1-2ix) \equiv 1 \bmod 4$ for $n=3$. It follows that the 2-adic order of the coefficient is equal to that of $\binom{(2a-b)2^{n-2}-1}{(a-b)2^{n-1}}$, similarly to the proof of (9).

The proof for a general $u > 0$ follows by writing u as $t2^q$ with $q=\nu_2(u) < n$ and some odd $t, 1 \leq t < 2^{n-q}$. Therefore, for example, $a2^n+u = (a2^{n-q}+t)2^q$, and thus, in identity (14), the parameters q,

$a2^{n-q}+t$, and $b2^{n-q}+t$ can play the role of n, a, and b, respectively. In fact, with these values, we get that $g(a2^{n-q}+t, b2^{n-q}+t) = d((a-b)2^{n-q}) + d(b2^{n-q}+t-1) - d((2a-b)2^{n-q}+t-1)$ which simplifies to $d((a-b)2^{n-q})+d(b2^{n-q})-d((2a-b)2^{n-q}) = d(a-b)+d(b)-d(2a-b) = g(a+1, b+1)$. □

Theorem 5 seems to be a powerful tool for tackling the cases with n sufficiently large as is demonstrated in the following proof. Note that the second part of Theorems 6 and 7 can also be handled via this theorem similarly to the

Proof of Theorem 3: We write $a = t2^{n-q}$ with an odd $t : 1 \leq t \leq 2^{q-1}$ and $1 \leq q \leq n$. We also write $c = o2^r + 1$ with an odd o and $r = \nu_2(c-1) \geq 1$. We set $A = (o2^r+1)2^q$ and $B = 2^q + t$, and apply Theorem 5 by replacing its parameters a, b and n with A, B and $n-q$, respectively. Note that $c2^n = A2^{n-q}$ and $2^n = B2^{n-q}$.

In fact, for a sufficiently large $n-q$ we have $\nu_2\big(S(A2^{n-q}, B2^{n-q})\big) = d(A-B)+d(B-1)-d(2A-B-1) = d(o2^{r+q}-t)+d(2^q+t-1)-d(o2^{r+q+1}+2^{q+1}-2^q-t-1) = (d(o)-1+r+q-d(t)+1)+(1+d(t)-1)-(d(o)+q-d(t)+1-1) = r+d(t) = \nu_2(c-1)+d(a)$. We note that Theorem 5 assumes that $n-q = \nu_2(a) > \max\{2, g(A,B)+1\} = d(a)+\nu_2(c-1)+1$. □

In the next theorem, we obtain a lower bound on $\nu_2(S(c2^n+u, 2^n))$ for any positive integer u. This also extends relation (13) for $m = n$, in some sense. It is worth noting that $\nu_2(S(c2^n, 2^n)) = 0$ has a very different nature.

Theorem 6 *Let $n, u, c \in \mathbb{N}$, then $\nu_2(S(c2^n+u, 2^n)) \geq n-1-\lfloor \log_2 u \rfloor$. If $u = 2^m$ is a power of two, with some integer $m, 0 \leq m \leq n-1$, then $\nu_2(S(c2^n+2^m, 2^n)) = n-1-m$.*

We note that with the specialization $u = 2^{n-a}, a \geq 1$ integer, we get that $\nu_2(S(c'2^{n-a}, 2^n)) = a-1$ for any integer $c' \geq 2^a$, which includes the fact that $S(c'2^{n-1}, 2^n)$ is odd for $c' \geq 2$.

The previous theorem can be extended to other values to obtain

Theorem 7 *Let $n, k, u, c \in \mathbb{N}$, $1 \leq k \leq 2^n$, and $u \leq 2^{\nu_2(k)}$, then $\nu_2(S(c2^n+u, k)) \geq \nu_2(k) - \lfloor \log_2 u \rfloor + d(k) - 2$. Furthermore, if $u = 2^m$ is a power of two, with some integer $m, 0 \leq m \leq \nu_2(k)-1$, then $\nu_2(S(c2^n+2^m, k)) = \nu_2(k) - m + d(k) - 2$.*

We might as well focus on the tth least significant binary digit of k and obtain the following theorem (which includes the first part of the previous theorem in the special case $t = 1$ which yields that $\nu_2(k) = m_{r-t+1}$).

Theorem 8 *Let $n, k, u, c, t \in \mathbb{N}$, $1 \leq k \leq 2^n$, $1 \leq t \leq r = d(k)$, and $u \leq 2^{m_{r-t+1}}$ given the binary expansion $k = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_r}$ with $m_1 > m_2 > \cdots > m_r \geq 0$. Then $\nu_2(S(c2^n+u, k)) \geq d(k) - t + m_{r-t+1} - \lfloor \log_2 u \rfloor - 1$.*

Remark 4 *In fact, for a given u, within the scope of this theorem, we can freely pick t as long as $u \leq 2^{m_{r-t+1}}$ (thus, it will not apply if $u > k$). Now we find that the largest lower bound on the 2-adic order is achieved at $t = d(k)$, i.e., $\nu_2(S(c2^n+u, k)) \geq m_1 - 1 - \lfloor \log_2 u \rfloor$ for $u \leq 2^{m_1}$.*

4 Differences of Stirling numbers

Another interesting property is related to the difference $S(c2^{n+1},k)-S(c2^n,k)$. It appears that its 2-adic order increases by one as n increases by one, provided that n is large enough. As a consequence, this would imply that $\nu_2(S(c2^n,k))$ becomes fix for some large n without explicitly indicating how small this n can be. Of course, Theorem 2 and Remark 1 take care of answering this question. We note that there are some conjectures on the structure of the sets $\{\nu_2(S(c2^n+u,k))\}_{c\geq c_0}$, with c_0 being minimum in order to guarantee $c_0 2^n+u\geq k$, as a function of u for any fixed n and k in Amdeberhan et al. (2008). We state

Conjecture 2 *Let $n,k,a,b\in\mathbb{N}$, $3\leq k\leq 2^n$, and $c\geq 1$ be an odd integer, then*

$$\nu_2(S(c2^{n+1},k)-S(c2^n,k))=n+1-f(k) \tag{15}$$

and

$$\nu_2(S(a2^n,k)-S(b2^n,k))=n+1+\nu_2(a-b)-f(k) \tag{16}$$

for some function $f(k)$ which is independent of n (for any sufficiently large n).

Remark 5 *The cases with $k=1$ and 2 are rather different but trivial. In fact, $S(n_1,1)-S(n_2,1)=0$ for $n_1,n_2\in\mathbb{N}$ and $S(n_1,2)-S(n_2,2)=2^{n_2-1}(2^{n_1-n_2}-1)$ if $n_2<n_1$, thus $\nu_2(S(n_1,2)-S(n_2,2))=n_2-1$. The case with $k=4$ follows by identity (6).*

Remark 6 *To illustrate the above conjecture, we prove a little more for $k=3$. Observe that*

$$S(n,3)=\frac{1}{2}(3^{n-1}-2^n+1),\quad n\geq 1.$$

Let us assume that $a\geq b$. For $n\geq 3$, the Lemma 1 below implies that

$$\nu_2\left(S(a2^n,3)-S(b2^n,3)\right)=-1+\nu_2\left(3^{(a-b)2^n}-1\right)=n+1+\nu_2(a-b),$$

and moreover, for $n\geq 3$ and any nonnegative integer u

$$\nu_2\left(S(a2^n+u,3)-S(b2^n+u,3)\right)=n+1+\nu_2(a-b).$$

It appears that there are only very few exceptions to (15) and (16) requiring the proviso on the large size of n (and perhaps, there is none if we require that $1\leq k\leq 2^{n-1}$). Relations similar to (15) seem to apply to $\nu_2(S(c2^{n+1}+u,k)-S(c2^n+u,k))$ for many nonnegative even integers u (cf. Remark 7 as an illustration to this in a special case).

We are not able to prove Conjecture 2, except for small values of k, e.g., $f(3)=0$ (cf. Remark 6), $f(4)=0, f(5)=2$, and $f(6)=2$ (by evaluating the expressions (20) and (22) using the method in the proofs of Theorems 9 and 10). However, we have the supporting evidence given by Theorem 9 which also suggests that $f(k)\leq\nu_2(k!)-1$ if the conjectured identity (15) holds, and Theorem 11 guarantees the much stronger $f(k)\leq\lceil\log_2 k\rceil-1$. For small values of k, numerical experimentation suggests that

$$f(k)=1+\lceil\log_2 k\rceil-d(k)-\delta(k), \tag{17}$$

with $\delta(4)=2$ and otherwise it is zero except if k is a power of two or one less, in which cases $\delta(k)=1$. This would imply that $f(k)\geq 0$. It appears that $f(2^m)=m-1$ for $m\geq 3$. Note that $\lceil\log_2 k\rceil-d(k)$ is the number of zeros in the binary expansion of k, unless k is a power of two.

Theorem 9 *Let $n,k\in\mathbb{N}, 3\le k\le 2^n$, u be a nonnegative integer, and $c\ge 1$ be an odd integer, then*

$$\nu_2(S(c2^{n+1}+u,k)-S(c2^n+u,k))\ge n+2-\nu_2(k!).$$

In the proof we use the following

Lemma 1 *Let $n,m\in\mathbb{N}$, and $c\ge 1$ be an odd integer, then*

$$\nu_2\left((2m+1)^{c2^n}-1\right)=n+2+\nu_2\left(\binom{m+1}{2}\right). \tag{18}$$

Proof of Lemma 1: We factor the expression on the left side of (18):

$$\begin{aligned}(2m+1)^{c2^n}-1 &= \left((2m+1)^{c2^{n-1}}-1\right)\left((2m+1)^{c2^{n-1}}+1\right)\\ &= \left((2m+1)^{2c}-1\right)\prod_{i=1}^{n-1}\left((2m+1)^{c2^i}+1\right).\end{aligned} \tag{19}$$

By the binomial expansion, each factor of the product can be rewritten as

$$(2m+1)^{c2^i}+1=1+2m\binom{c2^i}{1}+(2m)^2\binom{c2^i}{2}+\cdots+1\equiv 2 \bmod 4.$$

This implies that each factor contributes one to the 2-adic order. On the other hand, for the first factor of the last expression in (19), we get that $\nu_2\left((2m+1)^{2c}-1\right)=\nu_2((2m+1)^c-1)+\nu_2\left((2m+1)^c+1\right)=\nu_2(m)+1+\nu_2\left((2m+1)^c+1\right)=\nu_2(m)+1+\nu_2(m+1)+1$ by binomial expansion and $(2m+1)^c+1=((2m+1)+1)((2m+1)^{c-1}-(2m+1)^{c-2}+\cdots+1)$. Putting together the factors of (19), the 2-adic order becomes $n+1+\nu_2(m)+\nu_2(m+1)$. The proof is now complete. $\square$

By the well-known identity (cf. Comtet (1974)) for $S(n,k)$

$$k!S(n,k)=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$$

it follows that

$$k!\left(S(c2^{n+1},k)-S(c2^n,k)\right)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^{c2^n}\left((k-i)^{c2^n}-1\right). \tag{20}$$

We note that Theorem 9 is the special case of

Theorem 10 *Let $n,k,a,b\in\mathbb{N}$, $3\le k\le 2^n$, and u be a nonnegative integer, then*

$$\nu_2(S(a2^n+u,k)-S(b2^n+u,k))\ge n+\nu_2(a-b)+2-\nu_2(k!). \tag{21}$$

Its proof is similar to that of the previous theorem. Assuming that $a\ge b$ we can replace (20) by

$$k!\left(S(a2^n+u,k)-S(b2^n+u,k)\right)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^{b2^n+u}\left((k-i)^{(a-b)2^n}-1\right), \tag{22}$$

and the statement follows by Lemma 1.

5 Towards the proof of the Conjecture 2

We cannot prove Conjecture 2 but we do make some progress in that direction, and at the same time, we improve previously stated results, in general, and for the case when k is a power of two, in particular. We note that for a fixed value of k, the smallest value of n with $1 \le k \le 2^n$ is $\lceil \log_2 k \rceil$, so by Theorem 2, the inequalities $\nu_2(S(c2^{n+1}, k) - S(c2^n, k)) \ge n - \lceil \log_2 k \rceil + d(k)$ and $\nu_2(S(a2^n, k) - S(b2^n, k)) \ge n - \lceil \log_2 k \rceil + d(k)$ hold for this n. Moreover, by Theorem 4 and Remark 6, we have that $\nu_2(S(c2^{n+1}, k) - S(c2^n, k)) \ge n - \lceil \log_2 k \rceil + d(k) + \delta(k) = n + 1 - f(k)$ for this n. This agrees with (17) although in terms of a lower bound rather than the equality in (15).

One possibility for proving Conjecture 2 might be to use differences based on identity (7) or on the congruence by Junod (2002)

$$B_{m+np^\nu}(x) \equiv \sum_{j=0}^{n} \binom{n}{j} (x^p + x^{p^2} + \cdots + x^{p^\nu})^{n-j} B_{m+j}(x) \pmod{\frac{np}{2}\mathbb{Z}_p[x]} \tag{23}$$

with $p = 2$ and proper specializations of the parameters m, n and ν ($m, n \ge 0$ and $\nu \ge 1$ integers), where the Bell polynomials are defined (cf. Junod (2002)) by

$$B_n(x) = \sum_{k=0}^{n} S(n, k) x^k, n \ge 0.$$

We now prove one of our main results, the following weaker version of Conjecture 2, which still improves Theorems 9 and 10 for $k \ge 3$, and it puts us within $d(k) + \delta(k) - 2 < \log_2 k$ of the conjecture (although with some restriction in case of equation (16)).

Note that Theorems 12 and 13 completely prove the conjecture for $k \ge 5$ if $d(k) \le 2$ and $u = 0$. (In this case equation holds in (24).) The cases with $k \le 6$ are taken care of by the comments made on $f(k)$ after Remark 6.

Theorem 11 *Let $n, k \in \mathbb{N}, 3 \le k \le 2^n$, u be a nonnegative integer, and $c \ge 1$ be an odd integer, then*

$$\nu_2(S(c2^{n+1} + u, k) - S(c2^n + u, k)) \ge n - \lceil \log_2 k \rceil + 2. \tag{24}$$

Moreover, let $a, b \in \mathbb{N}$ and $a/2 \le b < a$, then

$$\nu_2(S(a2^n + u, k) - S(b2^n + u, k)) \ge n + \nu_2(a - b) - \lceil \log_2 k \rceil + 2. \tag{25}$$

Proof of Theorem 11: To prove (24), we use (23) with $p = 2, m = u, \nu = 1$, and n replaced by $c2^n$, and consider the coefficients of x^k:

$$\begin{aligned} &S(c2^{n+1} + u, k) \\ &\quad \equiv \textstyle\sum_{j=0}^{c2^n} \binom{c2^n}{j} S(j + u, k - 2(c2^n - j)) \\ &\quad \equiv S(c2^n + u, k) + \textstyle\sum_{j=c2^n - \lceil \frac{k}{2} \rceil + 1}^{c2^n - 1} \binom{c2^n}{j} S(j + u, k - 2(c2^n - j)) \bmod 2^n, \end{aligned} \tag{26}$$

since we observe that $k-2(c2^n-j)>0$ implies that $j>c2^n-\lceil\frac{k}{2}\rceil$. Clearly, in the given range of values $j=c2^n-\lceil\frac{k}{2}\rceil+v, 1\le v<\lceil\frac{k}{2}\rceil\le 2^{n-1}$, we have $\nu_2\big(\binom{c2^n}{j}\big)=\nu_2\big(\binom{c2^n}{\lceil\frac{k}{2}\rceil-v}\big)=n-\nu_2(\lceil\frac{k}{2}\rceil-v)\ge n-(\lceil\log_2 k\rceil-2)$. We note that if $u=0, k\ge 5$, and $d(k)\le 2$ then equality holds in (24) by Theorems 12 and 13.

This proof also applies to (25) with $p=2, m=(2b-a)2^n+u, \nu=1$, and n replaced by $(a-b)2^n$. Again, we consider the coefficients of x^k and get that

$$\begin{aligned}S(a2^n+u,k)\equiv S(b2^n+u,k)+\textstyle\sum_{j=(a-b)2^n-\lceil\frac{k}{2}\rceil+1}^{(a-b)2^n-1}\binom{(a-b)2^n}{j}\times\\ \times S(j+(2b-a)2^n+u,k-2((a-b)2^n-j)) \bmod 2^{n+\nu_2(a-b)},\end{aligned}$$

and the proof follows as above with $j=(a-b)2^n-\lceil\frac{k}{2}\rceil+v, 1\le v<\lceil\frac{k}{2}\rceil\le 2^{n-1}$ and $\nu_2\big(\binom{(a-b)2^n}{j}\big)=\nu_2\big(\binom{(a-b)2^n}{\lceil\frac{k}{2}\rceil-v}\big)=n+\nu_2(a-b)-\nu_2(\lceil\frac{k}{2}\rceil-v)\ge n+\nu_2(a-b)-(\lceil\log_2 k\rceil-2)$. Note that $k\le 2^{n+\nu_2(a-b)}$ suffices. $\square$

Now we illustrate a more involved application of (23) to prove equation (15) of Conjecture 2 if $k\ge 8$ is a power of two. (Other powers of two are settled in Remark 5.) We note that this provides a refinement of a direct consequence of equation (8) of Theorem 4.

Theorem 12 *Let $m\ge 3$ be an integer, then*

$$\nu_2(S(2^{m+1},2^m)-S(2^m,2^m))=2, \tag{27}$$

and in general, for an integer $n\ge m\ge 3$ and odd integer $c\ge 1$, we get

$$\nu_2(S(c2^{n+1},2^m)-S(c2^n,2^m))=n-m+2. \tag{28}$$

We mention that Conjecture 2 and equation (17) suggest that $\nu_2(S(c2^{n+1},2^m-1)-S(c2^n,2^m-1))=n+1$ for $n\ge m\ge 2$ and odd $c\ge 1$. Note the striking contrast to (28) in terms of m.

Proof of Theorem 12: To prove identity (27), we use (23) with $p=2, m=0, \nu=1$, and n replaced by 2^m, and consider the coefficients of x^{2^m} in

$$B_{2^{m+1}}(x)\equiv\sum_{j=0}^{2^m}\binom{2^m}{j}x^{2(2^m-j)}B_j(x) \bmod 2^m,$$

i.e., $S(2^{m+1},2^m)\equiv S(2^m,2^m)+\sum_{j=2^{m-1}+1}^{2^m-1}\binom{2^m}{j}S(j,2^m-2(2^m-j)) \bmod 2^m$. The 2-adic order of a general term of the summation with index j, provided that $\nu_2(j)=s<m-1$, is $m-s+\nu_2(S(c'2^s,c'2^{s+1}-2^m))\ge m-s$, with some odd $c'\ge 1$. The smallest such order is $m-(m-2)=2<m$ with the unique $j=3\cdot 2^{m-2}$ (by Theorem 6 with $c=1$, $n=m-1$, and $u=2^{m-2}$). Identity (27) follows.

In general, with $n\ge m$ and $c=1$, we use the above parameters in (23) except that now we replace n by 2^n rather than by 2^m. Similarly to the above proof, it can be shown that $(2^{n-m+2}-1)2^{m-2}=$

$2^n - 2^{m-2}$ is the unique index j that results in a term T with 2-adic valuation as small as $n - m + 2 < n$. In fact, $\nu_2\left(\binom{2^n}{(2^{n-m+2}-1)2^{m-2}}\right) = n - m + 2$, and T is an odd multiple of $2^{n-m+2}S((2^{n-m+2} - 1)2^{m-2}, 2^{m-1})$. This yields (28) by Theorem 6.

The proof with $n \geq m$ and a general odd $c \geq 1$ is similar to the previous case but now n is replaced by $c2^n$. Here $c2^n - 2^{m-2}$ is the unique index j between $c2^n - 2^{m-1} + 1$ and $c2^n - 1$ whose term achieves the smallest valuation $n - m + 2$.

We note that the structure of the 2-adic valuation of the terms shows a remarkably simple pattern. □

Remark 7 *The above proof can be extended to apply to* $\nu_2(S(c2^{n+1} + u, 2^m) - S(c2^n + u, 2^m))$ *if* $u \geq 0$ *is an integer multiple of* 2^{m-2}*, i.e.,*

$$\nu_2(S(c2^{n+1} + d2^{m-2}, 2^m) - S(c2^n + d2^{m-2}, 2^m)) = n - m + 2,$$

for integers $n \geq m \geq 3$*,* $d \geq 0$*, and odd integer* $c \geq 1$*.*

The previous theorem can be modified to yield

Theorem 13 *For integers* $n > m_1 \geq 2$*,* $m_1 > m_2 \geq 0$*, and odd integer* $c \geq 1$*, we get*

$$\nu_2(S(c2^{n+1}, 2^{m_1} + 2^{m_2}) - S(c2^n, 2^{m_1} + 2^{m_2})) = n - m_1 + 1. \tag{29}$$

Proof of Theorem 13: The proof is similar to that of the previous theorem. Here $c2^n - 2^{m_1-1}$ is the unique index j between $c2^n - 2^{m_1-1} - 2^{m_2-1} + 1$ and $c2^n - 1$ whose term achieves the smallest valuation $n - m_1 + 1$. □

6 Other primes

In this paper, we have aimed at divisibility properties by $p = 2$. However, it is worth mentioning that some of the congruences of the previous section can be generalized. For example, for illustrative purposes, we prove the modification of Theorem 11.

Theorem 14 *Let* $p \geq 3$ *be a prime,* $c, n, k \in \mathbb{N}$ *with* $1 \leq k \leq p^n$ *and* $(c, p) = 1$*, and* u *be a nonnegative integer, then*

$$\nu_p(S(cp^{n+1} + u, k) - S(cp^n + u, k)) \geq n - \lceil \log_p k \rceil + 2. \tag{30}$$

Moreover, let $a, b \in \mathbb{N}$ *and* $a/p \leq b < a$*, then*

$$\nu_p(S(ap^n + u, k) - S(bp^n + u, k)) \geq n + \nu_p(a - b) - \lceil \log_p k \rceil + 2. \tag{31}$$

Proof of Theorem 14: We use identity (23) with $m = u, \nu = 1$, the actual prime p, and n replaced by cp^n. We consider the coefficients of x^k:

$$\begin{aligned} &S(cp^{n+1} + u, k) \\ &\quad \equiv \textstyle\sum_{j=0}^{cp^n} \binom{cp^n}{j} S\left(j + u, k - p\left(cp^n - j\right)\right) \\ &\quad \equiv S(cp^n + u, k) \\ &\quad + \textstyle\sum_{j=cp^n-\lceil \frac{k}{p} \rceil+1}^{cp^n-1} \binom{cp^n}{j} S\left(j + u, k - p\left(cp^n - j\right)\right) \bmod\ p^{n+1}, \end{aligned}$$

as we observe that $k - p(cp^n - j) > 0$ implies that $j > cp^n - \lceil \frac{k}{p} \rceil$. Clearly, in the given range of values $j = cp^n - \lceil \frac{k}{p} \rceil + v, 1 \le v < \lceil \frac{k}{p} \rceil \le p^{n-1}$, we have $\nu_p\big(\binom{cp^n}{j}\big) = \nu_p\big(\binom{cp^n}{\lceil \frac{k}{p} \rceil - v}\big) = n - \nu_p(\lceil \frac{k}{p} \rceil - v) \ge n - (\lceil \log_p k \rceil - 2)$.

The proof of inequality (31) is similar to that of (30) and (25). □

We note the relation to some results in Gessel and Lengyel (2001). In fact, Theorem 2 of Gessel and Lengyel (2001) claims that if $u = 0$, c is a multiple of $p - 1$, and k is an odd multiple of p then the lower bound in Theorem 14 can be improved.

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Chip-Firing And A Devil's Staircase

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The devil's staircase – a continuous function on the unit interval [0,1] which is not constant, yet is locally constant on an open dense set – is the sort of exotic creature a combinatorialist might never expect to encounter in "real life." We show how a devil's staircase arises from the combinatorial problem of parallel chip-firing on the complete graph. This staircase helps explain a previously observed "mode locking" phenomenon, as well as the surprising tendency of parallel chip-firing to find periodic states of small period.

Keywords: Circle map, fixed-energy sandpile, mode locking, non-ergodicity, parallel chip-firing, rotation number, short period attractors

1 Introduction

In this extended abstract, we summarize recent work relating the Poincaré rotation number of a circle map $S^1 \to S^1$ to the behavior of parallel chip-firing on the complete graph. We use this connection to shed light on two intriguing features of parallel chip-firing, *mode locking* and *short period attractors*. Ever since Bagnoli, Cecconi, Flammini, and Vespignani [1] found evidence of mode locking and short period attractors in numerical experiments in 2003, these two phenomena have called out for a mathematical explanation. The proofs omitted here can be found in [12].

In chip-firing on a finite graph, each vertex v starts with a pile of $\sigma(v) \geq 0$ chips. A vertex is *unstable* if it has at least as many chips as its degree, and can *fire* by sending one chip to each neighbor. In *parallel chip-firing*, at each time step, all unstable vertices fire simultaneously. If it is possible in finitely many firings to reach a stable configuration in which no vertex can fire, then this final configuration does not depend on the order of firings [5]. In this case, the parallel restriction does not affect the final outcome. However, our focus will be on chip configurations that do not stabilize.

Previous work on parallel chip-firing [3, 4, 10, 14] has focused on the periodicity of the dynamics: given a graph G, for which natural numbers q does there exist a parallel chip-firing state on G which first recurs after q time steps? We will have more to say about this question below. In the statistical physics literature, parallel chip-firing often goes by the name "fixed energy sandpile" [1, 6, 7, 15]. The term "sandpile" refers to the Bak-Tang-Wiesenfeld model of self-organized criticality [2], while "fixed energy" refers to the lack of a sink or boundary vertex where chips disappear.

We add loops to the complete graph K_n, so that a vertex with n or more chips is unstable, and fires by sending one chip to each vertex of K_n, including one chip to itself. The *parallel update* rule fires all unstable vertices simultaneously, yielding a new chip configuration $U\sigma$ given by

$$U\sigma(v) = \begin{cases} \sigma(v) + r(\sigma), & \sigma(v) \leq n-1 \\ \sigma(v) - n + r(\sigma), & \sigma(v) \geq n. \end{cases} \tag{1}$$

Here

$$r(\sigma) = \#\{v | \sigma(v) \geq n\}$$

is the number of unstable vertices. Write $U^0\sigma = \sigma$, and $U^t\sigma = U(U^{t-1}\sigma)$ for $t \geq 1$.

Note that the total number of chips in the system is conserved. In particular, only finitely many different states are reachable from the initial configuration σ, so the sequence $(U^t\sigma)_{t\geq 0}$ is eventually periodic: there exist integers $m \geq 1$ and $t_0 \geq 0$ such that

$$U^{t+m}\sigma = U^t\sigma \qquad \forall t \geq t_0. \tag{2}$$

The *activity* of σ is the limit

$$a(\sigma) = \lim_{t\to\infty} \frac{\alpha_t}{nt}. \tag{3}$$

where

$$\alpha_t = \sum_{s=0}^{t-1} r(U^s\sigma)$$

is the total number of firings performed in the first t updates. By (2), the limit in (3) exists and equals $\frac{1}{mn}(\alpha_{t_0+m} - \alpha_{t_0})$. Since $0 \leq \alpha_t \leq nt$, we have $0 \leq a(\sigma) \leq 1$.

Following [1], we ask: how does the activity change when chips are added to the system? If σ_n is a chip configuration on K_n, write $\sigma_n + k$ for the configuration obtained from σ_n by adding k chips at each vertex. The function

$$\tilde{s}_n(k) = a(\sigma_n + k)$$

is called the *activity phase diagram* of σ_n. Surprisingly, for many choices of σ_n, the function $\tilde{s}_n$ looks like a staircase, with long intervals of constancy punctuated by sudden jumps (Figure 1). This phenomenon is known as *mode locking*: if the system is in a preferred mode, corresponding to a wide stair in the staircase, then even a relatively large perturbation in the form of adding extra chips will not change the activity. For a general discussion of mode locking in dynamical systems, including examples from astronomy and physics, see [11].

To quantify the idea of mode locking in our setting, suppose we are given an infinite family of chip configurations $\sigma_2, \sigma_3, \ldots$ with σ_n defined on K_n. Suppose σ_n is stable, i.e.,

$$0 \leq \sigma_n(v) \leq n-1 \tag{4}$$

for all $v \in [n]$. Moreover, suppose that there is a continuous function $F : [0,1] \to [0,1]$, such that for all $0 \leq x \leq 1$

$$\frac{1}{n}\#\{v \in [n] \,|\, \sigma_n(v) < nx\} \to F(x) \tag{5}$$

as $n \to \infty$. Then according to Theorem 3.1, the activity phase diagrams $\tilde{s}_n$, suitably rescaled, converge pointwise to a continuous, nondecreasing function $s : [0,1] \to [0,1]$.

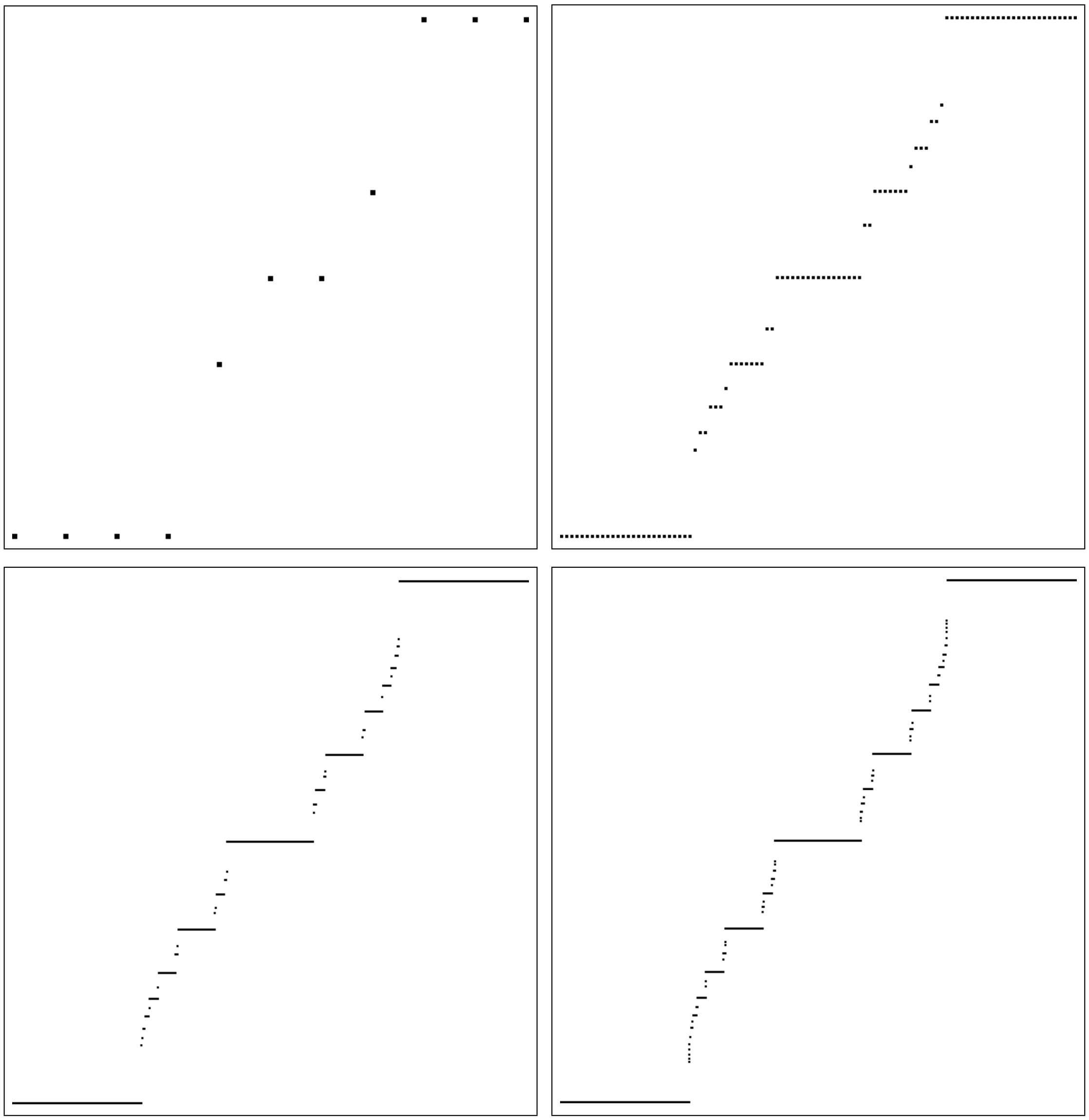

Fig. 1: The activity phase diagrams $a(\sigma_n + k)$, for $n = 10$ (top left), 100 (top right), 1000 (bottom left), and 10000, where σ_n is given by (6). On the horizontal axis, k runs from 0 to n. On the vertical axis, $a(\sigma_n + k)$ runs from 0 to 1.

Moreover, under a mild additional hypothesis, Proposition 3.2 says that this limiting function s is a *devil's staircase*: it is locally constant on an open dense subset of $[0,1]$. For each rational number $p/q \in [0,1]$ there is a stair of height p/q, that is, an interval of positive length on which s is constant and equal to p/q.

Related to mode locking, a second feature observed in simulations of parallel chip-firing is non-ergodicity: in trials performed with random initial configurations on the $n \times n$ torus, the activity observed in individual trials differs markedly from the average activity observed over many trials [15]. The experiments of [1] suggested a reason: the chip-firing states in locked modes, corresponding to stairs of the devil's staircase, tend to be periodic with very small period. We study these *short period attractors* in Theorem 4.6. Under the same hypotheses that yield a devil's staircase in Propositon 3.2, for each $q \in \mathbb{N}$, a constant fraction $c_q n$ of the states $\{\sigma_n + k\}_{k=0}^n$ have eventual period q.

To illustrate these results, consider the chip configuration σ_n on K_n defined by

$$\sigma_n(v) = \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{v-1}{2} \right\rfloor, \qquad v = 1, \ldots, n. \tag{6}$$

Here $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. This family of chip configurations satisfies (**??**) with

$$F(x) = \begin{cases} 0, & x \leq \frac{1}{4} \\ 2x - \frac{1}{2}, & \frac{1}{4} \leq x \leq \frac{3}{4} \\ 1, & x \geq \frac{3}{4}. \end{cases} \tag{7}$$

The activity phase diagrams of σ_n for $n = 10, 100, 1000, 10000$ are graphed in Figure 1. For example, when $n = 10$ we have

$$(a(\sigma_{10} + k))_{k=0}^{10} = (0,\ 0,\ 0,\ 0,\ 1/3,\ 1/2,\ 1/2,\ 2/3,\ 1,\ 1,\ 1)$$

and when $n = 100$, we have

$(a(\sigma_{100} + k))_{k=0}^{100}$ =(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/6, 1/5, 1/5, 1/4, 1/4, 1/4, 2/7, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 2/5, 2/5, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 3/5, 3/5, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 5/7, 3/4, 3/4, 3/4, 4/5, 4/5, 5/6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).

As n grows, the denominators of these rational numbers grow remarkably slowly: the largest denominator is 11 for $n = 1000$, and 13 for $n = 10000$. Moreover, for any fixed n the very smallest denominators occur with greatest frequency. For example, when $n = 10000$, there are 1667 values of k for which $a(\sigma_n + k) = \frac{1}{2}$, and 714 values of k for which $a(\sigma_n + k) = \frac{1}{3}$; but for each $p = 1, \ldots, 12$ there is just one value of k for which $a(\sigma_n + k) = \frac{p}{13}$. In Lemma 4.5, we relate these denominators to the periodicity: if $a(\sigma) = p/q$ in lowest terms, then σ has eventual period q.

The remainder of the paper is organized as follows. In section 2 we show how to construct, given a chip configuration σ on K_n, a circle map $f : S^1 \to S^1$ whose rotation number equals the activity of σ. This construction resembles the one-dimensional particle/barrier model of [9]. In section 3 we use the circle map to prove our main results on mode locking, Theorem 3.1 and Proposition 3.2. Short period attractors are studied in section 4, where we show that all states on K_n have eventual period at most n

(Proposition 4.4). Finally, in Theorem 4.11, we find a small "window" in which all states have eventual period two.

Many questions remain concerning parallel chip-firing on graphs other than K_n. If the underlying graph is a tree [4] or a cycle [7], then instead of a devil's staircase of infinitely many preferred modes, there are just three: activity 0, $\frac{1}{2}$ and 1. On the other hand, the numerical experiments of [1] for parallel chip-firing on the $n \times n$ torus suggest a devil's staircase in the large n limit. Our arguments rely quite strongly on the structure of the complete graph, whereas the mode locking phenomenon seems to be widespread. It would be very interesting to relate parallel chip-firing on other graphs to iteration of a circle map (or perhaps a map on a higher-dimensional manifold) in order to explain the ubiquity of mode locking.

2 Construction of the Circle Map

We first identify a certain class of chip configurations on K_n, the *confined states*, with the property that for any configuration σ of activity $a(\sigma) < 1$, we have $U^t\sigma$ confined for all sufficiently large t.

Definition. A chip configuration σ on K_n is *preconfined* if it satisfies

(i) $\sigma(v) \leq 2n - 1$ for all vertices v of K_n.

If, in addition, σ satisfies

(ii) $\max_v \sigma(v) - \min_v \sigma(v) \leq n - 1$

then σ is *confined*.

Lemma 2.1. *If σ is preconfined, then $U\sigma$ is confined.*

Lemma 2.2. *If $a(\sigma) < 1$, then $U^t\sigma$ is confined for all sufficiently large t.*

Note that from (1)

$$U\sigma(v) \equiv \sigma(v) + r(\sigma) \pmod{n}.$$

Iterating yields the congruence

$$U^t\sigma(v) \equiv \sigma(v) + \alpha_t \pmod{n} \tag{8}$$

where

$$\alpha_t = \sum_{s=0}^{t-1} r(U^s\sigma)$$

is the total number of firings before time t.

Our next lemma characterizes the vertices that fire at time $t+1$.

Lemma 2.3. *If $U^t\sigma$ is preconfined, then $U^{t+1}\sigma(v) \geq n$ if and only if*

$$\sigma(v) \equiv -j \ (\mathrm{mod}\ n)$$

for some $\alpha_t < j \leq \alpha_{t+1}$.

Let

$$\phi(j) = \#\{v \in [n] \,|\, \sigma(v) \equiv -j \pmod{n}\}. \tag{9}$$

By Lemma 2.3, if $U^t\sigma$ is preconfined, then the number of unstable vertices in $U^{t+1}\sigma$ is

$$r_{t+1} = \phi(\alpha_t + 1) + \ldots + \phi(\alpha_{t+1}),$$

hence

$$\alpha_{t+2} = \alpha_{t+1} + \sum_{j=\alpha_t+1}^{\alpha_{t+1}} \phi(j). \tag{10}$$

This gives a recurrence for α_t relating three consecutive terms α_t, α_{t+1} and α_{t+2}. Our next lemma simplifies this to a recurrence relating just two consecutive terms.

Lemma 2.4. *If σ is preconfined, then for all $t \geq 0$*

$$\alpha_{t+1} = g(\alpha_t),$$

where $g : \mathbb{N} \to \mathbb{N}$ is given by

$$g(k) = \alpha_1 + \sum_{j=1}^{k} \phi(j) \tag{11}$$

and ϕ is given by (9).

The function g appearing in Lemma 2.4 satisfies

$$\begin{aligned} g(k+n) &= g(k) + \sum_{j=k+1}^{k+n} \phi(j) \\ &= g(k) + \sum_{j=k+1}^{k+n} \#\{v \,|\, \sigma(v) \equiv -j \pmod{n}\} \\ &= g(k) + n. \end{aligned} \tag{12}$$

for all $k \in \mathbb{N}$. This suggests that a more natural domain of definition is the unit circle. First extend g to all of $\mathbb{Z}$ by defining

$$g(-k) = g(nk - k) - nk, \qquad k \in \mathbb{N}.$$

This is the unique extension with the property that $g - Id$ is periodic mod n. Now for $x \in \mathbb{R}$, let

$$f(x) = \frac{(1 - \{nx\})g(\lfloor nx \rfloor) + \{nx\}g(\lceil nx \rceil)}{n} \tag{13}$$

where $\lfloor y \rfloor$, $\lceil y \rceil$ and $\{y\}$ denote respectively the greatest integer $\leq y$, the least integer $\geq y$, and the fractional part of y.

By (12) we have for all $x \in \mathbb{R}$

$$\begin{aligned} f(x+1) &= \frac{(1 - \{nx\})g(\lfloor nx \rfloor + n) + \{nx\}g(\lceil nx \rceil + n)}{n} \\ &= f(x) + 1. \end{aligned}$$

Hence $f : \mathbb{R} \to \mathbb{R}$ descends to a circle map $\bar{f} : S^1 \to S^1$ (viewing S^1 as $\mathbb{R}/\mathbb{Z}$). Since f is nondecreasing, it has a well-defined *Poincaré rotation number* [8, 13]

$$\rho(f) = \lim_{t\to\infty} \frac{f^t(x)}{t} \tag{14}$$

which does not depend on x. Here f^t denotes the t-fold iterate $f^t(x) = f(f^{t-1}(x))$, with $f^0 = Id$. The rotation number measures the rate at which the sequence of points $x, \bar{f}(x), \bar{f}(\bar{f}(x)), \ldots$ winds around the circle.

Theorem 2.5. *If σ is preconfined, then $a(\sigma) = \rho(f)$.*

Note that the map g is defined in terms of α_1 and ϕ, both of which are easily read off from σ. So given a preconfined configuration σ, equations (11) and (13) give an explicit recipe for writing down a circle map f whose rotation number is the activity of σ.

One naturally wonders how to generalize this construction to chip-firing configurations on graphs other than K_n. A key step may involve identifying invariants of the dynamics. On K_n, these invariants take a very simple form: by (8), for any two vertices $v, w \in [n]$, the difference

$$U^t\sigma(v) - U^t\sigma(w) \mod n$$

does not depend on t. Analogous invariants for parallel chip-firing on the $n \times n$ torus are classified in [6].

3 Devil's Staircase

Let $\sigma_2, \sigma_3, \ldots$ be a sequence of chip configurations, with σ_n defined on K_n, satisfying the conditions (4) and (5). Extend F to all of $\mathbb{R}$ by setting

$$F(x+m) = F(x) + m, \qquad m \in \mathbb{Z}. \tag{15}$$

Note that (4) and (5) force $F(0) = 0$ and $F(1) = 1$, so this extension is continuous.

The *rescaled activity phase diagram* of σ_n is the function $s_n : [0,1] \to [0,1]$ defined by

$$s_n(y) = a(\sigma_n + \lfloor ny \rfloor).$$

As $n \to \infty$, the s_n have a pointwise limit identified in our next result. Here and in what follows, $\rho(\cdot)$ denotes the rotation number (14).

Theorem 3.1. *If (4) and (5) hold, then for each $y \in [0,1]$ we have*

$$s_n(y) \to s(y) := \rho(R_y \circ \Phi)$$

as $n \to \infty$, where $\Phi(x) = -F(-x)$, and $R_y(x) = x + y$.

Write $\Phi_y = R_y \circ \Phi$, and let $\bar{\Phi}_y : S^1 \to S^1$ be the corresponding circle map. We will call a function $s : [0,1] \to [0,1]$ a *devil's staircase* if it is continuous, nondecreasing, nonconstant, and locally constant on an open dense set. Next we show that if

$$\left(\bar{\Phi}_y\right)^q \neq Id \quad \text{for all } y \in S^1 \text{ and all } q \in \mathbb{N}, \tag{16}$$

then the limiting function $s(y)$ in Theorem 3.1 is a devil's staircase. Examples of these staircases for different choices of F are shown in Figure 2.

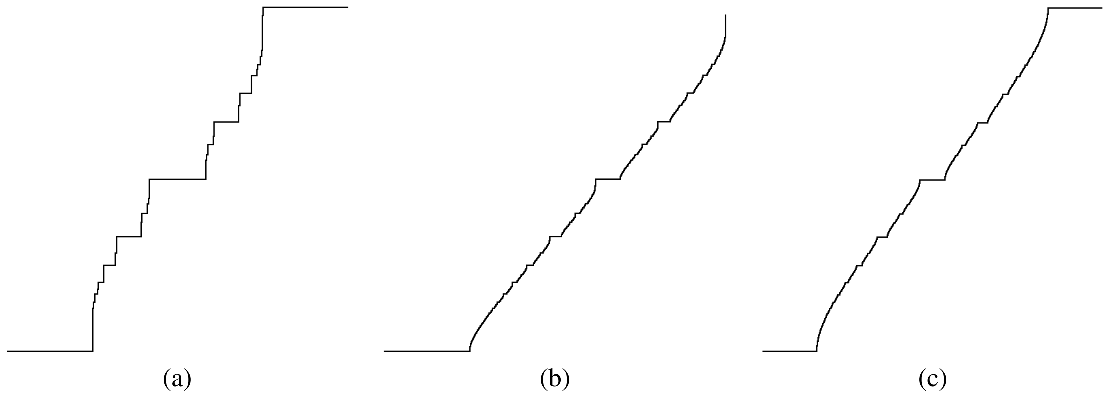

Fig. 2: The devil's staircase $s(y)$, when (a) $F(x)$ is given by (7); (b) $F(x) = \sqrt{x}$ for $x \in [0,1]$; and (c) $F(x) = x + \frac{1}{2\pi}\sin 2\pi x$. On the horizontal axis y runs from 0 to 1, and on the vertical axis $s(y)$ runs from 0 to 1.

Proposition 3.2. *The function $s(y) = \rho(\Phi_y)$ continuous and nondecreasing in y. If $z \in [0,1]$ is irrational, then $s^{-1}(z)$ is a point. Moreover, if (16) holds, then for every rational number $p/q \in [0,1]$ the fiber $s^{-1}(p/q)$ is an interval of positive length.*

Our next result shows that in the interiors of the stairs, we in fact have $s_n(y) = s(y)$ for sufficiently large n.

Proposition 3.3. *Suppose that (4), (5) and (16) hold. If $s^{-1}(p/q) = [a,b]$, then for any $\epsilon > 0$*

$$[a+\epsilon, b-\epsilon] \subset s_n^{-1}(p/q)$$

for all sufficiently large n.

The results in this section follow from Theorem 2.5 along with a few well-known properties of the rotation number $\rho(f)$. To give a flavor of the proofs, we list here the properties we use. For more background on the rotation number, see [8, 13].

Call a map $f : \mathbb{R} \to \mathbb{R}$ a *monotone degree one lift* if f is continuous, nondecreasing and satisfies

$$f(x+1) = f(x) + 1 \tag{17}$$

for all $x \in \mathbb{R}$. Let f, f_n, g be monotone degree one lifts, and denote by $\bar{f}, \bar{f}_n, \bar{g}$ the corresponding circle maps $S^1 \to S^1$. Write $f \le g$ if $f(x) \le g(x)$ for all $x \in \mathbb{R}$, and $f < g$ if $f(x) < g(x)$ for all $x \in \mathbb{R}$.

- **Monotonicity**. If $f \le g$, then $\rho(f) \le \rho(g)$.
- **Continuity**. If $\sup |f_n - f| \to 0$, then $\rho(f_n) \to \rho(f)$.
- **Conjugation Invariance**. If g is strictly increasing, then $\rho(g \circ f \circ g^{-1}) = \rho(f)$.
- **Instability of an irrational rotation number**. If $\rho(f) \notin \mathbb{Q}$, and $f_1 < f < f_2$, then

$$\rho(f_1) < \rho(f) < \rho(f_2).$$

- **Stability of a rational rotation number**. If $\rho(f) = p/q \in \mathbb{Q}$, and $\bar{f}^q \neq Id : S^1 \to S^1$, then for sufficiently small $\epsilon > 0$, either

$$\rho(g) = p/q \text{ whenever } f \leq g \leq f + \epsilon,$$

or

$$\rho(g) = p/q \text{ whenever } f - \epsilon \leq g \leq f.$$

4 Short Period Attractors

For a chip configuration σ on K_n and a vertex $v \in [n]$, let

$$u_t(\sigma, v) = \#\{0 \leq s < t \,|\, U^s\sigma(v) \geq n\}$$

be the number of times v fires during the first t updates. During these updates, the vertex v emits a total of $n\, u_t(\sigma, v)$ chips and receives a total of $\alpha_t = \sum_w u_t(\sigma, w)$ chips, so that

$$U^t\sigma(v) - \sigma(v) = \alpha_t - n\, u_t(\sigma, v). \tag{18}$$

An easy consequence is the following.

Lemma 4.1. *A chip configuration σ on K_n satisfies $U^t\sigma = \sigma$ if and only if*

$$u_t(\sigma, v) = u_t(\sigma, w) \tag{19}$$

for all vertices v and w.

According to our next lemma, if σ is confined, then $u_t(\sigma, v)$ and $u_t(\sigma, w)$ differ by at most one.

Lemma 4.2. *If σ is confined, and $\sigma(v) \leq \sigma(w)$, then for all $t \geq 0$*

$$u_t(\sigma, v) \leq u_t(\sigma, w) \leq u_t(\sigma, v) + 1.$$

Lemma 4.3. *If σ is confined, then $U^t\sigma = \sigma$ if and only if $n|\alpha_t$.*

Let σ be a confined state on K_n. By the pigeonhole principle, there exist times $0 \leq s < t \leq n$ with

$$\alpha_s \equiv \alpha_t \pmod{n}.$$

By Lemma 4.3 it follows that $U^s\sigma = U^t\sigma$, so σ has eventual period at most n.

Our next result improves this bound a bit. Write $m(\sigma)$ for the eventual period of σ, and

$$\nu(\sigma) = \#\{\sigma(v) | v \in [n]\}$$

for the number of distinct heights in σ.

Proposition 4.4. *For any chip configuration σ on K_n,*

$$m(\sigma) \leq \nu(\sigma).$$

Bitar [3] conjectured that any parallel chip-firing configuration on a connected graph of n vertices has eventual period at most n. A counterexample was found by Kiwi et al. [10]. It would be interesting to investigate for what classes of graphs Bitar's conjecture does hold; for example, no counterexample seems to be known on a regular graph.

Next we relate the period to the activity.

Lemma 4.5. *If $a(\sigma) = p/q$ and $(p, q) = 1$, then $m(\sigma) = q$.*

The proof uses the fact that the rotation number of a circle map determines the periods of its periodic points: if $f : \mathbb{R} \to \mathbb{R}$ is a monotone degree one lift (17) with $\rho(f) = p/q$ in lowest terms, then all periodic points of $\bar{f} : S^1 \to S^1$ have period q; see Proposition 4.3.8 and Exercise 4.3.5 of [8].

Given $1 \leq p < q \leq n$ with $(p, q) = 1$ and $p/q \leq 1/2$, one can check that the chip configuration σ on K_n given by

$$\sigma(v) = \begin{cases} v + p - 1, & v \leq q - 1 - p \\ v + n + p - q - 1, & q - p \leq v \leq q - 1 \\ n + p - 1, & v \geq q. \end{cases}$$

has activity $a(\sigma) = p/q$. For a similar construction on more general graphs in the case $p = 1$, see [14, Prop. 6.8]. In particular, $m(\sigma) = q$ by Lemma 4.5. So for every integer $q = 1, \ldots, n$ there exists a chip configuration on K_n of period q.

Despite the existence of states of period as large as n, states of smaller period are in some sense more prevalent. One way to capture this is the following.

Theorem 4.6. *If $\sigma_2, \sigma_3, \ldots$ is a sequence of chip configurations satisfying (4), (5) and (16), then for each $q \in \mathbb{N}$ there is a constant $c = c_q > 0$ such that for all sufficiently large n, at least cn of the states $\{\sigma_n + k\}_{k=0}^n$ have eventual period q.*

The proof follows easily from Proposition 3.3, which shows that a constant fraction cn of the states $\sigma_n + k$ have activity $1/q$. By Lemma 4.5 these states have eventual period q. The devil's staircase $s(y)$ determines the best possible constant c_q, namely, the total length of all stairs whose height has denominator q. If $s^{-1}(p/q) = [a_p, b_p]$, then any constant

$$c_q < \sum_{p:(p,q)=1} (b_p - a_p)$$

satisfies the conclusion of the theorem.

The rest of this section outlines the proof of Theorem 4.11, which finds a *period* 2 *window*: any chip configuration on K_n with total number of chips strictly between $n^2 - n$ and n^2 has eventual period 2. The following lemma is a special case of [14, Prop. 6.2].

Lemma 4.7. *If σ and τ are chip configurations on K_n with $\sigma(v) + \tau(v) = 2n - 1$ for all v, then $a(\sigma) + a(\tau) = 1$.*

Given a chip configuration σ on K_n, for $j = 1, \ldots, n$ we define *conjugate* configurations

$$c^j\sigma(v) = \begin{cases} \sigma(v) + j - n, & v \leq j \\ \sigma(v) + j, & v > j. \end{cases}$$

Lemma 4.8. *Let σ be a chip configuration on K_n, and fix $j \in [n]$. For all $t \geq 0$, we have for $v \leq j$*

$$u_t(\sigma, v) - 1 \leq u_t(c^j\sigma, v) \leq u_t(\sigma, v),$$

while for $v > j$

$$u_t(\sigma, v) \leq u_t(c^j\sigma, v) \leq u_t(\sigma, v) + 1.$$

Corollary 4.9. *For any chip configuration σ on K_n and any $j \in [n]$,*

$$a(c^j\sigma) = a(\sigma).$$

It turns out that the circle maps corresponding to σ and $c^j\sigma$ are conjugate to one another by a rotation. This gives an alternative proof of the corollary, in the case when both σ and $c^j\sigma$ are preconfined.

Lemma 4.10. *Let σ be a chip configuration on K_n. If $u_2(\sigma, v) \geq 1$ for all v, then $u_{2t}(\sigma, v) \geq t$ for all v and all $t \geq 1$.*

.

Write

$$|\sigma| = \sum_{v=1}^{n} \sigma(v)$$

for the total number of chips in the system.

Theorem 4.11. *Every chip configuration σ on K_n with $n^2 - n < |\sigma| < n^2$ has eventual period 2.*

The outline of the proof runs as follows. Writing

$$\ell(\sigma) = \min\{\sigma(1), \ldots, \sigma(n)\}$$

and

$$r(\sigma) = \#\{v \in [n] : \sigma(v) \geq n\},$$

a straightforward calculation shows that if $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n)$ and $n^2 - n < |\sigma| < n^2$, then

$$\sum_{j=1}^{n} (\ell(c^j\sigma) + r(c^j\sigma)) > n^2 - n.$$

Since each term in the sum on the left is a nonnegative integer, we must have

$$\ell(c^j\sigma) + r(c^j\sigma) \geq n.$$

for some $j \in [n]$. Thus every vertex v fires at least once during the first two updates of $c^j\sigma$. By Corollary 4.9 and Lemma 4.10, this implies

$$a(\sigma) = a(c^j\sigma) \geq \frac{1}{2}.$$

The chip configuration $\tau(v) := 2n - 1 - \sigma(v)$ also satisfies $n^2 - n < |\tau| < n^2$, so $a(\tau) \geq \frac{1}{2}$ as well. By Lemma 4.7 we have $a(\sigma) + a(\tau) = 1$, so $a(\sigma) = a(\tau) = \frac{1}{2}$. Finally, from Lemma 4.5 we conclude that $m(\sigma) = 2$.

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Macdonald polynomials at $t = q^k$

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Abstract. We investigate the homogeneous symmetric Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ for the specialization $t = q^k$. We show an identity relying the polynomials $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$. As a consequence, we describe an operator whose eigenvalues characterize the polynomials $P_\lambda(\mathbb{X}; q, q^k)$.

Résumé. Nous nous intéressons aux propriétés des polynômes de Macdonald symétriques $P_\lambda(\mathbb{X}; q, t)$ pour la spécialisation $t = q^k$. En particulier nous montrons une égalité reliant les polynômes $P_\lambda(\mathbb{X}; q, q^k)$ et $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$. Nous en déduisons la description d'un opérateur dont les valeurs propres caractérisent les polynômes $P_\lambda(\mathbb{X}; q, q^k)$.

Keywords: Symmetric functions, Macdonald polynomials, q-discriminant

1 Introduction

The Macdonald polynomials are (q, t)-deformations of the Schur functions which play an important rôle in the representation theory of the double affine Hecke algebra [11, 13] since they are the eigenfunctions of the Cherednik elements. More precisely, the non-symmetric Macdonald polynomials are the eigenfunctions of the Cherednik elements, but the symmetric Macdonald polynomials are the eigenfunctions of the symmetric functions in the Cherednik elements. The polynomials considered here are the homogeneous symmetric Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ and are the eigenfunctions of the Sekiguchi-Debiard-Macdonald operator $\mathfrak{M}_1$. For (q, t) generic, the dimension of each eigenspace equals 1 and each Macdonald polynomial is characterized (up to a multiplicative constant) by the associated eigenvalue of $\mathfrak{M}_1$. That is no longer true when t is specialized to a rational power of q (note that the case of the specialization $t^n q^m = 1$ - n and m being integer - has been investigated by Feigin et *al.* [4] in their study of ideals of symmetric functions defined by vanishing conditions). Hence, it is more convenient to characterize the Macdonald (homogeneous symmetric) polynomials by orthogonality (*w.r.t.* a (q, t)-deformation of the usual scalar product on symmetric functions) and by some conditions on their dominant monomials (see *e.g.* [12]). In this paper, we consider the specialization $t = q^k$ where k is a (strictly) positive integer. One of our motivations is to generalize an identity of [1], which shows that even powers of the discriminant are rectangular Jack polynomials. Here, we show that this property follows from deeper relations between the Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$ (in the λ-ring notation). This result is interesting in the context of the fractional quantum Hall effect [8], since it implies properties of the expansion of the powers of the discriminant in the Schur basis [3, 6, 14]. It implies also that the

Macdonald polynomials (at $t = q^k$) are characterized by the eigenvalues of an operator $\mathfrak{M}$ (described in terms of isobaric divided differences) whose eigenspaces are of dimension 1.

The paper is organized as follows. After recalling notations and background (Section 2) related to Macdonald polynomials, we give, in Section 3, some properties of the operator which substitutes a complete function to each power of a letter. These properties allow us to show our main result in Section 4 which is an identity involving the polynomial $P_\lambda(\mathbb{X}; q, q^k)$ and $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right)$. As a consequence, we describe (Section 5) an operator $\mathfrak{M}$ whose eigenvalues characterize the Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$. Finally, in Section 6, we give an expression of $\mathfrak{M}$ in terms of the Cherednik elements.

2 Notations and background

We recall here the basic definitions and classical properties of the symmetric functions and the Macdonald polynomials.

2.1 Symmetric functions

Consider an alphabet $\mathbb{X}$ (potentially infinite). Following [10] we define the symmetric functions on $\mathbb{X}$ by the generating functions of the complete homogeneous functions $S^p(\mathbb{X})$,

$$\sigma_z(\mathbb{X}) := \sum_i S^i(\mathbb{X}) z^i = \prod_{x \in \mathbb{X}} \frac{1}{1 - xz}.$$

The algebra Sym of symmetric functions has a λ-ring structure [10] and many properties of that structure can be understood by manipulating σ_z. For example, the sum of two alphabets $\mathbb{X} + \mathbb{Y}$ is defined by the product

$$\sigma_z(\mathbb{X} + \mathbb{Y}) := \sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sum_i S^i(\mathbb{X} + \mathbb{Y}) z^i.$$

In particular, if $\mathbb{X} = \mathbb{Y}$ one has $\sigma_z(2\mathbb{X}) = \sigma_z(\mathbb{X})^2$. This definition is extended to any complex number α by $\sigma_z(\alpha\mathbb{X}) = \sigma_z(\mathbb{X})^\alpha$. For example, the generating series of the elementary functions is

$$\begin{array}{rcl} \lambda_z(\mathbb{X}) & := & \sum \Lambda_i(\mathbb{X}) z^i = \prod_{x \in \mathbb{X}} (1 + xz) \\ & = & \sigma_{-z}(-\mathbb{X}) = \sum_i (-1)^i S^i(-\mathbb{X}) z^i. \end{array}$$

The complete functions of the product of two alphabets $\mathbb{X}\mathbb{Y}$ are given by the Cauchy kernel

$$K(\mathbb{X}, \mathbb{Y}) := \sigma_1(\mathbb{X}\mathbb{Y}) = \sum_i S^i(\mathbb{X}\mathbb{Y}) = \prod_{x \in \mathbb{X}} \prod_{y \in \mathbb{Y}} \frac{1}{1 - xy} = \sum_\lambda S_\lambda(\mathbb{X}) S_\lambda(\mathbb{Y}),$$

where S_λ denotes, as in [10], a Schur function. More generally, one has

$$K(\mathbb{X}, \mathbb{Y}) = \sum_\lambda A_\lambda(\mathbb{X}) B_\lambda(\mathbb{Y})$$

for any pair of bases $(A_\lambda)_\lambda$ and $(B_\lambda)_\lambda$ in duality for the usual scalar product $\langle\, ,\, \rangle$, *i.e.* $K(\mathbb{X}, \mathbb{Y})$ is the reproducing kernel associated to $\langle\, ,\, \rangle$.

2.2 Macdonald polynomials

The usual scalar product on symmetric functions admits a (q,t)-deformation (see *e.g.* [12]) defined for a pair of power sum functions Ψ^λ and Ψ^μ (in the notation of [10]) by

$$\langle \Psi^\lambda, \Psi^\mu \rangle_{q,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \tag{1}$$

where $\delta_{\lambda,\mu} = 1$ if $\lambda = \mu$ and 0 otherwise. The family of (symmetric homogeneous) Macdonald polynomials $(P_\lambda(\mathbb{X};q,t))_\lambda$ is the unique basis of the symmetric functions orthogonal *w.r.t.* $\langle \, , \, \rangle_{q,t}$ verifying

$$P_\lambda(\mathbb{X};q,t) = m_\lambda(\mathbb{X}) + \sum_{\mu \le \lambda} u_{\lambda\mu} m_\mu(\mathbb{X}), \tag{2}$$

where m_λ denotes, as usual, a monomial function [10, 12]. The reproducing kernel associated to this scalar product is

$$K_{q,t}(\mathbb{X},\mathbb{Y}) := \sum_\lambda \langle \Psi^\lambda, \Psi^\lambda \rangle_{q,t}^{-1} \Psi_\lambda(\mathbb{X}) \Psi_\lambda(\mathbb{Y}) = \sigma_1 \left(\frac{1-t}{1-q} \mathbb{X}\mathbb{Y} \right)$$

see *e.g.* [12, VI.2]. In particular, one has

$$K_{q,t}(\mathbb{X},\mathbb{Y}) = \sum_\lambda P_\lambda(\mathbb{X};q,t) Q_\lambda(\mathbb{Y};q,t), \tag{3}$$

where $Q_\lambda(\mathbb{X};q,t)$ is the dual basis of $P_\lambda(\mathbb{Y};q,t)$ with respect to $\langle \, , \, \rangle_{q,t}$,

$$Q_\lambda(\mathbb{X};q,t) = \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1} P_\lambda(\mathbb{X};q,t). \tag{4}$$

The coefficient $b_\lambda(q,t) = \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1}$ is known to be

$$b_\lambda(q,t) = \prod_{(i,j)\in\lambda} \frac{1-q^{\lambda_j-i+1} t^{\lambda'_i-j}}{1-q^{\lambda_j-i} t^{\lambda'_i-j+1}} \tag{5}$$

see [12, VI.6]. Writing

$$K_{q,t}\left(\left(\frac{1-q}{1-t} \right) \mathbb{X}, \mathbb{Y} \right) = K(\mathbb{X},\mathbb{Y}), \tag{6}$$

one finds that $\left(P_\lambda \left(\left(\frac{1-q}{1-t} \right) \mathbb{X}; q,t \right) \right)_\lambda$ is the dual basis of $(Q_\lambda(\mathbb{X};q,t))_\lambda$ with respect to the usual scalar product $\langle \, , \, \rangle$.

Note that there exists an other Kernel type formula which reads

$$\lambda_1(\mathbb{X}\mathbb{Y}) = \sum_\lambda P_{\lambda'}(\mathbb{X};t,q) P_\lambda(\mathbb{Y};q,t) = \sum_\lambda Q_{\lambda'}(\mathbb{X};t,q) Q_\lambda(\mathbb{Y};q,t). \tag{7}$$

where λ' denotes the conjugate partition of λ. This formula can be found in [12, VI.5 p329].

From equalities (6) and (3) , one has

$$\sigma_1(\mathbb{XY}) = K_{q,t}\left(\frac{1-q}{1-t}\mathbb{X},\mathbb{Y}\right) = \sum_\lambda Q_\lambda\left(\frac{1-q}{1-t}\mathbb{X};q,t\right)P_\lambda(\mathbb{Y};q,t). \tag{8}$$

Applying (7) to

$$\sigma_1(\mathbb{XY}) = \lambda_{-1}(-\mathbb{XY}),$$

one obtains

$$\sigma_1(\mathbb{XY}) = \sum_\lambda (-1)^{|\lambda|}Q_{\lambda'}(-\mathbb{X};t,q)Q_\lambda(\mathbb{Y};q,t). \tag{9}$$

Identifying the coefficient of $P_\lambda(\mathbb{Y};t,q)$ in (8) and (9), one finds the following property.

Lemma 2.1

$$Q_\lambda(-\mathbb{X};t,q) = (-1)^{|\lambda|}P_{\lambda'}\left(\frac{1-q}{1-t}\mathbb{X};q,t\right). \tag{10}$$

Unlike the usual ($q=t=1$) scalar product, there is no expression as a constant term for the product $\langle\ ,\ \rangle_{q,t}$ when $\mathbb{X}=\{x_1,\dots,x_n\}$ is finite. But the Macdonald polynomials are orthogonal with respect to an other scalar product defined by

$$\langle f,g\rangle'_{q,t;n} = \frac{1}{n!}\mathrm{C.T.}\{f(\mathbb{X})g(\mathbb{X}^\vee)\Delta_{q,t}(\mathbb{X})\} \tag{11}$$

where C.T. denotes the constant term *w.r.t.* the alphabet $\mathbb{X}$, $\Delta_{q,t}(\mathbb{X}) = \prod_{i\neq j}\frac{(x_ix_j^{-1};q)_\infty}{(tx_ix_j^{-1};q)_\infty}$, $(a;b)_\infty = \prod_{i\geq 0}(1-ab^i)$ and $\mathbb{X}^\vee = \{x_1^{-1},\dots,x_n^{-1}\}$. The expression of $\langle P_\lambda,Q_\lambda\rangle'_{q,t;n}$ is given by ([12, VI.9])

$$\langle P_\lambda,Q_\lambda\rangle'_{q,t;n} = \frac{1}{n!}\mathrm{C.T.}\{\Delta_{q,t}(\mathbb{X})\}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^it^{n-j}}. \tag{12}$$

2.3 Skew symmetric functions

Let us define as in [12, VI.7], the skew Macdonald functions $Q_{\lambda/\mu}$ by

$$\langle Q_{\lambda/\mu},P_\nu\rangle_{q,t} := \langle Q_\lambda,P_\mu P_\nu\rangle_{q,t}. \tag{13}$$

Straightforwardly, one has

$$Q_{\lambda/\mu}(\mathbb{X};q,t) = \sum_\nu \langle Q_\lambda,P_\nu P_\mu\rangle_{q,t}Q_\nu(\mathbb{X};q,t). \tag{14}$$

And classically, the following property holds (see *e.g.* [12, VI.7] for a short proof of this identity),

Proposition 2.2 *Let $\mathbb{X}$ and $\mathbb{Y}$ be two alphabets, one has*

$$Q_\lambda(\mathbb{X}+\mathbb{Y};q,t) = \sum_\mu Q_\mu(\mathbb{X};q,t)Q_{\lambda/\mu}(\mathbb{Y};q,t),$$

or equivalently

$$P_\lambda(\mathbb{X}+\mathbb{Y};q,t)=\sum_\mu P_\mu(\mathbb{X};q,t)P_{\lambda/\mu}(\mathbb{Y};q,t).$$

Equalities (3) and (7) are generalized by identities (15) and (16) as shown in [12, example 6 p.352],

$$\sum_\rho P_{\rho/\lambda}(\mathbb{X};q,t)Q_{\rho/\mu}(\mathbb{Y};q,t)=K_{qt}(\mathbb{X},\mathbb{Y})\sum_\rho P_{\mu/\rho}(\mathbb{X};q,t)Q_{\lambda/\rho}(\mathbb{Y};q,t), \tag{15}$$

$$\sum_\rho Q_{\rho'/\lambda'}(\mathbb{X};t,q)Q_{\rho/\mu}(\mathbb{Y};q,t)=\lambda_1(\mathbb{X}\mathbb{Y})\sum_\rho Q_{\mu'/\rho'}(\mathbb{X},t,q)Q_{\lambda/\rho}(\mathbb{Y};q,t). \tag{16}$$

3 The substitution $x^p \to S^p(\mathbb{Y})$ and the Macdonald polynomials

Let $\mathbb{X}=\{x_1,\dots,x_n\}$ be a finite alphabet and $\mathbb{Y}$ be an other (potentially infinite) alphabet. For simplicity we will denote by $\int_{\mathbb{Y}}$ the substitution

$$\int_{\mathbb{Y}}:\ x^p\to S^p(\mathbb{Y}), \tag{17}$$

for each $x\in\mathbb{X}$ and each $p\in\mathbb{Z}$. Let us define the symmetric function

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y};q,t):=\frac{1}{n!}\int_{\mathbb{Y}}P_\lambda(\mathbb{X};q,t)Q_\mu(\mathbb{X}^\vee;q,t)\Delta(\mathbb{X},q,t) \tag{18}$$

where $\mathbb{X}^\vee=\{x_1^{-1},\dots,x_n^{-1}\}$.

Set $\mathbb{Y}^{tq}:=\frac{1-t}{1-q}\mathbb{Y}$ and consider the substitution

$$\int_{\mathbb{Y}^{tq}}x^p=S^p\left(\mathbb{Y}^{tq}\right)=Q_p(\mathbb{Y};q,t). \tag{19}$$

The following result shows that $\mathfrak{H}^{n,k}_{\lambda/\mu}$ is a skew Macdonald polynomial on a suitable alphabet.

Theorem 3.1 *Let $\mathbb{X}=\{x_1,\dots,x_n\}$ be an alphabet, $\lambda=(\lambda_1,\dots,\lambda_n)$ be a partition and $\mu\subset\lambda$. The polynomial $\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq};q,t)$ is the Macdonald polynomial*

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(\mathbb{Y}^{tq};q,t)=\frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^it^{n-j}}\mathrm{C.T.}\{\Delta(\mathbb{X},q,t)\}P_{\lambda/\mu}(\mathbb{Y},q,t) \tag{20}$$

Set $\overline{\mathbb{Y}}=\{-y_1,\dots,-y_m,\dots\}$ if $\mathbb{Y}=\{y_1,\dots,y_m,\dots\}$ and note that the operation $\mathbb{Y}\to\overline{\mathbb{Y}}$ makes sense even for virtual alphabet since it sends any homogeneous symmetric polynomial $P(\mathbb{Y})$ of degree p to $(-1)^pP(\mathbb{Y})$. One observes the following phenomenon which is obtained from Theorem 3.1 by applying the operations of the λ-ring structure.

Corollary 3.2 *Let $\mathbb{X}=\{x_1,\dots,x_n\}$ be an alphabet, $\lambda=(\lambda_1,\dots,\lambda_n)$ be a partition and $\mu\subset\lambda$. One has*

$$\mathfrak{H}^{n,k}_{\lambda/\mu}(-\overline{\mathbb{Y}};q,t)=\frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^it^{n-j}}\mathrm{C.T.}\{\Delta(\mathbb{X},q,t)\}Q_{\lambda'/\mu'}(\mathbb{Y},t,q). \tag{21}$$

Note that in the case of partitions, one has

Corollary 3.3

$$\mathfrak{H}_{\lambda}^{n,k}(-\overline{\mathbb{Y}},q,t)=\frac{1}{n!}\prod_{(i,j)\in\lambda}\frac{1-q^{i-1}t^{n-j+1}}{1-q^{i}t^{n-j}}\mathrm{C.T.}\{\Delta(\mathbb{X},q,t)\}Q_{\lambda'}(\mathbb{Y},t,q) \tag{22}$$

Example 3.4 Consider the following equality

$$\mathfrak{H}_{41/3}^{2,3}(-\overline{\mathbb{Y}};q,t)=(*)\mathrm{C.T.}\{\Delta(\mathbb{X},q,t)\}Q_{2111/111}(\mathbb{Y};t,q).$$

where $\mathbb{X}=\{x_1,x_2\}$. The coefficient $(*)$ is computed as follows. One writes the partition $[41]$ in a rectangle of height 2 and length 4.

$\times$			
$\times$	$\times$	$\times$	$\times$

Each $\times$ of coordinates (i,j) is read as the fraction $[i,j]:=\frac{1-q^{i-1}t^{3-j}}{1-q^{i}t^{2-j}}$. Hence

$$(*)=[1,2][1,1][2,1][3,1][4,1]=\frac{(1-t)(1-t^2)(1-qt^2)(1-q^2t^2)(1-q^3t^2)}{(1-q)(1-qt)(1-q^2t)(1-q^3t)(1-q^4t)}$$

4 A formula involving the polynomials $P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X};q,q^k\right)$ and $P_\lambda\left(\mathbb{X};q,q^k\right)$

Now, we suppose that $t=q^k$ with $k\in\mathbb{N}$. In that case, the constant term $\mathrm{C.T.}\{\Delta(\mathbb{X},q,t)\}$ admits a closed form and Corollary 3.3 gives

Corollary 4.1

$$\mathfrak{H}_{\lambda}^{n,k}(-\overline{\mathbb{Y}},q,q^k)=\beta_{\lambda}^{n,k}(q)Q_{\lambda'}(\mathbb{Y};q^k,q). \tag{23}$$

where

$$\beta_{\lambda}^{n,k}(q)=\prod_{i=0}^{n-1}\begin{bmatrix}\lambda_{n-i}-1+k(i+1)\\ k-1\end{bmatrix}_q$$

and $\begin{bmatrix}n\\ p\end{bmatrix}_q=\frac{(1-q^n)\dots(1-q^{n-p+1})}{(1-q)\dots(1-q^r)}$ *denotes the q-binomial.*

Example 4.2 Set $k=2,n=3$ and consider the polynomial

$$\mathfrak{H}_{[320]}^{3,2}(-\overline{\mathbb{Y}};q,q^2)=\frac{1}{n!}\int_{-\overline{\mathbb{Y}}}P_{[32]}(x_1+x_2+x_3;q,q^2)\prod_{i\neq j}(1-x_ix_j^{-1})(1-qx_ix_j^{-1}).$$

One has

$$\mathfrak{H}_{[320]}^{3,2}(-\overline{\mathbb{Y}};q,q^2)=\frac{\left(1-q^5\right)\left(1-q^8\right)}{\left(1-q\right)^2}Q_{[221]}(\mathbb{Y};q^2,q).$$

Let

$$\Omega_S := \frac{1}{n!}\int_{\mathbb{X}} \prod_{i\neq j}(1 - x_i x_j^{-1}) \tag{24}$$

and for each $v \in \mathbb{Z}^n$,

$$\tilde{S}_v(\mathbb{X}) = \det\left(x_i^{v_j+n-j}\right)\prod_{i<j}(x_i - x_j)^{-1}.$$

Lemma 4.3 *If v is any vector in $\mathbb{Z}^n$, one has*

$$\Omega_S \tilde{S}_v(\mathbb{X}) = S_v(\mathbb{X}) := \det(S^{v_i-i+j}(\mathbb{X})) \tag{25}$$

In particular, Ω_S leaves invariant any symmetric polynomial. The operator

$$\mathfrak{A}_m := \Omega_S \Lambda^n(\mathbb{X})^{-m} \tag{26}$$

acts on symmetric polynomials by substracting m from each part of the partitions appearing in their expansion in the Schur basis.

Example 4.4 If $\mathbb{X} = \{x_1, x_2, x_3\}$ and $\lambda = [320]$, one has

$$P_{32}(\mathbb{X}; q, t) = S_{32}(\mathbb{X}) + \frac{(-q+t)\, S_{311}(\mathbb{X})}{qt-1} + \frac{(q+1)\left(qt^2-1\right)(-q+t)\, S_{221}(\mathbb{X})}{(qt-1)^2\,(qt+1)}.$$

Hence,

$$\begin{aligned}\mathfrak{A}_1 P_{32}(\mathbb{X}; q, t) &= \tfrac{(-q+t)S_2(\mathbb{X})}{qt-1} + \tfrac{(q+1)\left(qt^2-1\right)(-q+t)S_{11}(\mathbb{X})}{(qt-1)^2(qt+1)} \\ &= \tfrac{(-q+t)(t+1)\left(q^2t-1\right)P_{11}(\mathbb{X};q,t)}{(qt-1)^2(qt+1)} + \tfrac{(-q+t)P_2(\mathbb{X};q,t)}{qt-1}.\end{aligned}$$

Theorem 4.5 *If λ denotes a partition of length at most n, one has*

$$\mathfrak{A}_{(k-1)(n-1)} P_\lambda(\mathbb{X}; q, q^k) \prod_{l=1}^{k-1}\prod_{i\neq j}(x_i - q^l x_j) = \beta_\lambda^{n,k}(q) P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, q^k\right) \tag{27}$$

Example 4.6 Set $k = 2$, $n = 3$ and $\lambda = [2]$. One has

$$\begin{aligned}&P_{[2]}(x_1 + x_2 + x_3; q, q^2)\prod_{i\neq j}(x_i - qx_j) = -q^3 S_{[6,2]} + q^2\frac{q^3-1}{q-1}S_{[6,1,1]} \\ &+\frac{q^2(q^5-1)}{q^3-1}S_{[5,3]} - \frac{q(q^2+1)(q^5-1)}{q^3-1}S_{[5,2,1]} - \frac{q(q^7-1)}{q^3-1}S_{[4,3,1]} + \frac{q^7-1}{q-1}S_{[4,2,2]}.\end{aligned}$$

And,

$$\mathfrak{A}_2 P_{[2]}(x_1 + x_2 + x_3; q, q^2)\prod_{i\neq j}(x_i - qx_j) = \frac{q^7-1}{q-1}S_{[2]}.$$

Since,

$$P_{[2]}\left(\frac{x_1 + x_2 + x_3}{1+q}; q, q^2\right) = \frac{q-1}{q^3-1}S_{[2]}$$

one obtains

$$\mathfrak{A}_2 P_{[2]}(x_1+x_2+x_3;q,q^2)\prod_{i\neq j}(x_i-qx_j)=\begin{bmatrix}1\\1\end{bmatrix}_q\begin{bmatrix}3\\1\end{bmatrix}_q\begin{bmatrix}7\\1\end{bmatrix}_q P_{[2]}\left(\frac{x_1+x_2+x_3}{1+q};q,q^2\right).$$

As a consequence, one has

Corollary 4.7 *If* $\lambda=\mu+[((k-1)(n-1))^n]$,

$$P_\mu(\mathbb{X};q,q^k)\prod_{l=1}^{k-1}\prod_{i\neq j}(x_i-q^l x_j)=\beta_\lambda^{n,k}(q)P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X};q,q^k\right).$$

Example 4.8 Set $k=3$, $n=2$ and $\lambda=[5,2]$. One has

$$\begin{aligned}&P_{[5,2]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1)=\\&q^3S_{[9,2]}+\frac{(1-q^7)(1+q^4)}{1-q^5}S_{[7,4]}-\frac{(1-q^2)(1+q)(1+q^2)(1+q^4)}{1-q^5}S_{[8,3]}.\end{aligned}$$

This implies

$$\begin{aligned}&\mathfrak{A}_2 P_{[5,2]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1)=\\&(x_1x_2)^{-2}P_{[5,2]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1)=\\&P_{[3]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1).\end{aligned}$$

One verifies that

$$\begin{aligned}&P_{[3]}(x_1+x_2;q,q^3)(x_1-qx_2)(x_1-q^2x_2)(x_2-qx_1)(x_2-q^2x_1)=\\&\begin{bmatrix}4\\2\end{bmatrix}_q\begin{bmatrix}10\\2\end{bmatrix}_q P_{[5,2]}(\frac{x_1+x_2}{1+q+q^2};q,q^3).\end{aligned}$$

Remark 4.9 If μ is the empty partition, Corollary 4.7 gives

$$\prod_{l=1}^{k-1}\prod_{i\neq j}(x_i-q^l x_j)=\beta_\lambda^{n,k}(q)P_{[((k-1)(n-1))^n]}\left(\frac{1-q}{1-q^k}\mathbb{X};q,q^k\right). \tag{28}$$

This equality generalizes an identity given in [1]:

$$\prod_{i<j}(x_i-x_j)^{2(k-1)}=\frac{(-1)^{\frac{((k-1)n(n-1)}{2}}}{n!}\binom{kn}{k,\ldots,k}P^{(k)}_{n^{(n-1)(k-1)}}(-\mathbb{X}),$$

where $P^{(k)}_\lambda(\mathbb{X})=\lim_{q\to1}P^{(\alpha)}_\lambda(\mathbb{X};q,q^k)$ denotes a Jack polynomial (see *e.g.* [12]).

The expansion of the powers of the discriminant and their q-deformations in different basis of symmetric functions is a difficult problem having many applications, for example, in the study of Hua-type integrals (see *e.g.* [5, 7]) or in the context of the fractional quantum Hall effect (*e.g.* [3, 6, 8, 14]).

Note that in [2], we gave an expression of an other q-deformation of the powers of the discriminant as staircase Macdonald polynomials. This deformation is also relevant in the study of the expansion of $\prod_{i<j}(x_i-x_j)^{2k}$ in the Schur basis (for example, we generalized in [2] a result of [6]).

5 Macdonald polynomials at $t = q^k$ as eigenfunctions

Let $\mathbb{Y} = \{y_1, \dots, y_{kn}\}$ be an alphabet of cardinality kn with $y_1 = x_1, \dots, y_n = x_n$. One considers the symmetrizer π_ω defined by

$$\pi_\omega f(y_1, \dots, y_{kn}) = \prod_{i<j}(x_i - x_j)^{-1} \sum_{\sigma \in \mathfrak{S}_{kn}} \mathrm{sign}(\sigma) f(y_{\sigma(1)}, \dots, y_{\sigma(kn)}) y_{\sigma(1)}^{kn-1} \cdots y_{\sigma(kn-1)}.$$

Note that π_ω is the isobaric divided difference associated to the maximal permutation ω in $\mathfrak{S}_{kn}$.

This operator applied to a symmetric function of the alphabet $\mathbb{X}$ increases the alphabet from $\mathbb{X}$ to $\mathbb{Y}$ in its expansion in the Schur basis, since

$$\pi_\omega S_\lambda(\mathbb{X}) = S_\lambda(\mathbb{Y}). \tag{29}$$

Indeed, the image of the monomial $y_1^{i_1} \dots y_{kn}^{i_{kn}}$ is the Schur function $S_I(\mathbb{Y})$. Since

$$\pi_\omega S_\lambda(\mathbb{X}) = \pi_\omega x_1^{\lambda_1} \dots x_n^{\lambda_n} = \pi_\omega y_1^{\lambda_1} \dots y_n^{\lambda_n} y_{n+1}^0 \dots y_{kn}^0,$$

one recovers equality (29).

One defines the operator π^{tq} which consists in applying π_ω and specializing the result to the alphabet

$$\mathbb{X}^{tq} := \{x_1, \dots, x_n, qx_1, \dots, qx_n, \dots, q^{k-1}x_1, \dots, q^{k-1}x_n\}.$$

From equality (29), one has

$$\pi_\omega^{tq} S_\lambda(\mathbb{X}) = S_\lambda\left((1 + q + \dots + q^{k-1})\mathbb{X}\right), \tag{30}$$

for $l(\lambda) \leq n$. Furthermore, the expansion of $S_\lambda\left((1 + q + \dots + q^{k-1})\mathbb{X}\right)$ in the Schur basis is triangular, so the operator π^{tq} defines an automorphism of the space $Sym_{\leq n}$ generated by the Schur functions indexed by partitions whose length are less or equal to n, *i.e.* for each function $f \in Sym_{\leq n}$, one has

$$\pi^{tq} f(\mathbb{X}) = f(\mathbb{X}^{tq}). \tag{31}$$

In particular,

Lemma 5.1 *Let λ be a partition such that $l(\lambda) \leq n$ then*

$$\pi_\omega^{tq} P_\lambda\left(\frac{1-q}{1-q^k}\mathbb{X}; q, t = q^k\right) = P_\lambda(\mathbb{X}, q, q^k). \tag{32}$$

Consider the operator $\mathfrak{M} : f \to \mathfrak{M}f$ defined by

$$\mathfrak{M} := (x_1 \dots x_n)^{(k-1)(1-n)} \pi_\omega^{tq} \prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j).$$

The following theorem shows that the Macdonald polynomials are the eigenfunctions of the operator $\mathfrak{M}$.

Theorem 5.2 *The Macdonald polynomials $P_\lambda(\mathbb{X}; q, q^k)$ are eigenfunctions of $\mathfrak{M}$. The eigenvalue associated to $P_\mu(\mathbb{X}; q, q^k)$ is $\beta^{n,k}_{\mu+((k-1)(n-1))^n}(q)$. Furthermore, if $k > 1$, the dimension of each eigenspace is* 1.

Example 5.3 If $n = 5$, the eigenvalues associated to the partitions of 4 are

$$\begin{array}{lcl}
\beta^{4,k}_{[4k,4k-4,4k-4,4k-4,4k-4]} & = & \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-1 \\ k-1 \end{bmatrix}_q \ (\lambda = [4,0,0,0,0]), \\
\beta^{4,k}_{[4k-1,4k-3,4k-4,4k-4,4k-4]} & = & \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-2 \\ k-1 \end{bmatrix}_q \ (\lambda = [3,1,0,0,0]), \\
\beta^{4,k}_{[4k-2,4k-2,4k-4,4k-4,4k-4]} & = & \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-3 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-3 \\ k-1 \end{bmatrix}_q \ (\lambda = [2,2,0,0,0]), \\
\beta^{4,k}_{[4k-2,4k-3,4k-3,4k-4,4k-4]} & = & \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-3 \\ k-1 \end{bmatrix}_q \ (\lambda = [2,1,1,0,0]), \\
\beta^{4,k}_{[4k-3,4k-3,4k-3,4k-3,4k-4]} & = & \begin{bmatrix} 5k-5 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 6k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 7k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 8k-4 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 9k-4 \\ k-1 \end{bmatrix}_q \ (\lambda = [1,1,1,1,0]).
\end{array}$$

6 Expression of $\mathfrak{M}$ in terms of the Cherednik elements

In this paragraph, we restate Proposition 5.2 in terms of Cherednik operators. Cherednik's operators $\{\xi_i; i \in \{1,\dots,n\}\} =: \Xi$ are commutative elements of the double affine Hecke algebra. The Macdonald polynomials $P_\lambda(\mathbb{X}; q, t)$ are eigenfunctions of symmetric polynomials $f(\Xi)$ and the eigenvalues are obtained substituting each occurrence of ξ_i in $f(\Xi)$ by $q^{\lambda_i} t^{n-i}$ (see [11] for more details).
Suppose that $k > 1$ and consider the operator $\tilde{\mathfrak{M}} : f \to \tilde{\mathfrak{M}} f$ defined by

$$\tilde{\mathfrak{M}} := \prod_{i=1}^{k-1} (1 - q^i)^n \mathfrak{M}. \tag{33}$$

From Proposition 5.2, one has

$$\tilde{\mathfrak{M}} P_\lambda(\mathbb{X}; q, q^k) = \prod_{i=0}^{n-1} \prod_{j=1}^{k-1} (1 - q^{\lambda_{n-i} + k(i+1) - j}) P_\lambda(\mathbb{X}; q, q^k). \tag{34}$$

The following proposition shows that $\tilde{\mathfrak{M}}$ admits a closed expression in terms of the Cherednick elements.

Proposition 6.1 *One supposes that $k > 1$. For any symmetric function f, one has*

$$\tilde{\mathfrak{M}} f(\mathbb{X}) = \prod_{l=1}^{k-1} \prod_{i=1}^{n} (1 - q^{l+k} \xi_i) f(\mathbb{X}). \tag{35}$$

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A bijection between noncrossing and nonnesting partitions of types A and B

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Abstract. The total number of noncrossing partitions of type Ψ is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$ when $\Psi = A_{n-1}$, and the binomial coefficient $\binom{2n}{n}$ when $\Psi = B_n$, and these numbers coincide with the correspondent number of nonnesting partitions. For type A, there are several bijective proofs of this equality; in particular, the intuitive map, which locally converts each crossing to a nesting, is one of them. In this paper we present a bijection between nonnesting and noncrossing partitions of types A and B that generalizes the type A bijection that locally converts each crossing to a nesting.

Résumé. Le nombre total des partitions non-croisées du type Ψ est le n-ème nombre de Catalan $\frac{1}{n+1}\binom{2n}{n}$ si $\Psi = A_{n-1}$, et le coefficient binomial $\binom{2n}{n}$ si $\Psi = B_n$, et ces nombres son coïncidents avec le nombre correspondant des partitions non-emboîtées. Pour le type A, il y a plusieurs preuves bijectives de cette égalité; en particulier, la intuitive fonction, qui convertit localement chaque croisée en une emboîtée, c'est un d'entre eux. Dans ce papier nous présentons une bijection entre partitions non-croisées et non-emboîtées des types A et B qui généralise la bijection du type A qui localement convertit chaque croisée en une emboîtée.

Keywords: Root systems, noncrossing partitions, nonnesting partitions, bijection

1 Introduction

The poset of noncrossing partitions can be defined in a uniform way for any finite Coxeter group W. More precisely, for $u, w \in W$, let $u \leq w$ if there is a shortest factorization of w as a product of reflections in W having as prefix such a shortest factorization for u. This partial order turns W into a graded poset $Abs(W)$ having the identity 1 as its unique minimal element, where the rank of w is the length of the shortest factorization of w into reflections. Let c be a Coxeter element of W. Since all Coxeter elements in W are conjugate to each other, the interval $[1, c]$ in $Abs(W)$ is independent, up to isomorphism, of the choice of c. We denote this interval by $\mathrm{NC(W)}$ or by $\mathrm{NC}(\Psi)$, where Ψ is the Cartan-Killing type of W, and call it the *poset of noncrossing partitions* of W. It is a self-dual, graded lattice which reduces to the classical lattice of noncrossing partitions of the set $[n] = \{1, 2, \ldots, n\}$ defined by Kreweras in [9] when W is the symmetric group $\mathfrak{S}_n$ (the Coxeter group of type A_{n-1}), and to its type B analogue defined

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by Reiner in [10] when W is the hyperoctahedral group. The elements in $\mathrm{NC(W)}$ are counted by the generalized Catalan numbers,

$$Cat(W) = \prod_{i=1}^{k} \frac{d_i + h}{d_i},$$

where k is the number of simple reflections in W, h is the Coxeter number and $d_1, \ldots, d_k$ are the degrees of the fundamental invariants of W (see [1, 6, 7, 10] for details on the theory of Coxeter groups and noncrossing partitions). When W is the symmetric group $\mathfrak{S}_n$, the number $Cat(\mathfrak{S}_n)$ is just the usual nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$, and in type B_n this number is the binomial coefficient $\binom{2n}{n}$.

Nonnesting partitions were defined by Postnikov (see [10, Remark 2]) in a uniform way for all irreducible root systems associated with Weyl groups. If Φ is such a system, Φ^+ is a choice of positive roots, and Δ is the simple system in Φ^+, define the *root order* on Φ^+ by $\alpha \leq \beta$ if $\alpha, \beta \in \Phi^+$ and $\beta - \alpha$ is in the positive integer span of the simple roots in Δ. Equipped with this partial order, $(\Phi^+, \leq)$ is the *root poset* of the associated Weyl group W. A *nonnesting partition* on Φ is just an antichain in root poset $(\Phi^+, \leq)$. Denote by $\mathrm{NN(W)}$ or by $\mathrm{NN}(\Psi)$, where Ψ is the Cartan-Killing type of W, the set of all nonnesting partitions of W. Postnikov showed that the nonnesting partitions in $\mathrm{NN(W)}$ are also counted by the generalized Catalan number $Cat(W)$.

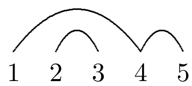
In the case of the root systems of type A, different bijective proofs of the equality between the cardinals $|\mathrm{NN(A_{n-1})}| = |\mathrm{NC(A_{n-1})}|$ are known (see [1, 2, 3, 8, 11]). Recently, Christian Stump [11] described a bijection between nonnesting and noncrossing partitions for type B, and simultaneously with our work, Alex Fink and Benjamin Giraldo [5] presented a different bijection for each classical group. Our contribution in this paper is to present a uniform proof that $|\mathrm{NN}(\Psi)| = |\mathrm{NC}(\Psi)|$, for $\Psi = A_{n-1}$ and $\Psi = B_n$, that generalizes the bijection presented by Armstrong in [1]. All three bijections are distinct, and preserves different statistics. While our bijection preserves the triples (op, cl, tr) formed by the openers, closers and transients (see the definitions below) of the partitions, and therefore also the number of blocks, the one by Alex Fink and Benjamin Giraldo preserves the type of the partitions but not the triples (op, cl, tr), and Stump's bijection does not preserve neither the type nor the triples (op, cl, tr).

2 Noncrossing and nonnesting partitions of types A and B

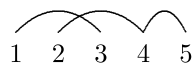
A partition of the set $[n]$ is a collection of nonempty disjoint subsets of $[n]$, called blocks, whose union is $[n]$. The type of a partition π of $[n]$ is the integer partition formed by the cardinals of the blocks of π. Let B be a block of π. Then, the least element of B is called an opener, the greatest element of B is said to be a closer, and the remaining elements of B are called transients. The sets of openers, closers and transients of π will be denoted by $op(\pi)$, $cl(\pi)$, and $tr(\pi)$, respectively. The triples $(op(\pi), tr(\pi), cl(\pi))$ encodes useful information about the partition π. For instance, the number of blocks is $|op(\pi)| = |cl(\pi)|$, and the number of blocks having only one element is $|op(\pi) \cap cl(\pi)|$. A partition can be graphically represented by placing the integers $1, 2, \ldots, n$ along a line and drawing arcs above the line between i and j whenever i and j lie in the same block and no other element between them does so.

A noncrossing partition of the set $[n]$ is a partition of $[n]$ such that there are no $a < b < c < d$, with a, c belonging to some block of the partition and b, d belonging to some other block. The set of noncrossing partitions of $[n]$, denoted by $\mathrm{NC}(n)$, is a lattice for the refinement order. A nonnesting partition of the set $[n]$ is a partition of $[n]$ such that if $a < b < c < d$ and a, d are consecutive elements of a block, then b and c are not both contained in some other block. The set of nonnesting partitions of $[n]$ will be denoted by

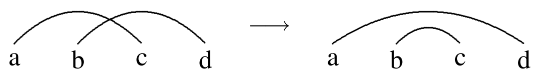
NN(n). Graphically, the noncrossing condition means that no two of the arcs cross, while the nonnesting condition means that no two arcs are nested one within the other. For instance, the noncrossing partition $\{\{2,3\},\{1,4,5\}\}$ and the nonnesting partition $\{\{1,3\},\{2,4,5\}\}$ are represented by



and



respectively. Both partitions have $\{1,2\}$ as set of openers, $\{3,5\}$ as set of closers and 4 is the only transient. As pointed out in [1], the map that locally converts each crossing to a nesting



defines a bijection from NN(n) to NC(n) that preserves the number of blocks. We will refer to this bijection as the L-map.

We will now review the usual combinatorial realizations of the Coxeter groups of types A and B, referring to [7] for any undefined terminology. The Coxeter group W of type A_{n-1} is realized combinatorially as the symmetric group $\mathfrak{S}_n$. The permutations in $\mathfrak{S}_n$ will be written in cycle notation. The simple generators of $\mathfrak{S}_n$ are the transpositions of adjacent integers $(i\ i+1)$, for $i = 1, \ldots, n-1$, and the reflections are the transpositions $(i\ j)$ for $1 \leq i < j \leq n$. To any permutation $\pi \in \mathfrak{S}_n$ we associate the partition of the set $[n]$ given by its cycle structure. This defines a isomorphism between the posets NC$(\mathfrak{S}_n)$ of noncrossing partitions of $\mathfrak{S}_n$, defined in the introduction, and NC(n), with respect to the Coxeter element $c = (12 \cdots n)$ [4, Theorem 1].

Denoting by $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$, the root system of type A_{n-1} consists of the set of vectors

$$\Phi = \{e_i - e_j : i \neq j, 1 \leq i, j \leq n\},$$

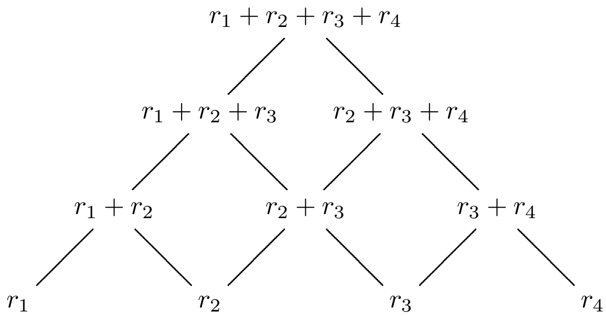
each root $e_i - e_j$ corresponding to the transposition $(i\ j)$. Take

$$\Phi^+ = \{e_i - e_j \in \Phi : i > j\}$$

for the set of positive roots and, defining $r_i := e_{i+1} - e_i$, $i = 1, \ldots, n-1$, we obtain the simple system $\Delta = \{r_1, \ldots, r_{n-1}\}$ for $\mathfrak{S}_n$. Note that

$$e_i - e_j = \sum_{k=j}^{i-1} r_k, \qquad \text{if } i > j\,.$$

The correspondence between the antichains in the root poset $(\Phi^+, \leq)$ and the set of nonnesting partitions of $[n]$ is given by the bijection which sends the positive root $e_i - e_j$ to the set partition of $[n]$ having an arc between vertices i and j. For instance, consider the root poset $(\Phi^+, \leq)$ of type A_4:



The antichain $r_1 + r_2 = e_3 - e_1$ corresponds to the transposition (13) in the symmetric group $\mathfrak{S}_5$, and thus to the nonnesting set partition $\{\{1,3\},\{2\},\{4\},\{5\}\}$, while the antichain $\{r_1 + r_2, r_2 + r_3, r_4\}$ corresponds to the product of transpositions $(13)(24)(45) = (13)(245)$ in $\mathfrak{S}_5$, and thus to the nonnesting set partition $\{\{1,3\},\{2,4,5\}\}$.

Given a positive root $\alpha = r_i + r_{i+1} + \cdots + r_j \in \Phi^+$, define the *support* of α as the set $\mathrm{supp}(\alpha) = \{r_i, r_{i+1}, \ldots, r_j\}$. The integers i and j will be called, respectively, the *first* and *last indices* of α, and the roots r_i and r_j the first and last elements of α, respectively. We have the following lemma.

Lemma 2.1 *Let α_1, α_2 be two roots in Φ^+ with first and last indices i_1, j_1 and i_2, j_2, respectively. Then, α_1, α_2 form an antichain if and only if $i_1 < i_2$ and $j_1 < j_2$.*

Consider now the Coxeter group W of type B_n, with its usual combinatorial realization as the hyperoctahedral group of signed permutations of

$$[\pm n] := \{\pm 1, \pm 2, \ldots, \pm n\}.$$

These are permutations of $[\pm n]$ which commute with the involution $i \mapsto -i$. We will write the elements of W in cycle notation, using commas between elements. The simple generators of W are the transposition $(-1,\ 1)$ and the pairs $(-i-1,\ -i)(i,\ i+1)$ for $i = 1, \ldots, n-1$. The reflections in W are the transpositions $(-i,\ i)$, for $i = 1, \ldots, n$, and the pairs of transpositions $(i,\ j)(-j,\ -i)$ for $i \neq j$. Identifying the sets $[\pm n]$ and $[2n]$ through the map $i \mapsto i$ for $i \in [n]$ and $i \mapsto n - i$ for $i \in \{-1, -2, \ldots, -n\}$, allows us to identify the hyperoctahedral group W with the subgroup U of $\mathfrak{S}_{2n}$ which commutes with the permutation $(1, n+1)(2, n+2) \cdots (n, 2n)$. For example, the signed permutations $(1,3)$ and $(2,-3)(-2,3)$ in the hyperoctahedral group of type B_3 correspond to the permutations $(1\ 3)$ and $(2\ 6)(5\ 3)$ in the symmetric group $\mathfrak{S}_6$. It follows that $\mathrm{NC}(U)$ is a sublattice of $\mathrm{NC}(\mathfrak{S}_{2n})$, isomorphic to $\mathrm{NC}(W)$ (see [1]).

The type B_n root system consists on the set of $2n^2$ vectors

$$\Phi = \{\pm e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : i \neq j, 1 \leq i, j \leq n\},$$

and we take

$$\Phi^+ = \{e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : 1 \leq j < i \leq n\}$$

as a choice of positive roots. Changing the notation slightly from the one used for $\mathfrak{S}_n$, let $r_1 := e_1$ and $r_i := e_i - e_{i-1}$, for $i = 2, \ldots, n$. The set

$$\Delta := \{r_1, r_2, \ldots, r_n\}$$

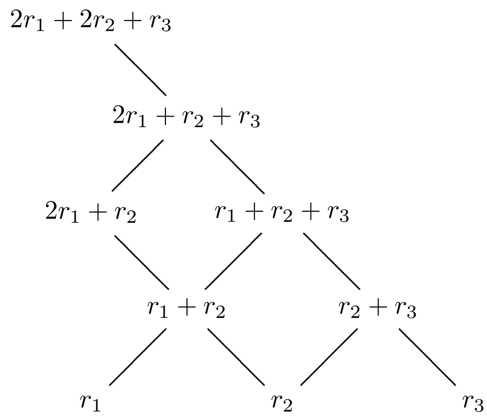
is a simple system for W, and easy computations show that

$$e_i = \sum_{k=1}^{i} r_k,$$

$$e_i - e_j = \sum_{k=j+1}^{i} r_k, \qquad \text{if } i > j$$

$$e_i + e_j = 2\sum_{k=1}^{j} r_k + \sum_{k=j+1}^{i} r_k, \qquad \text{if } i > j\,.$$

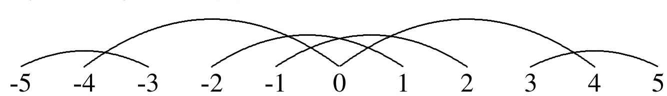
Each root $e_i, e_i - e_j$ and $e_i + e_j$ defines a reflection that acts on $\mathbb{R}^n$ as the permutation $(i,\ -i)$, $(i,\ j)(-i,\ -j)$ and $(i,\ -j)(-i,\ j)$, respectively, and we will identify the roots with the corresponding permutations. For example, consider the root poset of type B_3 displayed below:



The antichain $\{2r_1 + r_2, r_2 + r_3\}$ corresponds to the signed permutation $(1, 3, -2)(-1, -3, 2)$.

Using the inclusion $W \hookrightarrow \mathfrak{S}_{2n}$ specified above, we may represent noncrossing and nonnesting partitions of W graphically using the conventions made for its type A analogs. In these representations, we use the integers $-1, -2, \ldots, -n, 1, 2, \ldots, n$, or $-n, \ldots, -2, -1, 0, 1, 2, \ldots, n$, respectively for noncrossing and nonnesting partitions, instead of the usual $1, 2, \ldots, 2n$, where the presence of the zero in the ground set for nonnesting partitions is necessary to correctly represent (when present) the arc between a positive number i an its negative (see [2]).

Given a noncrossing or a nonnesting partition π of type B_n, let the set of openers $op(\pi)$ be formed by the least element of all blocks of π having only positive integers; let the set of closers $cl(\pi)$ be formed by the greatest element of all blocks of π having only positive integers and by the absolute values of the least and greatest elements of all blocks having positive and negative integers; and finally let the set of transients $tr(\pi)$ be formed by all elements of $[n]$ which are not in $op(\pi) \cup cl(\pi)$. For instance, if π is the nonnesting partition $\{\{-4, 4\}, \{-1, 2\}, \{-2, 1\}, \{3, 5\}, \{-3, -5\}\}$, represented below, then $op(\pi) = \{3\}$, $cl(\pi) = \{1, 2, 4, 5\}$ and $tr(\pi) = \emptyset$.



A factor $2r_i$ appearing in a positive root α will be called a *double root* of α. The *support* of a positive root $\alpha \in \Phi^+$ is the set of simple and double roots in α. Define also the set $\overline{\mathrm{supp}}(\alpha)$ as the set formed by the simple roots appearing in alpha as simple or double roots. For instance, for $\alpha = 2r_1 + \cdots + 2r_j + r_{j+1} + \cdots + r_i$ and $\beta = r_\ell + \cdots + r_k$ we have $\mathrm{supp}(\alpha) = \{2r_1, \ldots, 2r_j, r_{j+1}, \ldots, r_i\}$, $\overline{\mathrm{supp}}(\alpha) = \{r_1, \ldots, r_i\}$, and $\mathrm{supp}(\beta) = \overline{\mathrm{supp}}(\beta) = \{\mathrm{r}_\ell, \ldots, \mathrm{r_k}\}$. The first and last indices of α are, respectively $1, i$ and ℓ, k. The integer j will be called the last double index in α. Define also the set $D_\alpha := \{r_2, \ldots, r_j\}$ as the set of simple roots appearing in α as double roots, other than r_1. We have the following lemma.

Lemma 2.2 *Let α and β be two roots in Φ^+ with first and last indices i, j and i', j', respectively. If neither α nor β have double roots, then $\{\alpha, \beta\}$ is an antichain if and only if $i < i'$ and $j < j'$. If α has double roots, then $\{\alpha, \beta\}$ is an antichain if and only if $j < j'$ and the number of double roots in α is greater than the number of double roots in β.*

3 Main result

Let Φ denote a root system of type A or type B, and let Φ^+ and Δ be defined as above. In view of lemmas 2.1 and 2.2, we consider antichains $\{\alpha_1, \ldots, \alpha_m\}$ in Φ^+ as ordered m-tuples numbered so that if i_ℓ is the last index of α_ℓ, then $i_1 < \cdots < i_m$.

Definition 1 *Given two positive roots α and β, with β having no double roots, and such that the intersection of their supports is nonempty, define their union $\alpha \cup \beta$ and their intersection $\alpha \cap \beta$ as the positive roots with supports*

$$\mathrm{supp}(\alpha \cup \beta) := \mathrm{supp}(\alpha) \cup (\mathrm{supp}(\beta) \setminus \overline{\mathrm{supp}}(\alpha)) \quad \textit{and} \quad \mathrm{supp}(\alpha \cap \beta) := \overline{\mathrm{supp}}(\alpha) \cap \mathrm{supp}(\beta),$$

respectively. If moreover α has double roots, then define also their d-intersection $\alpha \cap^d \beta$ as the positive root with support $\mathrm{supp}(\alpha \cap^{\mathrm{d}} \beta) := \mathrm{D}_\alpha \cap \mathrm{supp}(\beta)$.

Example 2 *The union and the intersections of the type B_3 positive roots $\alpha = 2r_1 + 2r_2 + r_3$ and $\beta = r_2 + r_3 + r_4$ are $\alpha \cup \beta = 2r_1 + 2r_2 + r_3 + r_4$, $\alpha \cap^d \beta = r_2$, and $\alpha \cap \beta = r_2 + r_3$.*

An antichain $(\alpha_1, \ldots, \alpha_m)$ is said to be *connected* if the intersection of the supports of any two adjacent roots α_i, α_{i+1} is non empty. The connected components

$$(\alpha_1, \ldots, \alpha_i), (\alpha_{i+1}, \ldots, \alpha_j), \ldots, (\alpha_k, \ldots, \alpha_m)$$

of an antichain $\alpha = (\alpha_1, \ldots, \alpha_m)$ are the connected sub-antichains of α for which the supports of the union of the roots in any two distinct components are disjoint. For instance, the antichain $(r_1 + r_2, r_2 + r_3, r_4)$ has the connected components $(r_1 + r_2, r_2 + r_3)$ and r_4. We will use lower and upper arcs to match two roots in a connected antichain in a geometric manner. Two roots linked by a lower [respectively upper] arc are said to be l-linked [respectively, u-linked]. In what follows we will identify each root with the correspondent permutation.

Definition 3 *Define the map f from the set* $\mathrm{NN}(\Phi)$ *into* $\mathrm{NC}(\Phi)$ *recursively as follows. When α_1 is a positive root we set $f(\alpha_1) := \alpha_1$. If $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a connected antichain with $m > 2$, we have two cases:*

(a) *If there are no double roots in the antichain, define*

$$f(\alpha) := \left(\bigcup_{k=1}^{m} \alpha_k \right) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

where $\overline{\alpha}_k = \alpha_{k-1} \cap \alpha_k$ *for* $k = 2, \ldots, m$.

(b) *Assume now that* $\alpha_1, \ldots, \alpha_\ell$ *have double roots, for some* $\ell \geq 1$*, and* $\alpha_{\ell+1}, \ldots, \alpha_m$ *have none. Let* $\Gamma_d := (\alpha_1, \ldots, \alpha_\ell)$ *and* $\Gamma := (\alpha_{\ell+1}, \ldots, \alpha_m)$*. We start by introducing l-links as follows.*

Let m' *be the largest index of elements in* Γ *such that the following holds:* $\alpha_{m'}$ *has a first index* $i \neq 1$*, so that there is a rightmost element, say* α_k*, of* Γ_d *which has a term* $2r_i$*. If there is such an integer* m'*, l-link* α_k *with* $\alpha_{m'}$*. Then, ignore* α_k *and* $\alpha_{m'}$ *and proceed with the remaining roots as before. This procedure terminates after a finite number of steps (and not all elements of* α *need to be l-linked).*

Next proceed by introducing u-links in α*. The starting point of u-links, which we consider drawn from right to left, will be elements in* Γ *that have no first index* 1 *and are not l-linked. We will refer to these elements as admissible roots. So, let* m' *be the smallest integer such that the following holds:* $\alpha_{m'}$ *is an admissible root with first index* $i \neq 1$ *so that there is a leftmost element, say* α_k *which has* r_i *or* $2r_i$ *in its support and is not yet u-linked to an element on its right. If there is such an integer* m'*, u-link* α_k *with* $\alpha_{m'}$*. Remove* $\alpha_{m'}$ *from the set of admissible roots and proceed as before. Again this process terminates after a finite number of steps.*

Finally, let $T = \{t_1 < \cdots < t_p\}$ *be the collection of all last double indices of the roots in* Γ_d *not l-linked, and all the last indices of the roots in* α *not u-linked to an element on its right. Then, define*

$$f(\alpha) := \pi_1 \cdots \pi_\ell \pi_0 \, \theta_1 \, \cdots \theta_q f(\theta_{q+1}, \ldots, \theta_s),$$

where for $j = 1, \ldots, \ell$*,* $\pi_j = 2r_1 + \cdots + 2r_{j'} + r_{j'+1} + \cdots + r_{j''}$*, with* j' *and* j'' *respectively the leftmost and rightmost integers in* T *not considered yet;* π_0 *is either the root* $r_1 + \cdots + r_{i_j}$*, if the first index of* $\alpha_{\ell+1}$ *is* 1*, with* i_j *the only integer in* T *not used yet for defining the roots* π_j*, or the identity otherwise; each* θ_j*,* $j = 1, \ldots, q$ *is the d-intersection of l-linked roots, starting from the rightmost one in* Γ_d*, and each* θ_j*,* $j = q+1, \ldots, s$ *is the intersection of u-linked roots, starting from the leftmost one in* Γ*.*

(c) *For the general case, if* $(\alpha_1, \ldots, \alpha_i), (\alpha_{i+1}, \ldots, \alpha_j), \ldots, (\alpha_k, \ldots, \alpha_m)$ *are the connected components of* $(\alpha_1, \ldots, \alpha_m)$*, let*

$$f(\alpha_1, \ldots, \alpha_m) := f(\alpha_1, \ldots, \alpha_i) f(\alpha_{i+1}, \ldots, \alpha_j) \cdots f(\alpha_k, \ldots, \alpha_m).$$

Remark 4 (i) *Notice that in the type* A *case, the map* f *is defined only by conditions* (a) *and* (c) *of the above definition. Also, note that if all roots in* α *have double roots then condition* (b) *is vacuous and the map* f *reduces to the identity map. We point out that the number of roots in* $f(\alpha)$ *is equal to the number of roots in the antichain* α*.*

(ii) *The sequence* $(\overline{\alpha}_2, \ldots, \overline{\alpha}_m)$ *obtained in step* (a) *is a (not necessarily connected) antichain. It is easy to check that after all l-links and all u-links are settled, the set* T *has an odd number of elements if and only if the first index of* $\alpha_{\ell+1}$ *is* 1*. Thus, the root* π_0 *given in condition* (b) *is well defined.*

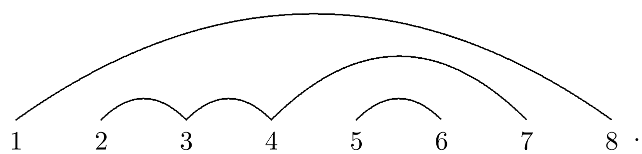
(iii) *A closer look at the construction of* f *shows that this map preserves the triples* $(op(\alpha), cl(\alpha), tr(\alpha))$ *for any antichain* α*.*

We will show that f establishes a bijection between the sets $\mathrm{NN}(\Psi)$ and $\mathrm{NC}(\Psi)$, for $\Psi = A_{n-1}$ or $\Psi = B_n$. Before, however, we present some examples.

Example 5 *Consider the antichain* $\alpha = (r_1 + r_2, r_2 + r_3, r_3 + r_4 + r_5, r_4 + r_5 + r_6, r_5 + r_6 + r_7)$ *in the root poset of type* A_7*, corresponding to the permutation* $(136)(247)(58)$ *in the symmetric group* $\mathfrak{S}_8$*. Applying the map* f *to* α*, we get the noncrossing partition*

$$\begin{aligned} f(\alpha) &= (r_1 + \cdots + r_7) f(r_2, r_3, r_4 + r_5, r_5 + r_6) \\ &= (r_1 + \cdots + r_7) r_2 r_3 f(r_4 + r_5, r_5 + r_6) \\ &= (r_1 + \cdots + r_7) r_2 r_3 (r_4 + r_5 + r_6) r_5 \\ &\equiv (18)(2347)(56), \end{aligned}$$

whose graphical representation is given below:



Example 6 *Consider now the antichain* $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ *in the root poset* B_9*, where*

$$\begin{aligned} \alpha_1 &= 2r_1 + 2r_2 + 2r_3 + 2r_4 + r_5, & \alpha_2 &= 2r_1 + 2r_2 + r_3 + r_4 + r_5 + r_6, \\ \alpha_3 &= r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7, & \alpha_4 &= r_3 + r_4 + r_5 + r_6 + r_7 + r_8, \\ \alpha_5 &= r_4 + r_5 + r_6 + r_7 + r_8 + r_9. \end{aligned}$$

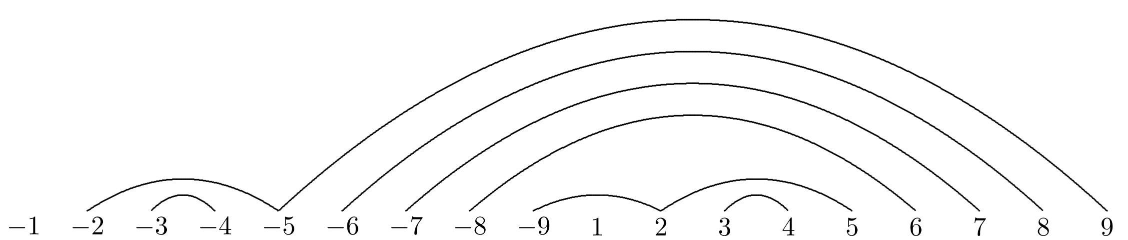
Following definition 3, we get the l-links and the u-links shown below:

$$\alpha = (\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \alpha_5) \qquad .$$

Therefore, $T = \{2, 6, 7, 8, 9\}$ *and the application of* f *to* α *yields:*

$$\begin{aligned} f(\alpha) &= (2r_1 + 2r_2 + r_3 + \cdots + r_9)(2r_1 + \cdots + 2r_6 + r_7 + r_8)(r_1 + \cdots + r_7) r_4 f(r_3 + r_4 + r_5) \\ &\equiv (2, -9)(-2, 9)(6, -8)(-6, 8)(7, -7)(3, 4)(-3, -4)(2, 5)(-2, -5) \\ &= (2, 5, -9)(-2, -5, 9)(6, -8)(-6, 8)(7, -7)(3, 4)(-3, -4). \end{aligned}$$

The image $f(\alpha)$ *is a noncrossing partition in* $[\pm 9]$*, as we may check in its representation:*



Lemma 3.1 *If* $\alpha \in \mathrm{NN}(B_n)$ then $f(\alpha) \in \mathrm{NC}(B_n)$.

Proof: Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an antichain in the root poset of type B_n, and let $(\alpha_1, \ldots, \alpha_w)$ be its first connected component. Start by assuming that none of the positive roots in α have the simple root r_1 nor the double root $2r_1$. We will use induction on $m \geq 1$ to show that in this case $f(\alpha)$ is a noncrossing partition on the set $\{i-1, \ldots, q, -(i-1), \ldots, -q\}$, where i is the fist index of α_1 and q is the last index of α_m, and such that each positive integer is sent to another positive integer. The result is clear when $m = 1$. So, let $m \geq 2$ and assume the result for antichains of length less than, or equal to $m-1$. Then, we may write

$$f(\alpha) = \left(\bigcup_{k=1}^{w} \alpha_k\right) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_w) f(\alpha_{w+1}, \ldots, \alpha_m),$$

where each $\overline{\alpha}_k = \alpha_{k-1} \cap \alpha_k$, for $k = 2, \ldots, w$. By the inductive step, $f(\overline{\alpha}_2, \ldots, \overline{\alpha}_w) \equiv \pi_1$ and $f(\alpha_{w+1}, \ldots, \alpha_m) \equiv \pi_2$ are noncrossing partitions on the sets

$$\{a-1, \ldots, b, -(a-1), \ldots, -b\} \text{ and } \{p-1, \ldots, q, -(p-1), \ldots, -q\},$$

respectively, where a and p are the first indices of $\overline{\alpha}_2$ and α_{w+1}, respectively, and b and q are the last indices of $\overline{\alpha}_w$ and α_m, respectively. Moreover, all positive integers are sent to positive ones by π_1 and π_2. Denoting by j the last index of α_w, we get $\bigcup_{k=1}^{w} \alpha_k = r_i + \cdots + r_j \equiv (i-1, j)(-(i-1), -j)$ with $i - 1 < a - 1 < b < j \leq p - 1 < q$. Therefore

$$f(\alpha) \equiv (i-1, j)(-(i-1), -j)\pi_1\pi_2$$

is a noncrossing partition on the set $\{i-1, \ldots, q, -(i-1), \ldots, -q\}$ sending each positive integer to another positive integer.

Note that for the rest of the proof, we may assume without loss of generality that α is connected, since none of the connected components of an antichain, except possible for the first one, have double roots, and therefore their images are noncrossing partitions sending each positive integer to another positive integer.

Suppose now that the first element of α_1 is r_1. We will show that $f(\alpha)$ is a noncrossing partition on the set $\{i-1, \ldots, q, -(i-1), \ldots, -q\}$, where i is the first index of α_2 and q is the last index of α_m, and such that one and only one positive integer is sent to a negative one. The result is certainly true for $m = 1$, and when $m > 1$ we have

$$f(\alpha) = \left(\bigcup_{k=1}^{m} \alpha_k\right) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

where $\bigcup_{k=1}^{m} \alpha_k \equiv (q, -q)$, and $\overline{\alpha}_k = \alpha_{k-1} \cap \alpha_k$ for $k = 2, \ldots, m$. By the previous case, $f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m) \equiv \pi$ is a noncrossing partition on the set $\{i-1, \ldots, j, -(i-1), \ldots, -j\}$, with i the first index of α_2 and $j < q$ the last index of α_{m-1}. Therefore, $f(\alpha) \equiv (q, -q)\pi$ is a noncrossing partition satisfying the desired conditions.

Next, assume that α satisfies condition (b) of definition 3, and consider its image

$$f(\alpha) = \pi_1 \cdots \pi_\ell \pi_0 \, \theta_1 \, \cdots \theta_q f(\theta_{q+1}, \ldots, \theta_s).$$

By the construction of the set T, it follows that each D_{α_j}, $j = 1, \ldots, \ell$, is contained in D_{π_i}, for some $i = 1, \ldots, \ell$, and that $\pi_1 \cdots \pi_\ell \pi_0$ is a noncrossing partition, sending each nonfixed positive integer to a negative one. Note also that the support of each θ_j, $j = 1, \ldots, q$, is contained in some D_{α_i}, $i = 1, \ldots, \ell$, and therefore, in some D_{π_i}, $i = 1, \ldots, \ell$. Moreover, the supports of any two roots θ_i and θ_j, $1 \leq i, j \leq q$, are either disjoint, or one of them is contained into the other one. Therefore $\theta_1 \cdots \theta_q$ is a noncrossing partition sending each nonfixed positive integer into another positive integer. By the previous cases, $f(\theta_{q+1}, \ldots, \theta_s)$ is also a noncrossing partition sending each nonfixed positive integer into another positive integer. Again by the construction of the set T, we find that the support of each θ_j, $j = q+1, \ldots, s$, is either contained in some D_{π_i}, or it does not intersect D_{π_ℓ}. For each $j = 1, \ldots, q$ and $i = q+1, \ldots, s$, either we have $\mathrm{supp}(\theta_\mathrm{i}) \cap \mathrm{supp}(\theta_\mathrm{j}) = \emptyset$, or $\mathrm{supp}(\theta_\mathrm{i}) \supseteq \mathrm{supp}(\theta_\mathrm{j})$, this last case happening when θ_i arises from the intersection of two u-linked roots $\alpha_u \in \Gamma_d$ and $\alpha_v \in \Gamma$, and there is some $\alpha_{v+k} \in \Gamma$, $k \geq 1$, l-linked to α_u, whose d-intersection gives θ_j. Therefore, it follows that $f(\alpha)$ is noncrossing. □

With some minor adaptations, the proof of lemma 3.1, in the case where neither the simple root r_1 nor the double root $2r_1$ are present in α, gives the type A analog of the previous result.

Corollary 3.2 *If $\alpha \in \mathrm{NN}(A_{n-1})$ then $f(\alpha) \in \mathrm{NC}(A_{n-1})$.*

We will now construct the inverse function of f, thus showing that f establishes a bijection between the sets $\mathrm{NN}(\Psi)$ and $\mathrm{NC}(\Psi)$, for $\Psi = A_{n-1}$ or $\Psi = B_n$. For that propose, recall the following property.

Lemma 3.3 *Two distinct transpositions (a, b) and (i, j) in $\mathfrak{S}_n$ commute if and only if the sets $\{i, j\}$ and $\{a, b\}$ are disjoint.*

If $\pi_1 \cdots \pi_p$ is the cycle structure of a signed permutation π, then for each cycle $\pi_i = (ij \cdots k)$ there is another cycle $\pi_j = (-i -j \cdots -k)$. Denote by π_i' the cycle in $\{\pi_i, \pi_j\}$ having the smallest positive integer (when $\pi_i = \pi_j$ then π_i' is just π_i), and call positive cycle structure to the subword of $\pi_1 \cdots \pi_p$ formed by the cycles π_i'. Extend this definition to permutations in $\mathfrak{S}_n$ by identifying positive cycle structure with cycle structure.

Theorem 3.4 *The map f is a bijection between the sets $\mathrm{NN}(\Psi)$ and $\mathrm{NC}(\Psi)$, for $\Psi = A_{n-1}$ or $\Psi = B_n$, which preserves the triples $(op(\pi), cl(\pi), tr(\pi))$.*

Proof: We will construct the inverse map $g : \mathrm{NC}(\Psi) \to \mathrm{NN}(\Psi)$ of f. Given $\pi \in \mathrm{NC}(\Psi)$, let $\pi_1 \cdots \pi_s$ be its positive cycle structure. Replace each cycle $\pi_i = (i_1 i_2 \cdots i_k)$ by $(i_1 i_2)(i_2 i_3) \cdots (i_{k-1} i_k)$, if $i_\ell > 0$ for $\ell = 1, \ldots, k$, or by

$$\pi_i = (i_1 i_{j+1})(i_1 i_2)(i_2 i_3) \cdots (i_{j-1} i_j)(i_{j+1} i_{j+2}) \cdots (i_{k-1} i_k),$$

if $i_\ell > 0$ for $\ell = 1, \ldots, j$, and $i_\ell < 0$ for $\ell = j+1, \ldots, k$. Next, baring in mind lemma 3.3 and recalling that π is noncrossing, move all transpositions (i, j), with $i > 0$ and $j < 0$ (if any), to the leftmost positions and order them by its least positive element, and order all remaining transpositions (i, j), with $i, j > 0$, by its least positive integer. Replace each transposition (ij) by its correspondent root in the root system of type Ψ, and let

$$(\alpha_1, \ldots, \alpha_k)(\alpha_{k+1}, \ldots, \alpha_\ell) \cdots (\alpha_m, \ldots, \alpha_n) \tag{1}$$

be the correspondent sequence of roots, divided by its connected components. Note that given two distinct roots in (1), the sets formed by the first and last indices, if there are no double roots, or by the last and last double indices, otherwise, are clearly disjoint.

We start by considering that the sequence (1) has only one connected component $(\alpha_1, \ldots, \alpha_k)$. Let $\Gamma_d = (\alpha_1, \ldots, \alpha_r)$ be the subsequence formed by the roots having double roots, and denote by $\Gamma = (\alpha_{r+1}, \ldots, \alpha_k)$ the remaining subsequence. Define $\Gamma' = \Gamma'_d = \emptyset$. If Γ_d is not empty and $r \neq k$, apply the following algorithm:

Let $\overline{\Gamma}$ be the subsequence of Γ obtained by striking out the root α_{r+1} if its first index is 1. While $\overline{\Gamma} \neq \emptyset$, repeat the following steps:

(i) Let α_i be the leftmost root in $\overline{\Gamma}$ and check if $\mathrm{supp}(\alpha_\mathrm{i}) \subseteq \mathrm{D}_{\alpha_\mathrm{j}}$, for some $\alpha_j \in \Gamma_d \setminus \Gamma'_d$.

(ii) If so, let α_{i_j} be the rightmost root in $\Gamma_d \setminus \Gamma'_d$ with this property. Update Γ' by including in it the rightmost root $\overline{\alpha}$ of $\overline{\Gamma}$ whose support is contained in $\mathrm{supp}(\alpha_\mathrm{i})$. Update $\overline{\Gamma}$ by striking out the root $\overline{\alpha}$ and update Γ'_d by including in this set the root α_{i_j}.

(iii) Otherwise, update $\overline{\Gamma}$ by striking out the root α_i.

Next, let $T = \{t_1 > \cdots > t_r\}$ be the set formed by all last double indices of the roots in $\Gamma_d \setminus \Gamma'_d$ and by the last indices of the roots in Γ'; let $F_{st} = \{f_{r+1} < \cdots < f_k\}$ be the set formed by the first indices of the roots in Γ, and let $L_{st} = \{\ell_1 < \cdots < \ell_k\}$ be the set formed by the last indices of the roots in $(\Gamma \setminus \Gamma') \cup \Gamma_d$ and by the last double indices of the roots in Γ'_d. By this construction, we have $f_i < \ell_i$ for $i = 1, \ldots, r$, and $f_i < \ell_i$, for $i = r+1, \ldots, k$. Then, define

$$g(\pi) = (\overline{\alpha}_1, \ldots, \overline{\alpha}_k),$$

where for $i = 1, \ldots, r$, $\overline{\alpha}_i = 2r_1 + \cdots + 2r_{t_i} + r_{t_i+1} + \cdots + r_{\ell_i}$, and for $i = r+1, \ldots, k$, $\overline{\alpha}_i = r_{f_i} + \cdots + r_{\ell_i}$.

For the general case define

$$g(\pi) = g(\alpha_1, \ldots, \alpha_k) g(\alpha_{k+1}, \ldots, \alpha_\ell) \cdots g(\alpha_m, \ldots, \alpha_n).$$

It is clear from this construction that $g(\pi)$ is an antichain in the root poset of type Ψ. Moreover, a closer look at the construction of the map f shows that g is the inverse of f. Thus, f (and g) establishes a bijection between nonnesting and noncrossing partitions of types A and B. $\square$

In the following examples we illustrate the application of the map g.

Example 7 *Consider the cycle structure of the noncrossing partition $\pi = (18)(2347)(56)$ in the symmetric group $\mathfrak{S}_8$ used in example 5. Following the proof of theorem 3.4, write*

$$\begin{aligned} \pi &\equiv (18)(2347)(56) \\ &= (18)(23)(34)(47)(56) \\ &\equiv (r_1 + \cdots + r_7) r_2 r_3 (r_4 + r_5 + r_6) r_5 \end{aligned}$$

Note that π has only one connected component, and there are no double roots. Next define the sets

$$F_{st} = \{1, 2, 3, 4, 5\}, \text{ and } L_{st} = \{2, 3, 5, 6, 7\}.$$

Thus, we find that the image of π by the map g is the antichain

$$g(\pi) = (r_1 + r_2, r_2 + r_3, r_3 + r_4 + r_5, r_4 + r_5 + r_6, r_5 + r_6 + r_7).$$

Example 8 *Consider now the noncrossing partition*

$$\pi = (2,5,-9)(-2,-5,9)(6,-8)(-6,8)(7,-7)(3,4)(-3,-4)$$

obtained in example 6. Its positive cycle structure is

$$(2,-9)(2,5)(6,-8)(7,-7)(3,4) = (2,-9)(6,-8)(7,-7)(2,5)(3,4),$$

and thus we get

$$\pi \equiv (2r_1 + 2r_2 + r_3 + \cdots + r_9, 2r_1 + \cdots + 2r_6 + r_7 + r_8, r_1 + \cdots + r_7, r_3 + r_4 + r_5, r_4).$$

Next, construct the sets

$$T = \{4,2\}, \quad F_{st} = \{1,3,4\}$$
$$L_{st} = \{5,6,7,8,9\}.$$

Therefore, the image of π by the map g is the antichain

$$(2r_1 + \cdots + 2r_4 + r_5, 2r_1 + 2r_2 + r_3 + \cdots + r_6, r_1 + \cdots + r_7, r_3 + \cdots + r_8, r_4 + \cdots + r_9).$$

Finally, in the next result we prove that the map f generalizes the bijection that locally converts each crossing to a nesting.

Theorem 3.5 *When restricted to the type A_{n-1} case, the map f coincides with the L-map.*

Proof: Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an antichain in the root poset of type A_{n-1}. The result will be handled by induction over $m \geq 1$. Without loss of generality, we may assume that α is connected, since otherwise there is an integer $1 < k < n-1$ such that each integer less (resp. greater) than k is sent by α to an integer that still is less (resp. greater) that k. Therefore, the same happens with the image of α by either the map f or the L-map.

The result is vacuous when $m = 1$, and when $m = 2$, the only connected nonnesting partition which does not stay invariant under the maps f and L is $\alpha = (r_i + \cdots + r_{i'})(r_j + \cdots + r_{j'})$, for some integers $1 \leq i < j < i' < j' \leq n-1$. In this case, the equality between f and the L-map is obvious. So, let $m > 2$ and assume the result for antichains of length $\leq m-1$. Let i and j be, respectively, the first and last indices of α_1 and α_m. Then,

$$f(\alpha) = (r_i + \cdots + r_j) f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

where each $\overline{\alpha}_k = \alpha_{k-1} \cap \alpha_k$ for $k \geq 2$, and the antichain $(\overline{\alpha}_2, \ldots, \overline{\alpha}_m)$ is clearly nonnesting, and not necessarily connected. By the inductive step, $f(\overline{\alpha}_2, \ldots, \overline{\alpha}_m) = L(\overline{\alpha}_2, \ldots, \overline{\alpha}_m)$. Moreover, note that converting, from left to right, each local crossing between the first root and the leftmost root in α whose arcs cross, into a nesting gives, precisely,

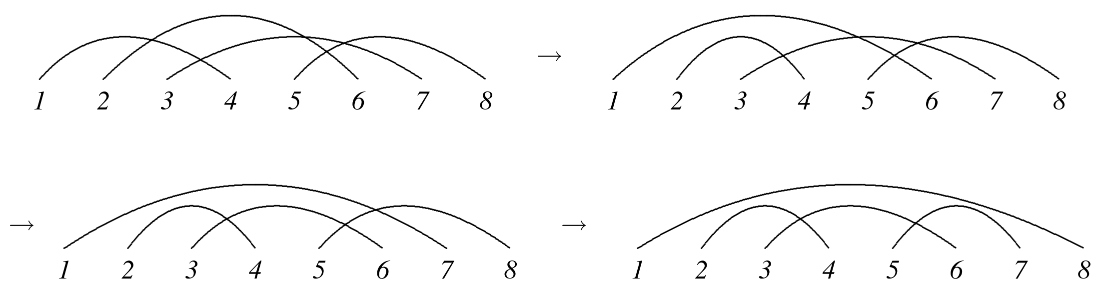
$$(r_i + \cdots + r_j) L(\overline{\alpha}_2, \ldots, \overline{\alpha}_m),$$

and this operation may be considered the first step of the L-map. Thus, we find that $f(\alpha) = L(\alpha)$. $\square$

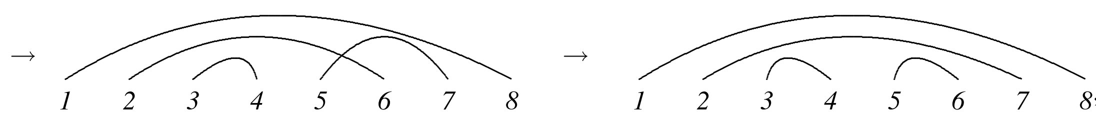
Example 9 *Consider the antichain* $\alpha = (r_1+r_2+r_3, r_2+r_3+r_4+r_5, r_3+r_4+r_5+r_6, r_5+r_6+r_7)$ *in the root poset of type* A_7*. Applying the map* f *we get*

$$\begin{aligned} f(\alpha) &= (r_1+\cdots+r_7)f(r_2+r_3, r_3+r_4+r_5, r_5+r_6) \\ &= (r_1+\cdots+r_7)(r_2+r_3+r_4+r_5+r_6)f(r_3,r_5) \\ &= (r_1+\cdots+r_7)(r_2+r_3+r_4+r_5+r_6)r_3r_5 \equiv (18)(27)(34)(56). \end{aligned}$$

On the other hand, applying the L-map to each crossing between the first root and the leftmost root in α *whose arcs cross, we get successively*



Thus, in the first step of the L-map, we get $L(\alpha) = (r_1+\cdots+r_7)L(r_2+r_3, r_3+r_4+r_5, r_5+r_6)$. *Continuing the application of the L-map, now replacing, by a nesting, each crossing between the second root and the leftmost root in* α *whose arcs cross, we get*



and therefore, we have $L(\alpha) = (r_1+\cdots+r_7)(r_2+r_3+r_4+r_5+r_6)r_3r_5 = f(\alpha)$.

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Election algorithms with random delays in trees

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The election is a classical problem in distributed algorithmic. It aims to design and to analyze a distributed algorithm choosing a node in a graph, here, in a tree. In this paper, a class of randomized algorithms for the election is studied. The election amounts to removing leaves one by one until the tree is reduced to a unique node which is then elected. The algorithm assigns to each leaf a probability distribution (that may depends on the information transmitted by the eliminated nodes) used by the leaf to generate its remaining random lifetime. In the general case, the probability of each node to be elected is given. For two categories of algorithms, close formulas are provided.

Keywords: Distributed Algorithm, Election Algorithm, Probabilistic Analysis, Random Process

1 Introduction

1.1 The problem

Starting from a configuration where all processors are in the same state, the goal of an election algorithm is to obtain a configuration where exactly one processor is in the state *leader*, the other ones being in the state *lost*. The (leader) election problem is often the first problem to solve in a distributed environment. A leader permits to centralize some information, to make some decisions, to coordinate the processors for subsequent tasks. Hence, the election problem – first posed by Le Lann in [6] – is one of the most studied problems in distributed algorithmic, and this under many different assumptions [9]. The graph encoding the relations between the processors can be a ring, a tree, a complete or a general connected graph. The system can be synchronous or asynchronous and processors may have access to a total or partial information of the geometry of the underlying graph, or of the current state of the system, etc.

In this paper we consider the case of election in trees, when the nodes have at time $t = 0$ a very partial information on the geometry of the tree: each node only knows its number of neighbors. A possible method for electing in a tree, introduced by Angluin ([1] Theorem 4.4), amounts to eliminating successively the leaves till only one node remains, the leader. In this paper, we investigate this method in the general case: assume that a node u being a leaf (was a leaf at time $t = 0$, or that becomes a leaf at time t) decides to live a random remaining time D_u before being eliminated; in other words, it is eliminated at time $t + D_u$ except if it is elected before this date. Starting with a given tree T_0 at time 0, denote by T_t the tree constituted with the non-eliminated nodes at time t. The family $(T_t)_{t\geq 0}$ is a random process taking its values in the set of trees. Given T_0, the distribution of $(T_t)_{t\geq 0}$ – and then, also the probability that a

given node is elected – depends on the way the nodes choose the distribution according to which they will compute their random remaining lifetime.
– In [7] the authors consider two elementary approaches. The first one is based on the assumption that *all sequences* of leaves elimination have the *same* probability (no distributed algorithm seems to have this property). Their second approach assumes that at each step *all leaves* have the same probability of being removed. This corresponds to the case where the $D'_u s$ are all exponentially distributed with parameter 1. The authors study thoroughly both approaches and prove many properties of resulting random processes.
– In [8], the authors show that if the nodes suitably choose their remaining random lifetime then a totally fair election process is possible, the nodes being elected equally likely (in Section 3.2 this example is revisited). In [4] and [3], the authors extend the result from [8] to a more general class of graphs: the polyominoid graphs. They also prove a conjecture: the expected value of the election duration is equal to $\log n$.

In this paper, we investigate the general case, namely, we consider the case where a leaf u generates its remaining lifetime D_u according to a distribution $\mathcal{D}_u$, where $\mathcal{D}_u$ may depend on all the information that u has at its disposal (see Remark 2 below). We warm the reader to distinguish the notation $\mathcal{D}_u$ and D_u.

Remark 1 *– In order to avoid that two nodes may disappear exactly at the same time, the distributions $\mathcal{D}_u$ need to avoid atoms (points with a positive mass). Even if not recalled in the statements, we assume that the distributions $\mathcal{D}_u$ have no atom. (In Section 3.3 a case where $\mathcal{D}_u$ maybe 0 with a positive probability arises and leads to problems).*
– It is assumed throughout the paper, that the nodes own independent random generators. This assumption is needed each time that the independence argument is used in the paper.

1.2 The general scheme

Throughout this paper $T = (V, E)$ is a tree in the graph theoretic sense: V is its set of nodes, E the set of edges. The graph T is acyclic and connected, and undirected. The *size* of T, denoted by $|T|$, is the number of nodes.

In the class of algorithms we study, a node u becoming a leaf at time t (or which was a leaf at time $t = 0$) disappears at time $t + D_u$ (except if it is elected before!); the quantity D_u, called the remaining lifetime of u, is computed locally by the leaf u. The description of the way u chooses the distribution $\mathcal{D}_u$ is crucial: this description is in fact equivalent to the description of an algorithm using the general method of elimination of leaves. We then enter into details here.

When a leaf is eliminated, it may transmit to its unique neighbor some *information* (this notion will be formalized below). During the execution of the algorithm, as a result of the successive eliminations of the leaves, each internal node u eventually becomes a leaf, say at time t_u. At this time, it may use the information received to compute the distribution $\mathcal{D}_u$: then, it generates a random variable D_u following $\mathcal{D}_u$ using a random generator. After this delay (at time $t_u + D_u$), u is eliminated: it may transmit some information to its (unique) neighbor, and disappears from the tree. The election goes on till eventually only one single node remains; this node is then elected.

As said above, the key point here is to understand that an algorithm (from the class we study) is parametrized by the way a node u chooses – according to the information it has – the distribution $\mathcal{D}_u$.

We here formalize more precisely what we understand by *information received and information transmitted*, this needed to be coherent with the distributed model we consider. This will straightforwardly leads to the formal definition of our class of algorithms.

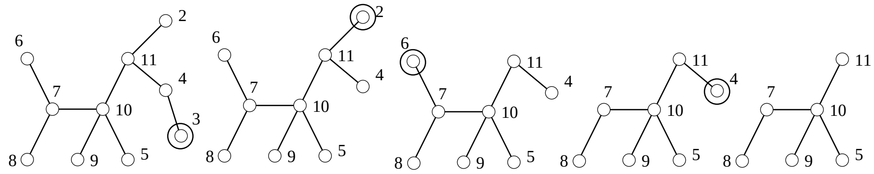

Fig. 1: On this example, are circled at each step the next leaf to disappear. On this example, the remaining lifetime of the leaf 11, according to an algorithm Δ is allowed to depend on the information given by the nodes 2 and 4; the information provided by 4 may include the information it received from the node 3. The total information received by 11 has a forest structure (a forest having 2,4 as roots, and having as set of nodes $2, 3, 4$, and possibly containing all the lifetimes, prescribed weights, and computed values of these nodes).

$a)$ The only information a node u has at time 0 is its degree $\deg_u$ and a prescribed weight w_u, which is an element of $\mathbb{R}$, $\mathbb{R}^d$ or any set (this may be viewed as a personal parameter),

$b)$ at its time of disappearance a leaf u transmits to its unique neighbor v all the information it has:
– the information it has received from its neighbors eliminated nodes,
– the 4-tuple $L_u = (\deg_u, D_u, w_u, \Gamma_u)$ which is the *local value* of u; the quantity Γ_u is computed by u using the information it has received and possibly the pair $(\deg_u, w_u)$. In the application we have, Γ_u is used to compute $\mathcal{D}_u$, and then we assume that Γ_u is not a function of D_u. We call Γ_u the *computed value* of u, it may belong to any set. See the remark below.

Assume that a node u becomes a leaf at time t when k of its $k+1$ neighbors $v_1, \dots, v_k$, have been eliminated. Denote by $I_1, \dots, I_k$ the information these nodes have transmitted to u. The node u has at its disposal the multiset $\{I_1, \dots, I_k\}$. Recursively, one sees that the structure of the information received by u is a forest with k rooted trees (a forest being here a multiset of trees) rooted at the v_i's and constituted with eliminated nodes; this forest has the geometry of the tree T fringed at the v_i's. The node u formally knows the local value of each of the nodes of this forest.

Remark 2 • *w_u and Γ_u are not used by each algorithm: when not used, they may be supposed to be 0.*
• *The notion of computed values aims to simplify the description of some algorithms, summing the needed information. Formally the transmission of this value is not necessary since it can be computed by a node having in hand all the other information.*
• *Let μ be a distribution on $\mathbb{R}$ with cumulative distribution function F. If U is uniform on $[0,1]$ then the law of $F^{-1}(U)$ is μ, where $F^{-1}(u) = \inf\{x \mid F(x) \geq u\}$ is the right continuous inverse of F; hence to simulate any distribution μ, a uniform random variable on $[0,1]$ is sufficient. We assume that the nodes have at their disposal some independent random generators providing uniform random values on $[0,1]$.*

Hence clearly, the information a node has received can be encoded without loss of information by a labelled forest f, where each node v is labelled by the 4-tuple L_v. The set of received information will then be identified with $\mathcal{F}$ the set of forests labelled by 4-tuple corresponding to the L_u's.

The other information at the disposal of a given node u that may be used to compute $\mathcal{D}_u$ is its own *local information* $L^\star_u = (\deg(u), w_u, \Gamma_u)$, where as said above Γ_u has been computed using $(\deg(u), w_u)$ and the received information. We denote by $\mathcal{L}^\star$ the set of local information.

An algorithm is then just parametrized by a function Δ

$$\begin{array}{rccc} \Delta: & \mathcal{F}\times\mathcal{L}^\star & \longrightarrow & \mathcal{M} \\ & (f,l^\star) & \longmapsto & \Delta(f,l^\star) \end{array}$$

where $\mathcal{M}$ is the set of probability measures having their support included in $[0,+\infty)$. The function Δ associates with a pair $(f,l^\star)$ a probability distribution $\Delta(f,l^\star)$. Any map Δ encodes an algorithm ALGO(Δ): when ALGO(Δ) is used, a node u becoming a leaf and having received the information f and having as local information $l_u^\star$, computes $\mathcal{D}_u = \Delta(f,l^\star)$ and generates D_u according to $\mathcal{D}_u$. The maps Δ exemplified below depend only on a part of the information received. The algorithms ALGO(Δ) are in the class of algorithms using the method of Angluin, and satisfy the constraints to be distributed.

Example 1 We translate into the form ALGO(Δ) the algorithm defined in Métivier & al. [8]. For each node u, $w_u = 1$. A node which is a leaf at time 0 computes $\Gamma_u = 1$. Let u be an internal node and $\Gamma_{v_1},\dots,\Gamma_{v_k}$ be the computed values of the eliminated neighbors of u. Then u computes:

$$\Gamma_u = 1 + \Gamma_{v_1} + \cdots + \Gamma_{v_k}. \tag{1}$$

Now the application Δ depends only on the computed values: suppose that u has received $(f,l^\star)$ and has computed Γ_u, then $\mathcal{D}_u = \Delta(f,l^\star)$ is simply $\mathbf{Expo}(\Gamma_u)$, the exponential distribution(i) with parameter Γ_u. Hence, $\mathcal{D}_u = \mathbf{Expo}(1)$ if u is a leaf at time 0, and if u becomes a leaf later, then $\mathcal{D}_u = \mathbf{Expo}(\Gamma_u)$, where Γ_u equals one plus the size of the forest of eliminated nodes leading to it (see Fig. 1). It turns out that in this case, each node is elected equally likely (for all tree T). We provide in Section 3.2 a new proof of this fact. Métivier et al. [8], [4] and [5] introduced election algorithms on trees, k-trees and polyominoids having also this property.

We address the question to compute according a general ALGO(Δ), the probability q_u that a given node u is eventually elected. In Section 2 we answer in the general case to this question, and express the result in terms of properties of some variables arising in a related problem of directed elimination.

In the sequel, we introduce and study two categories of algorithms in the class of algorithms ALGO(Δ). Before discussing their properties, we have to say that in order to get close formulas for $(q_u)_{u\in V}$, some stabilities in the computations are necessary, and this is not possible for general functions Δ. The two categories we propose raise on two different kinds of stability: the $(\max,+)$ algebra in distribution, and the stable distributions for the convolutions.

– The first one is built using the properties of the exponential distribution, and generalizes the computation of Métivier & al: the application Δ takes its values in the set of exponential distributions union the set of convolutions of such distributions. This category contains an algorithm ALGO(Δ) such that $(q_u)_{u\in V}$ is proportional to the prescribed weights $(w_u)_{u\in V}$. For technical reasons the prescribed weights $(w_u)_{u\in V}$ are to be integer valued. When the $(w_u)_{u\in V}$ are allowed to be real numbers, we propose an algorithm which elects proportionally to these weights in case of success, but which fails with a *low* probability,

– the second category may be less interesting from an algorithmic point of view, since the algorithms are more time consuming than the algorithms of the first category; it has however two main advantages: it clarify in some sense the properties needed to make the computation for a given function Δ, and it leads to a surprising proof of some mathematical identities involving the function $\arctan$.

(i) a random variable r.v. $\mathcal{E}$ has the distribution $\mathbf{Expo}(a)$, for some $a>0$ if $\mathbb{P}(\mathcal{E}\geq x)=\exp(-ax)$, for all $x\geq 0$.

2 General case: probability of a given node to be elected

In this section, we give a general formula giving $(q_u)_{u\in V}$ for ALGO(Δ). The proposition below is a generalization of a proposition of Métivier &. al [8] (the coupling argument we use is new).

The idea of the proof is to decompose the event $\{u \text{ is not elected }\}$ into disjoint events: if u is not elected, this means that u has become a leaf (or was a leaf at $t = 0$) and then has been eliminated. Let t be the time when u has become a leaf. At this time u had only one neighbor v, and since afterward u was not elected, this means that u has disappeared before v. If at time 0, u has k neighbors $v_1, \dots, v_k$ in the tree T, all of these nodes are possibly the last surviving node v evoked above: the family of events

$$E_i = \{u \text{ is not elected and the last neighbor of } u \text{ was } v_i\}. \tag{2}$$

are the "disjoint events" mentioned above. We just have to compute $\mathbb{P}(E_i)$.

Our idea to compute the probability of this event is to change of point of view, and to introduce a notion of *directed elimination*: if u is eliminated before v, this means that the sub-tree $T[u, v]$ – which is defined to be the tree rooted in u maximal for the inclusion in T which does not contain v (see Fig. 2)) – disappears entirely before $T[v, u]$; in the tree $T[u, v]$ the elimination is done from the leaves to the root u.

2.1 Directed elimination in rooted trees

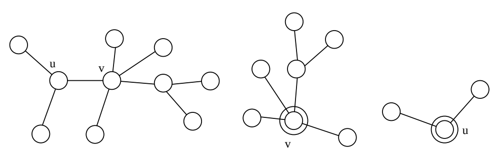

Fig. 2: A tree T, and the two rooted trees $T[v, u]$ and $T[u, v]$

We define an algorithm ALGO$^\star(\Delta)$ (very similar to ALGO(Δ)) which aims to eliminate all the nodes of a *rooted* tree, from the leaves to the root. We do not investigate the election since the last living node will be the root, but we are interested in the duration of the directed elimination of the whole tree.

We define ALGO$^\star(\Delta)$ recursively on a rooted tree τ. The only difference between ALGO(Δ) and ALGO$^\star(\Delta)$ is that with ALGO$^\star(\Delta)$ the root of τ is never considered as a leaf: using ALGO$^\star(\Delta)$

– the leaves of τ are eliminated as with ALGO(Δ), transmit and receive the same information, and compute their remaining lifetimes distribution with the same function Δ, but *the root of τ is not considered as a leaf*, even if it has only one child,

– when the root v of τ becomes alone, it has received some information from its neighbors (or none if it was yet alone at time 0), then it computes using Δ the distribution $\mathcal{D}^\star_v$, and generate $D^\star_v$ accordingly; in other words, the root once alone behaves as a leaf in ALGO(Δ). After the delay $D^\star_v$, v disappears.

We define the *duration* $D^\star(\tau)$ of the whole tree τ rooted in v according to ALGO$^\star(\Delta)$ as the date of disappearance of v. If τ is a rooted tree with root u, and such that the subtree of τ rooted at the children of u are $\tau_1, \dots, \tau_k$: one has

$$D^\star(\tau) = D^\star_u + \max_i D^\star(\tau_i); \tag{3}$$

$D^\star_u$ has a distribution given by Δ with the same rules as in ALGO(Δ).

We come back in the election problem in a (unrooted) tree T according to ALGO(Δ). Let u and v be two neighbors in a tree T; consider in one hand the event

$$E_{u,v} = \{u \text{ is not elected and the last neighbor of } u \text{ is } v\}$$

corresponding to a generic event E_i in (2). In the other hand, the two trees $T[u, v]$ and $T[v, u]$ are rooted trees, respectively in u and v; consider two independent directed eliminations on these trees as explained above, and denote by $D^\star(T[u, v])$ and $D^\star(T[v, u])$ their independent durations. It turns out that

Proposition 1 *The following identity holds true:*

$$\mathbb{P}(E_{u,v}) = \mathbb{P}\big(D^\star(T[u, v]) < D^\star(T[v, u])\big). \tag{4}$$

Proof: We propose a proof via a coupling argument. The idea is to compare the election process which takes place in T with the directed eliminations in $T[v, u]$ and $T[u, v]$, that are directed. The comparison is not immediate since these algorithms are not defined on the same probability space.

The algorithms ALGO(Δ) and ALGO$^\star(\Delta)$ allow each node u to choose a distribution $\mathcal{D}_u$ or $\mathcal{D}^\star_u$ depending on the information received, from which the nodes generate their lifetimes D_u or $D^\star_u$. According to Remark 2, a variable U uniform is sufficient to generate D_u or $D^\star_u$. Hence, we suppose that at time 0 each node w in the tree T has at its disposal a real number U_w obtained by a uniform random generator on $[0, 1]$. This is the key-point: a node w in T maybe considered also as a node in $T[v, u]$ or in $T[u, v]$, depending on which of these trees it belongs. If one now executes ALGO(Δ) on T and ALGO$^\star(\Delta)$ on $T[u, v]$ and $T[v, u]$ using the variable U_w for the generation of the D_w's and the $D^\star_w$'s, one can compare the events $\{E_{u,v}\}$ and $\{D^\star(T[u, v]) < D^\star(T[v, u])\}$, since they are now on the same probability space.

It turns out that for each assignment of the U_w's, we have $\{E_{u,v}\} = \{D^\star(T[u, v]) < D^\star(T[v, u])\}$. Indeed, since both algorithms use the U_w's, since the algorithms have the same constructions and the same rules concerning Δ, we see that the disappearance of leaves coincide in the two models till the disappearance of u or of v: after this time, the information transmitted are different, and then the two processes evolve in a non comparable manner. Now, in the election process ALGO(Δ) in T, if u is eliminated before v, then the tree $T[u, v]$ has lived a directed election, and thus $D^\star(T[u, v])$ coincides with the disappearance time of u (for ALGO(Δ)). At this time, since v is still alive, this means that the directed elimination in $T[v, u]$ is not finished, thus $D^\star(T[u, v]) < D^\star(T[v, u])$. Conversely, if $D^\star(T[u, v]) < D^\star(T[v, u])$, then u disappears before v according to ALGO(Δ), since till the time $\min(D^\star(T[u, v]), D^\star(T[v, u]))$ the two elimination processes coincide.

We then have construct a probability space (the one where are defined the U_w's) on which the two events $\{E_{u,v}\}$ and $\{D^\star(T[u, v]) < D^\star(T[v, u])\}$ coincide; thus, they have the same probability. $\square$

As a corollary we have

Corollary 1 *Let u be a node of a tree T and u_1,..., u_k its neighbors. Using ALGO(Δ)*

$$q_u = 1 - \sum_{1\le i\le k} \mathbb{P}\big(D^\star(T[u, u_i]) < D^\star(T[u_i, u])\big). \tag{5}$$

3 First category: around the $(\max, +)$ algebra

In this category, the distribution $\mathcal{D}_u$ are either the exponential distribution or a convolution of such distributions. We will see that this category contains the algorithm of Métivier &. al. allowing to elect uniformly in the tree, an algorithm electing proportionally to positive integer valued prescribed weights, some algorithms allowing to elect proportionally to some structural features of the tree.

Before doing this, we recall some classical facts. In the sequel $\mathcal{E}^{[a]}$ denote a r.v. having the $\mathbf{Expo}(a)$ distribution, and $M_n = \max_{1\leq i\leq n} \mathcal{E}_i^{[1]}$ is the maximum of n i.i.d. r.v. $\mathbf{Expo}(1)$ distributed. The distribution of M_n is denoted from now on by $\mathcal{M}_n$ (we have $\mathbb{P}(M_n \leq x) = (1 - \exp(-x))^n$, for any $x \geq 0$).

Lemma 1 *Let $\mathcal{E}^{[1]}, ..., \mathcal{E}^{[n]}$ be n independent exponential random variables with parameters $1, \dots, n$. The random variables $\mathcal{E}^{[1]} + ... + \mathcal{E}^{[n]}$ has distribution $\mathcal{M}_n$.*

Proof: Consider $(\hat{\mathcal{E}}_i,\ 1 \leq i \leq n)$, the order statistics of n i.i.d. $\mathbf{Expo}(1)$ random variables $\mathcal{E}_1^{[1]}, \dots, \mathcal{E}_n^{[1]}$, that is the sequence $(\mathcal{E}_i^{[1]},\ 1 \leq i \leq n)$, sorted in the increasing order. The variable $M_n = \max \mathcal{E}_i^{[1]}$ is also the sum of the random variables $\hat{\mathcal{E}}_i - \hat{\mathcal{E}}_{i-1}$, for $i = 1, \dots, n$ with the convention $\hat{\mathcal{E}}_0 = 0$. Using the memoryless property of the exponential distribution, one has $\hat{\mathcal{E}}_i - \hat{\mathcal{E}}_{i-1} \stackrel{d}{=} \mathcal{E}^{[n+1-i]}$ for all $i \in \{1, \dots, n\}$, and the variables $(\hat{\mathcal{E}}_i - \hat{\mathcal{E}}_{i-1})$ are independent (for more details, see Proposition p.19 in Feller [2]). $\square$

From the lemma we easily derive:

Corollary 2 $i)$ *Consider $k \geq 1$ positive integers $a_1, \dots, a_k$ summing to n. If the r.v. M_{a_i}'s are independent, and independent of $\mathcal{E}^{[n+1]}$ then $M_{n+1} \stackrel{d}{=} \mathcal{E}^{[n+1]} + \max_{1\leq i\leq k} M_{a_i}$.*
$ii)$ *For any $k \geq 1$ and $n \geq 1$, set*

$$Y_{n,k} \stackrel{d}{=} \mathcal{E}^{[n+1]} + \mathcal{E}^{[n+2]} + ... + \mathcal{E}^{[n+k]}, \tag{6}$$

where the variables $\mathcal{E}^{[n+i]}$ are independent. We have $M_{n+k} \stackrel{d}{=} M_n + Y_{n,k}$.

3.1 The algorithms of the first category

The first category of algorithms we design is based on Corollary 2. It may be more easily understood via the directed elimination ALGO$^\star(\Delta)$, where the duration of a rooted tree τ according to ALGO$^\star(\Delta)$ will have distribution $\mathcal{M}_n$, for some n. The application Δ will take its values in the set of distributions $\{\mathcal{Y}[n, k], n \geq 1, k \geq 1\}$, where $\mathcal{Y}[n, k]$ is the distribution of $Y_{n,k}$ (given in (6)).

The only difference between the algorithms of the first category is the computed values Γ_u's : the class of algorithm considered is then simply parametrized by the possible computed values Γ satisfying the constraint below. It is convenient to consider bi-dimensional computed values $\Gamma_u = (C_u, g_u)$ where C_u will be use to add some quantities coming from the received information, and g_u is used to make some local computations.

Here are in two points the description of all the algorithms of the first category:
– At time 0, the computed value Γ_u of any leaf u is $\Gamma_u = (0, g_u)$ where g_u is a positive integer. Then set

$$\mathcal{D}_u = \mathcal{Y}[0, g_u] \stackrel{d}{=} M_{C_u+g_u}. \tag{7}$$

– Let u be an internal node in T becoming a leaf; let f be the received information, and in particular let $\Gamma_1 = (C_1, g_1), \ldots, \Gamma_k = (C_k, g_k)$ be the computed values of its eliminated neighbors. Then the node u compute an integer value g_u according to its information (f and $L_u^\star$), and let $C_u = \sum_{i=1}^k C_i + g_i$. Then set $\mathcal{D}_u = \mathcal{Y}\left[C_u, g_u\right]$.

Let us think in terms of directed elimination. Recall that the notion of computed values are defined similarly in ALGO$^\star(\Delta)$ and in ALGO(Δ), but in the directed case, it is convenient to make appear the tree notation in the computed values instead of the node notation.

If a rooted tree τ is reduced to a leaf u, set $C(\tau) = 0, g(\tau) = g_u$. If τ has root u, and if the sub-trees rooted at the children of u are $\tau_1, \ldots, \tau_k$, then set $C(\tau) = \sum_{i=1}^k C(\tau_i) + g(\tau_i)$. The lifetime of the root of τ is then distributed as the maximum of the $D^\star(\tau_i)'s$ plus a random variable distributed as $\mathcal{Y}(C(\tau), g(\tau))$.

To simplify a bit the formula, for any rooted tree τ, let

$$\Theta(\tau) = g(\tau) + C(\tau). \tag{8}$$

Proposition 2 *For any algorithm ALGO$^\star(\Delta)$ of the first category the duration of a rooted tree τ satisfies*

$$D^\star(\tau) \stackrel{d}{=} M_{\Theta(\tau)}.$$

Proof: The lifetime of a tree τ reduced to a leaf is $\mathcal{Y}(0, g(\tau)) = M_{C(\tau)+g(\tau)} = M_{\Theta(\tau)}$. Assume by induction that the proposition is true for any rooted tree having less than n nodes. Consider now τ a rooted tree with n nodes and the τ_i defined as above. By recurrence $D^\star(\tau_i) \stackrel{d}{=} M_{\Theta(\tau_i)}$, and thus, by independence of the $M_{\Theta(\tau_i)}$'s, $D^\star(\tau) = \mathcal{Y}\left[\sum_i \Theta(\tau_i), g(\tau)\right] + \max_i M_{\Theta(\tau_i)}$ is in distribution equal to $M_{(\sum_i \Theta(\tau_i))+g(\tau)} \stackrel{d}{=} M_{\Theta(\tau)}$ by Corollary 2. □

As a corollary we have

Theorem 1 *For any algorithm ALGO(Δ) of the first category, any tree T,*

$$q_u = 1 - \sum_{1 \le i \le k} \frac{\Theta(T[u_i, u])}{\Theta(T[u, u_i]) + \Theta(T[u_i, u])} \tag{9}$$

Proof: This is a consequence of Propositions 1 and 2 and of the following identity: if M_a and M_b are independent, then $\mathbb{P}(M_a < M_b) = a/(a+b)$. □

This theorem has a direct consequence quite surprising, since it deals with very general function Γ. It is obtained by summing Equality (9) over all nodes:

Corollary 3 *For any tree T, any choice of positive integer values function $\Gamma_u = (C_u, g_u)$*

$$\sum_u \left[1 - \sum_i \frac{\Theta(T[u_i, u])}{\Theta(T[u_i, u]) + \Theta(T[u, u_i])}\right] = 1.$$

Remark 1 ensures that almost surely the election eventually succeeds. Indeed, each leaf eventually dies out with probability one, and then the election stops after a finite time. All the disappearance dates are different, since the lifetimes distributions have no atom: at the end it eventually remains only one leaving node which is elected.

Remark 3 *In general the denominator in the RHS of (9) depends on the node u and, thus, apart from the two first examples below where this denominator is constant, the formula (9) cannot be "simplified".*

3.2 Examples

1. The uniform electing algorithm (treated in Example 1) is a particular case of this model by letting $g_u = 1$ and, therefore, $\Theta(t) = |t|$, the total number of nodes in t. Since each node is either in $T[u, u_i]$ or in $T[u_i, u]$, by (9)

$$q_u = 1 - \sum_{1 \le i \le k} \frac{|T[u_i, u]|}{|T[u, u_i]| + |T[u_i, u]|} = 1 - \frac{|T \setminus \{u\}|}{|T|} = \frac{1}{|T|};$$

this is the uniform distribution on T, as found by Métivier & al.

2. Assume that all prescribed weights are positive integers. If $g_u = w_u$ for every nodes then $\Theta(t) = \sum_{u \in t} w_u$ the total weight of the rooted tree t. In this case $q_u = \frac{w_u}{w(T)}$ where $w(T) = \sum_{u \in T} w(u)$ is the total weight in T. Indeed, in the RHS of (9) the denominator is equal to $w(T)$ whatever is the value of i, and summing the numerators gives $w(T) - w_u$.

3. For $g_u = \deg(u)$, q_u becomes proportional to $\deg(u)$ (take $w_u = \deg(u)$ in the previous point 2).

4. In the case where $g_u = 1$ for the leaves and $g_u = |t|$ more generally for all the nodes, then $\Theta(t) = PLS(t) + |t|$ becomes the path length of (the rooted tree) t plus its size. Then Formula (9) gives the value of q_u.

3.3 Real-valued weights

In Example 3.2.2, we gave an algorithm of the first category such that q_u is proportional to w_u provided that the $w_u's$ are integers. The computations relying on Corollary 2, the weights have to be integer valued, or say have a known common divisor. A natural question arises: is there an algorithm such that q_u is proportional to general real-valued weights w_u's? We were not able to answer to this question, but using a randomized version of the algorithms of the first category, we provide an algorithm that may fail with a small probability, but such that conditionally on success, the q_u's are indeed proportional to the w_u's.

The difference with the algorithm described above is as follows. Instead of using its weight w_u as a parameter in a distribution $\mathcal{Y}(n, k)$, a node u becoming a leaf, uses its weight w_u as a parameter of a Poisson distribution: it generates W_u a r.v. following the Poisson(w_u) distribution and then uses this integer as its weight in the description of algorithms of the first category we gave. In other words, the computed value g_u instead of being simply w_u will take the value k with probability $\exp(-w_u)w_u^k/k!$. Let us discuss some points linked to the failure of the algorithm.

Remark 4 – *If the random generated W_u is zero for some u, then conditionally to W_u the remaining lifetime is $\mathbf{Expo}(0)$ distributed, that is zero almost surely: u is eliminated immediately.*
– If all nodes generate zero, then the algorithm fails: it terminates without choosing any node. The probability of failure for the algorithm is $e^{-w(T)}$ where $w(T) = \sum_{u \in V} w_u$ is the total weight. It becomes insignificant whenever $w(T)$ grows. To guarantee the success with a high probability, it suffices to multiply w by a great number c known by all nodes.

The following lemma, which is easily proved, simplifies the proof of the main proposition of this section.

Lemma 2 *Let $X_1, ..., X_n$ be n independent r.v. of Poisson distributions with parameters $\lambda_1, ..., \lambda_n$ respectively. For any $k > 0$, the distribution of X_1 conditionally on $X_1 + \cdots + X_n = k$ is binomial $B(k, \lambda_1/(\lambda_1 + \cdots + \lambda_n))$.*

Proposition 3 *Let T be any tree. The probability that the algorithm chooses a node u conditioned by the event that not all nodes generate 0 is proportional to w_u : $\mathbb{P}\Big(u \text{ elected } \Big| \sum_{v\in V} W_v > 0\Big) = w_u/w(T)$.*

Proof: Consider some integers $(k_v)_{v\in V}$, with at least one $k_v > 0$. Given the values $W_v = k_v$ according to Section 3.2, second example, we have:

$$\mathbb{P}\left(u \text{ elected} \mid W_v = k_v \text{ for any } v \text{ in } T\right) \quad = \quad k_u/(\sum_{v\in V} k_v).$$

Therefore the probability that the algorithm chooses u conditioned by $\sum_v W_v > 0$, is nothing but:

$$\mathbb{P}(u \text{ elected} \Big| \sum_v W_v > 0) \quad = \quad \mathbb{E}\left(\frac{W_u}{\sum_v W_v} \Big| \sum_v W_v > 0\right),$$

where $\mathbb{E}$ denotes the expected value. But then, according to the previous lemma, for a fixed $k > 0$,

$$\mathbb{E}\left(\frac{W_u}{\sum_v W_v} \mid \sum_v W_v = k\right) = \frac{w_u}{\sum_v w_v}.$$

This implies that if the sum of generated numbers is positive, whatever the values it takes, the probability of u to be elected is $\frac{w_u}{\sum_v w_v}$. The proposition follows. □

4 Second category: around the stable distributions

The second category relies on Formula (3). One sees that choosing a suitable $D^\star$ may let the $\max$ operator acting on the RHS disappears: the idea is to choose $D^\star_u$ under the form

$$D_u = X^u - \max_i D(\tau_i) + \sum_i D(\tau_i) \tag{10}$$

for some X^u whose distribution depends of the information received by u. In this case Formula (3) concerning the directed elimination becomes simply

$$D^\star(\tau) = X^u + \sum_i D^\star(\tau_i).$$

And the duration of a rooted tree satisfies:

$$D^\star(\tau) \quad = \quad X^u + \sum_i D^\star(\tau_i) = \sum_{v \text{ nodes in } \tau} X^v. \tag{11}$$

Once again, the involved variables X^v have a distribution that may depend on the history of the elimination of the sub-tree of τ rooted in v. The algorithms of the second category are parametrized by all the possible distribution for X^u (the variables X^u appearing in (10) and (11)).

In the case where the X^v are i.i.d, the distribution of $D^\star(\tau)$ is simple: it is a sum of $|\tau|$ i.i.d. random variables, and then it is indexed by the unique integer $|\tau|$. Denoting by S_n a sum of n i.i.d. copies of X^v, according to Corollary 1 we have for a node u having $u_1, \ldots, u_k$ as neighbors,

$$q_u = 1 - \sum_{1 \le i \le k} \mathbb{P}\big(S_{|T[u,u_i]|} < S_{|T[u_i,u]|}\big). \tag{12}$$

There is an interesting case where the computation in (12) can be made explicitly, and leads to close formulas: the case of the stable distribution with index $1/2$. The stable distributions are the families of distribution that are stable for the convolution (see Feller [2] for more information). We say that X has the stable distribution with index $1/2$ if the density of X is $f(t) = \mathbf{1}_{t\ge 0}\frac{e^{-1/(2t)}}{\sqrt{2\pi t^3}}$. If $X_1, \ldots, X_k$ are independent copies of X then $S_k = X_1 + \cdots + X_k \overset{d}{=} k^2 X$. Consider now S_m and S'_n two independent sums of m and n independent copies of X. One has

$$\mathbb{P}(S_m < S'_n) = \mathbb{P}(m^2 X \le n^2 X') \tag{13}$$

for two copies X and X' of X. Using the density of X and X', one gets $\mathbb{P}(S_m < S'_n) = \frac{2}{\pi}\arctan(n/m)$. Hence

Lemma 3 *For any tree T, for any node u having $u_1, \ldots, u_k$ as neighbors, under the algorithm presented above*

$$q_u = 1 - \sum_{1 \le i \le k} \frac{2}{\pi} \arctan\left(\frac{|T[u_i,u]|}{|T[u,u_i]|}\right).$$

In particular, since $\sum q_u = 1$ this gives for each tree a formula related to the arctan function. We review below some examples and derive formulas.

4.1 Applications: some identities involving the $\arctan$ function

Consider the star tree with n nodes: it is the tree where a node v has $n-1$ neighbors, say $v_1, \ldots, v_{n-1}$. By symmetry q_{v_i} does not depend on i; since v_i has for only neighbor v, by Lemma 3

$$q_{v_1} = 1 - (2/\pi)\arctan(n-1).$$

Using again Lemma 3, one has for the center of the star tree

$$q_v = 1 - \frac{2(n-1)}{\pi}\arctan\left(\frac{1}{n-1}\right).$$

Since $q_v + \sum_{i=1}^{n-1} q_{v_i} = 1$ (since a node is eventually elected with probability 1), we get for any $n \ge 2$,

$$\arctan(n-1) + \arctan(1/(n-1)) = \pi/2. \tag{14}$$

Consider now a sequence of trees T_n such that T_n is formed by two stars having $\alpha_n + 1$ and $\beta_n + 1$ nodes with center u and v, linked by an edge between u and v. The election probability of any leaf is $q_{v_i} = 1 - (2/\pi)\arctan(\alpha_n + \beta_{n+1})$, when

$$\begin{aligned} q_u &= 1 - \frac{2\alpha_n}{\pi}\arctan\left(\frac{1}{\alpha_n + \beta_{n+1}}\right) - \frac{2}{\pi}\arctan\left(\frac{\beta_n + 1}{\alpha_n + 1}\right) \\ q_v &= 1 - \frac{2\beta_n}{\pi}\arctan\left(\frac{1}{\alpha_n + \beta_{n+1}}\right) - \frac{2}{\pi}\arctan\left(\frac{\alpha_n + 1}{\beta_n + 1}\right). \end{aligned}$$

Using $(\alpha_n + \beta_n)q_{v_1} + q_u + q_v = 1$ and (14), we get

$$\frac{2}{\pi}\left(\arctan\left(\frac{\alpha_n+1}{\beta_n+1}\right) + \arctan\left(\frac{\beta_n+1}{\alpha_n+1}\right)\right) = 1.$$

If $\alpha_n/\beta_n \to x > 0$, by continuity of $\arctan$ one obtains the famous formula

$$\arctan(x) + \arctan(1/x) = \pi/2.$$

Going further, let T_n be the sequence of trees having a path of size k (k nodes $u_1, \ldots, u_k$ such that there is an edge between u_i and u_{i+1} and such that u_i has $\alpha_{n,i}$ other neighbors that are leaves). The probability of election of any of the $\sum \alpha_{n,i}$ leaves is $q_l = 1 - \frac{2}{\pi}\arctan(\sum \alpha_{n_i} + k - 1)$, that of u_i is

$$1-\frac{2}{\pi}\left[\alpha_{n,i}\arctan\left(\frac{1}{\sum \alpha_{n,i} + k - 1}\right) + \arctan\left(\frac{\sum_{j>i}(\alpha_{n,j}+1)}{\sum_{j\le i}(\alpha_{n,j}+1)}\right) + \arctan\left(\frac{\sum_{j<i}(\alpha_{n,j}+1)}{\sum_{j\ge i}(\alpha_{n,j}+1)}\right)\right].$$

Finally, assuming that for any i, $\alpha_{n,i} \to \alpha_i$ for some positive real number α_i, we get by continuity, and using that the sum of all events must be 1, that for any positive real number $\alpha_1, \ldots, \alpha_k$,

$$\sum_i \left[\arctan\left(\frac{\sum_{j>i}\alpha_j}{\sum_{j\le i}\alpha_j}\right) + \arctan\left(\frac{\sum_{j<i}\alpha_j}{\sum_{j\ge i}\alpha_j}\right)\right] = \frac{\pi}{2}(k-1). \tag{15}$$

Each simple finite tree used as a skeleton on which are grafted some packets of leaves (with size $\alpha_{n,k}$, k corresponding to a labeling of the nodes of the skeleton) will provide a formula similar to (15).

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Hopf algebras and the logarithm of the S-transform in free probability – Extended abstract

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This document is an extended abstract of the paper 'Hopf algebras and the logarithm of the S-transform in free probability' in which we introduce a Hopf algebraic approach to the study of the operation $\boxtimes$ (free multiplicative convolution) from free probability.

Keywords: Hopf algebra, free probability, non-crossing partition, symmetric function

1 Introduction

Two basic tools of free probability are the R-transform and the S-transform. These transforms were introduced by Voiculescu in the 1980s, and are used to understand the addition and the multiplication of two free random variables respectively. The R-transform has a natural and very useful multi-variable extension describing the addition of two free k-tuples of random variables, but the problem of finding such an extension for the S-transform is open.

The problem of the multi-variable S-transform can be re-phrased as the problem of understanding the structure of the group $(\mathcal{G}_k, \boxtimes)$, where $\mathcal{G}_k$ is a special set of joint distributions of noncommutative k-tuples (see precise definition in Equation (3.3) below), and where $\boxtimes$ ("free multiplicative convolution") is a binary operation on $\mathcal{G}_k$ which encodes the multiplication of free k-tuples. At present, the structure of $(\mathcal{G}_k, \boxtimes)$ is well-understood only in the special case $k = 1$; in this case, the S-transform of Voiculescu provides an isomorphism between $\mathcal{G}_1$ and a multiplicative group of power series in one variable. (A word of caution here: $\mathcal{G}_1$ is commutative, but it is easy to see that $\mathcal{G}_k$ is not commutative for any $k \geq 2$.)

In [4] we use Hopf algebra methods in order to study the multiplication of free k-tuples. Specifically, we construct a combinatorial Hopf algebra $\mathcal{Y}^{(k)}$ such that $(\mathcal{G}_k, \boxtimes)$ is naturally isomorphic to the the group $\mathbb{X}(\mathcal{Y}^{(k)})$ of characters of $\mathcal{Y}^{(k)}$. We then employ the log map from characters to infinitesimal characters of $\mathcal{Y}^{(k)}$, to introduce a transform LS_μ for distributions $\mu \in \mathcal{G}_k$. LS_μ is a power series in k non-commuting indeterminates $z_1, \ldots, z_k$; its coefficients can be computed from the coefficients of the R-transform of μ

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by using summations over chains in the lattices $NC(n)$ of non-crossing partitions. The LS-transform has the "linearizing" property that

$$LS_{\mu\boxtimes\nu} = LS_\mu + LS_\nu,\ \ \forall\, \mu,\nu \in \mathcal{G}_k \text{ such that } \mu\boxtimes\nu = \nu\boxtimes\mu.$$

If $k = 1$, then $\mathcal{Y}^{(1)}$ is naturally isomorphic to the Hopf algebra Sym of symmetric functions, and the LS-transform is related to the logarithm of the S-transform of Voiculescu, by the formula

$$LS_\mu(z) = -z\log S_\mu(z),\ \ \forall\,\mu \in \mathcal{G}_1.$$

In this case the group $(\mathcal{G}_1, \boxtimes)$ can be identified as the group of characters of Sym, in such a way that the S-transform, its reciprocal $1/S$ and its logarithm $\log S$ relate in a natural sense to the sequences of complete, elementary and power sum symmetric functions.

In [4] we connect several areas in mathematics: free probability, combinatorics of non-crossing partitions, and Hopf algebras. In this extended abstract emphasis is placed on reviewing concepts in these areas needed for understanding of our paper. This is done in Sections 2, 3, and 4. In Section 5 we define the Hopf algebra $\mathcal{Y}^{(k)}$ and describe the isomorphism $\mathcal{G}_k \simeq \mathbb{X}(\mathcal{Y}^{(k)})$. In Section 6 we explain the way logarithm of character on $\mathcal{Y}^{(k)}$ gives rise to the LS transform. Section 7 is then a review of what happens in the special case when $k = 1$. We do not include any of the proofs.

2 Notation: $NC(n)$ and power series

2.1 Non-crossing partitions

We will use the standard conventions of notation for non-crossing partitions (as in [8], or in Lecture 9 of [6]). For a positive integer n, the set of all non-crossing partitions of $\{1,\dots,n\}$ will be denoted by $NC(n)$. For $\pi \in NC(n)$, the number of blocks of π will be denoted by $|\pi|$. On $NC(n)$ we consider the partial order given by *reversed refinement*: for $\pi,\rho \in NC(n)$, we write "$\pi \le \rho$" to mean that every block of ρ is a union of blocks of π. The minimal and maximal element of $(NC(n),\le)$ are denoted by 0_n (the partition of $\{1,\dots,n\}$ into n blocks of 1 element each) and respectively 1_n (the partition of $\{1,\dots,n\}$ into 1 block of n elements).

Every partition $\pi \in NC(n)$ has associated to it a permutation of $\{1,\dots,n\}$, which is denoted by P_π, and is defined by the following prescription: for every block $B = \{b_1,\dots,b_m\}$ of π, with $b_1 < \dots < b_m$, one creates a cycle of P_π by putting

$$P_\pi(b_1) = b_2,\dots,P_\pi(b_{m-1}) = b_m, P_\pi(b_m) = b_1.$$

Note that in the particular case when $\pi = 0_n$ we have that P_{0_n} is the identity permutation of $\{1,\dots,n\}$, while for $\pi = 1_n$ we have that P_{1_n} is the cycle $1 \mapsto 2 \mapsto \dots \mapsto n \mapsto 1$.

The *Kreweras complementation map* is a special order-reversing bijection $K : NC(n) \to NC(n)$. In this paper we will use its description in terms of permutations associated to non-crossing partitions: for $\pi \in NC(n)$, the Kreweras complement of π is the partition $K(\pi) \in NC(n)$ uniquely determined by the fact that its associated permutation is

$$P_{K(\pi)} = P_\pi^{-1} P_{1_n}. \tag{2.1}$$

Formula (2.1) can be extended in order to cover the concept of *relative Kreweras complement* of π in ρ, for $\pi, \rho \in NC(n)$ such that $\pi \leq \rho$. This is the partition in $NC(n)$, denoted by $K_\rho(\pi)$, uniquely determined by the fact that the permutation associated to it is

$$P_{K_\rho(\pi)} = P_\pi^{-1} P_\rho. \tag{2.2}$$

Clearly, the Kreweras complementation map K from (2.1) is the relative complementation with respect to the maximal element 1_n of $NC(n)$.

The formulas (2.1), (2.2) do not follow exactly the original approach used by Kreweras in [3], but are easily seen to be equivalent to it (see e.g. [6], Exercise 18.25 on p. 301).

2.2 Power series and generalized coefficients

Let k be a positive integer. We use the notation $[k]^*$ for the set of all words of finite length over the alphabet $\{1, \dots, k\}$:

$$[k]^* := \bigcup_{n=0}^{\infty} \{1, \dots, k\}^n. \tag{2.3}$$

The length of a word $w \in [k]^*$ will be denoted by $|w|$. We follow the standard procedure of including into $[k]^*$ a unique word ϕ with $|\phi| = 0$.

We will use the notation $\mathbb{C}_0\langle\langle z_1, \dots, z_k\rangle\rangle$ for the set of power series with complex coefficients and with vanishing constant term in the non-commuting indeterminates $z_1, \dots, z_k$. The general form of a series $f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k\rangle\rangle$ is thus

$$f(z_1, \dots, z_k) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n = 1}^{k} \alpha_{(i_1, \dots, i_n)} z_{i_1} \cdots z_{i_n} = \sum_{\substack{w \in [k]^*, \\ |w| \geq 1}} \alpha_w z_w, \tag{2.4}$$

where the coefficients α_w are from $\mathbb{C}$ and where we write in short $z_w := z_{i_1} \cdots z_{i_n}$ for $w = (i_1, \dots, i_n) \in \{1, \dots, k\}^n$, $n \geq 1$.

For every word $w \in [k]^*$ with $|w| \geq 1$ we will denote by

$$\mathrm{Cf}_w : \mathbb{C}_0\langle\langle z_1, \dots, z_k\rangle\rangle \to \mathbb{C}$$

the linear functional which extracts the coefficient of z_w in a series $f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k\rangle\rangle$. Thus for f written as in Equation (2.4) we have $\mathrm{Cf}_w(f) = \alpha_w$.

Suppose we are given a positive integer n, a word $w = (i_1, \dots, i_n) \in \{1, \dots, k\}^n$ and a partition $\pi \in NC(n)$. We define a (generally non-linear) functional

$$\mathrm{Cf}_{w;\pi} : \mathbb{C}_0\langle\langle z_1, \dots, z_k\rangle\rangle \to \mathbb{C},$$

as follows. For every block $B = \{b_1, \dots, b_m\}$ of π, with $1 \leq b_1 < \cdots < b_m \leq n$, let us use the notation

$$w \,|\, B = (i_1, \dots, i_n)|B \;:=\; (i_{b_1}, \dots, i_{b_m}) \in \{1, \dots, k\}^m. \tag{2.5}$$

Then we define

$$\mathrm{Cf}_{w;\pi}(f) \;:=\; \prod_{B \text{ block of } \pi} \mathrm{Cf}_{w|B}(f), \;\; \forall\, f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k\rangle\rangle. \tag{2.6}$$

(For example if $w = (i_1, \dots, i_5)$ is a word of length 5 and if $\pi = \{\{1,4,5\},\{2,3\}\} \in NC(5)$, then the above formula comes to $\mathrm{Cf}_{(i_1,i_2,i_3,i_4,i_5);\pi}(f) = \mathrm{Cf}_{(i_1,i_4,i_5)}(f) \cdot \mathrm{Cf}_{(i_2,i_3)}(f)$, $f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k\rangle\rangle$.)

3 Free probability

3.1 Noncommutative probability space, random variables, and moments

Let $(\mathcal{A}, \varphi)$ be a *noncommutative probability space*, that is $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi: \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$. We refer to the elements of $\mathcal{A}$ as *random variables*. Given a random variable $a \in \mathcal{A}$, we refer to the number $\varphi(a^n)$ by calling it the *moment* of order n of a. The generating series for the moments of a,

$$M_a(z) = \sum_{n=1}^{\infty} \varphi(a^n) z^n, \tag{3.1}$$

is called the *moment series* of a. This terminology extends to the situation when we deal with a k-tuple $a_1, \ldots, a_k$ of elements of $\mathcal{A}$. Then family

$$\{\varphi(a_{i_1} \ldots a_{i_n}) \mid n \leq 1,\ 1 \leq i_1, \ldots, i_n \leq k\}$$

is called the family of *joint moments* of $a_1, \ldots, a_k$. These joint moments are the coefficients of a formal power series in k noncommuting indeterminates $z_1, \ldots, z_k$, which is denoted by $M_{a_1,\ldots,a_k}$ and is called the *joint moment series* of $a_1, \ldots, a_k$:

$$M_{a_1,\ldots,a_k}(z_1, \ldots, z_k) = \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{k} \varphi(a_{i_1} \ldots a_{i_n}) z_{i_1} \ldots z_{i_n}. \tag{3.2}$$

3.2 Free independence

We say that unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{A}$ are *freely independent* if $\varphi(a_1 \ldots a_m) = 0$ whenever we have the following

- $a_j \in \mathcal{A}_{i(j)}$ for $j = 1, \ldots, m$, where $i(1) \neq i(2), i(2) \neq i(3), \ldots, i(m-1) \neq i(m)$, and
- $\varphi(a_j) = 0$ for $j = 1, \ldots, m$.

We say that subsets $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of $\mathcal{A}$ are freely independent if the unital subalgebras $\mathcal{A}_i = \langle \mathcal{S}_i \rangle$, $i = 1, \ldots, n$, generated by these subsets are freely independent. In particular, we say that $\{a_1, \ldots, a_k\}$ is freely independent from $\{b_1, \ldots, b_k\}$, if unital subalgebras generated by these k-tuples are freely independent.

3.3 Distributions and free multiplicative convolution

If $a_1, \ldots, a_k$ is a k-tuple of elements of $\mathcal{A}$, then the *distribution* of $(a_1, \ldots, a_k)$ is the linear functional μ on the algebra of noncommutative polynomials $\mathbb{C}\langle X_1, \ldots, X_k \rangle$ defined by

$$\mu(X_{i_1} \cdots X_{i_n}) = \varphi(a_{i_1} \cdots a_{i_n}),\ \ \forall\, n \geq 0,\ \forall\, 1 \leq i_1, \ldots, i_n \leq k.$$

We denote by $\mathcal{D}_{\mathrm{alg}}(k)$ the set of linear functionals on $\mathbb{C}\langle X_1, \ldots, X_k \rangle$ that arise in this way. Clearly, this is just the set of all linear functionals on $\mathbb{C}\langle X_1, \ldots, X_k \rangle$ such that $\mu(1) = 1$. On $\mathcal{D}_{\mathrm{alg}}(k)$ we have a binary operation $\boxtimes$ which reflects the multiplication of two freely independent k-tuples in a noncommutative probability space. That is, $\boxtimes$ is well-defined and uniquely determined by the following requirement: if $a_1, \ldots, a_k, b_1, \ldots, b_k$ are elements in a noncommutative probability space $(\mathcal{A}, \varphi)$ such that $(a_1, \ldots, a_k)$ has distribution μ, $(b_1, \ldots, b_k)$ has distribution ν, and $\{a_1, \ldots, a_k\}$ is freely independent from $\{b_1, \ldots, b_k\}$, then it follows that the distribution of $(a_1 b_1, \ldots, a_k b_k)$ is equal to $\mu \boxtimes \nu$. The operation $\boxtimes$ on $\mathcal{D}_{\mathrm{alg}}(k)$ is associative and unital, where the unit is the functional $\mu_o \in \mathcal{D}_{\mathrm{alg}}(k)$ with $\mu_o(X_{i_1} \cdots X_{i_n}) = 1$ for all $n \geq 1$ and $1 \leq i_1, \ldots, i_n \leq k$. A distribution $\mu \in \mathcal{D}_{\mathrm{alg}}(k)$ is invertible with respect to $\boxtimes$ if and only if it satisfies $\mu(X_i) \neq 0, \forall\, 1 \leq i \leq k$; and moreover, the subset

$$\mathcal{G}_k := \{\mu \in \mathcal{D}_{\mathrm{alg}}(k) \mid \mu(X_i) = 1, \ \forall\, 1 \leq i \leq k\} \tag{3.3}$$

is a subgroup in the group of invertibles with respect to $\boxtimes$. For a basic introduction to free multiplicative convolution, we refer to Section 3.6 of [12] or to Lecture 14 in [6].

3.4 R-transform

Let μ be a distribution in $\mathcal{D}_{\mathrm{alg}}(k)$, that is, μ: $\mathbb{C}\langle X_1, \ldots, X_k\rangle \to \mathbb{C}$ is a linear functional such that $\mu(1) = 1$. The *R-transform* of μ is the series $R_\mu \in \mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$ uniquely determined by the requirement that for every $n \geq 1$ and every $1 \leq i_1, \ldots, i_n \leq k$ one has

$$\mu(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in NC(n)} \mathrm{Cf}_{(i_1, \ldots, i_n);\pi}(R_\mu). \tag{3.4}$$

If μ is the distribution corresponding to a k-tuple $a_1, \ldots, a_k$, then we often write $R_{a_1,\ldots,a_k} = R_\mu$.

It is easy to see that Equation (3.4) does indeed determine a unique series in $\mathbb{C}_0\langle\langle z_1, \ldots, z_k\rangle\rangle$. The coefficients of R_μ are called the free cumulants of μ, and because of this reason Equation (3.4) is sometimes referred to as "the moment–cumulant formula" – see Lectures 11 and 16 in [6].

An important point for the present paper is that the R-transform has a very nice behaviour under the operation $\boxtimes$. This is recorded in the next proposition.

Proposition 3.1 *Let μ, ν be distributions in $\mathcal{D}_{\mathrm{alg}}(k)$, and let w be a word in $[k]^*$, with $|w| \geq 1$. Then*

$$Cf_w\big(R_{\mu\boxtimes\nu}\big) = \sum_{\pi \in NC(n)} Cf_{w;\pi}\big(R_\mu\big) \cdot Cf_{w;K(\pi)}\big(R_\nu\big). \tag{3.5}$$

For the proof of Proposition 3.1 we refer to Theorem 14.4 and Proposition 17.2 of [6].

3.5 S-transform

An efficient method for computing the moments of a product of two freely independent random variables is via the S-transform S_μ given by

$$S_\mu(z) = \frac{1+z}{z} M_\mu^{<-1>}(z) = \frac{1}{z} R_\mu^{<-1>}(z) \tag{3.6}$$

(the superscript "$< -1 >$" refers to the inverse under composition). In [11] Voiculescu showed that one has the equation

$$S_{\mu\boxtimes\nu} = S_\mu \cdot S_\nu, \tag{3.7}$$

(the result was phrased in terms of products of freely independent random variables a, b in a noncommutative probability space $(\mathcal{A}, \varphi)$).

4 Graded connected Hopf algebras

We will work with graded bialgebras over $\mathbb{C}$ and we will use the standard conventions for notations regarding them (as in the monograph [9], for instance). Our review here does not aim at generality, but just covers the specialized Hopf algebras used in the present paper.

4.1 Notation

Let $\mathcal{B}$ be a graded bialgebra over $\mathbb{C}$. The comultiplication and counit of $\mathcal{B}$ will be denoted by Δ and respectively ε (or by $\Delta_{\mathcal{B}}$ and $\varepsilon_{\mathcal{B}}$ when necessary to distinguish $\mathcal{B}$ from other graded bialgebras that are considered at the same time).

For every $n \geq 0$, the vector subspace of $\mathcal{B}$ which consists of homogeneous elements of degree n will be denoted by $\mathcal{B}_n$. We thus have a direct sum decomposition $\mathcal{B} = \oplus_{n=0}^{\infty} \mathcal{B}_n$ where

$$\left\{ \begin{array}{l} \mathcal{B}_0 \ni 1_{\mathcal{B}} \text{ (the unit of } \mathcal{B}\text{)}, \\ \mathcal{B}_m \cdot \mathcal{B}_n \subseteq \mathcal{B}_{m+n}, \ \forall\, m,n \geq 0, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \varepsilon|_{\mathcal{B}_n} = 0, \ \forall\, n \geq 1, \\ \Delta(\mathcal{B}_n) \subseteq \oplus_{i=0}^{n} \mathcal{B}_i \otimes \mathcal{B}_{n-i}, \ \forall\, n \geq 0. \end{array} \right.$$

If the space $\mathcal{B}_0$ of homogeneous elements of degree 0 is equal to $\mathbb{C}1_{\mathcal{B}}$ then we say that the graded bialgebra $\mathcal{B}$ is *connected*.

4.2 Convolution Algebra

Let $\mathcal{B}$ be a graded connected bialgebra, let $\mathcal{M}$ be a unital algebra over $\mathbb{C}$, and let $L(\mathcal{B}, \mathcal{M})$ denote the vector space of all linear maps from $\mathcal{B}$ to $\mathcal{M}$. For $\xi, \eta \in L(\mathcal{B}, \mathcal{M})$ one can define their convolution product, denoted here simply as "$\xi\eta$", by the formula

$$\xi\eta := \text{Mult} \circ (\xi \otimes \eta) \circ \Delta, \tag{4.1}$$

where Mult : $\mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ is the linear map given by multiplication (Mult$(x \otimes y) = xy$ for $x, y \in \mathcal{M}$).

When endowed with its usual vector space structure and with the convolution product, $L(\mathcal{B}, \mathcal{M})$ becomes itself a unital algebra over $\mathbb{C}$. The unit of $L(\mathcal{B}, \mathcal{M})$ is the linear map $\mathcal{B} \ni b \mapsto \varepsilon(b)1_{\mathcal{M}}$, which by a slight abuse of notation is still denoted as ε (same notation as for the counit of $\mathcal{B}$).

4.3 Calculus in a convolution algebra

For every $\xi \in L(\mathcal{B}, \mathcal{M})$ it makes sense to form polynomial expressions in ξ, that is, expressions of the form

$$\sum_{\ell=0}^{n} t_\ell \xi^\ell \in L(\mathcal{B}, \mathcal{M}), \quad \text{for } n \geq 0 \text{ and } t_0, t_1, \ldots, t_n \in \mathbb{C},$$

where the ξ^ℓ $(0 \leq \ell \leq n)$ are convolution powers of ξ, and we make the convention that $\xi^0 := \varepsilon$ (the unit of $L(\mathcal{B}, \mathcal{M})$). An important point for the present paper is that if ξ is such that $\xi(1_{\mathcal{B}}) = 0$ then it also makes sense to define an element

$$\eta := \sum_{\ell=0}^{\infty} t_\ell \xi^\ell \in L(\mathcal{B}, \mathcal{M}), \tag{4.2}$$

for an arbitrary infinite sequence $(t_\ell)_{\ell \geq 0}$ in $\mathbb{C}$. Indeed, if $\xi(1_{\mathcal{B}}) = 0$ then due to the fact that Δ respects the grading one immediately sees that ξ^ℓ vanishes on $\mathcal{B}_n$ whenever $\ell > n$. Thus η from (4.2) can be defined as the unique linear map from $\mathcal{B}$ to $\mathcal{M}$ which satisfies

$$\eta \,|\, _{\mathcal{B}_n} = \Big(\sum_{\ell=0}^{N} t_\ell \xi^\ell \Big) \,|\, _{\mathcal{B}_n}, \ \ \forall\, N \geq n \geq 0. \tag{4.3}$$

4.4 The antipode

Every graded connected bialgebra $\mathcal{B}$ is in fact a Hopf algebra [5] – this means, by definition, that the identity map $\mathrm{id} : \mathcal{B} \to \mathcal{B}$ is an invertible element in the convolution algebra $L(\mathcal{B}, \mathcal{B})$. The inverse of id is called the *antipode* of $\mathcal{B}$ and is denoted by S. A reason why S is sure to exist is that one can introduce it via a series expansion as in (4.2) above, which mimics the geometric series expansion of $\big(\varepsilon - (\varepsilon - \mathrm{id})\big)^{-1}$. That is, one can put

$$S := \varepsilon + \sum_{\ell=1}^{\infty} (\varepsilon - \mathrm{id})^\ell \in L(\mathcal{B}, \mathcal{B}) \tag{4.4}$$

(which makes sense because $(\varepsilon - \mathrm{id})(1_{\mathcal{B}}) = 0$), and can then verify that S from (4.4) has indeed the property that $S\,\mathrm{id} = \varepsilon = \mathrm{id}\, S$. See Lemma 14 in [10].

4.5 The group of characters

A unital algebra homomorphism from $\mathcal{B}$ to $\mathbb{C}$ is called a *character*. The set of all characters of $\mathcal{B}$ will be denoted by $\mathbb{X}(\mathcal{B})$. It is easy to verify that the convolution product of two characters is again a character. Moreover, if η is a character then it is obvious that the functional $\eta \circ S \in L(\mathcal{B}, \mathbb{C})$ is a character as well, and it is easy to verify that $\eta\,(\eta \circ S) = \varepsilon = (\eta \circ S)\,\eta$. Hence $\mathbb{X}(\mathcal{B})$ is a subgroup of the group of invertibles of $L(\mathcal{B}, \mathbb{C})$, and is thus referred to as the *group of characters* of $\mathcal{B}$.

4.6 Exponentials and logarithms for functionals

Let $\mathcal{B}$ be a graded connected Hopf algebra. If ξ is a functional in $L(\mathcal{B}, \mathbb{C})$ such that $\xi(1_{\mathcal{B}}) = 0$, then it makes sense to define its exponential by the familiar formula

$$\exp \xi = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \xi^\ell. \tag{4.5}$$

It is easy to see that exp maps bijectively the set of functionals $\{\xi \in L(\mathcal{B}, \mathbb{C}) \mid \xi(1_{\mathcal{B}}) = 0\}$ onto $\{\eta \in L(\mathcal{B}, \mathbb{C}) \mid \eta(1_{\mathcal{B}}) = 1\}$; the inverse of this bijection is denoted as "log" and can be described by using the Taylor series expansion for logarithm:

$$\log \eta = -\sum_{\ell=1}^{\infty} \frac{1}{\ell} (\varepsilon - \eta)^\ell, \quad \text{for } \eta \in L(\mathcal{B}, \mathbb{C}) \text{ with } \eta(1_{\mathcal{B}}) = 1. \tag{4.6}$$

By adjusting the familiar argument for the exponential of a sum of two matrices, one finds that

$$\exp(\xi_1 + \xi_2) = \exp(\xi_1)\, \exp(\xi_2), \ \ \forall\, \xi_1, \xi_2 \in L(\mathcal{B}, \mathbb{C}) \ \ \begin{array}{l} \text{such that } \xi_1(1_{\mathcal{B}}) = \xi_2(1_{\mathcal{B}}) = 0 \\ \text{and such that } \xi_1 \xi_2 = \xi_2 \xi_1. \end{array} \tag{4.7}$$

As a consequence, in the opposite direction of the exp/log bijection one finds that

$$\log(\eta_1\eta_2) = \log(\eta_1) + \log(\eta_2), \ \ \forall\, \eta_1, \eta_2 \in L(\mathcal{B}, \mathbb{C}) \ \ \begin{array}{l}\text{such that } \eta_1(1_{\mathcal{B}}) = \eta_2(1_{\mathcal{B}}) = 1 \\ \text{and such that } \eta_1\eta_2 = \eta_2\eta_1.\end{array} \tag{4.8}$$

This exp/log bijection is a special case of standard general results from Appendix A in [7].

5 The Hopf algebra $\mathcal{Y}^{(k)}$

Throughout this section we fix a positive integer k. We use the notation $\mathcal{Y}^{(k)}$ for the commutative algebra of polynomials

$$\mathcal{Y}^{(k)} := \mathbb{C}\big[\, Y_w \mid w \in [k]^*, \ |w| \geq 2 \,\big] \tag{5.1}$$

In addition to that, we will also use the following conventions of notation.

Notation 5.1 1^o For a word $w \in [k]^*$ such that $|w| = 1$ (i.e. such that $w = (i)$ for some $1 \leq i \leq k$) we put $Y_w := 1$ (the unit of $\mathcal{Y}^{(k)}$).

2^o Let w be a word in $[k]^*$ with $|w| = n \geq 1$, and let $\pi = \{A_1, \dots, A_q\}$ be a partition in $NC(n)$. We will denote

$$Y_{w;\pi} := Y_{w_1} \cdots Y_{w_q} \in \mathcal{Y}^{(k)}, \tag{5.2}$$

where $w_j = w \,|\, A_j$ for $1 \leq j \leq q$ (and where the restriction $w \,|\, A$ of the word w to a non-empty subset $A \subseteq \{1, \dots, n\}$ is defined in the same way as in Equation (2.5)).

The comultiplication and counit of $\mathcal{Y}^{(k)}$ are defined as follows.

Definition 5.2 1^o Let $\Delta : \mathcal{Y}^{(k)} \to \mathcal{Y}^{(k)} \otimes \mathcal{Y}^{(k)}$ be the unital algebra homomorphism uniquely determined by the requirement that for every $w \in [k]^*$ with $|w| = n \geq 2$ we have

$$\Delta(Y_w) = \sum_{\pi \in NC(n)} Y_{w;\pi} \otimes Y_{w;K(\pi)}, \tag{5.3}$$

where we use the conventions of notation introduced above (cf. Equation (5.2)), and where $K(\pi)$ is the Kreweras complement of a partition $\pi \in NC(n)$. For example

$$\Delta(Y_{i_1i_2i_3}) = Y_{i_1i_2i_3} \otimes 1 + Y_{i_1i_2} \otimes Y_{i_2i_3} + Y_{i_1i_3} \otimes Y_{i_1i_2} + Y_{i_2i_3} \otimes Y_{i_1i_3} + 1 \otimes Y_{i_1i_2i_3}.$$

2^o Let $\varepsilon : \mathcal{Y}^{(k)} \to \mathbb{C}$ be the unital algebra homomorphism uniquely determined by the requirement that

$$\varepsilon(Y_w) = 0, \ \ \forall\, w \in [k]^* \text{ with } |w| \geq 2. \tag{5.4}$$

On $\mathcal{Y}^{(k)}$ we will also consider a grading, which is defined such that every generator Y_w of $\mathcal{Y}^{(k)}$ gets to be homogeneous of degree $|w| - 1$. More precisely, the homogeneous subspaces $\mathcal{Y}_n^{(k)}$ of $\mathcal{Y}^{(k)}$ are defined as follows.

Notation 5.3 For every $n \geq 0$ we denote

$$\mathcal{Y}_n^{(k)} := \text{span} \left\{ Y_{w_1} \cdots Y_{w_q} \ \middle| \ \begin{array}{l} q \geq 1, \ w_1, \dots, w_q \in [k]^* \text{ with} \\ |w_1|, \dots, |w_q| \geq 1 \text{ and } |w_1| + \cdots + |w_q| = n + q \end{array} \right\}. \tag{5.5}$$

Proposition 5.4 *With the comultiplication, counit and grading defined above, $\mathcal{Y}^{(k)}$ becomes a graded connected Hopf algebra.*

We conclude this section by describing an isomorphism from $\mathcal{G}_k$ to the group of characters of $\mathcal{Y}^{(k)}$.

Definition 5.5 Let μ be a distribution in $\mathcal{G}_k$ and consider the R-transform R_μ. The *character of $\mathcal{Y}^{(k)}$ associated to* μ is the character $\chi_\mu \in \mathbb{X}(\mathcal{Y}^{(k)})$ uniquely determined by the requirement that

$$\chi_\mu(Y_w) = \mathrm{Cf}_w(R_\mu), \ \ \forall\, w \in [k]^* \text{ such that } |w| \geq 2. \tag{5.6}$$

Theorem 5.6 *The map $\mu \mapsto \chi_\mu$ defined above is a group isomorphism from $\big(\mathcal{G}_k, \boxtimes\big)$ onto the group $\mathbb{X}(\mathcal{Y}^{(k)})$ of characters on $\mathcal{Y}^{(k)}$.*

6 The LS-transform

The LS-transform LS_μ is defined so that it stores the information about the functional $\log \chi_\mu$, as follows.

Definition 6.1 Let μ be a distribution in $\mathcal{G}_k$. The LS-transform of μ is the power series

$$LS_\mu(z_1, \ldots, z_k) := \sum_{\substack{w\in[k]^*,\\ |w|\geq 2}} \Big(\big(\log \chi_\mu\big)(Y_w) \Big) z_w, \tag{6.1}$$

where $\log \chi_\mu \colon \mathbb{X}(\mathcal{Y}^{(k)}) \to \mathbb{C}$ is as in Equation (4.8), and where the meaning of "z_w" is same as in Equation (2.4).

As a consequence of Equation (4.8), one obtains the following.

Theorem 6.2 *Let μ and ν be distributions in $\mathcal{G}_k$ such that $\mu \boxtimes \nu = \nu \boxtimes \mu$. Then*

$$LS_{\mu\boxtimes\nu} = LS_\mu + LS_\nu. \tag{6.2}$$

In particular, formula (6.2) always applies when one of μ, ν is the joint distribution of a repeated k-tuple $(a, a, \ldots, a)$, where a is a random variable in a non-commutative probability space $(\mathcal{A}, \varphi)$ (This special case is of particular relevance for the analytic framework of C^*-probability spaces, as explained in Example 5.2 of [4].)

The LS-transform was introduced above by using the Hopf algebra $\mathcal{Y}^{(k)}$, but it can also be described directly in combinatorial terms, by using summations over chains in lattices of non-crossing partitions. We next explain how this goes.

A chain in $NC(n)$ is an object of the form

$$\Gamma = (\pi_0, \pi_1, \ldots, \pi_\ell) \tag{6.3}$$

with $\pi_0, \pi_1, \ldots, \pi_\ell \in NC(n)$ such that $0_n = \pi_0 < \pi_1 < \cdots < \pi_\ell = 1_n$ (and where 0_n and 1_n denote the minimal and maximal element of $NC(n)$, respectively). For a chain Γ as in (6.3), the number ℓ is called the length of Γ and is denoted as $|\Gamma|$. Given a formal power series f in non-commuting indeterminates $z_1, \ldots, z_k$, one has a natural way of defining some "generalized coefficients"

$$\mathrm{Cf}_w^{(\Gamma)}(f) := \prod_{j=1}^{\ell} \mathrm{Cf}_{w;K_{\pi_j}(\pi_{j-1})}(f), \tag{6.4}$$

where $\Gamma = (\pi_0, \pi_1, \ldots, \pi_\ell)$ is a chain in $NC(n)$. By using the generalized coefficients (6.4), the combinatorial description of the LS-transform is stated as follows.

Theorem 6.3 *Let μ be a distribution in $\mathcal{G}_k$, and let $w = (i_1, \ldots, i_n)$ be a word in $[k]^*$, where $n \geq 2$. Consider (as in Equation (6.1) of Definition 6.1) the coefficient $\big(\log \chi_\mu\big)(Y_w)$ of z_w in the LS-transform of μ. This coefficient can be also expressed as*

$$\big(\log \chi_\mu\big)(Y_w) = \sum_{\substack{\Gamma \text{ chain} \\ \text{in } NC(n)}} \frac{(-1)^{1+|\Gamma|}}{|\Gamma|} Cf^{(\Gamma)}_{(i_1,\ldots,i_n)}(R_\mu), \tag{6.5}$$

where R_μ is the R-transform of μ.

7 Case of one variable

In the remaining part of the paper we look at the particular case when $k = 1$. In this case the notations are simplified due to the fact that words over the 1-letter alphabet $\{1\}$ are determined by their lengths. We make the convention to write simply "Y_n" instead of $Y_{(1,1,\ldots,1)}$ with n repetitions of 1 in the index; thus Equation (5.1) is now written in the form

$$\mathcal{Y}^{(1)} = \mathbb{C}\big[\, Y_n \mid n \geq 2 \,\big], \tag{7.1}$$

while Equation (6.1) defining LS_μ reduces to

$$LS_\mu(z) = \sum_{n=2}^{\infty} \Big((\log \chi_\mu)(Y_n) \Big) z^n. \tag{7.2}$$

A special feature of the case $k = 1$ (not holding for $k \geq 2$) is that the operation $\boxtimes$ is commutative. Hence for $k = 1$ the linearization property stated in Theorem 6.2 holds for all $\mu, \nu \in \mathcal{G}_1$. A large part of [4] is dedicated to proving that the one-dimensional LS-transform is related to the S-transform as follows.

Theorem 7.1 *For a distribution $\mu \in \mathcal{G}_1$, the power series S_μ and LS_μ are related by*

$$LS_\mu(z) = -z \log S_\mu(z). \tag{7.3}$$

The proof of Theorem 7.1 is obtained by following the connections that $\mathcal{Y}^{(1)}$ has with symmetric functions. Let

$$\mathrm{Sym} = \mathbb{C}[p_n | n \in \mathbb{N}] = \mathbb{C}[e_n | n \in \mathbb{N}] = \mathbb{C}[h_n | n \in \mathbb{N}],$$

be the Hopf algebra of symmetric functions, where $(e_n)_{n=1}^{\infty}$, $(h_n)_{n=1}^{\infty}$, and $(p_n)_{n=1}^{\infty}$ are sequences of elementary, complete homogeneous, and power sum symmetric functions, respectively. If we use the convention that $e_0 = h_0 = p_0 = 1$, then the comultiplication formulas for these sequences are as follows

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}, \qquad \Delta(h_n) = \sum_{i=0}^{n} h_i \otimes h_{n-i}, \qquad \Delta(p_n) = p_n \otimes 1 \,+\, 1 \otimes p_n,$$

for all $n \geq 1$. We introduce a new sequence of symmetric functions $(y_n)_{n=2}^{\infty}$ as follows:

$$y_n = \sum_{\substack{\pi=\{A_1,\ldots,A_q\} \\ \text{in } NC(n-1)}} e_{|A_1|} \cdots e_{|A_q|} \in \text{Sym}. \tag{7.4}$$

Clearly, every y_n is a homogeneous symmetric function of degree $n-1$. We also use the convention that $y_0 = y_1 = 1$.

Theorem 7.2 *The map $\Phi\colon \mathcal{Y}^{(1)} \to$ Sym, given by $Y_n \mapsto y_n$ is an isomorphism of graded connected Hopf algebras. Furthermore, if $\theta_\mu = \chi_\mu \circ \Phi^{-1}\colon$ Sym $\to \mathbb{C}$, then we have*

$$\begin{aligned} \theta_\mu(y_n) &= Cf_n(R_\mu), \\ \theta_\mu(h_n) &= (-1)^n Cf_n(S_\mu), \\ \theta_\mu(e_n) &= Cf_n(1/S_\mu), \\ \theta_\mu(p_n) &= (-1)^n n Cf_n(\log S_\mu). \end{aligned}$$

Remark 7.3 If we define a character $\zeta\colon \mathcal{Y}^{(1)} \to \mathbb{C}$ by $\zeta(Y_n) = 1$ for $n \in \mathbb{N}$, then the pair $(\mathcal{Y}^{(1)}, \zeta)$ is a combinatorial Hopf algebra in the sense of Aguiar, Bergeron and Sottile [1]. It turns out that the isomorphism $\Phi\colon \mathcal{Y}^{(1)} \to$ Sym above is the unique homomorphism of graded Hopf algebras satisfying $\zeta = \zeta_{\text{Sym}} \Phi$, which is guaranteed by Theorem 4.3 of [1].

Remark 7.4 Having placed the homomorphism Φ in the framework of combinatorial Hopf algebras leads to an interesting alternative description of the symmetric functions $\{y_n \mid n \geq 2\}$, as linear combinations of monomial quasi-symmetric functions. For every m-tuple of positive integers $(r_1, \ldots, r_m)$, the corresponding monomial quasi-symmetric function $M_{(r_1,\ldots,r_m)}$ is defined as

$$M_{(r_1,\ldots,r_m)} = \sum_{1 \leq i_1 < i_2 < \cdots < i_m} x_{i_1}^{r_1} x_{i_2}^{r_2} \cdots x_{i_m}^{r_m}, \tag{7.5}$$

where $\{x_i \mid i \geq 1\}$ is a family of commuting indeterminates. Using some results of Ehrenborg [2] we can show that

$$y_n = \sum_{\substack{\Gamma=(\pi_o,\pi_1,\ldots,\pi_\ell) \\ \text{chain in } NC(n)}} M_{(|\pi_0|-|\pi_1|,|\pi_1|-|\pi_2|,\ldots,|\pi_{\ell-1}|-|\pi_\ell|)}, \ \ \forall\, n \geq 2. \tag{7.6}$$

Remark 7.5 A 'direct' proof of the fact that the map Φ of Theorem 7.2 respects comultiplication would go as follows. Using recursion

$$y_n = \sum_{m=2}^{n} \Big(e_{m-1} \cdot \sum_{1=i_1<i_2<\cdots<i_m=n} y_{i_2-i_1}\, y_{i_3-i_2} \cdots y_{i_m-i_{m-1}} \Big). \tag{7.7}$$

one has to establish that

$$\Delta(y_n) = \sum_{\pi \in NC(n)} y_\pi \otimes y_{K(\pi)}, \ \ \forall\, n \geq 2. \tag{7.8}$$

This can be done by induction, using the formula for the comultiplication of e_n's. However this induction argument is quite lengthy and significantly more involved then the approach we take in [4]. There we observe that for $\mu \in \mathcal{G}_1$ we have $\theta_\mu \circ \Phi = \chi_\mu$ and that $\theta_\mu(h_n) = (-1)^n \mathrm{Cf}_n(S_\mu)$ (more precisely in [4] we define θ by $\theta_\mu(h_n) = (-1)^n \mathrm{Cf}_n(S_\mu)$ and use relationship between the R-transform and the S-transform to establish $\theta_\mu \circ \Phi = \chi_\mu$). Since due to the multiplicativity of the S-transform we know that $\theta\colon \mathcal{G}_1 \to \mathbb{X}(\mathrm{Sym})$, $\mu \mapsto \theta_\mu$ is a group isomorphism, and since due to the Cartier-Kostant-Milnor-Moore Theorem (see e.g. Theorem 13.0.1 on p. 274 of [9])) $\mathcal{Y}^{(1)}$ is isomorphic to Sym we then show that Φ must be a coalgebra map.

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Infinite log-concavity: developments and conjectures

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Abstract. Given a sequence $(a_k) = a_0, a_1, a_2, \ldots$ of real numbers, define a new sequence $\mathcal{L}(a_k) = (b_k)$ where $b_k = a_k^2 - a_{k-1}a_{k+1}$. So (a_k) is log-concave if and only if (b_k) is a nonnegative sequence. Call (a_k) *infinitely log-concave* if $\mathcal{L}^i(a_k)$ is nonnegative for all $i \geq 1$. Boros and Moll conjectured that the rows of Pascal's triangle are infinitely log-concave. Using a computer and a stronger version of log-concavity, we prove their conjecture for the nth row for all $n \leq 1450$. We can also use our methods to give a simple proof of a recent result of Uminsky and Yeats about regions of infinite log-concavity. We investigate related questions about the columns of Pascal's triangle, q-analogues, symmetric functions, real-rooted polynomials, and Toeplitz matrices. In addition, we offer several conjectures.

Résumé. Étant donné une suite $(a_k) = a_0, a_1, a_2, \ldots$ de nombres réels, on définit une nouvelle suite $\mathcal{L}(a_k) = (b_k)$ où $b_k = a_k^2 - a_{k-1}a_{k+1}$. Alors (a_k) est log-concave si et seulement si (b_k) est une suite non négative. On dit que (a_k) est *infiniment log-concave* si $\mathcal{L}^i(a_k)$ est non négative pour tout $i \geq 1$. Boros et Moll ont conjecturé que les lignes du triangle de Pascal sont infiniment log-concave. Utilisant un ordinateur et une version plus forte de log-concavité, on vérifie leur conjecture pour la $n$ième ligne, pour tout $n \leq 1450$. On peut aussi utiliser nos méthodes pour donner une preuve simple d'un résultat récent de Uminsky et Yeats à propos des régions de log-concavité infini. Reliées à ces idées, on examine des questions à propos des colonnes du triangle de Pascal, des q-analogues, des fonctions symétriques, des polynômes avec racines réelles, et des matrices de Toeplitz. De plus, on offre plusieurs conjectures.

Keywords: binomial coefficients, computer proof, Gaussian polynomial, infinite log-concavity, symmetric functions, real roots

1 Introduction

Let
$$(a_k) = (a_k)_{k\geq 0} = a_0, a_1, a_2, \ldots$$
be a sequence of real numbers. It will be convenient to extend the sequence to negative indices by letting $a_k = 0$ for $k < 0$. Also, if $(a_k) = a_0, a_1, \ldots, a_n$ is a finite sequence then we let $a_k = 0$ for $k > n$.

Define the $\mathcal{L}$*-operator* on sequences to be $\mathcal{L}(a_k) = (b_k)$ where $b_k = a_k^2 - a_{k-1}a_{k+1}$. Call a sequence i*-fold log-concave* if $\mathcal{L}^i(a_k)$ is a nonnegative sequence. So log-concavity in the ordinary sense is 1-fold log-concavity. Log-concave sequences arise in many areas of algebra, combinatorics, and geometry. See the survey articles of Stanley (20) and Brenti (7) for more information.

Boros and Moll (4, page 157) defined (a_k) to be *infinitely log-concave* if it is i-fold log-concave for all $i \geq 1$. They introduced this definition in conjunction with the study of a specialization of the Jacobi polynomials whose coefficient sequence they conjectured to be infinitely log-concave. Kauers and Paule (13) used a computer algebra package to prove this conjecture for ordinary log-concavity. Since the coefficients of these polynomials can be expressed in terms of binomial coefficients, Boros and Moll also made the statement:

"Prove that the binomial coefficients are ∞-logconcave."

We will take this to be a conjecture that the rows of Pascal's triangle are infinitely log-concave, although we will later discuss the columns and other lines. When given a function of more than one variable, we will subscript the $\mathcal{L}$-operator by the parameter which is varying to form the sequence. So $\mathcal{L}_k\binom{n}{k}$ would refer to the operator acting on the sequence $\binom{n}{k}_{k\geq 0}$. Note that we drop the sequence parentheses for sequences of binomial coefficients to improve readability. We now restate the Boros-Moll conjecture formally.

Conjecture 1.1 *The sequence $\binom{n}{k}_{k\geq 0}$ is infinitely log-concave for all $n \geq 0$.*

In the next section, we use a strengthened version of log-concavity and computer calculations to verify Conjecture 1.1 for all $n \leq 1450$. Uminsky and Yeats (25) set up a correspondence between certain symmetric sequences and points in $\mathbb{R}^m$. They then described an infinite region $\mathcal{R} \subset \mathbb{R}^m$ bounded by hypersurfaces and such that each sequence corresponding to a point of $\mathcal{R}$ is infinitely log-concave. In Section 3, we indicate how our methods can be used to give a simple derivation of one of their main theorems. We investigate infinite log-concavity of the columns and other lines of Pascal's triangle in Section 4. Section 5 is devoted to two q-analogues of the binomial coefficients. For the Gaussian polynomials, we show that certain analogues of some infinite log-concavity conjectures are false while others appear to be true. In contrast, our second q-analogue seems to retain all the log-concavity properties of the binomial coefficients. In Section 6, after showing why the sequence $(h_k)_{k\geq 0}$ of complete homogeneous symmetric is an appropriate analogue of sequences of binomial coefficients, we explore its log-concavity properties. We end with a section of related results and questions about real-rooted polynomials and Toeplitz matrices.

While one purpose of this article is to present our results, we have written it with two more targets in mind. The first is to convince our audience that infinite log-concavity is a fundamental concept. We hope that its definition as a natural extension of traditional log-concavity helps to make this case. Our second aspiration is to attract others to work on the subject; to that end, we have presented several open problems. These conjectures each represent fundamental questions in the area, so even solutions of special cases may be interesting.

2 Rows of Pascal's triangle

One of the difficulties with proving the Boros-Moll conjecture is that log-concavity is not preserved by the $\mathcal{L}$-operator. For example, the sequence $4, 5, 4$ is log-concave but $\mathcal{L}(4, 5, 4) = 16, 9, 16$ is not. So we will seek a condition stronger than log-concavity which is preserved by $\mathcal{L}$. Given $r \in \mathbb{R}$, we say that a sequence (a_k) is *r-factor log-concave* if

$$a_k^2 \geq r a_{k-1} a_{k+1} \tag{2.1}$$

for all k. Clearly this implies log-concavity if $r \geq 1$.

We seek an $r > 1$ such that (a_k) being r-factor log-concave implies that $(b_k) = \mathcal{L}(a_k)$ is as well. Assume the original sequence is nonnegative. Then expanding $r b_{k-1} b_{k+1} \leq b_k^2$ in terms of the a_k and rearranging the summands, we see that this is equivalent to proving

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_k^2 a_{k+1} \leq a_k^4 + r a_{k-2} a_k (a_{k+1}^2 - a_k a_{k+2}) + r a_{k-1}^2 a_k a_{k+2}.$$

By our assumptions, the two expressions with factors of r on the right are nonnegative, so it suffices to prove the inequality obtained when these are dropped. Applying (2.1) to the left-hand side gives

$$(r-1)a_{k-1}^2 a_{k+1}^2 + 2a_{k-1}a_k^2 a_{k+1} \leq \frac{r-1}{r^2} a_k^4 + \frac{2}{r} a_k^4.$$

So we will be done if

$$\frac{r-1}{r^2} + \frac{2}{r} = 1.$$

Finding the root $r_0 > 1$ of the corresponding quadratic equation finishes the proof of the first assertion of the following lemma, while the second assertion follows easily from the first.

Lemma 2.1 *Let (a_k) be a nonnegative sequence and let $r_0 = (3+\sqrt{5})/2$. Then (a_k) being r_0-factor log-concave implies that $\mathcal{L}(a_k)$ is too. So in this case (a_k) is infinitely log-concave.* $\square$

Now to prove that any row of Pascal's triangle is infinitely log-concave, one merely lets a computer find $\mathcal{L}_k^i \binom{n}{k}$ for i up to some bound I. If these sequences are all nonnegative and $\mathcal{L}_k^I \binom{n}{k}$ is r_0-factor log-concave, then the previous lemma shows that this row is infinitely log-concave. Using this technique, we have obtained the following theorem.

Theorem 2.2 *The sequence $\binom{n}{k}_{k\geq 0}$ is infinitely log-concave for all $n \leq 1450$.* $\square$

We note that the necessary value of I increases slowly with increasing n. As an example, when $n = 100$, our technique works with $I = 5$, while for $n = 1000$, we need $I = 8$.

Of course, the method developed in this section can be applied to any sequence such that $\mathcal{L}^i(a_k)$ is r_0-factor log-concave for some i. In particular, it is interesting to try it on the original sequence which motivated Boros and Moll (4) to define infinite log-concavity. They were studying the polynomial

$$P_m(x) = \sum_{\ell=0}^{m} d_\ell(m) x^\ell \tag{2.2}$$

where

$$d_\ell(m) = \sum_{j=\ell}^{m} 2^{j-2m} \binom{2m-2j}{m-j} \binom{m+j}{m} \binom{j}{\ell}.$$

Kauers [private communication] has used our technique to verify infinite log-concavity of the sequence $(d_\ell(m))_{\ell \geq 0}$ for $m \leq 129$. For such values of m, $\mathcal{L}_\ell^5$ applied to the sequence is r_0-factor log-concave.

3 A region of infinite log-concavity

Uminsky and Yeats (25) took a different approach to the Boros-Moll Conjecture as described in the Introduction. Since they were motivated by the rows of Pascal's triangle, they only considered real sequences $a_0, a_1, \ldots, a_n$ which are symmetric (in that $a_k = a_{n-k}$ for all k) and satisfy $a_0 = a_n = 1$. Each such sequence corresponds to a point $(a_1, \ldots, a_m) \in \mathbb{R}^m$ where $m = \lfloor n/2 \rfloor$.

Their region, $\mathcal{R}$, whose points all correspond to infinitely log-concave sequences, is bounded by m parametrically defined hypersurfaces. The parameters are x and $d_1, d_2, \ldots, d_m$ and it will be convenient to have the notation

$$s_k = \sum_{i=1}^{k} d_i.$$

We will also need $r_1 = (1+\sqrt{5})/2$. Note that $r_1^2 = r_0$. The kth hypersurface, $1 \le k < m$, is defined as

$$\begin{aligned}\mathcal{H}_k = \{(x^{s_1}, \ldots, x^{s_{k-1}}, r_1 x^{s_k}, x^{s_{k+1}+d_k-d_{k+1}}, \ldots, x^{s_m+d_k-d_{k+1}}) :\\ x \ge 1, \quad 1 = d_1 > \cdots > d_k > d_{k+2} > \cdots > d_m > 0\},\end{aligned}$$

while

$$\mathcal{H}_m = \{(x^{s_1}, \ldots, x^{s_{m-1}}, c x^{s_{m-1}}) :\ x \ge 1, \quad 1 = d_1 > \cdots > d_{m-1} > 0\},$$

where

$$c = \begin{cases} r_1 & \text{if } n = 2m, \\ 2 & \text{if } n = 2m+1. \end{cases}$$

Let us say that the *correct side* of $\mathcal{H}_k$ for $1 \le k \le m$ consists of those points in $\mathbb{R}^m$ that can be obtained from a point on $\mathcal{H}_k$ by increasing the kth coordinate. Then let $\mathcal{R}$ be the region of all points in $\mathbb{R}^m$ having increasing coordinates and lying on the correct side of $\mathcal{H}_k$ for all k. Our ideas of the previous section can be used to give a simple proof of one of Uminsky and Yeats' main theorems.

Theorem 3.1 ((25)) *Any sequence corresponding to a point of $\mathcal{R}$ is infinitely log-concave.*

The proof relies on the fact that, since $r_1^2 = r_0$, the conditions for containment in $\mathcal{R}$ are very close to the conditions for r_0-factor log-concavity.

4 Columns and other lines of Pascal's triangle

While we have treated Boros and Moll's statement about the infinite log-concavity of the binomial coefficients to be a statement about the rows of Pascal's triangle, their wording also suggests an examination of the columns.

Conjecture 4.1 *The sequence $\binom{n}{k}_{n \ge k}$ is infinitely log-concave for all fixed $k \ge 0$.*

We will give two pieces of evidence for this conjecture. First, it is not difficult to show infinite log-concavity for specific small values of k.

Proposition 4.2 *The sequence $\binom{n}{k}_{n \ge k}$ is infinitely log-concave for $0 \le k \le 2$.*

Secondly, some careful analysis shows that $\mathcal{L}_n^i \binom{n}{k}$ is nonnegative for certain values of i and all k.

Proposition 4.3 *The sequence $\mathcal{L}_n^i \binom{n}{k}$ is nonnegative for all k and for $0 \le i \le 4$.*

Kauers and Paule (13) proved that the rows of Pascal's triangle are i-fold log-concave for $i \leq 5$. Kauers [private communication] has used their techniques to confirm Proposition 4.3 and to also check the case $i = 5$ for the columns. For the latter case, Kauers used a computer to determine

$$\frac{(\mathcal{L}_n^5 \binom{n}{k})}{\binom{n}{k}^{2^5}} \tag{4.1}$$

explicitly, which is just a rational function in n and k. He then showed that (4.1) is nonnegative by means of cylindrical algebraic decomposition. We refer the interested reader to (13) and the references therein for more information on such techniques.

More generally, we can look at an arbitrary line in Pascal's triangle, i.e., consider the sequence

$$\binom{n+mu}{k+mv}_{m \geq 0}.$$

The unimodality and (1-fold) log-concavity of such sequences has been investigated in (3; 22; 23; 24). We do not require that u and v be coprime, so such sequences need not contain all of the binomial coefficients in which a geometric line would intersect Pascal's triangle, e.g., a sequence such as $\binom{n}{0}, \binom{n}{2}, \binom{n}{4}, \ldots$ would be included. By letting $u < 0$, one can get a finite truncation of a column. For example, if $n = 5$, $k = 3$, $u = -1$, and $v = 0$ then we get the sequence

$$\binom{5}{3}, \binom{4}{3}, \binom{3}{3}$$

which is not even 2-fold log-concave. So we will only consider $u \geq 0$. Also

$$\binom{n+mu}{k+mv} = \binom{n+mu}{n-k+m(u-v)}$$

so we can also assume $v \geq 0$.

We offer the following conjecture, which includes Conjecture 1.1 as a special case.

Conjecture 4.4 *Suppose that u and v are distinct nonnegative integers. Then $\binom{n+mu}{mv}_{m \geq 0}$ is infinitely log-concave for all $n \geq 0$ if and only if $u < v$ or $v = 0$.*

We first give a quick proof of the "only if" direction. Supposing that $u > v \geq 1$, we consider the sequence

$$\binom{0}{0}, \binom{u}{v}, \binom{2u}{2v}, \ldots$$

obtained when $n = 0$. We claim that this sequence is not even log-concave and that log-concavity fails at the second term. Indeed, the fact that $\binom{u}{v}^2 < \binom{2u}{2v}$ follows immediately from the identity

$$\binom{u}{0}\binom{u}{2v} + \binom{u}{1}\binom{u}{2v-1} + \cdots + \binom{u}{v}\binom{u}{v} + \cdots + \binom{u}{2v}\binom{u}{0} = \binom{2u}{2v},$$

which is a special case of Vandermonde's Convolution.

The proof just given shows that subsequences of the columns of Pascal's triangle are the only *infinite* sequences of the form $\binom{n+mu}{mv}_{m\geq 0}$ that can possibly be infinitely log-concave. We also note that the previous conjecture says nothing about what happens on the diagonal $u = v$. Of course, the case $u = v = 1$ is Conjecture 4.1. For other diagonal values, the evidence is conflicting. One can show by computer that $\binom{n+mu}{mu}_{m\geq 0}$ is not 4-fold log-concave for $n = 2$ and any $2 \leq u \leq 500$. However, this is the only *known* value of n for which $\binom{n+mu}{mu}_{m\geq 0}$ is not an infinitely log-concave sequence for some $u \geq 1$.

We conclude this section by offering considerable computational evidence in favor of the "if" direction of Conjecture 4.4. Theorem 2.2 provides such evidence when $u = 0$ and $v = 1$. Since all other sequences with $u < v$ have a finite number of nonzero entries, we can use the r_0-factor log-concavity technique for these sequences as well. For all $n \leq 500$, $2 \leq v \leq 20$ and $0 \leq u < v$, we have checked that $\binom{n+mu}{mv}_{m\geq 0}$ is infinitely log-concave.

5 q-analogues

This section will be devoted to discussing two q-analogues of binomial coefficients. For the Gaussian polynomials, we will see that the corresponding generalization of Conjecture 1.1 is false, and we show one exact reason why it fails. In contrast, the corresponding generalization of Conjecture 4.1 appears to be true. This shows how delicate these conjectures are and may in part explain why they seem to be difficult to prove. After introducing our second q-analogue, we conjecture that the corresponding generalizations of Conjectures 1.1, 4.1 and 4.4 are all true. This second q-analogue arises in the study of quantum groups; see, for example, the books of Jantzen (12) and Majid (17).

Let q be a variable and consider a polynomial $f(q) \in \mathbb{R}[q]$. Call $f(q)$ *q-nonnegative* if all the coefficients of $f(q)$ are nonnegative. Apply the $\mathcal{L}$-operator to sequences of polynomials $(f_k(q))$ in the obvious way. Call such a sequence *q-log-concave* if $\mathcal{L}(f_k(q))$ is a sequence of q-nonnegative polynomials, with i-fold q-log-concavity and infinite q-log-concavity defined similarly.

We will be particularly interested in the Gaussian polynomials. The standard *q-analogue of the nonnegative integer n* is

$$[n] = [n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Then, for $0 \leq k \leq n$, the *Gaussian polynomials* or *q-binomial coefficients* are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

where $[n]_q! = [1]_q[2]_q \cdots [n]_q$. For more information, including proofs of the assertions made in the next paragraph, see the book of Andrews (2).

Clearly substituting $q = 1$ gives $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$. Also, it is well known that the Gaussian polynomials are indeed q-nonnegative polynomials. In fact, they have various combinatorial interpretations, one of which we use. An (integer) *partition of n* is a weakly decreasing positive integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $|\lambda| \stackrel{\text{def}}{=} \sum_i \lambda_i = n$. The λ_i are called *parts*. We say that λ *fits inside an $s \times t$ box* if $\lambda_1 \leq t$ and $\ell \leq s$. Denote the set of all such partitions by $P(s, t)$. It is well known, and easy to prove by induction on n, that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\lambda \in P(n-k,k)} q^{|\lambda|}. \tag{5.1}$$

Using this combinatorial interpretation, we can prove that the q-analogue of the rows of Pascal's triangle are not 2-fold q-log-concave. More specifically, we have the following result. From this point on, we use the notation $L(a_k)$ for the kth element of the sequence $\mathcal{L}(a_k)$, and similarly for L_k and L_n.

Proposition 5.1 *For $n \geq 2$ and $k = \lfloor n/2 \rfloor$ we have*

$$L_k^2\left(\begin{bmatrix} n \\ k \end{bmatrix}\right) = -q^{n-2} + \textit{higher order terms}.$$

Consequently, $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{k\geq 0}$ is not 2-fold q-log-concave.

Given this, it may seem surprising that the following conjecture, which is a q-analogue of Conjecture 4.1, does seem to hold.

Conjecture 5.2 *The sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n\geq k}$ is infinitely q-log-concave for all fixed $k \geq 0$.*

As evidence, we give a q-analogue of Proposition 4.2 and an appropriate adaption of Proposition 4.3. The case $i = 2$ of Proposition 5.3(b) corresponds to the q-log-concavity of the q-Narayana numbers and is a result of (8).

Proposition 5.3

(a) The sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n\geq k}$ is infinitely q-log-concave for $0 \leq k \leq 2$.

(b) The sequence $\mathcal{L}_n^i\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)$ is q-nonnegative for all k and for $0 \leq i \leq 2$.

We conclude our discussion of the Gaussian polynomials by considering the sequence

$$\left(\begin{bmatrix} n+mu \\ mv \end{bmatrix}\right)_{m\geq 0} \tag{5.2}$$

for nonnegative integers u and v, as we did in Section 4 for the binomial coefficients. When $u > v$ the sequence has an infinite number of nonzero entries. We can use (5.1) to show that the highest degree term in $\begin{bmatrix} n+u \\ v \end{bmatrix}^2 - \begin{bmatrix} n+2u \\ 2v \end{bmatrix}$ has coefficient -1, so the sequence (5.2) is not even q-log-concave. When $u < v$, it seems to be the case that the sequence is not 2-fold q-log-concave, as shown for the rows in Proposition 5.1. When $u = v$, the evidence is conflicting, reflecting the behavior of the binomial coefficients. Since setting $q = 1$ in $\begin{bmatrix} n+mu \\ mu \end{bmatrix}$ yields $\binom{n+mu}{mu}$, we know that $\left(\begin{bmatrix} 2+mu \\ mu \end{bmatrix}\right)_{m\geq 0}$ is not always 4-fold q-log-concave. It also transpires that the case $n = 3$ is not always 5-fold q-log-concave. We have not encountered other values of n that fail to yield a q-log-concave sequence when $u = v$.

While the variety of behavior of the Gaussian polynomials is interesting, it would be desirable to have a q-analogue that better reflects the behavior of the binomial coefficients. A q-analogue that arises in the study of quantum groups serves this purpose. Let us replace the previous q-analogue of the nonnegative integer n with the expression

$$\langle n \rangle = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{1-n} + q^{3-n} + q^{5-n} + \cdots + q^{n-1}.$$

From this, we obtain a q-analogue of the binomial coefficients by proceeding as for the Gaussian polynomials: for $0 \leq k \leq n$, we define

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \frac{\langle n \rangle!}{\langle k \rangle! \langle n-k \rangle!}$$

where $\langle n\rangle! = \langle 1\rangle\langle 2\rangle\cdots\langle n\rangle$.

Letting $q \to 1$ in $\left\langle {n \atop k} \right\rangle$ gives $\binom{n}{k}$, and a straightforward calculation shows that

$$\left\langle {n \atop k} \right\rangle = \frac{1}{q^{nk-k^2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}. \tag{5.3}$$

So $\left\langle {n \atop k} \right\rangle$ is , in general, a Laurent polynomial in q with nonnegative coefficients. Our definitions of q-nonnegativity and q-log-concavity for polynomials in q extend to Laurent polynomials in the obvious way.

We offer the following generalizations of Conjectures 1.1, 4.1 and 4.4.

Conjecture 5.4

(a) The row sequence $\left(\left\langle {n \atop k} \right\rangle\right)_{k\geq 0}$ is infinitely q-log-concave for all $n \geq 0$.

(b) The column sequence $\left(\left\langle {n \atop k} \right\rangle\right)_{n\geq k}$ is infinitely q-log-concave for all fixed $k \geq 0$.

(c) For all integers $0 \leq u < v$, the sequence $\left(\left\langle {n+mu \atop mv} \right\rangle\right)_{m\geq 0}$ is infinitely q-log-concave for all $n \geq 0$.

Several remarks are in order. Suppose that for $f(g), g(q) \in \mathbb{R}[q, q^{-1}]$, we say $f(q) \leq g(q)$ if $g(q) - f(q)$ is q-nonnegative. Then the r-factor log-concavity ideas of Section 2 carry over to this setting, once we replace the term "log-concave" by "q-log-concave." Using these ideas, we have verified Conjecture 5.4(a) for all $n \leq 53$. Even though (a) is a special case of (c), we state it separately since (a) is the q-generalization of the Boros-Moll conjecture, the primary motivation for this paper. As evidence for Conjecture 5.4(b), it is not hard to prove the appropriate analogue of Proposition 5.3. Conjecture 5.4(c) has been verified for all $n \leq 24$ with $v \leq 10$. When $u > v$, we can use (5.3) to show that the lowest degree term in $\left\langle {n+u \atop v} \right\rangle^2 - \left\langle {n+2u \atop 2v} \right\rangle$ has coefficient -1, so the sequence is not even q-log-concave. When $u = v$, the quantum groups analogue has exactly the same behavior as we observed above for the Gaussian polynomials.

6 Symmetric functions

We now turn our attention to symmetric functions. We will demonstrate that the complete homogeneous symmetric functions $(h_k)_{k\geq 0}$ are a natural analogue of the rows and columns of Pascal's triangle. We show that the sequence $(h_k)_{k\geq 0}$ is i-fold log-concave in the appropriate sense for $i \leq 3$, but not 4-fold log-concave. Like the results of Section 5, this result underlines the difficulties and subtleties of Conjectures 1.1 and 4.1. In particular, it shows that any proof of Conjecture 1.1 or Conjecture 4.1 would need to use techniques that do not carry over to the sequence $(h_k)_{k\geq 0}$. For a more detailed exposition of the background material below, we refer the reader to the texts of Macdonald (16), Sagan (19) or Stanley (21).

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. For each $n \geq 0$, the elements of the symmetric group $\mathfrak{S}_n$ act on formal power series $f(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]]$ by permutation of variables (where x_i is left fixed if $i > n$). The algebra of symmetric functions, $\Lambda(\mathbf{x})$, is the set of all series left fixed by all symmetric groups and of bounded (total) degree.

The vector space of symmetric functions homogeneous of degree k has dimension equal to the number of partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of k. We will be interested in three bases for this vector space. The *monomial symmetric function* corresponding to λ, $m_\lambda = m_\lambda(\mathbf{x})$, is obtained by symmetrizing the monomial $x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$. The kth *complete homogeneous symmetric function*, h_k, is the sum of all monomials of degree k. For partitions, we then define

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}.$$

Finally, the *Schur function* corresponding to λ is

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq \ell}.$$

Our interest will be in the sequence just mentioned $(h_k)_{k \geq 0}$. Let $h_k(1^n)$ denote the integer obtained by substituting $x_1 = \cdots = x_n = 1$ and $x_i = 0$ for $i > n$ into $h_k = h_k(\mathbf{x})$. Then $h_k(1^n) = \binom{n+k-1}{k}$ (the number of ways of choosing k things from n things with repetition) and so the above sequence becomes a column of Pascal's triangle. By the same token $h_k(1^{n-k}) = \binom{n-1}{k}$ and so the sequence becomes a row.

For notational convenience, if a part k is repeated r times in a partition λ then we will denote this by writing k^r in the sequence for λ. Also, when we use λ as a subscript we will omit the parentheses. We need a result of Kirillov (14) about the product of Schur functions, which was proved bijectively by Kleber (15) and Fulmek and Kleber (11). This result can be obtained by applying the Desnanot-Jacobi Identity—also known as Dodgson's condensation formula—to the Jacobi-Trudi matrix for $s_{k^{r+1}}$.

Theorem 6.1 ((11; 14; 15)) *For positive integers k, r we have*

$$(s_{k^r})^2 - s_{(k-1)^r} s_{(k+1)^r} = s_{k^{r-1}} s_{k^{r+1}}.$$

To state our result, we need one more definition. If b_λ is a basis for $\Lambda(\mathbf{x})$ and $f \in \Lambda(\mathbf{x})$ then we say f is b_λ*-nonnegative* if the coefficient of b_λ in the expansion of f is nonnegative for all partitions λ. Note that m_λ-nonnegativity is the natural generalization to many variables of the q-nonnegativity definition for $\mathbb{R}[q]$. A well-known example of an m_λ-nonnegative symmetric function is s_μ, for any partition μ. Thus s_λ-nonnegativity implies m_λ-nonnegativity.

Theorem 6.2 *The sequence $\mathcal{L}^i(h_k)$ is s_λ-nonnegative for $0 \leq i \leq 3$. But the sequence $\mathcal{L}^4(h_k)$ is not m_λ-nonnegative.*

The proof involves determining $\mathcal{L}^i(h_k)$ explicitly for $0 \leq i \leq 3$, using Theorem 6.1 and various standard facts about symmetric functions to manipulate the expressions into sums of products of Schur functions; such sums are are always s_λ-nonnegative. By focussing on a suitable term in the expression for $L^4(h_k)$, one obtains the second assertion of the theorem.

7 Real roots and Toeplitz matrices

We now consider two other settings where, in contrast to the results of the previous section, Conjecture 1.1 does seem to generalize. In fact, this may be the right level of generality to find a proof.

Let $(a_k) = a_0, a_1, \ldots, a_n$ be a finite sequence of nonnegative real numbers. It was shown by Isaac Newton that if all the roots of the polynomial $p[a_k] \stackrel{\text{def}}{=} a_0 + a_1 x + \cdots a_n x^n$ are real, then the sequence (a_k) is log-concave. For example, since the polynomial $(1+x)^n$ has only real roots, the nth row of Pascal's triangle is log-concave. It is natural to ask if the real-rootedness property is preserved by the $\mathcal{L}$-operator. The literature includes a number of results about operations on polynomials which preserve real-rootedness; for example, see (5; 6; 7; 18; 26; 27).

Conjecture 7.1 *Let (a_k) be a finite sequence of nonnegative real numbers. If $p[a_k]$ has only real roots then the same is true of $p[\mathcal{L}(a_k)]$.*

This conjecture is due independently to Richard Stanley [private communication]. It is also one of a number of related conjectures made by Steve Fisk (10). If true, Conjecture 7.1 would immediately imply the original Boros-Moll Conjecture. As evidence for the conjecture, we have verified it by computer for a large number of randomly chosen real-rooted polynomials. We have also checked that $p[\mathcal{L}_k^i\binom{n}{k}]$ has only real roots for all $i \leq 10$ and $n \leq 40$.

Along with the rows of Pascal's triangle, it appears that applying $\mathcal{L}$ to the other finite lines we were considering in Section 4 also yields sequences with real-rooted generating functions. So we make the following conjecture which implies the "if" direction of Conjecture 4.4.

Conjecture 7.2 *For $0 \leq u < v$, the polynomial $p[\mathcal{L}_m^i(\binom{n+mu}{mv})]$ has only real roots for all $i \geq 0$.*

We have verified this assertion for all $n \leq 24$ with $i \leq 10$ and $v \leq 10$. In fact, it follows from a theorem of Yu (28) that the conjecture holds for $i = 0$ and all $0 \leq u < v$. So it will suffice to prove Conjecture 7.1 to obtain this result for all i.

We can obtain a matrix-theoretic perspective on problems of real-rootedness via the following renowned result of Aissen, Schoenberg and Whitney (1). A matrix A is said to be totally nonnegative if every minor of A is nonnegative. We can associate with any sequence (a_k) a corresponding (infinite) *Toeplitz matrix* $A = (a_{j-i})_{i,j\geq 0}$. In comparing the next theorem to Newton's result, note that for a real-rooted polynomial $p[a_k]$ the roots being nonpositive is equivalent to the sequence (a_k) being nonnegative.

Theorem 7.3 ((1)) *Let (a_k) be a finite sequence of real numbers. Then every root of $p[a_k]$ is a nonpositive real number if and only if the Toeplitz matrix $(a_{j-i})_{i,j\geq 0}$ is totally nonnegative.* □

To make a connection with the $\mathcal{L}$-operator, note that

$$a_k^2 - a_{k-1}a_{k+1} = \begin{vmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{vmatrix},$$

which is a minor of the Toeplitz matrix $A = (a_{j-i})_{i,j\geq 0}$. Call such a minor *adjacent* since its entries are adjacent in A. Now, for an arbitrary infinite matrix $A = (a_{i,j})_{i,j\geq 0}$, let us define the infinite matrix $\mathcal{L}(A)$ by

$$\mathcal{L}(A) = \left(\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} \right)_{i,j\geq 0}.$$

Note that if A is the Toeplitz matrix of (a_k) then $\mathcal{L}(A)$ is the Toeplitz matrix of $\mathcal{L}(a_k)$. Using Theorem 7.3, Conjecture 7.1 can now be strengthened as follows.

Conjecture 7.4 *For a sequence (a_k) of real numbers, if $A = (a_{j-i})_{i,j\geq 0}$ is totally nonnegative then $\mathcal{L}(A)$ is also totally nonnegative.*

Note that if (a_k) is finite, then Conjecture 7.4 is equivalent to Conjecture 7.1. As regards evidence for Conjecture 7.4, consider an arbitrary n-by-n matrix $A = (a_{i,j})_{i,j=1}^n$. For finite matrices, $\mathcal{L}(A)$ is defined in the obvious way to be the $(n-1)$-by-$(n-1)$ matrix consisting of the 2-by-2 adjacent minors of A. In (9, Theorem 6.5), Fallat, Herman, Gekhtman, and Johnson show that for $n \leq 4$, $\mathcal{L}(A)$ is totally nonnegative whenever A is. However, for $n = 5$, an example from their paper can be modified to show that if

$$A = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ t & t^2+1 & 2t & t^2 & 0 \\ t^2 & t^3+2t & 1+4t^2 & 2t^3+t & 0 \\ 0 & t^2 & 2t^3+2t & t^4+2t^2+1 & t \\ 0 & 0 & t^2 & t^3+t & t^2 \end{pmatrix}$$

then A is totally nonnegative for $t \geq 0$, but $\mathcal{L}(A)$ is not totally nonnegative for sufficiently large t ($t \geq \sqrt{2}$ will suffice). We conclude that the Toeplitz structure would be important to any affirmative answer to Conjecture 7.4.

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Triangulations of root polytopes and reduced forms (Extended abstract)

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Abstract. The type A_n root polytope $\mathcal{P}(A_n^+)$ is the convex hull in $\mathbb{R}^{n+1}$ of the origin and the points $e_i - e_j$ for $1 \leq i < j \leq n+1$. Given a tree T on vertex set $[n+1]$, the associated root polytope $\mathcal{P}(T)$ is the intersection of $\mathcal{P}(A_n^+)$ with the cone generated by the vectors $e_i - e_j$, where $(i,j) \in E(T)$, $i < j$. The reduced forms of a certain monomial $m[T]$ in commuting variables x_{ij} under the reduction $x_{ij}x_{jk} \rightarrow x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}$, can be interpreted as triangulations of $\mathcal{P}(T)$. If we allow variables x_{ij} and x_{kl} to commute only when i, j, k, l are distinct, then the reduced form of $m[T]$ is unique and yields a canonical triangulation of $\mathcal{P}(T)$ in which each simplex corresponds to a noncrossing alternating forest.

Résumé. Le polytope des racines $\mathcal{P}(A_n^+)$ de type A_n est l'enveloppe convexe dans $\mathbb{R}^{n+1}$ de l'origine et des points $e_i - e_j$ pour $1 \leq i < j \leq n+1$. Étant donné un arbre T sur l'ensemble des sommets $[n+1]$, le polytope des racines associé, $\mathcal{P}(T)$, est l'intersection de $\mathcal{P}(A_n^+)$ avec le cône engendré par les vecteurs $e_i - e_j$, où $(i,j) \in E(T)$, $i < j$. Les formes réduites d'un certain monôme $m[T]$ en les variables commutatives x_{ij} sous la reduction $x_{ij}x_{jk} \rightarrow x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}$ peuvent être interprétées comme des triangulations de $\mathcal{P}(T)$. Si on impose la restriction que les variables x_{ij} et x_{kl} commutent seulement lorsque les indices i, j, k, l sont distincts, alors la forme réduite de $m[T]$ est unique et produit une triangulation canonique de $\mathcal{P}(T)$ dans laquelle chaque simplexe correspond à une forêt alternée non croisée.

Keywords: root polytope, triangulation, volume, Ehrhart polynomial, reduced form, noncrossing alternating tree

1 Introduction

This work was initially inspired by an exercise in Stanley's Catalan Addendum (S, Exercise 6.C6), which calls on us to consider the monomial $w = x_{12}x_{23}\cdots x_{n,n+1}$ in commuting variables x_{ij}. Starting with $p_0 = w$, produce a sequence of polynomials $p_0, p_1, \ldots, p_m$ as follows. To obtain p_{r+1} from p_r, choose a term of p_r which is divisible by $x_{ij}x_{jk}$, for some i, j, k, and replace the factor $x_{ij}x_{jk}$ in this term with $x_{ik}(x_{ij} + x_{jk})$. Note that p_{r+1} has one more term than p_r. Continue this process until a polynomial p_m is obtained, in which no term is divisible by $x_{ij}x_{jk}$, for any i, j, k. Such a polynomial p_m is a **reduced form** of w. Remarkably, while the reduced form is not unique, it turns out that the number of terms in a reduced form is always the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Certain generalizations of this problem reach far beyond its setting in the world of polynomials. On one hand, the reductions can be interpreted in terms of root polytopes and their subdivisions, yielding a geometric, and subsequently a combinatorial, interpretation of reduced forms. On the other hand, using the combinatorial results obtained about the reduced forms, we obtain a method for calculating the volumes and Ehrhart polynomials of certain root polytopes. This "two way traffic" between the combinatorics of reduced forms and geometry is a satisfying outcome.

Root polytopes were defined by Postnikov in (P). The full root polytope $\mathcal{P}(A_n^+)$, which is the convex hull in $\mathbb{R}^{n+1}$ of the origin and points $e_i - e_j$ for $1 \leq i < j \leq n+1$, already made an appearance in the work of Gelfand, Graev and Postnikov (GGP), who gave a canonical triangulation of it in terms of noncrossing alternating trees on $[n+1]$. We obtain canonical triangulations for all acyclic root polytopes, of which $\mathcal{P}(A_n^+)$ is a special case.

We define **acyclic root polytopes** $\mathcal{P}(T)$ for a tree T on vertex set $[n+1]$ as the intersection of $\mathcal{P}(A_n^+)$ with a cone generated by the vectors $e_i - e_j$, where $(i,j) \in E(T)$, $i<j$. Let

$$\bar{G} = ([n+1], \{(i,j) \mid \text{there exist edges } (i,i_1)\ldots,(i_k,j) \text{ in } G \text{ such that } i < i_1 < \ldots < i_k < j\}),$$

denote the **transitive closure** of the graph G. Recall that a graph G on vertex set $[n+1]$ is said to be **noncrossing** if there are no vertices $i<j<k<l$ such that (i,k) and (j,l) are edges in G. A graph G on vertex set $[n+1]$ is said to be **alternating** if there are no vertices $i<j<k$ and (i,j) and (j,k) are edges in G. Alternating trees were introduced in (GGP). Gelfand, Graev and Postnikov (GGP) showed that the number of noncrossing alternating trees on $[n+1]$ is counted by the Catalan number C_n.

Theorem 1 *If T is a noncrossing tree on vertex set $[n+1]$ and $T_1,\ldots,T_k$ are the noncrossing alternating spanning trees of $\bar{T}$, then the root polytopes $\mathcal{P}(T_1),\ldots,\mathcal{P}(T_k)$ are n-dimensional simplices with disjoint interiors whose union is $\mathcal{P}(T)$. Furthermore,*

$$\mathrm{vol}\mathcal{P}(T) = f_{T,n}\frac{1}{n!},$$

where $f_{T,n}$ denotes the number of noncrossing alternating spanning trees of $\bar{T}$.

We can describe the intersections of the top dimensional simplices $\mathcal{P}(T_1),\ldots,\mathcal{P}(T_k)$ in Theorem 1 in terms of certain noncrossing alternating spanning forests of $\bar{T}$ and using this calculate the Ehrhart polynomial of the polytope $\mathcal{P}(T)$. Theorem 1 can also be generalized for any tree T, but we omit these details in this abstract.

The proof of Theorem 1 relies on relating the triangulations of a root polytope $\mathcal{P}(T)$ to reduced forms of a monomial $m[T]$ in variables x_{ij}, which we now define. Let A and B be two associative algebras over the polynomial ring $\mathbb{Q}[\beta]$, where β is a fixed constant, generated by the set of elements $\{x_{ij} \mid 1 \leq i < j \leq n+1\}$ modulo the relation $x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}$. Algebra A is commutative, i.e. it has additional relations $x_{ij}x_{kl} = x_{kl}x_{ij}$ for all i,j,k,l, while B is noncommutative and has additional relations $x_{ij}x_{kl} = x_{kl}x_{ij}$ for i,j,k,l distinct only.

We treat the first relation as a **reduction rule**:

$$x_{ij}x_{jk} \to x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}. \tag{1}$$

A **reduced form** of the monomial m in algebra A (algebra B) is a polynomial P_n^A (polynomial P_n^B) obtained by successive applications of reduction (1) until no further reduction is possible, where we allow commuting any two variables (commuting any two variables x_{ij} and x_{kl} where i, j, k, l are distinct) between reductions. Note that the reduced forms are not necessarily unique.

A possible sequence of reductions in algebra A yielding a reduced form of $x_{12}x_{23}x_{34}$ is given by

$$\begin{aligned} x_{12}\boldsymbol{x_{23}x_{34}} &\to \boldsymbol{x_{12}}x_{24}\boldsymbol{x_{23}} + \boldsymbol{x_{12}}x_{34}\boldsymbol{x_{24}} + \beta\boldsymbol{x_{12}x_{24}} \\ &\to \boldsymbol{x_{24}}x_{13}\boldsymbol{x_{12}} + x_{24}x_{23}x_{13} + \beta x_{24}x_{13} + x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} \\ &\quad +\beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} + \beta^2 x_{14} \\ &\to x_{13}x_{14}x_{12} + x_{13}x_{24}x_{14} + \beta x_{13}x_{14} + x_{24}x_{23}x_{13} + \beta x_{24}x_{13} \\ &\quad +x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} \\ &\quad +\beta^2 x_{14} \end{aligned} \tag{2}$$

where the pair of variables on which the reductions are performed is in boldface. The reductions are performed on each monomial separately.

Some of the reductions performed above are not allowed in the noncommutative algebra B. The following is an example of how to reduce $x_{12}x_{23}x_{34}$ in the noncommutative case.

$$\begin{aligned} x_{12}\boldsymbol{x_{23}x_{34}} &\to \boldsymbol{x_{12}x_{24}}x_{23} + \underline{x_{12}x_{34}}x_{24} + \beta\boldsymbol{x_{12}x_{24}} \\ &\to x_{14}\boldsymbol{x_{12}x_{23}} + x_{24}x_{14}x_{23} + \beta x_{14}x_{23} + x_{34}\boldsymbol{x_{12}x_{24}} + \beta x_{14}x_{12} \\ &\quad +\beta x_{24}x_{14} + \beta^2 x_{14} \\ &\to x_{14}x_{13}x_{12} + x_{14}x_{23}x_{13} + \beta x_{14}x_{13} + x_{24}x_{14}x_{23} + \beta x_{14}x_{23} \\ &\quad +x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} \\ &\quad +\beta^2 x_{14} \end{aligned} \tag{3}$$

In the example above the pair of variables on which the reductions are performed is in boldface, and the variables which we commute are underlined.

The "reason" for allowing x_{ij} and x_{kl} to commute only when i, j, k, l are distinct might not be apparent at first, but as we will prove in section 5 it insures that, unlike in the commutative case, there are unique reduced forms for a natural set of monomials. Kirillov (K) observed previously that the monomial $w = x_{12}x_{23}\cdots x_{n,n+1}$ has a unique reduced form in algebra B and asked for a nice combinatorial proof of this fact. We provide such a proof, as the uniqueness of the reduced form of w is a special case of our results.

Before we can state a simplified version of our main result on reduced forms, we need one more piece of notation. Given a graph G on vertex set $[n+1]$ we associate to it the monomial $m^A[G] = \prod_{(i,j)\in E(G)} x_{ij}$; if G is edge-labeled with labels $1, \ldots, k$, we can also associate to it the noncommutative monomial $m^B[G] = \prod_{a=1}^{k} x_{i_a,j_a}$, where $E(G) = \{(i_a, j_a)_a \mid a \in [k]\}$.

Theorem 2 *Let T be a noncrossing tree on vertex set $[n+1]$, and P_n^A a reduced form of $m^A[T]$. Then,*

$$P_n^A(x_{ij} = 1, \beta = 0) = f_{T,n},$$

where $f_{T,n}$ denotes the number of noncrossing alternating spanning trees of $\bar{T}$.

If we label the edges of T so that it becomes a good tree (to be defined in section 4), then the reduced form P_n^B of the monomial $m^B[T]$ is

$$P_n^B(x_{ij}, \beta = 0) = \sum_{T_0} x^{T_0},$$

where the sum runs over all noncrossing alternating spanning trees T_0 of $\bar{T}$ with lexicographic edge-labels (to be defined in section 5) and x^{T_0} is defined to be the noncommutative monomial $\prod_{l=1}^{n} x_{i_l, j_l}$ if T_0 contains the edges $(i_1, j_1)_1, \ldots, (i_n, j_n)_n$.

We can generalize Theorem 2 for any β; see sections 2 and 5. Theorem 2 can also be generalized for any tree T, but we omit these details in this abstract.

Our extended abstract is organized as follows. In section 2 we reformulate the reduction process in terms of graphs and elaborate further on Theorem 2. In section 3 we discuss acyclic root polytopes and relate them to reduced forms. In section 3 we also outline some of the steps for proving Theorems 1 and 2. The lemmas in section 4 indicate the significance of considering reduced forms in the noncommutative algebra B and prepares the ground for proving Theorem 2. In section 5 we conclude the proofs of Theorems 1 and 2.

2 Reductions in terms of graphs

We can phrase the reduction process described in section 1 in terms of graphs. This view will be useful throughout the abstract. Think of a monomial $m \in A$ as a directed graph G on the vertex set $[n+1]$ with an edge directed from i to j for each appearance of x_{ij} in m. Let $G^A[m]$ denote this graph. If, however, we are in the noncommutative version of the problem, and $m = \prod_{l=1}^{p} x_{i_l, j_l}$, then we can think of m as a directed graph G on the vertex set $[n+1]$ with p edge labels $1, \ldots, p$, such that the edge labeled l is directed from vertex i_l to j_l. Let $G^B[m]$ denote the edge-labeled graph just described. Let $(i,j)_a$ be the notation for an edge (i,j) labeled a. It is straightforward to reformulate the reduction rule (1) in terms of reductions on graphs. If $m \in A$, then it reads as follows.

The **reduction rule for graphs:** Given a graph G_0 on vertex set $[n+1]$ and $(i,j), (j,k) \in E(G_0)$ for some $i < j < k$, let G_1, G_2, G_3 be graphs on vertex set $[n+1]$ with edge sets

$$\begin{aligned} E(G_1) &= E(G_0) \backslash \{(j,k)\} \cup \{(i,k)\}, \\ E(G_2) &= E(G_0) \backslash \{(i,j)\} \cup \{(i,k)\}, \\ E(G_3) &= E(G_0) \backslash \{(i,j)\} \backslash \{(j,k)\} \cup \{(i,k)\}. \end{aligned} \tag{4}$$

We say that G_0 **reduces** to G_1, G_2, G_3 under the reduction rules defined by equations (4).

The reduction rule for graphs $G^B[m]$ with $m \in B$ is explained in section 4.

An A**-reduction tree** $\mathcal{T}_R^A$ for a monomial m_0, or equivalently, the graph $G^A[m_0]$, is constructed as follows. The root of $\mathcal{T}_R^A$ is labeled by $G^A[m_0]$. Each node $G^A[m]$ in $\mathcal{T}_R^A$ has three children, which depend on the choice of the edges of $G^A[m]$ on which we perform the reduction. Namely, if the reduction is performed on edges $(i,j), (j,k) \in E(G^A[m])$, then the three children of the node $G_0 = G^A[m]$ are labeled by the graphs G_1, G_2, G_3 as described by equation (4). For an example of an A-reduction tree, see Figure 1 (disregard the edge-labels).

Summing the monomials to which the graphs labeling the leaves of the reduction tree $\mathcal{T}_R^A$ correspond multiplied by suitable powers of β, we obtain a reduced form of m_0.

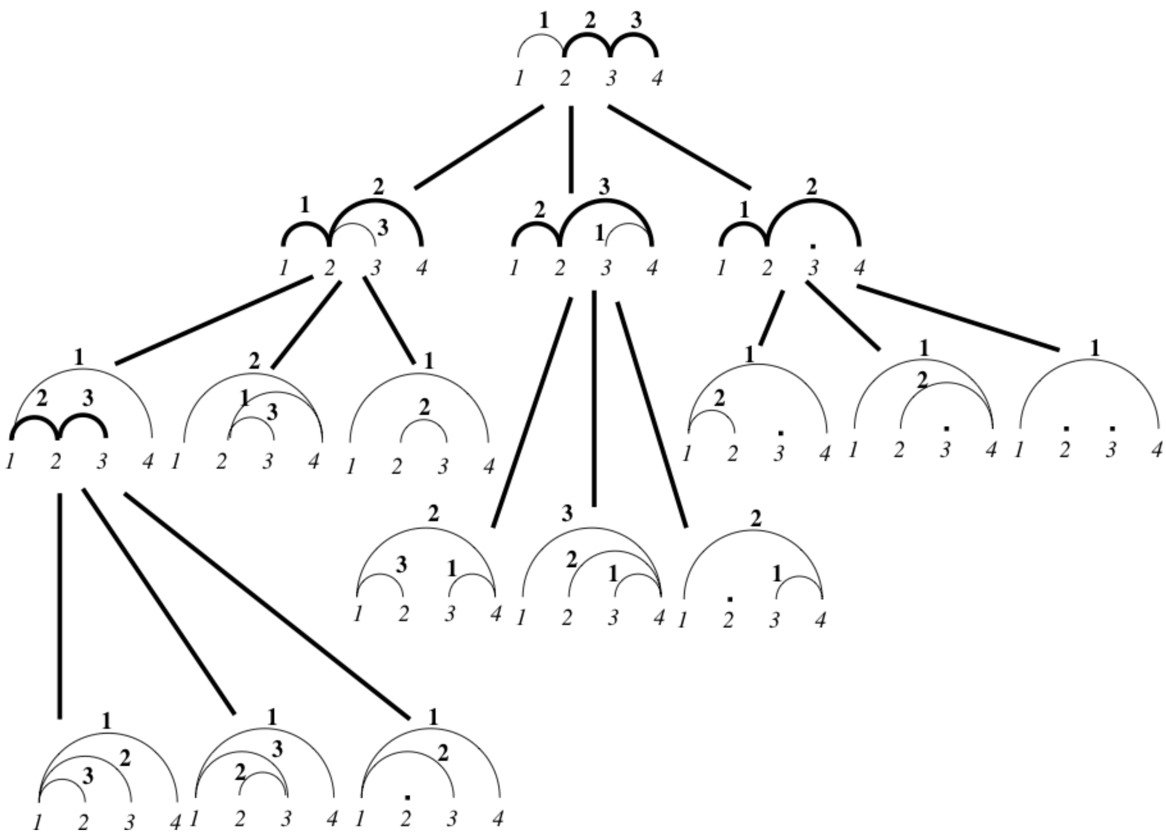

Fig. 1: This is an A-reduction tree with root $m^A[x_{12}x_{23}x_{34}]$, when the edge-labels are disregarded. The boldface edges indicate where the reduction is performed. We can read off the following reduced form of $x_{12}x_{23}x_{34}$ from the set of leaves: $x_{14}x_{13}x_{12} + x_{14}x_{23}x_{13} + \beta x_{14}x_{13} + x_{24}x_{14}x_{23} + \beta x_{14}x_{23} + x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} + \beta^2 x_{14}$. When the edge-labels are taken into account, this is the B-reduction tree corresponding to equation (3). Note that in the second child of the root we commuted edge-labels 1 and 2.

Let T be a noncrossing tree on vertex set $[n+1]$. In terms of reduction trees, Theorem 2 states that the number of leaves labeled by graphs with exactly n edges of an A-reduction tree with root labeled T is independent of the particular A-reduction tree. The generalization of Theorem 2 states that the number of leaves labeled by graphs with exactly k edges of an A-reduction tree with root labeled T, is independent of the particular A-reduction tree for any k. In terms of reduced forms we can write this as follows. If P_n^A is the reduced form of a monomial $m^A[T]$ for a noncrossing tree T, then

$$P_n^A(x_{ij}=1) = \sum_{m=0}^{n-1} f_{T,n-m}\beta^m,$$

where $f_{T,k}$ denotes the number of noncrossing alternating spanning forests of $\bar{T}$ with k edges and additional technical requirements. Also, if P_n^B is the reduced form of a monomial $m^B[T]$ for a noncrossing good tree T (defined in section 4), then

$$P_n^B(x_{ij}) = \sum_F x^F,$$

where the sum runs over all noncrossing alternating spanning forests F of $\bar{T}$ with lexicographic edge-labels (defined in section 5) and additional technical requirements.

If we consider the reduced forms of the path monomial $w = \prod_{i=1}^n x_{i,i+1}$, then $T = P = ([n+1], \{(i,i+1) \mid i \in [n]\})$, and $f_{P,k}$ is simply the number of noncrossing alternating spanning forests on $[n+1]$ with k edges containing edge $(1,n+1)$. Furthermore, $P_n^B(x_{ij}) = \sum_F x^F$, where the sum runs over all noncrossing alternating spanning forests F on $[n+1]$ with lexicographic edge-labels and containing edge $(1,n+1)$. Let $s_{n,k}$ denote the number of ways to draw k diagonals of a convex $(n+2)$-gon that do not intersect in their interiors.

Lemma 3 *With the notation above,* $f_{P,k+1} = s_{n,k}$.

Cayley (C) in 1890 showed that $s_{n,k} = \dfrac{1}{n+1}\dbinom{n+k+1}{n}\dbinom{n-1}{k}$. Using Lemma 3 and the expression by Cayley, we obtain $P_n^A(x_{ij}=1) = \sum_{m=0}^{n-1} s_{n,n-m-1}\beta^m$.

3 Acyclic root polytopes

In the terminology of (P), a root polytope of type A_n is the convex hull of the origin and some of the points $e_i - e_j$ for $1 \le i < j \le n+1$, where e_i denotes the i^{th} coordinate vector in $\mathbb{R}^{n+1}$. A very special root polytope is the full root polytope

$$\mathcal{P}(A_n^+) = \text{ConvHull}(0, e_{ij}^- \mid 1 \le i < j \le n+1),$$

where $e_{ij}^- = e_i - e_j$. We study a class of root polytopes including $\mathcal{P}(A_n^+)$, which we now discuss.

Let G be a connected simple graph on vertex set $[n+1]$. Define

$$\mathcal{V}_G = \{e_{ij}^- \mid (i,j) \in E(G), i<j\}, \text{ a set of vectors associated to } G;$$

$$\mathcal{C}_G = \langle \mathcal{V}_G \rangle := \{ \sum_{e_{ij}^- \in \mathcal{V}_G} c_{ij} e_{ij}^- \mid c_{ij} \ge 0\}, \text{ the cone associated to } G; \text{ and}$$

$$\bar{\mathcal{V}}_G = \Phi^+ \cap \mathcal{C}_G, \text{ all the positive roots of type } A_n \text{ contained in the cone associated to } G,$$

where $\Phi^+ = \{e^-_{ij} \mid 1 \leq i < j \leq n+1\}$ is the set of positive roots of type A_n. The idea to consider the positive roots of a root system inside a cone appeared earlier in Reiner's work (R1), (R2) on signed posets.

The root polytope $\mathcal{P}(G)$ associated to graph G is

$$\mathcal{P}(G) = \text{ConvHull}(0, e^-_{ij} \mid e^-_{ij} \in \bar{\mathcal{V}}_G).$$

Note that $\mathcal{P}(A_n^+) = \mathcal{P}(P)$ for the path graph $P = ([n], \{(i, i+1) \mid i \in [n]\})$. While the choice of G such that $\mathcal{P}(A_n^+) = \mathcal{P}(G)$ is not unique, it becomes unique if we require that for no edge $(i,j) \in E(G)$ can the corresponding vector e^-_{ij} be written as a nonnegative linear combination of the vectors corresponding to the edges $E(G)\backslash\{e\}$. Graph P satisfies this requirement.

Define

$$\mathcal{L}_n = \{G = ([n+1], E(G)) \mid \ G \text{ is an acyclic graph}\},$$

and

$$\mathcal{L}(A_n^+) = \{\mathcal{P}(G) \mid G \in \mathcal{L}_n\}, \text{the set of } \textbf{acyclic root polytopes}.$$

Note that the condition that G is an acyclic graph is equivalent to $\mathcal{V}_G$ being a set of linearly independent vectors. To avoid too much detail, in this extended abstract we restrict out attention to the acyclic root polytopes arising from noncrossing trees, however, our methods yield analogous results for all acyclic root polytopes.

The full root polytope $\mathcal{P}(A_n^+) \in \mathcal{L}(A_n^+)$, since the path graph P is acyclic. We show below how to obtain central triangulations for all polytopes $\mathcal{P} \in \mathcal{L}(A_n^+)$. A **central triangulation** of a d-dimensional root polytope $\mathcal{P}$ is a collection of d-dimensional simplices with disjoint interiors whose union is $\mathcal{P}$, the vertices of which are vertices of $\mathcal{P}$ and the origin is a vertex of all of them. Depending on the context we also take the intersections of these maximal simplices to be part of the triangulation.

Lemma 4 *For an acyclic root polytope $\mathcal{P}(G) = \mathcal{C}_G \cap \mathcal{P}(A_n^+)$.*

We now state the crucial lemma which relates root polytopes and algebras A and B defined in section 2.

Lemma 5 ***(Reduction Lemma)*** *Given a graph $G_0 \in \mathcal{L}_n$ with d edges let $(i,j),(j,k) \in E(G_0)$ for some $i < j < k$ and G_1, G_2, G_3 as described by equation (4). Then $G_1, G_2, G_3 \in \mathcal{L}_n$,*

$$\mathcal{P}(G_0) = \mathcal{P}(G_1) \cup \mathcal{P}(G_2)$$

where all polytopes $\mathcal{P}(G_0), \mathcal{P}(G_1), \mathcal{P}(G_2)$ are d-dimensional and

$$\mathcal{P}(G_3) = \mathcal{P}(G_1) \cap \mathcal{P}(G_2) \textit{ is } (d-1)\textit{-dimensional.}$$

What the Reduction Lemma really says is that performing a reduction on graph $G_0 \in \mathcal{L}_n$ is the same as "cutting" the d-dimensional polytope $\mathcal{P}(G_0)$ into two d-dimensional polytopes $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$, whose vertex sets are subsets of the vertex set of $\mathcal{P}(G_0)$, whose interiors are disjoint and whose union is $\mathcal{P}(G_0)$.

Proof Idea of Theorems 1 and 2: Let T be any tree on vertex set $[n+1]$ and consider any A-reduction tree $\mathcal{T}_R^A$ with root T. For simplicity only consider the nodes labeled by graphs with n edges, which corresponds to setting $\beta = 0$. Let $T_1, \dots, T_k$ be the trees with n edges labeling leaves of $\mathcal{T}_R^A$. Then, by applying the Reduction Lemma multiple times, we obtain that $\mathcal{P}(T_1), \dots, \mathcal{P}(T_k)$ are d-dimensional polytopes, whose vertex sets are subsets of the vertex set of $\mathcal{P}(T)$, whose interiors are disjoint and whose union is $\mathcal{P}(T)$. Clearly then

$$\mathrm{vol}\mathcal{P}(T) = \mathrm{vol}\mathcal{P}(T_1) + \cdots + \mathrm{vol}\mathcal{P}(T_k).$$

Since $T_1, \dots, T_k$ are leaves of $\mathcal{T}_R^A$, they must be alternating. Postnikov showed (P, Lemma 13.2) that for an alternating tree T_i on vertex set $[n+1]$, the polytope $\mathcal{P}(T_i)$ a simplex with $\mathrm{vol}\mathcal{P}(T_i) = \frac{1}{n!}$. His definitions in (P) differ from ours, but for this alternating tree case his Lemma 13.2 can be reformulated suitably. Thus, we obtain that $\mathrm{vol}\mathcal{P}(T) = \frac{k}{n!}$, where k is the number of leaves of $\mathcal{T}_R^A$ labeled by trees. Note, this shows that the number of leaves of an A-reduction tree labeled by trees is independent of which A-reduction tree we consider. As explained in section 2, summing the monomials to which the graphs labeling the leaves of a reduction tree $\mathcal{T}_R^A$ correspond multiplied by suitable powers of β, we obtain a reduced form of the monomial corresponding to the root of $\mathcal{T}_R^A$. Thus, we just showed that if P_n^A is this reduced form, then $P_n^A(x_{ij} = 1, \beta = 0) = k$, which is part of the statement of Theorem 2. What k exactly is, how to obtain the canonical triangulation described in Theorem 1 and how to express explicitly the reduced form of $m^B[T]$ as stated in Theorem 2 is all outlined in section 5. □

4 Reductions in the noncommutative case

In this section we state two crucial lemmas about reduction (1) in the noncommutative case necessary for proving Theorem 2. While in the commutative case reductions on $G^A[m]$ could result in crossing graphs, we prove that in the noncommutative case exactly those reductions from the commutative case are allowed which result in no crossing graphs, provided that $m = m^B[T]$ for a noncrossing tree T with suitable edge labels specified below. Furthermore, we also show that if there are any two edges (i, j) and (j, k) with $i < j < k$ in a successor of $G^B[m]$, then after suitably many commutations it is possible to apply reduction (1). Thus, once the reduction process terminates, the set of graphs obtained as leaves of the reduction tree are alternating forests. Now, unlike in the commutative case, they are also noncrossing. In fact, each noncrossing alternating spanning forest of $\bar{T}$ satisfying certain additional technical conditions occurs among the leaves of the reduction tree exactly once, yielding a complete combinatorial description of the reduced form of $m^B[T]$.

In terms of graphs the partial commutativity means that if G contains two edges $(i, j)_a$ and $(k, l)_{a+1}$ with i, j, k, l distinct, then we can replace these edges by $(i, j)_{a+1}$ and $(k, l)_a$, and vice versa. Reduction rule (1) on the other hand means that if there are two edges $(i, j)_a$ and $(j, k)_{a+1}$ in G_0, then we replace G_0 with three graphs G_1, G_2, G_3 on vertex set $[n+1]$ and edge sets

$$\begin{aligned}
E(G_1) &= E(G_0)\backslash\{(i,j)_a\}\backslash\{(j,k)_{a+1}\} \cup \{(i,k)_a\} \cup \{(i,j)_{a+1}\} \\
E(G_2) &= E(G_0)\backslash\{(i,j)_a\}\backslash\{(j,k)_{a+1}\} \cup \{(j,k)_a\} \cup \{(i,k)_{a+1}\} \\
E(G_3) &= (E(G_0)\backslash\{(i,j)_a\}\backslash\{(j,k)_{a+1}\})^a \cup \{(i,k)_a\}, \qquad (5)
\end{aligned}$$

where $(E(G_0)\backslash\{(i,j)_a\}\backslash\{(j,k)_{a+1}\})^a$ denotes the edges obtained from the edges $E(G_0)\backslash\{(i,j)_a\}\backslash\{(j,k)_{a+1}\}$ by reducing the label of each edge which has label greater than a by 1.

A B-**reduction tree** $\mathcal{T}_R^B$ is defined analogously to an A-reduction tree, except we use equation (5) to describe the children. See Figure 1 for an example. A graph H is called a B-**successor** of G if it is obtained by a series of reductions from G. For convenience, we refer to commutativity of x_{ij} and x_{kl} for distinct i, j, k, l as **reduction (2)**, by which we mean the rule $x_{ij}x_{kl} \leftrightarrow x_{kl}x_{ij}$, for i, j, k, l distinct, or, in the language of graphs, exchanging edges $(i, j)_a$ and $(k, l)_{a+1}$ with $(i, j)_{a+1}$ and $(k, l)_a$ for i, j, k, l distinct.

A forest H on vertex set $[n + 1]$ and k edges labeled $1, \ldots, k$ is **good** if it satisfies the following conditions:

(i) If edges $(i, j)_a$ and $(j, k)_b$ are in H, $i < j < k$, then $a < b$.

(ii) If edges $(i, j)_a$ and $(i, k)_b$ in H are such that $j < k$, then $a > b$.

(iii) If edges $(i, j)_a$ and $(k, j)_b$ in H are such that $i < k$, then $a > b$.

(iv) H is noncrossing.

No graph H with a cycle could satisfy all of $(i), (ii), (iii), (iv)$ simultaneously, which is why we only define good forests. Note, however, that any forest H has an edge-labeling that makes it a good forest.

Lemma 6 *If the root of a B-reduction tree is labeled by a good forest F, then all nodes of it are also labeled by good forests.*

A reduction applied to a noncrossing graph G is **noncrossing** if the graphs resulting from the reduction are also noncrossing.

The following is then a trivial corollary of Lemma 6.

Corollary 7 *If G is a good forest, then all reductions that can be applied to G and its B-successors are noncrossing.*

Lemma 8 *Let G be a good forest. Let $(i, j)_a$ and $(j, k)_b$ with $i < j < k$ be edges of G such that no edge of G crosses (i, k). Then after finitely many applications of reduction (2) we can apply reduction (1) to edges (i, j) and (j, k).*

Corollary 9 *If F labels a leaf of a B-reduction tree whose root is labeled by a good forest, then F is a noncrossing alternating forest.*

5 Proof Idea of Theorems 1 and 2

This section is devoted to understanding how to conclude the proofs of Theorems 1 and 2 started at the end of section 3. We first finish the sketch of the proof of Theorem 2, and then conclude with Theorem 1.

To prove Theorem 2 we describe the reduced form of $m^B[T]$ for a good graph T, which, unlike in the commutative case, is unique. For simplicity we lay out the exact details for the monomial $w_B = \prod_{i=1}^{n} x_{i,i+1}$. We index w by B to indicate that we are in the noncommutative algebra B.

Given a noncrossing alternating forest F on vertex set $[n+1]$ with k edges, the **lexicographic order** on its edges is as follows. Edge (i_1, j_1) is less than edge (i_2, j_2) in lexicographic order if $j_1 > j_2$, or $j_1 = j_2$ and $i_1 > i_2$. The forest F is said to have **lexicographic edge-labels** if its edges are labeled with integers $1, \ldots, k$ such that if edge (i_1, j_1) is less than edge (i_2, j_2) in lexicographic order, then the label of (i_1, j_1) is less than the label of (i_2, j_2) in the usual order on the integers. Clearly, given any graph G there is a unique edge-labeling of it which is lexicographic.

Lemma 10 *If a noncrossing alternating forest F is a B-successor of a good forest T, then upon some number of reductions (2) it is possible to obtain a noncrossing alternating forest F' with lexicographic edge-labels.*

Proposition 11 *By choosing the series of reductions suitably, the set of leaves of a B-reduction tree with root $G^B[w_B]$ can be all noncrossing alternating forests F on vertex set $[n+1]$ containing edge $(1, n+1)$ with lexicographic edge-labels.*

Idea of Proof: By Corollary 9 all leaves of a B-reduction tree are noncrossing alternating forests on vertex set $[n+1]$. It is easily seen that they all contain edge $(1, n+1)$. By the correspondence between the leaves of a B-reduction tree and simplices in a triangulation of $\mathcal{P}(G^B[w_B])$, it follows that no forest appears more than once among the leaves. Thus, it suffices to prove that any noncrossing alternating forest F on vertex set $[n+1]$ containing edge $(1, n+1)$ appears among the leaves of a B-reduction tree and that all these forests have lexicographic edge-labels. One can construct such a B-reduction tree inductively. □

Theorem 12 *The set of leaves of a B-reduction tree with root $G^B[w_B]$ is, up to applications of reduction (2), the set of all noncrossing alternating forests with lexicographic edge-labels on the vertex set $[n+1]$ containing edge $(1, n+1)$.*

Idea of Proof: By Proposition 11 there exists a B-reduction tree which satisfies the conditions above. Since the roots of type A_n are unimodular, it can be shown that the number of k-dimensional simplices in a central triangulation of a type A_n root polytope is fixed for any k. Thus, the number of forests on vertex set $[n+1]$ and k edges among the leaves of an B-reduction tree is fixed. Also, no vertex-labeled forest, with edge-labels disregarded, can appear twice among the leaves of a B-reduction tree. Together with Lemma 10 these imply the statement of Theorem 12. □

Using Theorem 12 we obtain the following characterziation of reduced forms of the noncommutative monomial w_B.

Theorem 13 *If the polynomial $P_n^B(x_{ij})$ is a reduced form of w_B, then*

$$P_n^B(x_{ij}) = \sum_F \beta^{n-|E(F)|} x^F,$$

where the sum runs over all noncrossing alternating forests F with lexicographic edge-labels on the vertex set $[n+1]$ containing edge $(1, n+1)$, and x^F is defined to be the noncommutative monomial $\prod_{l=1}^{k} x_{i_l, j_l}$ if F contains the edges $(i_1, j_1)_1, \ldots, (i_k, j_k)_k$.

As a corollary to Theorem 13 we obtain the other part of Theorem 2 for the commutative monomial $w_A = \prod_{i=1}^{n} x_{i,i+1}$.

Theorem 14 *If the polynomial $P_n^A(x_{ij})$ is a reduced form of w_A, then*

$$P_n^A(x_{ij} = 1) = \sum_{m=0}^{n-1} s_{n,n-m-1} \beta^m,$$

where $s_{n,k}$ is the number of noncrossing alternating forests on vertex set $[n+1]$ with $k+1$ edges, containing edge $(1, n+1)$.

Idea of Proof: While w_A may have many reduced forms, any reduced form arises from an A-reduction tree, which in turn gives a triangulation of $\mathcal{P}(A_n^+)$. A triangulation of $\mathcal{P}(A_n^+)$ can be shown to have a fixed number of simplices of a certain dimension, using that the positive roots of type A_n are unimodular. Using this it can be shown that there is a fixed number of leaves with k edges in any A-reduction tree. Using Theorem 13 we obtain that there are $s_{n,k}$ leaves with with $k+1$ edges in any A-reduction tree. □

Observe that the above theorems imply that the poset of all noncrossing alternating spanning forests on the vertex set $[n+1]$ containing the edge $(1, n+1)$ equals the face poset of the triangulation of the full type A_n root polytope $\mathcal{P}(A_n^+)$ obtained from the noncommutative process as described in Theorem 12. By face poset we mean the poset whose elements are the top dimensional simplices in the triangulation of $\mathcal{P}(A_n^+)$ and all their nonempty intersections and the order is given by inclusion.

The **Schröder numbers** s_n count the number of ways to draw any number of diagonals of a convex $(n+2)$-gon that do not intersect in their interiors. Recall that in section 2 $s_{n,k}$ denoted the number of ways to draw k diagonals of a convex $(n+2)$-gon that do not intersect in their interiors. Cayley (C) in 1890 showed that $s_{n,k} = \frac{1}{n+1}\binom{n+k+1}{n}\binom{n-1}{k}$. As indicated in Lemma 3, it is not by coincidence that we used $s_{n,k}$ to also denote the number of noncrossing alternating forests on vertex set $[n+1]$ and $k+1$ edges, containing edge $(1, n+1)$.

Theorems 13 and 14 imply Theorem 2 for the special case $T = P = ([n+1], \{(i, i+1) \mid i \in [n]\})$. We can generalize Theorems 12, 13 and 14 to monomials $m^B[T]$, where T is a good tree. Theorem 2 stated in the Introduction is a weaker version of these generalizations, but is easier to state. In the most general statements of Theorems 12, 13 and 14 we need to replace the condition "all noncrossing alternating forests on $[n+1]$ containing edge $(1, n+1)$" with "all noncrossing alternating forests on $[n+1]$ containing edge $(1, n+1)$ and certain technical requirements," the details of which we omit here. The proofs of the analogous statements use the statements of Theorems 12, 13 and 14 as base cases. If the polynomial $P_n^B(x_{ij})$ is a reduced form of $m^B[T]$ for a good tree T, then

$$P_n^B(x_{ij}) = \sum_F \beta^{n-|E(F)|} x^F,$$

where the sum runs over all noncrossing alternating spanning forests F of $\bar{T}$ with lexicographic edge-labels on the vertex set $[n+1]$ containing edge $(1, n+1)$ and satisfying some technical requirement. Also,

$$P_n^A(x_{ij} = 1) = \sum_{m=0}^{n-1} f_{T,n-m} \beta^m,$$

where $f_{T,n-m}$ is the number of noncrossing alternating forests on vertex set $[n+1]$ with $n-m$ edges, containing edge $(1, n+1)$ and satisfying some technical requirement.

We are now ready conclude the proof of Theorem 1. Recall that at the end of section 3 we proved that if $T_1, \ldots, T_k$ are the trees labeling leaves of $\mathcal{T}_R^A$ with root T, then $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ form a central triangulation of $\mathcal{P}(T)$. Note that the set of leaves of a B-reduction tree $\mathcal{T}_R^B$ can also be obtained as a set of leaves of some A-reduction tree $\mathcal{T}_R^A$, by simply disregarding the edge-labels of the graphs corresponding to the nodes of $\mathcal{T}_R^B$. The generalization of Theorem 12 implies that that the set of leaves of a B-reduction tree with root $m^B[T]$ which are trees are all noncrossing alternating spanning trees of $\bar{T}$ with lexicographic ordering. Thus, there is an A-reduction tree with root $m^A[T]$ whose leaves that are trees are all noncrossing alternating spanning trees of $\bar{T}$. Therefore, if $T_1, \ldots, T_k$ are all noncrossing alternating spanning trees of $\bar{T}$, then the root polytopes $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ are n-dimensional simplices with disjoint interiors whose union is $\mathcal{P}(T)$, yielding the canonical triangulation described in Theorem 1. Also, from this triangulation it follows that $\mathrm{vol}\mathcal{P}(T) = f_{T,n}\frac{1}{n!}$, where $f_{T,n}$ denotes the number of noncrossing alternating spanning trees of $\bar{T}$, since as noted at the end of section 3 each $\mathcal{P}(T_i)$ has volume $\frac{1}{n!}$. This concludes the proof of Theorem 1.

Theorem 1 can be generalized so that we not only describe the n-dimensional simplices in the triangulation of $\mathcal{P}(T)$, but also describe their intersections in terms of noncrossing alternating spanning forests in $\bar{T}$. Using the special property of Φ^+ that the vectors in it are unimodular, we can also calculate the Ehrhart polynomial of $\mathcal{P}(T)$ for any tree T. We now define Ehrhart polynomials for integer polytopes, and state our main result pertaining to them. For further background on the theory of Ehrhart polynomials see (BR).

Given a polytope $\mathcal{P} \subset \mathbb{R}^{n+1}$, the t^{th} **dilate** of $\mathcal{P}$ is

$$t\mathcal{P} = \{(tx_1, \ldots, tx_{n+1}) | (x_1, \ldots, x_{n+1}) \in \mathcal{P}\}.$$

The **Ehrhart polynomial** of an integer polytope $\mathcal{P} \subset \mathbb{R}^{n+1}$ is then defined to be

$$L_{\mathcal{P}}(t) = \#(t\mathcal{P} \cap \mathbb{Z}^{n+1}).$$

Theorem 15 *The Ehrhart polynomial of the polytope $\mathcal{P}(T)$, where T is a noncrossing tree on vertex set $[n+1]$, is*

$$L_{\mathcal{P}(T)}(t) = (-1)^n \sum_{i=0}^{n} (-1)^i f_{T,i} \binom{t+i}{i},$$

where $f_{T,i}$ is the number of noncrossing alternating forests on vertex set $[n+1]$ with i edges, containing edge $(1, n+1)$ and satisfying some technical requirement.

For $T = P = ([n], \{(i, i+1) \mid i \in [n]\})$ Theorem 15 specializes to the Ehrhart polynomial of $\mathcal{P}(P) = \mathcal{P}(A_n^+)$ with $f_{P,i} = s_{n,i-1}$. The Ehrhart polynomial of $\mathcal{P}(A_n^+)$ was previously calculated by Fong (F) by different methods.

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Bijective Enumeration of Bicolored Maps of Given Vertex Degree Distribution

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Abstract. We derive a new formula for the number of factorizations of a full cycle into an ordered product of two permutations of given cycle types. For the first time, a purely combinatorial argument involving a bijective description of bicolored maps of specified vertex degree distribution is used. All the previous results in the field rely either partially or totally on a character theoretic approach. The combinatorial proof relies on a new bijection extending the one in [G. Schaeffer and E. Vassilieva. *J. Comb. Theory Ser. A*, 115(6):903–924, 2008] that focused only on the number of cycles. As a salient ingredient, we introduce the notion of thorn trees of given vertex degree distribution which are recursive planar objects allowing simple description of maps of arbitrary genus.

Résumé. Nous démontrons une nouvelle formule exprimant le nombre de factorisations d'un long cycle en produit de deux permutations ayant un type cyclique donné. Pour la première fois, nous utilisons un argument purement combinatoire basé sur une description bijective des cartes bicolores dont la distribution des degrés des sommets est donnée. Tous les résultats précédents dans le domaine se basent soit partiellement soit totalement sur la théorie des caractères de groupe. La preuve combinatoire se fonde sur une nouvelle bijection généralisant celle introduite dans [G. Schaeffer and E. Vassilieva. *J. Comb. Theory Ser. A*, 115(6):903–924, 2008] ne s'intéressant qu'au nombre de cycles. L'ingrédient le plus saillant est l'introduction de la notion d'arbre épineux de structure cyclique donnée, des objets récursifs et planaires permettant une description simple des cartes de genus arbitraire.

Keywords: bicolored maps, full cycle factorization, vertex degree distribution

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1 Full cycle factorizations of given type

In what follows, we denote by $\lambda \vdash n$ an integer partition of n and $\ell(\lambda) = k$ the length or number of parts of λ. Thus, $\lambda = (\lambda_1, \ldots, \lambda_k)$ where $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ and $\sum \lambda_i = n$. If $m_i(\lambda)$ is the number of parts of λ that are equal to i, then we also write λ as $[1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \ldots]$ and let $Aut(\lambda) = \prod_i m_i(\lambda)!$. Then, for $\lambda \vdash n$, we use the monomial symmetric function $m_\lambda(x)$ which is the sum of all different monomials obtained by permuting the variables of $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$, and the power symmetric function $p_\lambda(x)$, defined multiplicatively as $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$ where $p_n(x) = m_n(x) = \sum_i x_i^n$ (see e.g. (10)).

In addition, we use π to denote a set partition of $[n] = \{1, 2, \ldots, n\}$ with blocks $\{\pi^1, \ldots, \pi^p\}$. The type of a set partition, $type(\pi) \vdash n$, is the integer partition of n obtained by considering the cardinalities of the blocks of π. Let S_n be the symmetric group on n elements, and $\mathcal{C}_\lambda$ be the conjugacy class in S_n of permutations with cycle type λ, where $\lambda \vdash n$. Given $\lambda, \mu, \nu \vdash n$, let $c_{\lambda\mu}^{\nu}$ be the number of ordered factorizations in S_n of a fixed permutation $\gamma \in \mathcal{C}_\nu$ as a product $\alpha\beta$ of two permutations $\alpha \in \mathcal{C}_\lambda$ and $\beta \in \mathcal{C}_\mu$. These numbers are called *connection coefficients* of the symmetric group. The problem of computing these coefficients has received significant attention and its history and references can be found in (3). We focus on the case $c_{\lambda\mu}^{n}$: when $\nu = (n) = n$ and γ is the long cycle $\gamma_n = (1, 2, \ldots, n)$. In this setting, we define the genus $g(\lambda, \mu)$ of a pair of partitions λ and μ by the equation $\ell(\lambda) + \ell(\mu) = n + 1 - 2g(\lambda, \mu)$. We can take $g(\lambda, \mu)$ to be a nonnegative integer, since otherwise it is easy to show that $c_{\lambda\mu}^{n} = 0$.

Regarding the evaluation of $c_{\lambda\mu}^{n}$, using an inductive argument Bédard and Goupil (1) first found a formula for the case $g(\lambda, \mu) = 0$, which was later reproved by Goulden and Jackson (2) via a bijection with a set of ordered rooted bicolored trees. Later, using characters of the symmetric group and a combinatorial development, Goupil and Schaeffer (3) derived an expression for connection coefficients of arbitrary genus as a sum of positive terms (see Biane (4) for a succinct algebraic derivation; and Poulalhon and Schaeffer (4), and Irving (7) for further generalizations). However, there are no fully bijective proofs for this expression unless the permutations are associated to set partitions (see Goulden and Nica (6)). In this paper, we follow the latter approach and introduce the notion of *partitioned bicolored maps* of given type to extend the work of Schaeffer and Vassilieva in (9), and derive a novel simpler formula thanks to a purely combinatorial argument. In the genus zero case, the argument reduces to the bijection with ordered rooted bicolored trees in (2). Our combinatorial result can be stated as follows:

Theorem 1 *The numbers $c_{\lambda\mu}^{n}$ of factorizations of the long cycle γ_n into an ordered product of two permutations of type λ and μ satisfy:*

$$\frac{1}{n} \sum_{\lambda,\mu \vdash n} c_{\lambda\mu}^{n} p_\lambda(x) p_\mu(y) = \sum_{\lambda,\mu \vdash n} \frac{(n - \ell(\lambda))!(n - \ell(\mu))!}{(n + 1 - \ell(\lambda) - \ell(\mu))!} m_\lambda(x) m_\mu(y) \tag{1}$$

We start in Section 2 by defining the main combinatorial structures used throughout the paper and reformulating Theorem 1. In Section 3 we introduce a mapping among these structures and in Section 4 we show it is a bijection.

2 Link with maps and partitioned maps of specified type

2.1 Unicellular partitioned bicolored maps

Definition 1 *Given partitions $\lambda, \mu \vdash n$, let $\mathcal{C}(\lambda, \mu)$ be the set of triples (π_1, π_2, α) such that $\alpha \in S_n$, π_1, π_2 are set partitions of $[n]$ with $type(\pi_1) = \lambda$ and $type(\pi_2) = \mu$, and each block of π_1 and π_2 is a union of cycles of α and $\beta = \alpha^{-1}\gamma_n$ respectively. Let $C(\lambda, \mu) = |\mathcal{C}(\lambda, \mu)|$.*

In accordance with the following graphical interpretation, each factorization $\alpha\beta = \gamma_n$ is called a *unicellular bicolored map*. Similarly, each triple (π_1, π_2, α) is called a *unicellular partitioned bicolored map* with n edges, blocks of type λ consisting of white vertices, and blocks of type μ consisting of black vertices. We represent these unicellular maps graphically using *ribbon graphs*:

A ribbon graph is a drawing of a graph in the plane such that any vertex of degree k has a neighborhood homeomorphic to a disk and the incident edges form a star with k branches. Edges can cross outside of these neighborhoods, and we call these *irrelevant crossings*. Moreover, given an edge e of a ribbon graph we assume it is oriented and identify the right hand side of e in the direction given by the orientation. We describe the boundary of the graph by moving along the edges. In doing so, we ignore irrelevant crossings and when we reach a vertex, we continue in the next branch of the star without crossing any edge in the neighborhood of the vertex. A graph is bicolored if its vertices are colored black and white such that each edge connects a black vertex with a white one. In this case the *right hand side of the edge* is the side of the edge which is on the right when going from a white vertex to a black one.

Then a unicellular partitioned bicolored map with n edges is represented by a labeled ribbon graph with the following six properties: (i) n edges with the labels $[n] = \{1, \ldots, n\}$, (ii) the cycles of α describe the white vertices, and (iii) the partition π_1 induces a partition of the white vertices of type λ . Properties (iv), (v), and (vi) are the analogues of (i), (ii), and (iii) for β, black vertices, π_2 and μ respectively. The fact that $\alpha\beta = \gamma_n$ means that if we start on the right hand side of the edge with label 1 and move along the edges as described above, we visit the right hand side of the edges $1, \ldots, n$ in this order.

Example 1 *Let* $n = 10$, $\lambda = [4, 6]$, $\mu = [2^2, 6]$, $\alpha = (2\,5)(3\,4\,6\,7)(1\,8\,9\,10) \in \mathcal{C}_{[2,4^2]}$ *and* $\beta = \alpha^{-1}\gamma_n = (3)(6)(1\,5\,4\,2\,7)(8)(9)(10) \in \mathcal{C}_{[1^5,5]}$, $\pi_1 = \{3\,4\,6\,7, 1\,2\,5\,8\,9\,10\}$, *and* $\pi_2 = \{1\,2\,4\,5\,7\,10, 6\,8, 3\,9\}$. *We easily check that* $g([2, 4^2], [1^5, 5]) = 1$ *and* $(\pi_1, \pi_2, \alpha) \in \mathcal{C}(\lambda, \mu)$. *We represent this unicellular partitioned bicolored map with the following ribbon graph with one irrelevant crossing and where each block is associated with a particular shape:*

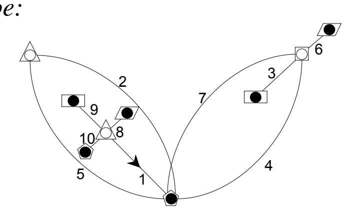

Fig. 1: A Partitioned Bicolored Map

2.2 Connection between $\mathcal{C}(\lambda, \mu)$ and $c^n_{\lambda\mu}$

Consider the partial order on integer partitions given by refinement. That is $\lambda \preceq \mu$ if and only if the parts of μ are unions of parts of λ, and we say that λ is a *refinement* of μ or that μ is *coarser* than λ. If $\lambda \preceq \mu$ let $\overline{R}_{\lambda\mu}$ be the number of ways to *coarse* λ to obtain μ. For example, if $\lambda = 1^2 2^2$ and $\mu = 123$ then $\overline{R}_{\lambda\mu} = 4$ since in λ any of the two 1-blocks can merge with any of the two 2-blocks. See (11) for more about the poset of integer partitions ordered by refinement.

Remark 1 *If* $m = \ell(\lambda)$ *and* $p = \ell(\mu)$ *then* $\overline{R}_{\lambda\mu}$ *is equal to the number of unordered partitions* $\pi = \{\pi_1, \ldots, \pi_p\}$ *of the set* $[m]$ *such that* $\mu_j = \sum_{i\in\pi_j} \lambda_i$ *for* $1 \leq j \leq p$. *Therefore, for the monomial and* *power symmetric functions, m_λ and p_λ, we have: $p_\lambda = \sum_{\mu \succeq \lambda} R_{\lambda\mu} m_\mu$, where $R_{\lambda\mu} = Aut(\mu)\overline{R}_{\lambda\mu}$ (10, Prop.7.7.1).*

We use this partial order on integer partitions to obtain a relation between $C(\lambda, \mu)$ and $c^n_{\lambda\mu}$.

Proposition 1

$$C(\nu, \rho) = \sum_{\lambda \preceq \nu, \mu \preceq \rho} \overline{R}_{\lambda\nu} \overline{R}_{\mu\rho} c^n_{\lambda\mu}. \tag{2}$$

Proof: Let $(\pi_1, \pi_2, \alpha) \in \mathcal{C}(\nu, \rho)$. If $\alpha \in \mathcal{C}_\lambda$ and $\beta = \alpha^{-1}\gamma_n \in \mathcal{C}_\mu$ then by definition of the set partitions we have that $type(\pi_1) = \nu \succeq \lambda$ and $type(\pi_2) = \rho \succeq \mu$. Thus, if

$$\mathcal{C}_{\lambda\mu}(\nu, \rho) = \{(\pi_1, \pi_2, \alpha) \in \mathcal{C}(\mu, \rho) \mid (\alpha, \alpha^{-1}\gamma_n) \in \mathcal{C}_\lambda \times \mathcal{C}_\mu\},$$

then $\mathcal{C}(\nu, \rho) = \bigcup_{\lambda \preceq \nu, \mu \preceq \rho} \mathcal{C}_{\lambda\mu}(\nu, \rho)$ where the union is disjoint. Finally, if $C_{\lambda\mu}(\nu, \rho) = |\mathcal{C}_{\lambda\mu}(\nu, \rho)|$ then it is easy to see that $C_{\lambda\mu}(\nu, \rho) = \overline{R}_{\lambda\nu} \overline{R}_{\mu\rho} c^n_{\lambda\mu}$. □

Using $p_\lambda = \sum_{\nu \succeq \lambda} R_{\lambda\nu} m_\nu$ we can rewrite Proposition 1 as:

$$\sum_{\lambda, \mu \vdash n} c^n_{\lambda\mu} p_\lambda(x) p_\mu(y) = \sum_{\lambda, \mu \vdash n} Aut(\lambda) Aut(\mu) C(\lambda, \mu) m_\lambda(x) m_\mu(y). \tag{3}$$

Remark 2 *Let $(\pi_1, \pi_2, \alpha) \in \mathcal{C}(\nu, \rho)$ with $\ell(\nu) + \ell(\rho) = n + 1$. If $\alpha \in \mathcal{C}_\lambda$ and $\beta \in \mathcal{C}_\mu$ then $\ell(\lambda) + \ell(\mu) = n + 1 - 2g(\lambda, \mu) \leq n + 1$. But $\ell(\lambda) \geq \ell(\nu)$ and $\ell(\mu) \geq \ell(\rho)$, therefore $\lambda = \nu$, $\mu = \rho$; and π_1 and π_2 are the underlying set partitions in the cycle decompositions of α and β respectively. In this case $C(\nu, \rho) = c^n_{\nu,\rho}$ ($g(\nu, \rho) = 0$).*

2.3 Ordered rooted bicolored thorn trees

We define the following sets of trees:

Definition 2 (Ordered rooted bicolored trees) *For $\lambda, \mu \vdash n$ such that $\ell(\lambda) + \ell(\mu) = n + 1$, let $\mathcal{BT}(\lambda, \mu)$ be the set of ordered rooted bicolored trees t with $\ell(\lambda)$ white vertices of degree distribution given by λ, and $\ell(\mu)$ black vertices of degree distribution given by μ. By convention, the root is a white vertex.*

If we are only interested in the number of vertices, let $\mathcal{BT}(p, q) = \bigcup_{\ell(\lambda) = p, \ell(\mu) = q} \mathcal{BT}(\lambda, \mu)$. As shown e.g. in (2), the cardinality of $\mathcal{BT}(\lambda, \mu)$ when $\ell(\lambda) + \ell(\mu) = n + 1$ is:

$$|\mathcal{BT}(\lambda, \mu)| = \frac{n}{Aut(\lambda) Aut(\mu)} (n - \ell(\lambda))! \, (n - \ell(\mu))!, \tag{4}$$

and they are in bijection with the ordered factorizations counted by $c^n_{\lambda\mu}$ when $g(\lambda, \mu) = 0$. By Remark 2, in this case we also have $C(\lambda, \mu) = c^n_{\lambda\mu}$. To get a combinatorial construction for $\mathcal{C}(\lambda, \mu)$ when $\ell(\lambda) + \ell(\mu) < n + 1$ we introduce the *ordered rooted bicolored thorn trees*:

Definition 3 (Ordered rooted bicolored thorn trees) *We call a thorn an edge connected to only one vertex. For $\lambda, \mu \vdash n$ such that $\ell(\lambda) + \ell(\mu) \leq n + 1$, we define $\widetilde{\mathcal{BT}}(\lambda, \mu)$ as the set of ordered rooted bicolored trees with $\ell(\lambda)$ white vertices, $\ell(\mu)$ black vertices, $n + 1 - \ell(\lambda) - \ell(\mu)$ thorns connected to the black vertices and $n + 1 - \ell(\lambda) - \ell(\mu)$ thorns connected to the white vertices. The white (respectively black) vertices' degree distribution (accounting the thorns) is specified by λ (respectively μ). The root is a white vertex.*

Adapting the Lagrange inversion developed in (2), it can be shown that:

$$|\widetilde{\mathcal{BT}}(\lambda,\mu)| = \frac{n}{Aut(\lambda)Aut(\mu)}\frac{(n-\ell(\lambda))!\,(n-\ell(\mu))!}{(n+1-\ell(\lambda)-\ell(\mu))!^{\,2}}. \tag{5}$$

Example 2 *The tree on Figure 2 belongs to* $\widetilde{\mathcal{BT}}(\lambda,\mu)$ *for* $n = 11$, $\lambda = [1^2, 2, 3, 4]$ *and* $\mu = [1, 2, 4^2]$ *(*$\ell(\lambda) + \ell(\mu) = 9 < 11$*).*

Remark 3 *One can notice the central role of the quantity* $n + 1 - \ell(\lambda) - \ell(\mu)(= 2g(\lambda,\mu)$ *in the case of* $\alpha \in \mathcal{C}_\lambda$ *and* $\alpha^{-1}\gamma_n \in \mathcal{C}_\mu$, i.e. *maps).*

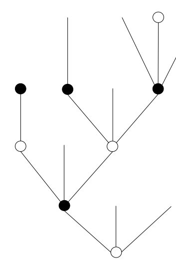

Fig. 2: An ordered rooted bicolored thorn Tree

2.4 New formulation of the main theorem

By equation (3), Theorem 1 is equivalent to proving the following formula:

$$C(\lambda,\mu) = \frac{n}{Aut(\lambda)Aut(\mu)}\frac{(n-\ell(\lambda))!\,(n-\ell(\mu))!}{(n+1-\ell(\lambda)-\ell(\mu))!^{\,2}}\times(n+1-\ell(\lambda)-\ell(\mu))! = |\widetilde{\mathcal{BT}}(\lambda,\mu)|\times|S_{n+1-\ell(\lambda)-\ell(\mu)}|.$$

In this paper we show this formula bijectively:

Theorem 2 *There is a direct bijection between partitioned bicolored maps* $\mathcal{C}(\lambda,\mu)$ *and pairs* (τ,σ) *of a bicolored thorn tree* $\tau \in \widetilde{\mathcal{BT}}(\lambda,\mu)$ *and a permutation* $\sigma \in S_{n+1-\ell(\lambda)-\ell(\mu)}$.

3 A mapping Ψ for bicolored partitioned maps of specified type

To prove Theorem 2, we first need to define some additional structures.

3.1 Reverse levels traversals of thorn trees and partial permutations

Definition 4 (Reverse white and black levels traversals) *For* τ *in* $\widetilde{\mathcal{BT}}(\lambda,\mu)$*, we define the reverse white levels traversal as the traversal going through all white vertices of* τ *and their descendants, either black vertices or thorns, in the following order (we assume that level* 1 *is the root's level):*

(i) The descendants (either black vertices or thorns if any) of the leftmost white vertex of the top white level are traversed from left to right.

(ii) Then the leftmost white vertex of the top white level is traversed.

(iii) *If i white vertices $(1, \ldots, i)$ (1 being the leftmost) of white level j ($j > 1$) have been traversed and there is a white vertex $i+1$ at level j on the right of i, the descendants of $i+1$ are traversed, then vertex $i+1$ is traversed. Otherwise, the descendants of vertex 1 of white level $j-1$ are traversed, followed by the white vertex itself.*

(iv) *The root vertex is the last to be traversed.*

The reverse black levels traversal is defined similarly except that the rightmost black descendant of the white root vertex is the last vertex to be traversed.

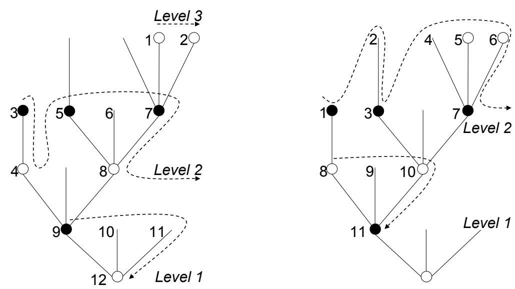

Fig. 3: Reverse white levels traversal (left) and reverse black levels traversal (right)

We also define partial permutations as in (9).

Definition 5 (Partial permutations) *Given two sets X and Y and a nonnegative integer m, let $\mathcal{PP}(X, Y, m)$ be the set of bijections from any m-subset of X to any m-subset of Y We call these bijections partial permutations. Then $|\mathcal{PP}(X, Y, m)| = \binom{|X|}{m}\binom{|Y|}{m} m!$. Trivially, a partial permutation $\widetilde{\sigma} \in \mathcal{PP}(X, Y, m)$ is also specified by a permutation $\sigma \in S_m$ and the complements of the domain and range.*

We proceed with the description of the bijective mapping Ψ,

$$\Psi_{n,\lambda,\mu} : \mathcal{C}(\lambda, \mu) \longrightarrow \widetilde{\mathcal{BT}}(\lambda, \mu) \times S_{n+1-\ell(\lambda)-\ell(\mu)}, \quad (\pi_1, \pi_2, \alpha) \longmapsto (\tau, \sigma).$$

3.2 The ordered bicolored thorn tree τ

Let $(\pi_1, \pi_2, \alpha) \in \mathcal{C}(\lambda, \mu)$. We construct an ordered bicolored thorn tree $\tau \in \widetilde{\mathcal{BT}}(\lambda, \mu)$ following the procedure below (in what follows, $p = \ell(\lambda)$ and $q = \ell(\mu)$).

(i) First step is to construct the *last passage unlabeled bicolor tree* t and two "relabeling" permutations θ_1 and θ_2 defined in the same way as in (9). We briefly describe this construction:
Let $\pi_1^{(1)}, \ldots, \pi_1^{(p)}$ and $\pi_2^{(1)}, \ldots, \pi_2^{(q)}$ be the blocks of the partitions π_1 and π_2 respectively. Denote by $m_1^{(i)}$ the maximal element of the block $\pi_1^{(i)}$ ($1 \leq i \leq p$) and by $m_2^{(j)}$ the maximal element of $\pi_2^{(j)}$ ($1 \leq j \leq q$). We assign the index p to the block of π_1 containing the element 1, and suppose that the indexing of all other blocks is arbitrary. We first construct the labeled ordered tree T in the following way: The white root is labeled p. For every $j = 1, \ldots, q$, the black vertex j is a

descendant of the white vertex i if the element $\beta(m_2^{(j)})$ belongs to the white block $\pi_1^{(i)}$. Similarly, for every $i = 1, \ldots, p-1$, a white vertex i is a descendant of a black vertex j if the element $m_1^{(i)}$ belongs to the black block $\pi_2^{(j)}$. If black vertices j, k are both descendants of a white vertex i, then j is to the left of k when $\beta(m_2^{(j)}) < \beta(m_2^{(k)})$; if white vertices i, l are both descendants of a black vertex j, then i is to the left of l when $\beta^{-1}(m_1^{(i)}) < \beta^{-1}(m_1^{(l)})$. It is not hard to show that this construction gives a tree (9) that we denote by T.

We remove the labels to obtain the unlabeled bicolored ordered tree t. Relabeling permutations θ_1 and θ_2 are defined by considering the *reverse-labelled* tree T' resulting from the labelling of t, based on two independent reverse-labelling procedures for white and black vertices. The root is labelled p, then going bottom up and right to left, we label the white vertices with labels $p-1, p-2$,... 1. An equivalent procedure applies to black vertices. Next step consists in relabeling the blocks by using the new indices from T'. If a white vertex is labeled i in T and i' in T', we set $\pi_1^{i'} = \pi_1^{(i)}$. Black blocks are relabeled in a similar fashion. Let ω^i, υ^j be the strings given by writing the elements of π_1^i, π_2^j in increasing order. Denote by $\omega = \omega^1 \ldots \omega^p$, $\upsilon = \upsilon^1 \ldots \upsilon^q$, the concatenations of the strings defined above. We define $\theta_1 \in S_n$ by setting ω as the first line and $[n]$ as the second line of the two-line representation of this permutation. Similarly, we define the relabeling permutation θ_2.

Example 3 *Let* $n = 8$*,* $\lambda = [2^4]$*,* $\mu = [2, 3^2]$*,* $\alpha = (23)(4)(15)(6)(78)$*,* $\beta = (2)(134)(568)(7)$*,* $\pi_1 = \{\pi_1^{(1)}, \pi_1^{(2)}, \pi_1^{(3)}, \pi_1^{(4)}\}$ *and* $\pi_2 = \{\pi_2^{(1)}, \pi_2^{(2)}, \pi_2^{(3)}\}$ *with*

$$\begin{aligned} &\pi_1^{(1)} = \{2,3\}, \quad \pi_1^{(2)} = \{4,6\}, \quad \pi_1^{(3)} = \{7,8\}, \quad \pi_1^{(4)} = \{1,5\}, \\ &\pi_2^{(1)} = \{1,3,4\}, \quad \pi_2^{(2)} = \{2,7\}, \quad \pi_2^{(3)} = \{5,6,8\} \end{aligned}$$

We associate shapes to the blocks as in Figure 4 and construct T*.*

We have that $\beta(m_2^{(2)}) = 7 \in \pi_1^{(3)}$*,* $\beta(m_2^{(1)}) = 1 \in \pi_1^{(4)}$*,* $\beta(m_2^{(3)}) = 5 \in \pi_1^{(4)}$*, and* $\beta(m_2^{(1)}) < \beta(m_2^{(3)})$*. Thus, the black vertex* 2 *is a descendant of the white vertex* 3*, and the descendants, from left to right, of the white vertex* 4 *are the black vertices* 1 *and* 3*. Also, we have that* $m_1^{(1)} = 3 \in \pi_2^{(1)}$*,* $m_1^{(2)} = 6 \in \pi_2^{(3)}$*,* $m_1^{(3)} = 8 \in \pi_2^{(3)}$ *and* $\beta^{-1}(m_1^{(2)}) < \beta^{-1}(m_1^{(3)})$*. Therefore, the white vertex* 1 *is a descendant of the black vertex* 1*, and the descendants, from left to right, of the black vertex* 3 *are the white vertices* 2 *and* 3*. Finally, we associate shapes to the blocks as in Figure 4. Then, by removing the labels we get the tree* t*. Reverse labeling of* t *gives* T' *and consequently the relabeling permutations:*

$$\theta_1 = \left(\begin{array}{cc|cc|cc|cc} 2 & 3 & 4 & 6 & 7 & 8 & 1 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right) \qquad \theta_2 = \left(\begin{array}{cc|ccc|ccc} 2 & 7 & 1 & 3 & 4 & 5 & 6 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right)$$

(ii) We use the sets

$$\{\theta_1(m_1^i)\}_{1 \le i \le p-1} \cup \{\theta_1(\beta(m_2^j))\}_{1 \le j \le q} \quad \text{and} \quad \{\theta_2(\beta^{-1}(m_1^i))\}_{1 \le i \le p-1} \cup \{\theta_2(m_2^j)\}_{1 \le j \le q}$$

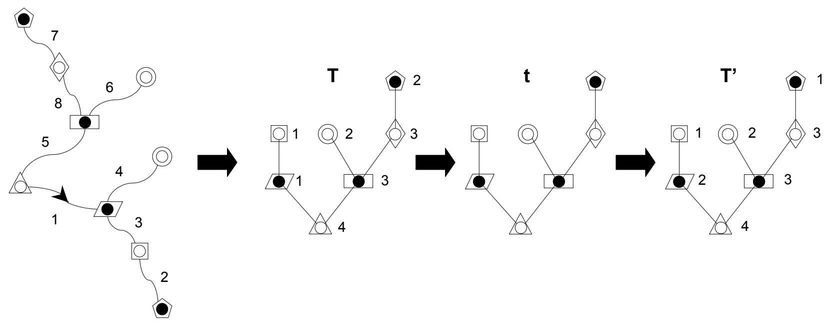

Fig. 4: Construction of T, t and T'

in order to double-label tree t. We assign $(\theta_1(m_1^i), \theta_2(\beta^{-1}(m_1^i)))$ to the white vertex indexed by $i, (i < p)$ in T' and $(\theta_1(\beta(m_2^j)), \theta_2(m_2^j))$ to black vertex indexed by j in T'. The root vertex is first labeled $n = \theta_1(m_1^p)$ and has no second label. We call T'' the resulting double-labeled tree (see figure 5 for an example).
Let $S_0 = \{m_1^1, \ldots, m_1^{p-1}\} \cup \{\beta(m_2^1), \ldots, \beta(m_2^q)\}$ and $\overline{\theta_1(S_0)}$ (resp. $\overline{\theta_2(\beta^{-1}(S_0))}$) be the ordered subset of $[n]$ obtained by arranging the elements of $\theta_1(S_0)$ (resp. $\theta_2(\beta^{-1}(S_0))$) in increasing order. We have the following lemmas:

Lemma 1 ((9)) *The set S_0 has $p+q-1$ elements,* i.e. $\{m_1^1, \ldots, m_1^{p-1}\} \cap \{\beta(m_2^1, \ldots, \beta(m_2^q)\} = \emptyset$.

Lemma 2 *Let $d = (d_1, d_2, \ldots, d_{p+q-1})$ (resp. $d' = (d'_1, d'_2, \ldots, d'_{p+q-1})$) be the ordered set of first (resp. second) labels obtained by traversing T'' up to, but not including, the root vertex according to the reverse white (resp. black) levels traversal defined in section 3. We have :*

$$d = \overline{\theta_1(S_0)}, \quad \textit{and} \quad d' = \overline{\theta_2(\beta^{-1}(S_0))}. \tag{6}$$

Proof: Let $\theta_1(m_1^0) = 0$. According to our construction, if a black vertex with first label $\theta_1(\beta(m_2^j))$ in T'' is a descendant of a white vertex with first label $\theta_1(m_1^i)$, $(1 \leq i \leq p)$, then $\beta(m_2^j)$ belongs to π_1^i and $\beta(m_2^j) \leq m_1^i$. As the image by θ_1 of any element of the white blocks $\{\pi_1^l\}_{1 \leq l < i}$ is strictly less than the image by θ_1 of any element of π_1^i, and θ_1 is an increasing function on each block $\{\pi_1^i\}_{1 \leq i \leq p}$ we have:

$$\theta_1(m_1^{i-1}) < \theta_1(\beta(m_2^j)) < \theta_1(m_1^i), \text{ where } \theta_1(m_1^0) = 0. \tag{7}$$

Suppose now that black vertices j and k with first labels $\theta_1(\beta(m_2^j))$ and $\theta_1(\beta(m_2^k))$ in T'' are both descendants of the same white vertex and that j is on the left of k. The construction of t and T'' implies that $\beta(m_2^j) < \beta(m_2^k)$. As θ_1 is increasing on the white blocks then $\theta_1(\beta(m_2^j)) < \theta_1(\beta(m_2^k))$. Finally, reverse white levels traversal of the first labels in T'' (up to but not including the root) yields $\overline{\theta_1(S_0)}$. Similarly, reverse black levels traversal of the second labels in T'' yields $\overline{\theta_2(\beta^{-1}(S_0))}$. $\square$

(iii) We add $n+1-p-q$ thorns to the white vertices and also $n+1-p-q$ thorns to the black vertices in T'' in the following fashion. Let d and d' be defined as in Lemma 2. We know from Lemma 1 and 2 that d and d' are strictly increasing subsequence of $[n]$ obtained by reverse levels traversals of T''. First, we add thorns to the white vertices in the following way:

- If $d_1 > 1$ and the vertex with first label d_1 is white, we connect $d_1 - 1$ thorns to it. If the vertex with first label d_1 is black, we connect $d_1 - 1$ thorns on the left of the ascending white vertex.

- For $1 < l < p+q-1$, if $d_l > d_{l-1}+1$ we follow exactly one of the four following cases:
  If d_l and d_{l-1} are both the first label of white vertices in T'', white vertex d_l (short for vertex corresponding to d_l) has no black descendant and it is the white vertex following d_{l-1} in the reverse white levels traversal of T''. We connect $d_l - d_{l-1} - 1$ thorns to vertex d_l.
  If d_l is the first label of a black vertex and d_{l-1} is the first label of a white one, then black vertex d_l is the leftmost descendant of the white vertex following d_{l-1}. We connect $d_l - d_{l-1} - 1$ thorns on the left of the ascending white vertex of d_l.
  If d_l is the first label of a white vertex and d_{l-1} is the first label of a black one, then the black vertex d_{l-1} is the rightmost descendant of vertex d_l. We connect $d_l - d_{l-1} - 1$ thorns to vertex d_l on the right of vertex d_{l-1}
  Finally, if d_l and d_{l-1} are both the first label of black vertices, these two vertices have the same white ascending vertex. We connect $d_l - d_{l-1} - 1$ thorns to the ascending white vertex between these two black vertices.

- If $d_{p+q-1} < n$, we connect $n - d_{p+q-1} - 1$ thorns to the root vertex on the right of its rightmost black descendant.

We can think of this as adding a thorn to the white vertices for each integer of $[n]$ not included in d. A similar construction is applied to add thorns to the black vertices following the sequence of integers d'. Finally, we remove all the labels to get the ordered bicolored thorn tree τ.

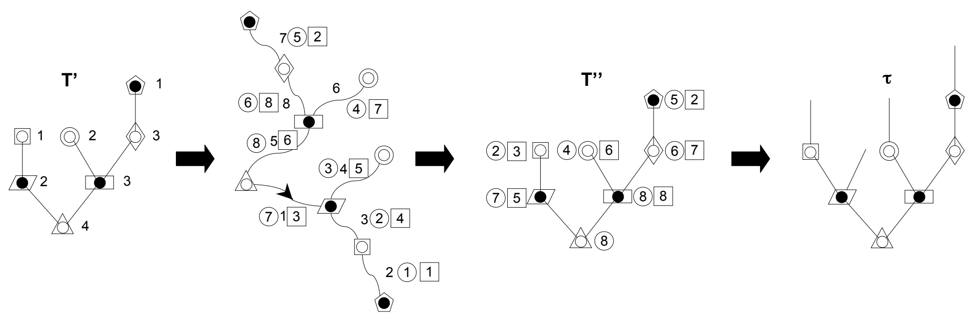

Fig. 5: Construction of T'' and τ. Circled numbers correspond to the first relabeling permutation θ_1 and the first labels in T''. Squared numbers correspond to the second relabeling permutation θ_2 and the second labels in T''.

Example 4 *Following the previous example, the corresponding T'' and τ are represented on figure 5. In order to get τ from T'', we use $d = (2, 4, 5, 6, 7, 8)$ and $d' = (2, 3, 5, 6, 7, 8)$. The missing integers in d are 1 and 3, so we connect one thorn to the white vertex with first label 2 (since $d_1 > 1$) and one to the white vertex with first label 4 (since $d_2 > d_1 + 1$). Similarly, we connect one thorn to the black vertex with second label 2 (since $d'_1 > 1$) and one to the right of the black vertex with second label 5 (since $d'_3 > d'_2 + 1$).*

Lemma 3 *τ as defined above belongs to $\widetilde{\mathcal{BT}}(\lambda, \mu)$*

Proof: As there are $p + q - 1$ distinct elements in d and d', exactly $n - |d| = n + 1 - p - q$ thorns are connected to the white vertices. Similarly $n+1-p-q$ thorns are connected to the black vertices. Then if we take two successive white vertices $i-1$ and i according to the reverse white level traversal of T'' with first labels $\theta_1(m_1^{i-1})$ and $\theta_1(m_1^i)$, $(i < p)$, a thorn is connected to i for each integer of $[\theta_1(m_1^{i-1}),\ \theta_1(m_1^i)]$ missing in d. The number of these missing integers is equal to $\theta_1(m_1^i) - 1 - \theta_1(m_1^{i-1}) - f_i$ where f_i is the number of black descendants of i. As i is not the root vertex, there is an edge between i and its ascendant so that the resulting degree v for i is:

$$\forall i \in [p-1], v(i) = \theta_1(m_1^i) - 1 - \theta_1(m_1^{i-1}) - f_i + f_i + 1 = \theta_1(m_1^i) - \theta_1(m_1^{i-1}) \tag{8}$$

Furthermore, $n - \theta_1(m_1^{p-1}) - f_p$ thorns are connected to the root vertex so that:

$$v(p) = n - \theta_1(m_1^{p-1}) \tag{9}$$

But, according to the construction of θ_1,

$$\begin{aligned} \theta_1(\pi_1^1) &= [\theta_1(m_1^1)] && (10)\\ \theta_1(\pi_1^i) &= [\theta_1(m_1^i)] \setminus [\theta_1(m_1^{i-1})],\ (2 \le i \le p-1) && (11)\\ \theta_1(\pi_1^p) &= [n] \setminus [\theta_1(m_1^{p-1})] && (12) \end{aligned}$$

Subsequently for $i \in [1, p]$, $v(i) = |\pi_1^i|$, and $\lambda = type(\pi)$ is the white vertex degree distribution of τ. In a similar fashion, μ is the black vertex degree distribution of τ. □

Lemma 4 *Assign a first set of labels $1, 2, \ldots, n$ to the vertices and the thorns connected to white vertices in τ in increasing order according to the reverse white levels traversal as well as a second set of labels $1, 2, \ldots, n$ to the vertices and the thorns connected to black vertices in increasing order according to the reverse black levels traversal. The double-labeling of the vertices is the same as in T''.*

Proof: According to the construction of τ, we add thorns to T'' when integers are missing in the reverse levels traversals of T'' so that the thorns would take these missing integers as labels when traversing the thorn tree. As a result, the labels of the vertices in the reverse levels traversals of τ are still d and d' and since they still appear in the same order, we have the desired result. □

3.3 The permutation σ

Let $S = [n] \setminus S_0$, we define the partial permutation $\widetilde{\sigma}$ on set $\theta_1(S)$ by:

$$\widetilde{\sigma} = \theta_2 \circ \beta^{-1} \circ \theta_1^{-1}|_{\theta_1(S)}. \tag{13}$$

Then let σ be the permutation in $S_{n+1-p-q}$ that is order-isomorphic to $\widetilde{\sigma}$. That is, we define the ordered set $\overline{\theta_1(S)}$ with all the elements of $\theta_1(S)$ sorted in increasing order and ρ_1 the labeling function which associates to each element of $\theta_1(S)$ its position index in $\overline{\theta_1(S)}$. As a direct consequence, we have $\rho_1(\theta_1(S)) = [n+1-p-q]$. Similarly we define ρ_2 to label the elements of $\theta_2(\beta^{-1}(S))$. Then σ is given by:

$$\sigma : [n+1-p-q] \longrightarrow [n+1-p-q], \quad u \longmapsto \rho_2 \circ \widetilde{\sigma} \circ \rho_1^{-1}(u)$$

Example 5 *Going ahead with our previous example, we find:*

$$\widetilde{\sigma} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad \textit{and} \qquad \sigma = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

4 Proof that the mapping Ψ is a bijection

Theorem 3 $\Psi_{n,\lambda,\mu}$ *is actually a bijection*

Proof: To show that $\Psi_{n,\lambda,\mu}$ is a one-to-one correspondence we take any element (τ, σ) in $\widetilde{\mathcal{BT}}(\lambda, \mu) \times S_{n+1-\ell(\lambda)-\ell(\mu)}$ and show that there is a unique element (π_1, π_2, α) in $\mathcal{C}(\lambda, \mu)$ such that $\Psi_{n,\lambda,\mu}(\pi_1, \pi_2, \alpha) = (\tau, \sigma)$. Let $p = \ell(\lambda)$ and $q = \ell(\mu)$. We proceed with a two step proof:

(i) The first step is to notice that (τ, σ) defines a unique unlabeled bicolored tree t and a unique partial permutation $\widetilde{\sigma}$ belonging to $\mathcal{BT}(p, q) \times \mathcal{PP}(n, n-1, n+1-p-q)$. Double-labeling of τ with $1, 2, \ldots, n$ in increasing order according to the reverse levels traversals and removing the two sets of $n+1-p-q$ thorns (together with their labels) gives a double-labeled tree T'' that leads to τ according to Ψ. This double-labeled tree is the unique one that can lead to τ since within Ψ, the constructions of τ and T'' have the same underlying tree structure, and according to Lemma 4, τ determines the labels of T''.
Then, using Lemma 2, the two series of labels (except the root's) in T'' are necessarily the missing elements in the domain (first labels) and the range (second labels) of $\widetilde{\sigma}$ sorted in increasing order (within the reverse levels traversals). Hence, T'' and σ uniquely determine t and $\widetilde{\sigma}$. Obviously, the process of using the labels in T'' as missing elements to reconstruct $\widetilde{\sigma}$ can always be performed. Within this process we have $\widetilde{\sigma}$ in $\mathcal{PP}(n, n-1, n+1-p-q)$ since two series of $p+q-1$ labels are used and the second label of the last black vertex traversed in the reverse black levels traversal of τ is n (since there are no other black vertices and no thorns connected to further black vertices either) and n is always a missing element of the image.

(ii) The bijection $\Theta_{n,p,q}$ in (9) is identical to the first steps (up to the construction of t and $\widetilde{\sigma}$) of $\Psi_{n,\lambda,\mu}$. There is therefore a unique triple (π_1, π_2, α) in $\mathcal{C}(p, q, n) = \bigcup_{\ell(\lambda)=p, \ell(\mu)=q} \mathcal{C}(\lambda, \mu)$ mapped to t and $\widetilde{\sigma}$ by Θ and equivalently by the first steps of Ψ. But, according to (9), the types of π_1 and π_2 can be recovered with the pair $(t, \widetilde{\sigma})$ via the missing elements in the domain and range of $\widetilde{\sigma}$ corresponding to the relabeling by θ_1 and θ_2 of the maximum elements of the blocks (that we can identify by assigning the missing elements in t using the reverse levels traversals). Furthermore, using Lemma 3, the vertex degree distribution of τ is equal to the type of the partitions encoded by the missing elements in $\widetilde{\sigma}$ corresponding to the relabeling of the maximum elements of the blocks. Finally, as the vertex degree distribution in τ is (λ, μ), so is the type of (π_1, π_2); obtained as part of the preimage of $(t, \widetilde{\sigma})$ under Θ. Therefore, (π_1, π_2, α) belongs to $\mathcal{C}(\lambda, \mu)$ as desired.

$\square$

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Cluster algebras of unpunctured surfaces and snake graphs

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Abstract. We study cluster algebras with principal coefficient systems that are associated to unpunctured surfaces. We give a direct formula for the Laurent polynomial expansion of cluster variables in these cluster algebras in terms of perfect matchings of a certain graph $G_{T,\gamma}$ that is constructed from the surface by recursive glueing of elementary pieces that we call tiles. We also give a second formula for these Laurent polynomial expansions in terms of subgraphs of the graph $G_{T,\gamma}$.

Résumé. Nous etudions des algebres amassees avec coefficients principaux associees aux surfaces. Nous presentons une formule directe pour les developpements de Laurent des variables amassees dans ces algebres en terme de couplages parfaits d'un certain graphe $G_{T,\gamma}$ que l'on construit a partir de la surface en recollant des pieces elementaires que l'on appelle carreaux. Nous donnons aussi une seconde formule pour ces developpements en termes de sous-graphes de $G_{T,\gamma}$.

Keywords: cluster algebra, triangulated surface, principal coefficients, F-polynomial, height function, snake graphs

1 Introduction

Cluster algebras, introduced in (FZ1), are commutative algebras equipped with a distinguished set of generators, the *cluster variables*. The cluster variables are grouped into sets of constant cardinality n, the *clusters*, and the integer n is called the *rank* of the cluster algebra. Starting with an initial cluster $\mathbf{x}$ (together with a skew symmetrizable integer $n \times n$ matrix $B = (b_{ij})$ and a coefficient vector $\mathbf{y} = (y_i)$ whose entries are elements of a torsion-free abelian group $\mathbb{P}$) the set of cluster variables is obtained by repeated application of so called *mutations*. To be more precise, let $\mathcal{F}$ be the field of rational functions in the indeterminates $x_1, x_2, \ldots, x_n$ over the quotient field of the integer group ring $\mathbb{ZP}$. Thus $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is a transcendence basis for $\mathcal{F}$. For every $k = 1, 2, \ldots, n$, the mutation $\mu_k(\mathbf{x})$ of the cluster $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is a new cluster $\mu_k(\mathbf{x}) = \mathbf{x} \setminus \{x_k\} \cup \{x'_k\}$ obtained from $\mathbf{x}$ by replacing the cluster variable x_k by the new cluster variable

$$x'_k = \frac{1}{x_k}\left(y_k^+ \prod_{b_{ki}>0} x_i^{b_{ki}} + y_k^- \prod_{b_{ki}<0} x_i^{-b_{ki}}\right) \tag{1}$$

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in $\mathcal{F}$, where y_k^+, y_k^- are certain monomials in $y_1, y_2, \ldots, y_n$. Mutations also change the attached matrix B as well as the coefficient vector $\mathbf{y}$, see (FZ1).

The set of all cluster variables is the union of all clusters obtained from an initial cluster $\mathbf{x}$ by repeated mutations. Note that this set may be infinite.

It is clear from the construction that every cluster variable is a rational function in the initial cluster variables $x_1, x_2, \ldots, x_n$. In (FZ1) it is shown that every cluster variable u is actually a Laurent polynomial in the x_i, that is, u can be written as a reduced fraction

$$u = \frac{f(x_1, x_2, \ldots, x_n)}{\prod_{i=1}^n x_i^{d_i}}, \tag{2}$$

where $f \in \mathbb{ZP}[x_1, x_2, \ldots, x_n]$ and $d_i \geq 0$. The right hand side of equation (2) is called the *cluster expansion* of u in $\mathbf{x}$.

The cluster algebra is determined by the initial matrix B and the choice of the coefficient system. A canonical choice of coefficients is the *principal* coefficient system, introduced in (FZ2), which means that the coefficient group $\mathbb{P}$ is the free abelian group on n generators $y_1, y_2, \ldots, y_n$, and the initial coefficient tuple $\mathbf{y} = \{y_1, y_2, \ldots, y_n\}$ consists of these n generators. In (FZ2), the authors show that knowing the expansion formulas in the case where the cluster algebra has principal coefficients allows one to compute the expansion formulas for arbitrary coefficient systems.

Inspired by the work of Fock and Goncharov (FG1; FG2; FG3) and Gekhtman, Shapiro and Vainshtein (GSV1; GSV2) which discovered cluster structures in the context of Teichmüller theory, Fomin, Shapiro and Thurston (FST; FT) initiated a systematic study of the cluster algebras arising from triangulations of a surface with boundary and marked points. In this approach, cluster variables in the cluster algebra correspond to arcs in the surface, and clusters correspond to triangulations. In (S2), building on earlier results in (S1; ST), this model was used to give a direct expansion formula for cluster variables in cluster algebras associated to unpunctured surfaces, with arbitrary coefficients, in terms of certain paths on the triangulation.

Our first main result in this paper is a new parametrization of this formula in terms of perfect matchings of a certain weighted graph that is constructed from the surface by recursive glueing of elementary pieces that we call tiles. To be more precise, let x_γ be a cluster variable corresponding to an arc γ in the unpunctured surface and let d be the number of crossings between γ and the triangulation T of the surface. Then γ runs through $d+1$ triangles of T and each pair of consecutive triangles forms a quadrilateral which we call a tile. So we obtain d tiles, each of which is a weighted graph, whose weights are given by the cluster variables x_τ associated to the arcs τ of the triangulation T.

We obtain a weighted graph $G_{T,\gamma}$ by glueing the d tiles in a specific way and then deleting the diagonal in each tile. To any perfect matching M of this graph we associate its weight $w(M)$ which is the product of the weights of its edges, hence a product of cluster variables. We prove the following cluster expansion formula:

Theorem 1.1.

$$x_\gamma = \sum_M \frac{w(M)\, y(M)}{x_{i_1} x_{i_2} \ldots x_{i_d}},$$

where the sum is over all perfect matchings M of $G_{T,\gamma}$, $w(M)$ is the weight of M, and $y(M)$ is a monomial in $\mathbf{y}$.

We also give a formula for the coefficients $y(M)$ in terms of perfect matchings as follows. The F-polynomial F_γ, introduced in (FZ2) is obtained from the Laurent polynomial x_γ (with principal coefficients) by substituting 1 for each of the cluster variables $x_1, x_2, \ldots, x_n$. By (S2, Theorem 6.2, Corollary 6.4), the F-polynomial has constant term 1 and a unique term of maximal degree that is divisible by all the other occurring monomials. The two corresponding matchings are the unique two matchings that have all their edges on the boundary of the graph $G_{T,\gamma}$. We denote by M_- the one with $y(M_-) = 1$ and the other by M_+. Now, for an arbitrary perfect matching M, the coefficient $y(M)$ is determined by the set of edges of the symmetric difference $M_- \ominus M = (M_- \cup M) \setminus (M_- \cap M)$ as follows.

Theorem 1.2. *The set $M_- \ominus M$ is the set of boundary edges of a (possibly disconnected) subgraph G_M of $G_{T,\gamma}$ which is a union of tiles $G_M = \cup_{j\in J} S_j$. Moreover,*

$$y(M) = \prod_{j\in J} y_{i_j}.$$

As an immediate corollary, we see that the corresponding g-vector, introduced in (FZ2), is

$$g_\gamma = \deg\left(\frac{w(M_-)}{x_{i_1}\cdots x_{i_d}}\right).$$

This follows from the fact that $y(M_-) = 1$.

Our third main result is yet another description of the formula of Theorem 1.1 in terms of the graph $G_{T,\gamma}$ only. In order to state this result, we need some notation. If H is a graph, let $c(H)$ be the number of connected components of H, let $E(H)$ be the set of edges of H, and denote by ∂H the set of boundary edges of H. Define $\mathcal{H}_k$ to be the set of all subgraphs H of $G_{T,\gamma}$ such that H is a union of k tiles $H = S_{j_1} \cup \cdots \cup S_{j_k}$ and such that the number of edges of M_- that are contained in H is equal to $k + c(H)$. For $H \in \mathcal{H}_k$, let

$$y(H) = \prod_{S_{i_j} \text{ tile in } H} y_{i_j}.$$

Theorem 1.3. *The cluster expansion of the cluster variable x_γ is given by*

$$x_\gamma = \sum_{k=0}^{d} \sum_{H\in\mathcal{H}_k} \frac{w(\partial H \ominus M_-)\, y(H)}{x_{i_1} x_{i_2} \cdots x_{i_d}}.$$

Theorem 1.1 has interesting intersections with work of other people. In (CCS2), the authors obtained a formula for the denominators of the cluster expansion in types A, D and E, see also (BMR). In (CC; CK; CK2) an expansion formula was given in the case where the cluster algebra is acyclic and the cluster lies in an acyclic seed. Palu generalized this formula to arbitrary clusters in an acyclic cluster algebra (Pa). These formulas use the cluster category introduced in (BMRRT), and in (CCS) for type A, and do not give information about the coefficients.

Recently, Fu and Keller generalized this formula further to cluster algebras with principal coefficients that admit a categorification by a 2-Calabi-Yau category (FK), and, combining results of (A) and (ABCP; LF), such a categorification exists in the case of cluster algebras associated to unpunctured surfaces.

In (SZ; CZ; Z; MP) cluster expansions for cluster algebras of rank 2 are given, in (Pr1; CP; FZ3) the case A is considered. In section 4 of (Pr1), Propp describes two constructions of snake graphs, the latter of which are unweighted analogues for the case A of the graphs $G_{T,\gamma}$ that we present in this paper. Propp assigns a snake graph to each arc in the triangulation of an n-gon and shows that the numbers of matchings in these graphs satisfy the Conway-Coxeter frieze pattern induced by the Ptolemy relations on the n-gon. In (M) a cluster expansion for cluster algebras of classical type is given for clusters that lie in a bipartite seed, and the forthcoming work of (MSW) will concern cluster expansions for cluster algebras with principal coefficients arising from any surface (with or without punctures), for an arbitrary seed.

Remark 1.4. The formula for $y(M)$ given in Theorem 1.2 also can be formulated in terms of height functions, as found in literature such as (EKLP) or (Pr2). As described in section 3 of (Pr2), one way to define the height function on the faces of a bipartite planar graph G, covered by a perfect matching M, is to superimpose each matching with the fixed matching $M_{\hat{0}}$ (the unique matching of minimal height). In the case where G is a snake graph, we take $M_{\hat{0}}$ to be M_-, one of the two matchings of G only involving edges on the boundary. Color the vertices of G black and white so that no two adjacent vertices have the same color. In this superposition, we orient edges of M from black to white, and edges of M_- from white to black. We thereby obtain a spanning set of cycles, and removing the cycles of length two exactly corresponds to taking the symmetric difference $M \ominus M_-$. We can read the resulting graph as a relief-map, in which the altitude changes by $+1$ or -1 as one crosses over a contour line, according to whether the counter-line is directed clockwise or counter-clockwise. By this procedure, we obtain a height function $h_M : F(G) \to \mathbb{Z}$ which assigns integers to the faces of graph G. When G is a snake graph, the set of faces $F(G)$ is simply the set of tiles $\{S_j\}$ of G. Comparing with the definition of $y(M)$ in Theorem 1.2, we see that

$$y(M) = \prod_{S_j \in F(G)} y_j^{h_M(j)}.$$

An alternative defintion of height functions comes from (EKLP) by translating the matching problem into a domino tiling problem on a region colored as a checkerboard. We imagine an ant starting at an arbitrary vertex at height 0, walking along the boundary of each domino, and changing its height by $+1$ or -1 as it traverses the boundary of a black or white square, respectively. The values of the height function under these two formulations agree up to scaling by four.

The paper is organized as follows. In section 2, we recall the construction of cluster algebras from surfaces of (FST). Section 3 contains the construction of the graph $G_{T,\gamma}$ and the statement of the cluster expansion formula. Proofs of our results appear in sections 4-6 of (MS). We close with an example in section 4.

2 Cluster algebras from surfaces

In this section, we recall the construction of (FST) in the case of surfaces without punctures.

Let S be a connected oriented 2-dimensional Riemann surface with boundary and M a non-empty set of marked points in the closure of S with at least one marked point on each boundary component. The pair (S, M) is called a *bordered surface with marked points*. Marked points in the interior of S are called *punctures*.

In this paper we will only consider surfaces (S, M) such that all marked points lie on the boundary of S, and we will refer to (S, M) simply by *unpunctured surface*.

We say that two curves in S *do not cross* if they do not intersect each other except that endpoints may coincide. An *arc* γ in (S, M) is a curve in S such that

(a) the endpoints are in M,

(b) γ does not cross itself,

(c) the relative interior of γ is disjoint from M and from the boundary of S,

(d) γ does not cut out a monogon or a digon.

Curves that connect two marked points and lie entirely on the boundary of S without passing through a third marked point are called *boundary arcs*. Hence an arc is a curve between two marked points, which does not intersect itself nor the boundary except possibly at its endpoints and which is not homotopic to a point or a boundary arc.

Each arc is considered up to isotopy inside the class of such curves. Moreover, each arc is considered up to orientation, so if an arc has endpoints $a, b \in M$ then it can be represented by a curve that runs from a to b, as well as by a curve that runs from b to a.

For any two arcs γ, γ' in S, let $e(\gamma, \gamma')$ be the minimal number of crossings of γ and γ', that is, $e(\gamma, \gamma')$ is the minimum of the numbers of crossings of arcs α and α', where α is isotopic to γ and α' is isotopic to γ'. Two arcs γ, γ' are called *compatible* if $e(\gamma, \gamma') = 0$. A *triangulation* is a maximal collection of compatible arcs together with all boundary arcs. The arcs of a triangulation cut the surface into *triangles*. Since (S, M) is an unpunctured surface, the three sides of each triangle are distinct (in contrast to the case of surfaces with punctures). Any triangulation has $n + m$ elements, n of which are arcs in S, and the remaining m elements are boundary arcs. Note that the number of boundary arcs is equal to the number of marked points.

Proposition 2.1. *The number n of arcs in any triangulation is given by the formula $n = 6g + 3b + m - 6$, where g is the genus of S, b is the number of boundary components and $m = |M|$ is the number of marked points. The number n is called the* rank *of (S, M).*

Proof. (FST, 2.10) □

Note that $b > 0$ since the set M is not empty. Following (FST), we associate a cluster algebra to the unpunctured surface (S, M) as follows. Choose any triangulation T, let $\tau_1, \tau_2, \ldots, \tau_n$ be the n interior arcs of T and denote the m boundary arcs of the surface by $\tau_{n+1}, \tau_{n+2}, \ldots, \tau_{n+m}$. For any triangle Δ in T define a matrix $B^\Delta = (b^\Delta_{ij})_{1 \le i \le n, 1 \le j \le n}$ by

$$b^\Delta_{ij} = \begin{cases} 1 & \text{if } \tau_i \text{ and } \tau_j \text{ are sides of } \Delta \text{ with } \tau_j \text{ following } \tau_i \text{ in} \\ & \text{counter-clockwise order;} \\ -1 & \text{if } \tau_i \text{ and } \tau_j \text{ are sides of } \Delta \text{ with } \tau_j \text{ following } \tau_i \text{ in} \\ & \text{clockwise order;} \\ 0 & \text{otherwise.} \end{cases}$$

(Note that this sign convention agrees with that of (S2) and differs from that (FST).) Then define the matrix $B_T = (b_{ij})_{1 \le i \le n, 1 \le j \le n}$ by $b_{ij} = \sum_\Delta b^\Delta_{ij}$, where the sum is taken over all triangles in T. Note that the boundary arcs of the triangulation are ignored in the definition of B_T. Let $\tilde{B}_T = (b_{ij})_{1 \le i \le 2n, 1 \le j \le n}$ be the $2n \times n$ matrix whose upper $n \times n$ part is B_T and whose lower $n \times n$ part is the identity matrix. The matrix B_T is skew-symmetric and each of its entries b_{ij} is either $0, 1, -1, 2$, or -2, since every arc τ can be in at most two triangles.

Let $\mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, B_T)$ be the cluster algebra with principal coefficients in the triangulation T, that is, $\mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, B_T)$ is given by the seed $(\mathbf{x}_T, \mathbf{y}_T, B_T)$ where $\mathbf{x}_T = \{x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_n}\}$ is the cluster associated to the triangulation T, and the initial coefficient vector $\mathbf{y}_T = (y_1, y_2, \ldots, y_n)$ is the vector of generators of $\mathbb{P} = \text{Trop}(y_1, y_2, \ldots, y_n)$. For the boundary arcs we define $x_{\tau_k} = 1$, $k = n+1, n+2, \ldots, n+m$.

For each $k = 1, 2, \ldots, n$, there is a unique quadrilateral in $T \setminus \{\tau_k\}$ in which τ_k is one of the diagonals. Let τ_k' denote the other diagonal in that quadrilateral. Define the *flip* $\mu_k T$ to be the triangulation $T \setminus \{\tau_k\} \cup \{\tau_k'\}$. The mutation μ_k of the seed Σ_T in the cluster algebra $\mathcal{A}$ corresponds to the flip μ_k of the triangulation T in the following sense. The matrix $\mu_k(B_T)$ is the matrix corresponding to the triangulation $\mu_k T$, the cluster $\mu_k(\mathbf{x}_T)$ is $\mathbf{x}_T \setminus \{x_{\tau_k}\} \cup \{x_{\tau_k'}\}$, and the corresponding exchange relation is given by

$$x_{\tau_k} x_{\tau_k'} = x_{\rho_1} x_{\rho_2} y^+ + x_{\sigma_1} x_{\sigma_2} y^-,$$

where y^+, y^- are some coefficients, and $\rho_1, \sigma_1, \rho_2, \sigma_2$ are the sides of the quadrilateral in which τ_k and τ_k' are the diagonals, such that ρ_1, ρ_2 are opposite sides and σ_1, σ_2 are opposite sides too.

3 Expansion formula

In this section, we will present an expansion formula for the cluster variables in terms of perfect matchings of a graph that is constructed recursively using so-called *tiles*.

3.1 Tiles

For the purpose of this paper, a *tile* $\overline{S}_k$ is a planar four vertex graph with five weighted edges having the shape of two equilateral triangles that share one edge, see Figure 1(a). The weight on each edge of the tile $\overline{S}_k$ is a single variable. The unique interior edge is called *diagonal* and the four exterior edges are called *sides* of $\overline{S}_k$. We shall use S_k to denote the graph obtained from $\overline{S}_k$ by removing the diagonal.

Now let T be a triangulation of the unpunctured surface (S, M). If $\tau_k \in T$ is an interior arc, then τ_k lies in precisely two triangles in T, hence τ_k is the diagonal of a unique quadrilateral Q_{τ_k} in T. We associate to this quadrilateral a tile $\overline{S}_k$ by assigning the weight x_k to the diagonal and the weights x_a, x_b, x_c, x_d to the sides of $\overline{S}_k$ in such a way that there is a homeomorphism $\overline{S}_k \to Q_{\tau_k}$ which sends the edge with weight x_i to the arc labeled τ_i, $i = a, b, c, d, k$, see Figure 1(a).

3.2 The graph $\overline{G}_{T,\gamma}$

Let T be a triangulation of an unpunctured surface (S, M) and let γ be an arc in (S, M) which is not in T. Choose an orientation on γ and let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. We denote by

$$p_0 = s, p_1, p_2, \ldots, p_{d+1} = t$$

the points of intersection of γ and T in order. Let $i_1, i_2, \ldots, i_d$ be such that p_k lies on the arc $\tau_{i_k} \in T$. Note that i_k may be equal to i_j even if $k \neq j$. Let $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_d$ be a sequence of tiles so that $\tilde{S}_k$ is isomorphic to the tile $\overline{S}_{i_k}$, for $k = 1, 2, \ldots, d$.

For k from 0 to d, let γ_k denote the segment of the path γ from the point p_k to the point p_{k+1}. Each γ_k lies in exactly one triangle Δ_k in T, and if $1 \leq k \leq d-1$ then Δ_k is formed by the arcs $\tau_{i_k}, \tau_{i_{k+1}}$, and a third arc that we denote by $\tau_{[\gamma_k]}$.

We will define a graph $\overline{G}_{T,\gamma}$ by recursive glueing of tiles. Start with $\overline{G}_{T,\gamma,1} \cong \tilde{S}_1$, where we orient the tile $\tilde{S}_1$ so that the diagonal goes from northwest to southeast, and the starting point p_0 of γ is in the

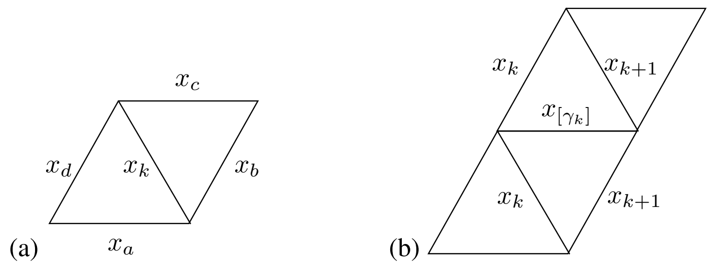

Fig. 1: (a) The tile $\overline{S}_k$; (b) Glueing tiles S_k and S_{k+1} along the edge weighted $x_{[\gamma_k]}$

southwest corner of $\tilde{S}_1$. For all $k = 1, 2, \ldots, d-1$ let $\overline{G}_{T,\gamma,k+1}$ be the graph obtained by adjoining the tile $\tilde{S}_{k+1}$ to the tile $\tilde{S}_k$ of the graph $\overline{G}_{T,\gamma,k}$ along the edge weighted $x_{[\gamma_k]}$, see Figure 1(b). We always orient the tiles so that the diagonals go from northwest to southeast. Note that the edge weighted $x_{[\gamma_k]}$ is either the northern or the eastern edge of the tile $\tilde{S}_k$. Finally, we define $\overline{G}_{T,\gamma}$ to be $\overline{G}_{T,\gamma,d}$.

Let $G_{T,\gamma}$ be the graph obtained from $\overline{G}_{T,\gamma}$ by removing the diagonal in each tile, that is, $G_{T,\gamma}$ is constructed in the same way as $\overline{G}_{T,\gamma}$ but using tiles S_{i_k} instead of $\overline{S}_{i_k}$.

A *perfect matching* of a graph is a subset of the edges so that each vertex is covered exactly once. We define the weight $w(M)$ of a perfect matching M to be the product of the weights of all edges in M.

3.3 Cluster expansion formula

Let (S, M) be an unpunctured surface with triangulation T, and let $\mathcal{A} = \mathcal{A}(\mathbf{x}_T, \mathbf{y}_T, B)$ be the cluster algebra with principal coefficients in the initial seed $(\mathbf{x}_T, \mathbf{y}_T, B)$ defined in section 2. Each cluster variable in $\mathcal{A}$ corresponds to an arc in (S, M). Let x_γ be an arbitrary cluster variable corresponding to an arc γ. Choose an orientation of γ, and let $\tau_{i_1}, \tau_{i_2} \ldots, \tau_{i_d}$ be the arcs of the triangulation that are crossed by γ in this order, with multiplicities possible. Let $G_{T,\gamma}$ be the graph constructed in section 3.2.

Theorem 1.1.

$$x_\gamma = \sum_M \frac{w(M)\, y(M)}{x_{i_1} x_{i_2} \ldots x_{i_d}},$$

where the sum is over all perfect matchings M of $G_{T,\gamma}$, $w(M)$ is the weight of M, and $y(M)$ is the monomial given in Theorem 1.2.

4 Example

We illustrate Theorem 1.1, Theorem 1.2 and Theorem 1.3 in an example. Let (S, M) be the annulus with two marked points on each of the two boundary components, and let $T = \{\tau_1, \ldots, \tau_8\}$ be the triangulation shown in Figure 2.

The corresponding cluster algebra has the following principal exchange matrix and quiver.

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

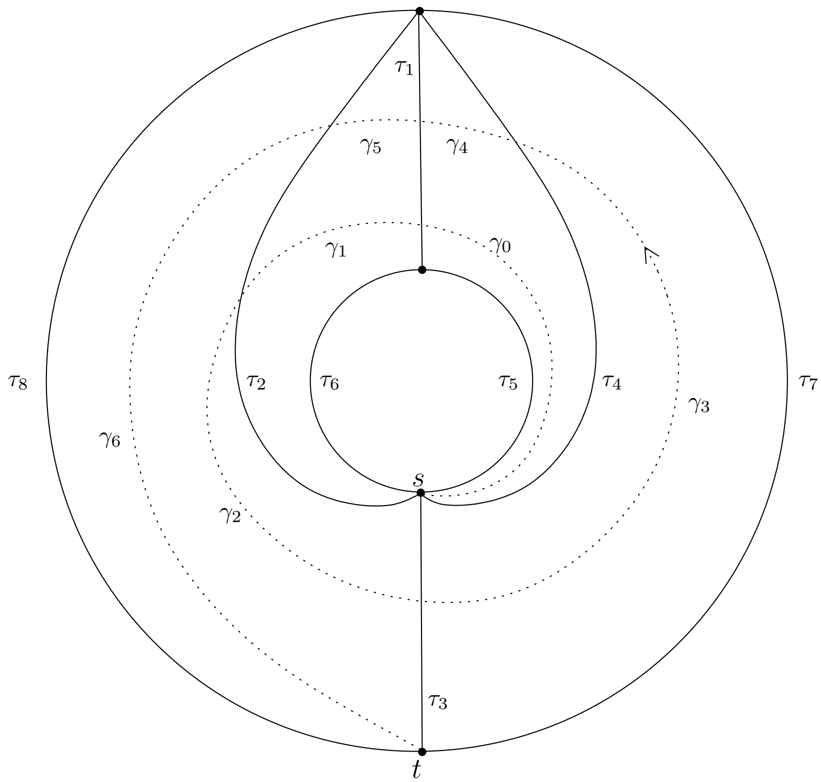

Fig. 2: Triangulated surface with dotted arc γ

Let γ be the dotted arc in Figure 2. It has $d = 6$ crossings with the triangulation. The sequence of crossed arcs $\tau_{i_1}, \ldots, \tau_{i_6}$ is $\tau_1, \tau_2, \tau_3, \tau_4, \tau_1, \tau_2$, and the corresponding segments $\gamma_0, \ldots, \gamma_6$ of the arc γ are labeled in the figure. Moreover, $\tau_{[\gamma_1]} = \tau_6$, $\tau_{[\gamma_2]} = \tau_8$, $\tau_{[\gamma_3]} = \tau_7$, $\tau_{[\gamma_4]} = \tau_5$ and $\tau_{[\gamma_5]} = \tau_6$.

The graph $G_{T,\gamma}$ is obtained by glueing the corresponding six tiles $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{S}_3$, $\tilde{S}_4$, $\tilde{S}_1$, and $\tilde{S}_2$. The result is shown in Figure 3.

Theorems 1.1 and 1.2 imply that $x_\gamma(x_{i_1} x_{i_2} \cdots x_{i_d})$ is equal to

$$\begin{array}{clcl}
 & x_5x_2x_2x_3x_1x_2x_8 & + & x_4x_6x_2x_3x_1x_2x_8\, y_1 \\
+ & x_5x_2x_2x_7x_5x_2x_8\, y_4 & + & x_4x_6x_2x_7x_5x_2x_8\, y_1y_4 \\
+ & x_5x_2x_8x_4x_5x_2x_8\, y_3y_4 & + & x_5x_2x_2x_7x_4x_6x_8\, y_4y_1 \\
+ & x_4x_6x_8x_4x_5x_2x_8\, y_1y_3y_4 & + & x_4x_6x_2x_7x_4x_6x_8\, y_1y_4y_1 \\
+ & x_5x_2x_8x_4x_4x_6x_8\, y_3y_4y_1 & + & x_5x_2x_2x_7x_4x_1x_3\, y_4y_1y_2 \\
+ & x_4x_1x_3x_4x_5x_2x_8\, y_1y_2y_3y_4 & + & x_4x_6x_8x_4x_4x_6x_8\, y_1y_3y_4y_1 \\
+ & x_4x_6x_2x_7x_4x_1x_3\, y_1y_4y_1y_2 & + & x_5x_2x_8x_4x_4x_1x_3\, y_3y_4y_1y_2 \\
+ & x_4x_1x_3x_4x_4x_6x_8\, y_1y_2y_3y_4y_1 & + & x_4x_6x_8x_4x_4x_1x_3\, y_1y_3y_4y_1y_2 \\
+ & x_4x_1x_3x_4x_4x_1x_3\, y_1y_2y_3y_4y_1y_2 & &
\end{array}$$

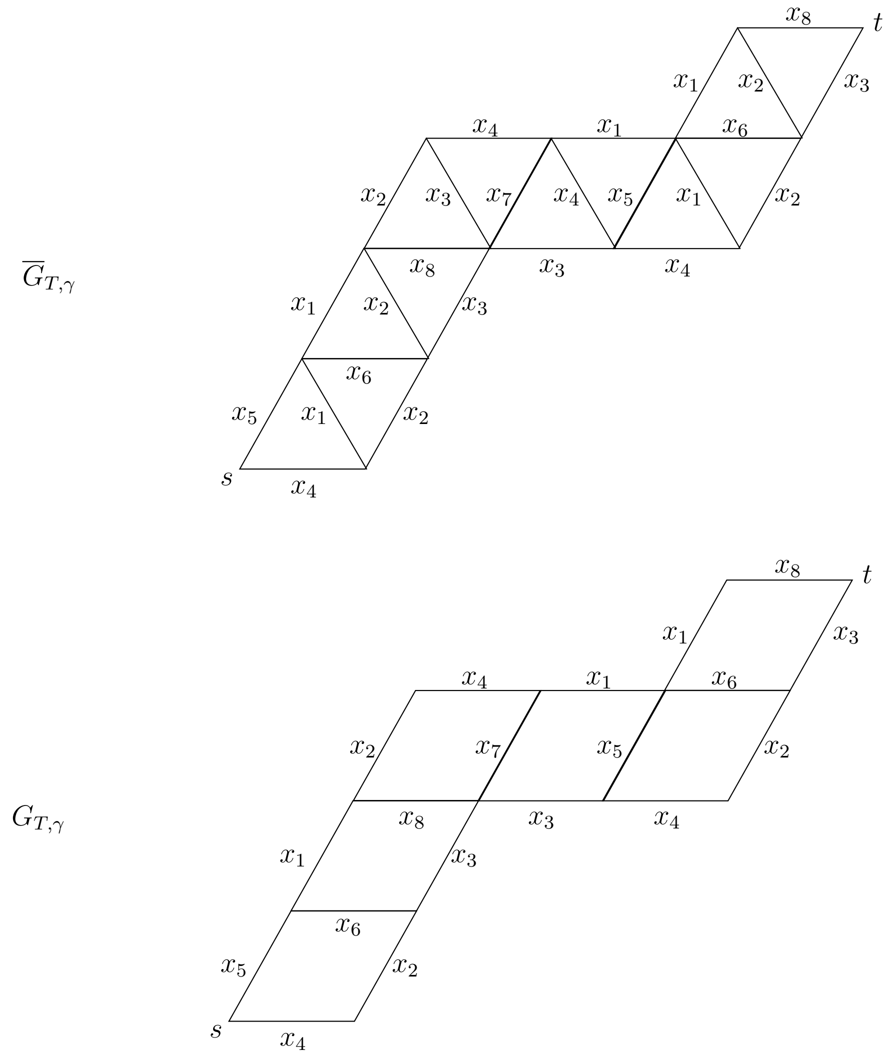

Fig. 3: Construction of the graphs $\overline{G}_{T,\gamma}$ and $G_{T,\gamma}$

which is equal to

$$\begin{array}{clcl}
 & x_1x_2^3x_3 & + & x_1x_2^2x_3x_4\,y_1 \\
+ & x_2^3\,y_4 & + & x_2^2x_4\,y_1y_4 \\
+ & x_2^2x_4\,y_3y_4 & + & x_2^2x_4\,y_1y_4 \\
+ & x_2x_4^2\,y_1y_3y_4 & + & x_2x_4^2\,y_1^2y_4 \\
+ & x_2x_4^2\,y_3y_4y_1 & + & x_1x_2^2x_3x_4\,y_1y_2y_4 \\
+ & x_1x_2x_3x_4^2\,y_1y_2y_3y_4 & + & x_4^3\,y_1^2y_3y_4 \\
+ & x_1x_2x_3x_4^2\,y_1^2y_2y_4 & + & x_1x_2x_3x_4^2\,y_3y_4y_1y_2 \\
+ & x_1x_3x_4^3\,y_1^2y_2y_3y_4 & + & x_1x_3x_4^3\,y_1^2y_2y_3y_4 \\
+ & x_1^2x_3^2x_4^2\,y_1^2y_2^2y_3y_4. & &
\end{array}$$

The first term corresponds to the matching M_- consisting of the boundary edges weighted x_5 and x_2 in the first tile, x_2 in the third tile, x_1 and x_3 in the forth, x_2 in the fifth and x_8 in the sixth tile. The twelfth term corresponds to the matching M consisting of the horizontal edges of the first three tiles and the horizontal edges of the last two tiles. Thus $M_- \ominus M = (M_- \cup M) \setminus (M_- \cap M)$ is the union of a cycle around the first tile and a cycle around the third, forth and fifth tiles, hence $y(M) = y_{i_1} y_{i_3} y_{i_4} y_{i_5} = y_1 y_3 y_4 y_1$.

To illustrate Theorem 1.3, let $k = 2$. Then $\mathcal{H}_k$ consists of the subgraphs H of $G_{T,\gamma}$ which are unions of two tiles and such that $E(H) \cap M_-$ has three elements if H is connected, respectively four elements if H has two connected components. Thus $\mathcal{H}_2$ has three elements

$$\mathcal{H}_2 = \{S_{i_3} \cup S_{i_4}, S_{i_4} \cup S_{i_5}, S_{i_1} \cup S_{i_4}\}$$

corresponding to the three terms

$$x_2^2 x_4 y_3 y_4,\, x_2^2 x_4 y_1 y_4 \text{ and } x_2^2 x_4 y_1 y_4.$$

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q-Hook formula of Gansner type for a generalized Young diagram

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Abstract. The purpose of this paper is to present the q-hook formula of Gansner type for a generalized Young diagram in the sense of D. Peterson and R. A. Proctor. This gives a far-reaching generalization of a hook length formula due to J. S. Frame, G. de B. Robinson, and R. M. Thrall. Furthurmore, we give a generalization of P. MacMahon's identity as an application of the q-hook formula.

Résumé. Le but de ce papier est présenter la q-hook formule de type Gansner pour un Young diagramme généralisé dans le sens de D. Peterson et R. A. Proctor. Cela donne une généralisation de grande envergure d'une hook length formule dû à J. S. Frame, G. de B. Robinson, et R. M. Thrall. Furthurmore, nous donnons une généralisation de l'identité de P. MacMahon comme une application de la q-hook formule.

Keywords: Generalized Young diagrams, Trace generating functions, q-hook formula, Kac-Moody Lie algebra, P. MacMahon's identity

1 Introduction

In [3], E. R. Gansner proved a *multivariable q-hook formula* for a Young diagram Y:

$$\sum_{\sigma:\text{reverse plane partition over } Y} \mathbf{q}^{\sigma} = \prod_{v \in Y} \frac{1}{1-\mathbf{q}^{\mathrm{H}(v)}}, \tag{1.1}$$

where $\mathrm{H}\,(v)$ denotes the *hook of a cell* $v \in Y$ (see section 2 and 3 for a precise definition). The identity (1.1) is a multi-q-refinement of the famous hook length formula [2]

$$\#\mathrm{STab}(Y) = \frac{(\#Y)!}{\prod_{v \in Y} \#\mathrm{H}\,(v)}, \tag{1.2}$$

due to J. S. Frame, G. de B. Robinson, and R. M. Thrall.

The purpose of this paper is to present a generalization of (1.1) for a *generalized Young diagram* in the sense of D. Peterson and R. A. Proctor. Our (multivariable) q-hook formula is:

$$\sum_{\sigma:\,(\mathrm{D}(\lambda)^{\vee};\,\leq)\text{-partition}} \mathbf{q}^{\sigma} = \prod_{\beta^{\vee} \in \mathrm{D}(\lambda)^{\vee}} \frac{1}{1-\mathbf{q}^{\mathrm{H}_{\lambda}(\beta^{\vee})}}, \tag{1.3}$$

where λ is a *finite pre-dominant integral weight* for a Kac-Moody Lie algebra and $\mathrm{D}(\lambda)^{\vee}$ is a certain set of real coroots (see section 6 for more details). We note that the identity 1.3 is equivalent with unpublished result by D. Peterson and R. A. Proctor [11].

Similarly, the identity (1.3) implies a hook length formula:

$$\mathcal{L}\left(\mathrm{D}(\lambda)^{\vee}\right) = \prod_{\beta^{\vee}\in\mathrm{D}(\lambda)^{\vee}} \frac{(\#\mathrm{D}(\lambda)^{\vee})!}{\prod_{\beta^{\vee}\in\mathrm{D}(\lambda)^{\vee}} \#\mathrm{H}_{\lambda}\left(\beta^{\vee}\right)}, \tag{1.4}$$

where $\mathcal{L}\left(\mathrm{D}(\lambda)^{\vee}\right)$ denotes the set of linear extensions (or reverse standard tableaux) of the poset $(\mathrm{D}(\lambda)^{\vee}; \leq\ )$. We note that the identity (1.4) is also equivalent with an unpublished result (see [1]) by D. Peterson, namely that:

$$\#\mathrm{Red}(w) = \frac{\ell(w)!}{\prod_{\beta\in\Phi(w)} \mathrm{ht}\,(\beta)},$$

a *hook formula* for the number of reduced decompositions of a minuscule elements w of the Kac-Moody Weyl group, where $\mathrm{Red}(w)$ denotes the set of reduced decompositions of w, $\ell(w)$ denotes the length of w, $\Phi\,(w)$ denotes the inversion set of w:

$$\Phi\,(w) = \left\{\, \beta \in \Phi_{+} \,\middle|\, w^{-1}(\beta) < 0 \,\right\},$$

and $\mathrm{ht}\,(\beta)$ denotes the height of β.

2 $(P;\leq)$-Partitions and $(c;I)$-Trace generating functions

Let $P = (P;\leq)$ be a finite partially ordered set.

Definition 2.1 *A map $\sigma : P \longrightarrow \mathbb{N} = \{0, 1, 2, \cdots\}$ is said to be a* $(P;\leq)$-partition *if:*

For each $u, v \in P$ such that $u \leq v$, we have $\sigma(u) \geq \sigma(v)$.

The set of $(P;\leq)$-partitions is denoted by $\mathsf{A}\,(P;\leq)$.

Let I be a finite *color-set* (just a set). Let $c : P \longrightarrow I$ be a *coloring* (just a map). Let q_i be an indeterminate indexed by a color $i \in I$. For each $\sigma \in \mathsf{A}\,(P;\leq)$, we define a monomial $\mathbf{q}^{\sigma}$ by:

$$\mathbf{q}^{\sigma} := \prod_{v\in P} q_{c(v)}^{\sigma(v)}.$$

We define a formal power series $T(P;\leq)$ by:

$$T(P;\leq) := \sum_{\sigma\in\mathsf{A}(P;\leq)} \mathbf{q}^{\sigma}.$$

We call $T(P;\leq)$ the $(c;I)$*-trace generating function of* $(P;\leq)$.

Definition 2.2 *Put $d := \#P$. A bijection $L : \{1, \cdots, d\} \longrightarrow P$ is said to be a* linear extension (or reverse standard tableau) of $(P;\leq)$ *if:*

$$L(k) \leq L(l) \textit{ implies } k \leq l, \quad k, l \in \{1, \cdots, d\}.$$

The set of linear extensions of $(P;\leq)$ is denoted by $\mathcal{L}\,(P;\leq)$.

Let q be another indeterminate. When we take the specialization $q_i \longmapsto q\ (i \in I)$, we denote $T(P;\leq)$ by $U(P;\leq)$.

Proposition 2.3 (R. P. Stanley [12]) *We have:*

$$U(P;\leq) = \frac{W(P;q)}{\prod_{k=1}^{d}(1-q^k)},$$

for some $W(P;q) \in \mathbb{Z}[q]$. Furthermore, we have $W(P;1) = \#\mathcal{L}(P;\leq)$.

Remark 2.4 *In section 7, we consider a certain* infinite *partially ordered set with a certain infinte color-set I. In such a situation, we define a notion of $(P;\leq)$-partitions as follows:*

We define a lattice Q by:

$$Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i,$$

where $\left\{ \alpha_i \,\middle|\, i \in I \right\}$ is a formal basis. A map $\sigma : P \longrightarrow \mathbb{N}$ is said to be a $(P;\leq)$-partition *if:*

1. *For each $u, v \in P$ such that $u \leq v$, we have $\sigma(u) \geq \sigma(v)$.*
2. *There exists at most finitely many $v \in P$ such that $\sigma(v) \geq 1$.*

The set of $(P;\leq)$-partitions is denoted by $\mathsf{A}(P;\leq)$. A (possibly infinite) partially ordered set $(P;\leq)$ is said to be a $(c;I)$-compatible poset *if:*

For each $\phi \in Q$, there exists at most finitely many $\sigma \in \mathsf{A}(P;\leq)$ such that $\sum_{v \in P} \sigma(v)\alpha_{c(v)} = \phi$.

3 Case of Young diagrams

When we draw a Young diagram, we use *nodes* instead of *cells* like FIGURE 3.1 (left) below:

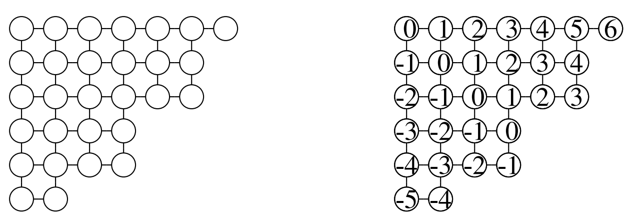


Fig. 3.1: a Young diagram and its coloring

Definition 3.1 *We equip the set $\mathbb{Y} := \mathbb{N} \times \mathbb{N}$ with the partial order:*

$$(i,j) \leq (i',j') \Longleftrightarrow i \geq i' \ \text{ and } \ j \geq j'.$$

A finite order filter Y of $\mathbb{Y}$ is called a Young diagram*.*

Definition 3.2 *Put $I := \mathbb{Z}$ as a color-set. For each node $v = (i, j) \in Y$, we attach the color $c(v)$ by:*

$$c(v) := j - i \in I.$$

see FIGURE 3.1 (right). The color $c(v)$ is known as the content of v.

In the case of above example, we can take

$$I = \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\},$$

as finite color-set.

Definition 3.3 *Let Y be a Young diagram. Let $v = (i, j) \in Y$. We define the subset* $\mathrm{H}\,(v)$ *of Y by:*

$$\begin{aligned} \mathrm{Arm}(v) &:= \left\{ (i', j') \in Y \,\middle|\, i = i' \text{ and } j < j' \right\}. \\ \mathrm{Leg}(v) &:= \left\{ (i', j') \in Y \,\middle|\, i < i' \text{ and } j = j' \right\}. \end{aligned}$$

$$\mathrm{H}\,(v) := \{v\} \sqcup \mathrm{Arm}(v) \sqcup \mathrm{Leg}(v).$$

The set $\mathrm{H}\,(v)$ *is called the* hook of $v \in Y$ *(see FIGURE 3.2).*

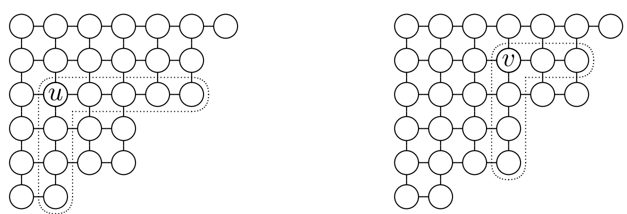


Fig. 3.2: Hooks of u and v

Then we have the following theorem:

Theorem 3.4 (E. R. Gansner [3]) *Let $Y = (Y; \leq)$ be a Young diagram. Then we have:*

$$T(Y; \leq) = \prod_{v \in Y} \frac{1}{1 - \mathbf{q}^{\mathrm{H}(v)}},$$

where $\mathbf{q}^{\mathrm{H}(v)} = \prod_{u \in \mathrm{H}(v)} q_{c(u)}$.

Remark 3.5 *A $(Y; \leq)$-partition is called a* reverse plane partition over Y.

4 Case of shifted Young diagrams

Definition 4.1 *We equip the* $\mathbb{S} := \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} \,\middle|\, i \leq j \right\}$ *with the partial order:*

$$(i, j) \leq (i', j') \Longleftrightarrow i \geq i' \text{ and } j \geq j'.$$

A finite order filter S of $\mathbb{S}$ is called a shifted Young diagram.

4.1 Case of Shifted Young Diagrams with standard hooks

Definition 4.2 *Put $I := \{\infty\} \cup \mathbb{N}$ as the color-set. For each node $v = (i, j) \in S$, we define the color $c(v)$ by:*

$$c(v) = \begin{cases} 0 & \text{if } i = j \text{ and } i \textit{ is even}, \\ \infty & \text{if } i = j \text{ and } i \textit{ is odd}, \\ j - i & \text{if } i < j. \end{cases}$$

see FIGURE 4.1.

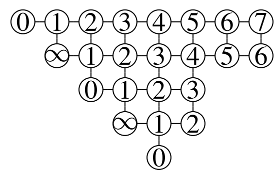


Fig. 4.1: Colors of the nodes of a shifted Young diagram

In the case of the above example, we can take $I = \{\infty, 0, 1, 2, 3, 4, 5, 6, 7\}$ as finite color-set.

Definition 4.3 *Let S be a shifted Young diagram. Let $v = (i, j) \in S$. We define the subset $\mathrm{H_D}(v)$ of S by:*

$$\mathrm{Arm_D}(v) := \left\{ (i', j') \in S \,\middle|\, i = i' \text{ and } j < j' \right\}.$$
$$\mathrm{Leg_D}(v) := \left\{ (i', j') \in S \,\middle|\, i < i' \text{ and } j = j' \right\}.$$
$$\mathrm{Tail_D}(v) := \left\{ (i', j') \in S \,\middle|\, j + 1 = i' \text{ and } j < j' \right\}.$$

$$\mathrm{H_D}(v) := \{v\} \sqcup \mathrm{Arm_D}(v) \sqcup \mathrm{Leg_D}(v) \sqcup \mathrm{Tail_D}(v).$$

The set $\mathrm{H_D}(v)$ is called the hook (of type D) of $v \in S$ *(see FIGURE 4.2).*

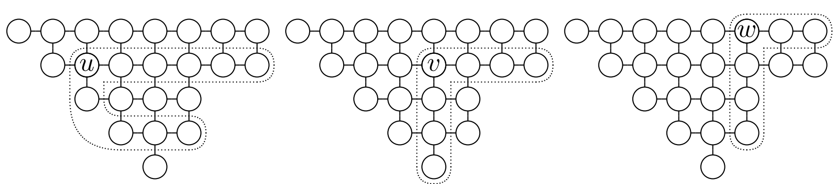


Fig. 4.2: Hooks of u, v , and w.

Then we have the following theorem:

Theorem 4.4 *Let $S = (S; \leq)$ be a shifted Young diagram with the coloring defined above. Then we have:*

$$T(S; \leq) = \prod_{v \in S} \frac{1}{1 - \mathbf{q}^{\mathrm{H_D}(v)}}.$$

Identifying q_∞ with q_0 in Theorem 4.4, we get the following theorem obtained by Gansner:

Theorem 4.5 (E. R. Gansner [3]) *Let $S = (S; \leq)$ be a shifted Young diagram. Then we have:*

$$T(S; \leq)|_{q_\infty = q_0} = \prod_{v \in S} \frac{1}{1 - \mathbf{q}^{\mathrm{H_D}(v)}}\bigg|_{q_\infty = q_0}.$$

Remark 4.6 *The proof of Theorem 4.5 by Gansner is by Hillman-Grassl algorithm [4] based on hooks of type D.*

4.2 Case of Shifted Young Diagrams with non-standard hooks

Definition 4.7 *Put $I := \mathbb{N}$ as the color-set. For each node $v = (i, j) \in S$, we define the color $c(v)$ by:*

$$c(v) = j - i,$$

see FIGURE 4.3.

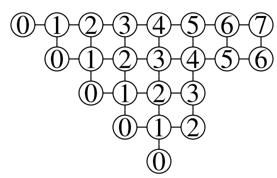


Fig. 4.3: Colors of the nodes of a shifted Young diagram

In the case of the above example, we can take $I = \{0, 1, 2, 3, 4, 5, 6, 7\}$ as a finite color-set.

Definition 4.8 *Let S be a shifted Young diagram. Let $v = (i, j) \in S$. We define a subset $\mathrm{H_B}(v)$ of S by:*

$$\begin{aligned}
\mathrm{Arm_B}(v) &:= \left\{ (i', j') \in S \,\middle|\, i = i' \text{ and } j < j' \right\}. \\
\mathrm{Leg_B}(v) &:= \left\{ (i', j') \in S \,\middle|\, i < i' \text{ and } j = j' \right\}. \\
\mathrm{Tail_B}(v) &:= \begin{cases} \{(i,i)\} \sqcup \left\{ (i', j') \in S \,\middle|\, j = i' \text{ and } j < j' \right\} & \text{if } i < j \text{ and } (j,j) \in S, \\ \varnothing & \textit{otherwise}. \end{cases}
\end{aligned}$$

$$\mathrm{H_B}(v) := \{v\} \sqcup \mathrm{Arm_B}(v) \sqcup \mathrm{Leg_B}(v) \sqcup \mathrm{Tail_B}(v).$$

The set $\mathrm{H_B}(v)$ is called a hook (of type B) of $v \in S$ *(see FIGURE 4.4).*

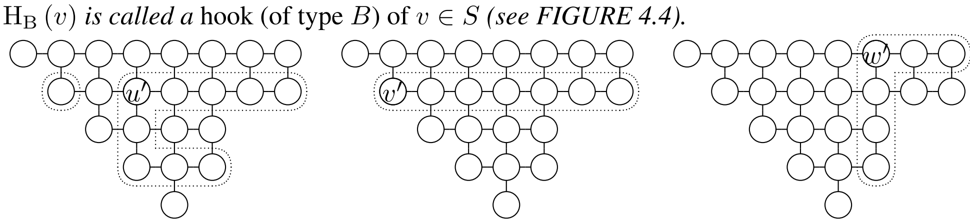


Fig. 4.4: Hooks of u', v', and w'.

Remark 4.9 *The nodes u', v', w' in FIGURE 4.4 corresponds to u, v, w in FIGURE 4.2 in the sense of Remark 4.11.*

Then we have the following theorem:

Theorem 4.10 *Let $S = (S; \leq)$ be a shifted Young diagram with the coloring defined above. Then we have:*

$$T(S; \leq) = \prod_{v \in S} \frac{1}{1 - \mathbf{q}^{\mathrm{H_B}(v)}}.$$

Remark 4.11 *Let $S = (S; \leq)$ be a shifted Young diagram. Then, there exists a bijection $S \ni v \mapsto v' \in S$ such that $\mathbf{q}^{\mathrm{H_B}(v')} = \mathbf{q}^{\mathrm{H_D}(v)}\big|_{q_\infty = q_0}$ $(v \in S)$. Hence, Theorem 4.10 is same as Theorem 4.5 except for "shapes" of hooks.*

5 Case of the bat

Definition 5.1 *Let* $\mathrm{Bat} = (\mathrm{Bat}; \leq)$ *be the poset depicted in FIGURE 5.1(left) below. The poset* Bat *is called the* bat.

Definition 5.2 *Put $I := \{1, 2, 3, 4, 5, 6, 7\}$ as the color-set. The color of each vertex is written in the vertex in FIGURE 5.1(right) below.*

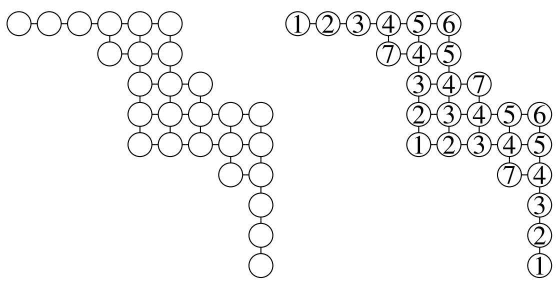


Fig. 5.1: The Bat, and the colors of the nodes of the Bat

Definition 5.3 *The hook $\mathrm{H_{Bat}}(v)$ of $v \in \mathrm{Bat}$ is defined as in FIGURE 5.2 below.*

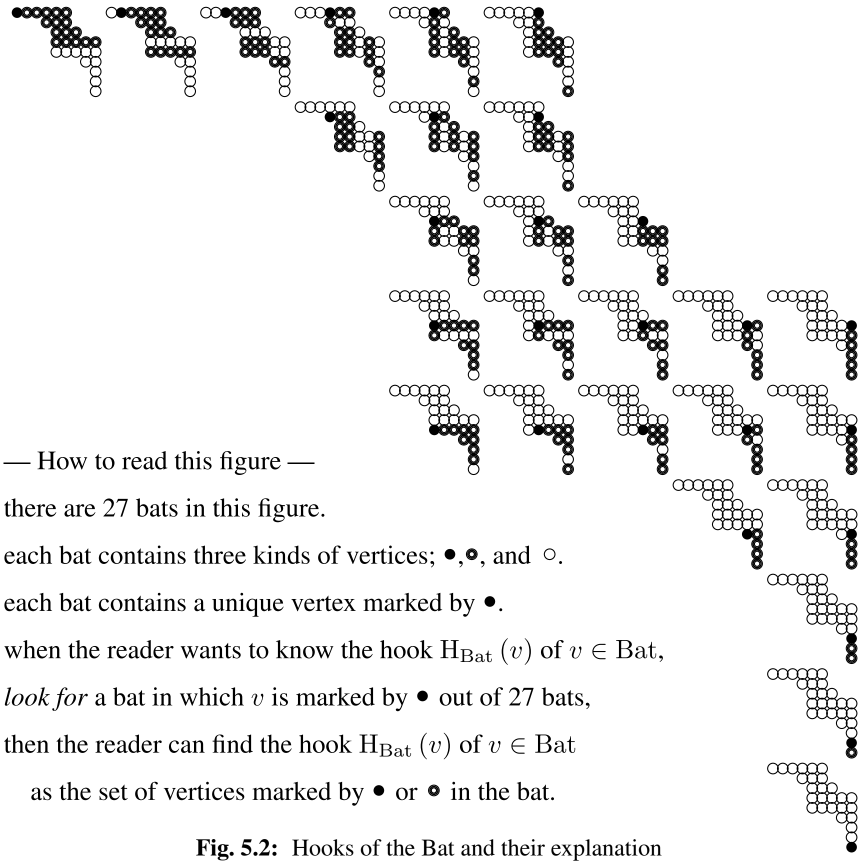
Then we have the following theorem:

Theorem 5.4 *The bat $\mathrm{Bat} = (\mathrm{Bat}; \leq)$ with the colors defined above satisfies:*

$$T(\mathrm{Bat}; \leq) = \prod_{v \in \mathrm{Bat}} \frac{1}{1 - \mathbf{q}^{\mathrm{H_{Bat}}(v)}}.$$

6 q-Hook formula of Gansner type for a generalized Young diagram

In this section, we fix a Kac-Moody Lie algebra $\mathfrak{g}$ with a simple root system $\Pi = \left\{ \alpha_i \,\middle|\, \in I \right\}$. For all undefined terminology in this section, we refer the reader to [5] [7].



— How to read this figure —

there are 27 bats in this figure.

each bat contains three kinds of vertices; •,∘, and ○.

each bat contains a unique vertex marked by •.

when the reader wants to know the hook $\mathrm{H}_{\mathrm{Bat}}(v)$ of $v \in \mathrm{Bat}$,

look for a bat in which v is marked by • out of 27 bats,

then the reader can find the hook $\mathrm{H}_{\mathrm{Bat}}(v)$ of $v \in \mathrm{Bat}$

as the set of vertices marked by • or ∘ in the bat.

Fig. 5.2: Hooks of the Bat and their explanation

Definition 6.1 *An integral weight λ is said to be* pre-dominant *if:*

$$\langle \lambda, \beta^\vee \rangle \geq -1 \quad \textit{for each } \beta^\vee \in \Phi_+^\vee,$$

where $\Phi_+^\vee$ denotes the set of positive real coroots. The set of pre-dominant integral weights is denoted by $P_{\geq -1}$. For $\lambda \in P_{\geq -1}$, we define the set $\mathrm{D}(\lambda)^\vee$ by:

$$\mathrm{D}(\lambda)^\vee := \left\{ \beta^\vee \in \Phi_+^\vee \,\middle|\, \langle \lambda, \beta^\vee \rangle = -1 \right\}.$$

The set $\mathrm{D}(\lambda)^\vee$ is called the shape of λ. *If $\#\mathrm{D}(\lambda)^\vee < \infty$, then λ is called* finite. *We regard the set $\mathrm{D}(\lambda)^\vee$ as a poset with coroot order $\leq$.*

6.1 Colors

We regard the index set I of simple roots as a color-set.

Definition 6.2 *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)^\vee$. A sequence $(\alpha_{i_1}, \cdots, \alpha_{i_d})$ of simple roots is said to be a* maximal λ-path *if:*

$$\langle s_{i_{k-1}} \cdots s_{i_1}(\lambda), \alpha_{i_k}^\vee \rangle = -1, \quad k = 1, \cdots, d.$$

The set of maximal λ-paths is denoted by $\mathrm{MPath}(\lambda)$.

Proposition 6.3 (see [8]) *Let $\lambda \in P_{\geq -1}$ be finite. Then*

1. *We have* $\mathrm{MPath}(\lambda) \neq \varnothing$.
2. *Let $\beta^\vee \in \mathrm{D}(\lambda)^\vee$. Let $(\alpha_{i_1}, \cdots, \alpha_{i_d}) \in \mathrm{MPath}(\lambda)$. Then there exists a unique $k \in \{1, \cdots, d\}$ such that*
$$s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}^\vee) = \beta^\vee.$$
3. *In Part (2), the index $i_k \in I$ only depends on $\beta^\vee$. (Namely, $i_k \in I$ is independent from the choice of maxmal λ-path.)*

Definition 6.4 *Let $\lambda \in P_{\geq -1}$ and $\beta^\vee \in \mathrm{D}(\lambda)^\vee$. Then we have a unique index $i \in I$ corresponding to $\beta^\vee$ in the sense of Proposition 6.3. We denote such $i \in I$ by $c_\lambda(\beta^\vee)$. We call $c_\lambda(\beta^\vee)$ the* color *of $\beta^\vee$.*

6.2 Hooks

Definition 6.5 *Let $\lambda \in P_{\geq -1}$ and $\beta^\vee \in \mathrm{D}(\lambda)^\vee$. We define the set $\mathrm{H}_\lambda\left(\beta\right)^\vee$ by:*

$$\mathrm{H}_\lambda\left(\beta^\vee\right) := \mathrm{D}(\lambda)^\vee \cap \Phi\left(s_\beta\right)^\vee.$$

where $\Phi\left(s_\beta\right)^\vee$ denotes the inversion set of the reflection corresponding to β:

$$\Phi\left(s_\beta\right)^\vee = \left\{\, \gamma^\vee \in \Phi_+^\vee \,\middle|\, s_\beta(\gamma^\vee) < 0 \,\right\}.$$

Proposition 6.6 (see [8]) *Let $\lambda \in P_{\geq -1}$ be finite and $\beta^\vee \in \mathrm{D}(\lambda)^\vee$. Then we have:*

1. $\sum_{\gamma^\vee \in \mathrm{H}_\lambda(\beta^\vee)} \alpha_{c_\lambda(\gamma^\vee)} = \beta$.
2. $\#\mathrm{H}_\lambda\left(\beta^\vee\right) = \mathrm{ht}\left(\beta\right)$.

6.3 Main Theorem and Corollaries

Theorem 6.7 (see [8]) *Let $\lambda \in P_{\geq -1}$ be finite. Then we have:*

$$T(\mathrm{D}(\lambda)^\vee; \leq) = \prod_{\beta^\vee \mathrm{D}(\lambda)^\vee} \frac{1}{1 - \mathbf{q}^{\mathrm{H}_\lambda(\beta^\vee)}} = \prod_{\beta^\vee \mathrm{D}(\lambda)^\vee} \frac{1}{1 - \mathbf{q}^{\beta}}.$$

Taking the specialization $q_i \longmapsto q$, we get:

Corollary 6.8 *Let $\lambda \in P_{\geq -1}$ be finite. Then we have:*

$$U(\mathrm{D}(\lambda)^\vee; \leq) = \prod_{\beta^\vee \mathrm{D}(\lambda)^\vee} \frac{1}{1 - q^{\#\mathrm{H}_\lambda(\beta^\vee)}} = \prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee} \frac{1}{1 - q^{\mathrm{ht}(\beta)}}.$$

Remark 6.9 *A statement equivalent with Corollary 6.8 is also given in [11].*

Hence, by Proposition 2.3, we get:

Corollary 6.10 *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)^\vee$. Then we have:*

1. $W(P;q) = \dfrac{\prod_{k=1}^{d}(1-q^k)}{\prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee}(1-q^{\#\mathrm{H}_\lambda(\beta^\vee)})} = \dfrac{\prod_{k=1}^{d}(1-q^k)}{\prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee}(1-q^{\mathrm{ht}(\beta)})}.$

2. $\#\mathcal{L}\left(\mathrm{D}(\lambda)^\vee;\le\right) = \dfrac{d!}{\prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee}\#\mathrm{H}_\lambda\left(\beta^\vee\right)} = \dfrac{d!}{\prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee}\mathrm{ht}\left(\beta\right)}.$

Remark 6.11 *All shapes explaind in section 3,4, and 5 are realized as shapes of some $\lambda \in P_{\ge -1}$ over some root systems of finite types. Furthermore, colors and hooks defined in section 3,4, and 5 are compatible with those defined in this section.*

- *A Young diagram is realized over a root system of type A.*
- *A shifted Young diagram with hooks of type D is realized over a root system of type D.*
- *A shifted Young diagram with hooks of type B is realized over a root system of type B.*
- *The bat (or an order filter of the bat) is realized over a root system of type E_7.*

There are 17 classes of generalized Young diagrams (15 of 17 are simply-laced). We note that many of them are realized over root systems of indefinite types (see [10] [13]).

Remark 6.12 *Corollary 6.10 (2) gives a proof of Peterson's hook formula. Another proof of Peterson's hook formula is given in [9].*

7 An application to infinite rank case

Although Theorem 6.7 holds for a finite pre-dominant integral weight, there exist several cases where Theorem 6.7 holds for an *infinite* pre-dominant integral weight.

Let A_∞ denote the Dynkin diagram depicted below:

$$\cdots - (-2) - (-1) - (0) - (1) - (2) - \cdots$$

Here, an integer in a vertex is the index of the vertex. Let ω_i denote the fundamental weight corresponding to an index $i \in \mathbb{Z}$. Let $\lambda = \sum_{i\in\mathbb{Z}} c_i\omega_i$ be an integral weight satisfying the following properties:

1. For each $i \in \mathbb{Z}$, we have $c_i \in \{1, 0, -1\}$.
2. Let $i, j \in \mathbb{Z}$ satisfy $i < j$ and $c_i = c_j = \pm 1$. Then we have $c_n = \mp 1$ for some $i < n < j$.
3. There exists at least one and at most finitely many $i \in \mathbb{Z}$ such that $c_i \neq 0$.
4. Let $i \in \mathbb{Z}$ be the minimum (or maximum) integer such that $c_i \neq 0$. Then we have $c_i = -1$.
5. (normalization) We have $\sum_{i\in\mathbb{Z}} c_i \cdot i = 0$.

Then λ is an infinite pre-dominant integral weight. The poset $(\mathrm{D}(\lambda)^\vee;\le)$ is $(c_\lambda;\mathbb{Z})$-compatible. Furthermore, the statement of Theorem 6.7 holds for this λ:

$$T(\mathrm{D}(\lambda)^\vee;\le) = \prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee} \frac{1}{1-\mathbf{q}^{\mathrm{H}_\lambda(\beta^\vee)}} = \prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee} \frac{1}{1-\mathbf{q}^{\beta}}. \tag{7.1}$$

Taking the specialization $q_i \longmapsto q \ (i \in \mathbb{Z})$ in (7.1), we get:

$$U(\mathrm{D}(\lambda)^\vee; \leq) = \prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee} \frac{1}{1 - q^{\#\mathrm{H}_\lambda(\beta^\vee)}} = \prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee} \frac{1}{1 - q^{\mathrm{ht}(\beta)}}. \tag{7.2}$$

In particular, we let $\lambda = -\omega_0$, which obviously satisfies the above properties. Then we have:

$$\mathrm{D}(-\omega_0)^\vee = \left\{ \sum_{n=i}^{j} \alpha_n^\vee \,\middle|\, i \leq 0 \leq j \right\}.$$

We can identify $(\mathrm{D}(-\omega_0)^\vee; \leq)$ with a poset $(\mathbb{Y}; \geq)$, and $(\mathrm{D}(-\omega_0)^\vee; \leq)$-partitions with *plane partitions*. Hence, the identity (7.1) can be rewritten as:

$$T(\mathbb{Y}; \geq) = T(\mathrm{D}(-\omega_0)^\vee; \leq) = \prod_{i \leq 0 \leq j} \frac{1}{1 - \mathbf{q}^{\sum_{i \leq n \leq j} \alpha_i}}. \tag{7.3}$$

Similarly, the identity (7.2) can be rewritten as:

$$U(\mathbb{Y}; \geq) = U(\mathrm{D}(-\omega_0)^\vee; \leq) = \prod_{i \leq 0 \leq j} \frac{1}{1 - q^{j-i+1}} = \prod_{n \geq 1} \left(\frac{1}{1 - q^n} \right)^n. \tag{7.4}$$

The identity (7.4) is known as MacMahon's identity (see [6]) for the generating function of plane partitions.

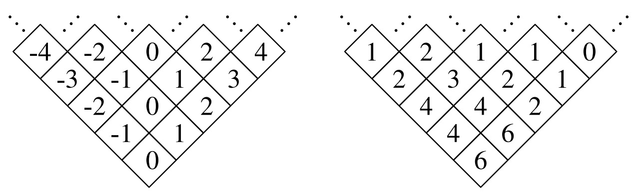


Fig. 7.1: The coloring of $(\mathbb{Y}; \geq)$ and an example of plane partition

Remark 7.1 *We get similar results for infinite Dynkin diagrams D_∞ and B_∞ below.*

These give the identities for the generating function of shifted *plane partitions. For example, a "shifted version" of the identity* (7.4) *can be written as:*

$$U(\mathbb{S}; \geq) = \prod_{n \geq 1} \left(\frac{1}{1 - q^n} \right)^{\lceil n/2 \rceil} = \prod_{n \geq 1} \left(\frac{1}{1 - q^{2n-1}} \frac{1}{1 - q^{2n}} \right)^n.$$

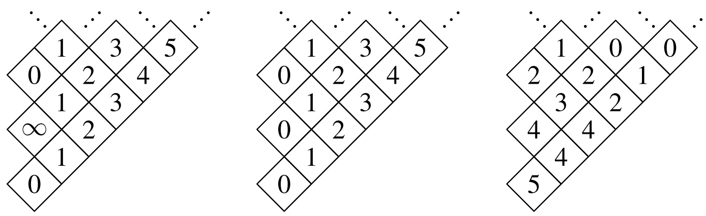


Fig. 7.2: The coloring of type D and of type B of $(\mathbb{S}; \geq)$, and an example of shifted plane partition

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Another bijection between 2-triangulations and pairs of non-crossing Dyck paths

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Abstract. A k-triangulation of the n-gon is a maximal set of diagonals of the n-gon containing no subset of $k+1$ mutually crossing diagonals. The number of k-triangulations of the n-gon, determined by Jakob Jonsson, is equal to a $k \times k$ Hankel determinant of Catalan numbers. This determinant is also equal to the number of k non-crossing Dyck paths of semi-length $n-2k$. This brings up the problem of finding a combinatorial bijection between these two sets. In FPSAC 2007, Elizalde presented such a bijection for the case $k=2$. We construct another bijection for this case that is stronger and simpler that Elizalde's. The bijection preserves two sets of parameters, degrees and generalized returns. As a corollary, we generalize Jonsson's formula for $k=2$ by counting the number of 2-triangulations of the n-gon with a given degree at a fixed vertex.

Résumé. Une k-triangulation du n-gon est un ensemble maximal de diagonales du n-gon ne contenant pas de sous-ensemble de $k+1$ diagonales mutuellement croisant. Le nombre de k-triangulations du n-gon, déterminé par Jakob Jonsson, est égal à un déterminant de Hankel $k \times k$ de nombres de Catalan. Ce déterminant est aussi égal au nombre de k chemins de Dyck de largo $n-2k$ que ne pas se croiser. Cela porte le problème de trouver une bijection de type combinatoire entre ces deux ensembles. À la FPSAC 2007, Elizalde a présenté une telle bijection pour le cas $k=2$. Nous construisons une autre bijection pour ce cas qui est plus forte et plus simple que de l'Elizalde. La bijection conserve deux ensembles de paramètres, les degré et les retours généralisée. De ce, nous généralisons la formule de Jonsson pour $k=2$ en comptant le nombre de 2-triangulations du n-gon avec un degré à un vertex fixe.

Keywords: k-triangulations, non-crossing Dyck paths, combinatorial bijection.

1 Introduction

The set of triangulations of n points in convex position on the plane has been studied for a long time because of its interesting combinatorial properties. In recent years, more general structures known as k-triangulations have been shown to satisfy many of the interesting properties of the classical triangulations.

A k-triangulation of the n-gon is a maximal set of diagonals of the n-gon containing no subset of $k+1$ mutually crossing diagonals. Note that the case $k=1$ corresponds to the standard triangulations of the n-gon.

This concept was introduced in 1992 by Capoyleas and Pach [3], who gave a tight bound for the number of diagonals in a k-triangulation. Later, Nakamigawa [13] and independently Dress, Kooolen and Moulton [5], showed that every k-triangulation attains that bound. Nakamigawa also showed that k-triangulations satisfy a flip property similar to the one of ordinary triangulations. This result has been recently strengthened and clarified with the discovery of the analogue of triangles for k-triangulations by Pilaud and Santos [16] and independently in [14].

In 2005, Jonsson [8] proved that the number of k-triangulations of the n-gon is equal to the following $k \times k$ Hankel determinant:

$$\det \begin{bmatrix} C_{n-2} & C_{n-3} & \iddots & \iddots & C_{n-k-1} \\ C_{n-3} & \iddots & \iddots & C_{n-k-1} & \iddots \\ \iddots & \iddots & \iddots & \iddots & \iddots \\ \iddots & C_{n-k-1} & \iddots & \iddots & C_{n-2k+1} \\ C_{n-k-1} & \iddots & \iddots & C_{n-2k+1} & C_{n-2k} \end{bmatrix}, \tag{1}$$

where C_n is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The set of all k non-crossing Dyck paths of semilength $n - 2k$ is counted by the same determinant. This follows from an almost direct application of the Lindström-Gessel-Viennot theorem ([12], [7]) for counting non-intersecting lattice paths (see [10] for details on the history of this idea). A combinatorial bijection between the set of k-triangulations of the n-gon and the corresponding set of k non-crossing Dyck paths would constitute a simpler proof of the formula for the number of k-triangulations. Elizalde [6] constructed such a bijection for the case $k = 2$. This problem was posed by Jonsson [8] and it has been restated by Krattenthaler [11] and Elizalde [6]. A combinatorial bijection for the general case (and for more general objects) has been constructed by Rubey [17]. Also Krattenthaler [11] found a combinatorial proof using growth diagrams.

Our goal is to construct another bijection for the case $k = 2$. Our bijection is stronger than that in [6] because it transforms two simple parameters for 2-triangulations, the degrees (number of neighbors) at two consecutive vertices, into two simple parameters for pairs of non-crossing Dyck paths, the number of (generalized) returns.

We begin by showing a way to recursively generate the set of k-triangulations and the set of k non-crossing Dyck paths. Then we introduce a k-tuple of parameters for the set of non-crossing Dyck paths that generalizes the usual returns of a single Dyck path. We show, by means of a family of involutions, that the distribution of non-crossing Dyck paths with respect to these generalized returns is independent of their order. In particular, we obtain a generalization for non-crossing Dyck paths of the fact that (single) Dyck paths have the same distribution with respect to the height of the first peak and with respect to the number of returns. This result is a particular case of a theorem originally found by Brak and Essam [2]. Krattenthaler [10] found a combinatorial proof of this theorem using semistandard tableaux. A determinantal formula for the number of non-crossing paths with a given number of returns in the lowermost path follows by applying the Lindström-Gessel-Viennot theorem. Closed product formulas for this determinant are shown in [15]. There are also product formulas for the determinant (1), see [10].

The involutions that we introduce for non-crossing Dyck paths to prove the equidistribution of the generalized returns are also essential in the construction of our bijection between 2-triangulations and pairs of non-crossing Dyck paths. The bijection sends the 2-triangulations having degrees c_0 and c_1 at two fixed consecutive vertices onto the set of all pairs of non-crossing Dyck paths having (generalized) returns c_0 and c_1. Finally, we obtain a formula for the number of 2-triangulations having degree c at a fixed vertex, which refines Jonsson's formula (1) for $k = 2$. We conjecture that a similar formula holds for every k.

2 The set of k-triangulations

Let S be a set of n points in convex position. A straight line segment joining two points in S is a diagonal of S. Two diagonals of S cross if they intersect in a point not in S. A k-triangulation of S is a maximal set of diagonals of S containing no subset of $k+1$ mutually crossing diagonals.

Starting with any point of S and proceeding in the counterclockwise direction, we label the points of S with the numbers $0, 1, \ldots, n-1$. In what follows, all operations on the labels of the vertices are performed modulo n, even if this is not explicitly stated. Also, we assume that $n \geq 2k+1$.

We identify the diagonals of S with the set of pairs of elements in S. In this way, the set of diagonals of S, denoted Σ_n is simply $\Sigma_n = \big\{\{i,j\} : \ 0 \leq i < j \leq n-1\big\}$.

It is clear that every diagonal in a set of $k+1$ mutually crossing diagonals must have at least k vertices of S on each side. Therefore, every k-triangulation contains the kn trivial diagonals of the form $\{i, i+h\}$ with $1 \leq h \leq k$. Of these diagonals, we want to keep only those of the form $\{i, i+k\}$.

Let $\Sigma_n^k = \Sigma_n - \big\{\{a, a+i\} : \ a \in \{0, \ldots, n-1\}, \quad i \in \{1, \ldots, k-1\}\big\}$, and $\Gamma_n^k = \big\{\{a, a+k\} : a \in \{0, \ldots, n-1\}\big\}$.

The decomposition theorems in [16] and [14] make clear that Γ_n^k is the natural boundary of a k-triangulation. In what follows, we assume that k-triangulations contain only these trivial diagonals. In other words, we redefine the k-triangulations of S as maximal subsets of Σ_n^k containing no subset of $k+1$ mutually crossing diagonals. We denote the set of all k-triangulations of S by $\mathcal{T}_n^k$.

Given a set of edges $U \subseteq \Sigma_n^k$, the set of neighbors in U of a vertex i is defined by $\mathsf{N}_i(U) = \{j : \ \{i,j\} \in U\}$. Also, for $0 \leq i \leq k-1$, we define the neighbors to the left and right of a given vertex a by $\mathsf{L}_i(a, U) = \{j \in \mathsf{N}_i(U) : \ i < j \leq a\}$ and $\mathsf{R}_i(a, U) = \{j \in \mathsf{N}_i(U) : \ a \leq j < i\}$.

Now we associate to every k-triangulation of S a partial order, by considering the sets of k mutually crossing diagonals incident to the vertices $0, \ldots, k-1$.

For $\Gamma_n^k \subseteq U \subseteq \Sigma_n^k$, define

$$\mathcal{C}(U) = \Big\{(a_0, \ldots, a_{k-1}) : \ \{0, a_0\}, \cdots, \{k-1, a_{k-1}\} \text{ are mutually crossing diagonals of } U \Big\}.$$

Note that $k \leq a_0 < \cdots < a_{k-1} \leq n-1$ for all $(a_0, \ldots, a_{k-1})$ in $\mathcal{C}(U)$.

The set $\mathcal{C}(U)$ is partially ordered by the (direct) product order $(a_0, \ldots, a_{k-1}) \leq (b_0, \ldots, b_{k-1})$ if and only if $a_i \leq b_i$ for all $i \in \{0, \ldots, k-1\}$. In fact, $\mathcal{C}(U)$ is a lattice with this order.

Now suppose we add a point, with label n, to the set S in such a way that the set $S' = S \cup \{n\}$ is in convex position and the point n is located between the points $n-1$ and 0 of S. The k-triangulations of S' can be obtained from those of S by applying a procedure that splits mutually crossing diagonals (incident to the vertices $0, \ldots, k-1$) in a k-triangulations of S. Let Σ_{n+1}^k and $\mathcal{T}_{n+1}^k$ be be the set of diagonals and the set of k-triangulations of S', respectively.

Let $\Gamma_n^k \subseteq W \subseteq \Sigma_n^k$. For $(a_0, \ldots, a_{k-1}) \in \mathcal{C}(W)$ define $\Psi_{\vec{a}}(W) \subseteq \Sigma_{n+1}^k$ by

$$\begin{aligned}\Psi_{\vec{a}}(W) = \quad & \big\{\{i,j\} \in W : \ k \leq i < j \leq n-1\big\} \quad \cup \\ & \big\{\{n,j\} : \ j \in \mathsf{R}_0(a_0, W)\big\} \quad \cup \\ & \big\{\{i,j\} : \ 0 \leq i \leq k-2, \quad j \in \mathsf{L}_i(a_i, W) \cup \mathsf{R}_{i+1}(a_{i+1}, W)\big\} \quad \cup \\ & \big\{\{k-1,j\} : \ j \in \mathsf{L}_{k-1}(a_{k-1}, W)\big\} \quad \cup \quad \big\{\{n, k-1\}\big\}.\end{aligned}$$

Let $\mathbf{T}_n : \bigcup_{T \in \mathcal{T}_n^k} \{T\} \times \mathcal{C}(T) \longrightarrow \Sigma_{n+1}^k$ be the function defined by $\mathbf{T}_n(T, \vec{a}) = \Psi_{\vec{a}}(T)$.

The importance of the function $\mathbf{T}_n$ is that it generates all the k-triangulations of S' in an injective way.

Theorem 1 *The function $\mathbf{T}_n$ establishes a bijection between $\bigcup_{T\in\mathcal{T}_n^k}\{T\}\times\mathcal{C}(T)$ and $\mathcal{T}_{n+1}^k$.*

Though with a different presentation, this fact was proved by Nakamigawa [13].

3 The set of non-crossing Dyck paths

For our purposes, it is convenient to define Dyck paths as integer functions as follows:

Given $m \geq 1$, a function $f : \{1,\ldots,m\} \longrightarrow \mathbb{N}$ is called a *Dyck path* of length m if it satisfies the following properties: (1) f is non-decreasing, (2) $i \leq f(i)$, for all $i \in \{1,\ldots,m\}$, and (3) $f(m)=m$.

For $m_1 \leq m_2$, let $\mathcal{F}_{m_1,m_2} = \{f : \{m_1,\ldots,m_2\} \longrightarrow \mathbb{N} : \quad f \text{ is non-decreasing}\}$. We write $f \leq g$ if $f(i) \leq g(i)$ for all $i \in \{m_1,\ldots,m_2\}$.

For any two non-decreasing functions f and g, a *(generalized) return* of g to f is a value i in the domain of f and g for which $f(i)=g(i)$. The set of all returns of g to f is denoted by $\text{ret}(g,f)$.

Define $\mathcal{D}_m^k = \{(f_0,\ldots,f_{k-1}) : \quad f_i \text{ is a Dyck path of length } m \text{ for all } i \text{ and} f_{i-1} \geq f_i \text{ for all } i\}$.

Note that the upper-most path is listed first in the k-tuple of paths. This is necessary to make the proof of Theorem 7 simpler.

For $i \in \{0,\ldots,k-1\}$ and $D=(f_0,\ldots,f_{k-1})$ define $\text{ret}_i(D) = \text{ret}(f_i,f_{i+1}) \cup \{0\}$, where f_k is the identity function, $f_k = \text{id}$.

The crossings among the returns of $D \in \mathcal{D}_m^k$ are defined by

$$\begin{aligned}\mathcal{C}(D) \quad = \quad \{(a_0,\ldots,a_{k-1}) : \quad & a_i \in \text{ret}_i(D) \text{ for } i \in \{0,\ldots,k-1\},\\ & a_{i-1} \leq a_i \text{ for } i \in \{1,\ldots,k-1\}\}.\end{aligned}$$

Clearly, $\mathcal{C}(D) \subseteq \text{ret}_0(D) \times \cdots \times \text{ret}_{k-1}(D)$, and $\mathcal{C}(D)$ is a lattice when regarded as a poset with the inherited product order. Note that for a k-triangulation T, $\mathcal{C}(T)$ consists of strictly increasing functions, while $\mathcal{C}(D)$ consists of non-decreasing functions.

Continuing the analogy with our treatment of k-triangulations, we introduce now the set of returns to the left and right of a given value: $\bar{\mathsf{L}}_i(a,D) = \{j \in \text{ret}_i(D) : \quad j \leq a\}$, and $\bar{\mathsf{R}}_i(a,D) = \{j \in \text{ret}_i(D) : a \leq j\}$.

For $D=(D_1,\ldots,D_{k-1}) \in \mathcal{D}_m^k$ and $\vec{a}=(a_0,\ldots,a_{k-1}) \in \mathcal{C}(D)$, define $\bar{\Psi}_{\vec{a}}(D) = (\hat{D}_1,\ldots,\hat{D}_{k-1})$ where $\hat{D}_i : \{0,\ldots,m+1\} \to \mathbb{N}$ is given by

$$\hat{D}_i(j) = \begin{cases} D_i(j) & \text{for} \quad j \in \{0,\ldots,a_i\},\\ D_i(j)+1 & \text{for} \quad j \in \{a_i+1,\ldots,m\},\\ m+1 & \text{if} \quad j = m+1.\end{cases}$$

Note that $\bar{\Psi}_{\vec{a}}(D) \in \mathcal{D}_{m+1}^k$, thanks to the condition $a_0 \leq \cdots \leq a_{k-1}$.

Finally, completing the analogy with k-triangulations, the functions $\bar{\Psi}_{\vec{a}}$ can be used to generate recursively the sets $\mathcal{D}_m^k$.

Let $\mathbf{D}_m : \bigcup_{D\in\mathcal{D}_m^k}\{D\}\times\mathcal{C}(D) \longrightarrow \mathcal{D}_{m+1}^k$ be defined by $\mathbf{D}_m(D,\vec{a}) = \bar{\Psi}_{\vec{a}}(D)$.

Theorem 2 *For every $m \geq 1$, $\mathbf{D}_m$ is a bijection.*

Now we study a family of involutions on the set $\mathcal{F}_{m_1,m_2}$ of non-decreasing integer functions on $\{m_1,\ldots,m_2\}$. For $f,h \in \mathcal{F}_{m_1,m_2}$, with $f \leq h$, the interval between f and g is given by $\mathcal{I}(f,h) = \{g \in \mathcal{F}_{m_1,m_2} : \quad f \leq g \leq h\}$.

Theorem 3 *For all $m_1 \leq m_2$, and for all $f, h \in \mathcal{F}_{m_1,m_2}$ with $f \leq h$, there exists an involution $\mu_{f,h}$ on $\mathcal{I}(f,h)$ such that*

$$\left|\mathrm{ret}(g,\ f)\right| = \left|\mathrm{ret}\big(h,\ \mu_{f,h}(g)\big)\right| \quad \textit{and} \quad \left|\mathrm{ret}(\mu_{f,h}(g),\ f)\right| = \left|\mathrm{ret}\big(h, g\big)\right|.$$

This theorem can be proved by applying induction on two parameters, the length of $m_2 - m_1 + 1$ of the domain of the functions, and the minimum distance $\min\big\{h(i) - f(i),\ m_1 \leq i \leq m_2\big\}$ between the functions, see [14].

The previous theorem implies that the number of chains $f_0 \geq \cdots \geq f_k$ of length k in $\mathcal{F}_{m_1,m_2}$ having a prescribed number of returns $\left|\mathrm{ret}(f_i, f_{i-1})\right| = c_i$, for $i \in \{1, \ldots, k\}$, does not depend on the order of the numbers c_i. We state this fact for chains with fixed endpoints. Define, for $m_1 \leq m_2$, $k \geq 1$, and $f_k \leq f_{-1} \in \mathcal{F}_{m_1,m_2}$,

$$\begin{array}{rcllr} \mathcal{H}^k_{f_{-1},f_k}(c_0,\ldots,c_k) & = & \big\{\ (f_0,\ldots,f_{k-1}) : & f_i \in \mathcal{I}(f_k, f_{-1}) & \text{for } i \in \{0,\ldots,k-1\}, \\ & & & f_{i-1} \geq f_i & \text{for } i \in \{0,\ldots,k\}, \\ & & & \left|\mathrm{ret}(f_{i-1}, f_i)\right| = c_i & \text{for } i \in \{0,\ldots,k\}\big\}. \end{array}$$

Corollary 4 *Let $m_1 \geq m_2$, $k \geq 1$ and $f_{-1}, f_k \in \mathcal{F}_{m_1,m_2}$. Then for every permutation σ on $\{0,\ldots,k\}$,*

$$\left|\mathcal{H}^k_{f_{-1},f_k}(c_0,\ldots,c_k)\right| = \left|\mathcal{H}^k_{f_{-1},f_k}(c_{\sigma(0)},\ldots,c_{\sigma(k)})\right|.$$

Returning to the study of Dyck paths, let $\mathrm{id}(i) = i$ and $\mathrm{cons}_m(i) = m$ for $m \geq 1$. Note that the set of all (single) Dyck paths of length m is equal to the interval $\mathcal{I}(\mathrm{id}, \mathrm{cons}_m)$ of $\mathcal{F}_{0,m}$.

It is well-known that Dyck paths have the same distribution with respect to the number of returns and with respect to the height of the first (or last) peak. This can be shown by means of an involution, see [4]. The following generalization of this property to sets of non-crossing Dyck paths is a simple consequence of Corollary 4. It tell us that the number of k non-crossing Dyck paths of length m for which the height of the first (or last) peak in the the upper-most path is c is equal to the number of paths having c returns in the lower-most path.

Theorem 5 *For all $m \geq 1$, $k \geq 1$ and $c \in \{0,\ldots,m\}$,*

$$\begin{aligned} &\left|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m : f_0(1) = c\ \Big\}\right| \\ &= \left|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m : \left|\{f_0(i) = m\}\right| = c\ \Big\}\right| \\ &= \left|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m : \left|\{f_{k-1}(i) = i\}\right| = c\ \Big\}\right|. \end{aligned}$$

Proof. The first equality is obvious. The second follows from Corollary 4:

$$\begin{aligned} &\left|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m :\ \left|\{f_0(i) = m\}\right| = c\ \Big\}\right| \\ &\quad = \sum_{r_0,\ldots,r_{k-1}} \left|\mathcal{H}^k_{\mathrm{id},\mathrm{cons}_m}(c, r_0,\ldots,r_{k-1})\right| = \sum_{r_0,\ldots,r_{k-1}} \left|\mathcal{H}^k_{\mathrm{id},\mathrm{cons}_m}(r_0,\ldots,r_{k-1},c)\right| \\ &\quad = \left|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m :\ \left|\{f_{k-1}(i) = i\}\right| = c\ \Big\}\right|. \end{aligned}$$

□

The previous theorem is a particular case of a result by Brak and Essam [2]. A combinatorial proof using semistandard tableaux is given by Krattenthaler [10].

It is not difficult to obtain determinantal formulas for the cardinality of the sets in our previous theorem. The ballot numbers (sometimes called generalized Catalan numbers) $B(n,m) = \frac{m-n+1}{m+1}\binom{m+n}{m}$ count the number of paths on the integer lattice having north and east steps, starting at $(0,0)$, ending at (n,m) and not going under the line $x=y$; see [9, 18]. Hence, ballot numbers count Dyck paths having a given height of the last peak. Combining this idea with the Lindström-Gessel-Viennot Theorem, it is easy to obtain a formula for the number of k non-crossing Dyck paths such that the height of the last peak for the top path is a given value c. By the previous theorem, this formula also counts the paths such that the lowest path has c returns. We state this formula in the following theorem.

Theorem 6 *The number of k non-crossing Dyck paths of length m such that the lowest path has exactly c returns is given by*

$$\det \begin{bmatrix} C_m & C_{m+1} & \iddots & \iddots & B^k_{m+k-1}(c) \\ C_{m+1} & \iddots & \iddots & C_{m+k-1} & \vdots \\ \iddots & \iddots & \iddots & \iddots & \vdots \\ \iddots & C_{m+k-1} & \iddots & \iddots & B^k_{m+2k-3}(c) \\ C_{m+k-1} & \iddots & \iddots & C_{m+2k-3} & B^k_{m+2k-2}(c) \end{bmatrix},$$

where $B^k_m(h) = \frac{2k+h-2}{m}\binom{2m-2k-h+1}{m-1}$.

Product formulas for the determinant in the previous theorem are found in [15] and [10].

4 A strong bijection between $\mathcal{T}^2_n$ and $\mathcal{D}^2_n$

In spite of the similarities that we have found between k-triangulations and non-crossing Dyck paths, we are able to construct a bijection between these sets only[(i)] for the case $k=2$. As pointed out in the introduction, a combinatorial bijection between $\mathcal{T}^2_n$ and $\mathcal{D}^2_n$ has already been found by Elizalde [6]. Our bijection presents the advantage of sending two simple parameters in $\mathcal{T}^2_n$, the degree (number of neighbors) at two consecutive vertices, into two simple parameter in $\mathcal{D}^2_n$, the number of returns of each path. From this property, we derive as a corollary a formula for the number of 2-triangulations with a given degree at a fixed vertex.

Theorem 7 *For all $n \geq 5$, there exists a bijection ω_n from the set $\mathcal{T}^2_n$ of all 2-triangulations of the n-gon onto the set $\mathcal{D}^2_{n-4}$ of all pairs of non-crossing Dyck paths of length $n-4$, such that for all $T \in \mathcal{T}^2_n$*

(1) $$\begin{aligned} \big|\mathsf{N}_0(T)\big| &= \big|\mathrm{ret}_0\big(\omega_n(T)\big)\big| \quad \textit{and} \\ \big|\mathsf{N}_1(T)\big| &= \big|\mathrm{ret}_1\big(\omega_n(T)\big)\big|. \end{aligned}$$

(2) Moreover, if $\beta^i_T : \mathsf{N}_i(T) \longrightarrow \mathsf{ret}_i\big(\omega_n(T)\big)$, for $i=0,1$, is the order-preserving bijection guaranteed to exist by (1), then $\beta_T : \mathcal{C}(T) \longrightarrow \mathcal{C}\big(\omega_n(T)\big)$ given by $\beta_T\big((i,j)\big) = \big(\beta^0_T(i), \beta^1_T(j)\big)$, is an isomorphism between the crossings of T and the crossings among the returns of $\omega_n(T)$.

(i) The case $k=1$ is elementary.

Proof. By induction on n:

If $n = 5$, then $\mathcal{T}_5^2$ and $\mathcal{D}_1^2$ contain one element each. Let T be the 2-triangulation in $\mathcal{T}_5^2$ and let D be the element of $\mathcal{D}_1^2$. Then, $\mathsf{N}_0(T) = \{2,3\}, \mathsf{N}_1(T) = \{3,4\}, \mathcal{C}(T) = \{(2,3),\ (2,4),\ (3,4)\}, \mathrm{ret}_0(D) = \{0,1\}, \mathrm{ret}_1(D) = \{0,1\}$ and $\mathcal{C}(D) = \{(0,0),(0,1),(1,1)\}$, so the map $\{T\} \to \{D\}$ satisfies properties (1) and (2).

Let $n \geq 5$, and suppose $\omega_n \colon \mathcal{T}_n^2 \longrightarrow \mathcal{D}_{n-4}^2$ is a bijection such that for all $T' \in \mathcal{T}_n^2$

I $$\begin{aligned} \big|\mathsf{N}_0(T')\big| &= \big|\mathrm{ret}_0\big(\omega_n(T')\big)\big| \\ \big|\mathsf{N}_1(T')\big| &= \big|\mathrm{ret}_1\big(\omega_n(T')\big)\big| \end{aligned}$$

II $\beta_{T'} \colon \mathcal{C}(T') \longrightarrow \mathcal{C}\big(\omega_n(T')\big)$, given by $\beta_{T'}\big(\,(i,j)\,\big) = (\beta^0_{T'}(i), \beta^1_{T'}(j))$, is an isomorphism between the sets of crossings of T' and $\omega_n(T')$, where $\beta^i_{T'} \colon\ \mathsf{N}_i(T') \longrightarrow \mathrm{ret}_i\big(\omega_n(T')\big)$, for $i = 0,1$, is the order-preserving bijection between these sets.

Define $\tilde{\omega}_{n+1} \colon \mathcal{T}_{n+1}^2 \longrightarrow \mathcal{D}_{n-3}^2$ by

$$\tilde{\omega}_{n+1}(T) = \bar{\Psi}_{\beta_{T'}(\vec{a})}\big(\omega_n(T')\big),$$

where $(T', \vec{a})$ is the unique element of $\bigcup_{T' \in \mathcal{T}_n^2} \{T'\} \times \mathcal{C}(T')$ such that $\Psi_{\vec{a}}(T') = T$.

By definition of $\tilde{\omega}_{n+1}$, the following diagram commutes:

$$\begin{array}{ccc} \big(T',\, \vec{a}\big) & \xrightarrow{(\omega_n,\, \beta_{T'})} & \big(\omega_n(T'),\, \beta_{T'}(\vec{a})\big) \\ \Big\downarrow \Psi_{\vec{a}} & & \Big\downarrow \bar{\Psi}_{\beta_{T'}(\vec{a})} \\ T & \xrightarrow{\tilde{\omega}_{n+1}} & \tilde{\omega}_{n+1}(T) \end{array}$$

Therefore, since the maps $\Psi_{\vec{a}}$, $(\omega_n,\ \beta_{T'})$ and $\bar{\Psi}_{\beta_{T'}(\vec{a})}$ are bijective, it follows that $\tilde{\omega}_{n+1}$ is bijective.

From the definitions of $\Psi_{\vec{a}}$ and $\bar{\Psi}_{\beta_{T'}(\vec{a})}$it is follows that

$$\begin{aligned} \mathsf{N}_0(T) &= \mathsf{L}_0(a_0,\, T') \cup \mathsf{R}_1(a_1,\, T'), \\ \mathsf{N}_1(T) &= \mathsf{L}_1(a_1,\, T') \cup \{n\}. \end{aligned} \tag{2}$$

On the other hand, taking $D' = \omega_n(T')$, $(\bar{a}_0, \bar{a}_1) = \beta_{T'}\big((a_0, a_1)\big)$,

$$\begin{aligned} \mathrm{ret}_0\big(\tilde{\omega}_{n+1}(T)\big) &= \bar{\mathsf{L}}_0(\bar{a}_0,\, D') \cup \bar{\mathsf{R}}_0(\bar{a}_1 + 1,\, D') \cup \{n-3\}, \\ \mathrm{ret}_1\big(\tilde{\omega}_{n+1}(T)\big) &= \bar{\mathsf{L}}_1(\bar{a}_1,\, D') \cup \{n-3\}. \end{aligned} \tag{3}$$

Comparison of (2) and (3) shows that $\tilde{\omega}_{n+1}$ does not have the desired properties (1) and (2). We need to apply the involutions μ_{m_1,m_2}.

For $D \in \mathcal{D}_{n-3}^2$, with $D = (D_0, D_1)$, let $m(D) = \max(\mathrm{ret}_1(D) - \{n-3\})$. Define $\Upsilon(D) \in \mathcal{D}_{n-3}^2$ by $\Upsilon(D) = (D_0, \hat{D}_1)$, where, for $i \in \{0, \ldots, n-3\}$,

$$\hat{D}_1(i) = \begin{cases} D_1(i) & \text{if } \ i \in \{0, \ldots, m(D)\}, \\ \mu_{\mathrm{id}+1, D'_0}\big(D'_1\big)(i) & \text{if } \ i \in \{m(D)+1, \ldots, n-4\}, \\ n-3 & \text{if } \ i = n-3, \end{cases}$$

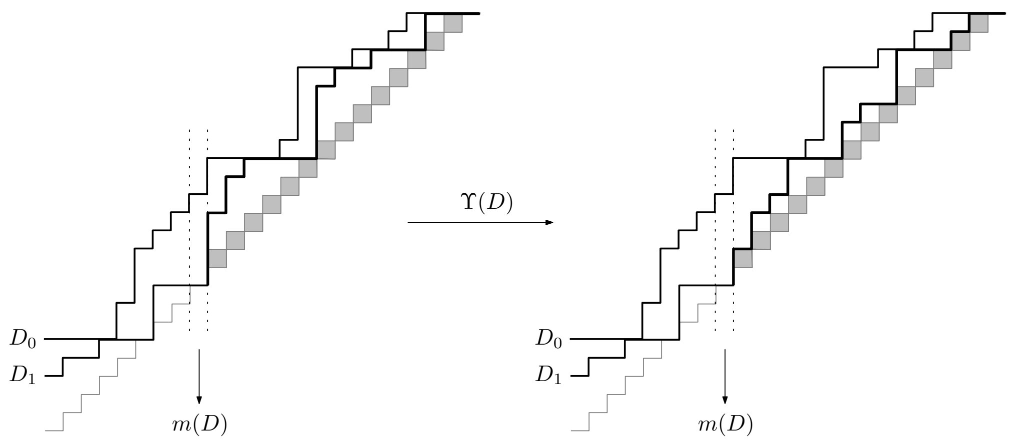

Fig. 1: An element of $\mathcal{D}^2_{n-3}$ and its image under Υ. The portion of the lower path between its last two returns is replaced with a path having a reversed number of intersections with the upper path and with the identity shifted up one unit.

and D'_0, D'_1 are the restrictions of D_0 and D_1 to the interval $\{m(D)+1,\ldots,n-4\}$.

Note that $\Upsilon\big((D_0,D_1)\big)$ is equal to (D_0,D_1) except for the portion of D_1 between its last two returns. This is replaced by a path with reversed number of intersections with D_0 and $\mathrm{id}+1$; see Figure 1.

Clearly, $\mathrm{ret}_1\big(\Upsilon(D)\big) = \mathrm{ret}_1(D)$, so $m\big(\Upsilon(D)\big) = m(D)$. Therefore, $\Upsilon\big(\Upsilon(D)\big) = D$, because the functions $\mu_{f,h}$ are involutions. Hence, Υ is a bijection from $\mathcal{D}^2_{n-3}$ onto $\mathcal{D}^2_{n-3}$.

By definition of Υ, the returns of $\Upsilon(D)$, where $D=(D_0,D_1)$, are

$$\begin{array}{rcl} \mathrm{ret}_0\big(\Upsilon(D)\big) &=& \bar{\mathsf{L}}_0\big(m(D),\,D\big)\cup S\cup\{n-3\},\\ \mathrm{ret}_1\big(\Upsilon(D)\big) &=& \mathrm{ret}_1(D), \end{array} \tag{4}$$

where S satisfies

$$\begin{array}{rcl} S &\subseteq& \{m(D)+1,\ldots,n-4\},\\ \big|S\big| &=& \big|\mathrm{ret}(\mathrm{id}+1,\,D_1)\cap\{m(D)+1,\ldots,n-4\}\big|. \end{array} \tag{5}$$

Taking $D=\tilde{\omega}_{n+1}(T)=\bar{\Psi}_{\beta_{T'}(\vec{a})}(D')$, it is clear that $m(D)=\bar{a}_1$. Also, $\bar{\mathsf{L}}_0(\bar{a}_0,\,D)=\bar{\mathsf{L}}_0(\bar{a}_1,\,D)$, which combined with (3), (4) and (5) gives us the returns of $\Upsilon\big(\tilde{\omega}_{n+1}(t)\big)$:

$$\begin{array}{rcl} \mathrm{ret}_0\big(\Upsilon(D)\big) &=& \bar{\mathsf{L}}_0\big(\bar{a}_0,\,D'\big)\cup S\cup\{n-3\},\\ \mathrm{ret}_1\big(\Upsilon(D)\big) &=& \bar{\mathsf{L}}_1\big(\bar{a}_1,\,D'\big)\cup\{n-3\}, \end{array}$$

where S satisfies

$$\begin{array}{rcl} S &\subseteq& \{\bar{a}_1+1,\ldots,n-4\},\\ \big|S\big| &=& \big|\mathrm{ret}(\mathrm{id}+1,\,D_1)\cap\{\bar{a}_1+1,\ldots,n-4\}\big|\\ &=& \bar{\mathsf{R}}_1\big(\bar{a}_1+1,\,D'\big). \end{array}$$

To summarize, if T, T', D and D' satisfy the following diagram

$$\begin{tikzcd} \big(T',\,(a_0,a_1)\big) \arrow[r, "(\omega_n,\,\beta_{T'})"] \arrow[d, "\Psi_{\vec{a}}"] & \big(D',\,(\bar{a}_0,\bar{a}_1)\big) \arrow[d, "\bar{\Psi}_{\beta_{T'}(\vec{a})}"] & \\ T \arrow[r, "\tilde{\omega}_{n+1}"] & D \arrow[r, "\Upsilon"] & \Upsilon(D) \end{tikzcd}$$

then the sets of neighbors of T satisfy

$$\begin{array}{rcl} \mathsf{N}_0(T) & = & \mathsf{L}_0(a_0,\,T') \cup \mathsf{R}_1(a_1,\,T'), \\ \mathsf{N}_1(T) & = & \mathsf{L}_1(a_1,\,T') \cup \{n\}. \end{array} \tag{6}$$

while the returns of $\Upsilon(D)$ are

$$\begin{array}{rcl} \mathrm{ret}_0\big(\Upsilon(D)\big) & = & \bar{\mathsf{L}}_0\big(\bar{a}_0,\,D'\big) \cup S \cup \{n-3\}, \\ \mathrm{ret}_1\big(\Upsilon(D)\big) & = & \bar{\mathsf{L}}_1\big(\bar{a}_1,\,D'\big) \cup \{n-3\}, \end{array} \tag{7}$$

where S satisfies

$$\begin{array}{rcl} S & \subseteq & \{\bar{a}_1+1,\ldots,n-4\}, \\ \big|S\big| & = & \bar{\mathsf{R}}_1\big(\bar{a}_1+1,\,D'\big). \end{array} \tag{8}$$

Therefore, the function $\omega_{n+1} = \Upsilon \circ \tilde{\omega}_{n+1}$ is a bijection from $\mathcal{T}^2_{n+1}$ onto $\mathcal{D}^2_{n-3}$, satisfying (1) and (2):

Note that $\big|\mathsf{L}_0(a_0,\,T')\big| = \big|\bar{\mathsf{L}}_0(\bar{a}_0,\,D')\big|$ because, according to inductive hypothesis I, $\big|\mathsf{N}_0(T')\big| = \big|\mathrm{ret}_0(D')\big|$ and the map $\beta^0_{T'}$, which sends a_0 to $\bar{a}_0$, is order-preserving. Similarly, $\big|\mathsf{R}_1(a_1,\,T')\big| = \big|\bar{\mathsf{R}}_1(\bar{a}_1,\,D')\big|$ because $\big|\mathsf{N}_1(T')\big| = \big|\mathrm{ret}_1(D')\big|$ and $\beta^1_{T'}$ is order-preserving.

But $\big|\bar{\mathsf{R}}_1(a_1+1,\,D')\big| = \big|\bar{\mathsf{R}}_1(\bar{a}_1,\,D')\big| - 1$, because $\bar{a}_1 \in \mathrm{ret}_1(D')$ since $(\bar{a}_0,\bar{a}_1)$ is an element of $\mathcal{C}(D')$. Hence, by (6), (7) and (8), $\big|\mathsf{N}_0(T)\big| = \big|\mathrm{ret}_0\big(\Upsilon(D)\big)\big|$. For similar reasons, $\big|\mathsf{L}_1(a_1,\,T')\big| = \big|\bar{\mathsf{L}}_1(\bar{a}_1,\,D')\big|$ and therefore we also have $\big|\mathsf{N}_1(T)\big| = \big|\mathrm{ret}_1\big(\Upsilon(D)\big)\big|$. Hence, ω_{n+1} satisfies (1).

Finally, let β^0_T and β^1_T be the order-preserving bijections from $\mathsf{N}_0(T)$ onto $\mathrm{ret}_0\big(\Upsilon(D)\big)$ and from $\mathsf{N}_1(T)$ onto $\mathrm{ret}_1\big(\Upsilon(D)\big)$, respectively. Let $\beta_T = (\beta^0_T, \beta^1_T)$. The fact that $(i,j) < (i',j')$ if and only if $\beta_T(i,j) < \beta_T(i',j')$ follows immediately because β^0_T and β^1_T are order-preserving.

Hence, we only need to show that β_T is a bijection between $\mathcal{C}(T)$ and $\mathcal{C}(\Upsilon(D))$ to obtain that it is an isomorphism between these sets.

But from (6),(7) and (8), we have the following partitions of $\mathcal{C}(T)$ and $\mathcal{C}(\Upsilon(D))$ into two disjoint sets:

$$\mathcal{C}(T) = \underbrace{\big\{(i,j) \le (a_0,a_1):\ (i,j) \in \mathcal{C}(T')\big\}}_{\text{(i)}} \ \cup\ \underbrace{\big\{(i,n):\ i \in \mathsf{N}_0(T)\big\}}_{\text{(ii)}}$$

and

$$\mathcal{C}\big(\Upsilon(D)\big) = \underbrace{\big\{(i,j) \le (\bar{a}_0,\bar{a}_1):\ (i,j) \in \mathcal{C}(D')\big\}}_{\text{(iii)}} \ \cup\ \underbrace{\big\{(i,n-3):\ i \in \mathrm{ret}_0\big(\Upsilon(D)\big)\big\}}_{\text{(iv)}}.$$

The sets (i) and (iii) are the lower ideals of (a_0,a_1) in $\mathcal{C}(T')$ and of $(\bar{a}_0,\bar{a}_1)$ in $\mathcal{C}(D')$, respectively. By inductive hypothesis II, they are isomorphic under $\beta_{T'}$. It is clear from (6) and (7) that β_T and $\beta_{T'}$ are equal when restricted to (i), so in particular β_T is a bijection between (i) and (iii). We have already seen that $\big|\mathsf{N}_0(T)\big| = \big|\mathrm{ret}_0\big(\Upsilon(D)\big)\big|$, so β_T is also a bijection between (ii) and (iv).

Therefore ω_{n+1} satisfies (2), which completes the proof. $\square$

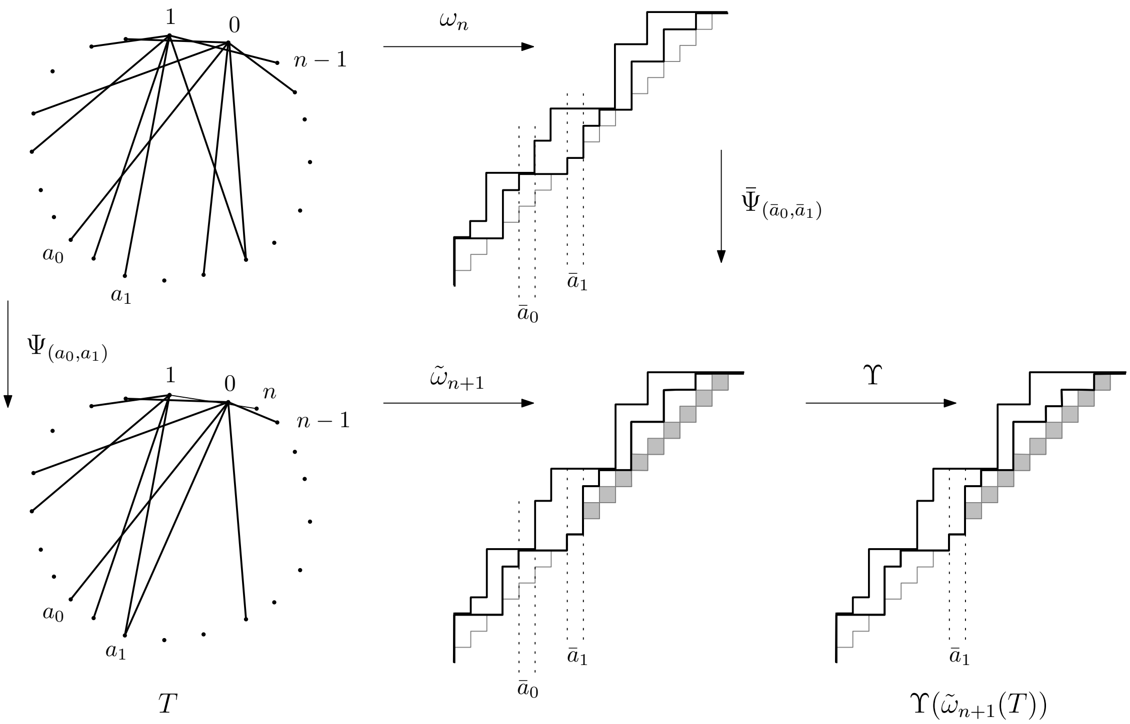

Fig. 2: An example where it can be verified that the poset $\mathcal{C}(T)$ is isomorphic to the poset $\mathcal{C}\big(\Upsilon(\tilde{\omega}_{n+1}(T))\big)$.

Using the property $\big|\mathsf{N}_1(T)\big| = \big|\mathrm{ret}_1\big(\omega_n(T)\big)\big|$ of the bijection ω_n obtained in the previous theorem, and the formula for $|\mathrm{ret}(\mathrm{id}, D_1)|$ from Theorem 6, we obtain a refinement, for the case $k = 2$, of the formula (1) for the number of 2-triangulations. The following formula gives the number of 2-triangulations according to the degree of a fixed vertex. Note that $|\mathrm{ret}(\mathrm{id}, D_1)| = \big|\mathrm{ret}_1\big(\omega_n(T)\big)\big| - 1$ because $0 \in \big|\mathrm{ret}_1\big(\omega_n(T)\big)\big|$. Also, by applying a rotation, the formula can be used to find the number of 2-triangulations with a given degree at any fixed vertex.

Corollary 8 *The number of* 2*-triangulations* T *of the* n*-gon with* $\big|\mathsf{N}_0(T)\big| = c$ *is given by*

$$\det\begin{vmatrix} C_{n-4} & B^2_{n-3}(c-1) \\ C_{n-3} & B^2_{n-2}(c-1) \end{vmatrix},$$

where $B^k_m(h) = \frac{2k+h-2}{m}\binom{2m-2k-h+1}{m-1}$.

The previous theorem cannot be generalized for $k > 2$. No such strong bijection can exist if $k > 2$, because an isomorphism between the crossings $\mathcal{C}(T)$ and $\mathcal{C}\big(\omega_n(T)\big)$ implies in particular that $\big|\mathcal{C}(T)\big| = \big|\mathcal{C}\big(\omega_n(T)\big)\big|$. But the distribution of k-triangulations and non-crossing Dyck paths is different for these parameters when $k > 2$. For example, for $k = 3$ and $n = 10$, $\mathcal{T}^3_{10}$ contains 24 triangulations T of the 10-gon such that $\big|\mathcal{C}(T)\big| = 9$, but there are 32 triples D of non-crossing Dyck paths of length 4 such that $\big|\mathcal{C}(D)\big| = 9$.

Surprisingly, computer experiments suggest that Theorem 7 can be generalized for $k > 2$ if we drop condition (2).

Conjecture 1 *For all $k \geq 1$ and $n \geq 2k+1$, there exists a bijection ω_n^k from the set $\mathcal{T}_n^k$ of all k-triangulations of the n-gon onto the set $\mathcal{D}_{n-2k}^k$ of all k non-crossing Dyck paths of length $n-2k$, such that for all $T \in \mathcal{T}_n^k$*

$$\left|\mathsf{N}_i(T)\right| = \left|\mathrm{ret}_i\big(\omega_n^k(T)\big)\right| \quad \textit{for all } i \in \{0, \ldots, k-1\}.$$

By Theorem 7, this conjecture is true for $k = 2$ (and $k = 1$). If it holds for all k then, by Theorem 6, there is an analogue to Corollary 8 for the general case. We conclude with this conjectured formula for the number of k-triangulations having a given degree at a fixed vertex.

Conjecture 2 *The number of k-triangulations $T \in \mathcal{T}_n^k$ with $\left|\mathsf{N}_0(T)\right| = c$ is given by*

$$\det \begin{bmatrix} C_{n-2k} & C_{n-2k+1} & \iddots & \iddots & B_{n-k-1}^k(c-1) \\ C_{n-2k+1} & \iddots & \iddots & C_{n-k-1} & \vdots \\ \iddots & \iddots & \iddots & \iddots & \vdots \\ \iddots & C_{n-k-1} & \iddots & \iddots & B_{n-3}^k(c-1) \\ C_{n-k-1} & \iddots & \iddots & C_{n-3} & B_{n-2}^k(c-1) \end{bmatrix},$$

where $B_m^k(h) = \frac{2k+h-2}{m}\binom{2m-2k-h+1}{m-1}$.

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Bounds of asymptotic occurrence rates of some patterns in binary words related to integer-valued logistic maps

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Abstract. In this article, we investigate the asymptotic occurrence rates of specific subwords in any infinite binary word. We prove that the asymptotic occurrence rate for the subwords is upper- and lower-bounded in the same way for every infinite binary word, in terms of the asymptotic occurrence rate of the zeros. We also show that both of the bounds are best-possible by constructing, for each bound, a concrete infinite binary word such that the bound is reached. Moreover, we apply the result to analyses of recently-proposed pseudorandom number generators that are based on integer-valued variants of logistic maps.

Résumé. Dans cet article, nous étudions les fréquences asymptotiques d'occurence de suites spécifiques dans tout mot binaire infini. Nous prouvons que la fréquence asymptotique d'occurence pour ces suites est borné supérieurement et inférieurement de la même façon pour chaque mot binaire infini, en termes des fréquences asymptotiques d'occurence de zéros. Nous montrons aussi que les deux limites sont les meilleures possibles en construisant concrètement, pour chaque limite, un mot binaire infini tel que la borne est atteinte à la limite. De plus, nous appliquons ce résultat à des analyses de générateurs de nombres pseudo-aléatoires proposés récemment qui sont basés sur des variantes des fonctions logistiques à valeurs entières.

Keywords: binary word, pattern occurrence rate, simple normality, logistic map, pseudorandom number generator

1 Introduction

In this article, we study the following very concrete problem: For infinite binary words, find relations between the sum of the asymptotic occurrence rates of three patterns 00, 0100, and 01010, and the asymptotic occurrence rates of 0s. (A motivation of this problem is explained in the next two paragraphs.) More precisely, for a finite or infinite binary word $x = x_1x_2x_3\cdots$ with $x_i \in \{0,1\}$ for every i, let $I(x)$ denote the set of indices i in x such that $x_i = 0$, and let $P(x)$ be the set of indices i in x that satisfy one of the following three conditions; (P1) $x_{i-1}x_i = 00$; (P2) $x_{i-1}x_ix_{i+1}x_{i+2} = 0100$; and (P3) $x_{i-1}x_ix_{i+1}x_{i+2}x_{i+3} = 01010$. For example, if $x = 0110101000$, that is the first 10 bits of the dyadic expansion of the fractional part $\mathrm{frac}(\sqrt{2})$ of $\sqrt{2}$ (*i.e.* $\sqrt{2}-1$), then we have $I(x) = \{1,4,6,8,9,10\}$, and $i = 9,7,5$ are examples of indices in $P(x)$ satisfying the conditions (P1), (P2), and (P3), respectively.

Let $x^{(n)}$ denote the initial subword of x of length n. In the above setting, the problem is of finding, for any infinite binary word x, relations between the quantities

$$r_{\inf}(x) = \liminf_{n\to\infty} \frac{|I(x^{(n)})|}{n} \quad \text{and} \quad R_{\inf}(x) = \liminf_{n\to\infty} \frac{|P(x^{(n)})|}{n} \ , \tag{1}$$

and those between the quantities

$$r_{\sup}(x) = \limsup_{n\to\infty} \frac{|I(x^{(n)})|}{n} \quad \text{and} \quad R_{\sup}(x) = \limsup_{n\to\infty} \frac{|P(x^{(n)})|}{n} \ . \tag{2}$$

In Section 2, we present simple upper and lower bounds of the quantities $R_{\inf}(x)$ and $R_{\sup}(x)$ in terms of $r_{\inf}(x)$ and $r_{\sup}(x)$, respectively. Moreover, we prove that these bounds are both "best possible". More precisely, for each of the lower and the upper bounds, we construct a concrete example of an infinite binary word that attains the equality in the bound. The first aim of this article is to describe the above combinatorial problem and its solution.

The problem presented in the previous paragraph, especially the specific choice of the three patterns, is motivated by analyses of pseudorandom number generators (PRNGs). To imitate random or chaotic behaviors of nature by using deterministic algorithms performed on computers is a ubiquitous and very fundamental task in several areas of science and information technology, such as computer simulation, statistics and cryptography; hence construction and analyses of PRNGs are one of the most active topics in information theory. One of the existing ideas to construct good PRNGs is to make use of the well-known chaotic behavior of the logistic maps $L(x) = \mu x(1-x)$, $0 < x < 1$, for some parameters μ (*e.g.* Wagner (1993); Phatak et al. (1995)). For example, the map $L(x)$ shows chaotic behavior by choosing a parameter $\mu = 4$. The PRNGs concerned in this article is the recently proposed schemes (see *e.g.* Araki et al. (2006)) that are based on some integer-valued variants of the map $L(x)$ with parameter $\mu = 4$. The corresponding integer-valued variant $L_n(x)$ with accuracy parameter $n \in \mathbb{Z}$, $n > 0$, is given by

$$L_n(x) = \left\lfloor \frac{4x(2^n - x)}{2^n} \right\rfloor = \left\lfloor \frac{x(2^n - x)}{2^{n-2}} \right\rfloor \text{ for } x \in X_n = \{1, 2, \dots, 2^n - 1\} \tag{3}$$

(e.g. Kuribayashi et al. (2005)), where $\lfloor z \rfloor$ denotes the integer part of $z \in \mathbb{R}$. The description of $L_n(x)$ is derived by first expanding the domain $(0,1) \subset \mathbb{R}$ of the original logistic map $L(x)$ proportionally to a larger interval $(0, 2^n)$ and then cutting off the fractional parts of real numbers in the latter interval. Now the PRNG mentioned above, that uses the map $L_n(x)$ as the updating function of internal states, is informally described as follows:

1. Choose an initial state $s = s_0 \in X_n$.
2. Update the internal state s by applying the map $L_n(x)$ K times (with K a parameter).
3. Output bits of the binary expression of s in some suitable positions.
4. Repeat Steps 2 and 3.

In some preceding works, appropriate choices of parameters for the scheme have been investigated (*e.g.* Araki et al. (2008)).

In particular, it has been pointed out (Miyazaki et al. (2007)) that, when we start from the initial value $s_0 = 2^{n-1}$, the internal state will be pushed out the domain X_n of the map $L_n(x)$ (*i.e.* $L_n(2^{n-1}) = 2^n$), thus the value 2^{n-1} should be excluded from the candidates of the initial value. (Note that, even if we tolerate the illegal input 2^n for $L_n(x)$, we then have $L_n(2^n) = 0$ and $L_n(0) = 0$, therefore the internal state s falls eventually into a stable value. This is also a very undesirable situation since it makes the outputs of the PRNG not random at all.) Moreover, it was also pointed out that even an initial value $s_0 \in X_n$ other than 2^{n-1} may lead the internal state to the excluded value 2^{n-1}, *i.e.* we may have $L_n(s_0) = 2^{n-1}$. Thus such an undesirable initial value should also be avoided in practical use of the PRNG. However, existence conditions for such an undesirable initial value $s_0 \neq 2^{n-1}$ have not been well investigated. The problem presented in the first paragraph arises from the author's recent research (Nuida (2008)) on conditions for the accuracy parameter n such that the corresponding $L_n(x)$ possesses an undesirable initial value $s_0 \neq 2^{n-1}$ (we call such a parameter n *dangerous*). More precisely, it is shown that lower bounds of the quantities $R_{\inf}(x)$ and $R_{\sup}(x)$ for x being the fractional part $\mathrm{frac}(\sqrt{2})_2$ of the dyadic expansion of $\sqrt{2}$ give lower bounds of the asymptotic rate of the dangerous parameters n in the positive integers (in terms of the values $r_{\inf}(x)$ and $r_{\sup}(x)$ for the same x). As a consequence, our analysis of the above PRNG is also deeply related to a long-standing conjecture on the quantities $r_{\inf}(\mathrm{frac}(\sqrt{2})_2)$ and $r_{\sup}(\mathrm{frac}(\sqrt{2})_2)$. See Section 3 for details.

This article is organized as follows. In Section 2, we present a solution of the problem described in the first paragraph. The solution itself is stated as Theorem 1 that is the main result of this article. The proof of Theorem 1 is divided into the following four parts; the lower bound, its best-possibility, the upper bound, and its best-possibility. Due to the limited pages, some lemmas in Section 2 are only accompanied with a sketchy proof; for the details of the proof, see a forthcoming full version of this article. In Section 3, we describe a relation between the above problem and analyses of PRNGs of the above types. Namely, we explain how a lower bound of the asymptotic occurrence rate of the dangerous parameters in the positive integers is derived from the result on the first problem. Finally, in Section 4, we propose open problems on possible improvements or generalizations of our results in this article.

2 Results

This section shows a solution of the problem presented in the first paragraph of the Introduction. The solution, that is the main theorem of this article, is the following:

Theorem 1 *For any infinite binary word $x = x_1x_2x_3\cdots$, let $r_{\inf}(x)$, $r_{\sup}(x)$, $R_{\inf}(x)$ and $R_{\sup}(x)$ be defined as in (1) and (2). Then we have*

$$\frac{5r_{\inf}(x)-2}{3} \leq R_{\inf}(x) \leq r_{\inf}(x) \quad \text{and} \quad \frac{5r_{\sup}(x)-2}{3} \leq R_{\sup}(x) \leq r_{\sup}(x) \ . \tag{4}$$

Moreover, all the bounds are best possible except trivial exceptions, in the following sense: For any $2/5 \leq r \leq 1$, there exists an infinite binary word x such that $r_{\inf}(x) = r_{\sup}(x) = r$ and $R_{\inf}(x) = R_{\sup}(x) = (5r-2)/3$. Similarly, for any $0 \leq r \leq 1$, there exists an infinite binary word x such that $r_{\inf}(x) = r_{\sup}(x) = r$ and $R_{\inf}(x) = R_{\sup}(x) = r$.

In what follows, we give a sketch of a proof of the theorem.

2.1 Lower Bounds

We demonstrate a sketchy proof of the lower bounds in Theorem 1. For any positive integer n, let W_n denote the set of binary words of length n. Let $\prec$ denote the lexicographic order on W_n (*e.g.* $1100 \succ 1011$). Let $\ell(x)$ denote the length of a word x. Let $\emptyset$ denote the empty word. For two words y and y', we write $y \subset y'$ if y appears in y' as a consecutive subword, and let $y^j = yy\cdots y$ (j repetition) for any $j \geq 0$. Then we define the following maps $\varphi_1, \varphi_2, \ldots, \varphi_7$ from W_n to itself, where w and w' signify some binary words:

$$\begin{aligned}
\varphi_1(x) &= \begin{cases} 1^p w 0 & \text{, if } x = w01^p,\ p \geq 1\ ; \\ x & \text{, otherwise,} \end{cases} \\
\varphi_2(x) &= \begin{cases} 1^{p+1} w 11 w' & \text{, if } x = 1^p w 111 w',\ p \geq 0,\ 111 \not\subset w \neq \emptyset,\ w_1 = w_{\ell(w)} = 0\ ; \\ x & \text{, otherwise,} \end{cases} \\
\varphi_3(x) &= \begin{cases} w0110^{p-1} w' & \text{, if } x = w0^p 11 w',\ p \geq 2,\ 0011 \not\subset w,\ w_{\ell(w)} \neq 0\ ; \\ x & \text{, otherwise,} \end{cases} \\
\varphi_4(x) &= \begin{cases} w01100 w' & \text{, if } x = w01010 w',\ 01010 \not\subset w010\ ; \\ x & \text{, otherwise,} \end{cases} \\
\varphi_5(x) &= \begin{cases} w10^{p+2} w' & \text{, if } x = w0^p 100 w',\ p \geq 1,\ 0100 \not\subset w0^p,\ w_{\ell(w)} \neq 0\ ; \\ x & \text{, otherwise,} \end{cases} \\
\varphi_6(x) &= \begin{cases} w010110^p w' & \text{, if } x = w0^p 10110 w',\ p \geq 2,\ 0010110 \not\subset w0^p,\ w_{\ell(w)} \neq 0; \\ x & \text{, otherwise,} \end{cases} \\
\varphi_7(x) &= \begin{cases} w1010110 w' & \text{, if } x = w0110110 w',\ 0110110 \not\subset w0110\ ; \\ x & \text{, otherwise.} \end{cases}
\end{aligned} \tag{5}$$

Note that these seven maps are all well-defined; namely, each φ_k transforms the leftmost consecutive subword of the specified form, and leaves the word unchanged if such a subword does not exist. Moreover, each φ_k leaves the quantity $|I(x)|$ invariant and is weakly increasing with respect to $\prec$ (*i.e.* $x \preceq \varphi_k(x)$). Then a case-by-case argument shows the following property:

Lemma 1 *For each φ_k, we have $|P(\varphi_k(x))| \leq |P(x)|$ for any $x \in W_n$.*

Let W_n^φ be the set of the common fixed points of $\varphi_1, \ldots, \varphi_7$ in W_n. Then, since each φ_k is weakly increasing with respect to the total order $\prec$, it follows that any word in W_n can be mapped into W_n^φ by applying $\varphi_1, \ldots, \varphi_7$ finitely many times. Let $\varphi(x)$ denote a (not necessarily unique) word in W_n^φ corresponding to $x \in W_n$. Note that $|I(\varphi(x))| = |I(x)|$ and $|P(\varphi(x))| \leq |P(x)|$ for any $x \in W_n$. Moreover, we have the following property of the fixed point set W_n^φ:

Lemma 2 *Any $x \in W_n^\varphi$ involves no consecutive subword listed in Table 1.*

Proof (Sketch): The excluded subword of Type k, $1 \leq k \leq 7$, is straightforwardly derived by the condition that $\varphi_k(x) = x$. On the other hand, each excluded subword of Type k, $k \geq 8$, is deduced from the preceding ones; for example, if $x \in W_n^\varphi$ contains a consecutive subword 001011 of Type 8,

Tab. 1: Excluded consecutive subwords for words in W_n^φ
Here w is some (possibly empty) word, and the symbol ")" in Type 1 denotes the end of the word.

type	subword	type	subword	type	subword
1	$0w1)$	2	$0w111$	3	0011
4	01010	5	0100	6	0010110
7	0110110	8	001011	9	00101
10	$0010w$ ($w \neq \emptyset$)	11	$001w$ ($w \neq 0$)		

then x must contain one of the words 0010110 (Type 6), 0010111 (Type 2) and $001011)$ (Type 1) as a consecutive subword. □

Now note that any $x \in W_n$ admits an expression of the following form:

$$x = 1^{p_0} 0^{q_1} 1^{p_1} \cdots 0^{q_k} 1^{p_k} 0^{q_{k+1}} \quad , \quad k \geq 0,\ q_i \geq 1,\ p_i \geq 1\ (1 \leq i \leq k) \ . \tag{6}$$

Then the next lemma follows from Lemma 2:

Lemma 3 *If $x \in W_n^\varphi$, then the expression (6) of x satisfies the following conditions:*

1. *If $k \geq 1$, then $q_{k+1} \geq 1$.*
2. *$p_i \leq 2$ for $i \geq 1$.*
3. *$q_i = 1$ for $1 \leq i \leq k-1$.*
4. *If $q_k \geq 2$, then $p_k = 1$ and $q_{k+1} = 1$.*
5. *If $k \geq 2$ and $q_k \geq 2$, then $p_{k-1} = 2$.*
6. *$(p_i, p_{i+1}) = (1,2)$ or $(2,1)$ for $1 \leq i \leq k-2$.*
7. *If $q_k = 1$, then $p_k = 2$ or $q_{k+1} = 1$.*

Proof (Sketch): The first and the second parts follow from the absence of subwords of Types 1 and 2, respectively. The third and the fourth parts both follow from the absence of subwords of Type 11. The fifth part follows from the second part and the absence of subwords of Type 5. The sixth part follows from the second part and the absence of subwords of Types 4 and 7. Finally, the seventh part follows from the first and the second parts, and the absence of subwords of Type 5. □

By Lemma 3, we obtain a complete list of words $x \in W_n^\varphi$ as in Table 2. In the table, Type 1 corresponds to the case that $k = 0$ in (6). Types 2 and 3 both correspond to the case that $k \geq 1$ and $q_k \geq 2$. Types 4 and 5 both correspond to the case that $k \geq 1$, $q_k = 1$ and $p_k = 2$. Finally, types 6 and 7 both correspond to the case that $k \geq 1$, $q_k = 1$ and $p_k = 1$. Table 2 also includes the values of n, $|I(x)|$, and $|P(x)|$, and the relations between $|I(x)|/n$ and $|P(x)|/n$ that play a key role in the proof of Theorem 1.

From now, we prove the lower bound of $R_{\inf}(x)$. This is trivial if $r_{\inf}(x) \leq 2/5$, therefore we assume that $r_{\inf}(x) > 2/5$. For each n, put $m_n = |P(x^{(n)})|$ and $y_n = \varphi(x^{(n)})$. Recall that $|I(y_n)| = |I(x^{(n)})|$ and $|P(y_n)| \leq |P(x^{(n)})|$. Now we have the following:

Tab. 2: Complete list of words in W_n^{φ}

Type 1	$x = 1^p 0^q \quad (p \geq 0, q \geq 0)$
	$n = p + q$ $\lvert I(x)\rvert = q$ $\lvert P(x)\rvert = q - 1$
	$\lvert I(x)\rvert/n = \lvert P(x)\rvert/n + 1/n$
Type 2	$x = 1^p (01011)^s 0^q 10 \quad (p \geq 0, q \geq 2, s \geq 0)$
	$n = 5s + p + q + 2$ $\lvert I(x)\rvert = 2s + q + 1$ $\lvert P(x)\rvert = q - 1$
	$\lvert I(x)\rvert/n = 3\lvert P(x)\rvert/(5n) + 2/5 + 4/(5n) - 2p/(5n)$
Type 3	$x = 1^p 011(01011)^s 0^q 10 \quad (p \geq 0, q \geq 2, s \geq 0)$
	$n = 5s + p + q + 5$ $\lvert I(x)\rvert = 2s + q + 2$ $\lvert P(x)\rvert = q - 1$
	$\lvert I(x)\rvert/n = 3\lvert P(x)\rvert/(5n) + 2/5 + 3/(5n) - 2p/(5n)$
Type 4	$x = 1^p (01011)^s 0^q \quad (p \geq 0, q \geq 1, s \geq 1)$
	$n = 5s + p + q$ $\lvert I(x)\rvert = 2s + q$ $\lvert P(x)\rvert = q - 1$
	$\lvert I(x)\rvert/n = 3\lvert P(x)\rvert/(5n) + 2/5 + 3/(5n) - 2p/(5n)$
Type 5	$x = 1^p 011(01011)^s 0^q \quad (p \geq 0, q \geq 1, s \geq 0)$
	$n = 5s + p + q + 3$ $\lvert I(x)\rvert = 2s + q + 1$ $\lvert P(x)\rvert = q - 1$
	$\lvert I(x)\rvert/n = 3\lvert P(x)\rvert/(5n) + 2/5 + 2/(5n) - 2p/(5n)$
Type 6	$x = 1^p (01011)^s 010 \quad (p \geq 0, s \geq 0)$
	$n = 5s + p + 3$ $\lvert I(x)\rvert = 2s + 2$ $\lvert P(x)\rvert = 0$
	$\lvert I(x)\rvert/n = 2/5 + 4/(5n) - 2p/(5n)$
Type 7	$x = 1^p 011(01011)^s 010 \quad (p \geq 0, s \geq 0)$
	$n = 5s + p + 6$ $\lvert I(x)\rvert = 2s + 3$ $\lvert P(x)\rvert = 0$
	$\lvert I(x)\rvert/n = 2/5 + 3/(5n) - 2p/(5n)$

Lemma 4 *If n is sufficiently large, then $y_n \in W_n^{\varphi}$ cannot be of Type 6 or 7 in Table 2.*

Proof: If y_n is of Type 6 or 7, then we have $|I(x^{(n)})|/n = |I(y_n)|/n \leq 2/5 + 4/(5n)$. Since $r_{\inf}(x) > 2/5$, there is a constant $c > 0$ such that $2/5 + 4/(5n) \leq r_{\inf}(x) - c$ and hence $|I(x^{(n)})|/n \leq r_{\inf}(x) - c$ for any sufficiently large n. Thus if y_n is of Type 6 or 7 for infinitely many n, then we have $r_{\inf}(x) \leq r_{\inf}(x) - c$, a contradiction. Hence the lemma holds. □

Now by Table 2, if y_n is of Types 1–5, then we have

$$\frac{m_n}{n} \geq \frac{|P(y_n)|}{n} \geq \frac{5|I(y_n)|}{3n} - \frac{2}{3} - \frac{4}{3n} = \frac{5|I(x^{(n)})|}{3n} - \frac{2}{3} - \frac{4}{3n} \ . \tag{7}$$

Thus Lemma 4 implies that

$$R_{\inf}(x) \geq \liminf_{n\to\infty} \left(\frac{5|I(x^{(n)})|}{3n} - \frac{2}{3} - \frac{4}{3n} \right) = \frac{5r_{\inf}(x) - 2}{3} \ , \tag{8}$$

as desired.

Secondly, we prove the lower bound of $R_{\sup}(x)$. This is trivial if $r_{\sup}(x) \leq 2/5$, therefore we assume that $r_{\sup}(x) > 2/5$. The task is to show that, for any $\varepsilon > 0$, there exist infinitely many indices n such that $m_n/n > (5r_{\sup}(x) - 2)/3 - \varepsilon$. Now take a positive ε' such that $\varepsilon' < 3\varepsilon/10$ and $\varepsilon' < r_{\sup}(x) - 2/5$. Then by the definition of $r_{\sup}(x)$, there exist infinitely many indices n such that $|I(x^{(n)})|/n > r_{\sup}(x) - \varepsilon'$. Let $\mathcal{N}$ denote the (infinite) set of the indices n with this property. Now we have the following:

Lemma 5 *$y_n \in W_n^{\varphi}$ is of Type 1–5 in Table 2 for any sufficiently large $n \in \mathcal{N}$.*

Proof: If $n \in \mathcal{N}$ and y_n is of Type 6 or 7, then $|I(y_n)|/n \leq 2/5 + 4/(5n)$ by Table 2, while $|I(y^{(n)})|/n = |I(x^{(n)})|/n > r_{\sup}(x) - \varepsilon'$ by the definition of $\mathcal{N}$. Thus we have $4/(5n) > r_{\sup}(x) - \varepsilon' - 2/5$ for such n. However, since $r_{\sup}(x) - \varepsilon' - 2/5 > 0$ by the choice of ε', the relation does not hold if n is sufficiently large. Hence the lemma holds. □

By Lemma 5 and Table 2, the inequality in (7) holds for any sufficiently large $n \in \mathcal{N}$. Thus by the definitions of $\mathcal{N}$ and ε', we have

$$\frac{m_n}{n} > \frac{5r_{\sup}(x) - 2}{3} - \frac{5\varepsilon'}{3} - \frac{4}{3n} > \frac{5r_2 - 2}{3} - \frac{\varepsilon}{2} - \frac{4}{3n} \tag{9}$$

for any sufficiently large $n \in \mathcal{N}$. Since the right-hand side of (9) is larger than $(5r_{\sup}(x) - 2)/3 - \varepsilon$ for any sufficiently large n, it follows that there exist infinitely many n with the desired property.

Hence the proof of the lower bounds in Theorem 1 is concluded.

2.2 Best-Possibility of Lower Bounds

In this subsection, we give an infinite binary word x for any $2/5 \leq r \leq 1$ such that $|I(x^{(n)})|/n$ and $|P(x^{(n)})|/n$ converge to r and $(5r - 2)/3$, respectively, when $n \to \infty$. This proves the best-possibility of the lower bounds in Theorem 1. Note that we can take $x = 0101101011\cdots$ (infinite repetition of 01011) and $x = 0000\cdots$ (infinite repetition of 0) for the cases $r = 2/5$ and $r = 1$, respectively. Thus we assume that $2/5 < r < 1$.

Our construction of the word x is as follows. First, put

$$p = \left\lceil \frac{5r-2}{1-r} \right\rceil \text{ and } \alpha = p + 5 - \frac{3}{1-r} = p - \frac{5r-2}{1-r} \ , \tag{10}$$

where $\lceil z \rceil$ denotes the smallest integer m such that $z \le m$, therefore $1 \le p < \infty$ and $0 \le \alpha < 1$. Let $\alpha = (0.\alpha_1\alpha_2\cdots)_2$ be the unique dyadic expansion of α with infinitely many 0s. Now we define finite binary sequences $x^{\langle 0\rangle}, x^{\langle 1\rangle}, \ldots$, such that $x^{\langle i\rangle}$ is a proper initial subword of $x^{\langle i+1\rangle}$, by

$$x^{\langle 0\rangle} = \emptyset \text{ and } x^{\langle i\rangle} = x^{\langle i-1\rangle}x^{\langle i-1\rangle}010110^{p-\alpha_i} \text{ for } i \ge 1 \ . \tag{11}$$

Put $\ell_i = \ell(x^{\langle i\rangle})$, $I_i = |I(x^{\langle i\rangle})|$, and $P_i = |P(x^{\langle i\rangle})|$ for each i. Let the word x be the limit of the sequence $x^{\langle 0\rangle}, x^{\langle 1\rangle}, \ldots$. Then an induction on i shows the following property:

Lemma 6 *For any $i \ge 1$, we have*

$$\begin{aligned} &\ell_i = (2^i-1)(p+5) - \sum_{j=1}^{i} 2^{i-j}\alpha_j \ , \quad I_i = (2^i-1)(p+2) - \sum_{j=1}^{i} 2^{i-j}\alpha_j \ , \\ &P_i = (2^i-1)p - 1 - \sum_{j=1}^{i} 2^{i-j}\alpha_j + \delta_{p,1}\alpha_i \ , \end{aligned} \tag{12}$$

where $\delta_{i,j}$ denotes the Kronecker delta.

By Lemma 6, we have

$$5I_i - 2\ell_i = 3P_i + 3 - 3\delta_{p,1}\alpha_i \text{ for any } i \ge 1 \ . \tag{13}$$

The following property is a key ingredient of the proof:

Lemma 7 *Each finite initial subword $x^{(n)}$ of the above word x is decomposed as*

$$x^{(n)} = x^{\langle i_1\rangle}x^{\langle i_2\rangle}\cdots x^{\langle i_{k-1}\rangle}(x^{\langle i_k\rangle})^{\lambda+1}y \ , \tag{14}$$

where $k \ge 1$, $i_1 > i_2 > \cdots > i_k \ge 0$, $\lambda \in \{0,1\}$, y is a (possibly empty) initial subword of $010110^{p-\alpha_{i_k+1}}$, and $i_k \ge 1$ if $k \ge 2$.

Proof: By the definition of x, it suffices to show that every initial subword x' of each $x^{\langle m\rangle}$ admits such a decomposition. We proceed the proof by induction on m. The case $m \le 1$ is obvious (take $k = 1$, $i_1 = 0$ and $y = x'$), therefore we consider the case $m \ge 2$. Now by the construction of $x^{\langle m\rangle}$, the last position of x' is contained in the first $x^{\langle m-1\rangle}$, in the second $x^{\langle m-1\rangle}$, or in the remaining part $010110^{p-\alpha_m}$. In the first case, the claim follows from the induction hypothesis. In the third case, the claim follows by taking $k = 1$, $i_k = m-1$ and $\lambda = 1$. Finally, in the second case, $x' = x^{\langle m-1\rangle}w$ for an initial subword w of $x^{\langle m-1\rangle}$. By the induction hypothesis, w admits a decomposition of the following form:

$$w = x^{\langle i'_1\rangle}x^{\langle i'_2\rangle}\cdots x^{\langle i'_{k'-1}\rangle}(x^{\langle i'_{k'}\rangle})^{\lambda'+1}y' \ , \tag{15}$$

where $m-2 \ge i'_1 > i'_2 > \cdots > i'_{k'}$. Now the claim follows by taking $k = k'+1$, $i_1 = m-1$, $i_j = i'_{j-1}$ for $2 \le j \le k$, $\lambda = \lambda'$, and $y = y'$. Hence the lemma holds in any case. $\square$

Now in the decomposition of $x^{(n)}$ in (14), for any $n \geq \ell_1$, we have $i_1 \geq 1$ and hence $i_k \geq 1$. Then a straightforward argument shows that, for any $n \geq \ell_1$, we have

$$\begin{aligned} n = \ell(x^{(n)}) = \sum_{j=1}^{k} \ell_{i_j} + \lambda \ell_{i_k} + \ell(y) \ , \ \ |I(x^{(n)})| = \sum_{j=1}^{k} I_{i_j} + \lambda I_{i_k} + |I(y)| \ , \\ |P(x^{(n)})| = \sum_{j=1}^{k} P_{i_j} + \lambda P_{i_k} + |P(y)| + k + (\lambda - \delta_{y,\emptyset})(1 - \delta_{p,1}\alpha_{i_k}) - \delta_{p,1} \sum_{j=1}^{k} \alpha_{i_j} \ . \end{aligned} \tag{16}$$

By these equalities, (12), and (13), an elementary argument shows that $|I(x^{(n)})|/n$ converges when $n \to \infty$ to $(p+2-\alpha)/(p+5-\alpha) = r$, and $|P(x^{(n)})|/n$ converges when $n \to \infty$ to $(5r-2)/3$.

Hence the lower bounds in Theorem 1 are best possible.

2.3 Upper Bounds

The proof of the upper bounds in Theorem 1 are much simpler than the case of lower bounds. In fact, for any finite binary word w, the map $i \mapsto i-1$ is an injection from $P(w)$ to $I(w)$, therefore $|P(w)| \leq |I(w)|$. Now the upper bounds are easy consequences of the inequality. Thus the nontrivial assertion on the upper bounds is only their best-possibility.

2.4 Best-Possibility of Upper Bounds

To prove the best-possibility of the upper bounds in Theorem 1, for any $0 \leq r \leq 1$, we construct an infinite binary word x such that both $|I(x)|/n$ and $|P(x)|/n$ converge to r when $n \to \infty$. Since $x = 000\cdots$ (infinite repetition of 0) satisfies the conditions when $r = 1$, we assume from now that $0 \leq r < 1$.

First, we define auxiliary values $\delta_k \in \{0,1\}$ for $k \geq 1$ inductively by

$$\delta_k = 1 \ \text{if} \ \frac{\sum_{i=1}^{k-1} \delta_i \cdot 2i + 2k}{k(k+1)} \leq r \ , \ \ \delta_k = 0 \ \text{otherwise.} \tag{17}$$

Then we have

$$\frac{\sum_{i=1}^{k} \delta_i \cdot 2i}{k(k+1)} \leq r \text{ for any } k \ . \tag{18}$$

Now let $x^{\langle k \rangle} = (1-\delta_k)^{2k}$ be the repetition of $1-\delta_k \in \{0,1\}$ of length $2k$ for each k, and define $x = x^{\langle 1 \rangle} x^{\langle 2 \rangle} x^{\langle 3 \rangle} \cdots$. Then for each k, we have

$$\ell(x^{\langle 1 \rangle} \cdots x^{\langle k \rangle}) = k(k+1) \ \text{and} \ |I(x^{\langle 1 \rangle} \cdots x^{\langle k \rangle})| = \sum_{i=1}^{k} \delta_i \cdot 2i \ . \tag{19}$$

Now by (18) and (19), an elementary argument shows that both $|I(x^{(n)})|/n$ and $|P(x^{(n)})|/n$ converge to r when $n \to \infty$. Thus the upper bounds in Theorem 1 are best possible.

Hence the proof of Theorem 1 is concluded.

3 Relations with PRNGs Based on Integer-Valued Logistic Maps

In this section, we explain a relation of the above problem with analyses of the PRNGs based on integer-valued logistic maps $L_n(x) = \lfloor x(2^n - x)/2^{n-2} \rfloor$ mentioned in the Introduction. A problem concerned in the analyses is to decide, for each parameter $n \in \mathbb{Z}$ with $n \geq 1$, whether there exists an initial value $s_0 \in \{1, 2, \dots, 2^n - 1\}$ such that $L_n(s_0) = 2^{n-1}$ (note that $L_n(2^{n-1}) \neq 2^{n-1}$). Recall from the Introduction that we call a parameter n dangerous if such an undesirable initial value $s_0 \neq 2^{n-1}$ for $L_n(x)$ exists. Now the problem is rephrased as the problem on existence of dangerous parameters n in the above sense.

The condition for an accuracy parameter n to be dangerous is equivalent to that there exists an integer $1 \leq x \leq 2^n - 1$ such that $2^{n-1} \leq x(2^n - x)/2^{n-2} < 2^{n-1} + 1$ (that is equivalent to $L_n(x) = 2^{n-1}$). By solving the inequalities, the relation $L_n(x) = 2^{n-1}$ is satisfied if and only if

$$\sqrt{2^{2n-3} - 2^{n-2}} < |2^{n-1} - x| \leq \sqrt{2^{2n-3}} \ . \tag{20}$$

Moreover, since

$$\sqrt{2^{2n-3}} - \sqrt{2^{2n-3} - 2^{n-2}} = \frac{2^{n-2}}{\sqrt{2^{2n-3}} + \sqrt{2^{2n-3} - 2^{n-2}}} > \frac{2^{n-2}}{2\sqrt{2^{2n-3}}} = \frac{\sqrt{2}}{4} \ , \tag{21}$$

(20) is satisfied if $2^{n-2}\sqrt{2} - \sqrt{2}/4 \leq |2^{n-1} - x| \leq 2^{n-2}\sqrt{2}$. Thus we have the following lemma:

Lemma 8 *A parameter n is dangerous if $2^{n-2}\sqrt{2} - \sqrt{2}/4 \leq m \leq 2^{n-2}\sqrt{2}$ for some integer m.*

From now, we rephrase the statement of Lemma 8 in terms of the dyadic expansion of $\sqrt{2}$. Namely, let $b = b_1 b_2 b_3 \cdots$ denote an infinite binary word that is the fractional part of the dyadic expansion of $\sqrt{2}$. For example, $b^{(10)} = 0110101000$ as mentioned in the Introduction. Then the fractional part of the dyadic expansion of $2^{n-2}\sqrt{2}$ is $(0.b_{n-1} b_n b_{n+1} \cdots)_2$, while the dyadic expansion of $\sqrt{2}/4$ is $(0.01 b_1 b_2 b_3 \cdots)_2$. Thus Lemma 8 implies the following:

Lemma 9 *A parameter n is dangerous if*

$$(0.b_{n-1} b_n b_{n+1} \cdots)_2 \leq (0.01 b_1 b_2 b_3 \cdots)_2 \ . \tag{22}$$

In particular, since $b_1 b_2 b_3 = 011$ as above, (22) is satisfied if $b_{n-1} b_n = 00$, $b_{n-1} b_n b_{n+1} b_{n+2} = 0100$, or $b_{n-1} b_n b_{n+1} b_{n+2} b_{n+3} = 01010$; that is, $n \in P(b)$ in the sense of Sections 1 and 2. Summarizing, we have the following result:

Proposition 1 *A parameter n is dangerous if $n \in P(b)$.*

Hence by Theorem 1, we obtain the following lower bound of the asymptotic occurrence rate of the dangerous parameters in the positive integers:

Theorem 2 *Let d_N denote the number of the dangerous parameters $n \leq N$. Then we have*

$$\liminf_{N \to \infty} \frac{d_N}{N} \geq \frac{5 r_{\mathrm{inf}}(b) - 2}{3} \text{ and } \limsup_{N \to \infty} \frac{d_N}{N} \geq \frac{5 r_{\mathrm{sup}}(b) - 2}{3} \ . \tag{23}$$

In particular, if $r_{\mathrm{sup}}(b) > 2/5$, then there exist infinitely many dangerous parameters n.

Regarding the quantities $r_{\mathrm{inf}}(b)$ and $r_{\mathrm{sup}}(b)$, it has been (even implicitly) conjectured that $r_{\mathrm{inf}}(b) = r_{\mathrm{sup}}(b) = 1/2$, that is, the asymptotic occurrence rates of 0s and 1s in the dyadic expansion of $\sqrt{2}$ coincide with each other. This property for a real number is called *simple normality* to base $q = 2$, while a stronger notion is *normality* to base q meaning that for each $\ell \geq 1$, the asymptotic occurrence rate of every subword of length ℓ in a given infinite q-ary word coincides with each other (see *e.g.* Borel (1909); Kuipers et al. (1974); Hertling (2002)). A motivation of the above conjecture on the simple normality of $\sqrt{2}$ to base 2 would come from a naive intuition that the dyadic expansion of $\sqrt{2}$ (and also of other several famous irrational numbers) looks very random, and also from a theorem by Borel (Borel (1909)) that almost every (in terms of Lebesgue measure) real number is normal to every base $q \geq 2$. We mention that recently Isaac posted a preprint (Isaac (2005)) to prove that every $\sqrt{s}$ with s an integer that is not perfect square is simply normal to base 2 (however, the author could not understand that his proof is completely correct). Moreover, several computer experiments also support the conjecture.

If this conjecture is true, then Theorem 2 implies that the asymptotic occurrence rate of the dangerous parameters n is at least $1/6$. On the other hand, even if the conjecture were not true, then a weaker assumption $r_{\mathrm{sup}}(b) > 2/5$ still could imply that infinitely many dangerous parameters n exist. This means that, to avoid to falsely choose a dangerous parameter n in a practical use of the above PRNG, a naive countermeasure of using sufficiently large parameters does very probably not solve the problem essentially.

4 Open Problems

In the previous sections, we have given in Proposition 1 a sufficient condition for a parameter n for the PRNG to be dangerous in terms of the dyadic expansion of $\sqrt{2}$. However, it is shown that the condition is not necessary. In fact, a direct calculation based on Lemma 9 shows that $n = 65$ is a dangerous parameter that does not satisfy $n \in P(b)$. Thus it is expected that we can obtain a better bound of the asymptotic occurrence rate of dangerous parameters than Theorem 2 by investigating the condition in Lemma 9, not only in Proposition 1 that is weaker than Lemma 9.

Here we describe a rough observation of the author for this problem. Put $b_0 = 1$ and $b_{-1} = 0$ for simplicity. Let $\mathcal{C}(b)$ be the set of all binary words of the form $b_{-1}b_0b_1 \cdots b_{k-1}0$ for any $k \geq -1$ with $b_k = 1$. Then a parameter n satisfies (22) if the infinite word $b_{n-1}b_nb_{n+1}\cdots$ involves a member of $\mathcal{C}(b)$ as an initial subword. Thus for any subset $\mathcal{C}'$ of $\mathcal{C}(b)$, a lower bound for *any* infinite binary word x of the number of indices n such that $x_{n-1}x_nx_{n+1}\cdots$ involves a member of $\mathcal{C}'$ yields a lower bound of the asymptotic rate of the dangerous parameters. For example, the argument in the previous sections deals with the subset $\mathcal{C}'$ with only three members 00, 0100, and 01010. Thus an immediate improvement of the bound in Theorem 2 would be derived by applying a similar argument to a larger subset $\mathcal{C}'$ of $\mathcal{C}(b)$.

Another possible generalization is a problem of finding a lower bound of the asymptotic rate of indices n that satisfy (22), not only for the above binary word b but also for an *arbitrary* infinite binary word x. A solution of this generalized problem yields another bound of the asymptotic rate of the dangerous parameters. Since an occurrence of a bit 1 in x yields a member of the set corresponding to $\mathcal{C}(b)$ in the previous paragraph, it seems possible that some nontrivial bound holds also in the generalized setting, at least when the asymptotic rate of 1s in x is not too low. Owing to the self-referential description, the author hopes that the problem involves certain mathematically interesting structure that is worthy to investigate.

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Combinatorics of Positroids

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Recently Postnikov gave a combinatorial description of the cells in a totally-nonnegative Grassmannian. These cells correspond to a special class of matroids called positroids. There are many interesting combinatorial objects associated to a positroid. We introduce some recent results, including the generalization and proof of the purity conjecture by Leclerc and Zelevinsky on weakly separated sets.

Keywords: Positroids, Total positivity, Grassmannian, Matroids, ⅃-diagrams, Decorated permutations, Weakly separated.

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1 Introduction

A *positroid* is a matroid that can be represented by a $k \times n$ matrix with nonnegative maximal minors. The classical theory of total positivity concerns matrices in which all minors are non-negative, and this subject was extended by Lusztig.

Lusztig introduced the totally non-negative variety $G \geq 0$ in an arbitrary reductive group G and the totally non-negative part $(G/P)_{\geq 0}$ of a real flag variety (G/P). He also conjectured that $(G/P)_{\geq 0}$ is made up of cells, and this was proved by Rietsch.

In this paper, we will restrict our attention to $(Gr_{kn})_{\geq 0}$, the *totally non-negative Grassmannian*. Then there is a more refined decomposition using matroid strata. Recently, Postnikov obtained a relationship between $(Gr_{kn})_{\geq 0}$ and certain planar bicolored graphs, producing a combinatorially explicit cell decomposition of $(Gr_{kn})_{\geq 0}$. The cells correspond to positroids.

In this extended abstract, We will briefly sketch some results about positroids, and introduce some open problems. No proof will be provided.

In section 3, we will go over describing matroidal operations on positroids via decorated permutations. In section 4, we will describe positroids by forbidden minors. In section 5 and 6, we will introduce the proof of the purity conjecture given in (OPS), and introduce related enumerative problems.

2 Preliminaries

An element in the Grassmannian Gr_{kn} can be understood as a collection of n vectors $v_1, \cdots, v_n \in \mathbb{R}^k$ spanning the space $\mathbb{R}^k$ modulo the simultaneous action of GL_k on the vectors. The vectors v_i are the

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columns of a $k \times n$-matrix A that represents the element of the Grassmannian. Then an element $V \in Gr_{kn}$ represented by A gives the matroid $\mathcal{M}_V$ whose bases are the k-subsets $I \subset [n]$ such that $\Delta_I(A) \neq 0$. Here, $\Delta_I(A)$ denotes the determinant of A_I, the k by k submatrix of A in the column set I.

Then Gr_{kn} has a subdivision into *matroid strata* $S_{\mathcal{M}}$ labelled by some matroids $\mathcal{M}$:

$$S_{\mathcal{M}} := \{V \in Gr_{kn} | \mathcal{M}_V = \mathcal{M}\}$$

The elements of the stratum $S_{\mathcal{M}}$ are represented by matrices A such that $\Delta_I(A) \neq 0$ if and only if $I \in \mathcal{M}$. Now we define the Schubert matroids, which correspond to the cells of the matroid strata.

Ordering $<_w, w \in S_n$ is defined as $a <_w b$ if $w^{-1}a < w^{-1}b$ for $a, b \in [n]$.

Definition 1 *Let $A, B \in \binom{[n]}{k}$, $w \in S_n$ where*

$$A = \{i_1, \cdots, i_k\}, i_1 <_w i_2 <_w \cdots <_w i_k$$

$$B = \{j_1, \cdots, j_k\}, j_1 <_w j_2 <_w \cdots <_w j_k$$

Then we set $A \leq_w B$ if and only if $i_1 \leq_w j_1, \cdots, i_k \leq_w j_k$. This ordering is called the Gale ordering on $\binom{[n]}{k}$ induced by w. We denote $\leq_t$ for $t \in [n]$ as $<_{c^{t-1}}$ where $c = (1, \cdots, n) \in S_n$.

We can also define matroids from the above ordering. See (G),(BGW).

Definition 2 *Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Then $\mathcal{M}$ is a matroid if and only if $\mathcal{M}$ satisfies the following property. For every $w \in S_n$, the collection $\mathcal{M}$ contains a unique member $A \in \mathcal{M}$ maximal in $\mathcal{M}$ with respect to the partial order $\leq_w$.*

Now we can define a Schubert matroid and a dual Schubert matroid using the partial order $\leq_w$.

Definition 3 *For $I = (i_1, \cdots, i_k)$, the Schubert Matroid SM_I^w consists of bases $H = (j_1, \cdots, j_k)$ such that $I \leq_w H$. The dual Schubert matroid $\tilde{SM}_I^w$ consists of bases $H = (j_1, \cdots, j_k)$ such that $I \geq_w H$.*

Let us define the totally nonnegative Grassmannian and its cells.

Definition 4 ((P)) *The* totally nonnegative Grassmannian $Gr_{kn}^{tnn} \subset Gr_{kn}$ *is the quotient* $Gr_{kn}^{tnn} = GL_k^+ \backslash Mat_{kn}^{tnn}$, *where Mat_{kn}^{tnn} is the set of real $k \times n$-matrices A of rank k with nonnegative* maximal *minors $\Delta_I(A) \geq 0$ and GL_k^+ is the group of $k \times k$-matrices with positive determinant.*

Definition 5 ((P)) Totally nonnegative Grassmann cells $S_{\mathcal{M}}^{tnn}$ *in Gr_{kn}^{tnn} are defined as $S_{\mathcal{M}}^{tnn} := S_{\mathcal{M}} \cap Gr_{kn}^{tnn}$. $\mathcal{M}$ is called a* positroid *if the cell $S_{\mathcal{M}}^{tnn}$ is nonempty.*

Note that from above definitions, we get

$$S_{\mathcal{M}}^{tnn} = \{GL_k^+ \bullet A \in Gr_{kn}^{tnn} | \Delta_I(A) > 0 \textbf{ for } I \in \mathcal{M}, \Delta_I(A) = 0 \textbf{ for } I \notin \mathcal{M}\}.$$

In (P), Postnikov showed a bijection between each cell and a combinatorial object called Grassmann necklace. He also showed that those necklaces can be represented as objects called decorated permutations. Let's first see how they are defined.

Definition 6 ((P)) *A* Grassmann necklace *is a sequence $\mathcal{I} = (I_1, \cdots, I_n)$ of subsets $I_r \subseteq [n]$ such that, for $i \in [n]$, if $i \in I_i$ then $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, for some $j \in [n]$; and if $i \in I_i$ then $I_{i+1} = I_i$. (Here the indices are taken modulo n.) In particular, we have $|I_1| = \cdots = |I_n|$.*

Definition 7 ((P)) *A decorated permutation $\pi^{:} = (\pi, col)$ is a permutation $\pi \in S_n$ together with a coloring function col from the set of fixed points $\{i|\pi(i) = i\}$ to $\{1, -1\}$. That is, a decorated permutation is a permutation with fixed points colored in two colors.*

It is easy to see the bijection between necklaces and decorated permutations. To go from a Grassmann necklace $\mathcal{I}$ to a decorated permutation $\pi^{:} = (\pi, col)$.

- if $I_{i+1} = (I_i \backslash \{i\}) \cup \{j\}$, $j \neq i$, then $\pi(i) = j$
- if $I_{i+1} = I_i$ and $i \notin I_i$ then $\pi(i) = i, col(i) = 1$
- if $I_{i+1} = I_i$ and $i \in I_i$ then $\pi(i) = i, col(i) = -1$.

To go from a decorated permutation $\pi^{:} = (\pi, col)$ to a Grassmann necklace $\mathcal{I}$,

$$I_r = \{i \in [n] | i <_r \pi^{-1}(i) \textbf{ or } (\pi(i) = i \textbf{ and } col(i) = -1)\}.$$

Let's look at a simple example. For decorated permutation $\pi^{:}$ with $\pi = 81425736$ and $col(5) = 1$, we get $I_1 = \{1,2,3,6\}, I_2 = \{2,3,6,8\}, I_3 = \{3,6,8,1\}, I_4 = \{4,6,8,1\}, I_5 = \{6,8,1,2\}, I_6 = \{6,8,1,2\}, I_7 = \{7,8,1,2\}, I_8 = \{8,1,2,3\}$.

Recall that we have defined $<_r$ to be a total order on $[n]$ such that $r <_r r+1 <_r \cdots <_r n <_r 1 <_r \cdots <_r r-1$. This ordering is the same as $<_{c^{r-1}}$ where $c = (1, \cdots, n) \in S_n$.

Lemma 8 ((P)) *For a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$ of rank k on the set $[n]$, let $\mathcal{I}_{\mathcal{M}} = (I_1, \cdots, I_n)$ be the sequence of subsets such that I_i is the minimal member of $\mathcal{M}$ with respect to $\leq_i$. Then $\mathcal{I}_{\mathcal{M}}$ is a Grassmann necklace.*

Theorem 9 ((P)) *Let $S^{tnn}_{\mathcal{M}}$ be a nonnegative Grassmann cell, and let $\mathcal{I}_{\mathcal{M}} = (I_1, \cdots, I_n)$ be the Grassmann necklace corresponding to $\mathcal{M}$. Then*

$$S^{tnn}_{\mathcal{M}} = \bigcap_{i=1}^{n} \Omega^{c^{i-1}}_{I_i} \cap Gr^{tnn}_{kn}$$

where $c = (1, \cdots, n) \in S_n$ and $\Omega^{c^{i-1}}_{I_i}$ is the permuted Schubert cell, which is the set of elements $V \in Gr_{kn}$ such that I_i is the lexicographically minimal base of M_V with respect to ordering $<_w$ on $[n]$.

This theorem implies that bases of a positroid are included in each Schubert matroids corresponding to the Grassmann necklace, but it does not imply that they are equal. Postnikov therefore conjectured that each positroid is exactly the intersection of Schubert matroids.

Theorem 10 ((O1)) *$\mathcal{M}$ is a positroid if and only if for some Grassmann necklace $(I_1, \cdots, I_n)$,*

$$\mathcal{M} = \bigcap_{i=1}^{n} SM^{c^{i-1}}_{I_i}.$$

In other words, $\mathcal{M}$ is a positroid if and only if the following holds : $H \in \mathcal{M}$ if and only if $H \geq_t I_t$ for any $t \in [n]$.

Let's see an example. Let $\mathcal{M}$ be a positroid indexed by a decorated permutation $[5,3,2,1,4,6]$, with $col(6) = -1$. Then we get : $I_1 = \{1,2,4,6\}, I_2 = \{2,4,5,6\}, I_3 = \{3,4,5,6\}, I_4 = \{4,5,6,2\}, I_5 = \{5,6,1,2\}, I_6 = \{1,2,4,6\}$.

$$\mathcal{M} = \{\{1,2,4,6\}, \{1,2,5,6\}, \{1,3,4,6\}, \{1,3,5,6\}, \{2,4,5,6\}, \{3,4,5,6\}\}.$$

3 Matroid operations via decorated permutations

In this section we will see that matroidal contraction, deletion and dual for positroids can be described via operations on decorated permutations. In this section, we will use $-$ for the set subtraction.

Definition 11 *Given a matroid $\mathcal{M}$ on set E. The* contraction *of $T \subset E$ from $\mathcal{M}$ is defined as*

$$\mathcal{M}/T = \{I - T : T \subset I \in \mathcal{M}\}.$$

The deletion *of $T \subset E$ from $\mathcal{M}$ is defined as*

$$\mathcal{M}\backslash T = \{I \in \mathcal{M} : I \subset (E - T)\}.$$

The restriction *of $\mathcal{M}$ to $T \subset E$ is defined as*

$$\mathcal{M}|_T = \mathcal{M}\backslash(E - T).$$

Fix a decorated permutation $\pi^{:} = (\pi, col)$. Let $(I_1, \cdots, I_n)$ be the corresponding Grassmannian necklace and $\mathcal{M}$ the corresponding positroid. Now look at $(J_1 := \pi^{-1}(I_1), \cdots, J_n := \pi^{-1}(I_n))$. They also form a necklace. We will call this the *upper Grassmann necklace* of π. So we have the following theorem.

Theorem 12 ((O1)) *Pick a decorated permutation. $\pi^{:}$. Then we have the corresponding Grassmann necklace and upper Grassmann necklace $\mathcal{I} = (I_1, \cdots, I_n), \mathcal{J} = (J_1, \cdots, J_n)$. Then $J_i = \pi^{-1}(I_i)$ for all $i \in [n]$ where $\pi^{:} = (\pi, col)$. And we have the equality*

$$\bigcap_{i=1}^{n} SM_{I_i}^{c^{i-1}} = \bigcap_{i=1}^{n} \tilde{SM}_{J_i}^{c^{i-1}}.$$

As a corollary, we now know how to describe taking the dual of positroids in terms of decorated permutations.

Corollary 13 *Let $\mathcal{M}$ be a positroid indexed by $\pi^{:} = (\pi, col)$. Let $\mathcal{M}'$ be a matroid obtained by taking the dual of $\mathcal{M}$. Then $\mathcal{M}'$ is a positroid and $\mathcal{M}'$ is indexed by $(\pi^{-1}, -col)$.*

Now let's show how to describe contractions of positroids via decorated permutations. Let $\mathcal{M}$ be a positroid indexed by $\pi^{:}$. Denote $\mathcal{M}'$ be $\mathcal{M}/\{j\}$, $j \in [n]$. Let $\mu^{:}$ be the decorated permutation indexing $\mathcal{M}'$. We have an algorithm to obtain $\mu^{:}$ directly from $\pi^{:}$.

For $i \neq j$, if $col(i)$ is defined, $col'(i)$ is defined as the same value. For other $i \in [n]$, if $\mu(i) = i$, $col'(i) = +1$. Let's try out the algorithm for $\pi = [6, 1, 4, 8, 2, 7, 3, 5]$ and $j = 3$.

1. $\mu = [6, 1, 4, 8, 2, 7, 3, 5]$.
2. $\mu(3) = 3, q = 4$. Now $a = 4, \mu = [6, 1, 3, 8, 2, 7, 3, 5]$.
3. Since $q = a = 4$, $\mu(4) = 4, q = \pi(4) = 8$. Now $a = 5, \mu = [6, 1, 3, 4, 2, 7, 3, 5]$.
4. Since $q \neq a$, $\pi(5) = 2 <_6 3$, $\mu(5) = min_6(2, 8) = 8, q = max_6(2, 8) = 2$. Now $a = 6, \mu = [6, 1, 3, 4, 8, 7, 3, 5]$.

Algorithm 1 When $\pi(j) \neq j$ and $\mathcal{M}_{\mu^:} = \mathcal{M}_{\pi^:}/\{j\}$, obtaining $\mu^: = (\mu, col')$ from $\pi^: = (\pi, col)$.

```
μ ⇐ π
col′ ⇐ col
μ(j) ⇐ j
col′(j) = 1
a ⇐ j + 1
q ⇐ π(j)
while π(a) ≠ j do
  if q = a or q <_{a+1} π(a) <_{a+1} j then
    μ(a) ⇐ q
    q ⇐ π(a)
    if μ(a) = a then
      col′(a) = 1
    end if
  end if
  a ⇐ a + 1
end while
μ(a) ⇐ q
```

5. Since $q \neq a$, $\pi(6) = 7 <_7 3$, $\mu(6) = min_7(2,7) = 7, q = max_7(2,7) = 2$. Now $a = 7, \mu = [6,1,3,4,8,7,3,5]$.

6. Since $\pi(7) = 3$, $\mu(7) = 2$. We are finished and the result is $\mu = [6,1,3,4,8,7,2,5]$.

7. $col'(3) = +1, col'(4) = +1$.

The deletion of $\mathcal{M}$ by $\{j\}$ can also be obtained directly by an algorithm. See (O2). It will be interesting to check how the contraction and restriction of positroids can be described in terms of ⅃-diagrams, which we will show the definition in the next section.

Problem 14 *How can contraction and restriction of positroids described in terms of ⅃-diagrams?*

4 Positroids and Forbidden Minors

Another important property of positroids is that they can be characterized via forbidden minors. A matroid is called a *minor* of $\mathcal{M}$ if it can be obtained by sequence of restrictions and contractions from $\mathcal{M}$.

Lemma 15 *Let $\mathcal{M}$ be a matroid of rank k over $[n]$. $\mathcal{M}$ is a positroid if and only if it satisfies the following condition:*

Let T be any $k-2$ element subset of $[n]$. For each $a,b,c,d \in [n]-T$ be such that $a <_t b <_t c <_t d$ for some $t \in [n]$, the following relation holds. $T \cup \{a,c\}, T \cup \{b,d\} \in \mathcal{M}$ if and only $T \cup \{a,b\}, T \cup \{c,d\} \in \mathcal{M}$ or $T \cup \{a,d\}, T \cup \{b,c\} \in \mathcal{M}$. See Figure 1.

Notice that the above condition can also be written as the following. Let T be any $k-2$ element subset of $[n]$. For any 4 element subset $Q \subseteq [n]-T$, $(\mathcal{M}/T)|_Q$ is a positroid.

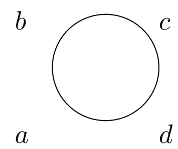

Fig. 1: $a <_t b <_t c <_t d$

Let's find all the matroids of rank 2 over $[4]$ that are not positroids:

$$\{12, 34, 13, 23, 14\}, \{12, 34, 14, 23, 24\}, \{12, 34, 14, 23\}.$$

By Lemma 15 and remark following it, we can conclude as the following.

Theorem 16 ((O1)) *A matroid is a positroid if and only if it has no minors among the above list.*

We can look at the nonnegative Grassmannian over finite fields. The positive part of a $\mathbb{F}_q$ when q is prime is defined as set of elements that can be expressed as a square of a nonzero element inside that field. Denote this as $\mathbb{F}_q^+$. So we can define cells in the nonnegative Grassmannian of $\mathbb{F}_q$. Using similar notation as before,

$$S_{\mathcal{M}}^{tnn}(\mathbb{F}_q) := \{GL_k^+(\mathbb{F}_q) \bullet A \in Gr_{kn}(\mathbb{F}_q) | \Delta_I(A) \in \mathbb{F}_q^+ \textbf{ for } I \in \mathcal{M}, \Delta_I(A) = 0 \textbf{ for } I \notin \mathcal{M}\}.$$

$\mathbb{F}_q$-positroid is defined as matroid $\mathcal{M}$ such that $S_{\mathcal{M}}^{tnn}(\mathbb{F}_q)$ is nonempty. The Plücker relations behave nicely in $\mathbb{F}_3$ and $\mathbb{F}_7$. $\mathbb{F}_3, \mathbb{F}_7$ positroids are special cases of positroids, and can also be expressed in terms of forbidden minors.

Proposition 17 *A positroid is a $\mathbb{F}_3, \mathbb{F}_7$-positroid if and only if it avoids minor* $\{12, 34, 13, 23, 14, 24\}$.

5 Plabic graphs and ⅃-diagrams.

In (P), Postnikov defined plabic graphs and ⅃-diagrams. There are a number of plabic graphs that can be used to paramatrize the corresponding positroid strata. And for each positroid strata, there exists a unique plabic graph that corresponds to a combinatorial object called the ⅃-diagram. So there is a bijection between positroids and ⅃-diagrams. In (OPS), we used plabic graphs to describe collection of maximal weakly separated sets of Leclerc and Zelevinsky. And in the next section, we will use ⅃-diagrams to count a related invariant on weakly separated sets. For more details on plabic graphs and ⅃-diagrams, see (P).

Definition 18 *A planar bicolored graph, or simply a plabic graph is a planar undirected graph G drawn inside a disk. The vertices on the boundary are called boundary vertices. All vertices in the graph are colored either white or black.*

A plabic graph is associated with a decorated permutation. The positroid corresponding to the decorated permutation is exactly the positroid that the plabic graph gives a parametrization of the corresponding strata.

Definition 19 *For a plabic graph G, a trip(one-way trip in notion of (P)) is a directed path T in G such that T joins two boundary vertices and satisfies the following rules of the road. Turn right at a black vertex, and turn left at a white vertex. Trip permutation $\pi_G \in S_n$ is defined such that $\pi_G(i) = j$ whenever trip that starts at the boundary vertex labeled i ends at boundary vertex j. Decorated trip permutation $\pi^{:}_G$ is defined similarly.*

We have following 3 moves on plabic graphs.

(M1) Pick a square with vertices alternating in colors. Then we can switch the colors of all the vertices. See Figure 2.

Fig. 2: (M1) Square move

(M2) For two adjoint vertices of the same color, we can contract them into one vertice. See Figure 3.

Fig. 3: (M2) Unicolored edge contraction

(M3) We can insert or remove a vertice inside any edge. See Figure 4.

Fig. 4: (M3) Middle vertex insertion/removal

Now we define reducedness of plabic graphs.

Definition 20 *A strand $i \in [n]$ for a plabic graph is given by the trip starting at $\pi^{-1}(i)$ and ending at i. Then the plabic graph is reduced if it satisfies the following two properties:*

1. *Each strand does not self-intersect.*
2. *For any two strands that intersect, the direction they are heading should be opposite of each other. That is, if they intersect at points a and b, the one should be heading from a to b and another from b to a.*

Really nice about the property of being a plabic graph is that it is closed under certain combinatorial moves. The same goes for being a reduced plabic graph.

Theorem 21 ((P)) *Let G and G' be two (reduced) plabic graphs with the same number of boundary vertices. Then the following claims are equivalent:*

- *G can be obtained from G' by moves (M1)-(M3).*
- *These two graphs have the same decorated trip permutation $\pi^{:}_{G} = \pi^{:}_{G'}$.*

We now define what is called a ⅃-diagram, which encodes one unique reduced plabic graph of the corresponding positroid. They are nice combinatorial objects that can be used to easily compute various invariants of the corresponding positroid. One example is that the dimension of the strata equals the number of dots plus one. Another example will be introduced in the next section.

Definition 22 *A Young diagram of shape λ is called a ⅃-diagram if it satisfies the following property. Each box is either empty or filled with one dot. For any three boxes indexed $(i,j),(i',j),(i,j')$, where $i' < i$ and $j' < j$, if boxes on position (i',j) and (i,j') contain a dot inside, then the box on (i,j) also contains a dot. This property is called the ⅃-property.*

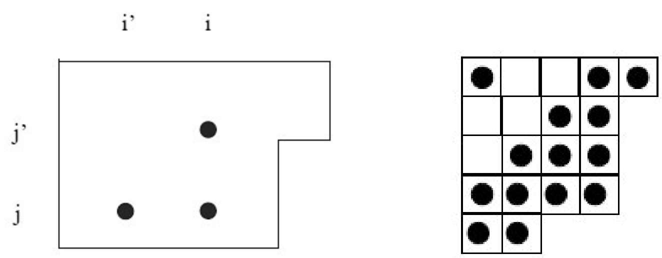

Fig. 5: ⅃-property and an example of a ⅃-diagram

The boundary of the diagram forms a lattice path from the upper-right corner to lower-left corner. Label the n steps in this path by numbers $1, \cdots, n$ consecutively. Define $I(\lambda)$ as the set of lables of k vertical steps in the path. Put dots on each edge of the boundary path. Connect all dots on same row and connect all dots on the same column. Then we get a ⅃-graph.

Definition 23 *⅃-graph is obtained from a ⅃-diagram in the following way. Put a dot on center of each edge of the boxes on the southeast boundary of the diagram, and label them $1, 2, \cdots$ starting from northeast to southwest. Call these the boundary vertices. Now for each dot inside the ⅃-diagram, draw a horizontal line to its right, and vertical line to its bottom until it reaches the boundary of the diagram.*

Now below is the method to check which decorated permuation the ⅃-diagram corresponds to. That is, it tells us which positroid the ⅃-diagram corresponds to.

Theorem 24 ((P)) *Define a map χ that sends a ⅃-diagram to a decorated permutation $\pi^{:} = (\pi, col)$ defined as following. Set $\pi(i) = j$ where we reach j when we start from i and follow the rules of the road in Figure 7. If $\pi(i) = i$, set $col(i) = -1$ if i is on a horizontal edge, $col(i) = 1$ if otherwise. Then χ is a bijection between ⅃-diagrams having lower boundary I and decorated permutations having $I_1 = I$, where $\mathcal{I} = (I_1, \cdots, I_n)$ is the Grassmann necklace of the decorated permutation.*

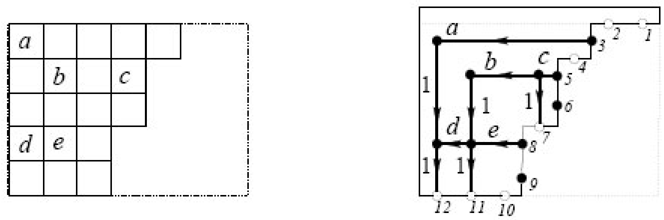

Fig. 6: Example of a ⅃-graph

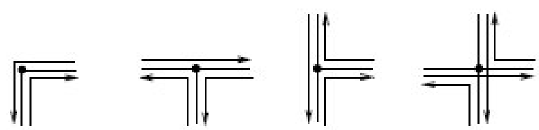

Fig. 7: Rules of the road

6 Weak Separation and the conjectures of Leclerc and Zelevinsky

In this section, we introduce Leclerc and Zelevinsky's conjecture on weakly separated sets, which was recently generalized and proved in (OPS) using positroids and plabic graphs. And then, we show a nice invariant related to this problem, counted using ⅃-diagrams.

We say that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are circularly ordered if there exists $\alpha \in [n]$ such that $\alpha_1 <_\alpha \cdots <_\alpha \alpha_n$.

Definition 25 ((LZ)) *Let $I, J \in \binom{[n]}{k}$. We say that I and J are weakly separated if there does not exist $i_1, i_2 \in I \setminus J, j_1, j_2 \in J \setminus I$ such that i_1, j_1, i_2, j_2 are circularly ordered. For $\mathcal{C} \subset \binom{[n]}{k}$, we call $\mathcal{C}$ a maximal weakly separated collection if it is maximal among collections such that each of its elements are pairwise weakly separated.*

Leclerc and Zelevinsky observed the following:

Proposition 26 ((LZ)) *Let $S \in \binom{[n]}{k-2}$ and let $\alpha, \beta, \gamma, \delta \in [n] \setminus S$ be circularly ordered. Suppose that a maximal weakly separated collection $\mathcal{C}$ contains $S\alpha\beta, S\beta\gamma, S\gamma\delta, S\delta\alpha$ and $S\alpha\gamma$. Then $\mathcal{C}' := \mathcal{C} \setminus \{S\alpha\gamma\} \cup \{S\beta\delta\}$ is also a maximal weakly separated collection.*

In the above proposition, we say that $\mathcal{C}'$ and $\mathcal{C}$ are obtained from each other by a flip. If $\mathcal{C}'$ can be obtained from $\mathcal{C}$ by sequence of flips, we say that $\mathcal{C}'$ and $\mathcal{C}$ are flip-connected.

Leclerc and Zelevinsky formulated two challenging conjectures on maximal weakly separated collections and they were recently proved in (OPS). The main tool is to relate maximal weakly separated collection and plabic graphs. There is a nice way to label the faces of plabic graphs corresponding to positroid $\mathcal{M} \subseteq \binom{[n]}{k}$ with elements of $\binom{[n]}{k}$. Since each strand divides a disk into two parts, each face is either on the left side or the right side of a strand. For each face we label that face with $J \in \binom{[n]}{k}$ such

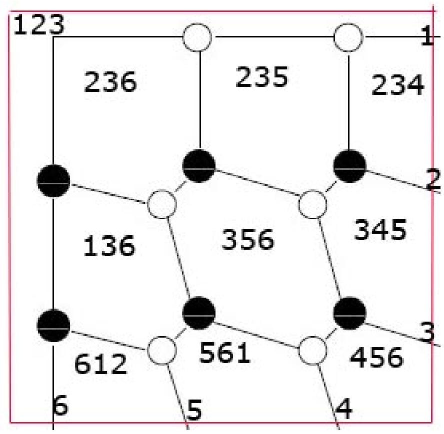

Fig. 8: Labeling faces of a plabic graph

that $i \in J$ if and only if there is a strand containing that face on its left side. See figure 8 for an example. So given a plabic graph $\mathcal{G}$, we define $\mathcal{F}_{\mathcal{G}}$ as the set of labels that occur on each face of that graph.

Theorem 27 ((OPS)) *$\mathcal{C} \subseteq \binom{[n]}{k}$ is a maximal weakly separated collection if and only if it is $\mathcal{F}_{\mathcal{G}}$ for some $\mathcal{G}$ a reduced plabic graph of positroid $\mathcal{M} = \binom{[n]}{k}$.*

For example, $\{123, 234, 345, 456, 561, 612, 236, 235, 136, 356\}$ is a maximal weakly separated collection of $\binom{[6]}{3}$. And one can check from Figure 8 that this collection comes from labeling the faces of a plabic graph of the positroid $\binom{[6]}{3}$.

In (P), it shown that the number of faces of reduced plabic graphs of the same positroid are the same, and that they equal the dimension of the corresponding strata. So as a corollary of the theorem, the two conjectures of Leclerc and Zelevinsky follows.

Corollary 28 ((OPS)) *Every maximal weakly separated collection in $\binom{[n]}{k}$ has cardinality $k(n-k)+1$. Any such two maximal weakly separated collections are flip-connected.*

One can observe that $\binom{[n]}{k}$ is a positroid corresponding to the top cell of the nonnegative part of the grassmannian. The corresponding strata has dimension $k(n-k)+1$. This number can also be read from the corresponding $\lrcorner$-diagram, which looks like a k by $n-k$ rectangle with dots inside all the boxes.

Definition 29 *Let $\mathcal{M}$ be a positroid having grassmann necklace $\mathcal{I} = (I_1, \cdots, I_n)$. $\mathcal{C} \subseteq \mathcal{M}$ is called a weakly separated collection inside $\mathcal{M}$ if it contains all I_i for $i \in [n]$ and each pair of its elements are weakly separated. Maximal weakly separated collection is defined as maximal collections among weakly separated collections of $\mathcal{M}$.*

Then we have the following generalization for general positroids:

Theorem 30 ((OPS)) *Fix any positroid $\mathcal{M}$. $\mathcal{C} \subseteq \mathcal{M}$ is a maximal weakly separated collection if and only if it is $\mathcal{F}_{\mathcal{G}}$ for some $\mathcal{G}$ a plabic graph of $\mathcal{M}$.*

Corollary 31 ((OPS)) *Every maximal weakly separated collection of $\mathcal{M}$ has cardinality $dim(S_{\mathcal{M}})$. Any two maximal weakly separated collections of $\mathcal{M}$ are flip-connected.*

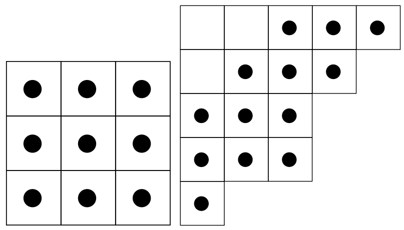

Fig. 9: Labeling the dots of a ⅃-diagram cooresponding to the top cell, a lattice path matroid.

Now let's define $\mathcal{F}_{\mathcal{M}}$ as $\bigcup_{\mathcal{G},\textbf{plabic graph of}\mathcal{M}} \mathcal{F}_{\mathcal{G}}$. This set is interesting because when Scott showed that the coordinate ring of the Grassmannian has a cluster algebra structure in (S), one of the two main key steps was to show that $\mathcal{F}_{\binom{[n]}{k}} = \binom{[n]}{k}$. Now what follows from the above theorem is describtion of $\mathcal{F}_{\mathcal{M}}$:

Corollary 32 *$I \in F_{\mathcal{M}}$ if and only if I is weakly separated with $I_1, \cdots, I_n$, where $\mathcal{I} = (I_1, \cdots, I_n)$ is the grassmann necklace of $\mathcal{M}$.*

Now we will see a nice method to compute the cardinality of $F_{\mathcal{M}}$ using ⅃-diagrams.

Definition 33 *Let us be given a ⅃-diagram. Choose k-columns and k-rows. Then we get k^2 positions, and if there are dots in every positions, we call that a k-by-k full-minor. We say that every ⅃-diagram has exactly one 0-by-0 full-minor.*

Theorem 34 *Fix a positroid $\mathcal{M}$. Let m_i denote the number of i-by-i full-minors inside the corresponding ⅃-diagram. Then $|F_{\mathcal{M}}| = \sum_{i=0}^{\infty} m_i$.*

For each dot in the ⅃-diagram, if we label the dot with sum of number of all i-by-i minors having that dot as the upper-left corner for all i, we get something more interesting. For top cells and lattice path matroids(When ⅃-diagram consists of a skew-young diagram with dots inside all the boxes.), we get the Pascal's triangle. This labeling gives us another way to compute $|F_{\mathcal{M}}|$.

Proposition 35 *$|F_{\mathcal{M}}|$ can also be obtained by adding 1 to the sum of all the labels of the dots given as above.*

Look at Figure 9 for an example. If we look at the left figure corresponding to the top cell $\binom{[6]}{3}$, we can get $|F_{\binom{[6]}{3}}| = 1 + 5 * 1 + 2 * 1 + 3 * 2 + 6 = 20$.

It would be interesting to find a way to count the number of plabic graphs of a given cell. That is, a way to count the number of maximal weakly separated collection of a positroid.

Problem 36 *What is the number of maximal weakly separated collection of a positroid $\mathcal{M}$?*

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Blocks in Constrained Random Graphs with Fixed Average Degree

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This work is devoted to the study of typical properties of random graphs from classes with structural constraints, like for example planar graphs, with the additional restriction that the average degree is fixed. More precisely, within a general analytic framework, we provide sharp concentration results for the number of blocks (maximal biconnected subgraphs) in a random graph from the class in question. Among other results, we discover that essentially such a random graph belongs with high probability to only one of two possible types: it either has blocks of at most logarithmic size, or there is a *giant* block that contains linearly many vertices, and all other blocks are significantly smaller. This extends and generalizes the results in the previous work [K. Panagiotou and A. Steger. Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '09), pp. 432-440, 2009], where similar statements were shown without the restriction on the average degree.

Keywords: Random Graphs, Boltzmann Sampling, Generating Functions

1 Introduction

In the early 60's Erdős and Rényi (5) introduced the random graph $G_{n,M}$, which is obtained by adding M random edges to an initially empty graph with n labeled vertices. Since then, the $G_{n,M}$ has been studied extensively, and in the meanwhile there have been thousands of papers devoted to the analysis of its typical properties. One property that has been of particular interest is the *evolution* of $G_{n,M}$. Of course, when $M = 0$ then $G_{n,M}$ is just the empty graph, and when $M = \binom{n}{2}$ then $G_{n,M}$ is the complete graph – but are there *critical* values in between where interesting changes happen? The answer is yes, and many such critical values of M have been discovered. Let us mention here only an example (the famous "*phase transition*"), and we refer the reader to the excellent monograph (3) of Bollobás for many other exciting results. Let $M = cn$. If $c < 1$, then $G_{n,M}$ has with high probability[(i)] connected components of size $\mathcal{O}(\log n)$. On the other hand, if $c > 1$, then the largest connected component of $G_{n,M}$ containts whp $\Theta(n)$ vertices, and the second largest component contains only $\mathcal{O}(\log n)$ vertices. In other words, the connectivity structure of $G_{n,M}$ changes dramatically when we "pass" the critical value $c = 1$.

Much less is known if we turn our attention to graph classes with structural constraints, like for example planar graphs. How different does a random planar graph with $\frac{3}{2}n$ edges look like from a random planar graph with $\frac{5}{2}n$ edges? In (11) the authors Gerke, McDiarmid, Steger and Weissl showed that for all

[(i)] whp, with probability tending to 1 when $n \to \infty$

$m = \lfloor cn \rfloor$, where $1 < c < 3$, a random planar graph with m edges contains linearly many copies of any given planar graph. Moreover, they showed that the probability of connectedness is bounded away from 0 and 1. Except of these results we currently have only very sparse information about how the *evolution* of a random planar graph does look like, and how random planar graphs with fixed average degree typically behave.

The goal of this paper is to present a unified analytic framework, which allows us to make precise statements about random graphs with fixed average degree drawn from a specified graph class. Our framework includes specifically the classes of labeled outerplanar, series-parallel, and planar graphs, and more generally every class for which we have sufficient information about the generating function enumerating this class. The parameter that we will study here is the *block structure*. Denote for any graph G by $b(\ell;\, G)$ the number of blocks, i.e. maximal biconnected subgraphs, that contain exactly ℓ vertices in G, and abbreviate $b(\ell_0 \dots \ell_1;\, G) = \sum_{\ell=\ell_0}^{\ell_1} b(\ell;\, G)$. Moreover, denote by $lb(G)$ the number of vertices in the largest block of G. Let $\mathcal{C}$ be a graph class, and let $\mathsf{C}_{n,m}$ be a random graph from $\mathcal{C}$ with n vertices and m edges. We shall show that whp $\mathsf{C}_{n,m}$ belongs to one of the following categories, which differ vastly in complexity:

- $lb(\mathsf{C}_{n,m}) \sim_n cn$, where $c = c(\mathcal{C}) > 0$ is given explicitly, and the second largest block contains n^α vertices, where $\alpha = \alpha(\mathcal{C}) < 1$ (where $x \sim_n y$ means $x = (1 + o(1))y$ for large n), or
- $lb(\mathsf{C}_{n,m}) = \mathcal{O}(\log n)$.

Moreover, we will show sharp concentration results for the quantities $b(\ell;\, \mathsf{C}_{n,m})$ for all $2 \le \ell \le n$. As a corollary we will obtain that for all $c \in (1, 3)$ random planar graphs with $\lfloor cn \rfloor$ edges belong to the first category, while random outerplanar and series-parallel graphs with fixed average degree belong to the second category. Finally, we will demonstrate that there are graph classes such that there a exists a *critical* density c_0 where the category to which a random graph with cn edges belongs to is different for $c > c_0$ and $c < c_0$. We shall discuss this and related issues in more detail later.

Before we present our results in detail we need a technical definition. For any set C of complex numbers and any $\delta > 0$ let $N(C, \delta)$ be the set of all complex numbers that are closer than δ to some point of C. We shall say that a function $F(x, y)$ is of *algebraic type* for y in a compact subset S of $(0, +\infty)$ if there exist $\delta > 0$ and $0 < \theta < \pi/2$ such that

$$F(x,y) = P(x,y) + \left(1 - \frac{x}{\rho(y)}\right)^{-\alpha} \cdot (g(y) + E(x,y)), \tag{1}$$

where

- $P(x, y)$ is a polynomial,
- $g(y)$ is analytic in $N(S, \delta)$, and $g(y) \neq 0$,
- $\rho(y)$ has continuous third partial derivatives in $N(S, \delta)$, and $\alpha \notin \mathbb{Z}_{\le 0}$,
- $E(x, y)$ is analytic in $\Delta \setminus \{\rho(y)\}$ and $E(x, y) = o(1)$ as $x \to \rho(y)$ uniformly for all $y \in N(S, \delta)$, where
$$\Delta = \{z : |z| \le \rho(y) + \delta,\ |\arg(z - \rho(y))| \ge \theta\}.$$

Functions of the algebraic type are commonly encountered in modern analytic combinatorics, and the above assumptions, although quite technical, are needed to unfold the full power of the available machinery. We refer the reader to Flajolet's and Sedgewick's book (7) for an excellent treatment and numerous applications. All functions that we shall consider in this work are of algebraic type.

The following definition describes precisely the graph classes that will be of interest in this paper, and is a generalization of a similar definition in (13).

Definition 1 *Let $\mathcal{C}$ be a class of labeled connected graphs and let $\mathcal{B} \subset \mathcal{C}$ be the class of biconnected graphs in $\mathcal{C}$. The class $\mathcal{C}$ is called nice if it has the following two properties.*

(i) Let $C \in \mathcal{C}$ and $B \in \mathcal{B}$. Then the graph obtained by identifying any vertex of C with any vertex from B is in $\mathcal{C}$. Moreover, all graphs in $\mathcal{C} \setminus \mathcal{B}$ can be constructed in such a way. Finally, the graph consisting of a single isolated vertex is in $\mathcal{C}$.

(ii) Let $B(x, y)$ be the exponential generating function enumerating the graphs in $\mathcal{C}$, where x marks the vertices, and y marks the edges. Then $\frac{\partial}{\partial x}B(x, y)$ is of algebraic type for $y \in [0, +\infty)$, where we will write $R(y) := \rho(y)$, $\alpha_B := \alpha$, and $g_B(y) := g(y)$.

The following statement says that the egf enumerating a nice class is also of algebraic type. We will use it without any further reference.

Proposition 1 *Let $\mathcal{C}$ be a nice class, and let S_C be a subinterval of S_B such that for all $y \in S_C$ it holds $R(y)B''(R(y), y) \neq 1$. Then $x\frac{\partial}{\partial x}C(x, y)$ is of algebraic type in S_C.*

In the proposition above we show that $x\frac{\partial}{\partial x}C(x, y)$, which is the exponential generating function enumerating vertex-rooted graphs from $\mathcal{C}$, is of algebraic type. We do this solely for technical convenience (instead of making a similar statement for $C(x, y)$): in what follows, we will study asymptotic properties of random graphs from $\mathcal{C}_n$, which do not depend on the root label. Hence, as there are exactly n distinct ways to root each graph in $\mathcal{C}_n$, any random variable defined on rooted graphs from $\mathcal{C}_n$ will be identically distributed with the corresponding random variable defined on graphs from $\mathcal{C}_n$.

Let us define some important **notation** that we shall use in the remainder of the paper without any further reference. We will denote by R, ρ the singularities of $\frac{\partial}{\partial x}B(x, y)$ and $x\frac{\partial}{\partial x}C(x, y)$, and by α_B, α the critical exponents of the corresponding singular expansions (see (1)). Finally, we will write g_B, g for the function g in the definition (1), for $\frac{\partial}{\partial x}B(x, y)$ respectively $x\frac{\partial}{\partial x}C(x, y)$.

In order to formulate our main result we need one additional technical definition. For any function $f(u)$ we will write $\partial_u f(u) = \frac{d}{du}f(u)$. Given a such a function $f(u)$, which is analytic at $u = 1$ and assumed to satisfy $f(1) \neq 0$, we set

$$\mathfrak{m}(f) = \frac{\partial_u f(1)}{f(1)} \text{ , and } \mathfrak{v}(f) = \frac{\partial_u^2 f(1)}{f(1)} + \mathfrak{m}(f) - \mathfrak{m}(f)^2. \tag{2}$$

Theorem 1 *Let $\mathcal{C}$ be a nice class. Suppose that for all $\beta \in S_C$*

- *$\mathfrak{v}(\rho_\beta) \neq 0$, where $\rho_\beta(y) := \rho(\beta y)$ and*
- *$\rho_\beta(1) < |\rho_\beta(u)|$ for all $u \in \{z \mid |z| = 1, z \notin N(S, \delta)\}$.*

Let $m = \lfloor \frac{-\beta\rho'(\beta)}{\rho(\beta)}n + \frac{\beta g'(\beta)}{g(\beta)} \rfloor$, and let $\mathsf{C}_{n,m}$ be a random graph from $\mathcal{C}_{n,m}$. Let $c(\beta) = R(\beta)B''(R(\beta), \beta)$. Then the following is true with high probability.

(I) If $c(\beta) > 1$ then let $0 < \tau(\beta) < R(\beta)$ be given by $\tau B''(\tau, \beta) = 1$, and set $\xi(\beta) = \tau(\beta)/R(\beta)$. Then we have for all $\varepsilon > 0$:

1. $lb(\mathsf{C}_{n,m}) \leq (|\alpha| + \varepsilon) \log_{1/\xi(\beta)} n$.

2. $b(\ell;\, \mathsf{C}_{n,m}) \sim b_\ell n$ *for all* $2 \le \ell \le (1-\varepsilon)\log_{1/\xi(\beta)} n$*, where*

$$b_\ell = [x^{\ell-1}]B'(x,\beta)\cdot \tau^{\ell-1} \;\sim_\ell\; \frac{g_B(\beta)}{\Gamma(\alpha_B)} \cdot \ell^{\alpha_B-1}\xi(\beta)^{\ell-1}, \tag{3}$$

3. $b\big((1-\varepsilon)\log_{1/\xi(\beta)} n \ldots (|\alpha|+\varepsilon)\log_{1/\xi(\beta)} n;\, \mathsf{C}_{n,m}\big) \le n^{2\varepsilon}$.

(II) If $c(\beta) < 1$*, then* $lb(\mathsf{C}_{n,m}) \sim (1-c(\beta))n$*. Moreover, we have* $\alpha_B < -1$ *and*

1. $b(\ell;\, \mathsf{C}_{n,m}) = 0$ *for all* $\ell = \omega(n^{-1/\alpha_B})$ *and* $\ell < lb(\mathsf{C}_n)$*,*
2. $b(\ell;\, \mathsf{C}_{n,m}) \sim b_\ell n$ *for all* $2 \le \ell$ *and* $\ell = o((\frac{n}{\log n})^{1/(1-\alpha_B)})$*, where*

$$b_\ell = [x^{\ell-1}]B'(x,\beta)\cdot R(\beta)^{\ell-1} \;\sim_\ell\; \frac{g_B(\beta)}{\Gamma(\alpha_B)} \cdot \ell^{\alpha_B-1}, \tag{4}$$

3. $b(\ell \ldots \delta\ell;\, \mathsf{C}_{n,m}) \sim b_{\ell,\delta} n$ *for all* $2 \le \ell$ *and* $\ell = o((\frac{n}{\log n})^{-1/\alpha_B})$*, where* $\delta > 1$ *and*

$$b_{\ell,\delta} = \sum_{s=\ell}^{\delta\ell}[x^{s-1}]B'(x)\cdot R(\beta)^{s-1} \;\sim_\ell\; \frac{g_B(\beta)}{\Gamma(\alpha_B+1)}\cdot(1-\delta^{\alpha_B})\ell^{\alpha_B}. \tag{5}$$

Let us discuss a few implications of the theorem above. Exploiting the results in (12) we obtain as a corollary the following result for random planar graphs with fixed average degree.

Corollary 1 *Let* $\mathsf{P}_{n,m}$ *be a graph drawn uniformly at random from the set of all labeled connected planar graphs with* n *vertices and* m *edges, where* $m = \lfloor cn \rfloor$ *and* $c \in (1,3)$*. Then* $\mathsf{P}_{n,m}$ *is with high probability of type (II).*

By applying the results in (2) we obtain statements about random outerplanar and series-parallel graphs.

Corollary 2 *Let* $\mathsf{O}_{n,m}$ *be a graph drawn uniformly at random from the set of all labeled connected outerplanar graphs with* n *vertices and* m *edges, where* $m = \lfloor cn \rfloor$ *and* $c \in (1,2)$*. Then* $\mathsf{O}_{n,m}$ *is with high probability of type (I). The same is true for random series-parallel graphs.*

One natural question that arises in the context of Theorem 1 is the following: is there a nice graph class such that there a exists a *critical* density c_0 where the type of a random graph with cn edges is different for $c > c_0$ and $c < c_0$? The following result gives an affirmative answer.

Theorem 2 *Let* $\mathcal{B}$ *be the class of biconnected planar graphs, and set* $\widetilde{\mathcal{B}} = \mathcal{B} \cup \{K_8\}$*, where* K_8 *is the complete graph with 8 vertices. Then the class* $\widetilde{\mathcal{C}}$ *in which every graph contains blocks only from* $\widetilde{\mathcal{B}}$ *is nice, and there is a* $c_0 \approx 3.9995$ *such that*

- *if* $c > c_0$*, then* $\widetilde{\mathsf{C}}_{n,\lfloor cn \rfloor}$ *is with high probability of type (I),*
- *if* $c < c_0$*, then* $\widetilde{\mathsf{C}}_{n,\lfloor cn \rfloor}$ *is with high probability of type (II).*

In fact, classes with *two* critical densities can be constructed, such that the type is different in each two neighboring intervals. We do not give the explicit construction here, but these graph classes seem to be very artificial. This raises the following questions. First, are there graph classes with *arbitrarily* many critical densities? And second, are there *natural* classes with more than one critical density? The definition of the term "natural" is of course a matter of taste – a possible candidate would be to require the class to be hereditary, i.e., closed under taking subgraphs.

Remark The discussion in this paper can easily be adapted to cover an even wider class of functions. Following (10), we say that a function is of algebraic-logarithmic type, if

$$F(x,y) = g(y) \cdot \left(1 - \frac{x}{\rho(y)}\right)^{\alpha(y)} \cdot \left(\frac{x}{\rho(y)} \log\left(1 - \frac{z}{\rho(y)}\right)\right)^{\beta(y)} \cdot (1 + E(x,y)),$$

where, in addition to the properties of (1), $\alpha(y)$ and $\beta(y)$ have continuous third partial derivatives in $N(S,\delta)$. With a little more technical work, and using the local limit theorems provided in (10), a more general version of Theorem 1 can be proved. We leave the straightforward but tedious details to the reader.

Notation We shall fix some additional notation that we will use throughout without further reference. Let G be any graph. We will denote by v_G the number of vertices in G, and by e_G the number of edges in G. Moreover, for a graph class $\mathcal{C}$ we will denote by $\mathcal{C}^\bullet$ the class that contains vertex-rooted graphs from $\mathcal{C}$, i.e., pairs (C,v), where $C \in \mathcal{C}$, and v is a vertex of C. Finally, we will denote by $C^\bullet(x,y)$ the egf for $\mathcal{C}^\bullet$, i.e., $C^\bullet(x,y) = x\frac{\partial}{\partial x}C(x,y)$.

2 Preliminaries

In this section we collect some well-known facts and make some observations that we will exploit later. The following theorem gives us information about the coefficients of algebraico-logarithmic functions.

Theorem 3 ((6)) *Let $F(x,y)$ be as in* (1). *Then, uniformly for $y \in N(S,\delta)$,*

$$[x^n]F(x,y) \sim \frac{g(y)}{\Gamma(\alpha)} \cdot n^{\alpha-1} \cdot \rho(y)^{-n},$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ denotes the Gamma-function.

The next statement describes an important combinatorial property of nice graph classes, and is taken from (13). Let us introduce some notation first. We denote by $\mathcal{Z}$ the graph class consisting just of one graph that contains a single labeled vertex. For two graph classes $\mathcal{X}$ and $\mathcal{Y}$, we write "$\mathcal{G} = \mathcal{X} \times \mathcal{Y}$" if there is a 1-1 correspondence between the graphs in $\mathcal{G}$ and the class formed by the *cartesian product* of $\mathcal{X}$ and $\mathcal{Y}$, followed by a relabeling step, so as to guarantee that all labels are distinct for an object in $\mathcal{X} \times \mathcal{Y}$. Moreover, $\mathsf{Set}(\mathcal{X})$ is the class in which every graph can be represented by *set* of graphs in $\mathcal{X}$. Finally, the class $\mathcal{X} \circ \mathcal{Y}$ consists of all graphs that are obtained from graphs from $\mathcal{X}$, where each vertex is replaced by a graph from $\mathcal{Y}$. This set of combinatorial operators (cartesian product, set, and substitution) appears frequently in modern theories of combinatorial analysis (4; 7), as well as in systematic approaches to random generation of combinatorial objects (4; 8). For a very detailed description of these operators and numerous applications we refer to (7) and further references therein.

Lemma 1 *Let $\mathcal{C}$ be a graph class having property (i) of Definition 1. Then*

$$\mathcal{C}^\bullet = \mathcal{Z} \times \mathsf{Set}(\mathcal{B}' \circ \mathcal{C}^\bullet) \quad \textit{and} \quad C^\bullet(x,y) = xe^{B'(C^\bullet(x,y),y)}.$$

2.1 Central & Local Limit Theorems

Let $(p_{n,k})_{n\geq 1}$ be a sequence of discrete probability distributions with mean μ_n and standard deviation σ_n. We say that the $p_{n,k}$ obey a *central limit theorem (CLT)* if there is a sequence $\varepsilon_n \to 0$ such that

$$\sup_{x\in\mathbb{R}} \left| \sigma_n \sum_{k\leq \mu_n + x\sigma_n} p_{n,k} - \frac{1}{\sqrt{2\pi}} \int_{t\leq x} e^{-t^2/2} dt \right| \leq \varepsilon_n. \quad (6)$$

Note that a central CLT gives us very precise information about the (cumulative) distribution of the sequences $p_{n,k}$. However, in general it fails to give us information about the *individual* probabilities. In this case we are interested in a *local limit theorem (LLT)*, which shows *pointwise* convergence. More precisely, the sequence $(p_{n,k})_{n\geq 1}$ is said to obey a LLT if there is a sequence $\varepsilon_n \to 0$ such that

$$\sup_{x\in\mathbb{R}} \left| \sigma_n p_{n,\lfloor \mu_n + x\sigma_n \rfloor} - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| \leq \varepsilon_n. \quad (7)$$

The statement below provides us with a CLT in a general setting, that is commonly encountered in the context of analytic combinatorics.

Theorem 4 *(7, Theorem IX.8) Let $(X_n)_{n\geq 1}$ be a sequence of discrete random variables supported by $\mathbb{N}$, with associated probability generating functions $p_n(u)$. Assume that, uniformly in a fixed complex neighborhood Ω of 1, for sequences $\beta_n, \kappa_n \to +\infty$, there holds*

$$p_n(u) = A(u)B(u)^{\beta_n} \left(1 + \mathcal{O}(\kappa_n^{-1})\right),$$

where $A(u), B(u)$ are analytic at $u = 1$, and $A(1) = B(1) = 1$. Moreover, assume that $\mathfrak{v}(B) \neq 0$. Then, X_n satisfies a CLT with $\varepsilon_n = \mathcal{O}(\kappa_n^{-1} + \beta_n^{-1/2})$, and

$$\mu_n = \beta_n \mathfrak{m}(B) + \mathfrak{m}(A) + \mathcal{O}(\kappa_n^{-1})$$
$$\sigma_n^2 = \beta_n \mathfrak{v}(B) + \mathfrak{v}(A) + \mathcal{O}(\kappa_n^{-1}).$$

Under a very light additional technical assumption, a similar LLT theorem can be shown. This assumption will be fulfilled in all our applications, and is typical in the context of analytic combinatorics.

Theorem 5 *(7, Theorem IX.14) Suppose that a random variable satisfies all conditions of Theorem 4. Moreover, assume the existence of a uniform bound*

$$|p_n(u)| \leq K^{-\beta_n} \quad (8)$$

for some $K > 1$ and all $u \in \{|z| = 1 \mid z \notin \Omega\}$. Then, X_n satisfies a LLT with μ_n, σ_n and ε_n as given in Theorem 4.

2.2 Bounding Tail Probabilities

In our proofs we will often bound the probability that certain random variables assume values far away from their expectation. The next lemma states the well-known Chernoff bounds.

Lemma 2 *Let X be a binomially distributed variable. Then, for every $0 < \varepsilon < 1$ we have*

$$\Pr\left[X \notin (1 \pm \varepsilon)\mathbb{E}\left[X\right]\right] \leq 2e^{-\varepsilon^2 \mathbb{E}[X]/3}.$$

The same bounds are true for Poisson distributed random variables.

Lemma 3 *Lemma 2 is true when X is distributed like a Poisson variable.*

3 Sampling & Asymptotics

Let $\mathcal{C}$ be a nice graph class such that $C^\bullet(x,y)$ is of algebraic type in a compact set $S \subseteq (0,+\infty)$. Recall that for every fixed $y \in S$ the quantity $\rho(y)$ denotes the singularity of $C^\bullet(x,y)$. Set

$$\lambda_{\mathcal{C}}(y) = B'\big(C^\bullet(\rho(y),y),\, y\big), \tag{9}$$

where $B(x,y)$ is the exponential generating function enumerating the biconnected graphs in $\mathcal{C}$. Note that *a priori* it is not clear whether $\lambda_{\mathcal{C}}(y)$ exists for all $y \in S$. However, as we shall argue later, the existence is an inherent property of nice classes. Moreover, let $\Gamma B'(x,y)$ be a randomized algorithm that generates graphs from $\mathcal{B}'$ according to the following distribution:

$$\forall B' \in \mathcal{B}' : \;\; \Pr\left[\Gamma B'(x,y) = B'\right] = \frac{x^{v_{B'}} y^{e_{B'}}}{B'(x,y)}, \tag{10}$$

provided that $B'(x,y)$ exists. The above distribution is called the *Boltzmann distribution* (or *Gibbs distribution*), and was introduced in the context of the random generation of combinatorial objects by Douchon, Flajolet, Louchard and Schaeffer in 2004, see (4). With this notation consider the following algorithm.

$\Gamma C^\bullet(\beta)$:
- $\gamma \leftarrow$ a single node r
- $k \leftarrow \mathrm{Po}(\lambda_{\mathcal{C}}(\beta))$ $(\star)$
- **for** $j = 1, \ldots, k$
  - $\gamma' \leftarrow \Gamma B'\big(C^\bullet(\rho(\beta),\beta),\, \beta\big)$, discard the labels of γ' $(\star\star)$
  - $\gamma \leftarrow$ merge γ and γ' at their roots
- **foreach** vertex $v \neq r$ of γ
  - $\gamma_v \leftarrow \Gamma C^\bullet(\beta)$, discard the labels of γ_v
- replace all nodes $v \neq r$ of γ with γ_v
- **return** γ, where the vertices are labeled uniformly at random

A similar version of this algorithm, for the special case $\beta = 1$, was studied already in (13). There the authors determined the number of blocks in random graphs with constraints, but they did not consider any restriction on the average degree. Here we are interested in general β, which makes the analysis more involved. The following lemma will be one of the main tools in our analysis, and says that with some reasonable probability the algorithm above will output a graph from $\mathcal{C}^\bullet_{n,m}$, for a very specific $m = m(\beta)$.

Lemma 4 *Let $\mathcal{C}$ be a nice graph class satisfying the assertions of Theorem 1. For any $\beta \in S_C$ there is a $c > 0$ such that*

$$\Pr[\Gamma C^\bullet(\beta) \in \mathcal{C}^\bullet_{n,\lfloor \mu(\beta;\, n)\rfloor}] \sim c n^{\alpha - 3/2}\text{ , where }\; \mu(\beta;\, n) = -\frac{\beta\rho'(\beta)}{\rho(\beta)} n + \frac{\beta g'(\beta)}{g(\beta)}. \tag{11}$$

Proof: The proof consists of three parts. First, we will show that $\Gamma C^\bullet$ is well-defined for any $\beta \in S_C$, i.e., we will show that $\lambda_{\mathcal{C}}(\beta)$ and $B'(C^\bullet(\rho(\beta),\beta),\beta)$ exist. Second, we argue that for any $\gamma \in \mathcal{C}^\bullet$

$$\Pr[\Gamma C^\bullet(\beta) = \gamma] = \frac{1}{C^\bullet(\rho(\beta),\beta)} \cdot \frac{\rho(\beta)^{v_\gamma} \cdot \beta^{e_\gamma}}{v_\gamma !}. \tag{12}$$

Finally, we show that there is a constant $C > 0$ such that

$$|C^\bullet_{n,\lfloor\mu(\beta;\,n)\rfloor}| \sim C \cdot n^{\alpha-3/2} \cdot \rho(\beta)^{-n} \cdot \beta^{-\lfloor\mu(\beta;\,n)\rfloor} \cdot n!. \tag{13}$$

Putting all three facts together proves then the statement. To see the first claim, apply Lemma 1 and note that $\psi(u) = ue^{-B'(u,y)}$ is the functional inverse of $C^\bullet(x, y)$. Let $R(y)$ be the singularity of $B'(x, y)$. By the Analytic and Singular Inversion Lemmas in (7, Lemma IV.2 and IV.3), for any $y \in S_C$, the singularity of $C^\bullet(x, y)$ depends on whether $\psi(u)$ is strictly monotone:

- if there is a unique $0 < \tau(y) < R(y)$ such that $\psi'(\tau(y)) = 0$, or equivalently, $\tau(y)B''(\tau(y), y) = 1$, then $\rho(y) = \psi(\tau(y))$ and $C^\bullet(\rho(y), y) = \tau(y)$.
- Otherwise, ψ is strictly monotone in $[0, R(y))$, and then $\rho(y) = \psi(R(y)) = R(y)e^{-B'(R(y),y)}$. Moreover, we have in this case $R(y)B''(R(y), y) \leq 1$ and $C^\bullet(\rho(y), y) = R(y)$.

Note that in both cases we have $C^\bullet(\rho(y), y) < \infty$. Moreover, in the first case we obviously have $C^\bullet(\rho(y), y) < R(y)$, which implies that $\lambda_{\mathcal{C}}(y)$ and $B'(C^\bullet(\rho(y), y), y)$ are well-defined. Finally, in the second case we have that $B''(R(y), y) < \infty$, as $R(y) > 0$. But then, also $B'(R(y), y) < \infty$ is true, which implies that also in this case $\lambda_{\mathcal{C}}(y)$ and $B'(C^\bullet(\rho(y), y), y)$ are well-defined. This completes the proof of the first part.

The identity (12) follows directly from the composition rules for Boltzmann samplers in (9) and (4), and the decomposition of nice classes provided in Lemma 1. To prove (13) consider the function $C_\beta(x, y) = C^\bullet(x, \beta y)$, and note that for any y such that $\beta y \in S_C$ its singularity is given by $\rho_\beta(y) = \rho(\beta y)$. Now, consider a random variable X with probability generating function

$$p_n(u) = \frac{[x^n]C_\beta(x, u)}{[x^n]C_\beta(x, 1)},$$

and note that

$$[u^s]p_n(u) \;=\; \Pr[X = s] \;=\; \frac{|C^\bullet_{n,s}|\frac{1}{n!} \cdot \beta^s}{[x^n]C_\beta(x, 1)}. \tag{14}$$

In the remainder we will estimate $[u^s]p_n(u)$ and $[x^n]C_\beta(x, 1)$ directly, which will yield (13). By applying Theorem 3 we obtain uniformly for u such that $\beta u \in S_C$

$$p_n(u) \sim \frac{g(\beta u)}{g(\beta)} \left(\frac{\rho_\beta(1)}{\rho_\beta(u)}\right)^n.$$

Note that the assertions of Theorem 4 are fulfilled, if we choose $A(u) = \frac{g(\beta u)}{g(\beta)}$, $B(u) = \frac{\rho_\beta(1)}{\rho_\beta(u)}$ and $\kappa_n = \omega(1)$, due to our assumptions. Moreover, from our assumptions follows that $|\rho_\beta(u)| = |\rho(\beta u)| > \rho(\beta) = \rho_\beta(1)$ for $u \in \{z \mid |z| = 1, z \notin N(S, \delta)\}$, and we may infer that there is a $K > 1$ such that for all such u

$$p_n(u) < K^{-n}.$$

All in all, the assertions of Theorem 5 are fulfilled, and we may conclude that

$$\Pr\left[X_n = \lfloor\mathbb{E}\left[X_n\right]\rfloor\right] \sim (2\pi\sigma_n)^{-1/2},$$

where

$$\mathbb{E}[X] = n\mathfrak{m}(B) + \mathfrak{m}(A) + o(1) = \frac{-\beta\rho'(\beta)}{\rho(\beta)}n + \frac{\beta g'(\beta)}{g(\beta)} + o(1),$$

and $\sigma_n = \mathrm{Var}[X_n] = n\mathfrak{v}(B) + \mathfrak{v}(A)$. This calculations determine the left-hand side of (14). Moreover, by applying Theorem 3 to the expansion of $C_\beta(x,1) = C^\bullet(x,\beta)$ we readily obtain that there is a constant $C' > 0$ such that

$$[x^n]C_\beta(x,1) \sim g(\beta)\frac{n^{\alpha-1}}{\Gamma(\alpha)}\rho(\beta)^{-n}.$$

By plugging this and the above estimate for $\Pr[X_n = \lfloor \mathbb{E}[X_n] \rfloor]$ into (14) we obtain (13). □

The following lemma is essentially taken from (13), where the special case "$\beta = 1$" was considered. As the proof is completely analogous, we refer the reader to (13). Before we state it let us introduce a little additional notation. We follow an approach first used in (14; 1) and consider a sampler that *simulates* an execution of $\Gamma C^\bullet$. Observe that $\Gamma C^\bullet$ makes twice a random choice: first, when it chooses a random value according to a Poisson distribution in the line marked with $(\star)$, and second, when it calls $\Gamma B'$ in the line marked with $(\star\star)$. We now consider an algorithm that takes as input a sequence of non-negative integers and a sequence of graphs from $\mathcal{B}'$ and uses them instead of making the random choices. More precisely, let K be an infinite sequence of numbers in $\mathbb{N}_0$, and let B' be an infinite sequence of graphs from $\mathcal{B}'$. Then the algorithm $\Gamma C^\bullet(\beta;\, K, B')$, which simulates the execution of $\Gamma C^\bullet$ by using the next unused value from the provided lists, generates obviously every graph from $\mathcal{C}^\bullet$ with the same probability as $\Gamma C^\bullet$, *provided* that the values in K and the graphs B' are generated independently and according to the appropriate probability distributions. In the sequel we will therefore assume that the notation $\Gamma C^\bullet(\beta)$ in fact denotes the sampler $\Gamma C^\bullet(\beta;\, K, B')$, where we often will omit the lists (K, B').

Lemma 5 *Let $K = \{k_1, k_2, \dots\}$ be an infinite sequence of non-negative integers and let $B' = \{B'_1, B'_2, \dots\}$ be an infinite sequence of graphs from $\mathcal{B}'$. Suppose that $\Gamma C^\bullet(\beta;\, K, B')$ used the first n values in K and the first m graphs in B' to generate a graph $\gamma \in \mathcal{C}^\bullet$. Then the following statements are true.*

(1) $n = |\gamma|$.
(2) $m = \sum_{j=1}^n k_j$.
(3) $m = \sum_{\ell \ge 2} b(\ell;\, \gamma)$.
(4) For any $\ell \ge 2$ we have that $b(\ell;\, \gamma) = \left|\left\{1 \le i \le m \;\middle|\; |B'_i| = \ell - 1\right\}\right|$.

4 Blocks With ℓ Vertices in $\mathsf{C}_{n,m}$

Let $\mathsf{C}_{n,m}$ be a graph with n vertices and m edges, drawn uniformly at random from $\mathcal{C}_{n,m}$, where $\mathcal{C}$ is nice. First, we apply Lemma 5 to deduce some information on the number of not too "large" blocks.

Lemma 6 *Let $\mathcal{C}$ be a nice class satisfying the assertions of Theorem 1. Let $\beta \in S_C$, $n \in \mathbb{N}$, and set $m = \lfloor -\frac{\beta\rho'(\beta)}{\rho(\beta)}n + \frac{\beta g'(\beta)}{g(\beta)} \rfloor$ and $\eta = C^\bullet(\rho(\beta), \beta)$. Moreover, let $0 < \varepsilon = \varepsilon(n) < 1$. For $\ell \ge 2$ define the quantities*

$$b_\ell = [x^{\ell-1}]B'(x,\beta) \cdot \eta^{\ell-1} \quad \textit{and} \quad \ell_0 = \ell_0(n,\varepsilon) = \max\left\{\ell \mid b_\ell n \ge 50\varepsilon^{-2}\alpha \log n\right\}.$$

Then we have for all $2 \le \ell \le \ell_0$ and sufficiently large n

$$\Pr[b(\ell;\, \mathsf{C}_{n,m}) \notin (1 \pm \varepsilon)b_\ell n] \le e^{-\frac{\varepsilon^2}{40}b_\ell n}. \tag{15}$$

Proof: The proof is similar to the proof of the analogous lemma in (13, Lemma 3.1); the sole difference is that we have to deal here with the fixed number of edges. We give this proof in full detail.

Let $\ell \in [2, \ell_0]$ and let $\mathcal{S} \subset \mathcal{C}_{n,m}$ denote the set of labeled graphs in $\mathcal{C}_{n,m}$ whose number of blocks of size ℓ is not in the interval $(1 \pm \varepsilon)b_\ell n$. Using Lemma 4 we obtain that there exists a constant $\hat{c} > 0$ such that for all large enough n we have

$$\Pr\left[\mathsf{C}_{n,m} \in \mathcal{S}\right] = \Pr\left[\Gamma C^\bullet \in \mathcal{S} \mid \Gamma C^\bullet \in \mathcal{C}^\bullet_{n,m}\right] \le \hat{c} n^{\alpha+3/2} \Pr\left[\Gamma C^\bullet \in \mathcal{S} \text{ and } \Gamma C^\bullet \in \mathcal{C}^\bullet_{n,m}\right]. \qquad (16)$$

We write $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $\mathcal{S}_1$ contains all graphs that satisfy $\sum_{\ell \ge 2} b(\ell;\, G) \notin (1 \pm \frac{\varepsilon}{3})\lambda_{\mathcal{C}}(\beta)n$, and $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$. By using Lemma 5, statements (1)-(3), the event "$\Gamma C^\bullet \in \mathcal{S}_1$ and $\Gamma C^\bullet \in \mathcal{C}^\bullet_{n,m}$" implies that the sum of n independent variables distributed like $\mathrm{Po}(\lambda_{\mathcal{C}}(\beta))$ is not in $(1 \pm \frac{\varepsilon}{3})\lambda_{\mathcal{C}}(\beta)n$. But this probability is easily seen to be less than $e^{-\frac{\varepsilon^2}{30}\lambda_{\mathcal{C}}(\beta)n}$, by applying Lemma 3.

Moreover, again by Lemma 5, this time Statement (4), the event "$\Gamma C^\bullet \in \mathcal{S}_2$ and $\Gamma C^\bullet \in \mathcal{C}^\bullet_{n,m}$" implies that a sequence of $N = (1 \pm \frac{\varepsilon}{3})\lambda_{\mathcal{C}}(\beta)n$ independent random graphs, which are drawn from $\mathcal{B}'$ according to the distribution (10) with parameters $x = C^\bullet(\rho(\beta), \beta) = \eta$ and $y = \beta$, contains less than $(1-\varepsilon)b_\ell n$ or more than $(1+\varepsilon)b_\ell n$ graphs with $\ell - 1$ non-virtual vertices. The probability that a single such random graph has exactly $\ell - 1$ non-virtual vertices is precisely

$$t_\ell := [x^{\ell-1}]B'(x,\beta) \cdot \frac{\eta^{\ell-1}}{B'(\eta,\beta)}. \qquad (17)$$

Hence, by applying the Chernoff bounds from Lemma 2 we deduce that the number of graphs with $\ell - 1$ non-virtual vertices among N independently drawn random graphs is less than $(1 - \frac{\varepsilon}{3})t_\ell N$ or more than $(1 + \frac{\varepsilon}{3})t_\ell N$ with probability at most $e^{-\frac{\varepsilon^2}{30}t_\ell N}$. The proof completes with $N \in (1 \pm \frac{\varepsilon}{3})\lambda_{\mathcal{C}}(\beta)n$, as $\Gamma C^\bullet \in \mathcal{S}_2$, and the assumptions on ℓ_0 and ε. $\square$

Lemma 7 *Let $\mathcal{C}$ be a nice class satisfying the assertions of Theorem 1. Let $\beta \in S_C$, $n \in \mathbb{N}$, and set $m = \lfloor -\frac{\beta\rho'(\beta)}{\rho(\beta)}n + \frac{\beta g'(\beta)}{g(\beta)} \rfloor$. Moreover, let $0 < \varepsilon = \varepsilon(n) < 1$. For $\ell \ge 1$ and $\delta > 1$ define the quantities*

$$b_{\ell,\delta} = \sum_{s=\ell}^{\delta\ell} [x^{s-1}]B'(x) \cdot R(\beta)^{s-1} \ \sim_\ell \ \frac{g_B(\beta)}{\Gamma(\alpha_B + 1)} \cdot (1 - \delta^{\alpha_B})\ell^{\alpha_B}.$$

Set $\ell_0 = \ell_0(\delta) = \max\left\{\ell \mid b_{\ell,\delta} n \ge 50\varepsilon^{-2}\alpha\alpha_B \log n\right\}$. If $R(\beta)B''(R(\beta),\beta) < 1$, then we have for all $1 \ll \ell \le \ell_0$ and sufficiently large n for a graph $\mathsf{C}_{n,m}$ drawn uniformly at random from $\mathcal{C}_{n,m}$

$$\Pr\left[b(\ell \ldots \delta\ell;\, \mathsf{C}_{n,m}) \notin (1 \pm \varepsilon)b_{\ell,\delta} n\right] \le e^{-\frac{\varepsilon^2}{40}b_{\ell,\delta} n}.$$

Proof: Note that $R(\beta)B''(R(\beta),\beta) < 1$ implies that $\eta = R(\beta)$ (see e.g. the discussion after (13)), and that d. Now, by using exactly the same arguments as in Lemma 6 we can prove the first claim; the sole modification has to be made in (17), where we use $t_\ell = b_{\ell,\delta}$ instead. To see the second claim we apply Theorem 3 to the singular expansion of B', and use straightforward Euler-McLaurin summation. $\square$

Lemma 8 *Let $\mathcal{C}$ be a nice class satisfying the assertions of Theorem 1. Let $\beta \in S_C$, $n \in \mathbb{N}$, and set $m = \lfloor -\frac{\beta\rho'(\beta)}{\rho(\beta)}n + \frac{\beta g'(\beta)}{g(\beta)} \rfloor$. If $R(\beta)B''(R(\beta),\beta) < 1$, then for sufficiently large n we have asymptotically almost surely for a graph $\mathsf{C}_{n,m}$ drawn uniformly at random from $\mathcal{C}_{n.m}$ that $lb(\mathsf{C}_{n,m}) \sim c(\beta)n$, where*

$$c(\beta) = 1 - R(\beta)B''(R(\beta),\beta).$$

Moreover, let ω_n be a function satisfying $\lim_{n\to\infty} \omega_n = \infty$. Then, for all $n^{-1/\alpha_B}\omega_n \leq \ell < lb(\mathsf{C}_{n,m})$ we have $b(\ell;\, \mathsf{C}_{n,m}) = 0$.

Proof: The proof proceeds in two steps: first, apply Lemmas 6 and 7 to count the number of vertices in blocks with size at most $n^{-1/\alpha_B}\omega_n$. Then, show by tedious but straightforward counting that the most probable case is that the remaining vertices form exactly one block. The details are similar to the details in the proof of Lemma 3.4 in (13), and are omitted due to space limitations. □

5 Graph Classes With Critical Densities

In this section we shall prove Corollary 1 and Theorem 2. Let us recall a few basic facts from (12). Let $B(x,y)$ be the egf enumerating biconnected labeled planar graphs, and let $C(x,y)$ be the egf enumerating labeled connected planar graphs. In (12) the authors showed that $B'(x,y)$ and $C^\bullet(x,y)$ are of the algebraic type for $y \in (0,\infty)$, where

$$B'(x,y) \sim -\frac{1}{R(y)}\left(B_2(y) + 2B_4(y)\Big(1 - \frac{x}{R(y)}\Big) + \frac{5}{2}B_5(y)\Big(1 - \frac{x}{R(y)}\Big)^{3/2}\right),$$

and

$$C^\bullet(x,y) \sim R(y) + \frac{R(y)^2}{2B_4(y) - R(y)}\Big(1 - \frac{x}{\rho(y)}\Big) - \frac{5}{2}B_5(y)\Big(1 - \frac{2B_4(y)}{R(y)}\Big)^{-5/2}\Big(1 - \frac{x}{\rho(y)}\Big)^{3/2},$$

where $\rho(y) = R(y)e^{B_2(y)/R(y)}$, and they also gave explicit expressions for $R(y)$, $B_2(y)$, $B_4(y)$ and $B_5(y)$. Moreover, they showed that ((12, Claim 2)) for all $y \in (0,\infty)$ it holds $R(y)B''(R(y),y) = \frac{2B_4(y)}{R(y)} < 1$. With those facts at hand Corollary 1 follows immediately.

Now we turn to the proof of Theorem 2. Recall that we set $\widetilde{\mathcal{B}} = \mathcal{B} \cup \{K_8\}$. Note that the singularity of $\widetilde{B}(x,y)$ is the same as the singularity of $B(x,y)$, i.e., $R(y)$. Observe that

$$R(y)\widetilde{B}''(R(y),y) = \frac{2B_4(y)}{R(y)} + \frac{R(y)^7y^{28}}{6!}$$

Using the explicit expressions for all involved functions we readily obtain that for $y \in (0,y_0)$ it holds $R(y)\widetilde{B}''(R(y),y) < 1$, while for $y \in (y_0,\infty)$ we have $R(y)\widetilde{B}''(R(y),y) > 1$, where $y_0 \approx 25.671$. Moreover, for $y \in (0,y_0)$ we have that

$$\widetilde{C}^\bullet(x,y) \sim R(y) + \widetilde{C}_2(y)\left(1 - \frac{x}{\rho_1(y)}\right) + \widetilde{C}_3(y)\left(1 - \frac{x}{\rho_1(y)}\right)^{3/2},$$

where the $\widetilde{C}_i(y)$ are given as functions of the $B_i(y)$ and $R(y)$, and $\rho_1(y) = R(y)e^{B_2(y)/R(y) - \frac{R(y)^7y^{28}}{7!}}$. Additionally, for $y \in (y_0,\infty)$ we obtain by applying Theorem VI.6 in (7)

$$\widetilde{C}^{\bullet}(x,y) \ \sim \ R(y) - \widetilde{C}_1(y)\left(1 - \frac{x}{\rho_2(y)}\right)^{1/2},$$

where $\rho_2(y) = \tau(y)e^{-B'(\tau(y),y)}$, $0 < \tau(y) < R(y)$ is given by the solution of $\tau(y)\widetilde{B}''(\tau(y),y) = 1$, and $\widetilde{C}_1(y)$ is analytically given. To obtain c_0 we determine $\lim_{y\to y_0^-} \frac{-y\rho_1'(y)}{\rho_1(y)y} = \lim_{y\to y_0^+} \frac{-y\tilde{\rho}_2'(y)}{\tilde{\rho}_2(y)} \approx 3.9995$. All numerical calculations performed in this section can be easily performed with MAPLE.

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*Noncrossing partitions and the shard intersection order**

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We define a new lattice structure $(W, \preceq)$ on the elements of a finite Coxeter group W. This lattice, called the *shard intersection order*, is weaker than the weak order and has the noncrossing partition lattice $\mathrm{NC}(W)$ as a sublattice. The new construction of $\mathrm{NC}(W)$ yields a new proof that $\mathrm{NC}(W)$ is a lattice. The shard intersection order is graded and its rank generating function is the W-Eulerian polynomial. Many order-theoretic properties of $(W, \preceq)$, like Möbius number, number of maximal chains, etc., are exactly analogous to the corresponding properties of $\mathrm{NC}(W)$. There is a natural dimension-preserving bijection between simplices in the order complex of $(W, \preceq)$ (i.e. chains in $(W, \preceq)$) and simplices in a certain pulling triangulation of the W-permutohedron. Restricting the bijection to the order complex of $\mathrm{NC}(W)$ yields a bijection to simplices in a pulling triangulation of the W-associahedron.

The lattice $(W, \preceq)$ is defined indirectly via the polyhedral geometry of the reflecting hyperplanes of W. Indeed, most of the results of the paper are proven in the more general setting of simplicial hyperplane arrangements.

Keywords: lattice congruence, noncrossing partition, shard, weak order

1 Introduction

The (classical) noncrossing partitions were introduced by Kreweras in [13]. Work of Athanasiadis, Bessis, Biane, Brady, Reiner and Watt [1, 2, 3, 5, 27] led to the recognition that the classical noncrossing partitions are a special case ($W = S_n$) of a combinatorial construction which yields a noncrossing partition lattice $\mathrm{NC}(W)$ for each finite Coxeter group W.

Besides the interesting algebraic combinatorics of the W-noncrossing partition lattice, there is a strong motivation for this definition arising from geometric group theory. In that context, $\mathrm{NC}(W)$ is a tool for studying the Artin group associated to W. (As an example, the Artin group associated to S_n is the braid group.) For the purposes of Artin groups, a key property of $\mathrm{NC}(W)$ is the fact that it is a lattice. This was first proved uniformly (i.e. without a type-by-type check of the classification of finite Coxeter groups) by Brady and Watt [6]. Another proof, for crystallographic W, was later given by Ingalls and Thomas [12].

The motivation for the present work is a new construction of $\mathrm{NC}(W)$ leading to a new proof that $\mathrm{NC}(W)$ is a lattice. The usual definition constructs $\mathrm{NC}(W)$ as an interval in a non-lattice (the absolute

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* This is an extended abstract, outlining the results of a paper [24] which will be submitted elsewhere. Some parts of this extended abstract are direct quotes from the complete paper.

order) on W; we define a new lattice structure $(W, \preceq)$ on all of W and identify a sublattice of $(W, \preceq)$ isomorphic to $\mathrm{NC}(W)$. No part of this construction—other than proving that the sublattice is isomorphic to $\mathrm{NC}(W)$—relies on previously known properties of $\mathrm{NC}(W)$. Thus, one can take the new construction as a definition of $\mathrm{NC}(W)$. The proof that $\mathrm{NC}(W)$ can be embedded as a sublattice of $(W, \preceq)$ draws on nontrivial results about *sortable elements* established in [21, 22, 25, 26].

Beyond the initial motivation for defining $(W, \preceq)$—to construct $\mathrm{NC}(W)$ and prove that it is a lattice—the lattice $(W, \preceq)$ turns out to have very interesting properties. In particular, many of the properties of $(W, \preceq)$ are precisely analogous to the properties of $\mathrm{NC}(W)$.

The lattice $(W, \preceq)$ is defined in terms of the polyhedral geometry of *shards*, certain codimension-1 cones introduced and studied in [16, 17, 18, 22]. Shards were used to give a geometric description of lattice congruences of the weak order. In this paper, we consider the collection Ψ of arbitrary intersections of shards, which forms a lattice under reverse containment. Surprisingly, Ψ is in bijection with W. The lattice $(W, \preceq)$ is defined to be the partial order induced on W, via this bijection, by the lattice $(\Psi, \supseteq)$. Thus we call $(W, \preceq)$ the *shard intersection order* on W.

For the remainder of this extended abstract, we will fill in some additional details about the constructions and results summarized above and in Table 1. We also illustrate the case $W = S_4$.

2 Shards and intersections of shards

In this section we define shards and discuss the lattice $(\Psi(W), \supseteq)$, where $\Psi(W)$ is the collection of arbitrary intersections of shards. We then describe a bijection between $\Psi(W)$ and W, and use this bijection to define a partial order $(W, \preceq)$ isomorphic to $(\Psi(W), \supseteq)$. The motivation for the definition of shards arises from the study of lattice congruences of the weak order, and will be discussed in Section 4.

Finite Coxeter groups correspond to finite reflection groups: finite groups of orthogonal transformations of $\mathbb{R}^n$ generated by reflections. Given a finite reflection group W, let T be the set of elements of W that act as reflections and let $\mathcal{A}$ be the collection of reflecting hyperplanes of elements of T. The set $\mathbb{R}^n \setminus (\cup\mathcal{A})$ consists of connected components which are called *regions*. Each region is an n-dimensional simplicial cone. Fixing some region D to represent the identity element, the map $w \mapsto wD$ is a bijection between W and the set of regions.

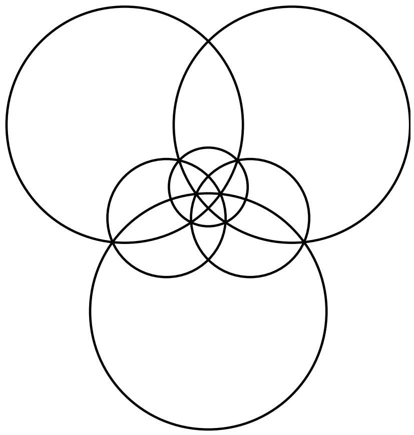
Example 2.1 As a running example, consider the Coxeter group $W = S_4$. This is the group of reflective symmetries of the regular tetrahedron. Exactly six elements of S_4 act as reflections (the six transpositions). Thus $\mathcal{A}$ consists of six reflecting planes in $\mathbb{R}^3$. To visualize this collection of planes, first take the intersection of $\mathcal{A}$ with the unit sphere to obtain a collection of six great circles on the sphere. Then stereographically project the unit sphere to the plane. The great circles map to circles in the plane. The result of this construction appears as Figure 1.a. Each of the 24 curvilinear triangles, including the outer triangle, represents a region. Each region is a triangular cone.

The shards are defined[(i)] as follows: For each hyperplane H, we describe a collection of *cutting subspaces* of H. These cutting subspaces are of codimension-1 in H (and thus of codimension-2 in the ambient vector space). The shards contained in H are the (closed) regions of this arrangement of codimension-1 subspaces of H. We will say that the cutting subspaces cut H into shards. The complete collection of shards in $\mathcal{A}$ consists of all of the shards in all of the hyperplanes of $\mathcal{A}$. The definition of shards will depend on the choice of D, but only up to symmetry.

[(i)] See [24, Section 3] for a different phrasing of the definition.

Tab. 1: Properties of $(W, \preceq)$ and $\mathrm{NC}(W)$.

$(W, \preceq)$ is a lattice.	$\mathrm{NC}(W)$ is a lattice—a sublattice of $(W, \preceq)$.
$(W, \preceq)$ is atomic and coatomic.	$\mathrm{NC}(W)$ is atomic and coatomic.
$(W, \preceq)$ is graded, with rank numbers given by the W-Eulerian numbers.	$\mathrm{NC}(W)$ is graded, with rank numbers given by the W-Narayana numbers.
$(W, \preceq)$ is **not** self-dual.	$\mathrm{NC}(W)$ is self-dual.
$(W, \preceq)$ is weaker than weak order.	$\mathrm{NC}(W)$ is weaker than the Cambrian lattice. (See [20] for a definition.)
Every lower interval $[1, w]_\preceq$ of $(W, \preceq)$ is isomorphic to $(W_J, \preceq)$ for some standard parabolic subgroup W_J depending on w.	Similarly, lower intervals of $\mathrm{NC}(W)$ are isomorphic to noncrossing partition lattices $\mathrm{NC}(W_J)$.
The Möbius number of $(W, \preceq)$ is equal, up to a sign, to the number elements of W that are not contained in any proper standard parabolic subgroup of W.	The Möbius number of $\mathrm{NC}(W)$ is equal, up to a sign, to the number of elements of $\mathrm{NC}(W)$ that are not contained in any proper standard parabolic subgroup of W.
Maximal chains in $(W, \preceq)$ are in bijection with maximal simplices in a certain triangulation of the W-permutohedron. Loday [14] described the triangulation in the case $W = S_n$. The bijection between maximal chains and maximal simplices is new for every W.	A similar bijection holds for $\mathrm{NC}(W)$ and a triangulation of the W-associahedron. Loday [14] described the triangulation and established the bijection in the case $W = S_n$. The bijection between chains and simplices is new for every other W.
More generally, for each k, there is a bijection between k-chains in $(W, \preceq)$ and k-simplices in the same triangulation of the W-permutohedron. This is especially surprising because the triangulation and the order complex of $(W, \preceq)$ have different topology. This result is new for all W.	The same is true of k-chains in $\mathrm{NC}(W)$ and k-simplices in the same triangulation of the W-associahedron. This result is also new for all W.
There is a recursion counting maximal chains in $(W, \preceq)$ by summing the number of maximal chains in $(W_J, \preceq)$ for each maximal proper standard parabolic subgroup W_J. With $\langle s \rangle = S \setminus \{s\}$, $$\mathrm{MC}(W, \preceq) = \sum_{s \in S} \left(\frac{\lvert W \rvert}{\lvert W_{\langle s \rangle} \rvert} - 1 \right) \mathrm{MC}(W_{\langle s \rangle}, \preceq).$$	There is a similar recursion [23, Corollary 3.1] counting maximal chains in $\mathrm{NC}(W)$. $$\mathrm{MC}(\mathrm{NC}(W)) = \frac{h}{2} \sum_{s \in S} \mathrm{MC}(\mathrm{NC}(W_{\langle s \rangle})).$$

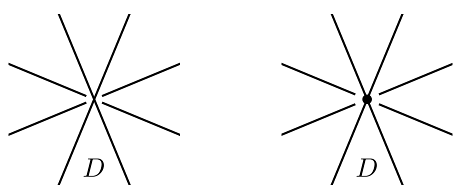


(a) (b)

Fig. 1: a: The reflecting planes of S_4, in stereographic projection. **b:** Shards in the case $W = S_4$.

Each cutting subspace of H will be the intersection of H with some other hyperplane in $\mathcal{A}$. Given H and H' in $\mathcal{A}$, let $\mathcal{A}'$ be the set of hyperplanes in $\mathcal{A}$ containing $H \cap H'$. Exactly one of the regions defined by $\mathcal{A}'$ contains the fixed region D; let D' denote the $\mathcal{A}'$-region containing D. The two hyperplanes in $\mathcal{A}'$ defining the facets of the cone D' are called the *basic* hyperplanes of $\mathcal{A}'$. The subspace $H' \cap H$ is a cutting subspace of H if and only if H is not basic in $\mathcal{A}'$.

Example 2.2 Figure 2.a illustrates the definition of shards in the case where W is of type B_2. In this case, for any distinct hyperplanes H and H', we have $\mathcal{A}' = \mathcal{A}$. Thus the two hyperplanes defining facets of D are never cut, but each of the other hyperplanes is cut at the origin. All of the shards are closed cones containing the origin; however, some shards in the picture are offset slightly to indicate that they do not continue through the origin.



(a) (b)

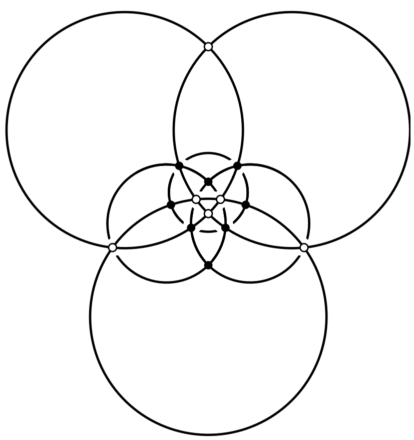
Fig. 2: (a) Shards in the case $W = B_2$. (b) $\Psi(B_2)$.

Example 2.3 The shards, for the case $W = S_4$, are pictured in Figure 1.b. This figure is a stereographic projection as explained in Example 2.1. The cone D is the small triangular region which is inside the three largest circles. The shards are closed two-dimensional cones (which in some cases are entire planes). Thus they appear as full circles or as circular arcs in the figure. To clarify the picture, we continue the convention of Figure 2.a: Where shards intersect, certain shards are offset slightly from the intersection to indicate that they do not continue through the intersection.

Let $(\Psi(W), \supseteq)$ be the set of arbitrary intersections of shards, partially ordered by reverse containment. It is immediate that $(\Psi(W), \supseteq)$ is a join semilattice; the join operation is intersection. Interpreting the empty intersection of shards to be the ambient vector space, we see that $(\Psi(W), \supseteq)$ is a lattice.

Example 2.4 This example continues Example 2.2. The set $\Psi(W)$, for the case where W is of type B_2, is pictured in Figure 2.b. The elements of $\Psi(W)$ are closed cones in $\mathbb{R}^2$, namely the origin, the six shards, and the whole space $\mathbb{R}^2$ (arising as the intersection of the empty set of shards).

Example 2.5 This example continues Example 2.3. The set $\Psi(S_4)$ is pictured in Figure 3.a. The elements of $\Psi(S_4)$ are closed cones in $\mathbb{R}^3$, namely the origin, eleven one-dimensional cones (three of which are entire lines), eleven shards (two-dimensional cones, three of which are entire planes) and the whole space $\mathbb{R}^3$. Each cone intersects the unit sphere in one of six ways: an empty intersection, a single point, a pair of antipodal points, an arc of a great circle, a great circle, or the entire sphere. Figure 3.a depicts these intersections in a stereographic projection onto the plane. Thus the shards are shown as circles or circular arcs and the one-dimensional cones are pictured as points or pairs of points. A white dot indicates a point which is paired with its antipodal point. (To find antipodal points, note that any two of the circles shown intersect in a pair of antipodal points.)



(a) (b)

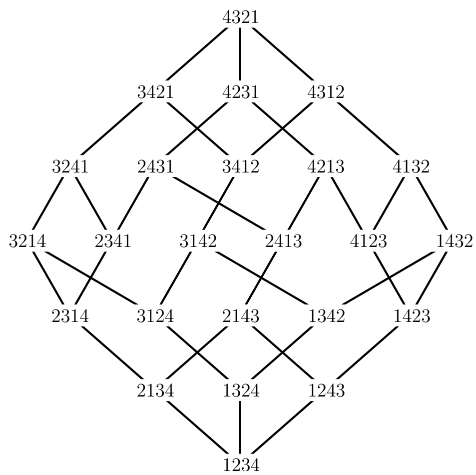
Fig. 3: (a) $\Psi(S_4)$. (b) Shards and join-irreducible elements in the case $W = S_4$.

The most important fact about the set $\Psi(W)$ is that it is in bijection with the elements of the group W. The bijection employs the weak order on W and will be explained in Section 3.

3 The weak order

In this section, we review the weak order on a finite Coxeter group. The weak order is a partial order on the elements of a Coxeter group W. When W is finite, this partial order is a lattice [4]. The weak order is relevant to the present discussion for at least two reasons: to motivate the definition of shards and to explain the bijection between intersections of shards and elements of W.

Example 3.1 When W is the symmetric group S_n, the weak order has a simple description in terms of the one-line notation for permutations: A cover relation in the weak order corresponds to swapping two adjacent entries. Going "up" in the cover relation means placing the two entries out of numerical order. The weak order on S_4 is illustrated in Figure 4.a.



(a) (b)

Fig. 4: Two views of the weak order on S_4.

The weak order also has a geometric description in terms of the arrangement $\mathcal{A}$ of reflecting hyperplanes. Recall that, once a region D is chosen to represent the identity, the elements of W are in bijection with the regions defined by $\mathcal{A}$. A cover relation in the weak order relates two adjacent regions. If H is the hyperplane separating the two then the lower region is on the same side of H as the identity region D.

Example 3.2 The weak order on W can be visualized in the stereographic projection of Figure 1.a. A dot representing each region and an edge representing each cover relation combine to form a "radial Hasse diagram," shown in Figure 4.b. Here the unique minimal element is the central element (contained in the shaded region). The upper vertex of a cover relation is a greater distance in the plane from the center than the lower vertex of the cover. The unique maximal element is the point at infinity.

For general finite W, the shards in $\mathcal{A}$ are in bijection [17, Proposition 2.2] with the join-irreducible elements of the weak order: the elements $j \in W$ covering exactly one other element $j_* \in W$. The region jD representing j is separated from the region j_*D by a common facet of both. The bijection sends j to the unique shard $\Sigma(j)$ containing the common facet. We will write the inverse map as $\Sigma \mapsto j(\Sigma)$.

Example 3.3 This example continues Example 2.3. Figure 3.b again shows the shards for $W = S_4$. The shaded triangles correspond to join-irreducible elements. Each such triangle has two convex sides and one concave side. The bijection between join-irreducible elements and shards sends the triangle to the shard containing its concave side.

The bijection between join-irreducible elements and shards is the restriction of the bijection between group elements and intersections of shards. We now describe the latter.

Each element w of W has [26, Theorem 8.1] a *canonical join representation* in the weak order on W. A join representation is an expression for w as an irredundant join of join-irreducible elements. The canonical join-representation of w is the unique minimal (i.e. lowest in the partial order) join representation for W, in a sense that can be made precise. For finite lattices, the property that each element has a canonical join-representation and a canonical meet representation is equivalent to the property of semi-distributivity [11, Theorem 2.24].

Let $\mathrm{Can}(w)$ be the set of join-irreducible elements occurring in the canonical join representation of w. Define a map $\psi : W \to \Psi(W)$ and a map $\omega : \Psi(W) \to W$ by setting

$$\psi(w) = \bigcap_{j \in \mathrm{Can}(w)} \Sigma(j), \qquad \text{and} \qquad \omega(C) = \bigvee_{\Sigma \supseteq C} j(\Sigma).$$

In the latter formula, the sum is over shards Σ containing C and the join is taken in the weak order on W.

Proposition 3.4 *Let W be a finite Coxeter group. Then:*

(i) *ψ is a bijection from W to $\Psi(W)$ with inverse map ω.*

(ii) *ω is an order-preserving map from $(\Psi, \supseteq)$ to the weak order $(W, \leq)$.*

(iii) *The number of right descents of $w \in W$ equals the codimension of $\psi(w)$.*

The right descents of w are the simple generators $s \in S$ such that $\ell(ws) < \ell(w)$. The proof of Proposition 3.4 employs geometric results about the cutting subspaces of hyperplanes as well as lattice-theoretic results about the weak order.

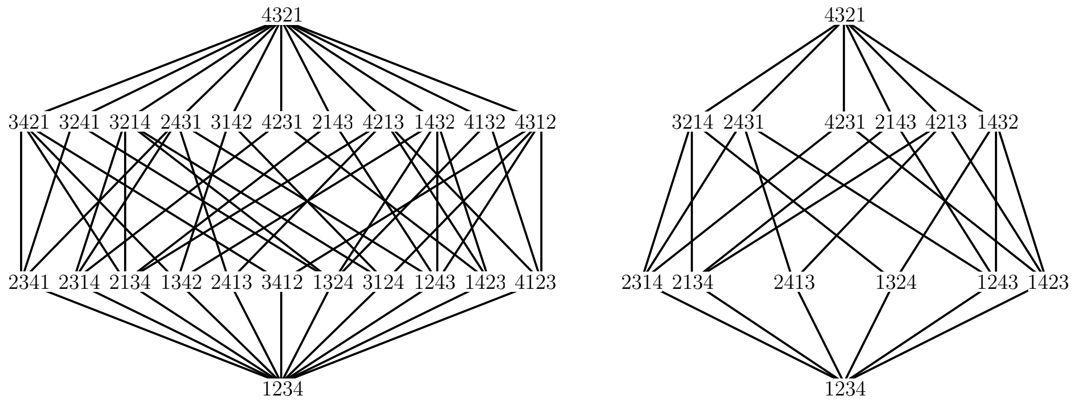
Let $(W, \preceq)$ denote the lattice induced on W, via the bijection of Proposition 3.4, from $(\Psi(W), \supseteq)$. We can give a direct characterization of $\preceq$ as follows: Given $x \lessdot y$ in the weak order, let $\Sigma(x \lessdot y)$ be the shard containing the common facet of xD and yD. Let $j(x \lessdot y) = j(\Sigma(x \lessdot y))$. Given $w \in W$, define $b(w)$ to be the meet of the elements covered by w and define $A(w) = \{j(x \lessdot y) : b(w) \leq x \lessdot y \leq w\}$. Then $v \preceq w$ if and only if $A(v) \subseteq A(w)$. Up to now, the geometric definition of $(W, \preceq)$ has been much more useful in proofs than this direct combinatorial approach.

Example 3.5 Continuing Example 2.5, the lattice $(W, \preceq)$ is shown in Figure 5.a for the case $W = S_4$. Readers wishing to work through the details of this example will be aided by [19, Proposition 6.4], where the map $(x \lessdot y) \mapsto j(x \lessdot y)$ is described explicitly in the case $W = S_n$.

In this section, we provide more detail on some of the properties of the lattice $(W, \preceq)$ listed in Table 1.

Proposition 3.6 *The lattice $(W, \preceq)$ is graded, with the rank of $w \in W$ equal to the number of right descents of w. Alternately, the rank of a cone $C \in \Psi(W)$ is the codimension of C.*

Theorem 3.7 *For any $w \in W$, the lower interval $[1, w]_\preceq$ is isomorphic to $(W_J, \preceq)$, where $J = \mathrm{Des}(w)$ is the set of right descents of w.*



(a) (b)

Fig. 5: (a) $(S_4, \preceq)$. (b) $(S_4, \preceq)$ restricted to c-sortable elements.

Theorem 3.7 should sound geometrically plausible, once it is translated into the context of $(\Psi(W), \supseteq)$. The lower interval below a cone C in $(\Psi(W), \supseteq)$ is the set of shard intersections containing C. This is analogous to the interval below a subspace in the intersection lattice of $\mathcal{A}$. The latter interval is isomorphic to the intersection lattice of $\mathcal{A}_J$, the arrangement of reflecting hyperplanes of W_J, for J as in Theorem 3.7.

Theorem 3.7 allows us to determine the Möbius number of $(W, \preceq)$: Up to a sign, it is the number of elements of W not contained in any proper standard parabolic subgroup. The identity element of W is the unique minimal element of $(W, \preceq)$ and the longest element w_0 is the unique maximal element.

Theorem 3.8 *The Möbius function of* $(W, \preceq)$ *satisfies* $\mu_\preceq(1, w_0) = \sum_{J \subseteq S} (-1)^{|J|} |W_J|$.

Proof: In light of Theorem 3.7, it is enough to show that the following sum vanishes:

$$\sum_{w \in W} \sum_{J \subseteq \mathrm{Des}(w)} (-1)^{|J|} |W_J| = \sum_{J \subseteq S} (-1)^{|J|} |W_J| \sum_{\substack{w \in W \\ J \subseteq \mathrm{Des}(w)}} 1.$$

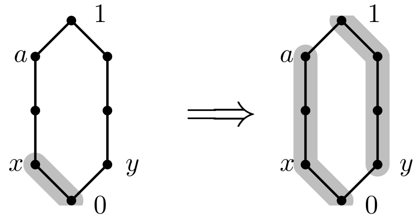
The inner sum is the number of maximal-length representatives of cosets of W_J in W. This number is $|W| / |W_J|$, so the sum reduces to zero. □

The fact that the proof of Theorem 3.8 is so simple is an indication that the poset $(W, \preceq)$ is a natural partial order on W. Theorem 3.7 also allows us to give a recursive formula for $\mathrm{MC}(W, \preceq)$, the number of maximal chains in $(W, \preceq)$. Recall that for $s \in S$, the symbol $\langle s \rangle$ stands for $S \setminus \{s\}$.

Theorem 3.9 *For any finite Coxeter group* W *with simple generators* S*,*

$$\mathrm{MC}(W, \preceq) = \sum_{s \in S} \left(\frac{|W|}{|W_{\langle s \rangle}|} - 1 \right) \mathrm{MC}(W_{\langle s \rangle}, \preceq).$$

Proof: The number of maximal chains in $(W, \preceq)$ is the sum over all coatoms w of $(W, \preceq)$ of the number of maximal chains in $[1, w]$. In light of Proposition 3.6, every coatom w is a maximal-length coset representative of the subgroup $W_{\langle s \rangle}$ for some unique $s \in S$. On the other hand, for each $s \in S$, every coset of



$$\begin{array}{rrcl} & x & \equiv & 0 \\ \Longrightarrow & x \vee y & \equiv & 0 \vee y \\ \text{i.e.} & 1 & \equiv & y \\ \Longrightarrow & a \wedge 1 & \equiv & a \wedge y \\ \text{i.e.} & a & \equiv & 0. \end{array}$$

Fig. 6: Forcing in a polygonal lattice.

$W_{\langle s \rangle}$ has a unique maximal-length coset representative. This representative w has $\mathrm{rank}(W) - 1$ descents and thus is a coatom of $(W, \preceq)$, except if w is w_0, which has $\mathrm{rank}(W)$ descents. For each $s \in S$, there are exactly $|W| / \left|W_{\langle s \rangle}\right|$ cosets of $W_{\langle s \rangle}$, and exactly one of these cosets has w_0 as its maximal length representative. The proposition follows. □

Remark 3.10 Recursions involving sums over maximal proper parabolic subgroups (such as the recursion appearing in Theorem 3.9) are very natural in the context of Coxeter groups/root systems. We have seen another such recursion in Table 1, counting maximal chains in $\mathrm{NC}(W)$. Another example is a a recursive formula for the face numbers of generalized associahedra [10, Proposition 3.7] [9, Proposition 8.3]. Yet another is a formula for the volume of the W-permutohedron which can be obtained by simple manipulations from Postnikov's formula [15, Theorem 18.3] expressing volume in terms of Φ-trees.

4 Lattice congruences of the weak order

In this section, we discuss lattice congruences of the weak order. The goal is to motivate the definition of shards in Section 2 and to lay the groundwork for the discussion of $\mathrm{NC}(W)$ in Section **??**.

A congruence on a finite lattice L is an equivalence relation $\equiv$ that respects the operations $\vee$ (least upper bound) and $\wedge$ (greatest lower bound). It is easy to verify that congruence classes are always intervals in L. Therefore, the relation $\equiv$ is determined by transitivity, once one knows all equivalences of the form $x \equiv y$ for $x \lessdot y$. We say that $\equiv$ *squashes* the edge $x \lessdot y$ if $x \equiv y$.

Let us consider "building" a congruence by squashing one edge at a time. As one might expect, edges cannot be squashed independently. Rather, there are some *forcing* relations.

As an example, consider a *polygonal lattice* L. That is, L is composed of two chains of length at least two, with the tops of the two chains identified and the bottoms of the two chains identified.

A "side" edge of the polygon is an edge that is not incident to the top element or the bottom element. A bottom edge is an edge incident to the bottom element, and a top edge is an edge incident to the top element. (Since L is constructed from chains of length at least two, no edge can be both a top edge and a bottom edge.) Edge forcing in a polygonal lattice is described as follows: One easily verifies that side edges can be squashed independently. That is, for any side edge, there is a lattice congruence that squashes that edge an no other edge. In contrast, squashing a bottom edge forces the opposite top edge and all side edges to be squashed, as illustrated in Figure 6. In the figure, squashed edges are highlighted. Dually, squashing a top edge forces the opposite bottom edge and all side edges.

Many of the intervals in the weak order on S_4 are polygonal intervals; this is true for the weak order in general. The polygonal intervals are the key to edge-forcings for congruences on the weak order.

Theorem 4.1 *Let W be a finite Coxeter group. Then all edge forcings for lattice congruences of the weak order on W are determined locally within polygonal intervals.*

For general lattices, forcing is much more complicated. The local property in Theorem 4.1 is equivalent to the assertion that the weak order is *congruence normal* in the sense of Day [8]. Thus Theorem 4.1 follows from [7, Theorem 6] (Cf. [16, Theorem 27]), where a stronger property, *congruence uniformity*, is established for the weak order.

Lattice congruences on the weak order have nice geometric properties. Interpret a lattice congruence as an equivalence relation on the regions cut out by the reflecting hyperplanes. For each congruence class C, let $\cup C$ denote the union of the regions in C. The following is part of [19, Theorem 1.1]

Theorem 4.2 *The sets $\cup C$ are the maximal cones of a complete fan.*

Shards arise naturally in the context of congruences on the weak order. When we interpret a lattice congruence as an equivalence on regions, and then glue together equivalence classes of regions, squashing edges means removing the common facet (or *wall*) separating two adjacent regions. A shard is a maximal collections of walls which must always be removed together in a lattice congruence, because of edge-forcing. Edge-forcing also implies some forcing relations among shards. In particular, choosing a lattice congruence on the weak order corresponds to removing a collection of shards that is closed under forcing.

Example 4.3 Consider Example 2.2 and Figure 2.a, and notice that the lattice in Figure 4 is the weak order on B_2. The discussion above about forcing in polygonal lattices explains why each line bounding the region D is not cut into two shards: a top edge is squashed if and only if the opposite bottom edge is squashed. The other two lines are cut because side edges can be squashed independently. Removing either of the shards bounding D forces each of the four shards not bounding D to be removed also.

We conclude this extended abstract by returning to the original motivation for the study of $(W, \preceq)$. We review the usual construction of W-noncrossing partitions and then explain how the noncrossing partition lattice $\mathrm{NC}(W)$ arises as a sublattice of $(W, \preceq)$.

Let S be the set of simple generators for W and let T be the set of reflections. A *Coxeter element* c of W is the product, in any order, of the elements of S. A *reduced T-word* for $w \in W$ is a shortest possible word for w in the alphabet T. (This contrasts with the usual notion of a *reduced word* for W, a shortest possible word for w in the alphabet S.) The *absolute order* on W is the prefix order on reduced T-words: we set $u \leq v$ if and only if any reduced T-word for u occurs as a prefix of some reduced T-word for v. The *W-noncrossing partition lattice* $\mathrm{NC}(W)$ is the interval $[1, c]_T$ in the absolute order, where c is any Coxeter element of W. Up to isomorphism, this definition is independent of the choice of c.

Recall that lattice congruences are described by specifying a collection of shards to be "removed." Recall also that there are forcing relations among shards, so that removing one shard may force the removal of another. Let Θ be a lattice congruence on W and let $\Psi(W/\Theta)$ be the collection of all intersections of shards not removed by Θ. It is immediate that $(\Psi(W/\Theta), \supseteq)$ is a join-sublattice of $(\Psi(W), \supseteq)$.

The noncrossing partition lattice $\mathrm{NC}(W)$ can be realized as $(\Psi(W/\Theta), \supseteq)$ in the case where Θ is the *Cambrian congruence* introduced in [20] and studied in [22, 25, 26]. There is a small set R_c of shards (depending on a choice of Coxeter element c of W) such that the Cambrian congruence Θ_c corresponds to removing the shards in R_c and all other shards whose removal is then forced. The bijection between W and $\Psi(W)$ restricts to an isomorphism between $(\Psi(W/\Theta_c), \supseteq)$ and the restriction of $(W, \preceq)$ to the *c-sortable elements* defined in [21] and studied in [22, 25, 26]. In [21], a bijection nc_c was defined between c-sortable elements and $\mathrm{NC}_c(W) = [1, c]_T$. We can now take that result further:

Theorem 4.4 *The map* nc_c *is an isomorphism between the restriction of* $(W, \preceq)$ *to* c*-sortable elements and* $\mathrm{NC}_c(W)$.

As indicated above, the restriction of $(W, \preceq)$ to c-sortable elements is isomorphic to the join-sublattice $(\Psi(W/\Theta_c), \supseteq)$ of $(\Psi(W), \supseteq)$. In fact, one can show that $(\Psi(W/\Theta_c), \supseteq)$ is a sublattice of $(\Psi(W), \supseteq)$.

Corollary 4.5 *The poset* $\mathrm{NC}_c(W)$ *is a lattice.*

The first uniform proof [6] that $\mathrm{NC}_c(W)$ is a lattice also used the polyhedral geometry of cones. That proof is, in a sense, dual to the proof discussed here (in the broadest outlines but not in any of the details).

Example 4.6 Figure 5.b shows the restriction of $(S_4, \preceq)$ to c-sortable elements, with $c = (1\ 2)(3\ 4)(2\ 3)$.

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The shifted plactic monoid (extended abstract)

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We introduce a shifted analog of the plactic monoid of Lascoux and Schützenberger, the *shifted plactic monoid*. It can be defined in two different ways: via the *shifted Knuth relations*, or using Haiman's mixed insertion.

Applications include: a new combinatorial derivation (and a new version of) the shifted Littlewood-Richardson Rule; similar results for the coefficients in the Schur expansion of a Schur P-function; a shifted counterpart of the Lascoux-Schützenberger theory of noncommutative Schur functions in plactic variables; a characterization of shifted tableau words; and more.

Keywords: plactic monoid, shifted tableau, mixed insertion, Schur P-function, shifted Littlewood-Richardson rule.

> *[...] pour affirmer la nécessité d'installer le monoïde plaxique parmi les structures remarquables.*
>
> – *M.-P. Schützenberger* (16)

Introduction

The (shifted) plactic monoid. The celebrated *Robinson-Schensted-Knuth correspondence* (14) is a bijection between words in a linearly ordered alphabet $X = \{1 < 2 < 3 < \cdots\}$ and pairs of *Young tableaux* with entries in X. More precisely, each word corresponds to a pair consisting of a semistandard *insertion tableau* and a standard *recording tableau*. The words producing a given insertion tableau form a *plactic class*. A. Lascoux and M. P. Schützenberger (11) made a crucial observation based on a result by D. E. Knuth (6): the plactic classes $[u]$ and $[v]$ of two words u and v uniquely determine the plactic class $[uv]$ of their concatenation. This gives the set of all plactic classes (equivalently, the set of all semistandard Young tableaux) the structure of a *plactic monoid* $\mathbf{P} = \mathbf{P}(X)$. This monoid has important applications in representation theory and the theory of symmetric functions; see, e.g., (10).

The main goal of this paper is to construct and study a proper analog of the plactic monoid for (semistandard) *shifted Young tableaux*, with similar properties and similar applications. The problem of developing

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such a theory was already posed more than 20 years ago by B. Sagan (12). Shifted Young tableaux are certain fillings of a *shifted shape* (a shifted Young diagram associated with a strict partition) with letters in an alphabet $X' = \{1' < 1 < 2' < 2 < \cdots\}$; see, e.g., (13). M. Haiman (5) defined the (shifted) *mixed insertion correspondence*, a beautiful bijection between permutations and pairs of standard shifted Young tableaux; each pair consists of the mixed insertion tableau and the mixed recording tableau. Haiman's correspondence is easily generalized to a bijection between words in the alphabet X and pairs consisting of a semistandard shifted mixed insertion tableau and a standard shifted mixed recording tableau. (We emphasize that this bijection deals with words in the original alphabet X rather than the extended alphabet X'.) We define a *shifted plactic class* as the set of all words which have a given mixed insertion tableau. Thus, shifted plactic classes are in bijection with shifted semistandard Young tableaux. The following key property, analogous to that of Lascoux and Schützenberger's in the ordinary case, holds (Theorem 4): the shifted plactic class of the concatenation of two words u and v depends only on the shifted plactic classes of u and v. Consequently, one can define the *shifted plactic monoid* $\mathbf{S} = \mathbf{S}(X)$ in which the product is, again, given by concatenation. In analogy with the classical case, we obtain a presentation of $\mathbf{S}$ by the quartic *shifted Knuth (or shifted plactic) relations*. So two words are shifted Knuth-equivalent if and only if they have the same mixed insertion tableau.

Sagan (12) and Worley (20) have introduced the *Sagan-Worley correspondence*, another analog of Robinson-Schensted-Knuth correspondence for shifted tableaux. In the case of permutations, Haiman (5) proved that the mixed insertion correspondence is dual to Sagan-Worley's. We use a semistandard version of this duality to describe shifted plactic equivalence in yet another way, namely: two words u and v are shifted plactic equivalent if and only if the recording tableaux of their inverses (as biwords) are the same.

(Shifted) Plactic Schur functions. The *plactic algebra* $\mathbb{Q}\mathbf{P}$ is the semigroup algebra of the plactic monoid. The shape of a plactic class is the shape of the corresponding tableau. A *plactic Schur function* $\mathcal{S}_\lambda \in \mathbb{Q}\mathbf{P}$ is the sum of all plactic classes of shape λ; it can be viewed as a noncommutative version of the ordinary Schur function s_λ. This notion was used by Schützenberger (15) to obtain a proof of the Littlewood-Richardson rule along the following lines. It can be shown that the plactic Schur functions span the ring they generate. Furthermore, this ring is canonically isomorphic to the ordinary ring of symmetric functions: the isomorphism simply sends each Schur function s_λ to its plactic counterpart $\mathcal{S}_\lambda$. It follows that the Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$ is equal to the coefficient of a fixed plactic class T_λ of shape λ in the product of plactic Schur functions $\mathcal{S}_\mu\mathcal{S}_\nu$. In other words, $c^\lambda_{\mu,\nu}$ is equal to the number of pairs (T_μ, T_ν) of plactic classes of shapes μ and ν such that $T_\mu T_\nu = T_\lambda$.

We develop a shifted counterpart of this classical theory. The *shifted plactic algebra* $\mathbb{Q}\mathbf{S}$ is the semigroup algebra of the shifted plactic monoid, and a (shifted) *plactic Schur P-function* $\mathcal{P}_\lambda \in \mathbb{Q}\mathbf{S}$ is the sum of all shifted plactic classes of a given shifted shape. We prove that the plactic Schur P-functions span the ring they generate, and this ring is canonically isomorphic to the ring spanned/generated by the ordinary Schur P-functions. Again, the isomorphism sends each Schur P-function P_λ to its plactic counterpart $\mathcal{P}_\lambda$. This leads to a proof of the shifted Littlewood-Richardson rule (Corollary 16). Our version of the rule states that the coefficient $b^\lambda_{\mu,\nu}$ of P_λ in the product $P_\mu P_\nu$ is equal to the number of pairs (T_μ, T_ν) of shifted plactic classes of shapes μ and ν such that $T_\mu T_\nu = T_\lambda$, where T_λ is a fixed shifted plactic class of shape λ. The first version of the shifted Littlewood-Richardson rule was given by Stembridge (19). In Lemma 18 we relate our rule to Stembridge's by a simple bijection.

It turns out that the shifted plactic relations are a "relaxation" of the ordinary Knuth (plactic) relations.

More precisely, the tautological map $u \mapsto u$ that sends each word in the alphabet X to itself descends to a monoid homomorphism $\mathbf{S} \to \mathbf{P}$. By extending this map linearly, we obtain the following theorem (Corollary 21): For a shifted shape θ, the coefficient g_μ^θ of s_μ in the Schur expansion of P_θ is equal to the number of shifted plactic classes of shifted shape θ contained in a fixed plactic class of shape μ. A simple bijection (Theorem 23) recovers a theorem of Stembridge (19): g_μ^θ is equal to the number of standard Young tableaux of shape μ which rectify to a fixed standard shifted Young tableau of shape θ.

(Shifted) Tableau words. In the classical setting, an approach developed by Lascoux and his school begins with the plactic monoid as the original fundamental object, and identifies each tableau T with a distinguished canonical representative of the corresponding plactic class, the *reading word* $\mathrm{read}(T)$. This word is obtained by reading the rows of T from left to right, starting from the bottom row and moving up. A word w such that $w = \mathrm{read}(T)$ for some tableau T is called a *tableau word*. By construction, tableau words are characterized by the following property. Each of them is a concatenation of weakly increasing words $w = u_l u_{l-1} \cdots u_1$, such that

(A) for $1 \leq i \leq l-1$, the longest weakly increasing subword of $u_{i+1}u_i$ is u_i.

For a tableau word w, the lengths of the segments u_i are precisely the row lengths of the Young tableau corresponding to w.

We develop an analog of this approach in the shifted setting by taking the shifted plactic monoid as the fundamental object, and constructing a canonical representative for each shifted plactic class. Since shifted Young tableaux have primed entries while the words in their respective shifted plactic classes have not, the reading of a shifted Young tableau cannot be defined in as simple a manner as in the classical case. Instead, we define the *mixed reading word* $\mathrm{mread}(T)$ of a shifted tableau T as the unique word in the corresponding shifted plactic class that has a distinguished *special recording tableau*. The latter notion is a shifted counterpart of P. Edelman and C. Greene's *dual reading tableau* (1).

A word w such that $w = \mathrm{mread}(T)$ for some shifted Young tableau T is called a *shifted tableau word*. Such words have a characterizing property similar to (A), with weakly increasing words replaced by *hook words* (a hook word consists of a strictly decreasing segment followed by a weakly increasing one). We prove that w is a shifted tableau word if and only if

(B) for $1 \leq i \leq l-1$, the longest hook subword of $u_{i+1}u_i$ is u_i.

For a shifted tableau word w, the lengths of the segments u_i are precisely the row lengths of the shifted Young tableau corresponding to w.

Semistandard decomposition tableaux. The proofs of our main results make use of the following machinery. Building on the concept of *standard decomposition tableaux* introduced by W. Kraśkiewicz (7) and further developed by T. K. Lam (9), we define a (shifted) *semistandard decomposition tableau* (SSDT) R of shifted shape λ as a filling of λ by entries in X such that the rows $u_1, u_2, \ldots, u_l$ of R are hook words satisfying (B). We define the *reading word* of R by $\mathrm{read}(R) = u_l u_{l-1} \cdots u_1$, that is, by reading the rows of R from left to right, starting with the bottom row and moving up.

As a semistandard analog of Kraśkiewicz's correspondence (7), we develop the *SK correspondence*. This is a bijection between words in the alphabet X and pairs of tableaux with entries in X. Every word corresponds to a pair consisting of an SSDT called the *SK insertion tableau* and a standard shifted Young tableau called the *SK recording tableau*. We prove that the mixed recording tableau and the SK recording tableau of a word w are the same. Furthemore, we construct a bijection Φ between SSDT and shifted Young tableaux of the same shape that preserves the reading word: $\mathrm{read}(R) = \mathrm{mread}(\Phi(R))$. In light of the conditions (A) and (B) above, one can see that the counterpart of an SSDT in the ordinary case is nothing but a semistandard Young tableau.

This text is an extended abstract of the preprint (17), where complete proofs can be found.

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Main results

Preliminaries: shifted Young tableaux and the mixed insertion

A *strict partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{Z}^l$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. The *shifted diagram*, or *shifted shape* of λ is an array of square cells in which the i-th row has λ_i cells, and is shifted $i-1$ units to the right with respect to the top row.

Throughout this paper, we identify a shifted shape corresponding to a strict partition λ with λ itself.

The *size* of λ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$. We denote $\ell(\lambda) = l$, the number of rows.

To illustrate, the shifted shape $\lambda = (5,3,2)$, with $|\lambda| = 10$ and $\ell(\lambda) = 3$, is shown below:

$$\begin{array}{ccccc} \square & \square & \square & \square & \square \\ & \square & \square & \square & \\ & & \square & \square & \end{array} .$$

A *skew shifted diagram* (or shape) λ/μ is obtained by removing a shifted shape μ from a larger shape λ containing μ.

A *(semistandard) shifted Young tableaux* T of shape λ is a filling of a shifted shape λ with letters from the alphabet $X' = \{1' < 1 < 2' < 2 < \cdots\}$ such that:

- rows and columns of T are weakly increasing;
- each k appears at most once in every column;
- each k' appears at most once in every row;
- there are no primed entries on the main diagonal.

If T is a filling of a shape λ, we write $\mathrm{shape}(T) = \lambda$.

A *skew shifted Young tableau* is defined analogously.

The *content* of a tableau T is the vector $(a_1, a_2, \ldots)$, where a_i is the number of times the letters i and i' appear in T.

Example 1 *The shifted Young tableau*

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3' & 4 \\ \hline & 4 & 5 & 5 & \\ \cline{2-4} & & 6 & 9' & \\ \cline{3-4} \end{array}$$

has shape $\lambda = (5,3,2)$ *and content* $(2,1,1,2,2,1,0,0,1)$.

A tableau T of shape λ is called *standard* if it contains each of the entries $1, 2, \ldots, |\lambda|$ exactly once. In particular, standard shifted Young tableaux have no primed entries. Note that a standard shifted tableau has content $(1, 1, \ldots, 1)$.

M. Haiman (5) has introduced *shifted mixed insertion*, a remarkable correspondence between permutations and pairs of shifted Young tableaux. Haiman's construction can be viewed as a shifted analog of the Robinson-Schensted-Knuth correspondence.

The following is a semistandard generalization of shifted mixed insertion, which we call *semistandard shifted mixed insertion*. It is a correspondence between words in the alphabet X and pairs of shifted Young tableaux, one of them semistandard and one standard. Throughout this paper we refer to semistandard shifted mixed insertion simply as *mixed insertion*.

Definition 2 (Mixed insertion) *Let $w = w_1 \ldots w_n$ be a word in the alphabet X. We recursively construct a sequence $(T_0, U_0), \ldots, (T_n, U_n) = (T, U)$ of tableaux, where T_i is a shifted Young tableau, and U_i is a standard shifted Young tableau, as follows. Set $(T_0, U_0) = (\emptyset, \emptyset)$. For $i = 1, \ldots, n$, insert w_i into T_{i-1} in the following manner:*

Insert w_i into the first row, bumping out the smallest element a that is strictly greater than w_i (in the order given by the alphabet X').

1. *if a is not on the main diagonal, do as follows:*
   - *(a) if a is unprimed, then insert it in the next row, using step (1);*
   - *(b) if a is primed, insert it into the next column to the right, using the same procedure as in row insertion;*
2. *if a is on the main diagonal, then it must be unprimed. Prime it, and insert it into the next column to the right.*

The insertion process terminates once a letter is placed at the end of a row or column, bumping no new element. The resulting tableau is T_i.

The shapes of T_{i-1} and T_i differ by one box. Add that box to U_{i-1}, and write i into it to obtain U_i.

We call T the mixed insertion tableau *and U the* mixed recording tableau, *and denote them $P_{\mathrm{mix}}(w)$ and $Q_{\mathrm{mix}}(w)$, respectively.*

Example 3 *The word $u = 3415961254$ has the following mixed insertion and recording tableau*

$$P_{\mathrm{mix}}(u) = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3' & 4 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & 5 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 6 & 9' & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array} \qquad Q_{\mathrm{mix}}(u) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 9 \\ \hline \multicolumn{1}{c|}{} & 3 & 6 & 8 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{2}{c|}{} & 7 & 10 & \multicolumn{1}{c}{} \\ \cline{3-4} \end{array} .$$

The shifted plactic monoid

The following is a shifted analog of Knuth's Theorem (6). It can be considered a semistandard generalization of theorems by Haiman (5) and by Kraśkiewicz (7).

Theorem 4 *Two words u and v have the same mixed insertion tableau if and only if they are equivalent modulo the following relations:*

$$abdc \equiv adbc \quad \textit{for} \quad a \leq b \leq c < d \quad \textit{in } X; \tag{1}$$

$$acdb \equiv acbd \quad \textit{for} \quad a \le b < c \le d \quad \textit{in } X; \tag{2}$$

$$dacb \equiv adcb \quad \textit{for} \quad a \le b < c < d \quad \textit{in } X; \tag{3}$$

$$badc \equiv bdac \quad \textit{for} \quad a < b \le c < d \quad \textit{in } X; \tag{4}$$

$$cbda \equiv cdba \quad \textit{for} \quad a < b < c \le d \quad \textit{in } X; \tag{5}$$

$$dbca \equiv bdca \quad \textit{for} \quad a < b \le c < d \quad \textit{in } X; \tag{6}$$

$$bcda \equiv bcad \quad \textit{for} \quad a < b \le c \le d \quad \textit{in } X; \tag{7}$$

$$cadb \equiv cdab \quad \textit{for} \quad a \le b < c \le d \quad \textit{in } X. \tag{8}$$

See Remark 7 for a concise alternative description of relations (1)–(8).

Definition 5 *Two words u and v in the alphabet X are* shifted plactic equivalent*, denoted $u \equiv v$, if they have the same mixed insertion tableau. By Theorem 4, u and v are shifted plactic equivalent if they are equivalent modulo the* shifted plactic relations *(1)–(8).*

A shifted plactic class *is an equivalence class under $\equiv$. We can associate a shifted plactic class with its corresponding shifted Young tableau, or with any of the words in the class, which insert to the corresponding tableau. The shifted plactic class corresponding to the Young tableau T is denoted $[T]$, and the shifted plactic class that contains a word u is denoted $[u]$. The Appendix at the end of (17) shows all kinds of shifted plactic classes of 4-letter words.*

For a word $w = w_1 w_2 \cdots w_n$ in X, let $P_{\mathrm{RSK}}(w)$ be its Robinson-Schensted-Knuth insertion tableau. Two words u and v in the alphabet X are *plactic equivalent* if $P_{\mathrm{RSK}}(u) = P_{\mathrm{RSK}}(v)$. Knuth (6) has proved that the latter holds if and only if u and v are equivalent modulo the *plactic relations*

$$acb \sim cab \quad \text{for} \quad a \le b < c \quad \text{in } X, \tag{9}$$

$$bca \sim bac \quad \text{for} \quad a < b \le c \quad \text{in } X. \tag{10}$$

Remark 6 (cf. (16)) *Relations (9)–(10) can be restated as follows.*

Let us call w a line word *if*

$$w_1 > w_2 > \cdots > w_n$$

or

$$w_1 \le w_2 \le \cdots \le w_n.$$

Line words are precisely those words w for which the shape of $P_{\mathrm{RSK}}(w)$ is a single row or a single column.

Two 3-letter words w and w' in the alphabet X are plactic equivalent if and only if:

- *w and w' differ by an adjacent transposition, and*
- *neither w nor w' is a line word.*

Remark 7 *The shifted plactic relations can be described in a similar way. Define a* hook word *as a word $w = w_1 \cdots w_l$ such that for some $1 \leq k \leq l$, we have*

$$w_1 > w_2 > \cdots > w_k \leq w_{k+1} \leq \cdots \leq w_l. \tag{11}$$

It is easy to see that w is a hook word if and only if $P_{\mathrm{mix}}(w)$ consists of a single row.

Two 4-letter words w and w' in the alphabet X are shifted plactic equivalent if and only if:

- *w and w' are plactic equivalent, and*
- *neither w nor w' is a hook word.*

The following proposition can be verified by direct inspection.

Proposition 8 *Shifted plactic equivalence is a refinement of plactic equivalence. That is, each plactic class is a disjoint union of shifted plactic classes. To put it yet in another way: if two words are shifted plactic equivalent, then they are plactic equivalent.*

For 4-letter words, Proposition 8 is illustrated in the Appendix to (17).

Definition 9 *The* shifted plactic monoid *is the set of shifted plactic classes where multiplication is given by $[u][v] = [uv]$. Equivalently, the monoid is generated by the symbols in X subject to the relations (1)–(8).*

An alternative point of view is to identify the shifted plactic classes with the corresponding shifted Young tableaux, thus giving a notion of a (shifted plactic) product of shifted tableaux.

The *shape* of a shifted plactic class is defined as the shape of the corresponding shifted Young tableau.

The *shifted plactic algebra* $\mathbb{Q}\mathbf{S}$ is the semigroup algebra of the plactic monoid.

Example 10 *One can check that both words in each of the shifted plactic relations have the same mixed insertion tableau. For example, for relation (1),*

$$P_{\mathrm{mix}}(abdc) = P_{\mathrm{mix}}(adbc) = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline \multicolumn{1}{c|}{} & d & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}.$$

Example 11 *The words $u = 3415961254$ and $v = 3451196524$ are shifted Knuth equivalent, because $P_{\mathrm{mix}}(u) = P_{\mathrm{mix}}(v)$. (cf. Example 3.) Furthermore, one can obtain v from u by the following a sequence of shifted plactic relations (where the relation to be used is stated and highlighted in bold)*

$$\begin{array}{rcll} u & = & 341596\mathbf{1254} & (1) \\ & \equiv & 34159\mathbf{6152}4 & (3) \\ & \equiv & 3415\mathbf{9165}24 & (3) \\ & \equiv & \mathbf{3415}196524 & (7) \\ & \equiv & 3451196524 & \\ & = & v. & \end{array}$$

Plactic Schur P-functions and their applications

For a shifted Young tableau T, with content $(a_1, a_2, \ldots)$ define its corresponding monomial as $x^T = x_1^{a_1} x_2^{a_2} \cdots$.

For each strict partition λ, the *Schur P-function* is defined as the generating function for shifted Young tableaux of shape λ, namely

$$P_\lambda = P_\lambda(x_1, x_2, \ldots) = \sum_{\mathrm{shape}(T)=\lambda} x^T.$$

The *Schur Q-function* is given by

$$Q_\lambda = Q_\lambda(x_1, x_2, \ldots) = 2^{\ell(\lambda)} P_\lambda,$$

or equivalently, as the generating function for a different kind of shifted Young tableaux, namely those in which the elements in the main diagonal are allowed to be primed.

The *skew Schur P- and Q-functions* $P_{\lambda/\mu}$ and $Q_{\lambda/\mu} = 2^{\ell(\lambda)-\ell(\mu)} P_{\lambda/\mu}$ are defined similarly, on a skew shifted shape λ/μ.

The following is an example of a Schur P-function in two variables:

Example 12 *For $\lambda = (3, 1)$,*

$$\begin{array}{ccccccccc} P_\lambda(x_1, x_2) & = & x_1^3 x_2 & + & x_1^2 x_2^2 & + & x_1^2 x_2^2 & + & x_1 x_2^3. \\ & & \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2' \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 2' & 2 \\ \hline \end{array} \\ & & \;\;\boxed{2}\;\;\; & & \;\;\boxed{2}\;\;\; & & \;\;\boxed{2}\;\;\; & & \;\;\boxed{2}\;\;\; \end{array}$$

The Schur P- and Q-Schur functions form bases for an important subring Ω of the ring Λ of symmetric functions.

The *shifted Littlewood-Richardson coefficients*, $b_{\mu,\nu}^\lambda$ are of great importance in combinatorics, algebraic geometry, and representation theory. They appear in the expansion of the product of two Schur P-functions,

$$P_\mu P_\nu = \sum_\lambda b_{\mu,\nu}^\lambda P_\lambda$$

and also in the expansion of a skew Schur Q-function

$$Q_{\lambda/\mu} = \sum_\nu b_{\mu,\nu}^\lambda Q_\nu.$$

The latter can be rewritten as

$$P_{\lambda/\mu} = \sum_\nu 2^{\ell(\mu)+\ell(\nu)-\ell(\lambda)} b_{\mu,\nu}^\lambda P_\nu.$$

Definition 13 *A* shifted plactic Schur P-function $\mathcal{P}_\lambda \in \mathbb{Q}\mathbf{S}$ *is defined as the sum of all shifted plactic classes of shape λ. More specifically,*

$$\mathcal{P}_\lambda = \sum_{\mathrm{shape}(T)=\lambda} [T].$$

Example 14 *We represent each shifted plactic class as $[w]$, for some representative w, to obtain*

$$
\begin{array}{ccccccccc}
\mathcal{P}_{(3,1)} & = & [1211] & + & [2211] & + & [1212] & + & [2212]. \\
 & & \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array} & & \begin{array}{|c|c|c|}\hline 1 & 2' & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{|c}{} \\ \cline{2-2} \end{array}
\end{array}
$$

The reader can check that each word gets mixed inserted into the tableau underneath, making it a valid representative of its corresponding plactic class.

One can see that the $\mathcal{P}_\lambda$ are noncommutative analogs of the Schur P-functions. In the last example, $\mathcal{P}_{(3,1)}$ is the noncommutative analog of

$$P_{(3,1)}(x_1, x_2) = x_1^3 x_2 + 2x_1^2 x_2^2 + x_1 x_2^3 = s_{3,1}(x_1, x_2) + s_{2,2}(x_1, x_2). \tag{12}$$

Theorem 15 *The map $P_\lambda \mapsto \mathcal{P}_\lambda$ extends to a canonical isomorphism between the algebra generated by the ordinary and shifted plactic Schur P-functions, respectively. As a result, the $\mathcal{P}_\lambda$ commute pairwise, span the ring they generate, and multiply according to the shifted Littlewood-Richardson rule. Namely,*

$$\mathcal{P}_\mu \mathcal{P}_\nu = \sum_\lambda b_{\mu,\nu}^\lambda \mathcal{P}_\lambda. \tag{13}$$

Sagan (12) has extended the concept of *jeu de taquin* to shifted tableaux, and proved that, just as in the ordinary case, the result of applying a sequence of (shifted) jeu de taquin moves is independent from the order in which they are done. Throughout this paper we only apply shifted jeu de taquin to standard skew tableaux, for which the process is exactly as it is done in the ordinary case. For pairs of standard skew tableaux T and U, we say that T rectifies to U if U can be obtained from T by a sequence of shifted jeu de taquin moves.

Our first application of Theorem 15 is a new proof (and a new version of) the *shifted Littlewood-Richardson rule*. Stembridge (19) proved that the shifted Littlewood-Richardson number $b_{\mu,\nu}^\lambda$ is equal to the number of standard shifted Young skew tableaux of shape λ/μ which rectify to a fixed standard shifted Young tableau of shape ν.

By taking the coefficient of the shifted plactic class $[T]$ corresponding to a fixed tableau T of shape λ on both sides of (13), one obtains the following:

Corollary 16 (Shifted Littlewood-Richardson rule) *Fix a shifted plactic class $[T]$ of shape λ. The shifted Littlewood-Richardson coefficient $b_{\mu,\nu}^\lambda$ is equal to the number of pairs of shifted plactic classes $[U]$ and $[V]$ of shapes μ and ν, respectively, such that $[U][V] = [T]$.*

Remark 17 *This rule can be restated in the language of words as follows. In Chapter 2 of (17) we introduce a canonical representative of the shifted plactic class $[T]$ corresponding to the tableau T. This representative is called the* mixed reading word *of T, and denoted $\mathrm{mread}(T)$. A word w is called a* shifted tableau word *if $w = \mathrm{mread}(T)$ for some shifted Young tableau T. The shape of a shifted tableau word is given by the shape of the corresponding tableau.*

With this terminology, the shifted Littlewood-Richardson rule can be restated as follows: Fix a shifted tableau word w of shape λ. The shifted Littlewood-Richardson coefficient $b_{\mu,\nu}^\lambda$ is equal to the number of pairs of shifted tableau words u, v of shapes μ, ν, respectively, such that $w \equiv uv$.

Lemma 18 *Fix a shifted tableau word w of shape λ and fix a standard shifted tableau Q of shape ν. The number of pairs of shifted tableau words u, v of shapes μ and ν, respectively, such that $uv = w$ is equal to the number of standard shifted skew tableaux of shape λ/μ which rectify to Q.*

As a corollary, we obtain the original result of Stembridge (19).

Corollary 19 *Fix a standard shifted tableau Q of shape ν. The coefficient $b^{\lambda}_{\mu,\nu}$ is equal to the number of standard shifted skew tableaux of shape λ/μ which rectify to Q.*

Example 20 *We compute $b^{21}_{2,1} = 1$. For this, we fix the shifted tableau word $w = 132$, associated to the shifted Young tableau $T = \begin{array}{cc}\boxed{1}&\boxed{2}\\ &\boxed{3}\end{array}$. The only way to express $w = uv$ where u and v are reading words of shapes (2) and (1), respectively, is with $u = 13$, associated to the tableau $U = \boxed{1}\boxed{3}$, and $v = 2$, associated to the tableau $V = \boxed{2}$.*

$$\begin{array}{ccccc} 1\,32 & \equiv & 13 & \cdot & 2 \\ \begin{array}{cc}\boxed{1}&\boxed{2}\\ &\boxed{3}\end{array} & & \boxed{1}\boxed{3} & & \boxed{2} \end{array}\ .$$

The second application is a new proof (and a new version of) the *Schur expansion of a Schur P-function*. Stembridge (19) has found a combinatorial interpretation for the coefficient g^{θ}_{μ} in the sum

$$P_{\theta} = \sum_{\mu} g^{\theta}_{\mu} s_{\mu}.$$

We find a different interpretation for the g^{θ}_{μ} in terms of shifted plactic classes. Lascoux and Schützenberger (11) have defined the *plactic monoid* $\mathbf{P}$ as follows. Two words are *plactic equivalent* if they have the same *Robinson-Schensted-Knuth insertion tableau*. A *plactic class* is an equivalence class under plactic equivalence. The plactic class of a word u in the alphabet X is denoted $\langle u\rangle$. $\mathbf{P}$ is the set of plactic classes where multiplication is given by $\langle u\rangle\langle v\rangle = \langle uv\rangle$. Equivalently, it is generated by the symbols in X subject to the Knuth relations (9)–(10).

Recall, by Proposition 8, any two shifted plactic equivalent words are plactic equivalent, or in other words, plactic classes decompose into a union of shifted plactic classes. This yields the natural projection

$$\pi : \mathbf{S} \to \mathbf{P},$$

in which the shifted plactic class $[u]$ gets mapped to the plactic class $\langle u\rangle$.

We now consider the image of a plactic Schur P-function under π.

Theorem 21 *The plactic Schur P-function $\mathcal{P}_{\theta}$ gets mapped under π to a sum of plactic Schur functions $\mathcal{S}_{\mu}$. The coefficients g^{θ}_{μ} are the same as those in*

$$\pi(\mathcal{P}_{\theta}) = \sum_{\mu} g^{\theta}_{\mu} \mathcal{S}_{\mu}.$$

Moreover, g^{θ}_{μ} is equal to the number of shifted plactic classes $[u]$ of shifted shape θ such that $\pi([u]) = \langle v\rangle$ for some fixed plactic class $\langle v\rangle$ of shape μ.

Example 22 *Let μ be the ordinary shape $(3,1)$, and θ be the shifted shape $(3,1)$. We compute the coefficient $g_\mu^\theta = 1$; this is the coefficient of s_μ in P_θ (cf. (12)). For this, we fix $\langle u \rangle = \langle 2134 \rangle$, namely, the plactic class corresponding to the Young tableau $U = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$. Note that the words in $\langle u \rangle$ are $2134, 2314$, and 2341. These get split into two shifted plactic classes, namely $[2134]$ corresponding to the shifted Young tableau $\begin{array}{|c|c|c|c|}\hline 1 & 2' & 3 & 4 \\ \hline\end{array}$, and $[2314] = [2341]$ corresponding to the shifted Young tableau $\begin{array}{|c|c|c|}\hline 1 & 2' & 4 \\ \hline \multicolumn{1}{c|}{} & 3 \\ \cline{2-2}\end{array}$. Since the only one of these plactic classes has shape μ, namely $\langle 2314 \rangle$, we get $g_\mu^\theta = 1$.*

Theorem 23 *Let θ be a shifted shape, and U_θ a fixed standard shifted tableau of shape θ. Fix a plactic class $\langle T_\mu \rangle$ of shape μ. Let $\mathcal{G}_\mu^\theta$ be the set of shifted plactic classes $[T_\theta]$ of shape theta for which $\pi([T_\theta]) = \langle T_\mu \rangle$. Let $\mathcal{H}_\mu^\theta$ be the set of standard Young tableaux of shape μ which rectify to U_θ. Then the sets $\mathcal{G}_\mu^\theta$ and $\mathcal{H}_\mu^\theta$ are in bijection.*

As a corollary, we obtain the original result of Stembridge (19).

Corollary 24 *The coefficient g_μ^θ is equal to the number of standard Young tableaux Q_μ of shape μ which rectify to a fixed standard shifted Young tableau Q_θ of shape θ.*

For ordinary Young tableau, one uses the concept of rectification (under jeu de taquin) to obtain the Littlewood-Richardson coefficients in the Schur expansion of a skew Schur function.

We have been unable to construct an analog of a jeu de taquin slide for skew semistandard shifted tableaux, but nonetheless, we can define the rectification $\mathrm{rect}(T)$ of such a tableau T; see (17, Section 2.1). (In the notation of (17, Lemma 2.11), $\mathrm{rect}(T) = P_{\mathrm{mix}}(\mathrm{mread}(T))$.) We then define the *shifted plactic skew Schur P-function* of shape λ/μ as the following element of $\mathbb{Q}\mathbf{S}$:

$$\mathcal{P}_{\lambda/\nu} = \sum_{\mathrm{shape}(T)=\lambda/\mu} [\mathrm{rect}(T)].$$

Conjecture 25 *$\mathcal{P}_{\lambda/\mu}$ belongs to the ring generated by the plactic Schur P-functions.*

Corollary 26 *Fix a shifted Young tableau U of shape ν. The coefficient of P_ν in $P_{\lambda/\mu}$ is equal to the number of skew shifted Young tableaux T with $\mathrm{rect}(T) = U$.*

Remark 27 *For the moment we can prove a slightly weaker statement than Conjecture 25. The projection $\pi(\mathcal{P}_{\lambda/\mu})$ (which lives in $\mathbb{Q}\mathbf{P}$) belongs to the ring generated by the plactic Schur functions $\mathcal{S}_\mu$. This will enable us to find a combinatorial interpretation for the coefficients in the Schur expansion of the skew Schur P-functions.*

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k-Parabolic Subspace Arrangements

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Abstract. In this paper, we study k-parabolic arrangements, a generalization of the k-equal arrangement for any finite real reflection group. When $k = 2$, these arrangements correspond to the well-studied Coxeter arrangements. Brieskorn (1971) showed that the fundamental group of the complement of the type W Coxeter arrangement (over $\mathbb{C}$) is isomorphic to the pure Artin group of type W. Khovanov (1996) gave an algebraic description for the fundamental group of the complement of the 3-equal arrangement (over $\mathbb{R}$). We generalize Khovanov's result to obtain an algebraic description of the fundamental group of the complement of the 3-parabolic arrangement for arbitrary finite reflection group. Our description is a real analogue to Brieskorn's description.

Résumé. Nous généralisons les arrangements k-égaux à tous les groupes de réflexions finis réels. Les arrangements ainsi obtenus sont dits k-paraboliques. Dans le cas où k = 2 nous retrouvons les arrangements de Coxeter qui sont bien connus. En 1971, Brieskorn démontra que le groupe fondamental associé au complément (complexe) de l'arrangement de Coxeter de type W est en fait isomorphe au groupe pure d'Artin de type W . En 1996, Khovanov donne une description algébrique du groupe fondamental du complément (réel) de larrangement 3-égaux. Nous généralisons le résultat de Khovanov et obtenons une description algébrique du groupe fondamental de l'espace complément d'un arrangement k-parabolique pour tous les groupes de réflexions finis et réels. Il se trouve que notre description est l'analogue réel de la description de Brieskorn.

Keywords: Subspace Arrangements, Coxeter Groups, Discrete Homotopy Theory

1 Introduction

A *subspace arrangement* $\mathscr{A}$ is a collection of linear subspaces of a finite-dimensional vector space V, such that there are no proper containments among the subspaces. Examples of subspace arrangements include real and complex *hyperplane arrangements*. One of the main questions regarding subspace arrangements is to study the structure of the *complement* $\mathcal{M}(\mathscr{A}) = V - \cup_{X \in \mathscr{A}} X$. A combinatorial tool that has proven useful in studying the complement is the intersection lattice, $\mathcal{L}(\mathscr{A})$, which is the lattice of intersections of subspaces, ordered by inclusion. Many results regarding the homology and homotopy theory of $\mathcal{M}(\mathscr{A})$ can be found in the book by Orlik and Terao [17], when $\mathscr{A}$ is a real or complex *hyperplane* arrangement.

There are two interesting problems regarding homotopy of $\mathcal{M}(\mathscr{A})$ that we will concern ourselves with. The first problem is determining whether or not $\mathcal{M}(\mathscr{A})$ is an Eilenberg-MacLane space. An Eilenberg-MacLane space (or $K(\pi, m)$-space) is a space X such that $\pi_k(X) = 0$ for $i \neq m$ and $\pi_m(X) = \pi$. A $K(\pi, 1)$ subspace arrangement is an arrangement whose complement is a $K(\pi, 1)$ space. It is worth noting that not all complex hyperplane arrangements are $K(\pi, 1)$-spaces. The second problem is to find a presentation for the fundamental group of $\mathcal{M}(\mathscr{A})$.

We will look at several motivating examples where both questions have been answered. One example of a complex $K(\pi, 1)$ hyperplane arrangement is the braid arrangement, which is the collection of "diagonals" $z_i = z_j$ for $1 \leq i < j \leq n$ from a complex n-dimensional vector space. In 1963, Fox and Neuwirth [14] showed that the fundamental group of the complement is isomorphic to the pure braid group. It was also shown by Fadell and Neuwirth [12] that the higher homotopy groups of the complement are trivial. Thus this is an example of a $K(\pi, 1)$-arrangement.

An example of a real $K(\pi, 1)$ subspace arrangement is the 3-equal arrangement, which is the collection of all subspaces of the form $x_i = x_j = x_k$ for $1 \leq i < j < k \leq n$ in a real n-dimensional vector space. It was Khovanov, in 1996, who proved that this is a $K(\pi, 1)$ subspace arrangement [16]. He also gave a presentation for the fundamental group of the complement. The presentation of this group, as well as the presentation of the pure braid group, use the symmetric group in their construction. It is well known that the symmetric group is generated by adjacent transpositions $s_i = (i, i + 1)$, $i \in [n - 1]$, subject to the following relations:

1. $s_i^2 = 1$
2. $s_i s_j = s_j s_i$, if $|i - j| > 1$
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

The braid group has presentation given by the same generating set, but subject only to relations 2 and 3. The pure braid group is the kernel of the surjective homomorphism, φ, from the braid group to the symmetric group, given by $\varphi(s_i) = s_i$ for all $i \in [n - 1]$. Khovanov's presentation of the fundamental group of the complement of the 3-equal arrangement is very similar. He defines the *triplet group*, which we shall denote A'_{n-1}. This group has a presentation given by the generators s_i, but subject only to relations 1 and 2, and he defines the *pure triplet group* to be the kernel of the surjective homomorphism, $\varphi' : A'_{n-1} \to A_{n-1}$, given by $\varphi'(s_i) = s_i$ for all $i \in [n - 1]$. Khovanov showed that the fundamental group of the complement of the 3-equal arrangement is isomorphic to the pure triplet group. Thus, Khovanov found a "real analogue" to the results of Fadell, Fox and Neuwirth.

The work of Fadell and Neuwirth has been generalized to other hyperplane arrangements. A simplicial hyperplane arrangement is an arrangement whose regions are simplicial cones. In 1972, Deligne [11] showed that the complexification of any simplicial hyperplane arrangement is a $K(\pi, 1)$-arrangement. Given a finite real reflection group W, consider the complexification of the reflection arrangement associated to W. Since reflection arrangements are simplicial, their complexifications are $K(\pi, 1)$-arrangements. Moreover, in 1971 Brieskorn [9] found that the fundamental group of the complement is isomorphic to the pure Artin group of type W. We review the definition of Artin groups in Section 4.

Our primary interest is to give "real analogues" of these results for subspace arrangements in $\mathbb{R}^n$ that correspond to finite real reflection groups. In particular, given a finite real reflection group W, we define a family of (real) subspace arrangements which we call k-parabolic arrangements. We show in Theorem 4.1 that the fundamental group of the complement of a 3-parabolic arrangement has the following simple description. We construct a new Coxeter group W' on the same generating set S as W, but we relax all relations of W that are not commutative relations nor involutions. Then the fundamental group is the kernel of a surjective homomorphism $\varphi' : W' \to W$ given by $\varphi'(s) = s$ for all $s \in S$. It turns out that the 3-parabolic arrangement is also a $K(\pi, 1)$ arrangement, a result due to Davis et al. (Theorem 0.1.9 in [10]).

Our primary tool for finding our presentation is the notion of discrete homotopy theory. Discrete homotopy theory is a theory that was developed in [2]. The theory involves constructing a bigraded sequence of groups defined on an abstract simplicial complex that are invariants of a combinatorial nature. Instead of being defined on the topological space of a geometric realization of a simplicial complex, the discrete homotopy groups are defined in terms of the *combinatorial* connectivity of the complex. That is, we are interested in how simplices intersect. In this paper, we show that the discrete fundamental group of the Coxeter complex is isomorphic to π_1 of the complement of the 3-parabolic arrangement. Thus, our result shows that sometimes we can replace a group defined in terms of the topology of the space with a group defined in terms of the combinatorial structure of the space.

In Section 2 we give a definition of the k-parabolic arrangement, and review some necessary definitions related to Coxeter groups. We also relate k-parabolic arrangements to previous analogues of the k-equal arrangement given by Björner and Sagan [7] for types B and D. In Section 3, we give a brief overview of discrete homotopy theory and the definition of the Coxeter complex. Then we give an isomorphism between the classical fundamental group of the complement of the 3-parabolic arrangement and the discrete fundamental group of the corresponding Coxeter complex. In Section 4, we use this isomorphism and a study of discrete homotopy loops in the Coxeter complex to obtain our algebraic description of the fundamental group of the complement of the 3-parabolic arrangement. In Section 5 we conclude with some open questions related to $\mathscr{W}_{n,k}$-arrangements as well as a discussion on the $K(\pi, 1)$ problem.

2 Definition of the $\mathcal{W}_{n,k}$-arrangement

Let W be a finite real reflection group acting on $\mathbb{R}^n$ and fix a root system Φ associated to W. Let $\Pi \subset \Phi$ be a fixed simple system. Finally, let S be the set of simple reflections associated to Π. Assume that Π spans $\mathbb{R}^n$. We let $m(s,t)$ denote the order of st in W. We know that $m(s,s) = 1$ and $m(s,t) = m(t,s)$ for all $s, t \in S$. Finally, given a root α, let s_α denote the corresponding reflection, and let $(\cdot, \cdot)$ denote the standard inner product.

Recall that there is a hyperplane arrangement associated to W, called the *Coxeter arrangement* $\mathscr{H}(W)$, which consists of hyperplanes $H_\alpha = \{x \in \mathbb{R}^n : (x, \alpha) = 0\}$ for each $\alpha \in \Phi^+$. Since Π spans $\mathbb{R}^n$, the Coxeter arrangement is central and essential, which implies that the intersection of all the hyperplanes is the origin.

Since we are generalizing the k-equal arrangement, which corresponds to the case $W = A_n$, we use it as our motivation. For this paper, we will actually work with the essentialized k-equal arrangement. The k-equal arrangement, $\mathscr{A}_{n,k}$, is the collection of all subspaces given by $x_{i_1} = x_{i_2} = \ldots = x_{i_k}$ over all indices $\{i_1, \ldots, i_k\} \subset [n+1]$, with the relation $\sum_1^{n+1} x_i = 0$. The k-equal arrangement is an arrangement that has been studied extensively ([6], [8], [16]). We note that the intersection poset $\mathcal{L}(\mathscr{A}_{n,k})$ is a subposet of $\mathcal{L}(\mathscr{H}(A_n))$. There is already a well-known combinatorial description of both of these posets. The poset of all set partitions of $[n+1]$ ordered by refinement is isomorphic to $\mathcal{L}(\mathscr{H}(A_n))$, and under this isomorphism, $\mathcal{L}(\mathscr{A}_{n,k})$ is the subposet of set partitions where each block is either a singleton, or has size at least k. However, our generalization relies on a lesser-known description of these posets in terms of parabolic subgroups.

Definition 2.1 *A subgroup $G \subseteq W$ is a* parabolic subgroup *if there exists a subset $T \subseteq S$ of simple reflections, and an element $w \in W$ such that $G = < wTw^{-1} >$. If w can be taken to be the identity, then G is a* standard *parabolic subgroup. We view (G, wTw^{-1}) as a Coxeter system, and call G* irreducible *if (G, wTw^{-1}) is an irreducible system.*

It is well known that the lattice of *standard* parabolic subgroups, ordered by inclusion, is isomorphic to the Boolean lattice. However, the lattice of *all* parabolic subgroups, $\mathscr{P}(W)$, ordered by inclusion, was shown by Barcelo and Ihrig [1] to be isomorphic to $\mathcal{L}(\mathscr{H}(W))$. Since this isomorphism is essential to our generalization, we review it. The isomorphism is given by sending a parabolic subgroup G to $Fix(G) = \{x \in \mathbb{R}^n : wx = x, \forall w \in G\}$, and the inverse is given by sending an intersection of hyperplanes X to $Gal(X) = \{w \in W : wx = x, \forall x \in X\}$.

This Galois correspondence gives a description of $\mathcal{L}(\mathscr{H}(A_n))$ in terms of parabolic subgroups of A_n. We also obtain another description of $\mathcal{L}(\mathscr{A}_{n,k})$ under this correspondence.

Proposition 2.2 *The Galois correspondence gives a bijection between subspaces of $\mathscr{A}_{n,k}$ and irreducible parabolic subgroups of A_n of rank $k-1$.*

Proof: Let X be a subspace of $\mathbb{R}^{n+1}$ given by $x_1 = \ldots = x_k$. The k-equal arrangement is the orbit of X under the action of $A_n = S_{n+1}$, and $Gal(X) =< (1,2), ..., (k-1,k) >$, hence is irreducible. For $w \in A_n$, $Gal(wX) = wGal(X)w^{-1}$, so all of the subspaces in the k-equal arrangement have irreducible Galois groups.

Conversely, every irreducible parabolic subgroup of rank $k-1$ in A_n is the Galois group of some subspace in the k-equal arrangement. To see this, consider an irreducible parabolic subgroup G of rank $k-1$. Then there exists a standard parabolic subgroup H and an element $w \in W$ such that $G = wHw^{-1}$. Since H is an irreducible *standard* parabolic subgroup, $H =< (i, i+1), ..., (i+k-1, i+k) >$ for some $1 \le i \le n+1-k$. Thus, $Fix(H)$ is given by $x_i = \ldots = x_k$, and $Fix(G) = Fix(wHw^{-1}) = wFix(G)$ is given by $x_{w(i)} = \ldots x_{w(k)}$, which is a subspace in the k-equal arrangement. □

With this proposition as motivation, we give the following definition for a k-parabolic arrangement.

Definition 2.3 *Let W be an finite real reflection group of rank n. Let $\mathscr{P}_{n,k}(W)$ be the collection of* all *irreducible parabolic subgroups of W of rank $k-1$.*

Then the k-parabolic arrangement $\mathscr{W}_{n,k}$ is the collection of subspaces

$$\{Fix(G) : G \in \mathscr{P}_{n,k}(W)\}.$$

The k-parabolic arrangements have many properties in common with the k-equal arrangements. Both of these arrangements can be embedded in the corresponding Coxeter arrangement. That is, every subspace in these arrangements can be given by intersections of hyperplanes of the Coxeter arrangements. Moreover, $\mathcal{L}(\mathscr{W}_{n,k})$ is a subposet of $\mathcal{L}(\mathscr{H}(W)) = \mathcal{L}(\mathscr{W}_{n,2})$, and these arrangements are invariant under the action of W. Indeed, consider a subspace X in $\mathscr{W}_{n,k}$ and an element $w \in W$. Since X is in $\mathscr{W}_{n,k}$, $Gal(X)$ is an irreducble parabolic subgroup of rank $k-1$. It is clear that $Gal(wX) = wGal(X)w^{-1}$, so $Gal(wX)$ is also an irreducible parabolic subgroup of rank $k-1$, whence $Gal(wX) \in \mathscr{P}_{n,k}(W)$. Since $Fix(Gal(wX)) = wX$, it follows that $wX \in \mathcal{W}_{n,k}$.

When W is of type A, we see that we have recovered the k-equal arrangement. To see what happens when W is type B or D, first we recall type B and D analogues of the k-equal arrangement. In 1996, Björner and Sagan defined a class of subspace arrangements of type B and D [7], which they call the $\mathscr{B}_{n,k,h}$-arrangements and $\mathscr{D}_{n,k}$-arrangements.

Definition 2.4 *The $\mathscr{D}_{n,k}$-arrangement consists of subspaces given by $\pm x_{i_1} = \pm x_{i_2} = \ldots = \pm x_{i_k}$, over distinct indices $i_1, \ldots, i_k$. The $\mathscr{B}_{n,k,h}$-arrangements are obtained from the $\mathscr{D}_{n,k}$-arrangements by including subspaces given by $x_{i_1} = \ldots = x_{i_h} = 0$ over distinct indices $i_1, \ldots, i_h$, with $h < k$.*

The Betti numbers of $\mathcal{M}(\mathscr{B}_{n,k,h})$ were computed by Björner and Sagan in [7], while the Betti numbers of $\mathcal{M}(\mathscr{D}_{n,k})$ were computed by Kozlov and Feichtner in [13].

Example 2.5 (When W is of type B) *When W is of type B, the k-parabolic arrangement is the $\mathscr{B}_{n,k,k-1}$-arrangement of Björner and Sagan [7]. Recall that B_n has presentation given by generators $s_i, 0 \leq i \leq n$, such that $< s_1, \ldots, s_n >$ generate the symmetric group, $(s_0 s_1)^4 = 1$, and $s_0 s_i = s_i s_0$ for $i > 1$. It is well-known that the $\mathscr{B}_{n,k,k-1}$-arrangement is the orbit of two subspaces given by $x_1 = \ldots = x_k$ and $x_1 = \ldots = x_{k-1} = 0$, under the action of B_n. Clearly the Galois groups of these two spaces are given by $< s_1, \ldots, s_{k-1} >$ and $< s_0, \ldots, s_{k-2} >$. These are both irreducible parabolic subgroups of rank $k-1$, so every subspace of the $\mathscr{B}_{n,k,k-1}$-arrangement corresponds to an irreducible parabolic subgroup of rank $k-1$. Similarly, given an irreducible parabolic subgroup of rank $k-1$, it is not hard to show that this subgroup corresponds to a subspace in the $\mathscr{B}_{n,k,k-1}$-arrangement. The argument is similar to the case for type A, and we omit the details.*

3 Discrete Homotopy Theory

To facilitate the proofs of our algebraic description for $\pi_1(\mathcal{M}(\mathscr{W}_{n,k}))$, first we give a combinatorial description of $\pi_1(\mathcal{M}(\mathscr{W}_{n,k}))$ in terms of discrete homotopy theory of the Coxeter complex for W. As motivation, we mention the following result:

Theorem 3.1 *Let $\mathcal{M}(\mathscr{A}_{n,k})$ be the complement of the k-equal arrangement $\mathscr{A}_{n,k}$. Let $\mathscr{C}(A_n)$ be the order complex of the Boolean lattice.*

Then $\pi_1(\mathcal{M}(\mathscr{A}_{n,k})) \cong A_1^{n-k+1}(\mathscr{C}(A_n))$, where A_1^q is a discrete homotopy group, to be defined below.

This result was shown independently by Björner [5] and Babson (appears in [3]) in 2001). It turns out that the order complex of the Boolean lattice is the Coxeter complex of type A, which explains our choice of notation.

One of the original motivations for discrete homotopy theory was to create a sequence of groups for studying social networks being modeled as simplicial complexes. However, as Theorem 3.1 shows, discrete homotopy theory has applications in other areas of mathematics. We will show that there is an isomorphism between $\pi_1(\mathcal{M}(\mathscr{W}_{n,k}))$ and the discrete fundamental group, A_1^{n-k+1}, of the Coxeter complex, a combinatorial structure associated to the Coxeter arrangement. Essentially, we are replacing a topologically defined group with a combinatorially defined group. First, however, we give an overview of some of the needed basic definitions and results from discrete homotopy theory. Many details and background history of discrete homotopy theory can be found in [2].

Fix a positive integer d. Let Δ be a simplicial complex of dimension d, fix $0 \leq q \leq d$, and let $\sigma_0 \in \Delta$ be maximal with dimension $\geq q$. Two simplicies σ and τ are *q-near* if they share $q+1$ elements. A *q-chain* is a sequence $\sigma_1, \ldots, \sigma_k$, such that σ_i, σ_{i+1} are q-near for all i. A *q-loop* based at σ_0 is a q-chain with $\sigma_1 = \sigma_k = \sigma_0$.

Definition 3.2 *We define an equivalence relation, $\simeq$ on q-loops with the following conditions:*

1. *The q-loop*
$$(\sigma) = (\sigma_0, \sigma_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0)$$
is equivalent to the q-loop
$$(\sigma)' = (\sigma_0, \sigma_1, \ldots, \sigma_i, \sigma_i, \sigma_{i+1}, \ldots, \sigma_n, \sigma_0),$$

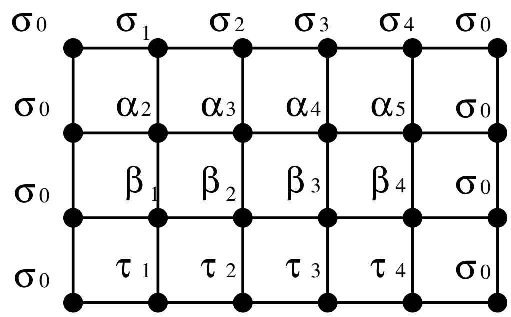


Fig. 1: An example of a homotopy grid

which we refer to as stretching.

2. *If (σ) and (τ) are two q-loops that have the same length then they are equivalent if there is a diagram as in figure 1. The vertices represent simplices, and two vertices are connected by an edge if and only if the corresponding simplices are q-near. Thus, every row is a q-loop based at σ_0, and every column is a q-chain. Such a diagram is called a (discrete) homotopy between (σ) and (τ).*

Define $A_1^q(\Delta, \sigma_0)$ to be the collection of equivalence classes of q-loops based at σ_0. Then the operation of concatenation of q-loops gives a group operation on $A_1^q(\Delta, \sigma_0)$, the *discrete homotopy group* of Δ. The identity is the equivalence class containing the trivial loop (σ_0), and given an equivalence class $[\sigma]$ for the q-loop $(\sigma) = (\sigma_0, \sigma_1, \ldots, \sigma_k, \sigma_0)$, the inverse $[\sigma]^{-1}$ is the equivalence class of $(\sigma_0, \sigma_k, \sigma_{k-1}, \ldots, \sigma_2, \sigma_1, \sigma_0)$. As in classical topology, if a pair of maximal simplices σ, τ of dimension at least q in Δ are q-connected, then $A_1^q(\Delta, \sigma) \cong A_1^q(\Delta, \tau)$. Thus, in the case Δ is q-connected, we will set $A_1^q(\Delta) = A_1^q(\Delta, \sigma_0)$ for any maximal simplex $\sigma_0 \in \Delta$ of dimension at least q.

Before we use discrete homotopy theory, we need a result from [2] that relates discrete homotopy theory of a simplicial complex to classical homotopy theory of a related space. Given $0 \leq q \leq d$, let $\Gamma^q(\Delta)$ be a graph whose vertices are maximal simplices of Δ of size at least q, and with edges between two simplices σ, τ, if and only if σ and τ are q-near. Then the following result relates $A_1^q(\Delta, \sigma_0)$ in terms of a cell complex related to $\Gamma^q(\Delta)$.

Proposition 3.3 (Proposition 5.12 in [2])

$$A_1^q(\Delta, \sigma_0) \cong \pi_1(X_\Gamma, \sigma_0)$$

where X_Γ is a cell complex obtained by gluing a 2-cell on each 3- and 4-cycle of $\Gamma = \Gamma^q(\Delta)$.

Let $W, \mathcal{H}(W), \Phi, \Pi, S$ be as in section 2. As mentioned previously, we study the discrete homotopy groups of the *Coxeter complex* associated to W, and relate them to $\pi_1(\mathcal{M}(\mathscr{W}_{n,k}))$. The majority of these details can be found in Section 1.14 in Humphrey's book on Coxeter groups [15]. The concepts regarding fans and zonotopes can be found in Chapter 7 of Zeigler's book on polytopes [18].

For a given set $I \subseteq S$, let $W_I =< I >$, and $\Pi_I = \{\alpha \in \Pi : s_\alpha \in I\}$. We can associate to W_I the set of points $C_I = \{x \in \mathbb{R}^n : (x, \alpha) = 0, \forall \alpha \in \Pi_I,$ and $(x, \alpha) > 0, \forall \alpha \in \Pi - \Pi_I\}$. The set C_I is the intersection of hyperplanes H_α for $\alpha \in \Pi_I$ with certain open half-spaces. We see that $C_\emptyset$ corresponds to the interior of a fundamental region, and C_S is the origin.

For a given coset wW_I, we can associate the set of points wC_I. The collection $\mathscr{C}(W)$ of wC_I for all $w \in W$, and all $I \subseteq S$ partitions $\mathbb{R}^n$, and is called the *Coxeter complex* of W. The face poset of the Coxeter complex can be viewed as the collection of cosets wW_I for any $w \in W, I \subseteq S$, ordered by *reverse inclusion*. We note that this poset is *not* $\mathcal{L}(\mathscr{H}(W))$. For $W = A_n$, we have already mentioned that $\mathcal{L}(\mathscr{H}(W))$ is isomorphic to the partition lattice. The face poset of the braid arrangement, however, is isomorphic to the order complex of the boolean lattice. Since chains in the boolean lattice are in one-to-one correspondence with *ordered* set partitions, these two posets are related, but are very different.

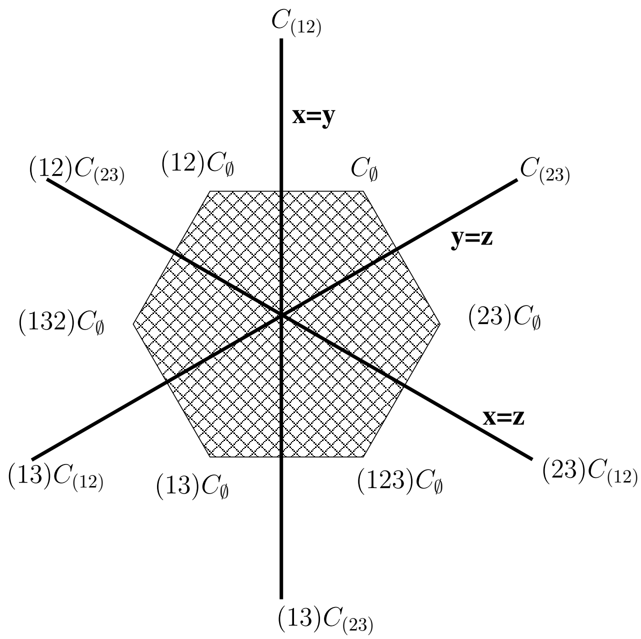


Fig. 2: Coxeter Complex and Zonotope for $W = A_3$. Note that C_S is the origin.

For a given w, I, the closure of wC_I is a convex polyhedral cone. In fact, the collection of all $w\bar{C}_I$ forms a fan of $\mathbb{R}^n$, which is the fan associated to $\mathscr{H}(W)$. Under this view, the sets $w\bar{C}_I$ are the faces of the arrangement.

Recall that we can associate a *zonotope* to a hyperplane arrangement. That is, given line segments of unit length normal to the hyperplanes, one can form a polytope by taking the Minkowski sum of these line segments. For a Coxeter arrangement of type W this zonotope is called the W-Permutahedron. Also, the fan of the arrangement is the normal fan of the zonotope. Thus, we can label the faces of the W-Permutahedron by cosets wW_I, where a face F gets the label wW_I if the normal cone for F is $w\bar{C}_I$. Under this labeling, the face poset of the W-Permutahedron is indexed by cosets wW_I for all $w \in W, I \subseteq S$, ordered by *inclusion*.

We observe that in the W-Permutahedron, the vertices correspond to elements of W, and two vertices share an edge if and only if the corresponding regions share an $(n-1)$-dimensional boundary, that is if and only if the corresponding elements of W differ by multiplication on the right by a simple reflection.

From this it follows that the graph (one-skeleton) of the W-Permutahedron is the graph $\Gamma^{n-2}(\mathscr{C}(W))$ defined before Proposition 3.3.

We also characterize the cycles that are boundaries of 2-faces in the W-Permutahedron. Given a 2-dimensional face F and a vertex w in F, we see that one edge adjacent to w in F is of the form w, ws for some $s \in S$. Likewise, one of the two edges of F incident to the edge w, ws is the edge ws, wst, where $t \in S - s$. Thus we see that the coset associated to the normal cone of F contains both wW_s and wsW_t. Likewise, it is the smallest coset to contain these two cosets, so the corresponding coset is given by $wW_{\{s,t\}}$. The cycle that is the boundary of F is seen to have length $2m(s,t)$. This means that the graph has no 3-cycles, and 4-cycles are boundaries of faces which correspond to a coset of $W_{\{s,t\}}$, where $s, t \in S$ and $m(s,t) = 2$. The fact that the graph has no 3-cycles will turn out to be useful in section 4.

Now we turn to the main result of this section.

Theorem 3.4 *Let $\mathcal{M}(\mathscr{W}_{n,k})$ be the complement of the k-parabolic arrangement $\mathscr{W}_{n,k}$. Then $\pi_1(\mathcal{M}(\mathscr{W}_{n,k})) \cong A_1^{n-k+1}(\mathscr{C}(W))$.*

The proof is given in the full version of the paper [4].

4 An algebraic description of $\pi_1(\mathcal{M}(\mathscr{W}_{n,3}))$

In this section, we give a description of $\pi_1(\mathcal{M}(\mathscr{W}_{n,k}))$ that is similar to the idea of a pure Artin group. In our case, the group we consider is a (possibly infinite) Coxeter group. Recall that W affords the following presentation: W is generated by S subject to the relations:

1. $s^2 = 1, \forall s \in S$
2. $st = ts, \forall s, t \in S$ such that $m(s,t) = 2$
3. $sts = tst, \forall s, t \in S$, such that $m(s,t) = 3$

$$\vdots$$

i. $\underbrace{stst\cdots}_{i} = \underbrace{tsts\cdots}_{i}, \forall s, t \in S$, such that $m(s,t) = i$

$$\vdots$$

where of course we have no relation of the form $st\cdots = ts\cdots$ if $m(s,t) = \infty$.

If G is a group generated by S subject to every relation except relations of type 1, then G is an *Artin group*. There is a surjective homomorphism $\varphi : G \to W$ given by $\varphi(s) = s$ for all $s \in S$. The kernel of φ is the *pure* Artin group. As stated in the introduction, the pure Artin group is isomorphic to the fundamental group of the complement of the complexification of the Coxeter arrangement for W. The goal of this section is to give a real analogue of this result for the $\mathscr{W}_{n,3}$-arrangements.

In our case, let W' be a group on S subject to only the relations of type 1 and 2. Equivalently, W' is subject to $s^2 = 1$ for all $s \in S$, and two elements $s, t \in S$ commute in W' if and only if they commute in W. In essence, given the Dynkin diagram D for W, W' is obtained by replacing all the edge labels in D with the edge label ∞, and letting W' be the resulting Coxeter group.

Consider the surjective homomorphism $\varphi' : W' \to W$ given by $\varphi'(s) = s$ for all $s \in S$. Then the following result holds:

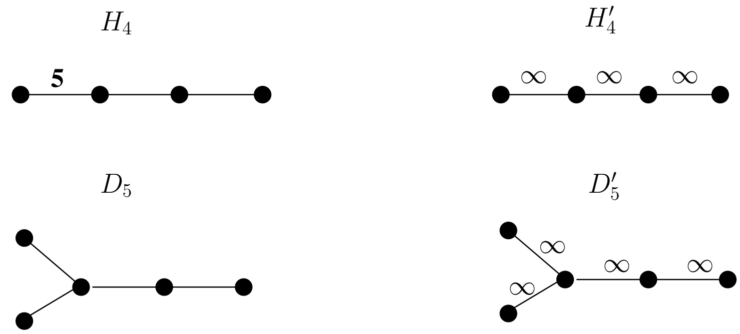


Fig. 3: Dynkin diagrams for W and W'

Theorem 4.1 $\pi_1(\mathcal{M}(\mathscr{W}_{n,3})) \cong \ker \varphi'$.

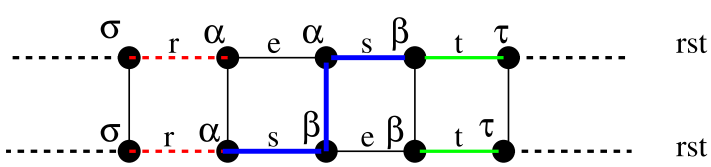
When W is of type A or B, Theorem 4.1 was shown by Khovanov [16], where the arrangements are referred to using different terminology. However, we give a proof for any finite real reflection group.

As a result of Theorem 3.4, we know that we can study $\pi_1(\mathcal{M}(\mathscr{W}_{n,k}))$ using discrete homotopy theory of $\mathscr{C}(W)$. Before we prove Theorem 4.1 we first investigate the structure of $(n-2)$-loops in $\mathscr{C}(W)$ in more detail. For the duration of the section we will use the term loop to mean $(n-2)$-loop. To any such loop $(\sigma) = (\sigma_0, \ldots, \sigma_\ell, \sigma_0)$ in $\mathscr{C}(W)$ we associate a sequence of elements of $S \cup \{1\}$ of length ℓ in the following way: For any $i \in [\ell]$, if $\sigma_i = \sigma_{i-1}$, let $s_i = 1$. Otherwise let s_i be the unique element s of S for which $\sigma_{i-1}s = \sigma_i$. Thus we associate a word $f(\sigma)$ in S^* to (σ): the product of the elements of the corresponding sequence in order.

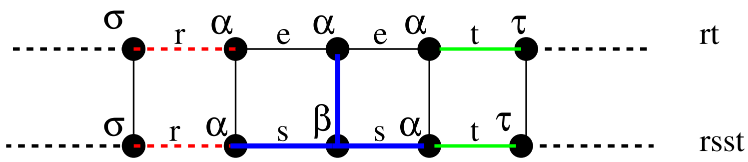
We note that if (σ) is a loop, then $f(\sigma) = 1$ in W. This implies that when viewing $f(\sigma)$ as a product in W', $f(\sigma) \in \ker \varphi'$. We also note that to any element $w = s_1 \cdots s_k$ in S^* we can associate a chain $g(w) = (\sigma_0, \sigma_0 s_1, \ldots, \sigma_0 s_1 \cdots s_k)$, where the elements $s_1 \cdots s_i$ are being viewed as elements of W. If $w = 1$ when viewed as an element of W, then $g(w)$ is actually a loop. It is easy to see that for two loops $(\sigma), (\tau)$, $f((\sigma) * (\tau)) = f(\sigma)f(\tau)$, and if $u, v \in S^*$, $u = v = 1$ in W, then $g(uv) = g(u) * g(v)$.

Suppose there is a homotopy between two loops (σ) and (τ) of the same length. Since $\Gamma^{n-2}(\mathcal{C}(W))$ does not have any 3-cycles, it turns out that there is a (discrete) homotopy between them where adjacent rows in the grid follow one of the three following *discrete homotopy operations*. In each case, we also show how the associated words differ between the adjacent rows. Finally, e refers to the identity element of W.

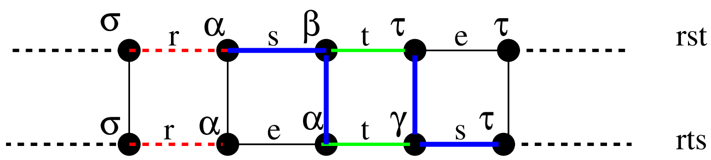
(T1) Repeating simplices. A simplex α is repeated consecutively on the top row, and a different simplex β is repeated consecutively on the bottom row. Note that this results in no change in the associated words.



(T2) Inserting or removing a simplex. On one row there are three adjacent identical simplices α, and on the bottom row the middle simplex of this triple is replaced with a new simplex β that is $(n-2)$-near α. Note that the corresponding words differ by an involution relation.



(T3) Exchanging pairs that are $(n-2)$-near. We happen to know that $(\alpha, \beta, \tau, \gamma)$ is a loop of distinct simplices. We construct a discrete homotopy as shown in the figure. We note that the resulting words differ by an application of a commutative relation. It is also worth noting this operation can only be performed when s, t commute.



Thus for any discrete homotopy operation, the corresponding words are either equal, or differ by one of the generating relations of W'. In the full paper [4], we use this observation to prove the following lemma.

Lemma 4.2 *1. Let $(\sigma), (\tau)$ be loops. If $(\sigma) \simeq (\tau)$ then $f(\sigma) = f(\tau)$ in W'.*

2. Let $w \in S^*$. If $w = 1$ in W', then $g(w)$ is contractible.

3. Let $w, v \in S^*$. If $w = v$ in W', then $g(w) \simeq g(v)$.

Proof of Theorem 4.1:

The isomorphism is given by sending the equivalence class with representative (σ) to $f(\sigma)$, and the inverse is given by sending an element $w \in W'$, expressed as $s_1 s_2 \cdots s_k$, $k \in \mathbb{N}, s_1, \ldots, s_k \in S$, to $g(s_1 \cdots s_k)$. The details that these functions are well-defined isomorphisms is given in the full paper [4]. $\square$

5 Conclusion and Open Problems

It follows as a result of Corollary 5 in [8] that for $k > 3$, the k-parabolic arrangements are not $K(\pi, 1)$. However, the $\mathscr{W}_{n,3}$ -arrangement is a $K(\pi, 1)$-arrangement. As a result of Davis, Januszkiewicz and Scott, if $\mathscr{A}$ is any collection of codimension 2 subspaces of $\mathscr{H}(W)$ that are invariant under the action of W , then $\mathscr{A}$ is a $K(\pi, 1)$-arrangement (Theorem 0.1.9 in [10]).

Currently there is no presentation for the fundamental groups of the complement of such W-invariant arrangements. Motivated by our results, and the work of Khovanov [16], we give the following conjectured presentation.

Conjecture 5.1 *Let $\mathscr{P}$ be a collection of rank 2 parabolic subgroups of a finite real reflection group W such that $\mathscr{P}$ is closed under conjugation, and let $\mathscr{W} = \{Fix(G) : G \in \mathscr{P}\}$ Define a new Coxeter group W' with the same generating set S as W, and subject to:*

$$m'(s,t) = \begin{cases} \infty & \textit{if } < s,t > \in \mathscr{P} \\ m(s,t) & \textit{else} \end{cases}$$

and let $\varphi : W' \to W$ be given by sending $s \to s$ for all $s \in S$. Then $\pi_1(\mathcal{M}(\mathscr{W})) \cong \ker \varphi$.

In [2], a definition is given for higher discrete homotopy groups, which are denote $A^q_m(\Delta, \sigma_0)$. A natural question is whether or not these groups are related to the higher homotopy groups of $\mathcal{M}(\mathscr{W}_{n,k})$.

Conjecture 5.2 *Let $\mathcal{M}(\mathscr{W}_{n,k})$ be the complement of the k-parabolic arrangement $\mathscr{W}_{n,k}$. Then $\pi_m(\mathcal{M}(\mathscr{W}_{n,k})) \cong A^{n-k+1}_m(\mathscr{C}(W))$.*

For $m < k$, it would suffice to show that $A^{n-k+1}_m(\mathscr{C}(W))$ is trivial. The conjecture becomes interesting for $k > 3, m = k$, because in this case the k-th homology group of $\mathcal{M}(\mathscr{W}_{n,k})$ is isomorphic to the k-th homotopy group. Thus, one could find the formulas for the first non-zero Betti numbers using discrete homotopy theory. Determining the Betti numbers for the k-parabolic arrangements is also an open problem, in the case that W is an exceptional groups.

Finally, one may if it is possible to generalize Theorem 3.4 to other hyperplane arrangements. That is, given a hyperplane arrangement $\mathscr{H}$, let $\mathscr{C}(\mathscr{H})$ be the face complex of $\mathscr{H}$. Is there a subspace arrangement $\mathscr{A}$ for which $\pi_1(\mathcal{M}(\mathscr{A})) \cong A^{n-2}_1(\mathscr{C}(\mathscr{H}))$? This would be an example of using discrete homotopy theory of a complex that arises from geometry to study a topological space related to the original complex.

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The Discrete Fundamental Group of the Associahedron

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Abstract. The associahedron is an object that has been well studied and has numerous applications, particularly in the theory of operads, the study of non-crossing partitions, lattice theory and more recently in the study of cluster algebras. We approach the associahedron from the point of view of discrete homotopy theory, that is we consider 5-cycles in the 1-skeleton of the associahedron to be combinatorial holes, but 4-cycles to be contractible. We give a simple description of the equivalence classes of 5-cycles in the 1-skeleton and then identify a set of 5-cycles from which we may produce all other cycles. This set of 5-cycle equivalence classes turns out to be the generating set for the abelianization of the discrete fundamental group of the associahedron. In this paper we provide presentations for the discrete fundamental group and the abelianization of the discrete fundamental group. We also discuss applications to cluster algebras as well as generalizations to type B and D associahedra.

Résumé. L'associahèdre est un objet bien etudié que l'on retrouve dans plusieurs contextes. Par exemple, il est associé à la théorie des opérades, à l'étude des partitions non-croisées, à la théorie des treillis et plus récemment aux algèbres dámas. Nous étudions cet objet par le biais de la théorie des homotopies discretes. En bref cette théorie signifie qu'un cycle de longueur 5 (sur le squelette de l'associahèdre) est considéré comme étant le bord d'un trou combinatoire, alors qu'un cycle de longueur 4 peut être contracté sans problème. Les classes d'homotopies discrètes sont donc des classes d'équivalence de cycles de longueurs 5. Nous donnons une description simple de ces classes d'équivalence et identifions un ensemble de générateurs du groupe correspondant (abélien) d'homotopies discrètes. Nous d'ecrivons également les liens entre notre construction et les algèbres d'amas.

Keywords: associahedron, discrete fundamental group, conic arrangements

1 Introduction

Let $\mathcal{T}_n$ be the abstract simplicial complex on the set of all diagonals of a regular $(n+3)$-gon whose maximal simplices, T_i, correspond to triangulations of the regular $(n+3)$-gon. It is well known that if we (partially) order the simplices of $\mathcal{T}_n$ by reverse inclusion then we have a poset that is isomorphic to the face poset of the associahedron [14]. There is a wealth of recent literature focusing on the associahedron and its generalizations, [4, 6, 5, 9, 16, 12, 15]. Simion, in [14], gives an excellent description of the origins and early study of the associahedron. It is our intention to study the associahedron through the lens of the discrete homotopy theory, or A-theory, of Barcelo, Kramer, Laubenbacher and Weaver [1, 2]. This approach highlights some interesting combinatorial properties of the associahedron and provides a framework to study several of the generalizations of the associahedron in the same manner.

Our approach is not completely novel; we are motivated by the study of the discrete fundamental group of the permutahedron done by Barcelo and Smith. It had been shown previously by Babson [2] and independently by Björner that that the discrete fundamental group of the permutahedron is isomorphic to the classical homotopy group of the real complement of the k-equal arrangement, $M_{n,k}$. In [3], the authors provide a combinatorial method for calculating the abelianization of the discrete fundamental group of the permutahedron, which in turn gives a purely combinatorial method of calculating the Betti number of $M_{n,k}$. In our case, due to the structure of the associahedron we do not have a resulting subspace arrangement but we will give a link to what we call a *conic arrangement* as well as a connection to cluster algebras.

We present a short overview of the facts about A-theory needed here but for a more thorough understanding we refer the reader to [1, 2].

Recall that a triangulation of an $(n+3)$-gon contains n non crossing diagonals. Given two maximal simplices (triangulations) T_1, T_2 in $\mathfrak{T}_n$, we say they are *near* if $|T_1 \cap T_2| = n-1$. We may also restate this as: T_1 and T_2 are near if they differ by a diagonal flip. A sequence of maximal simplices, $T_1 - T_2 - \cdots - T_k$ is called a *chain* if T_i, T_{i+1} are near for all $0 \leq i \leq k$ and a chain that starts and ends with the same simplex is called a *loop*. Note that in the general discrete homotopy theory, one can vary the definition of near by adjusting a parameter q. This parameter q is fixed in our case to be $n-2$.

There is an equivalence relation, $\simeq_A$, that may be placed on the set of all loops based at T_0. A full description of this relation can be found in [1, 2].

Let $A_1^{n-2}(\mathfrak{T}_n, T_0)$ be the set of equivalence classes of loops based at T_0 (the superscript $n-2$ is the parameter q mentioned previously). By Proposition 2.3 in [1], a group structure can be imposed on $A_1^{n-2}(\mathfrak{T}_n, T_0)$ by adding the operation of concatenation of loops. We call $A_1^{n-2}(\mathfrak{T}_n, T_0)$ with this group structure, the *discrete fundamental group* of $\mathfrak{T}_n$. A grid between loops in $A_1^{n-2}(\mathfrak{T}_n, T_0)$ can be thought of as analogous to a continuous deformation of one curve to another in classical homotopy theory. The structure of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ gives us information about $\mathfrak{T}_n$ in the same way the classical homotopy group gives us information about a topological space.

Given the complex $\mathfrak{T}_n$, we may also define a graph, $\Gamma^{n-2}(\mathfrak{T}_n)$, where the vertex set of $\Gamma^{n-2}(\mathfrak{T}_n)$ is in bijection with the set of maximal simplices of $\mathfrak{T}_n$ and we put an edge between T_i and T_j if they are near. It is shown in [1] that closed walks based at T_0 in $\Gamma^{n-2}(\mathfrak{T}_n)$ are in bijection with loops using elements from $\mathfrak{T}_n$, and in fact two closed based walks in $\Gamma^{n-2}(\mathfrak{T}_n)$ are homotopic if they differ by 3- and 4-cycles only. Thus we may think of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ as being the group of equivalence classes of closed based walks in $\Gamma^{n-2}(\mathfrak{T}_n)$ with the obvious operation of concatenation and the identity and inverses just as in the previous description of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ in terms of loops. It is well known [14] that the graph $\Gamma^{n-2}(\mathfrak{T}_n)$ is the 1-skeleton of the associahedron, hereafter referred to as Asc_n to reinforce the connection between $\mathfrak{T}_n$ and the associahedron in the mind of the reader. Hence when we discuss $A_1^{n-2}(\mathfrak{T}_n, T_0)$, we may think of elements in terms of walks in the 1-skeleton of the associahedron.

By Proposition 5.12 in [1], we know that $A_1^{n-2}(\mathfrak{T}_n, T_0) \simeq \pi_1(X_\Gamma)$, where X_Γ is the topological space obtained by attaching a 2-cell to every 3- and 4-cycle of $\Gamma^{n-2}(\mathfrak{T}_n)$. We refer to cycles in Asc_n that bound a 2-face of the associahedron as *basic cycles*. If we continue our analogy between discrete and classical homotopy theory, we can see now that a hole in $\mathfrak{T}_n$ corresponds to a basic cycle in Asc_n of length ≥ 5. However, because X_Γ is not a graph, it is not guaranteed that $A_1^{n-2}(\mathfrak{T}_n, T_0)$ is free and we show that there are in fact commutivity relations between the generators of $A_1^{n-2}(\mathfrak{T}_n, T_0)$.

When we move on to the abelianization of $A_1^{n-2}(\mathfrak{T}_n, T_0)$, we are considering the equivalence classes of holes, corresponding to 5-cycles in Asc_n, but we are able to show that although there are $\binom{n+3}{5}$ equivalence classes, we may recover all of the equivalence classes of 5-cycles using only a set of $\binom{n+2}{4}$ equivalence classes. This leads to the main result of Section 4.

Theorem 1.1 *The abelianization of $A_1^{n-2}(\mathcal{T}_n, T_0)$ is a free abelian group of rank $\binom{n+2}{4}$.*

Although the classical fundamental group of a convex polytope is always trivial, the discrete fundamental group is not, and seems to provide an indication of the complexity of the polytope as compared to the n-simplex. In [10], the authors provide an excellent view of the associahedron as a a truncation of the n-simplex and the permutahedron as a truncation of the associahedron. At each step in the truncation process, the number of generators of the abelianization of the discrete fundamental group increases, going from trivial in the case of the n-simplex to $\binom{n+2}{4}$ for the associahedron and $2^{n-3}(n^2 - 5n + 8) - 1$ for the permutahedron.

In Section 2 we establish a labeling scheme for edges of Asc_n and a set of words whose letters are the labels of edges in Asc_n. It is shown in [1] that loops based at T_0 are in one-to-one correspondence with closed walks in $\Gamma^{n-2}(\mathcal{T}_n)$ based at T_0 (we abuse notation here and use T_0 to refer to both a maximal simplex of $\mathcal{T}_n$ and a vertex of $\Gamma^{n-2}(\mathcal{T}_n)$). Thus we may think of $A_1^{n-2}(\mathcal{T}_n, T_0)$ both as the group on equivalence classes of loops and as a group on equivalence classes of based walks in Asc_n. This will allow us to work entirely with closed walks in Asc_n and words constructed from those walks.

In Section 3 we give a generating set for the group $A_1^{n-2}(\mathcal{T}_n, T_0)$ using a description of the classical fundamental group in terms of the cycles that bound 2-faces in the associahedron. This approach is informative in that it gives us a combinatorial description of the generating set for $A_1^{n-2}(\mathcal{T}_n, T_0)$.

Section 4 contains the proof of Theorem 1.1 as well as a series of lemmas needed to prove the result. We have omitted many of the proofs of these lemmas due to space considerations, however many of them follow easily from the definitions and ideas in Section 2. In proving Theorem 1.1 we give a simple combinatorial description of the generators.

Finally, we conclude in Section 5 with a description of applications and two directions for future study.

Due to space considerations we have omitted some details of proofs and background, however all of the material here appears in full detail in the first author's PhD thesis ([13]).

2 Properties of Asc_n

As noted in the introduction, the associahedron has been very well studied. For an overview of the basic properties and facts of this object we refer the reader to the list of references presented in the introduction. In this section we establish a labeling scheme for the edges of Asc_n. Using this new labeling scheme as an alphabet, we are able to translate a loop of simplices, or a walk in Asc_n, to a word. The use of words makes our proofs in the following sections more clear and gives an algebraic framework for our discussions of discrete homotopy theory.

We also look more closely at the equivalence relation $\simeq_A$, giving a shorter description as in [3]. Due to the lack of triangles in Asc_n, we may write any discrete homotopy between loops as a series of three fundamental changes to the loop; stretching at a simplex, inserting a new simplex, and commuting two simplices.

We begin with the edge labels for Asc_n. Fix a regular $(n+3)$-gon and label the vertices clockwise in order with $1, \ldots, n+3$. Recall that an edge in Asc_n corresponds to changing one diagonal between two triangulations of an $(n+3)$-gon, or a diagonal flip. We may use this flip to label the edge in a distinct way.

Definition 2.1 *Let e be an edge in Asc_n which corresponds to changing the diagonal ac to the diagonal bd. Define the label set of e, $L(e)$ to be the set $\{a, b, c, d\}$, where a, b, c, d are elements of $\{1, \ldots, n+3\}$ corresponding to the vertices of the $(n+3)$-gon.*

Note that while the every edge has exactly one label set, many edges may share the same label set.

We also may derive the label for an edge by considering its corresponding simplex in $\mathcal{T}_n$. Edges in Asc_n correspond to a simplex S with $n-1$ diagonals, so we may take the $(n+3)$-gon and add all of the diagonals in S. When we have added all of the diagonals in S we have one region inside the $(n+3)$-gon which has not been triangulated. This region is a quadrilateral and the vertices that bound it are exactly the label set of the edge corresponding to S. This method of determining the label set is very easy to understand with an illustration, so we have provided the graph Asc_2 with the triangulated 5-gon corresponding to each vertex, and each edge labeled with our scheme in Figure 1. Observe that each diagonal flip occurs inside a fixed quadrilateral and we may read off the edge label from the vertices of that quadrilateral.

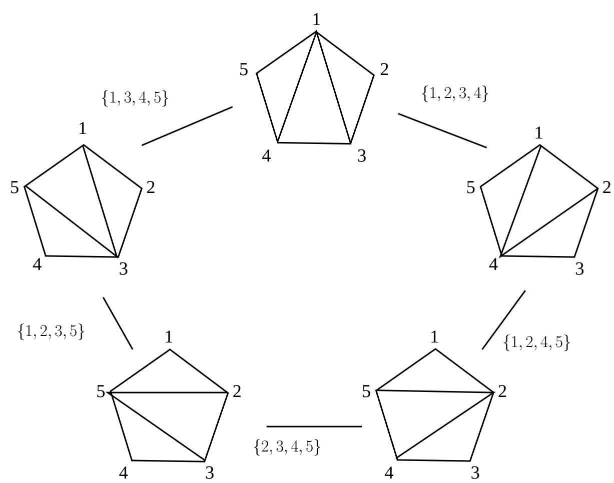

Fig. 1: The graph Asc_2 with vertices as triangulations of a regular 5-gon and edges labeled.

Just as we have label sets for the edges of Asc_n, we also introduce the notion of basic cycle label sets. The basic cycle label is a natural extension of the edge label obtained in a very similar manner.

Definition 2.2 *Let C be a basic cycle in Asc_n and let e and f be two edges on C, with $L(e) \neq L(f)$. Define the label set of C, $L(C)$ to be the set $L(e) \cup L(f)$.*

We also have an intuitive way to see the label set of a basic cycle given its corresponding partial triangulation of an $(n+3)$-gon. As in the case of edge label sets, we consider the simplex S that corresponds to a basic cycle. This simplex has $(n-2)$ diagonals and so when we add these diagonals to the $(n+3)$-gon we have a partial triangulation. Each missing diagonal gives us a quadrilateral region inside the $(n+3)$-gon. The boundary vertices of these two regions give us the label set of the cycle. An illustration of the regions corresponding to a basic 4-cycle and a basic 5-cycle can be seen in Figure 2.

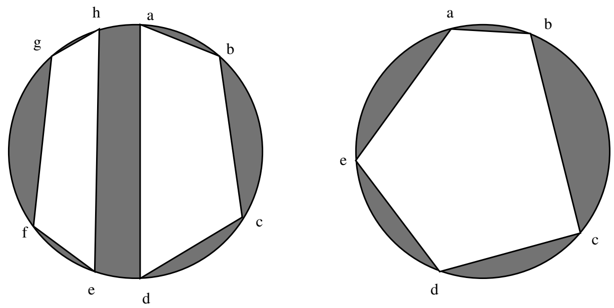

Fig. 2: Regions inside a regular $(n+3)$-gon corresponding to a basic 4-cycle and basic 5-cycle respectively. Shading indicates a region is triangulated.

In the case that the two regions have interiors that overlap it must be the case that they share three boundary vertices and hence their intersection is a pentagon. As seen in Figure 1, there are five ways to triangulate a pentagon and so we obtain a corresponding basic 5-cycle in Asc_n. If the regions do not have intersecting interiors then it is easy to see we may triangulate them each in two different ways, giving us the four vertices of a basic 4-cycle.

If a basic cycle is a 4-cycle then the opposite edges on the cycle must have the same label set, since we are performing two separate flips in sequence and then performing the exact same flips in the same sequence again, in effect undoing them. If a basic cycle is a 5-cycle, then the edges of the cycle must have distinct label sets. This can be observed in Figure 1. In fact, given a basic 5-cycle we know even more about the label sets of the edges.

Proposition 2.3 *At least one of the edges of a basic 5-cycle has a label set that does not contain the element 1.*

Proof: Let C be a basic 5-cycle with label set $L(C)$. Then if $e_1, \ldots, e_5$ are the edges of C, the label sets $L(e_1), \ldots, L(e_5)$ are the five subsets of $L(C)$ of size 4. If $1 \in L(C)$ then one subset of $L(C)$ of size 4 must not contain 1 (If $1 \notin L(C)$ it is clear no subset of size 4 will contain 1). $\Box$

Now that we have established labels for the edges of Asc_n, we may think of elements of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ in terms of words on the alphabet of edge labels. Due to the fact that the elements of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ are closed walks based at T_0, we have a very easy way to write down the corresponding words.

Definition 2.4 *Let $W = e_1 \ldots e_n$ be a closed walk in Asc_n based at vertex T_0. The word for the walk W is $w = L(e_1) \cdots L(e_n)$.*

Although we still consider elements of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ primarily as closed walks based at T_0, it will be useful to operate on the words corresponding to the walks. To do this we must establish how the relation $\simeq_A$ affects words. We make use of a result from [3] that because Asc_n is triangle free, there are only 3 possible changes we may make to walks and words that will preserve $\simeq_A$. We list the three changes (T1)-(T3) briefly and refer the reader to the reference above for further information due to space considerations.

- **(T1) Stretch.** We may stretch a loop by repeating a vertex one or more times.

$$\ell = T_0 - \cdots - T_i - \cdots - T_0 \simeq T_0 - \cdots - T_i - T_i - \cdots - T_0$$

In Asc_n we have not traversed any new edges so the walk stays the same. We may think of this operation as holding at a vertex. This operation also does not change a word since there is no new edge label added.

- **(T2) Insertion.** This change consists of inserting a new simplex in a loop. Suppose we have already stretched at T_i and suppose T_j and T_i are near. Then

$$\ell = T_0 - \cdots - T_i - T_i - \cdots - T_0 \simeq T_0 - \cdots - T_i - T_j - T_i - \cdots - T_0.$$

In Asc_n this change corresponds to traversing an edge e from T_i to a new vertex T_j, then traversing e in the opposite direction to return to the original walk. In the word corresponding to the walk we have added the letter $L(e)$ twice.

- **(T3) Switch.** Let $\ell = T_0 - \cdots - T_{i-1} - T_i - T_{i+1} - T_{i+1} - \cdots - T_0$ be a loop and let T_j be near to both T_{i-1} and T_{i+1}. Then we may switch T_j for T_i and have

$$\begin{aligned}\ell = T_0 - \cdots - T_{i-1} - T_i - T_{i+1} - T_{i+1} - \cdots - T_0 \\ \simeq T_0 - \cdots - T_{i-1} - T_{i-1} - T_j - T_{i+1} - \cdots - T_0.\end{aligned}$$

In Asc_n we have a 4-cycle $T_{i-1}, T_i, T_{i+1}, T_j$ with the edges e_1, e_2, e_3, e_4 respectively, and we change the walk from traversing edges e_1, e_2 to edges e_4, e_3. Recall that the opposite edges in a 4-cycle have the same label set, hence $L(e_1) = L(e_3)$ $L(e_2) = L(e_4)$. Thus in the word corresponding to the walk, we have commuted the letters $L(e_1)$ and $L(e_2)$.

Remark 2.5 *The change (T3) tells us that two letters commute if their associated edge label sets are adjacent on some 4-cycle in Asc_n. Recall that the label set of an edge e, $L(e)$, gives the boundary vertices of a quadrilateral region inside an $(n+3)$-gon. Thus, given a letter $L(e)$, we may commute it with any letter $L(f)$ as long as the region inside an $(n+3)$-gon bounded by the elements of $L(e)$ does not intersect the region bounded by the elements of $L(f)$. This implies that $|L(e) \cup L(f)| \leq 2$, but it is not a sufficient condition for $L(e)$ and $L(f)$ to commute as letters.*

Another important fact about (T2)-(T3) concerns their effect on the parity of letters in a word w.

Remark 2.6 *The changes (T2)-(T3) preserve the parity of letters in a word.*

It should also be noted that we may apply the relation $\simeq_A$ to paths in Asc_n that have the same start and end point. This is equivalent to comparing two chains of simplices that have the same start and end simplices. All of the changes (T1)-(T3) make sense in this case and the underlying idea of having a grid between the two chains works the same. We use this type of comparison of paths in Section 4.

3 A Description of $A_1^{n-2}(\mathfrak{T}_n, T_0)$

We now give a description of $A_1^{n-2}(\mathfrak{T}_n, T_0)$. Though we are primarily concerned in this abstract with the abelianization of $A_1^{n-2}(\mathfrak{T}_n, T_0)$, we feel that exposing the structure of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ is still a rewarding exercise in and of itself. We omit the majority of the proofs due to space considerations but provide a sketch of our main result as it gives us insight into the generators of $A_1^{n-2}(\mathfrak{T}_n, T_0)$..

In [7] the authors show that the classical fundamental group of the 1-skeleton of the associahedron is generated by all of the basic 4- and 5-cycles, pinned down to a base point. This result follows from two theorems in Massey ([11]). We note that the theorems in Massey allow us to choose the paths from the base point to a basic cycle and we make use of this to choose paths such that any two cycles with the same label will have corresponding loops that are homotopic.

Given a basic cycle label class, we fix a representative C of that class by fixing a partial triangulation such that all of the diagonals outside of the embedded pentaton are connected to the smallest labeled vertex in the region of the $(n+3)$-gon that they are in. We then fix a path P from the base vertex v_0 to the basic cycle C. Now, for every other cycle C' with the same label we fix a path PQ to the cycle, such that P is the path from v_0 to C and Q is a path from C to C'. Such a path exists by Lemma 4.1. Also, we may use Theorem 4.3 to conclude that any two loops that use basic cycles with the same label set are homotopic.

Theorem 3.1 *A generating set of $A_1^{n-2}(\mathfrak{T}_n, T_0)$ is given by $\{PCP^{-1}\}$, where C ranges over all fixed representatives of label set equivalence classes such that 1 is in the label set. P is as described above. There are $\binom{n+2}{4}$ such loops.*

Proof: This result follows from the description of the classical homotopy group of Asc_n given above, the relationship between $\pi_1(Asc_n)$ and $A_1^{n-2}(\mathfrak{T}_n, T_0)$, and Theorem 4.3. Full details are provided in [13]. □

We note that we have not given a nice description of the relations between the generators here. Doing so is much more complicated and loses some of the elegance of this description, however in [13] we do give a full presentation of $A_1^{n-2}(\mathfrak{T}_n, T_0)$.

4 The abelianization of $A_1^{n-2}(\mathfrak{T}_n, T_0)$

Just as in [3], in order to find $A_1^{n-2}(\mathfrak{T}_n, T_0)^{ab}$ we must count the equivalence classes of basic cycles under the relation $\simeq_A$. Recall that Asc_n has only basic 4- and 5-cycles, and that under $\simeq_A$ 4-cycles are contractible, so our goal may be reduced to counting the equivalence classes of basic 5-cycles in Asc_n. In the case of Asc_3, which is shown in Figure 3, we can see that there are six basic 5-cycles, however we know from a simple computation that we may write the outside basic 5-cycle as a product of those inside.

It does not suffice to count the classes of basic 5-cycles; we must also provide a minimal generating set. It turns out that there is a very simple combinatorial description of the equivalence classes of basic 5-cycles using the cycle labels introduced in Section 2, and that a minimal generating set for $A_1^{n-2}(\mathfrak{T}_n, T_0)^{ab}$ contains only the equivalence classes of basic 5-cycles whose label set contain 1.

We start with some results needed to prove Theorem 1.1.

Lemma 4.1 *Let $C = e_1, \ldots, e_5$ and $C' = e'_1, \ldots, e'_5$ be two basic 5-cycles with $L(C) = L(C')$. Then there is a series of edges $p_1, \ldots, p_k$ in Asc_n between e_i and e'_i such that $L(p_j)$ (taken as letters in a word) commute with $L(e_i)$ and $L(e'_i)$ for every i and j.*

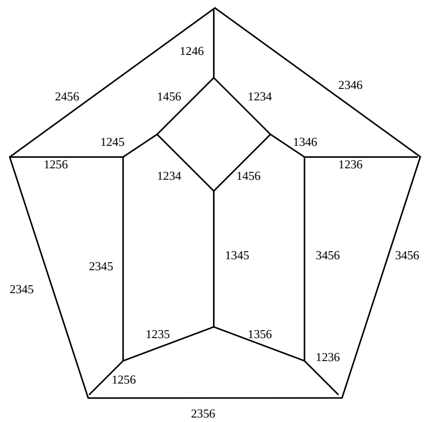

Fig. 3: Asc_3 with edge labels.

Proof: This result is obtained by flipping diagonals outside of the pentagon region given by the label set of both C and C' in the partially triangulated $(n+3)$-gon. We have omitted the full details but they are available in [13]. □

Lemma 4.2 *Given two basic 5-cycles $C = e_1, \ldots, e_5$ and $C' = e'_1, \ldots, e'_5$ in Asc_n, if $L(C) \neq L(C')$ then, there is at most one pair, e_i, e'_j such that $L(e_i) = L(e'_j)$.*

Proof: This result follows very easily from Definition 2.2. Full details are available in [13]. □

We are now ready to provide a necessary and sufficient condition for two basic 5-cycles to be equivalent under $\simeq_A$.

Theorem 4.3 *Let C and C' be basic 5-cycles in Asc_n. Then $L(C) = L(C)'$ if and only if $C \simeq_A C'$.*

Proof: We keep this proof in its entirety as we feel that it provides a method to visualize the homotopies between basic 5-cycles in Asc_n.

We start by showing that if $L(C) = L(C')$, then $C \simeq_A C'$. By Lemma 4.1 we know there is a sequence of edges between C and C' whose associated letters commute with the letters of C and C'. Let C have associated word $w = L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)$. Using changes (T2) and (T3) we can inductively construct a new word w' which is equivalent to w and has associated 5-cycle C'.

Suppose the sequence of edges is length 1 with associated letter $L(x)$. We change w as follows:

$$
\begin{aligned}
L(e_1)L(e_2)L(e_3)L(e_4)L(e_5) &\simeq_A L(e_1)L(e_2)L(e_3)L(x)L(x)L(e_4)L(e_5) && (T2) && (1)\\
&\simeq_A L(e_1)L(e_2)L(x)L(e_3)L(e_4)L(x)L(e_5) && (T3) && (2)\\
&\simeq_A L(e_1)L(x)L(e_2)L(e_3)L(e_4)L(e_5)L(x) && (T3) && (3)\\
&\simeq_A L(x)L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)L(x) && (T3). && (4)
\end{aligned}
$$

Now suppose the sequence is of length k with associated letters $L(x_1), \ldots, L(x_k)$ and assume that the hypothesis holds for a sequence of length $k-1$. Then we use the hypothesis to insert letters $L(x_1), \ldots, L(x_{k-1})$ and commute them so we have a word

$$(L(x_1) \cdots L(x_{k-1}))L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)(L(x_1) \cdots L(x_{k-1}))^{-1} \simeq_A w$$

Using the same argument above, we can then insert $L(x_k)$ and obtain a new equivalent word $w' = (L(x_1) \cdots L(x_k))L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)(x_1 \cdots x_k)^{-1}$. This new word corresponds to a path that goes around C' and is equivalent to the path around C.

What we are doing is forming a net of basic 4-cycles between the two basic 5-cycles with the same label set. At each step on the path we have a new basic 5-cycle with the same label set and the homotopy relation can be read off immediately.

For the other direction we proceed by contradiction. Assume that we have two 5-cycles, C and C' that do not have the same label set.

Let $w = L(e_1)L(e_2)L(e_3)L(e_4)L(e_5)$ and $w' = L(e_1')L(e_2')L(e_3')L(e_4')L(e_5')$ be the words associated to C and C'. By Lemma 4.2, if $L(C) \neq L(C')$ then they share at most one edge label, and hence their words share at most one letter. However, by Remark 2.6, we are unable to change the parity of letters by using (T1), (T2) and (T3), so w has a letter of odd parity that can only have even parity in w'. This implies we cannot change w to w' using (T1), (T2) and (T3), which contradicts $C \simeq_A C'$.

□

Now that we have established that the equivalence classes of basic 5-cycles are in bijection with the label sets we may count the equivalence classes easily. We obtain a label set for a basic 5-cycle by choosing five vertices on the $(n+3)$-gon to form a pentagon region and then we may triangulate all regions outside that pentagon to arrive at a specific cycle with that label set. There are $\binom{n+3}{5}$ label sets for basic 5-cycles and thus $\binom{n+3}{5}$ equivalence classes of basic 5-cycles. If we stipulate that a label set must contain the element 1, then we have $\binom{n+2}{4}$ equivalence classes of 5-cycles with 1 in their label set, and now we show that in fact any equivalence class of basic 5-cycles without 1 in its label set can be written as a product of those that do have 1 in their label set.

We first consider Asc_3 (Figure 3). It is easy to show (though we do not do so here due to space considerations) that we can write the outside 5-cycle of Asc_3 as a product homotopic to the product the inner 5-cycles. Having done this for Asc_3, we show that there is an isomorphic copy of Asc_3 in Asc_n for $n > 3$ and so we may reduce to that case for all n.

Due to the structure of Asc_n, we can find an isomorphic copy of Asc_3 where the outside basic 5-cycle can have any label set that does not include 1. The basic 5-cycles inside still have 1 in their label sets and in this way we may write any basic 5-cycle without 1 in its label set as a product of basic 5-cycles that have 1 in their label sets.

Lemma 4.4 *Any basic 5-cycle C in Asc_n such that $1 \notin L(C)$ can be written as a product of basic 5-cycles whose label set do contain 1.*

Proof: The result follows easily however since we may create a copy of Asc_3 inside Asc_n with the desired labels by choosing an appropriate hexagon region inside the $(n+3)$-gon and triangulating it in all possible ways. Full details available in [13]. □

We now know that any basic 5-cycle whose label set does not contain 1 may be written as a product of those that do contain 1, so we need at most $\binom{n+2}{4}$ classes of basic 5-cycles to generate $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$. In fact, we cannot reduce the number of generators below $\binom{n+2}{4}$.

Lemma 4.5 *The $\binom{n+2}{4}$ equivalence classes of basic 5-cycles whose label set contain 1 is a minimal generating set for $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$.*

Proof: This result follows similarly to the latter half of the proof of Theorem 4.3. □

We have a minimal generating set for $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$ and we can see that there are no relations between the generators outside of commutivity. Theorem 1.1 follows; that is, $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$ is free abelian and of rank $\binom{n+2}{4}$.

5 Applications and Future Directions

We have provided a study of the discrete fundamental group of the complex $\mathcal{T}_n$, and now we give a sketch of the applications of this study as well as two areas for future research.

We first consider an application to cluster algebras. It is well known that the complex $\mathcal{T}_n$ is a cluster complex and its exchange graph is Asc_n [8]. In the same paper, it is also noted that the first szygy module of the cluster algebra is generated by all of the edges of Asc_n. It is easy then to see that $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$ is giving us a quotient of the second szygy module of the cluster algebra. The basic 5-cycles that generate this module correspond exactly to occurences of the pentagon recurrence noted in [8] and in taking a quotient by the 4-cycles we are removing any basic cycles that do not correspond to this pentagon recurrence. The equivalence classes of $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$ correspond exactly to equivalences in the pentagon reccurence as well. That is, two cycles are homotopic if and only if they give the same recurrence. By studying $A_1^{n-2}(\mathcal{T}_n, T_0)^{ab}$, we are able to identify all the ways the pentagon recurrence occurs in the cluster algebra and classify them combinatorially.

A second application involves what we will define as a *conic arrangement*. Let $\mathcal{F}$ be the normal fan of the associahedron and $\mathcal{C} \subset \mathcal{F}$ be the set of cones corresponding to the 2-faces bounded by basic 5-cycles. Then, in a similar fashion to [3], we are able to show a link between $A_1^{n-2}(\mathcal{T}_n, T_0)$ and the classical fundamental group of the topological space obtained by removing all of the cones in $\mathcal{C}$ from $\mathbb{R}^n$.

Finally, we are led to two natural expansions of our study of $A_1^{n-2}(\mathcal{T}_n, T_0)$. First, we note that the associahedron is a graph associahedra. In [5], [12] and [16] the authors give methods of constructing and realizing graph associahedra as well as many of their properties. Our initial investigations suggest that many of the results obtained in the study of $A_1^{n-2}(\mathcal{T}_n, T_0)$ may be applied to this generalization.

A second expansion brings us back to cluster algebras. In [13], we study the discrete fundamental group of the cluster complexes of type B and D. These complexes have combinatorial descriptions similar to those of the associahedron and provide similar insight into the syzygy modules of the corresponding cluster algebras.

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A further correspondence between $(bc, \bar{b})$*-parking functions and* $(bc, \bar{b})$*-forests*

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Abstract. For a fixed sequence of n positive integers $(a, \bar{b}) := (a, b, b, \ldots, b)$, an $(a, \bar{b})$-parking function of length n is a sequence $(p_1, p_2, \ldots, p_n)$ of positive integers whose nondecreasing rearrangement $q_1 \leq q_2 \leq \cdots \leq q_n$ satisfies $q_i \leq a + (i-1)b$ for any $i = 1, \ldots, n$. A $(a, \bar{b})$-forest on n-set is a rooted vertex-colored forests on n-set whose roots are colored with the colors $0, 1, \ldots, a-1$ and the other vertices are colored with the colors $0, 1, \ldots, b-1$. In this paper, we construct a bijection between $(bc, \bar{b})$-parking functions of length n and $(bc, \bar{b})$-forests on n-set with some interesting properties. As applications, we obtain a generalization of Gessel and Seo's result about $(c, \bar{1})$-parking functions [Ira M. Gessel and Seunghyun Seo, Electron. J. Combin. **11**(2)R27, 2004] and a refinement of Yan's identity [Catherine H. Yan, Adv. Appl. Math. **27**(2–3):641–670, 2001] between an inversion enumerator for $(bc, \bar{b})$-forests and a complement enumerator for $(bc, \bar{b})$-parking functions.

Résumé. Soit $(a, \bar{b}) := (a, b, b, \ldots, b)$ une suite d'entiers positifs. Une $(a, \bar{b})$-fonction de parking est une suite $(p_1, p_2, \ldots, p_n)$ d'entiers positives telle que son réarrangement non décroissant $q_1 \leq q_2 \leq \cdots \leq q_n$ satisfait $q_i \leq a + (i-1)b$ pour tout $i = 1, \ldots, n$. Une $(a, \bar{b})$-forêt enracinée sur un n-ensemble est une forêt enracinée dont les racines sont colorées avec les couleurs $0, 1, \ldots, a-1$ et les autres sommets sont colorés avec les couleurs $0, 1, \ldots, b-1$. Dans cet article, on construit une bijection entre $(bc, \bar{b})$-fonctions de parking et $(bc, \bar{b})$-forêts avec des des propriétés intéressantes. Comme applications, on obtient une généralisation d'un résultat de Gessel-Seo sur $(c, \bar{1})$-fonctions de parking [Ira M. Gessel and Seunghyun Seo, Electron. J. Combin. **11**(2)R27, 2004] et une extension de l'identité de Yan [Catherine H. Yan, Adv. Appl. Math. **27**(2–3):641–670, 2001] entre l'énumérateur d'inversion de $(bc, \bar{b})$-forêts et l'énumérateur complémentaire de $(bc, \bar{b})$-fonctions de parking.

Keywords: Bijection, Forests, Parking functions

1 Introduction

It is well-known [Sta99] that parking functions and (rooted) forests on n-set are both counted by Cayley's formula $(n+1)^{n-1}$. Foata and Riordan [FR74] gave the first bijection between these two equinumerous sets. In the past years, many generalizations and refinements of this result were obtained (See [MR68,

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Kre80, Yan01, SP02, KY03, GS06]). In particular, Stanley and Pitman [SP02] introduced the notion of $(a,\bar{b})$-parking functions where a and b are two positive integers.

Recall that an $(a,\bar{b})$-*parking function (of length n)* (see [SP02]) is a sequence $(p_1, p_2, \ldots, p_n)$ of positive integers whose nondecreasing rearrangement $q_1 \le q_2 \le \cdots \le q_n$ satisfies $q_i \le a + (i-1)b$ for $1 \le i \le n$. It is shown [SP02] that the number of $(a,\bar{b})$-parking functions is

$$a(a+bn)^{n-1}.$$

Looking for its forest counter parts, Yan [Yan01] defined a *(rooted)* $(a,\bar{b})$-*forest* (see section 2.2) to be a vertex-colored forest in which all roots are colored with the colors $0, 1, \ldots, a-1$ and the other vertices are colored with the colors $0, 1, \ldots, b-1$. She proved that the enumerator $\bar{P}_n^{(a,\bar{b})}(q)$ of complements of $(a,\bar{b})$-parking functions and the enumerator $I_n^{(a,\bar{b})}(q)$ of $(a,\bar{b})$-forests by the number of their inversions are identical, i.e.,

$$I_n^{(a,\bar{b})}(q) = \bar{P}_n^{(a,\bar{b})}(q). \tag{1}$$

It is an open problem to give a bijective proof of the identity (1). Generalizing a bijection of Foata and Riordan [FR74], Yan [Yan01] did give a bijection between $(a,\bar{b})$-forests and $(a,\bar{b})$-parking functions which is a bijective proof of (1) for $q=1$, but this bijection does not keep track of the statistics involved in (1) even in ordinary $a=b=1$ case. Note that Eu et al. [EFL05] were able to extend the bijection of Foata and Riordan to enumerate $(a,\bar{b})$-parking functions by their leading terms. Recently, Shin [Shi08] gave a bijective proof of (1) when $a=b=1$.

A different refinement of Cayley's formula was given by Gessel and Seo [GS06]. Using generating functions, they showed that the enumerator of forests with respect to proper vertices and the number of trees and the lucky enumerator of $(a,\bar{1})$-parking function are both equal to

$$au\prod_{i=1}^{n-1}(i+u(n-i+a)).$$

Bijective proof of above results for $a=1$ have been given by Seo and Shin [SS07] and Shin [Shi08].

In this paper, we prove three main results. First, in Theorem 1, we establish a bijection between $(bc,\bar{b})$-parking functions and $(bc,\bar{b})$-forests, which is a generalization of the first author's recent bijection [Shi08]. Secondly, in Theorem 4, we generalize the aforementioned formula of Gessel and Seo to $(bc,\bar{b})$ case. Finally, in Theorem 5, we extend Gessel and Seo's hook-length formula [GS06, Corollay 6.3] for forests to $(a,\bar{b})$-forests.

The rest of this paper is organized as follows: In Section 2, we introduce definitions of various statistics on general parking functions and forests. The main theorems of this paper are presented in Section 3. The proofs of main theorems are given in Sections 4, 5, 6.

2 Definitions

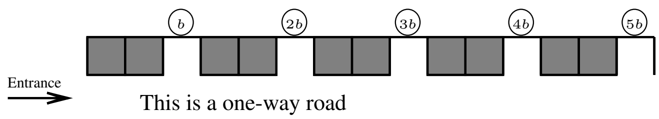
2.1 Statistics on $(bc,\bar{b})$-parking functions

From now on, we fix $a=bc$. We define a parking algorithm for $(bc,\bar{b})$-parking functions by generalizing algorithm in [GS06] for $(c,\bar{1})$-parking functions. Suppose that there are $1, 2, \ldots, (n+c-1)b$ parking lots with only $n+c-1$ available parking spaces at $b, 2b, \ldots, (n+c-1)b$, that means the positions are multiples of b.

Parking Space	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
Cars' Number, c	$\square$	$\square$	3	$\square$	$\square$	1	$\square$	$\square$	5	$\square$	$\square$	$\emptyset$	$\square$	$\square$	4	$\square$	$\square$	2	
$\mathrm{jump}(P;c)$			$\boxed{0}$			$\boxed{1}$			$\boxed{7}$						$\boxed{0}$			$\boxed{2}$	$\mathrm{jump}(P)=10$
$\mathrm{block}(P;c)$			$\boxed{1}$			$\boxed{1}$			$\boxed{1}$						$\boxed{0}$			$\boxed{0}$	$\mathrm{block}(P)=3$
$\mathrm{jump}_{(6,\bar{3})}(P;c)$			$\boxed{3}$			$\boxed{4}$			$\boxed{10}$						$\boxed{0}$			$\boxed{2}$	$\mathrm{jump}_{(6,\bar{3})}(P)=19$
lucky car			✓												✓				$\mathrm{lucky}(P)=2$
critical car									✓						✓			✓	$\mathrm{crit}(P)=3$

$$\mathbf{JUMP}_{(6,\bar{3})}(P)=\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

Fig. 1: A $(bc,\bar{b})$-parking function $P=(5,16,3,15,2)$ of length 5 and statistics of P for $b=3$, $c=2$ where circled numbers are available parking spaces



Given a $(bc,\bar{b})$-parking function $P=(p_1,p_2,\ldots,p_n)$ of length n, suppose that Cars $1,2,\ldots,n$ come to the parking lots in this order and car i prefers parking space p_i. We can park the n cars with $n+c-1$ parking spaces by the following *parking algorithm*: If p_i is occupied or non-available, then car i takes the next available space. If q_i be the actual parking space with i-th car for $i=1,\ldots,n$, we define

$$\mathrm{park}(p_1,\ldots,p_n)=(q_1,\ldots,q_n).$$

In Figure 1, we give an example of a $(bc,\bar{b})$-parking function $(5,16,3,15,2)$ for $b=3$ and $c=2$. By the Parking Algorithm, we get a sequence with length 5,

$$\mathrm{park}(5,16,3,15,2)=(6,18,3,15,9).$$

The difference between the favorite parking space p_i and the actual parking space q_i is called the *jump* of car i, and denoted by $\mathrm{jump}(P;i)$, that is,

$$\mathrm{jump}(P;i)=q_i-p_i \quad \text{if} \quad \mathrm{park}(p_1,\ldots,p_n)=(q_1,\ldots,q_n).$$

Let $\mathrm{jump}(P)$ denote the sum of the jumps of P, that is,

$$\mathrm{jump}(P)=\sum_i \mathrm{jump}(P;i).$$

Clearly $\mathrm{jump}(P;i)\geq 0$. We say that car i is *lucky* if $\mathrm{jump}(P;i)=0$. Denote the number of lucky cars of P by $\mathrm{lucky}(P)$.

After parking all the n cars, there are $c-1$ non-occupied parking spaces which divide the parking lots into c *blocks* of parking lots. Let $\mathrm{block}(P;i)$ be the number of non-occupied parking spaces on the right of car i after running parking algorithm. Let $\mathrm{block}(P)$ denote the sum of blocks of all cars, i.e., $\mathrm{block}(P) = \sum_i \mathrm{block}(P;i)$. We define $(bc,\bar{b})$-jump of $(bc,\bar{b})$-parking function

$$\begin{aligned}\mathrm{jump}_{(bc,\bar{b})}(P;i) &= \mathrm{jump}(P;i) + b\cdot \mathrm{block}(P;i),\\ \mathrm{jump}_{(bc,\bar{b})}(P) &= \mathrm{jump}(P) + b\cdot \mathrm{block}(P) = bcn + \binom{n}{2}b - |P|,\end{aligned}$$

where $|P| = \sum p_i$. Note that $(bc,\bar{b})$-jump is identical to the complement of $|P|$ in [Yan01].

Let $\mathrm{lucky}_{j,k}(P)$ denote the number of cars i such that $\mathrm{block}(P;i) = j$ and $\mathrm{jump}(P;i) = k$. We define the multi-statistic $\mathbf{JUMP}_{(bc,\bar{b})}$ by

$$\mathbf{JUMP}_{(bc,\bar{b})}(P) = \begin{pmatrix} \mathrm{lucky}_{0,0}(P) & \mathrm{lucky}_{0,1}(P) & \cdots & \mathrm{lucky}_{0,N}(P)\\ \mathrm{lucky}_{1,0}(P) & \mathrm{lucky}_{1,1}(P) & \cdots & \mathrm{lucky}_{1,N}(P)\\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{lucky}_{c-1,0}(P) & \mathrm{lucky}_{c-1,1}(P) & \cdots & \mathrm{lucky}_{c-1,N}(P)\end{pmatrix},$$

where $N = \binom{n+1}{2}b - n$.

A car c is called *critical* if there are only former cars parked on the right of the block containing c after parking. If car c is critical in a $(bc,\bar{b})$-parking function P, $\mathrm{crit}(P;c) = 1$. Otherwise, $\mathrm{crit}(P;c) = 0$. Denote the number of critical cars in a $(bc,\bar{b})$-parking function P by $\mathrm{crit}(P)$.

As an example, a $(bc,\bar{b})$-parking function is given in Figure 1 for $b = 3$ and $c = 2$ in order to illustrate different statistics.

2.2 Statistics on $(bc,\bar{b})$-Forests

A *(rooted) forest* is a simple graph on $[n] = \{1,\dots,n\}$ without cycles, whose every connected component has a distinguished vertex, called a *root*. A *(rooted)* $(a,\bar{b})$*-forest* on $[n]$ is a pair (F,κ) where F is a forest on $[n]$, κ is a mapping from the set of vertices in F to non-negative integers such that $\kappa(v) < a$ if v is a root and $\kappa(v) < b$, otherwise.

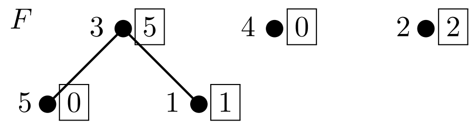
In a rooted forest F, a vertex j is called a *descendant* of a vertex i if the vertex i lies on the unique path from the root to the vertex j. In particular, every vertex is a descendant of itself. Denote the set of descendants of a vertex v by $D_F(v)$. The *hook-length* h_v of v is defined by the number of descendants of v in a forest. A vertex v is a parent of u if v and u are connected by one edge and u is a descendant of v.

As defined by Mallows and Riordan [MR68], an *inversion* in a rooted forest is an ordered pair (i,j) such that $i > j$ and j is a descendant of i. Let $\mathrm{Inv}(F;v)$ denote the set of ordered pairs (v,x) such that $v > x$ and $x \in D_F(v)$. Denote the number of all inversions in a rooted forest F by $\mathrm{inv}(F)$. We need to generalize the notion of inversions to $(bc,\bar{b})$-forests as follows: Let $\bar{\kappa}(v)$ denote the remainder of $\kappa(v)$ modulo b, i.e.,

$$\kappa(v) \equiv \bar{\kappa}(v) \mod b \quad \text{with } 0 \le \bar{\kappa}(v) \le b-1.$$

Define the inversion $\mathrm{inv}(F;v)$ of a $(bc,\bar{b})$-forest F by

$$\mathrm{inv}(F;v) = |\mathrm{Inv}(F;v)| + \bar{\kappa}(v)\cdot |D_F(v)|.$$



Vertex v	1	2	3	4	5	
$\kappa(v)$	1	2	5	0	0	
$\lvert\mathrm{Inv}(F;v)\rvert$	0	0	1	0	0	
$\bar{\kappa}(v)$	1	2	2	0	0	
$\lvert D_F(v)\rvert$	1	1	3	1	1	
$\mathrm{inv}(F;v)$	1	2	7	0	0	$\mathrm{inv}(F)=10$
$\mathrm{tcol}(F;v)$	1	0	1	0	1	$\mathrm{tcol}(F)=3$
$\mathrm{inv}_{(6,\bar{3})}(F;v)$	4	2	10	0	3	$\mathrm{inv}_{(6,\bar{3})}(F)=19$
proper vertex				✓	✓	$\mathrm{prop}(F)=2$
root		✓	✓	✓		$\mathrm{tree}(F)=3$

$$\mathbf{INV}_{(6,\bar{3})}(F)=\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

Fig. 2: A $(bc,\bar{b})$-forest F on $[5]$ and statistics of F for $b=3$, $c=2$ where $\kappa(v)$ is boxed

Let $\mathrm{inv}(F)$ denote the sum of $\mathrm{inv}(F;v)$ over all vertices v of F, i.e.,

$$\mathrm{inv}(F)=\sum_v \mathrm{inv}(F;v).$$

Given a $(bc,\bar{b})$-forest F, a vertex v is called a *proper vertex* if the vertex v is the smallest among all its descendants and its color is a multiple of b, that is, $\mathrm{inv}(F;v)=0$. Let $\mathrm{prop}(F)$ denote the number of all proper vertices in a rooted forest F. By definition, every leaf v with $\bar{\kappa}(v)=0$ is a proper vertex.

Denote the root of the tree including a vertex v in an $(bc,\bar{b})$-forest F by $R(v)$. A *tree-color* $\mathrm{tcol}(F;v)$ of a vertex v in a $(bc,\bar{b})$-forest F is defined by $\mathrm{tcol}(F;v)=\lfloor\frac{\kappa(R(v))}{b}\rfloor$. Let $\mathrm{tcol}(F)$ denotes the sum of root colors of all vertices, i.e., $\mathrm{tcol}(F)=\sum_v \mathrm{tcol}(F;v)$. We define the $(bc,\bar{b})$-inversion of $(bc,\bar{b})$-forest F by

$$\begin{aligned}\mathrm{inv}_{(bc,\bar{b})}(F;v) &= \mathrm{inv}(F;v)+b\cdot\mathrm{tcol}(F;v),\\ \mathrm{inv}_{(bc,\bar{b})}(F) &= \mathrm{inv}(F)+b\cdot\mathrm{tcol}(F).\end{aligned}$$

Note that $(bc,\bar{b})$-inversion is identical to the (bc,b)-inversion in [Yan01].

Let $\mathrm{prop}_{j,k}(F)$ denote the number of vertices such that $\mathrm{tcol}(F;v)=j$ and $\mathrm{inv}(F;v)=k$. We define the multi-statistic $\mathbf{INV}_{(bc,\bar{b})}$ by

$$\mathbf{INV}_{(bc,\bar{b})}(F)=\begin{pmatrix} \mathrm{prop}_{0,0}(F) & \mathrm{prop}_{0,1}(F) & \cdots & \mathrm{prop}_{0,N}(F) \\ \mathrm{prop}_{1,0}(F) & \mathrm{prop}_{1,1}(F) & \cdots & \mathrm{prop}_{1,N}(F) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{prop}_{c-1,0}(F) & \mathrm{prop}_{c-1,1}(F) & \cdots & \mathrm{prop}_{c-1,N}(F) \end{pmatrix},$$

where $N = \binom{n+1}{2}b - n$.

If a vertex v is a root of a forest F, we define $\mathrm{tree}(f;v) = 1$. Otherwise, $\mathrm{tree}(f;v) = 0$. Denote the number of trees (or roots) in a $(bc,\bar{b})$-forest F by $\mathrm{tree}(F)$.

In Figure 2, an example of a $(bc,\bar{b})$-forest F on n-set is given for $b = 3$ and $c = 2$ in order to illustrate different statistics.

3 Main Results

Let $PF_{(bc,\bar{b})}$ be the set of $(bc,\bar{b})$-parking functions of length n and $F_{(bc,\bar{b})}$ be the set of $(bc,\bar{b})$-forests on $[n]$. First of all, we recall the mapping $\varphi : F_{(1,\bar{1})} \to PF_{(1,\bar{1})}$ between forests and ordinary parking functions in [Shi08]. Given a forest $F \in F_{(1,\bar{1})}$ and a vertex $v \in [n]$, let h_v be the number of descendants of v in F and $D_F(v) = \{d_1, d_2, \ldots, d_{h_v}\}$ is the set of descendants of v in F. We define a cyclic permutation θ_v on $D_F(v)$ by

$$\theta_v = (d_1 d_2 \cdots d_{k-1} v)$$

where $d_1 > d_2 > \ldots > d_{k-1}$ are all the descendants of $v \in V(F)$ greater than v and $\theta_v(d_i) = d_{i+1}$ for $1 \le i \le k-1$ and $\theta_v(v) = d_1$. Let $\theta_F = \theta_1\theta_2\cdots\theta_n$. We attach to each vertex v in F a triple of labels

$$(\theta_F(v), \mathrm{inv}(F : v), \mathrm{post}(\theta_F(F) : \theta_F(v)))$$

where $\theta_F(F)$ is a forest by relabeling v by $\theta_F(v)$ and $\mathrm{post}(F : v)$ is a postorder index of v in F. We define the mapping $f : [n] \to [n]$ by

$$v \mapsto \mathrm{post}(\theta_F(F) : \theta_F(v)) - \mathrm{inv}(F : v)$$

for every vertex v. The bijection $\varphi : F_{(1,\bar{1})} \to PF_{(1,\bar{1})}$ is defined by

$$\varphi(F) = (f(\theta_F^{-1}(1)), f(\theta_F^{-1}(2)), \ldots, f(\theta_F^{-1}(n))). \tag{2}$$

Now we generalize the mapping φ to a bijection between $(bc,\bar{b})$-forests and $(bc,\bar{b})$-parking functions. We define the mapping $\varphi : F_{(bc,\bar{b})} \to PF_{(bc,\bar{b})}$ as follows: Given a $F \in F_{(bc,\bar{b})}$, the connected components of a forest F can be classified according to tree-colors. Let F_k be the sub-forests of F satisfying

$$\mathrm{tcol}(F : v) = k$$

for all $v \in F_k$. We define a cyclic permutation θ_v on $D_F(v)$ as above. When we define a postorder index $\mathrm{post}(F : v)$ of v in F, forests $F_{c-1}, F_{c-2}, \ldots, F_0$ are traversed in this order. We attach to each vertex v in F a quadruple of labels

$$(\theta_F(v), \mathrm{inv}(F : v), \mathrm{post}(\theta_F(F) : \theta_F(v)), \mathrm{tcol}(F : v))$$

where $\theta_F(F)$ is a forest by relabeling v by $\theta_F(v)$. After that, we define the mapping $f : [n] \to [n]$ by

$$v \mapsto (\mathrm{post}(\theta_F(F) : \theta_F(v)) + c - 1 - \mathrm{tcol}(F : v))b - \mathrm{inv}(F : v)$$

on every vertex v. The mapping $\varphi : F_{(bc,\bar{b})} \to PF_{(bc,\bar{b})}$ is also defined by (2). For example, the forest F in Figure 2 goes to the parking function P in Figure 1 by the mapping φ.

Theorem 1 (Main Theorem) *The mapping φ is a bijection between $(bc, \bar{b})$-forests and $(bc, \bar{b})$-parking functions satisfying*

$$(\mathbf{INV}_{(bc,\bar{b})}, \mathrm{tree})(F) = (\mathbf{JUMP}_{(bc,\bar{b})}, \mathrm{crit})\varphi(F),$$

for all $(bc, \bar{b})$-forests F.

By definitions, the statistics $\mathrm{inv}_{(bc,\bar{b})}$, inv, tcol, and prop can be written as follows:

$$\begin{aligned}
\mathrm{inv}_{(bc,\bar{b})}(F) &= \mathrm{inv}(F) + b \cdot \mathrm{tcol}(F),\\
\mathrm{inv}(F) &= (1,1,1,\ldots,1)\,\mathbf{INV}_{(bc,\bar{b})}(F)(0,1,2,\ldots,N)^T,\\
\mathrm{tcol}(F) &= (0,1,2,\ldots,(c-1))\,\mathbf{INV}_{(bc,\bar{b})}(F)(1,1,1,\ldots,1)^T,\\
\mathrm{prop}(F) &= (1,1,1,\ldots,1)\,\mathbf{INV}_{(bc,\bar{b})}(F)(1,0,0,\ldots,0)^T.
\end{aligned}$$

Similarly, the statistics $\mathrm{jump}_{(bc,\bar{b})}$, jump, block, and lucky can also be written as follows:

$$\begin{aligned}
\mathrm{jump}_{(bc,\bar{b})}(P) &= \mathrm{jump}(P) + b \cdot \mathrm{block}(P),\\
\mathrm{jump}(P) &= (1,1,1,\ldots,1)\,\mathbf{JUMP}_{(bc,\bar{b})}(P)(0,1,2,\ldots,N)^T,\\
\mathrm{block}(P) &= (0,1,2,\ldots,(c-1))\,\mathbf{JUMP}_{(bc,\bar{b})}(P)(1,1,1,\ldots,1)^T,\\
\mathrm{lucky}(P) &= (1,1,1,\ldots,1)\,\mathbf{JUMP}_{(bc,\bar{b})}(P)(1,0,0,\ldots,0)^T.
\end{aligned}$$

As a consequence, we derive the following corollary from Theorem 1.

Corollary 2 *The bijection $\varphi : F_{(bc,\bar{b})} \to PF_{(bc,\bar{b})}$ has the following property:*

$$(\mathrm{inv}_{(bc,\bar{b})}, \mathrm{inv}, \mathrm{tcol}, \mathrm{prop}, \mathrm{tree})(F) = (\mathrm{jump}_{(bc,\bar{b})}, \mathrm{jump}, \mathrm{block}, \mathrm{lucky}, \mathrm{crit})\varphi(F),$$

for $F \in F_{(bc,\bar{b})}$.

Introduce the following enumerators of $(bc, \bar{b})$-forest and $(bc, \bar{b})$-parking functions:

$$\begin{aligned}
I_n^{(bc,\bar{b})}(q,u,t) &= \sum_{F \in F_{(bc,\bar{b})}} q^{\mathrm{inv}_{(bc,\bar{b})}(F)} u^{\mathrm{prop}(F)} t^{\mathrm{tree}(F)},\\
\bar{P}_n^{(bc,\bar{b})}(q,u,t) &= \sum_{P \in PF_{(bc,\bar{b})}} q^{\mathrm{jump}_{(bc,\bar{b})}(P)} u^{\mathrm{lucky}(P)} t^{\mathrm{crit}(P)}.
\end{aligned}$$

Then we can derive a partial refinement of (1) from Corollary 2.

Corollary 3 *We have*

$$I_n^{(bc,\bar{b})}(q,u,t) = \bar{P}_n^{(bc,\bar{b})}(q,u,t).$$

Define the homogeneous polynomial

$$P_n(a,b,c) = c \prod_{i=1}^{n-1} (ai + b(n-i) + c).$$

Theorem 4 *We have*

$$\sum_{P\in PF_{(bc,\bar{b})}} u^{\mathrm{lucky}(P)}t^{\mathrm{crit}(P)} = \sum_{F\in F_{(bc,\bar{b})}} u^{\mathrm{prop}(F)}t^{\mathrm{tree}(F)} = P_n(b, b-1+u, ct(b-1+u)). \quad (3)$$

Remark. For $b = c = 1$ and $b = t = 1$, we recover, respectively, two results of Gessel and Seo [GS06, Theorem 6.1 and Corollay 10.2].

Theorem 5 *We have the hook-length formula of $(a,\bar{b})$-forests*

$$\sum_{F\in F_{(a,\bar{b})}} c^{\mathrm{tree}(F)} \prod_v \left(1+\frac{\alpha}{h_v}\right) = P_n(b, b(1+\alpha), ac(1+\alpha)), \quad (4)$$

where the sum is over all $(a,\bar{b})$-forests on n-set.

Remark. For $a = b = 1$ this is Gessel and Seo's hook-length formula [GS06, Corollay 6.3].

4 Proof of Theorem 1

The inverse map of the extended mapping φ can be defined like the method in the paper [Shi08]: Given a $(bc,\bar{b})$-parking function P, all cars are parked by the parking algorithm. At that time, we record the $\mathrm{jump}(P;c)$ for every car in next row. After finishing, we draw an edge between the car c and the closest car on its right which is larger than c in its same block. We get the forest-structure on the cars as vertices. That is a forest D. By defining

$$|\mathrm{inv}(F;v)| \equiv \mathrm{jump}(P;c) \mod |D_F(v)|,$$

we can recover two forests I and F. By $\bar{\kappa}(v) := \lfloor\frac{\mathrm{jump}(P;c)}{b}\rfloor$, we can recover the color of v in F where $\theta_F(v) = c$.

We can prove that φ is weight preserving by the following lemma.

Lemma 6 *There is a bijection $\varphi : F_{(bc,\bar{b})} \to PF_{(bc,\bar{b})}$ between $(bc,\bar{b})$-forests and $(bc,\bar{b})$-parking functions such that*

$$(\mathrm{inv}, \mathrm{tcol}, \mathrm{tree})(F;v) = (\mathrm{jump}, \mathrm{block}, \mathrm{crit})(\varphi(F);\theta_F(v)),$$

for all $(bc,\bar{b})$-forests F and all vertices $v \in F$.

Proof: If we use the function $d \mapsto (g+c-1-k)b$ instead of $d \mapsto (g+c-1-k)b-i$, all cars are lucky since all images of f are different. So using the original function $d \mapsto ((g+c-1-k)b-i)$, the value of $\mathrm{jump}(P:c)$ increases by $\mathrm{inv}(T:v)$ where $\theta_F(v) = c$. Thus $\mathrm{inv}(F:v) = \mathrm{jump}(\varphi(F):\theta_F(v))$.

Suppose that $\mathrm{tcol}(F;v) = k$, which means that a vertex v is in F_k. So a label of $\theta_F(v)$ is also in D_k. Then car $\theta_F(v)$ is parked actually in a k-th block. Then $\mathrm{block}(\varphi(F);\theta_F(v)) = k$.

If a vertex v is a root of a tree in F, a parent of $\theta_F(v)$ is the root of D. So there is no car larger than the car $\theta_F(v)$ on its right in same block. Hence the car $\theta_F(v)$ is critical. $\square$

5 Proof of Theorem 4

The first equality follows from Corollary 3 for $q = 1$, i.e.,

$$\sum_{F \in F_{(bc,\bar{b})}} u^{\mathrm{prop}(F)} t^{\mathrm{tree}(F)} = \sum_{P \in PF_{(bc,\bar{b})}} u^{\mathrm{lucky}(P)} t^{\mathrm{crit}(P)}.$$

To prove the second equality in Theorem 4, we need to appear for two Prüfer-like algorithms: the colored Prüfer code [CKSS04] and reverse Prüfer algorithm in [SS07]. Given a $(bc,\bar{b})$-forest F, deleting the largest leaves successively $v_n, \ldots, v_1$ where σ_i is the parent of v_i or $\sigma_i = -\,\mathrm{tcol}(F : v_i)$ if v_i is a root and the color $c_i = \bar{\kappa}(v_i)$. Then the *colored Prüfer code* of F is defined by

$$\sigma = \begin{pmatrix} \sigma_n & \sigma_{n-1} & \cdots & \sigma_1 \\ c_n & c_{n-1} & \cdots & c_1 \end{pmatrix} \in \binom{\{-(c-1), \ldots, n\}}{\{0, \ldots, b-1\}}^{n-1} \times \binom{\{-(c-1), \ldots, 0\}}{\{0, \ldots, b-1\}}.$$

In order to count the number of proper vertices, we define the *reverse colored Prüfer algorithm* as follows: Starting from a colored Prüfer code $\sigma = \begin{pmatrix} \sigma_n & \sigma_{n-1} & \cdots & \sigma_1 \\ c_n & c_{n-1} & \cdots & c_1 \end{pmatrix}$. Let F_1 be the forest with unlabeled single vertex v_1 by $\mathrm{tcol}(F : v_1) = -\sigma_1$. For each $i = 2, \ldots, n$, we assume that F_{i-1} is the forest obtained from the subcode $\begin{pmatrix} \sigma_{i-1} & \sigma_{i-2} & \cdots & \sigma_1 \\ & c_{i-2} & \cdots & c_1 \end{pmatrix}$. Let ℓ be the minimal element in $[n]$ which does not appear in F_{i-1}. To construct F_i from F_{i-1} and (σ_i, c_{i-1}), we should consider the following two cases.

1. Suppose that σ_i appears in F_{i-1}. Then the unlabeled vertex v in F_{i-1} is labeled by ℓ with color c_{i-1} in T_i. Since the new label ℓ is minimal among the unused labels in T_{i-1}, the vertex v with the color c_{i-1} is a proper vertex in T if and only if $c_{i-1} = 0$.

2. Suppose that σ_i does not appear in T_{i-1}. Then the unlabeled vertex v in F_{i-1} is labeled by σ_i in F_i.

   (a) If $\sigma_i \leq 0$, then the vertex v is a proper vertex in F, as in case (1) and the unlabeled vertex in F_i becomes a root in F.

   (b) If $\sigma_i = l$, then the vertex v is a proper vertex in F, as in case (1).

   (c) If $\sigma_i \neq l$, then the vertex v will have a descendant labeled by ℓ. Thus, the vertex v is not proper vertex in F.

So there are exactly $i - 1 + c$ choices of σ_i and one choice of c_{i-1} in case (1), case (2a), and case (2b), such that the newly labeled vertex v is a proper vertex in F. Because the number of i's such that $\sigma_i \leq 0$ in a colored Prüfer-code equals the number of the roots in F, $\mathrm{tree}(F)$ is enumerated by nonpositive number in the colored Prüfer-code of a forest F. Thus we have the following formula:

$$\begin{aligned}\sum_{F\in F_{(bc,\bar{b})}} u^{\mathrm{prop}(F)} t^{\mathrm{tree}(F)} &= ct && \text{by } \sigma_1 \in \{0,-1,\dots,-(c-1)\} \\ &\times \prod_{i=2}^{n} (b(n-i+1) + (i-1+ct)(b-1+u)) && \text{by } (\sigma_i, c_{i-1}) \\ &\times (b-1+u) && \text{by } c_{n-1} \\ &= P_n(b, b-1+u, ct(b-1+u)).\end{aligned}$$

This completes the bijective proof of equation (3).

6 Proof of Theorem 5

By Theorem 4, the right side of (4) is

$$\sum_{F\in F_{(b,\bar{b})}} (1+b\alpha)^{\mathrm{prop}(F)} \left(\frac{ac}{b}\right)^{\mathrm{tree}(F)}.$$

Replacing α by α/b in (4), it suffices to prove the identity:

$$\sum_{F\in F_{(a,\bar{b})}} c^{\mathrm{tree}(F)} \prod_{v} \left(1 + \frac{\alpha}{bh_v}\right) = \sum_{F\in F_{(b,\bar{b})}} (1+\alpha)^{\mathrm{prop}(F)} \left(\frac{ac}{b}\right)^{\mathrm{tree}(F)}. \tag{5}$$

We follow Gessel and Seo's proof [GS06] in the case of $a = b = 1$. For each (unlabeled) forest $\tilde{F}$ on n sets, a *labeling* of $\tilde{F}$ is a bijection from $V(\tilde{F})$ to $[n]$ and $(a,\bar{b})$-*coloring* κ is a mapping from $V(\tilde{F})$ to nonnegative numbers such that $\kappa(v) < a$ if v is a root and $\kappa(v) < b$ otherwise. Define the set of $(a,\bar{b})$-forests

$$L_{(a,\bar{b})}(\tilde{F}) = \left\{(L,\kappa) : L \text{ is a labeling and } \kappa \text{ is a } (a,\bar{b})\text{-coloring of } \tilde{F}\right\}.$$

Lemma 7 *Let $\tilde{F}$ be a (unlabeled) forest with n vertices. If S is a subset of $V(\tilde{F})$, then the number of labelings $L \in L_{(b,\bar{b})}(\tilde{F})$ such that every vertex in S is a proper vertex is*

$$\frac{n!b^n}{\prod_{v\in S}(bh_v)}. \tag{6}$$

Proof: Clearly the cardinality of $L_{(b,\bar{b})}(\tilde{F})$ is $n!b^n$. Among the elements of $L_{(b,\bar{b})}(\tilde{F})$, the probability that some vertex $v \in S$ is a proper vertex equals $\frac{1}{bh_v}$. In other words, the number of labelings $L \in L_{(b,\bar{b})}(\tilde{F})$ such that every vertex in S is a proper vertex is $\frac{1}{bh_v}$ times the number of labelings in which every vertex in $S \setminus \{v\}$ is a proper vertex. By induction on $|S|$, we are done. □

Let us consider the formula

$$\sum_{L\in L_{(b,\bar{b})}(\tilde{F})} (1+\alpha)^{\mathrm{prop}(L)}(\frac{ac}{b})^{\mathrm{tree}(L)} = \sum_{L\in L_{(b,\bar{b})}(\tilde{F})} \sum_{S} \alpha^{|S|}(\frac{ac}{b})^{\mathrm{tree}(L)},$$

where S runs over the subsets of the set of proper vertices of L. Reversing the order of two summations, it follows by Lemma 7 that

$$\begin{aligned}\sum_{S\subset V(\tilde{F})} (\frac{ac}{b})^{\mathrm{tree}(\tilde{F})} \sum_{L} \alpha^{|S|} &= \sum_{S\subset V(\tilde{F})} (\frac{ac}{b})^{\mathrm{tree}(\tilde{F})} \frac{n!b^n}{\prod_{v\in S}(bh_v)} \alpha^{|S|} \\ &= n!b^n(\frac{ac}{b})^{\mathrm{tree}(\tilde{F})} \prod_{v\in V(\tilde{F})} (1+\frac{\alpha}{bh_v}),\end{aligned}$$

where $L \in L_{(b,\bar{b})}(\tilde{F})$ such that every vertex in S is a proper vertex. Therefore,

$$\sum_{L\in L_{(b,\bar{b})}(\tilde{F})} (1+\alpha)^{\mathrm{prop}(L)}(\frac{ac}{b})^{\mathrm{tree}(L)} = n!b^n(\frac{ac}{b})^{\mathrm{tree}(\tilde{F})} \prod_{v\in V(\tilde{F})} (1+\frac{\alpha}{bh_v}) \tag{7}$$

Let us say that two labelings with colorings of a forest $\tilde{F}$ are *equivalent* if there is an automorphism of $\tilde{F}$ that takes one labeling with coloring to the other. Let $\tilde{F}$ be a forest on n set with automorphism group G. Then the $n!b^{n-\mathrm{tree}(\tilde{F})}a^{\mathrm{tree}(\tilde{F})}$ labelings with colorings of F fall into $n!b^{n-\mathrm{tree}(\tilde{F})}a^{\mathrm{tree}(\tilde{F})}/\,|G|$ equivalence classes. Define

$$\tilde{L}_{(a,\bar{b})}(\tilde{F}) = \left\{L \in F_{(a,\bar{b})} : \text{The underlying graph of } L \text{ is } \tilde{F}\right\}.$$

Clearly $\left|\tilde{L}_{(a,\bar{b})}(\tilde{F})\right| = n!b^{n-\mathrm{tree}(\tilde{F})}a^{\mathrm{tree}(\tilde{F})}/\,|G|$ and equivalent labelings with coloring have the same number of proper vertices of trees, dividing (7) by $|G|$, so we obtain the following.

$$\sum_{L\in \tilde{L}_{(b,\bar{b})}(\tilde{F})} (1+\alpha)^{\mathrm{prop}(L)}(\frac{ac}{b})^{\mathrm{tree}(L)} = \left|\tilde{L}_{(a,\bar{b})}(\tilde{F})\right| c^{\mathrm{tree}(\tilde{F})} \prod_{v\in V(\tilde{F})} (1+\frac{\alpha}{bh_v}).$$

Summing over all (unlabeled) forests $\tilde{F}$ yields

$$\sum_{\tilde{F}} \sum_{L\in \tilde{L}_{(b,\bar{b})}(\tilde{F})} (1+\alpha)^{\mathrm{prop}(L)}(\frac{ac}{b})^{\mathrm{tree}(L)} = \sum_{\tilde{F}} \left|\tilde{L}_{(a,\bar{b})}(\tilde{F})\right| c^{\mathrm{tree}(\tilde{F})} \prod_{v\in V(\tilde{F})} (1+\frac{\alpha}{bh_v}).$$

As $F_{(a,\bar{b})} = \bigcup_{\tilde{F}} \tilde{L}_{(a,\bar{b})}(\tilde{F})$, we obtain (5).

7 Concluding Remarks

In this paper, we give a bijective proof of (1) in the $(bc, \bar{b})$ case. The problem of giving a bijective proof of (1) in the general $(a, \bar{b})$ case is still open. It seems that the construct of such a bijection in the $(1, \bar{b})$ case is crucial.

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A Combinatorial Approach to Multiplicity-Free Richardson Subvarieties of the Grassmannian

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Abstract. We consider Buch's rule for K-theory of the Grassmannian, in the Schur multiplicity-free cases classified by Stembridge. Using a result of Knutson, one sees that Buch's coefficients are related to Möbius inversion. We give a direct combinatorial proof of this by considering the product expansion for Grassmannian Grothendieck polynomials. We end with an extension to the multiplicity-free cases of Thomas and Yong.

R'esum'e. On examine la règle de Buch pour la K-théorie de la variété grassmannienne dans les cas sans multiplicité de Schur, qui ont étés classifiés par Stembridge. En utilisant un résultat de Knutson, on démontre que les coefficients de Buch sont liés à l'inversion de Möbius. On en fait une preuve directe et combinatoire qui passe par le developpement de produits de polynômes de Grothendieck. Pour conclure, on donne une application de cette théorie aux cas sans multiplicité de Thomas et Yong.

Keywords: Grassmannian, Richardson varieties, Grothendieck polynomials, Schur multiplicity free

1 Motivation

1.1 Schubert and Richardson varieties

We consider the **Grassmannian** $Gr_k\mathbb{C}^n := \{V \leq \mathbb{C}^n \mid dim(V) = k\}$. For a partition λ contained in a $k \times (n-k)$ box, consider the path from the northeast corner to its southwest corner of the box that traces the partition. For the standard flag ($C_i = (*_1, \ldots, *_i, 0, \ldots 0)$), we define the **Schubert variety** as

$$X_\lambda = \{V \in Gr_k\mathbb{C}^n \mid dim(V \cap C_i) \geq \#(\text{ south steps in the first } i \text{ steps of the path })\}.$$

We denote the **Schubert class** in cohomology as $S_\lambda := [X_\lambda]_H \in H^\star(Gr_k\mathbb{C}^n)$. The set

$$\{S_\lambda \mid \lambda \subset k \times (n-k) \text{ box}\}$$

forms a $\mathbb{Z}-$basis for $H^\star(Gr_k\mathbb{C}^n)$, where

$$S_\lambda S_\mu = \sum c^\nu_{\lambda\mu} S_\nu$$

for $|\nu| = |\lambda| + |\mu|$, $\nu \subset k \times (n-k)$ box, and $c^{\nu}_{\lambda\mu}$ the Littlewood-Richardson coefficients. This follows from the surjective homomorphism

$$\{\text{ring of symmetric polynomials}\} \twoheadrightarrow \{H^{\star}(Gr_k\mathbb{C}^n)\}$$

$$s_\lambda \longmapsto \begin{cases} S_\lambda, & \text{if } \lambda \text{ fits in } k \times (n-k) \text{ box;} \\ 0, & \text{otherwise.} \end{cases}$$

for Schur functions s_λ.

The **Möbius function** $\mu_{\mathcal{P}}(\nu)$ is defined recursively on a poset $\mathcal{P}$ as the unique function satisfying

$$\sum_{\alpha \geq_{\mathcal{P}} \nu} \mu_{\mathcal{P}}(\alpha) = 1$$

The connection of this definition to K-classes is shown in [Kn08]. Since we are primarily interested in working in K-theory, we will use $[A]$ to denote the K-class of a subscheme of A, and $[A]_H$ to denote its homology class. Any subvariety X of a flag manifold is rationally equivalent to a linear combination of Schubert cycles with uniquely determined non-negative integer coefficients [Br03]. We say X is **multiplicity-free** if these coefficients are 0 or 1.

Theorem 1 *[Kn08] Let X be a multiplicity-free irreducible subvariety of G/P, in the sense of [Br03], with $[X]_H = \sum_{d\in D}[X_d]_H$, D a subset of the Bruhat order, and P a parabolic subgroup. Let $\mathcal{P} \subseteq W/W_P$ be the set of Schubert varieties contained in $\cup_{d\in D} X_d$ (an order ideal in the Bruhat order on W/W_P). Then as an element of $K(G/P)$,*

$$[X] = \sum_{X_e \subseteq \bigcup_{d\in D} X_d} \mu_{\mathcal{P}}(X_e)\,[X_e].$$

We will give an independent combinatorial proof of this fact in the case that X is a multiplicity-free **Richardson variety** in a Grassmannian, the intersection of a Schubert variety X_λ with an opposite Schubert variety $w_0 \cdot X_\mu$, for w_0 the longest word. For any $X_\lambda \subset Gr_k\mathbb{C}^n$, let $G_\lambda := [X_\lambda]$. We have that $\{G_\lambda \mid \lambda \subset k \times (n-k) \text{ box}\}$ form a basis for $K(Gr_k\mathbb{C}^n)$. For certain symmetric polynomials g_λ which we will define in the next section, we have a surjective homomorphism [Bu02]:

$$\{\text{ring of symmetric functions}\} \twoheadrightarrow \{K(Gr_k\mathbb{C}^n)\}$$

$$g_\lambda \longmapsto \begin{cases} G_\lambda, & \text{if } \lambda \text{ fits in } k \times (n-k) \text{ box;} \\ 0, & \text{otherwise.} \end{cases}$$

Our main theorem will show that, for a poset $\mathcal{P}$ that we will define,

$$G_\lambda \cdot G_\mu = \sum_{\nu} \mu_{\mathcal{P}}(G_\nu)\, G_\nu$$

where the sum is over ν such that $\nu \subseteq (k \times n-k)$ box and $|\nu| \geq |\lambda| + |\mu|$. Our proof will proceed with sign-reversing involutions on this poset, and many reductions in the sizes of the partitions in the product.

1.2 Grothendieck Polynomials

For finite non-empty sets in $\mathbb{Z}^+$, a and b, we say $a < b$ if $\max(a) < \min(b)$, and $a \leq b$ if $\max(a) \leq \min(b)$. For a partition λ, Buch defined a **set-valued (English) tableau (SVT)** as a filling of a Young diagram with nonempty sets in $\mathbb{Z}^+$ [Bu02]. If each box has a single entry, it is a **Young tableau**. A tableau is a **semistandard tableau (SS)** if it is weakly increasing across rows and strictly increasing down columns. The **superstandard** filling of a tableau is the one in which each box (i, j) has a single entry, i (its row). In all of our examples, we will use numbers smaller than 10, so we can avoid the use of set notation: we use 45 to denote the set $\{4, 5\}$.

Recall the combinatorial definition for the Schur polynomials,

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^T.$$

We consider the **Grothendieck polynomials** g_λ of Lascoux and Schützenberger [LS82]. For λ a partition, Buch [Bu02] gives the formula

$$g_\lambda = \sum_{T \in SS-SVT(\lambda)} (-1)^{|T|-|\lambda|} x^T$$

where

$$|T| = \sum_{i,j} |T(i, j)|.$$

He proves that this is a special case of the Lascoux-Schützenberger formula (which we will not need) for g_π in the case when π is a Grassmannian permutation. These are the $\{g_\lambda\}$ representing the G_λ in section 1.1. As with the Schur polynomials, it is not obvious from the combinatorial definition that these polynomials are in fact symmetric and a basis for the symmetric polynomials [Bu02]. Linear independence follows from the fact that the lowest homogeneous component of g_λ is s_λ.

We define the **word of a tableau** $w(T)$ to be the entries read right to left, top to bottom. Note that entries in a set are listed in increasing order, so that they occur in decreasing order in the word. A word is called a **reverse lattice word (RLW)** if for any initial string (e.g. 001 in 001110101),

$$\mathrm{m}ultiplicity(i) \geq \mathrm{m}ultiplicity(i+1) \ \forall \ i \geq 1$$

A word that satisfies this condition is sometimes called an **election word** (or ballot sequence). For tableaux of shape λ and μ, we define the shape $\lambda \times \mu$ as the skew tableau formed by placing μ directly southwest of λ. When we refer to a filling of the shape $\lambda \times \mu$, we will call λ the "northeast" partition, and μ the "southwest" partition. Buch [Bu02] gives a combinatorial rule for the product of two Grothendieck polynomials:

$$g_\lambda g_\mu = \sum c'^{\nu}_{\lambda\mu} g_\nu$$

where the coefficients are given by

$$c'^{\nu}_{\lambda\mu} = (-1)^{|\nu|-|\lambda|-|\mu|} \#(T)$$

for SS-SVT T of shape $\lambda \times \mu$, content ν, with $w(T)$ a RLW. We call these the K-theoretic Littlewood-Richardson numbers, since if $|\nu| = |\lambda| + |\mu|$, then ${c'}^{\nu}_{\lambda\mu} = c^{\nu}_{\lambda\mu}$, the usual Littlewood-Richardson number. Figure 1 shows the calculation of the expansion of $g_1 \cdot g_1$.

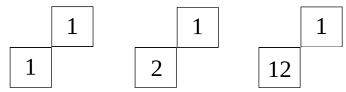

Fig. 1: $g_1 g_1 = g_2 + g_{11} - g_{21}$

First, we note that the reverse lattice word condition requires that the filling of the northeast tableau λ always be superstandard. We will construct a poset out of all of the allowed fillings of the southwest tableau μ, where each vertex is labeled with all tableaux of a given content, and for vertices ν, ν', $\nu \leq_{\mathcal{P}} \nu'$ if $content(\nu) \supset content(\nu')$. We are interested in a poset, since its Möbius function will allow us to compute structure constants. Note that for each tableau, the row in the poset corresponds to the number of "extra" elements in the filling (e.g. the top row has only semistandard Young tableaux, corresponding to the product $s_\lambda \cdot s_\mu$).

Example 1 *Consider the product $G_{2,1} \cdot G_{2,2}$ and its poset in Figure 2. From the second line of this poset, we can see that the coefficient of G_{431} is 2 since there are two tableaux on the corresponding vertex, while G_{422} has coefficient 1. Note that this product is H-multiplicity-free, but not K-multiplicity-free. The K-multiplicity-free cases are extremely rare, occurring only when both partitions λ and μ are rectangles or one of them is a single box or empty ([Bu02, Proposition 7.2]).*

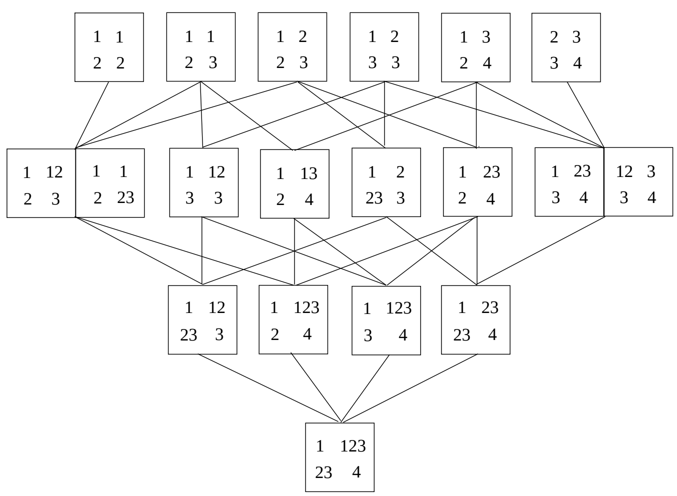

Fig. 2: The poset corresponding to $G_{2,1} \cdot G_{2,2}$: the product satisfies Stembridge cases (3) and (4) from Theorem 3.

We will consider our products as being inside an ambient box of size $k \times (n-k)$. That is, we limit the terms in the expansion to those indexed by partitions that fit inside this box. We note that this restriction gives us a sub-poset of the full poset. The Möbius function on the remaining terms is unaffected by the removal of vertices with content exceeding the box size, since all terms above a vertex ν have content contained in the content of ν. That is, for a given vertex ν with content in the ambient box, no vertex in its upwards order ideal will have content exceeding the ambient box. We are interested in cases in which the terms in the Grothendieck expansion which correspond to the Schur expansion are multiplicity-free, i.e. that their coefficients are 0 or 1.

Then Theorem 1 implies the following:

Theorem 2 *Consider partitions $\lambda = (\lambda_1^{\beta_1}, \ldots, \lambda_l^{\beta_l})$ and $\mu = (\mu_1^{\alpha_1}, \ldots, \mu_m^{\alpha_m})$ such that $G_\lambda \cdot G_\mu$ in a $k \times (n-k)$ box is a Schur-multiplicity-free product. In the corresponding poset, for each vertex ν', $\mu(\nu')$ gives the coefficient of G_ν in the Buch expansion of the product, where $\nu = \nu' \bigcup (1^{\lambda_1}, \ldots, l^{\lambda_l})$.*

These Schur-multiplicity-free cases have been classified by Stembridge [St01] as follows, and our proof explicitly uses his analysis. We begin by recalling Stembridge's definitions and classification of Schur-multiplicity-free cases.

Definition 1 *[St01] A partition μ with at most one part size (i.e., empty, or of the form (c^r) for suitable $c, r > 0$) is said to be a **rectangle**. If it has k rows or k columns (i.e., $k = r$ or $k = c$), then we say ν is a **k-line rectangle**. A partition μ with exactly two part sizes (i.e., $\mu = (b^r c^s)$ for suitable $b > c > 0$ and $r, s > 0$) is said to be a **fat hook**. If it is possible to obtain a rectangle by deleting a single row or column from the fat hook μ, then we say that μ is a **near rectangle**.*

We will call these **top, bottom, left,** or **right** near rectangles, to denote the location of the extra row or column. We say that a product of Schur functions is **multiplicity-free** if all of the Littlewood-Richardson coefficients of the expansion are 0 or 1.

Theorem 3 *[St01] The product of Schur functions $s_\lambda \cdot s_\mu$ is multiplicity-free if and only if*

1. *λ and μ are rectangles, or*
2. *(Pieri rule) λ is arbitrary, and μ is a 1-line rectangle*
3. *λ is a rectangle and μ is a near-rectangle*
4. *μ is a fat hook and λ is a 2-line rectangle*

We now mention some speculative geometry that motivated our *combinatorial* proof of Theorem 2. Buch shows that the expansion of $X_\lambda \cap (w_0 \cdot X_\mu)$ into Schubert classes has signs that alternate with dimension ([Bu02]). This suggests that there exists an exact sequence on sheaves

$$0 \to \mathcal{O}_{\bigcup_{\nu \in \mathcal{P}} X_\nu} \to \bigoplus_{T, |T| = |\nu|+1} \mathcal{O}_{X_{content(T)}} \to \cdots \to \bigoplus_{T, |T| = |\nu|+k-1} \mathcal{O}_{X_{content(T)}} \to \cdots \tag{1}$$

where the k^{th} nonzero term sums over Buch Littlewood-Richardson tableaux with $k-1$ extra entries. This leads to a sequence for the point in the Grassmannian corresponding to each λ,

$$0 \to \mathbb{C}^1 \to \cdots \to \bigoplus_{T, |T| = |\nu|+k-1, content(T) \subseteq \lambda} \mathbb{C}^1 \to \cdots$$

One can hope that this sequence is in fact exact. Our main result is

Theorem 4 *There exists such an exact sequence of vector spaces, and it can be explicitly constructed as a direct sum of exact sequences with exactly two non-zero terms.*

The proof requires an involution which pairs terms differing in size by one. In some cases, we provide a single rule that matches all terms required. In other cases however, we must resort to a multistage divide and conquer approach, where the involution is defined differently on several disjoint subsets. Assuming Theorem 4, we can prove Theorem 2 as a corollary.

Proof of Theorem 2: The exactness of the sequence (1) gives us that the alternating sum of dimensions is 0. Thus the sum of the coefficients of the pairs of Buch tableaux, with signs alternating with number of extra numbers (i.e. how many more than a single entry), is also 0. Together with the extra 1 from the single fixed point tableau, this is equivalent to the statement that the coefficient of ν' is given by the Möbius function. □

2 An Extension to the Thomas-Yong Cases

Consider partitions λ and μ in a $(k \times (n-k))$ box. We will review the notation introduced in [TY07]. We call $R = (\lambda, \mu, k \times (n-k))$ a **Richardson quadruple**, and use the notation $poset(R)$ to denote the associated poset of allowed fillings of μ. Place λ in the upper left corner of the box, then rotate μ by $180°$ (call this **rotate(**μ**)**) and place it in the lower right corner. This quadruple $(\lambda, \mu, k \times (n-k))$ is called **basic** if $\lambda \bigcup \text{rotate}(\mu)$ does not contain any full rows or columns. If it is not basic, we can remove all full rows and columns to get a **basic demolition** $\widetilde{R} = (\widetilde{\lambda}, \widetilde{\mu}, \widetilde{k} \times (\widetilde{n} - \widetilde{k}))$. We call each row (column) removal a **row (column) demolition**. Notice that if $\lambda \bigcap \text{rotate}(\mu) \neq \emptyset$, then $G_\lambda \cdot G_\mu = 0$, so the demolition is undefined. In order to determine whether a Richardson quadruple is multiplicity-free, we consider its basic demolition.

Theorem 5 *[TY07] A Richardson quadruple is multiplicity-free if and only if its basic Richardson quadruple is multiplicity-free. If a the basic demoltion of a Richardson quadruple $(\lambda, \mu, k \times n - k)$ is multiplicity-free, then it must be in the cases classified by [St01].*

For example, consider the case $((4,4,2,2,1),(4,3,2,1),5 \times 5)$. This product is not multiplicity-free, but has a basic demolition of $(1,1,2 \times 2)$, which is multiplicity-free.

We will show that our analysis of the [St01] multiplicity-free cases extends to this larger class of products by showing that the posets of a Richardson quadruple and its basic demolition are isomorphic. Let us define the **accessible word** w_A as the independent values of λ read in increasing order, or equivalently

$$w_A(j) = 1 + \#(\text{rows of } \lambda \text{ in column } n-k-j).$$

Lemma 1 (Column Demolition Lemma) *For $R = (\lambda, \mu, k \times (n-k))$, if column c is full, let $\widetilde{R}$ be the quadruple with column c removed. Then $poset(\widetilde{R})$ is isomorphic to $poset(R)$.*

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$. There is a full row in the diagram if and only if $\lambda_r + \mu_{k-r+1} = n-k$.

Lemma 2 *For $R = (\lambda, \mu, k \times (n-k))$, if $\lambda_1 = n-k$, let $\widetilde{R}$ be the quadruple with λ_1 removed. Then $poset(R)$ is isomorphic to $poset(\widetilde{R})$ for $\widetilde{R} = ((\lambda_2, \ldots, \lambda_l), \mu, (k-1) \times (n-k))$.*

Lemma 3 (Row Demolition Lemma) *For $R = (\lambda, \mu, k \times (n-k))$, if row r is full, then the $poset(\widetilde{R})$ is isomorphic to $poset(R)$.*

We note that basic demolition is well defined, i.e. independent of the order of full column/row removal, thus so are the corresponding isomorphisms between posets.

Figure 3 is an example of a case with both a full row and column. (This product is Stembridge multiplicity-free in any ambient box, but is a good example of the row and column demolition commutativity.)

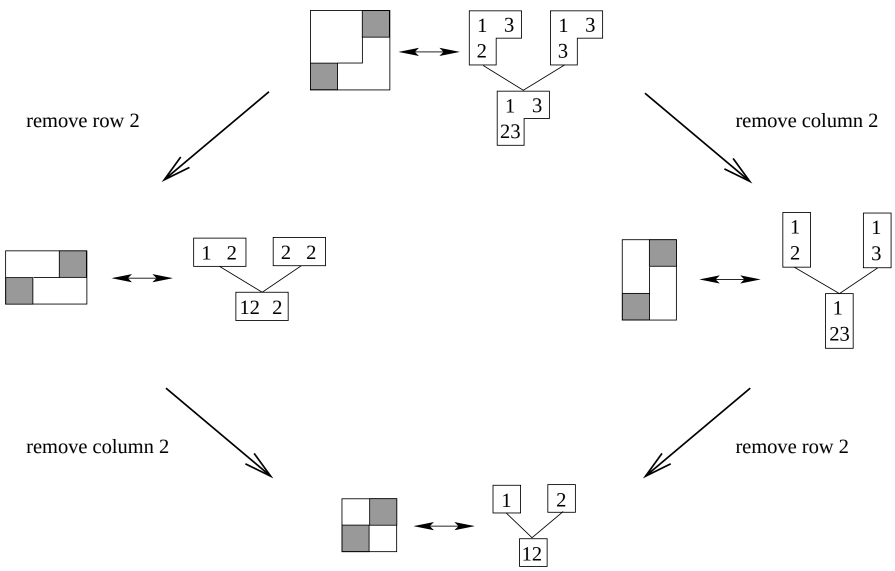

Fig. 3: Two demolition paths of $((2,2),(2,1),3\times 3)$ to $((1),(1),2\times 2)$.

Proposition 1 *Theorem 4 holds for any Richardson quadruple whose basic demolition is a Stembridge case.*

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The poset perspective on alternating sign matrices

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Abstract. Alternating sign matrices (ASMs) are square matrices with entries 0, 1, or -1 whose rows and columns sum to 1 and whose nonzero entries alternate in sign. We put ASMs into a larger context by studying the order ideals of subposets of a certain poset, proving that they are in bijection with a variety of interesting combinatorial objects, including ASMs, totally symmetric self–complementary plane partitions (TSSCPPs), Catalan objects, tournaments, semistandard Young tableaux, and totally symmetric plane partitions. We use this perspective to prove an expansion of the tournament generating function as a sum over TSSCPPs which is analogous to a known formula involving ASMs.

Résumé. Les matrices à signe alternant (ASMs) sont des matrices carrées dont les coefficients sont $0, 1$ ou -1, telles que dans chaque ligne et chaque colonne la somme des entrées vaut 1 et les entrées non nulles ont des signes qui alternent. Nous incluons les ASMs dans un cadre plus vaste, en étudiant les idéaux des sous-posets d'un certain poset, dont nous prouvons qu'ils sont en bijection avec de nombreux objets combinatoires intéressants, tels que les ASMs, les partitions planes totalement symétriques autocomplémentaires (TSSCPPs), des objets comptés par les nombres de Catalan, les tournois, les tableaux semistandards, ou les partitions planes totalement symétriques. Nous utilisons ce point de vue pour démontrer un développement de la série génératrice des tournois en une somme portant sur les TSSCPPs, analogue à une formule déjà connue faisant apparaître les ASMs.

Keywords: alternating sign matrices, posets, plane partitions, order ideals, Catalan numbers, tournaments

1 Introduction

Alternating sign matrices (ASMs) are simply defined as square matrices with entries 0, 1, or -1 whose rows and columns sum to 1 and alternate in sign, but have proved quite difficult to understand (and even count). Totally symmetric self–complementary plane partitions (TSSCPPs) are plane partitions, each equal to its complement and invariant under all permutations of the coordinate axes. TSSCPPs inside a $2n \times 2n \times 2n$ box are equinumerous with $n \times n$ ASMs, but no explicit bijection between these two sets of objects is known. In this paper we present a new perspective which sheds light on ASMs and TSSCPPs and brings us closer to constructing a explicit ASM–TSSCPP bijection.

2 The tetrahedral poset

Given an $n \times n$ ASM A, consider the following bijection to objects called monotone triangles of order n [2]. For each row of A note which columns have a partial sum (from the top) of 1 in that row.

Record the numbers of the columns in which this occurs in increasing order. This gives a triangular array of numbers 1 to n. This process can be easily reversed, and is thus a bijection. Monotone triangles can be defined as objects in their own right as follows [2].

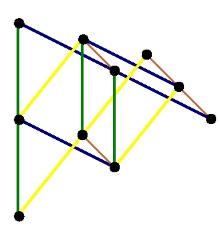
Definition 2.1 *Monotone triangles of order n are all triangular arrays of integers with bottom row 1 2 3 ... n and integer entries a_{ij} such that $a_{i,j} \leq a_{i-1,j} \leq a_{i,j+1}$ and $a_{ij} < a_{i,j+1}$.*

Note that the bottom row of a monotone triangle of order n is always $1\ 2\ 3\ \ldots\ n$. If we rotate the monotone triangle clockwise by $\frac{\pi}{4}$ we obtain a semistandard Young tableau (SSYT) of staircase shape $\delta_n = n\ (n-1)\ (n-2) \ldots 3\ 2\ 1$ whose northeast to southwest diagonals are weakly increasing. Thus we have the following theorem.

$$\begin{array}{ccccc} 4 \times 4 \text{ ASM} & & \text{Monotone triangle of order 4} & & \text{Rotated array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & \Longleftrightarrow & \begin{array}{ccccccc} & & & 2 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array} & \Longleftrightarrow & \begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & \\ 3 & 4 & & \\ 4 & & & \end{array} \end{array}$$

Theorem 2.2 *$n \times n$ alternating sign matrices are in bijection with SSYT of staircase shape δ_n with entries $y_{i,j}$ at most n such that $y_{i,j} \leq y_{i+1,j-1}$. Denote this set as SSA_n.*

Ordered by componentwise comparison of the entries, SSA_n forms a distributive lattice $J(P)$ where the Hasse diagram of the poset of join-irreducibles P (for $n = 4$) is shown below:

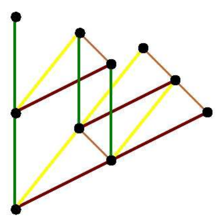


Given a TSSCPP $t = \{t_{i,j}\}_{1 \leq i,j \leq 2n}$ take a fundamental domain consisting of the triangular array of integers $\{t_{i,j}\}_{n+1 \leq i \leq j \leq 2n}$. In this triangular array $t_{i,j} \geq t_{i+1,j} \geq t_{i+1,j+1}$ since t is a plane partition. Also for these values of i and j the entries $t_{i,j}$ satisfy $0 \leq t_{i,j} \leq 2n + 1 - i$. Now if we reflect this array about a vertical line then rotate clockwise by $\frac{\pi}{4}$ we obtain a staircase shape array x whose entries $x_{i,j}$ satisfy the conditions $x_{i,j} \leq x_{i,j+1} \leq x_{i+1,j}$ and $0 \leq x_{i,j} \leq j$. Now add i to each entry in row i. This gives us the following theorem.

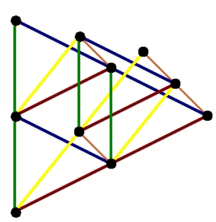
Theorem 2.3 *Totally symmetric self–complementary plane partitions inside a $2n \times 2n \times 2n$ box are in bijection with SSYT of staircase shape δ_n with entries $y_{i,j}$ at most n such that $y_{i,j} \leq y_{i-1,j+1} + 1$. Denote this set as SST_n.*

$$
\begin{array}{c}
\text{TSSCPP}\\
\begin{array}{cccccccc}
8&8&8&8&6&6&4&4\\
8&8&8&8&6&5&4&4\\
8&8&7&6&5&4&3&2\\
8&8&6&5&4&3&2&2\\
6&6&5&4&3&2&\cdot&\cdot\\
6&5&4&3&2&1&\cdot&\cdot\\
4&4&3&2&\cdot&\cdot&\cdot&\cdot\\
4&4&2&2&\cdot&\cdot&\cdot&\cdot
\end{array}
\end{array}
\iff
\begin{array}{l}
\text{Fundamental}\\
\text{domain}\\
\begin{array}{llll}
3&&&\\
2&1&&\\
\cdot&\cdot&\cdot&\\
\cdot&\cdot&\cdot&\cdot
\end{array}
\end{array}
\iff
\begin{array}{l}
\text{Reflected}\\
\text{\& rotated}\\
\begin{array}{llll}
\cdot&\cdot&1&3\\
\cdot&\cdot&2&\\
\cdot&\cdot&&\\
\cdot&&&
\end{array}
\end{array}
\iff
\begin{array}{l}
i\text{ added}\\
\text{to row } i\\
\begin{array}{llll}
1&1&2&4\\
2&2&4&\\
3&3&&\\
4&&&
\end{array}
\end{array}
$$

Ordered by componentwise comparison of the entries, SST_n forms a distributive lattice $J(Q)$ where the Hasse diagram of the poset of join-irreducibles Q (for $n = 4$) is shown below. (Note that in this paper we will extend the definition of a Hasse diagram slightly by at times drawing edges in the Hasse diagram from x to y when $x < y$ but y does not cover x, like the yellow edges below.)

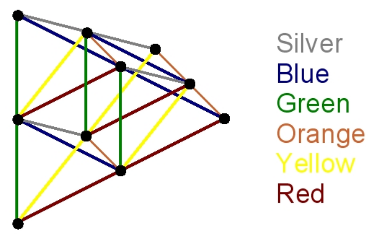


Suppose we put the posets P and Q together and consider SSYT with both conditions on the diagonals. The Hasse diagram of our new poset looks like a tetrahedron with one direction of edges missing:



Inserting those extra edges yields a tetrahedral poset, denoted T_n, whose lattice of order ideals we find to be in bijection with totally symmetric plane partitions (TSPPs) inside an $(n-1) \times (n-1) \times (n-1)$ box.

We define T_n precisely as follows. Define the unit vectors $\overrightarrow{r} = (\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$, $\overrightarrow{g} = (0, 1, 0)$, and $\overrightarrow{b} = (-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$, $\overrightarrow{y} = (\frac{\sqrt{3}}{6}, \frac{1}{2}, \frac{\sqrt{6}}{3})$, $\overrightarrow{o} = (\frac{-\sqrt{3}}{3}, 0, \frac{\sqrt{6}}{3})$, $\overrightarrow{s} = (\frac{-\sqrt{3}}{6}, \frac{1}{2}, -\frac{\sqrt{6}}{3})$. Let the elements of T_n be defined as the coordinates of all the points reached by linear combinations of $\overrightarrow{r}$, $\overrightarrow{g}$, and $\overrightarrow{y}$. Thus as a set $T_n = \{c_1 \overrightarrow{r} + c_2 \overrightarrow{g} + c_3 \overrightarrow{y},\ c_1, c_2, c_3 \in \mathbb{Z}_{\geq 0},\ c_1 + c_2 + c_3 \leq n - 2\}$. Let all the vectors $\overrightarrow{r}$, $\overrightarrow{g}$, and $\overrightarrow{y}$ used to define the elements of T_n be directed edges in the Hasse diagram of colors red, green, and yellow. Additionally draw as edges of colors blue, orange, and silver the vectors $\overrightarrow{b}$, $\overrightarrow{o}$, and $\overrightarrow{s}$ between poset elements wherever possible. The partial order of T_n is defined so that the corner vertex with edges colored red, green, and yellow is the smallest element, the corner vertex with edges colored silver, green, and blue is the largest element, and the other two corner vertices are ordered such that the one with silver, yellow, and orange edges is above the one with orange, red, and blue edges.



Since the ASM and TSSCPP posets appear as subposets of T_n with certain edge colors, we now investigate the subposets of T_n made up of all the different combinations of edge colors. Surprisingly, for almost all of these posets, there exists a nice product formula for the number of order ideals and a bijection between these order ideals and an interesting set of combinatorial objects. We wish to consider only subsets of the colors which include all the colors whose covering relations are induced by combinations of other colors, which are the *admissible* subsets of the following definition.

Definition 2.4 *Let a subset S of the six colors $\{$red, blue, green, orange, yellow, silver$\}$ (abbreviated $\{r, b, g, o, y, s\}$) be* admissible *if all of the following are true: If $\{r, b\} \subseteq S$ then $g \in S$, if $\{o, s\} \subseteq S$ then $b \in S$, if $\{s, y\} \subseteq S$ then $g \in S$, and if $\{r, o\} \subseteq S$ then $y \in S$.*

Given an admissible subset S of the colors $\{r, b, g, o, y, s\}$, let $T_n(S)$ denote the poset formed by the vertices of T_n together with all the edges whose colors are in S. The induced colors will be in parentheses.

We give a bijection between order ideals of $T_n(S)$, S an admissible subset of $\{r, b, g, y, o, s\}$, and arrays of integers with certain inequality conditions.

Definition 2.5 *Let S be an admissible subset of $\{r, b, g, y, o, s\}$ and suppose $g \in S$. Define $Y_n(S)$ to be the set of all integer arrays x of staircase shape δ_n with entries $x_{i,j}$, $1 \leq i \leq n$, $0 \leq j \leq n - i$ satisfying both $i \leq x_{i,j} \leq j + i$ and the following inequality conditions corresponding to the additional colors in S: orange: $x_{i,j} < x_{i+1,j}$, red: $x_{i,j} \leq x_{i-1,j+1} + 1$, yellow: $x_{i,j} \leq x_{i,j+1}$, blue: $x_{i,j} \leq x_{i+1,j-1}$, silver: $x_{i,j} \leq x_{i,j-1} + 1$*

For the proof of the following proposition we will need to note the following: $T_n(\{r, b, (g)\})$ is a disjoint poset, whose connected components we will call, from smallest to largest, $P_2, P_3, \ldots P_n$. Thus T_n can be thought of as the poset which results from beginning with P_n, overlaying $P_{n-1}, P_{n-2}, \ldots, P_3, P_2$ successively, and connecting each P_i to P_{i-1} by the orange, yellow, and silver edges.

Proposition 2.6 *If S is an admissible subset of $\{r, b, g, y, o, s\}$ and $g \in S$ then $Y_n(S)$ is in weight–preserving bijection with $J(T_n(S))$ where the weight of $x \in Y_n(S)$ is given by $\sum_{i=1}^{n-1} \sum_{j=0}^{n-i} (x_{i,j} - i)$ and the weight of $I \in J(T_n(S))$ equals $|I|$.*

Proof: Let S be an admissible subset of $\{r, b, g, y, o, s\}$ and suppose $g \in S$. Recall that T_n is made up of the layers P_k where $2 \leq k \leq n$. Since $g \in S$, P_k contains $k - 1$ green–edged chains of length $k - 1, \ldots, 2, 1$. For each $P_k \subseteq T_n$ let the $k - 1$ green chains inside P_k determine the entries $x_{i,j}$ ($j \neq 0$) of an integer array on the diagonal where $i + j = k$. In particular, given an order ideal I of $T_n(S)$ form an array x by setting $x_{i,j}$ equal to i plus the number of elements in the induced order ideal of the length j

green chain inside P_{i+j} (in column 0 $x_{i,0} = i$). This defines x as an integer array of staircase shape δ_n whose entries satisfy $i \leq x_{i,j} \leq j + i$. Also since each entry $x_{i,j}$ is given by an induced order ideal and since each element of T_n is in exactly one green chain we know that $|I| = \sum_{i,j} x_{i,j} - i$. Thus the weight is preserved.

Now it is left to determine what the other colors mean in terms of the array entries. Since the colors red and blue connect green chains from the same P_k we see that inequalities corresponding to red and blue should relate entries of x on the same northeast to southwest diagonal of x. So if $r \in S$ then $x_{i,j} \leq x_{i-1,j+1} + 1$ and if $b \in S$ then $x_{i,j} \leq x_{i+1,j-1}$. The colors yellow, orange, and silver connect P_k to P_{k+1} for $2 \leq k \leq n-1$. So from our construction we see that if $o \in S$ then $x_{i,j} \leq x_{i+1,j}$, if $y \in S$ then $x_{i,j} \leq x_{i,j+1}$, and if $s \in S$ then $x_{i,j} \leq x_{i,j-1} + 1$. $\Box$

3 Combinatorial objects as subposet order ideals

We will now give product formulas for the number of order ideals of $T_n(S)$ for S an admissible set of colors along with the rank generating functions wherever we have them, where $F(P, q)$ denotes the rank generating function for the poset P. For the sake of comparison we have also written each formula as a product over the same indices $1 \leq i \leq j \leq k \leq n-1$ in a way which is reminiscent of the MacMahon box formula. See Figure 1 for the big picture of inclusions and bijections between these order ideals. For a more detailed discussion, see [6].

Theorem 3.1 *For any color $x \in \{r, b, y, g, o, s\}$*

$$F(J(T_n(\{x\})), q) = \prod_{j=1}^{n} j!_q = \prod_{1\leq i\leq j\leq k\leq n-1} \frac{[i+1]_q}{[i]_q}. \tag{1}$$

Proof: $T_n(\{x\})$ is the disjoint sum of $n-j$ chains of length $j-1$ as j goes from 1 to $n-1$. So the number of order ideals is the product of the number of order ideals of each chain. $\Box$

Theorem 3.2 *If $S \in \{\{g, o\}, \{r, s\}, \{b, y\}\}$ then*

$$F(J(T_n(S)), q) = \prod_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q = \prod_{1\leq i\leq j\leq k\leq n-1} \frac{[j+1]_q}{[j]_q}. \tag{2}$$

Proof: The arrays $Y_n(\{g, o\})$ strictly decrease down columns and have no conditions on the rows. Thus in a column of length j there must be j distinct integers between 1 and n; this is counted by $\binom{n}{j}$. If we give a weight to each of these integers of q to the power of that integer minus its row, we have a set q–enumerated by the q–binomial coefficient $\begin{bmatrix} n \\ j \end{bmatrix}_q$. Thus $\prod_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q$ is the generating function of the arrays $Y_n(\{g, o\})$ and also of the order ideals $F(J(T_n(\{g, o\})), q)$. The posets $T_n(\{g, o\})$, $T_n(\{r, s\})$, and $T_n(\{b, y\}\}$ are all isomorphic, thus the result follows by poset isomorphism. $\Box$

Theorem 3.3 *If $S_1 \in \{\{b, g\}, \{b, s\}, \{y, o\}, \{g, s\}\}$ and $S_2 \in \{\{r, y\}, \{r, g\}, \{y, g\}, \{b, o\}\}$ then*

$$|J(T_n(S_1))| = |J(T_n(S_2))| = \prod_{j=1}^{n} C_j = \prod_{j=1}^{n} \frac{1}{j+1}\binom{2j}{j} = \prod_{1\leq i\leq j\leq k\leq n-1} \frac{i+j+2}{i+j} \tag{3}$$

$$F(J(T_n(S_1)), q) = F(J(T_n^*(S_2)), q) = \prod_{j=1}^{n} C_j(q) \tag{4}$$

where * *is poset dual,* C_j *is the* j*th Catalan number, and* $C_j(q)$ *is the Carlitz–Riordan q–Catalan number defined by the recurrence* $C_j(q) = \sum_{k=1}^{j} q^{k-1} C_{k-1}(q) C_{j-k}(q)$ *with initial conditions* $C_0(q) = C_1(q) = 1$.

Proof: $T_n(\{b, g\})$ is isomorphic to the disjoint sum of posets $P_j(\{b, g\})$ for $2 \le j \le n$ with rank generating functions $C_j(q)$. This can be shown using an easy bijection between these order ideals and Dyck paths from $(0, 0)$ to $(2j, 0)$. Thus the number of order ideals $|J(T_n(\{b, g\}))|$ equals the product $\prod_{j=1}^{n} C_j$ and the rank generating function $F(J(T_n(\{b, g\})), q)$ equals the product $\prod_{j=1}^{n} C_j(q)$. Finally, the posets $T_n(S_1)$ for any choice of $S_1 \in \{\{b, g\}, \{b, s\}, \{y, o\}, \{g, s\}\}$ and the posets $T_n^*(S_2)$ for any $S_2 \in \{\{r, y\}, \{r, g\}, \{y, g\}, \{b, o\}\}$ are all isomorphic, thus the result follows by poset isomorphism. □

Theorem 3.4 *If* S *is an admissible subset of* $\{r, b, g, o, y, s\}$, $|S| = 3$, *and* $S \notin \{\{r, g, y\}, \{s, b, r\}\}$ *then*

$$F(J(T_n(S)), q) = \prod_{j=1}^{n-1} (1 + q^j)^{n-j} = \prod_{1 \le i \le j \le k \le n-1} \frac{[i + j]_q}{[i + j - 1]_q}. \tag{5}$$

Thus if we set $q = 1$ *we have* $|J(T_n(S))| = 2^{\binom{n}{2}}$.

We will prove Theorem 3.4 using two lemmas since there are two nonisomorphic classes of posets $T_n(S)$ where S is admissible, $|S| = 3$, and $S \notin \{\{r, g, y\}, \{s, b, r\}\}$. The first lemma is the case where $T_n(S)$ is a disjoint sum of posets and the second lemma is the case where $T_n(S)$ is a connected poset.

Lemma 3.5 *Suppose* $S \in \{\{o, s, (b)\}, \{s, y, (g)\}, \{o, r, (y)\}, \{b, r, (g)\}\}$. *Then*

$$F(J(T_n(S)), q) = \prod_{j=1}^{n-1} (1 + q^j)^{n-j} = \prod_{1 \le i \le j \le k \le n-1} \frac{[i + j]_q}{[i + j - 1]_q}.$$

Proof: $T_n(\{b, r, (g)\})$ is a disjoint sum of the P_j posets for $2 \le j \le n$. The order ideals of P_j are counted by 2^{j-1} and the rank generating function of $J(P_j)$ is given by $\prod_{i=1}^{j-1}(1 + q^i)$, both of which are proved by induction. Thus $F(T_n(\{b, r, (g)\}), q)$ is the product of $\prod_{i=1}^{j-1}(1 + q^i)$ for $2 \le j \le n$. Rewriting the product we obtain $\prod_{j=2}^{n} \prod_{i=1}^{j-1}(1 + q^i) = \prod_{j=1}^{n-1}(1 + q^j)^{n-j}$. The posets $T_n(S)$ where $S \in \{\{o, s, (b)\}, \{s, y, (g)\}, \{o, r, (y)\}\}$ are isomorphic to $T_n(\{b, r, (g)\})$ so the result follows by poset isomorphism. □

Lemma 3.6 *Suppose* $S \in \{\{r, g, s\}, \{o, b, y\}, \{y, g, o\}, \{b, g, o\}, \{y, g, b\}\}$. *Then*

$$F(J(T_n(S)), q) = \prod_{j=1}^{n-1} (1 + q^j)^{n-j} = \prod_{1 \le i \le j \le k \le n-1} \frac{[i + j]_q}{[i + j - 1]_q}.$$

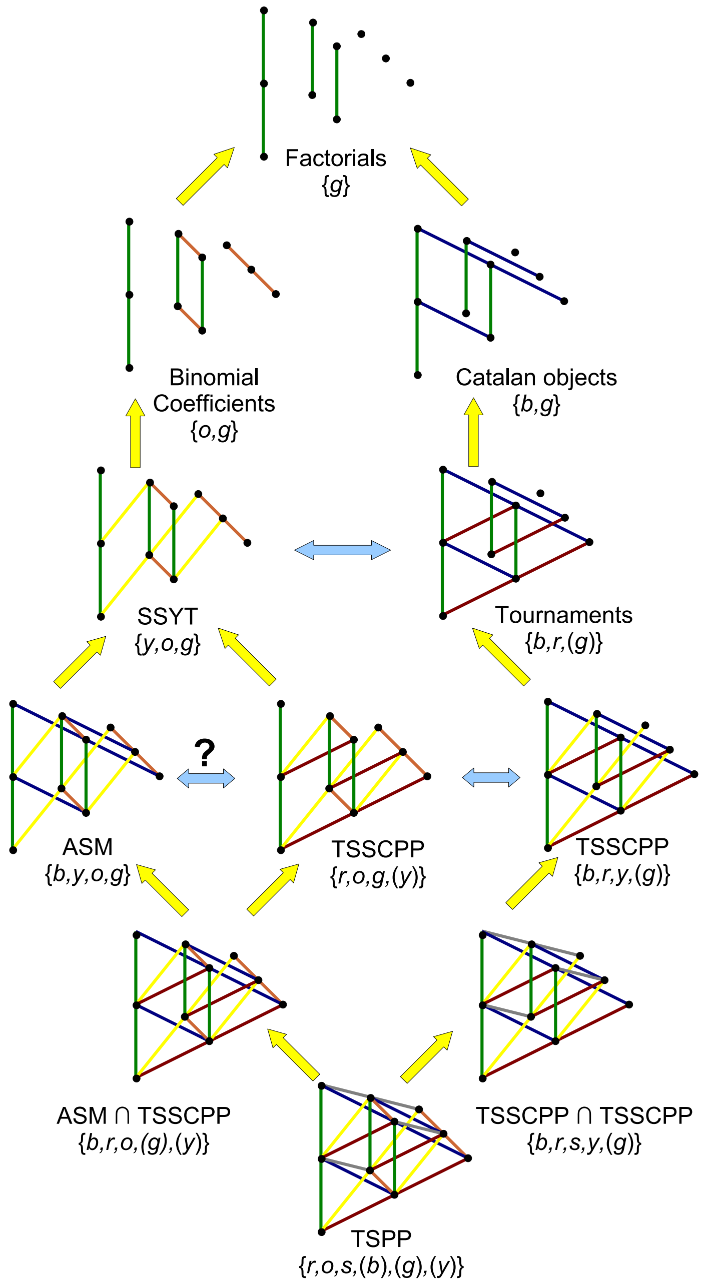


Fig. 1: The big picture of inclusions and bijections between order ideals $J(T_n(S))$. The one sided arrows represent inclusions of one set of order ideals into another. The two sided arrows represent bijections between sets of order ideals. The bijection between the order ideals of the three color posets is in Section 4 and the bijections between TSSCPP posets is by poset isomorphism. The only missing bijection between sets of order ideals of the same size is between ASM and TSSCPP.

Proof: The arrays $Y_n(\{g,y,o\})$ are by Definition 2.5 equivalent to SSYT of staircase shape δ_n, thus their generating function is given by the Schur function $s_{\delta_n}(x_1,x_2,\ldots,x_n)$. Now

$$s_{\delta_n}(x_1,x_2,\ldots,x_n)=\frac{\det(x_i^{2(n-j)})_{i,j=1}^n}{\det(x_i^{n-j})_{i,j=1}^n}=\prod_{1\le i<j\le n}\frac{x_i^2-x_j^2}{x_i-x_j}=\prod_{1\le i<j\le n}(x_i+x_j)$$

using the algebraic Schur function definition and the Vandermonde determinant. The principle specialization of this generating function yields the q–generating function $\prod_{j=1}^{n-1}(1+q^j)^{n-j}$. The posets $T_n(S)$ where $S\in\{\{r,g,s\},\{o,b,y\},\{b,g,o\},\{y,g,b\}\}$ are isomorphic to $T_n(\{g,y,o\})$ so the result follows by poset isomorphism. $\square$

Proof of Theorem 3.4: By Lemma 3.5, if $S\in\{\{o,s,(b)\}$, $\{s,y,(g)\}$, $\{o,r,(y)\}$, $\{b,r,(g)\}\}$ then the generating function $F(J(T_n(S)),q)$ is as above. By Lemma 3.6, if $S\in\{\{r,g,s\}$, $\{o,b,y\}$, $\{y,g,o\}$, $\{b,g,o\}$, $\{y,g,b\}\}$ then generating function $F(J(T_n(S)),q)$ is as above. These are the only admissible subsets S of $\{r,b,g,o,y,s\}$ with $|S|=3$ and $S\notin\{\{r,g,y\},\{s,b,r\}\}$. $\square$

There seems to be no nice product formula for the number of order ideals of the dual posets $T_n(\{r,g,y\})$ and $T_n(\{s,b,r\})$. The number of order ideals up to $n=6$ are: 1, 2, 9, 96, 2498, 161422.

Theorem 3.7 *If S is an admissible subset of $\{r,b,g,o,y,s\}$ and $|S|=4$ then*

$$|J(T_n(S))|=\prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!}=\prod_{1\le i\le j\le k\le n-1}\frac{i+j+k+1}{i+j+k-1}. \tag{6}$$

Proof: The posets $T_n(S)$ for S admissible and $|S|=4$ are $T_n(\{g,y,b,o\})$, the three isomorphic posets $T_n(\{r,o,(y),g\})$, $T_n(\{r,b,(g),y\})$, and $T_n(\{y,s,(g),r\})$, and the three posets dual to these, $T_n(\{y,s,(g),b\})$, $T_n(\{o,s,(b),g\})$, $T_n(\{r,b,(g),s\})$. In Theorem 2.2 we showed that the order ideals of $T_n(\{g,y,b,o\})$ are in bijection with $n\times n$ ASMs and in Theorem 2.3 we showed that the order ideals of $T_n(\{r,o,(y),g\})$ are in bijection with TSSCPPs inside a $2n\times 2n\times 2n$ box. Therefore by poset isomorphism TSSCPPs inside a $2n\times 2n\times 2n$ box are in bijection with the order ideals of any of $T_n(\{r,o,(y),g\})$, $T_n(\{r,b,(g),y\})$, $T_n(\{y,s,(g),r\})$, $T_n^*(\{y,s,(g),b\})$, $T_n^*(\{o,s,(b),g\})$, or $T_n^*(\{r,b,(g),s\})$. Thus by the enumeration of ASMs in [8] and [3] and the enumeration of TSSCPPs in [1] we have the above formula for the number of order ideals. $\square$

There are two different cases for five colors: one case consists of the dual posets $T_n(\{(g),(b),o,y,s\})$ and $T_n(\{r,b,(g),o,(y)\})$ and the other case is $T_n(\{r,b,s,(y),g\})$. A nice product formula has not yet been found for either case.

Theorem 3.8

$$|J(T_n)|=\prod_{1\le i\le j\le n-1}\frac{i+j+n-2}{i+2j-2}=\prod_{1\le i\le j\le k\le n-1}\frac{i+j+k-1}{i+j+k-2} \tag{7}$$

Proof: Totally symmetric plane partitions are plane partitions which are symmetric with respect to all permutations of the x,y,z axes. Thus we can take as a fundamental domain the wedge where $x\ge y\ge z$.

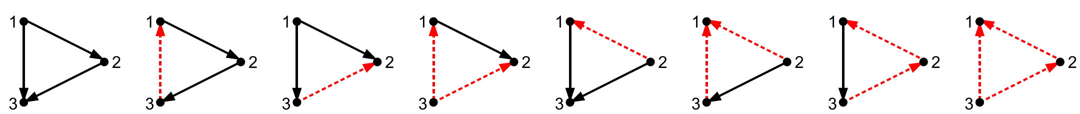
Then if we draw the lattice points in this wedge (inside a fixed bounding box of size $n-1$) as a poset with edges in the x, y, and z directions, we obtain the poset T_n where the x direction corresponds to the red edges of T_n, the y direction to the orange edges, and the z direction to the silver edges. All other colors of edges in T_n are induced by the colors red, silver, and orange. Thus TSPPs inside an $(n-1)\times(n-1)\times(n-1)$ box are in bijection with the order ideals of T_n. Thus by the enumeration of TSPPs in [5] the number of order ideals $|J(T_n)|$ is given by the above formula. $\square$

4 Bijections with tournaments

Theorem 3.4 states that the order ideals of the three color posets $J(T_n(S))$ where S is admissible, $|S|=3$, and $S \notin \{\{r,g,y\},\{s,b,r\}\}$ are counted by $2^{\binom{n}{2}}$. This is also the number of graphs on n labeled vertices and equivalently the number of tournaments on n vertices. A tournament is a complete directed graph with labeled vertices. We now discuss bijections between these order ideals and tournaments.

Theorem 4.1 *There exists an explicit bijection between the order ideals of the poset $T_n(\{b,r,(g)\})$ and tournaments on n vertices.*

$$\begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{array} \quad \begin{array}{lll} 1 & 1 & 2 \\ 2 & 2 & \\ 3 & & \end{array} \quad \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \quad \begin{array}{lll} 1 & 1 & 3 \\ 2 & 3 & \\ 3 & & \end{array} \quad \begin{array}{lll} 1 & 2 & 1 \\ 2 & 2 & \\ 3 & & \end{array} \quad \begin{array}{lll} 1 & 2 & 2 \\ 2 & 2 & \\ 3 & & \end{array} \quad \begin{array}{lll} 1 & 2 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \quad \begin{array}{lll} 1 & 2 & 3 \\ 2 & 3 & \\ 3 & & \end{array}$$



Proof: The colors blue and red correspond to inequalities on $Y_n(\{b,r,(g)\})$ such that as one goes up the southwest to northeast diagonals at each step the next entry has the choice between staying the same and decreasing by one. Therefore since each of the $\binom{n}{2}$ entries of the array not in the 0th column has exactly two choices of values given the value of the entry to the southwest, we may consider each array entry $\alpha_{i,j}$ with $j \geq 1$ to symbolize the outcome of the game between i and $i+j$ in a tournament. If $\alpha_{i,j} = \alpha_{i+1,j-1}$ say the outcome of the game between i and $i+j$ is an upset and otherwise not. Thus tournaments on n vertices are in bijection with the arrays $Y_n(\{b,r,(g)\})$ and also with the order ideals of $T_n(\{b,r,(g)\})$. $\square$

The bijection between the order ideals of the connected three color posets of Lemma 3.6 and tournaments is due to Sundquist in [7] and involves repeated use of jeu de taquin and column deletion to go from SSYT of shape δ_n and largest entry n to certain tableaux in bijection with tournaments on n vertices.

Next we describe which subsets of tournaments correspond to TSSCPPs.

Theorem 4.2 *TSSCPPs inside a $2n \times 2n \times 2n$ box are in bijection with tournaments on vertices labeled $1, 2, \ldots, n$ which satisfy the following condition on the upsets: if vertex v has k upsets with vertices in $\{u, u+1, \ldots, v-1\}$ then vertex $v-1$ has at most k upsets with vertices in $\{u, u+1, \ldots, v-2\}$.*

Proof: We have seen in Theorem 4.1 the bijection between the order ideals of $T_n(\{r,b,(g)\})$ and tournaments on n vertices. Thus if we consider the TSSCPP arrays $Y_n(\{r,b,(g),y\})$ we need only find an interpretation for the yellow edges in terms of tournaments. Recall that yellow corresponds to a weak increase across the rows of α. To satisfy this condition, for each choice of $i \in \{1, \ldots, n-1\}$ and

$j \in \{1, \ldots, n-i-1\}$ the number of diagonal equalities to the southwest of $\alpha_{i,j}$ must be less than or equal to the number of diagonal equalities to the southwest of $\alpha_{i,j+1}$. So in terms of tournaments, the number of upsets between $i+j+1$ and vertices greater than or equal to i must be greater than or equal to the number of upsets between $i+j$ and vertices greater than or equal to i. □

5 Connections between ASMs, TSSCPPs, and tournaments

In this section we discuss the expansion of the tournament generating function as a sum over ASMs and derive a new expansion as a sum over TSSCPPs. We begin with the following theorem of Robbins and Rumsey [4]. We need the following notion: the inversion number of an ASM A is defined as $I(A) = \sum A_{ij} A_{k\ell}$ where the sum is over all i, j, k, ℓ such that $i > k$ and $j < \ell$.

Theorem 5.1 (Robbins–Rumsey) *Let A_n be the set of $n \times n$ alternating sign matrices, and for $A \in A_n$ let $I(A)$ denote the inversion number of A and $N(A)$ the number of -1 entries in A, then*

$$\prod_{1 \le i < j \le n} (x_i + \lambda x_j) = \sum_{A \in A_n} \lambda^{I(A)} \left(1 + \lambda^{-1}\right)^{N(A)} \prod_{i,j=1}^{n} x_j^{(n-i)A_{ij}}. \tag{8}$$

Note that the left–hand side is the generating function for tournaments on n vertices where each factor of $(x_i + \lambda x_j)$ represents the outcome of the game between i and j in the tournament. If x_i is chosen then the expected winner, i, is the actual winner, and if λx_j is chosen then j is the unexpected winner and the game is an upset. Thus in each monomial in the expansion of $\prod_{1 \le i < j \le n} (x_i + \lambda x_j)$ the power of λ equals the number of upsets and the power of x_k equals the number of wins of k.

We rewrite Theorem 5.1 in different notation which will also be needed later. For any staircase shape integer array $\alpha \in Y_n(S)$ let $E_{i,k}(\alpha)$ be the number of entries of value k in row i equal to their southwest diagonal neighbor, $E^i(\alpha)$ be the number of entries in (southwest to northeast) diagonal i equal to their southwest diagonal neighbor, and $E_i(\alpha)$ be the number of entries in row i equal to their southwest diagonal neighbor, that is, $E_i(\alpha) = \sum_k E_{i,k}(\alpha)$. Also let $E(\alpha)$ be the total number of entries of α equal to their southwest diagonal neighbor, that is, $E(\alpha) = \sum_i E_i(\alpha) = \sum_i E^i(\alpha)$. We now define variables for the content of α. Let $C_{i,k}(\alpha)$ be the number of entries in row i with value k and let $C_k(\alpha)$ be the total number of entries of α equal to k, that is, $C_k(\alpha) = \sum_i C_{i,k}(\alpha)$. Let $N(\alpha)$ be the number of entries of α strictly greater than their neighbor to the west and strictly less than their neighbor to the southwest. When $\alpha \in Y_n(\{b, y, o, g\})$ then $N(\alpha)$ equals the number of -1 entries in the corresponding ASM.

Using this notation we reformulate Theorem 5.1 in the following way.

Theorem 5.2 *The generating function for tournaments on n vertices can be expanded as a sum over the ASM arrays $Y_n(\{b, y, o, g\})$ in the following way.*

$$\prod_{1 \le i < j \le n} (x_i + \lambda x_j) = \sum_{\alpha \in Y_n(\{b,y,o,g\})} \lambda^{E(\alpha)} (1 + \lambda)^{N(\alpha)} \prod_{k=1}^{n} x_k^{C_k(\alpha)-1} \tag{9}$$

Proof: First we rewrite Equation 8 by factoring out λ^{-1} from each $\left(1 + \lambda^{-1}\right)$.

$$\prod_{1 \le i < j \le n} (x_i + \lambda x_j) = \sum_{A \in A_n} \lambda^{I(A) - N(A)} \left(1 + \lambda\right)^{N(A)} \prod_{i,j=1}^{n} x_j^{(n-i)A_{ij}}$$

Let $\alpha \in Y_n(\{b,y,o,g\})$ be the array which corresponds to A. It is left to show that $I(A)-N(A)=E(\alpha)$ and $\prod_{i,j=1}^n x_j^{(n-i)A_{ij}} = \prod_{j=1}^n x_j^{C_j(\alpha)-1}$. In the latter equality take the product over i of the left hand side: $\prod_{i,j=1}^n x_j^{(n-i)A_{ij}} = \prod_{j=1}^n x_j^{\sum_{i=1}^n (n-i)A_{ij}}$. We wish to show $C_j(\alpha)-1 = \sum_{i=1}^n (n-i)A_{ij}$. $C_j(\alpha)$ equals the number of entries of α with value j, so $C_j(\alpha)-1$ equals the number of entries of α with value j not counting the j in the 0th column. Now the number of js in columns 1 through $n-1$ of α equals the number of 1s in column j of A plus the number of zeros in column j of A which are south of a 1 with no -1s in between. This is precisely what $\sum_{i=1}^n (n-i)A_{ij}$ counts by taking a positive contribution from every 1 and every entry below that 1 in column j and then subtracting one for every -1 and every entry below that -1 in column j. Thus $C_j(\alpha)-1 = \sum_{i=1}^n (n-i)A_{ij}$ so that $\prod_{i,j=1}^n x_j^{(n-i)A_{ij}} = \prod_{j=1}^n x_j^{C_j(\alpha)-1}$.

Now we wish to show that $I(A)-N(A)=E(\alpha)$. Fix i, j, and ℓ and consider $\sum_{k<i} A_{ij}A_{k\ell}$. Let k' be the row of the southernmost nonzero entry in column ℓ such that $k' < i$. If there exists no such k' (that is, $A_{k\ell} = 0 \,\forall\, k < i$) or if $A_{k'\ell} = -1$ then $\sum_{k>i} A_{ij}A_{k\ell} = 0$ since there must be an even number of nonzero entries in $\{A_{k\ell}, k < i\}$ half of which are 1 and half of which are -1. If $A_{k'\ell} = 1$ then $\sum_{k<i} A_{ij}A_{k\ell} = A_{ij}$. Thus $I(A) = \sum_{i,j} \alpha_{ij}A_{ij}$ where α_{ij} equals the number of columns east of column j such that $A_{k'\ell}$ with $k' > i$ exists and equals 1. Let column ℓ' be one of the columns counted by α_{ij}. Then $A_{i\ell'}$ cannot equal 1, otherwise $A_{k'\ell'}$ would either not exist or equal -1. If $A_{i\ell'} = 0$ then in α there is a corresponding diagonal equality. If $A_{i\ell'} = -1$ then there is no diagonal equality in α. Thus $I(A) = E(\alpha) + N(A)$. $\square$

Many people have wondered what the TSSCPP analogue of the -1 in an ASM may be. The following theorem does not give a direct analogue, but rather expands the left–hand side of (9) as a sum over TSSCPPs instead of ASMs.

Theorem 5.3 *The generating function for tournaments on n vertices can be expanded as a sum over the TSSCPP arrays $Y_n(\{b,r,(g),y\})$ in the following way.*

$$\prod_{1\le i<j\le n}(x_i+\lambda x_j) = \sum_{\alpha\in Y_n(\{b,r,(g),y\})} \lambda^{E(\alpha)} \prod_{i=1}^{n-1} x_i^{n-i-E_i(\alpha)} \sum_{\textit{row shuffles } \alpha' \textit{ of } \alpha} \prod_{j=1}^{n-1} x_j^{E^j(\alpha')} \tag{10}$$

where a row shuffle α' of $\alpha \in Y_n(\{b,r,(g),y\}$ is an array obtained by reordering the entries in the rows of α in such a way that $\alpha' \in Y_n(\{b,r,(g)\}$. Also, setting the x's to 1 we have

$$(1+\lambda)^{\binom{n}{2}} = \sum_{\alpha\in Y_n(\{b,r,(g),y\})} \lambda^{E(\alpha)} \prod_{1\le i\le k\le n-1} \binom{C_{i+1,k}(\alpha)}{E_{i,k}(\alpha)}. \tag{11}$$

Proof: We begin with the set $Y_n(\{b,r,(g),y\})$ and remove the inequality restriction corresponding to the color yellow to obtain the arrays $Y_n(\{b,r,(g)\})$ (which are in bijection with tournaments). We use the following algorithm for turning any $\alpha \in Y_n(\{b,r,(g)\})$ into an element of $Y_n(\{b,r,(g),y\})$ thus grouping all the elements of $Y_n(\{b,r,(g)\})$ into fibers over the elements of $Y_n(\{b,r,(g),y\})$. Assume each row of α below row i is weakly increasing. Thus $\alpha_{i+1,j} \le \alpha_{i+1,j+1}$. If $\alpha_{i+1,j} < \alpha_{i+1,j+1}$ then $\alpha_{i,j+1} \le \alpha_{i+1,j+2}$ since $\alpha_{i,j+1} \in \{\alpha_{i+1,j}, \alpha_{i+1,j}-1\}$ and $\alpha_{i,j+2} \in \{\alpha_{i+1,j+1}, \alpha_{i+1,j+1}-1\}$ by the inequalities corresponding to red and blue. So the only entries which may be out of order in row i are those for which their southwest neighbors are equal. If $\alpha_{i+1,j} = \alpha_{i+1,j+1}$ but $\alpha_{i,j+1} > \alpha_{i,j+2}$ it must be that $\alpha_{i,j+1} = \alpha_{i+1,j}$ and $\alpha_{i,j+2} = \alpha_{i+1,j+1} - 1$. So we may swap $\alpha_{i,j+1}$ and $\alpha_{i,j+2}$ along with their entire northeast diagonals while not violating the red and blue inequalities. By completing this process for all rows we obtain an array with weakly increasing rows which is thus in $Y_n(\{b, r, (g), y\})$.

Now we do a weighted count of how many arrays in $Y_n(\{b, r, (g)\})$ are mapped to a given array in $Y_n(\{b, r, (g), y\})$. Again we rely on the fact that entries in a row can be reordered only when their southwest neighbors are equal. Thus to find the weight of all the $Y_n(\{b, r, (g)\})$ arrays corresponding to a single $Y_n(\{b, r, (g), y\})$ array we simply need to find the set of diagonals containing equalities. The diagonal equalities give a weight dependent on which diagonal they are in, whereas the diagonal inequalities give a weight according to their row (which remains constant). Thus if we are keeping track of the x_i weight we can do no better than to write this as a sum over all the allowable shuffles of the rows of α with the x weight of the diagonal equalities dependent on the position. Thus we have Equation (10).

If we set $x_i = 1$ for all i and only keep track of the λ we can make a more precise statement. The above proof shows that the λ's result from the diagonal equalities, and the number of different reorderings of the rows tell us the number of different elements of $Y_n(\{b, r, (g)\})$ which correspond to a given element of $Y_n(\{b, r, (g), y\})$. We count this number of allowable reorderings as a product over all rows i and all array values k as $\binom{C_{i+1,k}(\alpha)}{E_{i,k}(\alpha)}$. This yields Equation (11). $\square$

The difference in the weighting of ASMs and TSSCPPs in Theorems 5.2 and 5.3 is substantial. For ASMs the more complicated part of the formula arises in the power of λ and for TSSCPPs the complication comes from the x variables. These theorems are also strangely similar. They show that the tournament generating function can be expanded as a sum over either ASMs or TSSCPPs, but we still have no direct reason why the number of summands should be the same. The combination of Theorems 5.2 and 5.3 may contribute toward finding a bijection between ASMs and TSSCPPs, but the differences between these expansions show why a bijection is not obvious.

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Combinatorial formulas for ⅃-coordinates in a totally nonnegative Grassmannian, extended abstract

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Postnikov constructed a decomposition of a totally nonnegative Grassmannian $(\mathrm{Gr}_{kn})_{\geq 0}$ into positroid cells. We provide combinatorial formulas that allow one to decide which cell a given point in $(\mathrm{Gr}_{kn})_{\geq 0}$ belongs to and to determine affine coordinates of the point within this cell. This simplifies Postnikov's description of the inverse boundary measurement map and generalizes formulas for the top cell given by Speyer and Williams. In addition, we identify a particular subset of Plücker coordinates as a totally positive base for the set of non-vanishing Plücker coordinates for a given positroid cell.

Keywords: positroid, totally nonnegative Grassmannian, Le-diagram

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Postnikov [4] has described a cell decomposition of a totally nonnegative Grassmannian into positroid cells, which are indexed by ⅃-diagrams; this decomposition is analogous to the matroid stratification of a real Grassmannian given by Gel'fand, Goresky, MacPherson, and Serganova [2]. Postnikov also introduced a parametrization of each positroid cell using a collection of parameters which we call ⅃-coordinates.

In this extended abstract, we give an informal description of the main results of [8], in which the reader will find rigorous formulations and proofs. Specifically, we give an explicit criterion for determining which positroid cell contains a given point in a totally nonnegative Grassmannian and explicit combinatorial formulas for the ⅃-coordinates of a point. This generalizes the formulas of Speyer and Williams given for the top dimensional positroid cell [5], and provides a simpler description of Postnikov's inverse boundary measurement map, which was given recursively in [4]. For a fixed positroid cell, our formulas are written in terms of a minimal set of Plücker coordinates, and this minimal set forms a totally positive base (in the sense of Fomin and Zelevinsky [1]) for the set of Plücker coordinates which do not vanish on the specified cell.

1 Positroid stratification and the boundary measurement map

In this section, we review Postnikov's positroid stratification of a totally nonnegative Grassmannian and boundary measurement map.

Let Gr_{kn} denote the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. A point $x \in \mathrm{Gr}_{kn}$ can be described by a collection of (projective) Plücker coordinates $(P_J(x))$, indexed by the k-element subsets

$J \subset [n]$. The *totally nonnegative Grassmannian* $(\mathrm{Gr}_{kn})_{\geq 0}$ is the subset of points $x \in \mathrm{Gr}_{kn}$ such that all Plücker coordinates $P_J(x)$ can be chosen to be simultaneously nonnegative.

In [2], the authors gave a decomposition of the Grassmannian Gr_{kn} into *matroid strata*. More precisely, for a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$, let $S_{\mathcal{M}}$ denote the subset of points $x \in \mathrm{Gr}_{kn}$ such that $P_J(x) \neq 0$ if and only if $J \in \mathcal{M}$. In particular, each possible vanishing pattern of Plücker coordinates is given by a unique (realizable) matroid $\mathcal{M}$. In [4], Postnikov studies a natural analogue of the matroid stratification for the totally nonnegative Grassmannian, a decomposition into disjoint *positroid cells* taking the form $(S_{\mathcal{M}})_{\geq 0} = S_{\mathcal{M}} \cap (\mathrm{Gr}_{kn})_{\geq 0}$.

Definition 1.1. A ⅃-*diagram* is a partition λ together with a filling of the boxes of the Young diagram of λ with entries 0 and $+$ satisfying the ⅃-*property*: there is no 0 which has a $+$ above it (in the same column) and a $+$ to its left (in the same row).

Replacing the boxes labeled $+$ in a ⅃-diagram with positive real numbers, called ⅃-*coordinates*, we obtain a ⅃-*tableau*. Let $\mathbf{T}_L$ denote the set of ⅃-tableaux whose vanishing pattern is determined by the ⅃-diagram L. Note that $\mathbf{T}_L$ is an affine space whose dimension is equal to the number of "+" entries in L, which we denote by $|L|$.

For a box B in λ, we let L_B and T_B denote the labels of the box B in the ⅃-diagram L and the ⅃-tableau T, respectively.

In the positroid cell decomposition of $(\mathrm{Gr}_{kn})_{\geq 0}$ given in [4], the positroid cells are indexed by ⅃-diagrams L which fit inside a $k \times (n-k)$ rectangle. Further, the positroid corresponding to a fixed ⅃-diagram L is parametrized by the ⅃-tableaux $T \in \mathbf{T}_L$, i.e., those with vanishing pattern given by L.

T_{17}	0	0	0	0	0	0
0	0	0	T_{24}	0	0	0
0	T_{36}	0	T_{34}	T_{33}	0	T_{31}
0	0	T_{45}	T_{44}	T_{43}	0	
T_{57}	T_{56}	T_{55}	T_{54}			

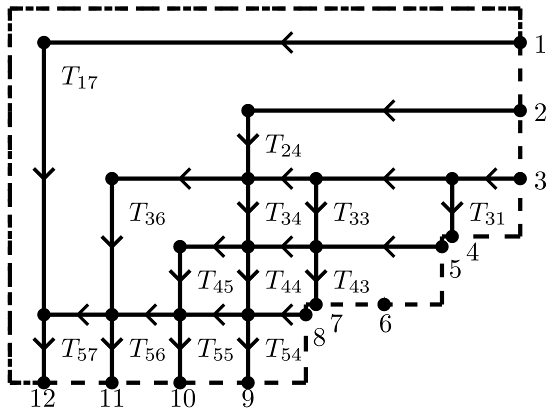

Fig. 1: The ⅃-tableau T and Γ-network N_T for a point in $(Gr_{5,12})_{\geq 0}$. We have shape $\lambda = (7,7,7,6,4)$ and boundary sources $I = \{1,2,3,5,8\}$.

The parametrization described below is a special case of Postnikov's boundary measurement map. To give a formula for this parametrization, we need to introduce certain planar networks, called Γ-networks, which are in bijection with ⅃-tableaux. As shown in Figure 1, given a ⅃-tableau T, we start with a boundary disk of shape λ, draw a hook for each positive entry L_B, and give the resulting face under that hook the weight L_B. Further, if T has an empty row or column, we place an isolated vertex on the east or south boundary of the disk. All edges are directed from east to west or north to south. We now have one *boundary source* for each row of λ and one *boundary sink* for each column of lambda. Let I denote the set of boundary sources.

In the special case of Γ-networks, the definition of Postnikov's map given in [4] can be viewed as an instance of the classical formula of Lindström [3]. This formula is usually given in terms of weights of edges; we apply Postnikov's transformation from edge weights to face weights [4] to obtain the following restatement of his definition.

Definition 1.2. For each Ꞁ-diagram L which fits in a $k \times (n-k)$ rectangle, the *boundary measurement map* $\mathrm{Meas}_L : \mathbf{T}_L \to (\mathrm{Gr}_{kn})_{\geq 0}$ is defined by

$$P_J(\mathrm{Meas}_L(T)) = \sum_{A \in \mathcal{A}_J(N_T)} \mathrm{wt}(A), \text{ where}$$

- N_T is the Γ-network corresponding to the Ꞁ-tableau T, and its boundary source set is labeled by I,
- $\mathcal{A}_J(N_T)$ is the collection of non-intersecting path families $A = \{A_i\}_{i \in I}$ in N_T from the boundary sources I to the boundary destinations J,
- $\mathrm{wt}(A) = \prod_{i \in I} \mathrm{wt}(A_i)$, and
- the weight $\mathrm{wt}(A_i)$ of a path A_i in the family A is the product of the weights of the faces of N_T which lie southeast of A_i.

For a Ꞁ-diagram L, let G_L be the corresponding Γ-graph. Let us define the set $\mathcal{M}_L \subseteq \binom{[n]}{k}$ by the condition that $J \in \mathcal{M}_L$ if and only if there exists a non-intersecting path collection in G_L with sources I and destinations J. It can be shown that $\mathcal{M}_L$ has the structure of a matroid, but this is not necessary for our purposes. Further, it is easily verified that for distinct Ꞁ-diagrams L and L^*, we have $\mathcal{M}_L \neq \mathcal{M}_{L^*}$.

Theorem 1.3. *[4] For each Ꞁ-diagram L which fits in a $k \times (n-k)$ rectangle, the map $\mathrm{Meas}_L : \mathbf{T}_L \to (\mathrm{Gr}_{kn})_{\geq 0}$ is injective, and the image $\mathrm{Meas}_L(\mathbf{T}_L)$ is exactly the positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$.*

These positroid cells are pairwise disjoint, and the union $\bigcup_L (S_{\mathcal{M}_L})_{\geq 0}$, taken over all Ꞁ-diagrams L which fit inside the $k \times (n-k)$ rectangle, is the entire totally nonnegative Grassmannian $(\mathrm{Gr}_{kn})_{\geq 0}$. Each positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$ is a topological cell; that is, $(S_{\mathcal{M}_L})_{\geq 0}$ is isomorphic to $\mathbb{R}^{|L|}$, where $|L|$ is the number of "+" entries in L. Thus, the positroid cells form a cell decomposition of $(\mathrm{Gr}_{kn})_{\geq 0}$.

In Postnikov's work [4], this result is proved by giving a recursive algorithm for finding the Ꞁ-tableau T corresponding to a given point in $(\mathrm{Gr}_{kn})_{\geq 0}$. In [8], we obtain explicit combinatorial formulas solving the same problem. This is done in two stages. First, we give an explicit rule for determining which positroid cell contains a given point. Next, we give two combinatorial formulas for the inverse of each particular map Meas_L (i.e., formulas for the corresponding Ꞁ-coordinates) in terms of the relevant Plücker coordinates.

2 Determining the positroid cell of a point in $(\mathrm{Gr}_{kn})_{\geq 0}$

In this section, we give an explicit formula for the ⅃-tableau $L(x)$ that determines which positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$ a given point $x \in (\mathrm{Gr}_{kn})_{\geq 0}$ belongs to. Let $x \in (\mathrm{Gr}_{kn})_{\geq 0}$ be given by its Plücker coordinates

$$\left(P_J(x) : J \in \binom{[n]}{k}\right).$$

Order the k-subsets of $[n]$ lexicographically. That is, a k-subset $A = \{a_1 < a_2 < \cdots < a_k\}$ is less than or equal to a k-subset $B = \{b_1 < b_2 < \cdots < b_k\}$ if at the smallest index m for which $a_m \neq b_m$, we have $a_m < b_m$.

Set $\mathcal{M}(x) = \{J \in \binom{[n]}{k} : P_J(x) \neq 0\}$. Let $I = \{i_1 < i_2 < \cdots < i_k\}$ be the lexicographically minimum set in $\mathcal{M}(x)$. Let $\lambda(x)$ be the partition in the $k \times (n-k)$ rectangle whose southeastern border is given by the path from the northeast corner of the $k \times (n-k)$ rectangle to its southwest corner which has edges to the south in positions I and edges to the west in positions $[n] \setminus I$. Then $\lambda(x)$ is the shape of the ⅃-diagram corresponding to x.

Next, let $A_{r,c} = \{1, 2, \ldots, i_r\} \cup \{j_c, j_c + 1 \ldots, n\}$. As an intermediate step, we set $M'(B) = \mathrm{lexmax}\{J \in \mathcal{M}(x) : J \cap A_{r,c} = I \cap A_{r,c}\}$. In plain language, this says that we are taking the maximum over sets J which contain all of the sources outside the open interval from i_r to j_c and none of the sinks, i.e., those sets whose interesting behavior happens *inside* the interval. This lexicographically maximal set gives the destinations of the non-intersecting path collection which is nested as far northwest as possible (strictly) under the hook along row r and column c.

Let $M(B) = (M'(B) \setminus \{i_r\}) \cup \{j_c\}$. This corresponds to adding the hook along row r and column c to the path collection above.

Theorem 2.1. *For $x \in (\mathrm{Gr}_{kn})_{\geq 0}$. Then the filling of $\lambda(\mathcal{M}(x))$ given by*

$$L(x)_B = \begin{cases} 0 & \text{if } P_{M(B)}(x) = 0; \\ + & \text{if } P_{M(B)}(x) \neq 0. \end{cases}$$

is a ⅃-diagram, and x lies in the positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$.

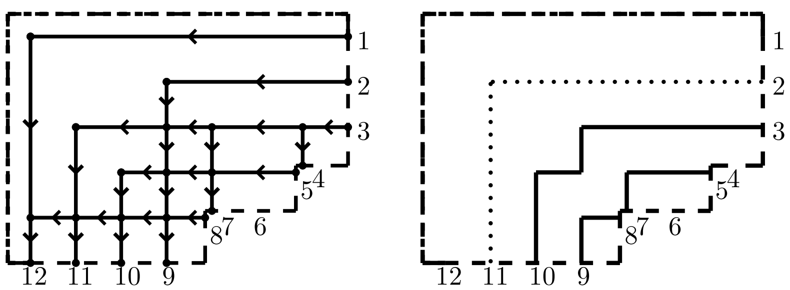

Fig. 2: The Γ-graph of an example in $(\mathrm{Gr}_{5,12})_{\geq 0}$ and the path families corresponding to $M'((2,6),\mathcal{M}_L)$ and $M((2,6),\mathcal{M}_L)$.

Example 2.2. On the left in Figure 2, we have the Γ-graph of the example in Figure 1. We see that $M'((2,6),\mathcal{M}_L) = \{1,2,7,9,10\}$, corresponding to the solid path collection on the right in Figure 2. Adding in the potential (dotted) hook from $i_2 = 2$ to $j_6 = 11$, we have $M((2,6),\mathcal{M}_L) = \{1,7,9,10,11\}$. Since this hook does not occur in the Γ-graph, we must have $P_{M((2,6),\mathcal{M}_L)}(x) = 0$ for this point.

3 The ⅃-tableau associated with a point in $(S_{\mathcal{M}_L})_{\geq 0}$

In Postnikov's original work, the map from $(\mathrm{Gr}_{kn})_{\geq 0}$ to $\bigcup_L \mathbf{T}_L$ is given recursively. In this section, we provide an explicit description of that map. More precisely, given a point $x \in (S_{\mathcal{M}_L})_{\geq 0}$, we give combinatorial formulas for the entries of the parametrizing ⅃-tableau, which we call ⅃-*coordinates* for x.

Informally, for a directed path W in a Γ-graph, we let $\mathcal{OC}(W)$ index the boxes where W turns to the south, and $\mathcal{IC}(W)$ index the boxes where W turns to the west, as in Figure 3. We call these boxes *outer corners* and *inner corners*, respectively.

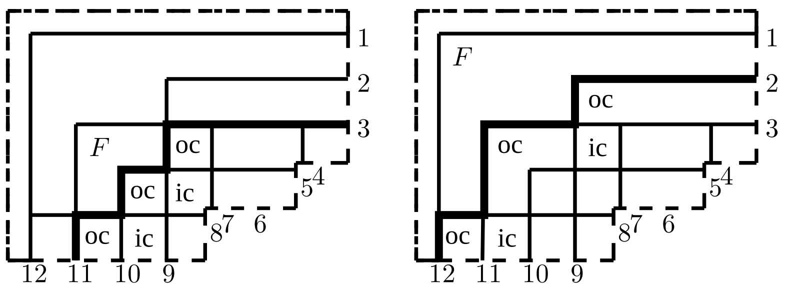

Fig. 3: Finding the outer corners (marked "oc") and the inner corners (marked "ic") of the two paths in bold.

For a face $F = F(B)$ with the box B in its northwest corner, let U_F be the unique hook which determines the northwest boundary of F, and let D_F be the unique path which has the same endpoints as U_F and determines the southeast boundary of F. (If F touches the boundary of the disk, D_F may consist of a union of non-intersecting paths.)

Definition 3.1. For any two faces $F_1 = F(B_1)$ and $F_2 = F(B_2)$ of G_L, we have

$$\mu_L(F_1,F_2) = \begin{cases} 1 & \text{if } F_1 = F_2 \text{ or } B_2 \in \mathcal{IC}(D_{F_1}) \\ -1 & \text{if } B_2 \in \mathcal{OC}(D_{F_1}) \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.2. *Suppose $x \in (S_{\mathcal{M}_L})_{\geq 0}$. Then the ⅃-coordinates of x are the entries of the ⅃-tableau $T(x) \in \mathbf{T}_L$ defined below. That is, $\mathrm{Meas}_L(T(x)) = x$, and $T(x)$ is the unique ⅃-tableau whose image under Meas_L is x.*

$$T(x)_B = \begin{cases} 0 & \text{if } P_{M(B)}(x) = 0; \\ \prod_{L_C=+} \left(\frac{P_{M(C)}(x)}{P_{M'(C)}(x)}\right)^{\mu(B,C)} & \text{if } P_{M(B)}(x) \neq 0. \end{cases}$$

4 ⅃-coordinates of a positroid cell in terms of a minimal set of Plücker coordinates

By Theorem 1.3, the dimension of a positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$ is $|L|$, the number of "+" entries in the corresponding ⅃-diagram L. However, finding the ⅃-coordinates of a point $x \in (S_{\mathcal{M}_L})_{\geq 0}$ via equation (4.1) may require roughly twice this many Plücker variables. In this section, we give a formula for the map from $(S_{\mathcal{M}_L})_{\geq 0}$ to $\mathbf{T}_L$, using precisely $|L|$ Plücker variables. This formula is, of course, equivalent to our first formula modulo Plücker relations, but we now use exactly the desired number of parameters.

Suppose $x \in (S_{\mathcal{M}_L})_{\geq 0}$ and $\mathrm{Meas}_L(T) = x$. As in the previous section, let U_F and D_F denote paths determining the upper and lower boundaries of the face F. Let U'_F and D'_F be the northwest-most paths lying strictly southeast of U_F and D_F, respectively. (Again, it is possible that these are unions of disjoint paths.)

For a path W in a Γ-network N and a box B in λ, we set

$$\varepsilon_W(B) = \begin{cases} 1 & \text{if } B \in \mathcal{OC}(W); \\ -1 & \text{if } B \in \mathcal{IC}(W); \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Suppose $x \in (S_{\mathcal{M}_L})_{\geq 0}$ and $\mathrm{Meas}_L(T) = x$. Then the ⅃-coordinates of x may be written in the alternate form*

$$T_B = \begin{cases} 0 & \text{if } P_{M(B)}(x) = 0; \\ \prod_{L_C=+} (P_{M(C)}(x))^{\varepsilon(C)} & \text{if } P_{M(B)}(x) \neq 0, \end{cases}$$

where $\varepsilon(C) = [\varepsilon_{U_F}(C) - \varepsilon_{U'_F}(C)] - [\varepsilon_{D_F}(C) - \varepsilon_{D'_F}(C)]$.

While this formula may look complicated, it is very easy to use in practice: we simply trace out four easily defined paths, keeping track of where they turn.

The following corollary uses the *totally positive bases* of [1].

Corollary 4.2. *The set of Plücker coordinates*

$$\mathcal{P}_L = \{P_{M(B)} : L_B = +\}$$

forms a totally positive base *for the non-vanishing Plücker coordinates $\{P_J : J \in \mathcal{M}_L\}$ of the positroid cell $(S_{\mathcal{M}_L})_{\geq 0}$. That is, every Plücker coordinate P_J with $J \in \mathcal{M}_L$ can be written as a subtraction-free rational expression (i.e., a ratio of two polynomials with nonnegative integer coefficients) in the elements of $\mathcal{P}_L$, and $\mathcal{P}_L$ is a minimal set (with respect to inclusion) with this property. Further, each P_J with $J \in \mathcal{M}_L$ is a Laurent polynomial in the elements of $\mathcal{P}_L$, with nonnegative coefficients.*

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Type B plactic relations for r-domino tableaux

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The recent work of Bonnafé et al. (2007) shows through two conjectures that r-domino tableaux have an important role in Kazhdan-Lusztig theory of type B with unequal parameters. In this paper we provide plactic relations on signed permutations which determine whether given two signed permutations have the same insertion r-domino tableaux in Garfinkle's algorithm (1990). Moreover, we show that a particular extension of these relations can describe Garfinkle's equivalence relation on r-domino tableaux which is given through the notion of open cycles. With these results we enunciate the conjectures of Bonnafé et al. and provide necessary tool for their proofs.

Keywords: insertion algorithm, domino tableaux, plactic relations

1 Introduction

Let W be a finite Coxeter group and let $L : W \mapsto \mathbb{Z}_{\geq 0}$ be a weight function such that

$$L(uw) = L(u) + L(w) \text{ if and only if } l(uw) = l(u) + l(w)$$

where $l : W \mapsto \mathbb{Z}_{\geq 0}$ is the usual length function on W. As it is described by Lusztig in (15) every weight function determines an Iwahori-Hecke algebra and some preorders on W whose equivalence classes are called *left, right* and *two-sided cells*. The importance of these cells lies in the fact that they carry representations of W and its corresponding Iwahori-Hecke algebra $\mathcal{H}$. Furthermore they have an important role in the representation theory of reductive algebraic groups over finite or p-adic fields (15) and in the study of rational Cherednik algebras (8) and the Calogero-Moser spaces (9).

The case $L = l$ is in fact first introduced by Kazhdan and Lusztig in (11) as a purely combinatorial tool for the theory of primitive ideals in the universal enveloping algebras of semisimple complex Lie algebras. In this case the combinatorial characterizations of cells are well known, where Knuth (or plactic) relations appear as the mediating tool. Namely, when W is type A then each right (left) cell corresponds to the plactic (respectively coplactic) class of some standard Young tableau, whereas each two-sided cell consists of those permutations which lie in the plactic classes of tableaux of the same shape. This characterizations depend on Joseph's classification of primitive ideals in type A, where Knuth (plactic) relations play a crucial role.

In the types B,C and D, on the other hand the emerging combinatorial objects are standard domino tableaux. The connection is first revealed in the work of Barbash and Vogan (1) where they provide

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necessary conditions for the characterizations of primitive ideals through an algorithm which uses the palindrome representations of signed permutations in order to assign to every signed permutation α a pair of same shape standard r-domino tableaux $(P^r(\alpha), Q^r(\alpha))$ bijectively, for $r = 0$ or $r = 1$. Meanwhile, an analog of Knuth relations provided by Joseph in (10) established the sufficient conditions. On the other hand Garfinkle (4; 5; 6) finalized classification problem for these types by showing through her two algorithms on domino tableaux that these two sets of relations are in fact equivalent. Her first algorithm assigns any signed permutation to a pair of same shape standard r-domino tableaux for r equals to 0 or 1 and the second defines an equivalence relation between domino tableaux through the notion of *open cycles*. We remark that the extension of Garfinkle and Barbash-Vogan algorithm for larger r is given in (14) and (3) respectively.

The case $L \neq l$ is also known as unequal parameters Kazhdan-Lusztig theory and it appears for the types B_n, $I_2(n)$ and F_4, where the classification problem for the latter two can be dealt with computational methods, see (7). For type B_n, the weight function is determined by two integers $a, b > 0$ such that $L(s_i) = a$ if $1 \leq i \leq n-1$ and it is equal to b if $i = 0$ where s_0 is the transposition $(-1, 1)$ and $\{s_i = (i, i+1) | 1 \leq i \leq n-1\}$ are the type A generators of B_n. Recently, the role of r-domino tableaux in this theory is revealed in the work of Bonnafé, Geck, Iancu, and Lam (3) through two main conjectures:

- *Conjecture A:* If $ra < b < (r+1)a$ for some $r \geq 0$ then two signed permutations lie in the same Kazhdan Lusztig right (left) cell if and only if their insertion (recording) r-domino tableau are the same.
- *Conjecture B:* If $b = ra$ for some $r \geq 1$ then two signed permutations lie in the same Kazhdan Lusztig right (left) cell if and only if their insertion (recording) r-domino tableau or $(r-1)$-domino tableau are the same.

On the other hand, in order to establish the proofs of these conjecture one definitely needs the plactic relations between signed permutations which determines when the insertion r-domino tableaux of two signed permutations are the same. Our aim here is to fill this gap.

Definition 1.1 *For $\alpha = \alpha_1 \ldots \alpha_n \in B_n$ and $r \geq 0$ consider the following relations:*

D_1^r: *If $\alpha_i < \alpha_{i+2} < \alpha_{i+1}$ or $\alpha_i < \alpha_{i-1} < \alpha_{i+1}$ for some i, then*

$$\alpha = \alpha_1 \ldots \alpha_{i-1} \, (\alpha_i \, \alpha_{i+1}) \, \alpha_{i+2} \ldots \alpha_n \;\sim\; \alpha_1 \ldots \alpha_{i-1} \, (\alpha_{i+1} \, \alpha_i) \, \alpha_{i+2} \ldots \alpha_n$$

D_2^r: *If there exists $0 < j \leq r$ such that $\alpha_j > 0$ and $\alpha_{j+1} < 0$ (or $\alpha_j < 0$ and $\alpha_{j+1} > 0$) and $\alpha_1 \ldots \alpha_j \, \alpha_{j+1}$ is a shuffle of some positive decreasing and negative increasing sequence ending with α_j and α_{j+1} (or respectively α_{j+1} and α_j) then*

$$\alpha = \alpha_1 \ldots (\alpha_j \, \alpha_{j+1}) \ldots \alpha_{r+2} \ldots \alpha_n \;\sim\; \alpha_1 \ldots (\alpha_{j+1} \, \alpha_j) \ldots \alpha_{r+2} \ldots \alpha_n$$

D_3^r: *If $|\alpha_1| > |\alpha_i|$ for all $2 \leq i \leq r+2$ and $\alpha_2 \ldots \alpha_{r+2}$ is a shuffle of some positive decreasing and negative increasing sequences, then*

$$\alpha = \alpha_1 \, \alpha_2 \ldots \alpha_{r+2} \ldots \alpha_n \;\sim\; (-\alpha_1) \, \alpha_2 \ldots \alpha_{r+2} \ldots \alpha_n$$

Now we are ready to state our results.

Theorem 1.2 *α and β in B_n are equivalent through a series of D_1^r, D_2^r or D_3^r relations if and only if they have the same insertion r-domino tableaux.*

Remark 1.3 *A set of relations for r-domino tableaux is defined in (3), but as it is already discussed there it is far from being sufficient for the characterization. In fact the plactic relation in (3, Section 3.8) can be shown to be equivalent to the one given with D_1^r and D_2^r here. Recently T. Pietraho has independently found another set of generators (17). Finally depending on his result and earlier version (20) of this paper, Bonnafé proved that r-plactic and r-cycle equivalence are sufficient for Conjecture A and B respectively, see (2).*

Remark 1.4 *Recall that for a signed permutation $\alpha = \alpha_1 \dots \alpha_n$ in B_n, its palindrome representation is given by $\overline{\alpha}_n \dots \overline{\alpha}_1 0 \alpha_1 \dots \alpha_n$ where $\overline{\alpha}_i = -\alpha_i$. Then D_1^r is just the usual Knuth (plactic) relation on the palindrome representation of α for any non negative integer r. On the other hand it is easy to see that when $r = 1$, D_2^r and D_3^r are also usual Knuth relation on the palindrome representation of α.*

In this paper, the descriptions of Barbash-Vogan and Garfinkle's algorithms can be found in Section 2 together with some lemmas which are essential in the proofs of our results. Section 3 is devoted to the proof of Theorem 1.2.

2 Related background

A sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ is a *partition* of n, denoted by $\lambda \vdash n$, if $\sum_{i=1}^k \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1} > 0$ where its *Ferrers diagram* consists of left justified arrows of boxes such that the i-th row has λ_i boxes.

A partition $\lambda = (\lambda_1, \dots, \lambda_k)$ can be also seen as a set of integer pairs (i, j) such that $1 \leq i \leq k$ and $1 \leq j \leq \lambda_i$. Therefore for two partitions λ and μ, we can define usual set operations such as $\lambda \cup \mu$, $\lambda \cap \mu$, $\lambda \subset \mu$, $\lambda - \mu$, but the resulting sets do not necessarily correspond to some partitions.

Definition 2.1 *For two partitions λ and μ satisfying $\mu \subset \lambda$ we define $\lambda/\mu = \lambda - \mu$ to be the skew partition determined by λ and μ.*

Definition 2.2 *Let γ and γ' be two skew shapes.*

1. *If $\gamma \cap \gamma' = \emptyset$ and $\gamma \cup \gamma'$ also corresponds a skew shape then we define $\gamma \oplus \gamma' = \gamma \cup \gamma'$.*
2. *If $\gamma' \subset \gamma$ and $\gamma - \gamma'$ also corresponds a skew shape then we define $\gamma \ominus \gamma' = \gamma - \gamma'$.*

Definition 2.3 *Let λ be a partition and $(i, j) \in \lambda$.*

1. *If $(i, j) \in \lambda$ and $\lambda \ominus (i, j)$ is also a partition then (i, j) is called a corner of λ.*
2. *If $(i, j) \notin \lambda$ and $\lambda \oplus (i, j)$ is also a partition then (i, j) is called an empty corner of λ.*

Definition 2.4 *A skew tableau T of shape λ/μ is obtained by labeling the cells of λ/μ with non repeating positive integers such that the numbers increase from left to right and from top to bottom. If $\mu = \emptyset$ then T is called a Young tableau. We denote by $\mathrm{label}(T)$ the set of numbers labeling each box of T and by $\mathrm{shape}(T)$ the partition underlying T. If the size of $\mathrm{shape}(T) = n$ and $\mathrm{label}(T) = \{1, 2, \dots, n\}$ then T is called a standard skew or standard Young tableau according to the shape of T. We denote by SYT_n the set of all standard Young tableaux of n cells.*

There is an important connection, between standard Young tableaux SYT_n and the symmetric group S_n, known as the *Robinson-Schensted correspondence (RSK)*, which was realized by Robinson and Schensted independently. In this correspondence, every permutation $w \in S_n$ is assigned bijectively to a pair of same shape tableaux $(P(w), Q(w))$ in $SYT_n \times SYT_n$ through *insertion* and *recording* algorithms. There are two equivalence relations introduced by Knuth which have very important applications in the combinatorics of tableaux.

Definition 2.5 *For $u \in S_n$ consider the following relation: If $u_i < u_{i+2} < u_{i+1}$ or $u_i < u_{i-1} < u_{i+1}$ for some i then $u = u_1 \dots u_{i-1}(u_i\ u_{i+1})\ u_{i+2} \dots u_n \overset{K}{\sim} u_1 \dots u_{i-1}(u_{i+1}\ u_i)\ u_{i+2} \dots u_n = u'$.*

We say $u, w \in S_n$ are Knuth equivalent, $u \overset{K}{\sim} w$, if w can be obtained from u by applying a sequence of $\overset{K}{\sim}$ relations. On the other hand if $u^{-1} \overset{K}{\sim} w^{-1}$ then u and w are called dual Knuth equivalent, $u \overset{K^*}{\sim} w$.

The following theorem given by Knuth (12) is fundamental.

Theorem 2.6 *Let $u, w \in S_n$. Then $u \overset{K}{\sim} w \iff P(u) = P(w)$ and $u \overset{K^*}{\sim} w \iff Q(u) = Q(w)$.*

Definition 2.7 *The set of two adjacent cells $A = \{(i, j), (i, j+1)\}$ (or $A = \{(i, j), (i+1, j)\}$) is called a horizontal (or respectively vertical) domino cell. By a labeling of domino cell A we mean a pair of positive numbers (a, a') which label the boxes of A such that $a \leq a'$ and a labels the cell of A which is smaller in the lexicographic order. When we want to indicate the domino cell A with its labeling, we use the notation*

$$[A, (a, a')]$$

so that $\mathrm{shape}([A, (a, a')]) = A$ *and* $\mathrm{label}([A, (a, a')]) = (a, a')$.

Let λ be a partition and A be a domino cell. If $\lambda \oplus A$ is a partition then A is called an *empty domino corner* of λ whereas if $\lambda \ominus A$ is also a partition then A is called a *domino corner* of λ. Clearly, if a partition has no domino corner then it must be a r-staircase shape $(r, \dots, 2, 1)$ for some $r > 0$. On the other hand it is easy to see that any partition λ can be reduced uniquely to a r-staircase shape $(r, \dots, 2, 1)$ for some $r \geq 0$, by subsequent removal of existing domino corners one at a time. In this case we say λ has a 2-*core* equivalent to $(r, \dots, 2, 1)$ and we denote by $P(2n, r)$ the set of all such partitions of size $2n + r(r+1)/2$.

Definition 2.8 *A r-domino tableau T of shape $\lambda \in P(2n, r)$ is obtained by tiling the skew partition $\lambda/(r, \dots, 2, 1)$ with labeled horizontal or vertical dominos $\{[A_1, (a_1, a_1)], \dots [A_n, (a_n, a_n)]\}$ such that $a_i \neq a_j$ for $i \neq j$ and the labels increase from left to right and from top to bottom. In this case we have*

$$\mathrm{label}(T) = \{a_1, a_2, \dots, a_n\}.$$

A standard r-domino tableau T is a r-domino tableau which has $\mathrm{label}(T) = \{1, \dots, n\}$. *We denote by $SDT^r(n)$ the set of all standard r-domino tableaux of n dominos.*

Definition 2.9 *Let T be a r-domino tableau and $\lambda = \mathrm{shape}(T)$. For $a \in \mathrm{label}(T)$ and A is a domino cell in λ we define,*

1. $\mathrm{Dom}(T, a)$ *to be the domino cell of T whose both cells are labeled with a in T.*
2. $\mathrm{dom}(T, a) = \mathrm{shape}(\mathrm{Dom}(T, a))$.
3. $\mathrm{label}(T, A)$ *to be the pair of integers (a, a') which label the domino cell A in T, where $a \leq a'$.*

Example 2.10 *For example the following is a* 2*-domino tableau in* $SD^2(5)$*.*

$$T = \begin{array}{|c|c|c|c|c|}
\hline
 & & 1 & 1 & 5 \\
\hline
 & 3 & 4 & 4 & 5 \\
\hline
2 & 3 \\
\cline{1-2}
2 \\
\cline{1-1}
\end{array}$$

T has two domino corners: $A_1 = \{(1,5),(2,5)\}$ and $A_2 = \{(2,4),(2,5)\}$. $\mathrm{label}(T, A_1) = (5,5)$ *and* $\mathrm{label}(T, A_2) = (4,5)$. *On the other hand* $\mathrm{dom}(T,5) = A_1$ *and* $\mathrm{dom}(T,4) = \{(2,3),(2,4)\} \neq A_2$.

Definition 2.11 *For two r-domino tableau S and T satisfying $S \subset T$ we define $T/S = T - S$ to be the skew r-domino tableau determined by S and T.*

Definition 2.12 *Let R and R' be two skew r-domino tableaux with* $\mathrm{shape}(R) = \gamma$ *and* $\mathrm{shape}(R') = \gamma'$.

1. *If $\gamma \oplus \gamma'$ is defined and $R \cup R'$ corresponds to some skew r-domino tableau as a set then we define $R \oplus R' = R \cup R'$*

2. *If $\gamma \ominus \gamma'$ is defined and if $R - R'$ corresponds to some skew r-domino tableau as a set then we define $R \ominus R' = R - R'$*

Definition 2.13 *Let T be a (skew) r-domino tableau and $a \in$* $\mathrm{label}(T)$. *Then we define*

1. *$T_{<a}$ ($T_{\leq a}$) to be the r-domino tableau obtained by restricting T to its dominos which are labeled with integers less than (and equal to) a.*

2. *$T_{>a}$ ($T_{\geq a}$) to be the skew r-domino tableau obtained by restricting T to its dominos which are labeled with integers greater than (and equal to) a.*

2.1 Garfinkle's algorithm and reverse insertion

Recall that a signed permutation $\alpha \in B_n$ is a bijection of $[-n, +n]$ such that $\alpha(-i) = -\alpha(i)$. The *usual presentation* of $\alpha \in B_n$ is denoted as $\alpha = \alpha_1\alpha_2 \dots \alpha_n$ where $\alpha_i = \alpha(i)$ for $1 \leq i \leq n$ and $\{|\alpha_1|, |\alpha_2| \dots, |\alpha_n|\} = \{1, 2, \dots, n\}$.

In (4) Garfinkle provide an algorithm for $r = 0, 1$ by which any signed permutation $\alpha \in B_n$ is assigned bijectively to a pair of same shape standard r-domino tableau $(P^r(\alpha), Q^r(\alpha))$, where $P^r(\alpha)$ is called *insertion* and $Q^r(\alpha)$ is called *recording* tableau of α. Detailed explanations of this algorithm can be found in (13) and (21). Based on Garfinkle's algorithm we now describe the reverse-insertion of domino corners through the Corollary below. Then we will state several lemmas which are the main tool in the proof of Theorem 1.2.

Let T be a r-domino tableau and A be a domino corner in $\mathrm{shape}(T)$. We denote by $T^{\uparrow A}$ and $\eta(T^{\uparrow A})$ respectively the tableau which is obtained by the reverse-insertion of A, and the number which is bumped out of T as a result of this operation.

Corollary 2.14 *Let T be an r-domino tableau and A is a domino corner. Furthermore let A' be the domino cell which is pushed back by A in reverse insertion $T^{\uparrow A}$. Then,* *i)* *If* $A = \{(i,j),(i,j+1)\}$ *and* $\mathrm{label}(T,A) = (a,a)$ *then* $A' \subset \{(i-1,k) \mid k \geq j\}$.

ii) *If* $A = \{(i,j),(i,j+1)\}$ *and* $\mathrm{label}(T,A) = (a',a)$ *for some* $a' < a$ *then* $A' = \{(i-1,j),(i,j)\}$.

iii) *If* $A = \{(i,j),(i+1,j)\}$ *and* $\mathrm{label}(T,A) = (a,a)$ *then* $A' \subset \{(k,j-1) \mid k \geq i\}$.

iv) *If* $A = \{(i,j),(i+1,j)\}$ *and* $\mathrm{label}(T,A) = (a',a)$ *for some* $a' < a$ *then* $A' = \{(i,j-1),(i,j)\}$.

Example 2.15 *Let* $S \in SD^3(5)$ *as given below. We will show that* $\eta(S^{\uparrow A}) = 1$ *where* $A = \{(3,3),(4,3)\}$. *In the following the barred letters indicate the domino cell which is pushed back during the reverse insertion algorithm.*

$$\begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 3 & 3 \\ \cline{1-3} 2 & 4 & \bar{4} \\ \cline{1-3} 2 & 5 & \bar{5} \\ \cline{1-3}\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & 3 & 3 \\ \cline{1-3} 2 & \bar{4}/5 & \bar{4} \\ \cline{1-3} 2 & 5 \\ \cline{1-2}\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline & \bar{3}/4 & \bar{3}/4 \\ \cline{1-3} 2 & 5 \\ \cline{1-2} 2 & 5 \\ \cline{1-2}\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline & & \bar{1}/3 & \bar{1}/3 \\ \hline & 4 & 4 \\ \cline{1-3} 2 & 5 \\ \cline{1-2} 2 & 5 \\ \cline{1-2}\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline & & 3 & 3 \\ \hline & 4 & 4 \\ \cline{1-3} 2 & 5 \\ \cline{1-2} 2 & 5 \\ \cline{1-2}\end{array} = S^{\uparrow A}$$

Lemma 2.16 *Let* T *be a* r*-domino tableau and* A *be a domino corner of* $\mathrm{shape}(T)$. *Then* $T^{\uparrow A}$ *and* $\eta(T^{\uparrow A})$ *are unique.*

Definition 2.17 *Let* T *be a* r*-domino tableau and* A *be a domino corner of* $\mathrm{shape}(T)$ *such that* $A = \{(i,j),(i,j+1)\}$ *or* $A = \{(i,j),(i+1,j)\}$. *We denote by* (T,A,ne) *and* (T,A,ne) *the regions of* T *such that*

$$\begin{array}{rl}(T,A,\mathrm{ne}) & := \{(k,l) \mid k < i \text{ and } l \geq j\} \\ (T,A,\mathrm{sw}) & := \{(k,l) \mid k \geq i \text{ and } l < j\}\end{array}$$

as illustrated in Figure 1.

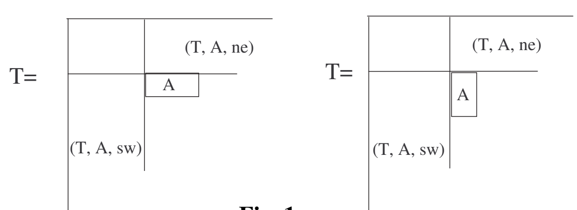

Fig. 1:

Lemma 2.18 *Let* A *and* B *be a domino corners of* $\mathrm{shape}(T)$ *and* $\mathrm{shape}(T^{\uparrow A})$ *respectively.*

i) *If* B *lies in the portion* (T,A,sw) *then* $\eta(T^{\uparrow A \uparrow B}) < \eta(T^{\uparrow A})$.

ii) *If* B *lies in the portion* (T,A,ne) *then* $\eta(T^{\uparrow A \uparrow B}) > \eta(T^{\uparrow A})$.

Proof: The proof is omitted for the sake of place. □

2.2 Barbash and Vogan algorithm and descents of r-domino tableaux.

We will now explain the algorithm which is provided by Barbash and Vogan in (1) to establish the bijection between signed permutations and standard r-domino tableaux for $r = 0, 1$ whereas its extension for larger cores is provided in (3). We also remark that the equivalence of Barbash-Vogan algorithm to Garfinkle's algorithm for $r = 0, 1$, is due to Van Leeuwen (14).

Recall that for $\alpha = \alpha_1\ \alpha_2 \ldots \alpha_n$ a signed permutation the palindrome representation of α is given by $\alpha^0 = \overline{\alpha}_n \ldots \overline{\alpha}_2\ \overline{\alpha}_1\ \alpha_1\ \alpha_2 \ldots \alpha_n$ if α lies in C_n, or $\alpha^1 = \overline{\alpha}_n \ldots \overline{\alpha}_2\ \overline{\alpha}_1\ 0\ \alpha_1\ \alpha_2 \ldots \alpha_n$ if α lies in B_n, where $\overline{\alpha}_i = -\alpha_i$. We will call α^0 and α^1 as 0-*core* and 1-*core* representation of α respectively. By following the approach of (3) let us describe how to extend this representation for larger cores. We first identify $\{1, 2, \ldots, r(r+1)/2\}$ with $\{0_1, 0_2, \ldots, 0_{r(r+1)/2}\}$ together with the total ordering $-n < \ldots < -2 < -1 < 0_1 < 0_2 < \ldots < 0_{r(r+1)/2} < 1 < 2 \ldots < n$. Let $w \in S_{r(r+1)/2}$ be a permutation under this identification, whose RSK insertion tableau is of shape $(r, r-1, \ldots, 1)$. Now for $\alpha \in B_n$ let r-*core representation* of α to be $\alpha^r = \overline{\alpha}_n \ldots \overline{\alpha}_2\ \overline{\alpha}_1\ w\ \alpha_1\ \alpha_2 \ldots \alpha_n$. The algorithm first applies RSK algorithm on α^r. Then starting from the lowest number $\bar{n}$, it vacates the negative integer $\overline{i}$ in the tableaux by jeu de taquin slides until it becomes adjacent to i, where the vacation is repeated for $\overline{i-1}$ until $i = 1$. The following example illustrates this algorithm for $r = 1$.

Example 2.19 *For $\alpha = 3\ \bar{1}\ 2 \in B_n$, we have $\alpha^1 = \bar{2}\ 1\ \bar{3}\ 0\ 3\ \bar{1}\ 2$ be its 1-core representation. Then*

$$P(\alpha^1) = \begin{array}{|c|c|c|}\hline \bar{3} & \bar{1} & 2 \\ \hline \bar{2} & 0 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline \bar{2} & \bar{1} & 2 \\ \hline 0 & \bar{3} & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline \bar{1} & \bar{2} & 2 \\ \hline 0 & \bar{3} & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline 0 & \bar{2} & 2 \\ \hline \bar{1} & \bar{3} & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \mapsto \begin{array}{|c|c|c|}\hline & 2 & 2 \\ \hline 1 & 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} = P^1(\alpha).$$

Theorem 2.20 ((3), Theorem 3.3) *Signed permutations α and β have the same insertion r-domino tableau if and only if α^r and β^r have the same RSK insertion tableau.*

The following proposition is a consequence of Theorem 2.20 and Theorem 2.6.

Proposition 2.21 *Let α and β be two signed permutations which differ by a single D_1^r relation. Then $P^r(\alpha) = P^r(\beta)$, in other words α and β have the same insertion r-domino tableau.*

Recall that B_n carries a Coxeter group structure with the generator set $S = \{s_0, s_1, \ldots, s_{n-1}\}$ where $\{s_i = (i, i+1) | 1 \leq i \leq n-1\}$ is the set of transpositions which also generates the symmetric group S_n and s_0 corresponds to the transposition $(-1, 1)$. Let $l(\alpha)$ denote the length of α, which is the minimum number of generators of α and let

$$\begin{aligned} \mathrm{Des}_L(\alpha) \quad &:= \{i \mid l(s_i\alpha) < l(\alpha) \text{ and } 0 \leq i \leq n-1\} \\ &= \{i \mid \text{if } \ 1 \leq i \leq n-1 \ \text{ and } \ i+1 \text{ comes before } i \text{ in } \alpha^0\} \cup \{0 \mid \ \text{if } \ 1 \text{ comes before } -1 \text{ in } \alpha^0\} \end{aligned}$$

denote respectively the sets of left and right descents of α. On the other hand the descent set of a r-domino tableau T is defined in the following way:

$$\begin{aligned} \mathrm{Des}(T) := \quad &\{i \mid \ \text{if the domino labeled with } \ (i+1, i+1) \ \text{ lies below the one labeled with } (i, i)\} \\ &\cup \{0 \mid \ \text{if the domino labeled with } \ (1, 1) \ \text{ is vertical}\} \end{aligned}$$

It is a well known property of RSK algorithm that $\mathrm{Des}_L(w) = \mathrm{Des}(P(w))$ for any $w \in S_n$ whereas the descent set of a (skew or Young) tableau T is defined by $\mathrm{Des}(T) = \{i \mid i+1 \text{ lies below } i \text{ in } T\}$. On the other hand it is easy to see that jeu de taquin slides do not change the descent sets of tableaux, therefore the following result is a consequence of Theorem 2.20.

Corollary 2.22 *For $\alpha \in B_n$ we have* $\mathrm{Des}_L(\alpha) = \mathrm{Des}(P^r(\alpha))$.

Observe that if α and β differ by a single D_1^r relations in B_n then $P^r(\alpha) = P^r(\beta)$ and we have either $\beta^{-1} = s_i \cdot \alpha^{-1}$ or $\beta^{-1} = s_{i+1} \cdot \alpha^{-1}$ and moreover we have either $i \in \mathrm{Des}_L(\alpha^{-1})$ but $i+1 \notin \mathrm{Des}_L(\alpha^{-1})$ or $i \notin \mathrm{Des}_L(\alpha^{-1})$ but $i+1 \in \mathrm{Des}_L(\alpha^{-1})$ for some $1 \leq i \leq n-2$. In the following we will follow Garfinkle's approach in (4) to study the effect of a single D_1^r relation on the recording tableaux.

For i, j two adjacent integers satisfying $1 \leq i, j \leq n-1$, consider the following sets:

$$\begin{aligned} D_{i,j}(B_n) &:= \quad \{\alpha \in B_n \mid i \in \mathrm{Des}_L(\alpha) \text{ but } j \notin \mathrm{Des}_L(\alpha)\} \\ D_{i,j}(SDT^r(n)) &:= \quad \{T \in SDT^r(n) \mid i \in \mathrm{Des}(T) \text{ but } j \notin \mathrm{Des}(T)\} \end{aligned}$$

together with the map $V_{i,j} : D_{i,j}(B_n) \mapsto D_{j,i}(B_n)$ where $V_{i,j}(\alpha) = \{s_i \cdot \alpha,\ s_j \cdot \alpha\} \cap D_{j,i}(B_n)$. We also define a map $V_{i,j} : D_{i,j}(SDT^r(n)) \mapsto D_{j,i}(SDT^r(n))$ in the following manner: Without loss of generality we assume that $j > i$, i.e., $j = i+1$. Observe that if $i \in \mathrm{Des}(T)$ but $i+1 \notin \mathrm{Des}(T)$ then $i+1$ lies strictly below i in T whereas $i+2$ lies strictly right to $i+1$ in T. On the other hand we have two cases according to the positions of dominos labeled with (i,i) and $(i+2, i+2)$ with respect to each other.

Case 1. We first assume that $i+2$ lies strictly below i in T. Since the $i+2$ lies strictly to the right of $i+1$ and $i+1$ lies below i we have two cases to consider: If the boundaries $\mathrm{Dom}(T, i+1)$ and $\mathrm{Dom}(T, i)$ intersect at most on a point then $V_{i,i+1}(T)$ is obtained by interchanging the labels i and $i+1$ in T. Otherwise there is only one possibility which satisfies $i+2$ lies below i and it lies to the right of $i+1$, in which T has the subtableau U as illustrated below and $V_{i,i+1}(T)$ is obtained by substituting U with U' in T.

$$U = \begin{array}{|c|c|}\hline i & i \\ \hline i+1 & i+2 \\ \hline i+1 & i+2 \\ \hline \end{array} \qquad U' = \begin{array}{|c|c|}\hline i & i+1 \\ \hline i & i+1 \\ \hline i+2 & i+2 \\ \hline \end{array}$$

Case 2. Now we assume $i+2$ lies strictly right to i in T. Again if the boundaries of $\mathrm{Dom}(T, i+1)$ and $\mathrm{Dom}(T, i+2)$ intersect at most on a point then $V_{i,i+1}(T)$ is obtained by interchanging the labels $i+1$ and $i+2$ in T. Otherwise there is only one possible case where T has the subtableau U given below and $V_{i,i+1}(T)$ is obtained by substituting U with U' in T.

$$U = \begin{array}{|c|c|c|}\hline i & i & i+2 \\ \hline i+1 & i+1 & i+2 \\ \hline \end{array} \qquad U' = \begin{array}{|c|c|c|}\hline i & i+1 & i+1 \\ \hline i & i+2 & i+2 \\ \hline \end{array}$$

Example 2.23 *We have $T_2 = V_{5,6}(T_1)$, $T_3 = V_{3,4}(T_2)$, and $T_4 = V_{4,5}(T_3) = V_{6,5}(T_3)$ for the following tableaux.*

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & \mathbf{5} \\ \hline 1 & 2 & \mathbf{5} \\ \hline 3 & 3 & \mathbf{7} \\ \hline 4 & \mathbf{6} & \mathbf{7} \\ \hline 4 & \mathbf{6} \\ \cline{1-2} \end{array} \quad T_2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 1 & 2 & 6 \\ \hline \mathbf{3} & \mathbf{3} & 7 \\ \hline \mathbf{4} & \mathbf{5} & 7 \\ \hline \mathbf{4} & \mathbf{5} \\ \cline{1-2} \end{array} \quad T_3 = \begin{array}{|c|c|c|}\hline 1 & 2 & \mathbf{6} \\ \hline 1 & 2 & \mathbf{6} \\ \hline 3 & \mathbf{4} & \mathbf{7} \\ \hline 3 & \mathbf{4} & \mathbf{7} \\ \hline \mathbf{5} & \mathbf{5} \\ \cline{1-2} \end{array} \quad T_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 3 & 4 & 7 \\ \hline 6 & 6 \\ \cline{1-2} \end{array}$$

Remark 2.24 *The map $V_{i,j}$ is first introduced on the symmetric group by Vogan (22), with the aim of classifying the primitive ideals in the universal enveloping algebra of complex semi simple Lie algebras. In fact when it is considered on the symmetric group the map $V_{i,j}$ produces nothing but the dual Knuth relation on the permutations and their insertion tableaux.*

Lemma 2.25 *Let i and j be two consecutive integers such that $1 \leq i, j \leq n-1$. Suppose $\alpha \in D_{i,j}(B_n)$. Then $P^r(\alpha) \in D_{i,j}(SDT^r(n))$ and $P^r(V_{i,j}(\alpha)) = V_{i,j}(P^r(\alpha))$.*

Proof: This result is first proven by Garfinkle (5, Theorem 2.1.19.) for $r = 0, 1$. On the other hand one can check that her proof does not depend on the specific value of r and it can easily be extended for any value of r. We omit the proof for the sake of space. □

The following result follows directly from Lemma 2.25 and it has an important role in the proof of Theorem 1.2.

Corollary 2.26 *Suppose $\alpha = \alpha_1 \ldots \alpha_{i-1}(\alpha_i\alpha_{i+1})\alpha_{i+2} \ldots \alpha_n$ and $\beta = \alpha_1 \ldots \alpha_{i-1}(\alpha_{i+1}\alpha_i)\alpha_{i+2} \ldots \alpha_n$ differ by a single D_1^r relation. Then one of the following is satisfied:*

i) $\alpha_i < \alpha_{i+2} < \alpha_{i+1}$ *for some* $i \leq n-2$ *and* $\beta^{-1} = V_{i+1,i}(\alpha^{-1})$ *and* $Q^r(\beta) = V_{i+1,i}(Q^r(\alpha))$.

ii) $\alpha_i > \alpha_{i+2} > \alpha_{i+1}$ *for some* $i \leq n-2$ *and* $\beta^{-1} = V_{i,i+1}(\alpha^{-1})$ *and* $Q^r(\beta) = V_{i,i+1}(Q^r(\alpha))$.

iii) $\alpha_i < \alpha_{i-1} < \alpha_{i+1}$ *for some* $i \leq n-1$ *and* $\beta^{-1} = V_{i-1,i}(\alpha^{-1})$ *and* $Q^r(\beta) = V_{i-1,i}(Q^r(\alpha))$.

iv) $\alpha_i > \alpha_{i-1} > \alpha_{i+1}$ *for some* $i \leq n-1$ *and* $\beta^{-1} = V_{i,i-1}(\alpha^{-1})$ *and* $Q^r(\beta) = V_{i,i-1}(Q^r(\alpha))$.

3 Plactic relations for r-domino tableaux

3.1 Proof of Theorem 1.2

In this section we will prove the main Theorem 1.2, i.e., we will show that the relations D_1^r,D_2^r and D_3^r from Definition 1.1 are sufficient and necessary to characterize plactic classes of standard r-domino tableaux.

Proof: (Proof of Theorem 1.2) Let α and β be two signed permutations which differ by a sequence of D_1^r, D_2^r or D_3^r relations. By using Garfinkle's insertion algorithm for r-domino tableau and Proposition 2.21 it is easy to check that $P^r(\alpha) = P^r(\alpha')$ if α and α' differs by a single D_i^r relation for $i = 1, 2, 3$. Therefore $P^r(\alpha)$ must be equal to $P^r(\beta)$.

Now we let $\alpha = \alpha_1 \ldots \alpha_{n-1}\alpha_n$ and $\beta = \beta_1 \ldots \beta_{n-1}\beta_n$ such that $T = P^r(\alpha) = P^r(\beta)$. We will show by induction that $\alpha \overset{p_r}{\sim} \beta$. Let $r \geq 0$ be arbitrary. If $n = 1$ there is nothing to prove. Therefore we assume that the statement holds for all signed permutations of size $n-1$.

If $P^r(\alpha_1 \ldots \alpha_{n-1}) = P^r(\beta_1 \ldots \beta_{n-1}) = T^{\uparrow A}$ for some domino corner A of $\mathrm{shape}(T)$ then $\alpha_n = \beta_n$ by Lemma 2.16. By induction we can assume that $\alpha_1 \ldots \alpha_{n-1} \overset{p_r}{\sim} \beta_1 \ldots \beta_{n-1}$. Therefore $\alpha \overset{p_r}{\sim} \beta$.

If $P^r(\alpha_1 \ldots \alpha_{n-1}) \neq P^r(\beta_1 \ldots \beta_{n-1})$ then there exist two different domino corners say A and B of T such that

$$\begin{array}{rl} T^{\uparrow A} = & P^r(\alpha_1 \ldots \alpha_{n-1}) \text{ and } \eta(T^{\uparrow A}) = \alpha_n \\ T^{\uparrow B} = & P^r(\beta_1 \ldots \beta_{n-1}) \text{ and } \eta(T^{\uparrow B}) = \beta_n. \end{array} \tag{1}$$

In the following we proceed according to the orientation of A and B as illustrated in Figure 2 where in the first four pictures $(T, A, \mathrm{ne}) \cap (T, B, \mathrm{sw})$ is represented with the shaded areas.

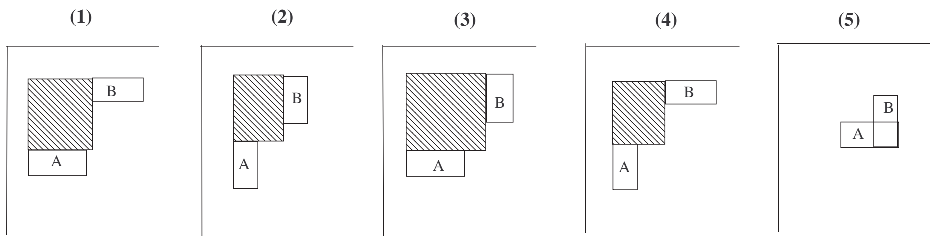

Fig. 2:

Cases (1),(2) and (3): We will first show that $\alpha \overset{p_r}{\sim} \beta$ for the first three cases of Figure 2. Consider the domino corner B of $T^{\uparrow A}$ and let $b = \eta(T^{\uparrow A \uparrow B})$. It is easy to see that there exists a domino corner, say C of $T^{\uparrow A \uparrow B}$ which lies in $(T, A, \mathrm{ne}) \cap (T, B, \mathrm{sw})$. Let $c = \eta(T^{\uparrow A \uparrow B \uparrow C})$ and $\tilde{u}$ be a signed word such that $P^r(\tilde{u}) = T^{\uparrow A \uparrow B \uparrow C}$. Therefore by Lemma 2.16 we have

$$P^r(\tilde{u}cb\alpha_n) = P^r(\tilde{u})^{\downarrow c \downarrow b \downarrow \alpha_n} = (T^{\uparrow A \uparrow B \uparrow C})^{\downarrow c \downarrow b \downarrow \alpha_n} = (T^{\uparrow A \uparrow B})^{\downarrow b \downarrow \alpha_n} = (T^{\uparrow A})^{\downarrow \alpha_n} = T$$

and by induction hypothesis $\tilde{u}cb \overset{p_r}{\sim} \alpha_1 \ldots \alpha_{n-1}$ since $P^r(\tilde{u}cb) = T^{\uparrow A} = P^r(\alpha_1 \ldots \alpha_{n-1})$. Therefore letting u denote the signed permutation $\tilde{u}cb\alpha_n$, we have $\alpha \overset{p_r}{\sim} u$.

Observe that since $P^r(\tilde{u}) = T^{\uparrow A \uparrow B \uparrow C}$, the recording tableau $Q^r(\tilde{u}cb\alpha_n)$ has its domino cells A, B and C labeled with (n, n), $(n-1, n-1)$ and $(n-2, n-2)$ respectively.

On the other hand having B in (T, A, ne) and C in (T, B, sw) yields by Lemma 2.18 that

$$b = \eta(T^{\uparrow A \uparrow B}) > \eta(T^{\uparrow A}) = \alpha_n \ \text{ and } \ b = \eta(T^{\uparrow A \uparrow B}) > \eta(T^{\uparrow A \uparrow B \uparrow C}) = c.$$

Therefore we have by Corallary 2.26

$$\begin{array}{rll} \text{either} & b > \alpha_n > c, & \text{and hence } \ u = \tilde{u}cb\alpha_n \overset{\mathrm{D}_1^r}{\sim} \tilde{u}bc\alpha_n = w \ \text{ and } \ V_{n-1,n-2}(Q^r(u)) = Q^r(w) \\ \text{or} & b > c > \alpha_n, & \text{and hence } \ u = \tilde{u}cb\alpha_n \overset{\mathrm{D}_1^r}{\sim} \tilde{u}c\alpha_n b = w \ \text{ and } \ V_{n-1,n-2}(Q^r(u)) = Q^r(w) \end{array}$$

The last argument implies that in both cases the signed permutation w has its recording tableau $Q^r(w)$ obtained by interchanging the labels (n, n) of A and $(n-1, n-1)$ of B in $Q^r(u)$ i.e., $Q^r(w)$ had the domino corner B labeled with (n, n). Then by Lemma 2.16 we have

$$P^r(w_1 \ldots w_{n-1}) = T^{\uparrow B} = P^r(\beta_1 \ldots \beta_{n-1}) \ \text{ and } w_n = \beta_n$$

and by induction $w_1 \ldots w_{n-1} \overset{p_r}{\sim} \beta_1 \ldots \beta_{n-1}$. Therefore $w \overset{p_r}{\sim} \beta$. and $\alpha \overset{p_r}{\sim} u \overset{p_r}{\sim} w \overset{p_r}{\sim} \beta$.

Case (4): For the fourth case of Figure 2, let $\alpha, \beta \in B_n$ as in 1. If there exist a domino corner in $(T, A, \mathrm{ne}) \cap (T, B, \mathrm{sw})$ then one can follow the same argument which is used for Cases (1),(2),(3). On the other hand it may happen that $(T, A, \mathrm{ne}) \cap (T, B, \mathrm{sw})$ is a staircase shape and in the following we consider several subcases as illustrated in Figure 3.

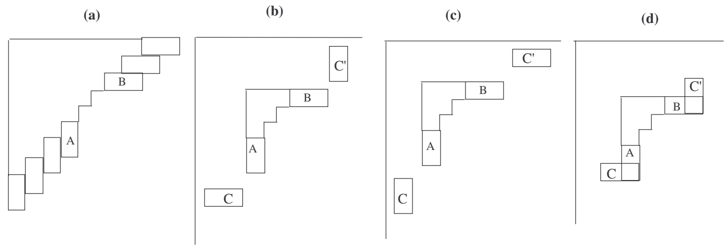

Fig. 3:

Observe that, in case T has the configuration of Figure 3(a), we have $n \leq r+1$, $\alpha_n < 0$, $\beta_n > 0$ and

$$\begin{aligned} \eta(T^{\uparrow A \uparrow B}) &= \beta_n \\ \eta(T^{\uparrow B \uparrow A}) &= \alpha_n \\ P^r(\alpha_1 \dots \alpha_{n-2}) = T^{\uparrow A \uparrow B} &= T^{\uparrow B \uparrow A} = P^r(\beta_1 \dots \beta_{n-2}). \end{aligned}$$

Let $\tilde{u}$ be a signed word such that $P^r(\tilde{u}) = T^{\uparrow A \uparrow B} = T^{\uparrow B \uparrow A}$. Clearly $\tilde{u}$ must be a shuffle of positive decreasing and negative increasing sequences and $P^r(\tilde{u}\alpha_n\beta_n) = T = P^r(\tilde{u}\beta_n\alpha_n)$. Therefore $\tilde{u}\alpha_n\beta_n \overset{\mathrm{D}_2^r}{\sim} \tilde{u}\beta_n\alpha_n$. On the other hand $P^r(\tilde{u}\beta_n) = T^{\uparrow A}$ and $P^r(\tilde{u}\alpha_n) = T^{\uparrow B}$ and by induction hypothesis we have $\alpha_1 \dots \alpha_{n-1} \overset{p_r}{\sim} \tilde{u}\beta_n$ and $\beta_1 \dots \beta_{n-1} \overset{p_r}{\sim} \tilde{u}\alpha_n$. Hence

$$\alpha = \alpha_1 \dots \alpha_{n-1}\alpha_n \overset{p_r}{\sim} \tilde{u}\beta_n\alpha_n \overset{\mathrm{D}_2^r}{\sim} \tilde{u}\alpha_n\beta_n \overset{p_r}{\sim} \beta_1 \dots \beta_{n-1}\beta_n = \beta.$$

Now we assume that T has the configuration of Figure 3(d) i.e. the corner C and A (or C' and B) intersect. Again we let $\sigma_1 \dots \sigma_n \in B_n$ such that $P^r(\sigma_1 \dots \sigma_{n-1}) = T^{\uparrow C}$. Observe that there is a domino corner in $(T, C, \mathrm{ne}) \cap (T, B, \mathrm{sw})$ therefore $\beta \overset{p_r}{\sim} \sigma$ follows. We only need to show $\alpha \overset{p_r}{\sim} \sigma$.

Observe that since T has the configuration of Figure 3(d) we have a domino corner A' of $T^{\uparrow A}$ and A'' of $T^{\uparrow A \uparrow A'}$ as it is illustrated in Figure 4 below.

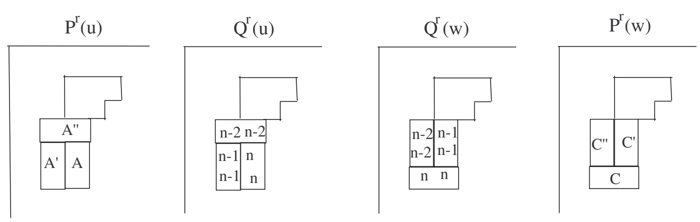

Fig. 4:

Let $a' = \eta(T^{\uparrow A \uparrow A'})$ and $a'' = \eta(T^{\uparrow A \uparrow A' \uparrow A''})$. Suppose $\tilde{u}$ be a signed word such that $P^r(\tilde{u}) = T^{\uparrow A \uparrow A' \uparrow A''}$. Then the signed permutation $u = \tilde{u}a''a'\alpha_n$ has $P^r(u) = T$ whereas its recording tableau $Q^r(u)$ must have the form as it is shown in Figure 4.

On the other hand since $P^r(\tilde{u}a''a) = T^{\uparrow A} = P^r(\alpha_1 \dots \alpha_{n-1})$ we have by induction $\tilde{u}a''a' \overset{p_r}{\sim} \alpha_1 \dots \alpha_{n-1}$ and therefore $u = \tilde{u}a''a'\alpha_n \overset{p_r}{\sim} \alpha_1 \dots \alpha_{n-1}\alpha_n = \alpha$.

Furthermore having A' in (T, A, sw) and A'' in (T, A', ne) yields $a' = \eta(T^{\uparrow A \uparrow A'}) < \eta(T^{\uparrow A}) = \alpha_n$ and $a' = \eta(T^{\uparrow A \uparrow A'}) < \eta(T^{\uparrow A \uparrow A' \uparrow A''}) = a''$ by Lemma 2.18. Therefore we have

$$\begin{array}{rll} \text{either} & a'' > \alpha_n > a', & \text{and hence } u = \tilde{u} a'' a' \alpha_n \overset{\mathrm{D}_1^r}{\sim} \tilde{u} a' a'' \alpha_n = w \text{ and } Q^r(w) = V_{n-2,n-1}(Q^r(u)) \\ \text{or} & \alpha_n > a'' > a', & \text{and hence } u = \tilde{u} a'' a' \alpha_n \overset{\mathrm{D}_1^r}{\sim} \tilde{u} a'' \alpha_n a' = w \text{ and } Q^r(w) = V_{n-2,n-1}(Q^r(u)). \end{array}$$

In both cases Corollary 2.26 yields that the recording tableau $Q^r(w)$ of w has the form illustrated in Figure 4 and by Lemma 2.16 we have

$$P^r(w_1 \ldots w_{n-1}) = T^{\uparrow C} = P^r(\sigma_1 \ldots \sigma_{n-1}) \text{ and } w_n = \beta_n.$$

Then by induction $w_1 \ldots w_{n-1} \overset{p_r}{\approx} \sigma_1 \ldots \sigma_{n-1}$ and hence $w \overset{p_r}{\approx} \sigma$. Hence as desired $\alpha \overset{p_r}{\approx} u \overset{\mathrm{D}_1^r}{\sim} w \overset{p_r}{\approx} \beta$.

If T has the configuration of Figure 3(b) or Figure 3(c), T may have a domino corner, say C lying in (T, A, ne). Let $\sigma = \sigma_1 \ldots \sigma_n \in B_n$ such that $= P^r(\sigma_1 \ldots \sigma_{n-1}) = T^{\uparrow C}$. Observe that there exist a domino corner in $(T, C, \mathrm{ne}) \cap (T, A, \mathrm{sw})$ and $(T, C, \mathrm{ne}) \cap (T, B, \mathrm{sw})$ therefore we can apply the argument which is used for Cases (1),(2) and (3) in order to get $\alpha \overset{p_r}{\approx} \sigma$ and $\beta \overset{p_r}{\approx} \sigma$ and hence $\alpha \overset{p_r}{\approx} \beta$. On the other hand if T has domino corner C' lying in (T, B, sw) the same argument applied on T^t gives the desired result.

Case (5): Again let $\alpha, \beta \in B_n$ as in 1 and suppose that T has the configuration of Figure 2(5) . We consider Figure 5 for several cases.

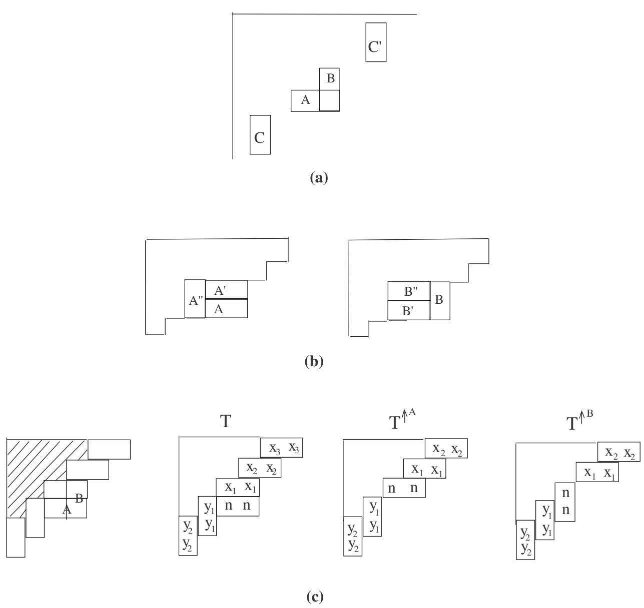

Fig. 5:

If T has a corner, say C, lying in (T, A, sw), as it is illustrated in Figure 5(a), let $\sigma_1 \dots \sigma_n \in B_n$ such that

$$P^r(\sigma_1 \dots \sigma_{n-1}) = T^{\uparrow C}.$$

Since there is a domino corner in $(T, C, \mathrm{ne}) \cap (T, B, \mathrm{sw})$ we have $\beta \overset{p_r}{\sim} \sigma$ as in the Cases (1),(2) and (3). If C is a vertical domino corner, the argument of the Case (4) applied on the domino corners A and C gives that $\alpha \overset{p_r}{\sim} \sigma$. On the other hand if C is a horizontal domino corner then the argument of the Cases (1),(2) and (3) applied on A and C gives $\alpha \overset{p_r}{\sim} \sigma$. Therefore $\alpha \overset{p_r}{\sim} \beta$. On the other hand T has a corner, say C', lying in (T, B, ne) one can use the same argument in the transpose of T.

If there no domino corner in T other then A and B there are two possibility as illustrated in Figure 5(b) and Figure 5(c). Observe that the case given in Figure 5(b) is just the transpose of the Case (4) illustrated in Figure 4(d), therefore it follows directly that $\alpha \overset{p_r}{\sim} \beta$.

For the latter case shown in Figure 5(c), observe that the shaded area is a r-staircase shape and we must have either the domino corner A or B of T labeled by (n, n). Here we assume A is labeled by (n, n) since for the other case one can use the same argument on the transpose tableau T^t. So as Figure 5(c) illustrates, let $x_1 \dots x_k$ be the labels of horizontal domino cells and $y_1 \dots y_l$ be the vertical domino cells which are both positive decreasing sequence such that $r + 1 = k + l$. Observe that $\eta(T^{\uparrow A}) = \eta(T^{\uparrow B}) = x_k > 0$ therefore $\alpha_n = \beta_n = x_k$. Let $\tilde{u}$ be a signed word which is a shuffle of $x_1 \dots x_{k-1}$ and $-y_1 \dots -y_l$. It is easy to see that $P^r(n\tilde{u}x_k) = T = P^r(-n\tilde{u}x_k)$, and $n\tilde{u}x_k \overset{\mathrm{D}_3^r}{\sim} (-n)\tilde{u}x_k$. On the other hand $P^r(n\tilde{u}) = T^{\uparrow A}$ and $P^r(-n\tilde{u}) = T^{\uparrow B}$ and by induction hypothesis we have $n\tilde{u} \overset{p_r}{\sim} \alpha_1 \dots \alpha_{n-1}$ and $(-n)\tilde{u} \overset{p_r}{\sim} \beta_1 \dots \beta_{n-1}$. Hence $\alpha = \alpha_1 \dots \alpha_{n-1} x_k \overset{p_r}{\sim} n\tilde{u}x_k \overset{\mathrm{D}_3^r}{\sim} (-n)\tilde{u}x_k \overset{p_r}{\sim} \beta_1 \dots \beta_{n-1} x_k = \beta$. $\square$

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Spanning forests, electrical networks, and a determinant identity[†]

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Abstract. We aim to generalize a theorem on the number of rooted spanning forests of a highly symmetric graph to the case of asymmetric graphs. We show that this can be achieved by means of an identity between the minor determinants of a Laplace matrix, for which we provide two different (combinatorial as well as algebraic) proofs in the simplest case. Furthermore, we discuss the connections to electrical networks and the enumeration of spanning trees in sequences of self-similar graphs.

Résumé. Nous visons à généraliser un théorème sur le nombre de forêts couvrantes d'un graphe fortement symétrique au cas des graphes asymétriques. Nous montrons que cela peut être obtenu au moyen d'une identité sur les determinants mineurs d'une matrice Laplacienne, pour laquelle nous donnons deux preuves différentes (combinatoire ou bien algébrique) dans le cas le plus simple. De plus, nous discutons les relations avec des réseaux électriques et l'énumération d'arbres couvrants dans de suites de graphes autosimilaires.

Keywords: spanning forest, electrical network, Laplace matrix, determinant identity

1 Introduction

It is known since Kirchhoff's days [10] that there is a close relationship between electrical networks, spanning trees, and the Laplace matrix of a graph. There is a vast amount of literature on spanning trees, electrical networks and related notions: see e.g. [1, 4, 8, 12, 13]. The relation to probability theory was studied in [9, 14]. The celebrated matrix-tree theorem is the most important tool for the enumeration of spanning trees, and it has been successfully used to find closed formulæ for the number of spanning trees in various classes of graphs. A version of the matrix-tree theorem considers all minors of the Laplace matrix of a graph G rather than just those that result from deleting one row and one column. It turns out that the determinants of smaller submatrices count *spanning forests* of G:

Theorem 1 *Let $G = (V, E)$ be a graph and $L = L_G$ its Laplace matrix. For a subset $R \subseteq V$, let $L(R)$ be the matrix that results from deleting all rows and columns that correspond to vertices in R. Then, the number $r(R) = r_G(R)$ of rooted spanning forests whose roots are precisely the vertices in R is given by*

$$r(R) = \det L(R).$$

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We refer the interested reader to [5, 6, 15] for a proof of this theorem. This important result was used in a recent paper by the authors [17], in which the following theorem was given as a byproduct:

Theorem 2 *Let G be a connected, finite (multi-)graph and let $D \subseteq V$ be a subset of θ distinguished vertices. Suppose that G is strongly symmetric with respect to D, i.e. the restriction of the automorphism group of G to D is either the entire symmetric group or the alternating group. Then we have*

$$r(R) = k\rho^{k-1}\theta^{1-k}t(G)$$

for all sets $R \subseteq D$ of cardinality k, where ρ is the resistance scaling factor of G with respect to D and $t(G)$ is the number of spanning trees of G.

A precise definition of the resistance scaling factor is given in Section 3. The above result was inspired by the problem of enumerating spanning trees in certain sequences of self-similar graphs which in turn was motivated by applications in statistical physics [7]. However, it appeared that the condition "strongly symmetric with respect to the distinguished vertices" is stronger than necessary, and experimentally, it seemed that it could be relaxed to "the automorphism group acts 2-homogeneously on the set of distinguished vertices". In this paper, we show that this will be a consequence of a certain determinant identity, thus providing a generalization to the case of graphs that lack symmetry. We prove this determinant identity in the simplest case (three distinguished vertices) in two different ways and discuss its implications to the theory of electrical networks and the aforementioned enumeration of spanning trees in sequences of self-similar graphs. The general form of the determinant identity is left as a conjecture to be proved at a later stage. This conjecture reads as follows:

Conjecture 3 *Let G be a (possibly edge-weighted, not necessarily connected) graph and L its weighted Laplace matrix. For a set R of vertices, we write $L(R)$ for the matrix that results from deleting all rows and columns corresponding to R as before. Furthermore, we set $r(R) = \det L(R)$, and $t(G)$ denotes the number of spanning trees of G (counted according to the weights). Then, the identity*

$$r(R)t(G)^{|R|-2} = \sum_{B} \alpha(B) \prod_{\{v,w\}\in E(B)} r(\{v,w\}) \tag{1}$$

holds for all sets R with $|R| \geq 2$, where the sum is taken over all graphs B with vertex set R and the following properties:

- *The number of edges of B is exactly $|R| - 1$,*
- *All components of B are either paths (possibly single vertices) or cycles (which includes the 2-cycle with two edges connecting the same vertices).*

The coefficient $\alpha(B)$ is then given by

$$\alpha(B) = \prod_{C\in\mathcal{C}(B)} \beta(C),$$

where $\mathcal{C}(B)$ is the set of all components of B and

$$\beta(C) = \begin{cases} 2^{1-\ell} & \text{if } C \text{ is a path of length } \ell > 0, \\ -2^{1-\ell} & \text{if } C \text{ is a cycle of length } \ell > 2, \\ 1 & \text{if } C \text{ is a single vertex,} \\ -\frac{1}{4} & \text{if } C \text{ is a 2-cycle.} \end{cases}$$

Remark 1 *Note that $t(G) = r(\{v\})$ for any vertex v. Hence,* (1) *remains true for $|R| = 1$ if the empty product is considered to be* 1.

If the graph G is connected (hence $t(G) > 0$), we may write Formula (1) in the form

$$\frac{r(R)}{t(G)} = \sum_B \alpha(B) \prod_{\{v,w\} \in E(B)} \frac{r(\{v,w\})}{t(G)}.$$

Thus the equation above relates the quotient $r(R)/t(G)$ for arbitrary root set R to the same quantities for root sets of size 2. The quantity $r(\{v,w\})/t(G)$ measures the effective resistance between v and w, see Section 3 for further information about this.

2 Proof of the special case

As mentioned in the introduction, we want to exhibit two different ways to prove our determinant identity in the case of three distinguished vertices. In this simple case, it reads as follows:

$$\begin{aligned} r(\{v,w,x\})r(\{v\}) = \tfrac{1}{2}\Big(r(\{v,w\})r(\{v,x\}) + r(\{v,w\})r(\{w,x\}) + r(\{v,x\})r(\{w,x\})\Big) \\ - \tfrac{1}{4}\Big(r(\{v,w\})^2 + r(\{v,x\})^2 + r(\{w,x\})^2\Big) \quad (2) \end{aligned}$$

for arbitrary vertices $v, w, x \in V$.

2.1 Combinatorial proof

First, we construct a graph H as follows: let G and G' be disjoint isomorphic copies of G, with an isomorphism $\phi : G \to G'$. The vertices in G' that correspond to v, w, x are denoted by v', w', x'. Now, we identify v and v', w and w', and x and x'. Furthermore, we impose an additional weight λ on all edges of G and an additional weight μ on all edges of G' (note that edges connecting v, w, x are doubled and thus receive a weight of $\lambda + \mu$). If the Laplace matrix of G has the shape

$$L_G = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix},$$

where L_1 and L_2 form the rows corresponding to v, w, x, and L_1 and L_3 form the respective columns, then the Laplace matrix of H has the shape

$$L_H = \begin{pmatrix} (\lambda+\mu)L_1 & \lambda L_2 & \mu L_2 \\ \lambda L_3 & \lambda L_4 & 0 \\ \mu L_3 & 0 & \mu L_4 \end{pmatrix}.$$

We delete the first row and column to obtain a matrix $\tilde{L}$ of the form

$$\tilde{L} = \begin{pmatrix} (\lambda+\mu)\tilde{L}_1 & \lambda\tilde{L}_2 & \mu\tilde{L}_2 \\ \lambda\tilde{L}_3 & \lambda L_4 & 0 \\ \mu\tilde{L}_3 & 0 & \mu L_4 \end{pmatrix}.$$

The weighted number of spanning trees of H is given by

$$\det \tilde{L} = \det \begin{pmatrix} (\lambda+\mu)\tilde{L}_1 & \lambda\tilde{L}_2 & \mu\tilde{L}_2 \\ \lambda\tilde{L}_3 & \lambda L_4 & 0 \\ 0 & -\mu L_4 & \mu L_4 \end{pmatrix} = \det \begin{pmatrix} (\lambda+\mu)\tilde{L}_1 & (\lambda+\mu)\tilde{L}_2 & \mu\tilde{L}_2 \\ \lambda\tilde{L}_3 & \lambda L_4 & 0 \\ 0 & 0 & \mu L_4 \end{pmatrix}$$
$$= (\lambda+\mu)^2 \lambda^{|V|-3}\mu^{|V|-3} \det \begin{pmatrix} \tilde{L}_1 & \tilde{L}_2 \\ \tilde{L}_3 & L_4 \end{pmatrix} \det L_4.$$

Note that the coefficient of $\lambda^{|V|-2}\mu^{|V|-2}$ gives those spanning trees which contain $|V|-2$ edges in G and $|V|-2$ edges in G' and thus induce two spanning forests with two components each on G and G'. From the above expression for the determinant, it is obvious that this coefficient is exactly

$$2 \det \begin{pmatrix} \tilde{L}_1 & \tilde{L}_2 \\ \tilde{L}_3 & L_4 \end{pmatrix} \det L_4 = 2r(\{v\})r(\{v,w,x\}).$$

This means that the left hand side of (2) is also the (weighted) number of unordered pairs (F_1, F_2) of spanning forests with two components in G resp. G' and the property that their union is a spanning tree in H (note that $\phi(F_1) \neq F_2$ for such a pair, since this would yield a cycle, and thus the number of unordered pairs is indeed just $\frac{1}{2}$ of the number of ordered pairs). We want to show that this is exactly the right hand side of (2). Each component of F_1 and F_2 has to contain at least one of the vertices v, w, x, since their union forms a spanning tree, and they are only joined at v, w, x. The right hand side of (2) only counts pairs of (rooted) spanning forests with this property by definition, hence it suffices to consider such spanning forests.

Now we only have to show that an unordered pair (F_1, F_2) of spanning forests with two components each of which contains at least one vertex of $\{v, w, x\}$ is counted with coefficient 1 on the right hand side of (2) if the union is a spanning tree and with coefficient 0 otherwise. We distinguish three cases:

- F_1 and F_2 induce distinct connections on the set $\{v, w, x\}$, so that the union forms a spanning tree. Without loss of generality, we assume that F_1 connects v and w, while F_2 connects v and x. Then, F_1 can be rooted at v and x or at w and x, and F_2 can be rooted at v and w or at w and x. The four possibilities yield a total coefficient of 1:

roots of F_1	roots of F_2	coefficient
v, x	v, w	$\frac{1}{2}$
v, x	w, x	$\frac{1}{2}$
w, x	v, w	$\frac{1}{2}$
w, x	w, x	$-2 \cdot \frac{1}{4}$

- F_1 and F_2 induce the same connections on the set $\{v, w, x\}$, so that a cycle is formed, but $\phi(F_1) \neq F_2$. Without loss of generality, we assume that F_1 and F_2 connect v and w. Again, we have to consider four possibilities:

roots of F_1	roots of F_2	coefficient
v, x	v, x	$-2 \cdot \frac{1}{4}$
v, x	w, x	$\frac{1}{2}$
w, x	v, x	$\frac{1}{2}$
w, x	w, x	$-2 \cdot \frac{1}{4}$

The total coefficient is 0, as desired.

- $\phi(F_1) = F_2$. Suppose for instance that F_1 connects v and w. As in the previous case, the union is not a spanning tree, and again, we obtain a coefficient 0:

roots of F_1	roots of F_2	coefficient
v, x	v, x	$-\frac{1}{4}$
v, x	w, x	$\frac{1}{2} \cdot \frac{1}{2}$
w, x	v, x	$\frac{1}{2} \cdot \frac{1}{2}$
w, x	w, x	$-\frac{1}{4}$

Putting everything together, we reach the desired result.

2.2 Algebraic proof

We are now going to derive Formula (2) using basic linear algebra and the Desnanot-Jacobi identity (also known as condensation formula, see for example [3]): For simplicity we assume that the vertex set V is given by $V = \{1, 2, \dots, n\}$ with $v = 1, w = 2, x = 3$. Furthermore, we write L_B^A to denote the submatrix of L obtained by deleting the rows in $A \subseteq V$ and columns in $B \subseteq V$ and set $D_B^A = \det(L_B^A)$. Then Formula (2) reads as follows:

$$D_1^1 D_{1,2,3}^{1,2,3} = \tfrac{1}{2}\Big(D_{1,2}^{1,2} D_{1,3}^{1,3} + D_{1,2}^{1,2} D_{2,3}^{2,3} + D_{1,3}^{1,3} D_{2,3}^{2,3}\Big) - \tfrac{1}{4}\Big(\big(D_{1,2}^{1,2}\big)^2 + \big(D_{1,3}^{1,3}\big)^2 + \big(D_{2,3}^{2,3}\big)^2\Big)$$

In order to prove this identity we start with the following simple observation: Let $b_1, b_2, \dots, b_n$ be the column vectors of $L^{1,2}$. Then $b_1 + b_2 + b_3 + b_4 + \cdots + b_n = 0$, since the sum of column vectors in L is equal to 0. Hence

$$\begin{aligned} 0 &= \det(b_1 + b_2 + b_3, b_4, \dots, b_n) \\ &= \det(b_3, b_4, \dots, b_n) + \det(b_2, b_4, \dots, b_n) + \det(b_1, b_4, \dots, b_n) = D_{1,2}^{1,2} + D_{1,3}^{1,2} + D_{2,3}^{1,2}. \end{aligned}$$

By symmetry of L the minors $D_{2,3}^{1,2}$ and $D_{1,2}^{2,3}$ are equal. Thus

$$D_{1,2}^{1,2} + D_{1,3}^{1,2} + D_{1,2}^{2,3} = 0.$$

Similarly, we find that

$$D_{1,3}^{1,2} + D_{1,3}^{1,3} + D_{1,3}^{2,3} = 0 \qquad \text{and} \qquad D_{1,2}^{2,3} + D_{1,3}^{2,3} + D_{2,3}^{2,3} = 0.$$

Adding the first two equations and subtracting the last one we obtain

$$2D_{1,3}^{1,2} = D_{2,3}^{2,3} - D_{1,2}^{1,2} - D_{1,3}^{1,3}. \tag{3}$$

By the Desnanot-Jacobi identity we have

$$D_{1,2,3}^{1,2,3} D_1^1 = D_{1,2}^{1,2} D_{1,3}^{1,3} - D_{1,3}^{1,2} D_{1,2}^{1,3} = D_{1,2}^{1,2} D_{1,3}^{1,3} - \left(D_{1,3}^{1,2}\right)^2, \tag{4}$$

where $D_{1,3}^{1,2} = D_{1,2}^{1,3}$ by symmetry of L. By inserting (3) into (4) we finally obtain the asserted identity.

Remark 2 *Let us note that we have verified Conjecture 3 for the case of four and five boundary vertices using a similar algebraic argument and a more general version of the Desnanot-Jacobi identity (see [11]).*

3 Electrical networks

Let $G = (V, E, c)$ be an edge-weighted graph (network) with weights (conductances) $c : E \to [0, \infty)$. The (weighted) Laplace matrix L is defined by its entries

$$L_{x,y} = \begin{cases} -c(\{x,y\}) & \text{if } x \neq y, \\ \sum_{z \sim x} c(\{x,z\}) & \text{if } x = y \end{cases}$$

for all vertices $x, y \in V$. We say that two networks $(V(G), E(G), c_G)$ and $(V(H), E(H), c_H)$ are *electrically equivalent* with respect to $D \subseteq V(G) \cap V(H)$, if they cannot be distinguished by applying voltages to D and measuring the resulting currents on D. By Kirchhoff's current law this means that the rows corresponding to D of $L_G H_D^{V(G)}$ and $L_H H_D^{V(H)}$ are equal, where $H_D^{V(G)}$ is the matrix associated to harmonic extension. If $u, v \in V(G)$ are vertices in G and H is the complete graph with vertex set $\{u, v\}$, then there exists a conductance $c_{\text{eff}}(u, v)$ on the single edge of H, so that $(V(G), E(G), c_G)$ and H equipped with $c_{\text{eff}}(u, v)$ are equivalent. The number $\rho_{\text{eff}}(u, v) = c_{\text{eff}}(u, v)^{-1}$ is called *effective resistance* of u and v.

In combinatorics unit conductances are of great interest because of the well-known relation between electrical networks and the number of spanning trees. Let G be a graph and c_G be unit conductances on the edges of G. We say that G has *resistance scaling factor* $\rho = \rho_D$ with respect to $D \subseteq V$, if (G, c_G) is electrically equivalent to $(H, \rho^{-1} c_H)$, where H is a complete graph with vertex set $V(H) = D$ and c_H are unit conductances on H. Note that the effective resistance of vertices u and v in a graph with unit conductances is exactly the resistance scaling factor with respect to $\{u, v\}$.

Theorem 2 implies that the effective resistance of two vertices u, v in a connected graph with unit conductances is given by

$$\rho_{\text{eff}}(u, v) = \frac{r_G(\{u, v\})}{t(G)}. \tag{5}$$

This can also be obtained from Kirchhoff's famous result connecting currents and spanning trees (see for example [2]). Now Conjecture 3 allows the following interpretation: given all effective resistances of a graph, we can determine all quotients of the form

$$\frac{r_G(R)}{t(G)}.$$

In particular, if two graphs G and H are electrically equivalent with respect to D, then

$$\frac{r_G(R)}{t(G)} = \frac{r_H(R)}{t(H)}$$

for all $R \subseteq D$ (note that Theorem 2 is the special case when $H = K_\theta$). If we pursue this thought to its climax, we finally end up with the following question: Given all effective resistances of a graph, can we reconstruct the original graph?

Of course, we may state this question more generally for networks: Let G be a complete graph on n vertices and conductances on the edges. Clearly the conductances comprise a tuple of $\binom{n}{2}$ non-negative numbers. Given the conductances we can compute all effective resistances in this network easily. The effective resistances also form a tuple of $\binom{n}{2}$ non-negative numbers. Hence we may ask whether it is possible to reverse this computation.

Conjecture 4 *Given effective resistances for each pair of vertices of a complete graph, there is exactly one tuple of conductances, which yields the given effective resistances, and there is a formula similar to* (1) *that determines them.*

It is plausible that this or similar problems have been considered in physics and related fields. Yet we were unable to find anything in the literature we studied, and the expert colleagues we discussed the problem with were not aware of any results in this direction either.

If we are given the numbers $t(G)$ and $r_G(\{u,v\})$ for all $u, v \in V(G)$ of a connected graph, we can compute all effective resistances of G by means of (5). Assuming that the conjecture above holds, we can now reconstruct the network and hence the graph. With full information it is finally easy to compute the numbers $r_G(R)$ for all $R \subseteq V(G)$. Hence Conjecture 3 is plausible, if Conjecture 4 holds.

Let us briefly discuss Conjecture 4 for $n = 3$: Let $V = \{u, v, w\}$. A simple computation yields that

$$\begin{aligned}
c_{\text{eff}}(u,v) &= c(\{u,v\}) + \frac{c(\{v,w\})c(\{w,u\})}{c(\{v,w\}) + c(\{w,u\})} = \frac{t(G)}{c(\{v,w\}) + c(\{w,u\})}, \\
c_{\text{eff}}(v,w) &= c(\{v,w\}) + \frac{c(\{w,u\})c(\{u,v\})}{c(\{w,u\}) + c(\{u,v\})} = \frac{t(G)}{c(\{w,u\}) + c(\{u,v\})}, \\
c_{\text{eff}}(w,u) &= c(\{w,u\}) + \frac{c(\{u,v\})c(\{v,w\})}{c(\{u,v\}) + c(\{v,w\})} = \frac{t(G)}{c(\{u,v\}) + c(\{v,w\})},
\end{aligned}$$

noting that $t(G) = c(\{u,v\})c(\{v,w\}) + c(\{v,w\})c(\{w,u\}) + c(\{w,u\})c(\{u,v\})$. From this it is easy to deduce that given effective conductances $c_{\text{eff}}(u,v)$, $c_{\text{eff}}(v,w)$, and $c_{\text{eff}}(w,u)$ there is at most one solution for the conductances $c(\{u,v\})$, $c(\{v,w\})$, and $c(\{w,u\})$ of the system above (that can be given explicitly). Finally, a simple manipulation shows that

$$c(\{u,v\}) = \tfrac{1}{2} t(G)\big(\rho_{\text{eff}}(\{u,w\}) + \rho_{\text{eff}}(\{v,w\}) - \rho_{\text{eff}}(\{u,v\})\big),$$

or

$$c(\{u,v\}) = \tfrac{1}{2}\big(r_G(\{u,w\}) + r_G(\{v,w\}) - r_G(\{u,v\})\big),$$

which shows a certain resemblance to Equation (1).

4 Enumeration of spanning trees

Recently, it was shown in two papers independently [7, 16] how the number of spanning trees in Sierpiński graphs (i.e., the finite approximations to the Sierpiński gasket) can be calculated. If S_n denotes the level-n Sierpiński graph (starting with $S_0 = K_3$, see Figure 1), the number of spanning trees is given by the formula

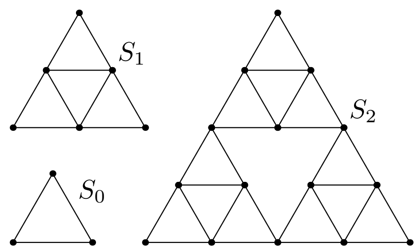

Fig. 1: Sierpiński graphs

$$t(S_n) = \sqrt[4]{\tfrac{3}{20}} \cdot \left(\tfrac{5}{3}\right)^{-n/2} \cdot \left(\sqrt[4]{540}\right)^{3^n}. \tag{6}$$

The proofs given in [7, 16] make extensive use of symmetry; in this section, we show that all that is essentially needed is electrical equivalence. To this end, we consider a modified version $T_0, T_1, T_2, \ldots$ of the Sierpiński graphs (see Figure 2). Obviously, the resulting graphs are not as symmetric as the Sierpiński graphs and we note that the arguments of [7, 16] are not applicable anymore. We do not only modify the initial graph but also change the number of subdivisions in the construction, since for the simpler construction rule of Sierpiński graphs not all possible phenomena occur. It is not difficult to see that the initial graph T_0 is electrically equivalent to a K_3 (with unit conductances) with respect to the three corner vertices, and thus this is also the case for all graphs T_n in the sequence (up to a resistance scaling factor of $\left(\frac{15}{7}\right)^n$, which is easily shown by induction). We write $x_{1,n}, x_{2,n}, x_{3,n}$ for the corner vertices of T_n; then, if H_n is the complete graph with vertices $x_{1,n}, x_{2,n}, x_{3,n}$ and edge weights (conductances) $\left(\frac{7}{15}\right)^n$, we have

$$\frac{r_{T_n}(R)}{t(T_n)} = \frac{r_{H_n}(R)}{t(H_n)}$$

for all subsets $R \subseteq \{x_{1,n}, x_{2,n}, x_{3,n}\}$ of cardinality 2, since the effective resistances are the same. But this is trivially true for subsets of cardinality 1, and the special case of Conjecture 3 for three vertices shows that it is also the case for $R = \{x_{1,n}, x_{2,n}, x_{3,n}\}$.

Now consider the graph T_{n+1}, which comprises of six copies of T_n. Fix one of these copies and call it C. The graph induced by the remaining edges is called B. Every spanning tree of T_{n+1} induces spanning forests on B and C. Now fix a spanning forest F on B that can be extended to a spanning tree of T_{n+1}. F induces certain connections on the corner vertices u, v, w of C: If the corner vertices of C are not connected at all by F, a spanning tree on C is needed to complete a spanning tree on T_{n+1}. If F connects precisely two of the corner vertices of C (say u and v), then we need a spanning forest with two components and the property that u and v are in different components. However, this can also be interpreted as a rooted spanning forest with roots u and v! If all corner vertices of C are connected

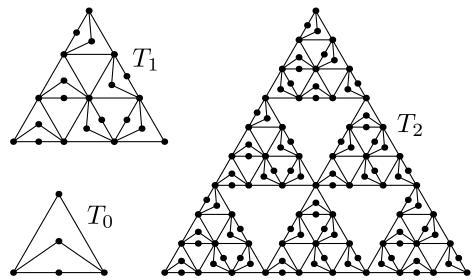

Fig. 2: Modified Sierpiński graphs with three subdivisions

"from the outside" by F, then we need a rooted spanning forest with three components on C to complete a spanning tree, where the roots are precisely the corner vertices again. Hence there are coefficients ν_R such that

$$t(T_{n+1}) = \sum_{R \subseteq \{u,v,w\}} \nu_R \cdot r_C(R),$$

and these coefficients only depend on B. Note that the coefficient $\nu_{\{u,v,w\}}$ is 0 in the case of ordinary Sierpiński graphs (Figure 1), which is the reason why we deal with three subdivisions instead of two. If we replace C by H_n now to obtain a graph T'_{n+1}, the above considerations show that

$$\begin{aligned} t(T'_{n+1}) &= \sum_{R \subseteq \{u,v,w\}} \nu_R \cdot r_{H_n}(R) = \frac{t(H_n)}{t(C)} \cdot \sum_{R \subseteq \{u,v,w\}} \nu_R \cdot r_C(R) \\ &= \frac{t(H_n)}{t(C)} \cdot t(T_{n+1}) = \frac{t(H_n)}{t(T_n)} \cdot t(T_{n+1}). \end{aligned}$$

Applying this procedure repeatedly for all three copies of T_n, we obtain

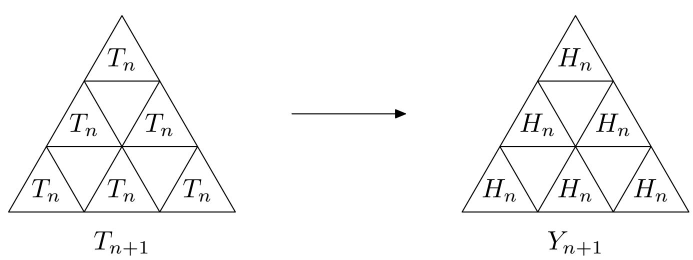

Fig. 3: Replacing T_n by H_n

$$t(T_{n+1}) = \left(\frac{t(T_n)}{t(H_n)} \right)^6 \cdot t(Y_{n+1}),$$

where Y_{n+1} comprises of three copies of H_n, as indicated in Figure 3. But H_n and Y_{n+1} are small graphs for which the (weighted) number of spanning trees is easily computed explicitly: one has

$$t(H_n) = 3 \cdot \left(\tfrac{7}{15}\right)^{2n} \qquad \text{and} \qquad t(Y_{n+1}) = 5292 \cdot \left(\tfrac{7}{15}\right)^{9n}$$

and thus

$$t(T_{n+1}) = \tfrac{196}{27} \cdot \left(\tfrac{15}{7}\right)^{3n} \cdot t(T_n)^6.$$

Now it is just an easy induction to show that

$$t(T_n) = \left(\frac{3^{12}}{2^{10} \cdot 5^3 \cdot 7^7}\right)^{1/25} \cdot \left(\frac{15}{7}\right)^{-3n/5} \cdot \left(\left(\frac{2^{10} \cdot 5^3 \cdot 7^7}{3^{12}}\right)^{1/25} t(T_0)\right)^{6^n}$$

In the case of the sequence depicted in Figure 2, we have $t(T_0) = 12$ and obtain

$$t(T_n) = \left(\frac{3^{12}}{2^{10} \cdot 5^3 \cdot 7^7}\right)^{1/25} \cdot \left(\frac{15}{7}\right)^{-3n/5} \cdot \left(2^{60} \cdot 3^{12} \cdot 5^3 \cdot 7^7\right)^{6^n/25}$$

In a similar way, one can derive Equation (6) for the number of spanning trees of the ordinary Sierpiński graphs. The essential point in this approach was the fact that the graphs $S_0, S_1, \ldots$ and $T_0, T_1, \ldots$ were electrically equivalent to simple graphs with resistances that could be determined explicitly. If this is not the case any more, things become more complicated, as can be seen from the final example below. Nonetheless, we believe that the technique of replacing subgraphs by electrically equivalent graphs can be very useful for the enumeration of spanning trees (and we also conjecture that it is applicable in general, not just in the case of three boundary vertices).

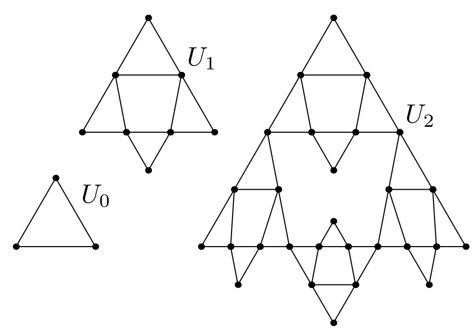

Fig. 4: Another modification of the Sierpiński graphs

Let us now consider the sequence of self-similar graphs depicted in Figure 4. We can still replace the four copies of U_n in U_{n+1} by simple complete graphs $H_n \simeq K_3$ to obtain a graph Y_{n+1}, but the conductances in H_n are not all equal any longer. The effective conductances in U_n can be found by iterating the map that is shown in Figure 5: starting with $(a_0, b_0) = (1, 1)$, one applies the recursion

$$(a_{n+1}, b_{n+1}) = \left(\frac{(2a_n + b_n)(3a_n^2 + 8a_n b_n + b_n^2)}{2(3a_n + 2b_n)(3a_n + 5b_n)}, \frac{b_n(2a_n + b_n)}{3a_n + 2b_n}\right)$$

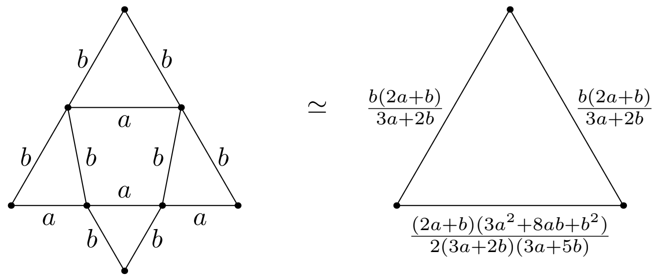

Fig. 5: The map that defines the conductances recursively

to obtain the effective conductances (a_{n+1}, b_{n+1}) of U_{n+1} from those of U_n. Arguing as in the previous example, one obtains

$$t(U_{n+1}) = \left(\frac{t(U_n)}{t(H_n)}\right)^4 \cdot t(Y_{n+1}).$$

Now one has

$$t(H_n) = b_n(2a_n + b_n) \qquad \text{and} \qquad t(Y_{n+1}) = 2b_n^3(2a_n + b_n)^3(a_n + 3b_n)$$

and thus

$$t(U_{n+1}) = \frac{2(a_n + 3b_n)}{b_n(2a_n + b_n)} \cdot t(U_n)^4.$$

There are no simple formulæ for a_n and b_n, but one can show that they behave asymptotically like

$$a_n = A \cdot \left(\tfrac{5}{9}\right)^n \left(1 + O\left(\left(\tfrac{2}{3}\right)^n\right)\right), \qquad b_n = 3A \cdot \left(\tfrac{5}{9}\right)^n \left(1 + O\left(\left(\tfrac{2}{3}\right)^n\right)\right)$$

for some constant A, which results in the following asymptotic behavior for $t(U_n)$:

$$t(U_n) \sim B \cdot \left(\tfrac{9}{5}\right)^{-n/3} \cdot C^{4^n}$$

for certain constants B and C. Note that the structure of this asymptotic formula is still the same as for the sequence of Sierpiński graphs.

Acknowledgment

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Branching rules in the ring of superclass functions of unipotent upper-triangular matrices

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It is becoming increasingly clear that the supercharacter theory of the finite group of unipotent upper-triangular matrices has a rich combinatorial structure built on set-partitions that is analogous to the partition combinatorics of the classical representation theory of the symmetric group. This paper begins by exploring a connection to the ring of symmetric functions in non-commuting variables that mirrors the symmetric group's relationship with the ring of symmetric functions. It then also investigates some of the representation theoretic structure constants arising from the restriction, tensor products and superinduction of supercharacters.

Keywords: set partitions, supercharacters, branching rules, symmetric functions

1 Introduction

The representation theory of the symmetric group S_n –with its connections to partition and tableaux combinatorics– has become a fundamental model in combinatorial representation theory. It has become clear in recent years that the representation theory finite group of unipotent upper-triangular matrix groups $U_n(q)$ can lead to a similarly rich combinatorial theory. While understanding the usual representation theory of $U_n(q)$ is a wild problem, André [1, 2, 3, 4] and Yan [23, 24] constructed a natural approximation to the representation theory that leads to a more computable theory. This approximation (known as a super-representation theory) now relies on set-partition combinatorics in the same way that the representation theory of the symmetric group relies on partition combinatorics.

A fundamental tool in symmetric group combinatorics is the ring of symmetric functions, which encodes the character theory of all symmetric groups simultaneously in a way that polynomial multiplication in the ring of symmetric functions becomes symmetric group induction from Young subgroups. This kind of a relationship has been extended to wreath products and type A finite groups of Lie type (for descriptions see for example [17, 21]). One of the purposes of this paper is to suggest an analogous relationship between the supercharacter theory of $U_n(q)$ and the ring on symmetric functions in non-commuting variables NCSym. In particular, Corollary 3.2 shows that there are a family of algebra isomorphisms from the ring of supercharacters to NCSym, where we replace induction from subgroups with its natural analogue superinduction from subgroups. Unfortunately, there does not yet seem to be a canonical choice (ideally, such a choice would take the Hopf structure of NCSym into account).

The other purpose of this paper is to use the combinatorics of set partitions to supply recursive algorithms for computing restrictions to a family of subgroups called parabolic subgroups. It turns out that if $k \leq n$, then there are many ways in which $U_k(q)$ sits inside $U_n(q)$ as a subgroup. In fact, for every subset $S \subseteq \{1, 2, \ldots, n\}$ with k elements, there is a distinct subgroup U_S of $U_n(q)$ isomorphic to $U_k(q)$. The restriction from $U_n(q)$ to U_S depends on S, and Theorem 4.4 sorts out the combinatorial differences for all possible subsets S. This result can then be easily extended to give restriction rules for all parabolic subgroups. These computations require knowledge of tensor product results that were previously done by André [1] for large prime and by Yan [23] for arbitrary primes. For completeness, this paper supplies an alternate proof that relates tensor products to restriction and a generalization of the inflation functor (see Lemma 4.5).

By Frobenius reciprocity we then also obtain the coefficients of superinduction from these subgroups. Corollary 4.10 concludes by stating that superinduced supercharacters from parabolic subgroups are essentially twisted super-permutation characters (again using the generalization of the inflation map). These results give the structure constants for the ring of superclass functions of the finite unipotent upper-triangular groups. However, the underlying coefficient ring is $\mathbb{Z}[q^{-1}]$, unlike in the case of the symmetric group where the ring is $\mathbb{Z}$.

Section 2 introduces some set-partition combinatorics; describes the parabolic subgroups that will replace Young subgroups in our theory; reviews the supercharacter theory of pattern groups (as defined in [12]); and recalls the ring of symmetric functions in non-commuting variables NCSym. We proceed in Section 3.2 by describing the family of isormorphisms between NCSym and the ring of supercharacters. Section 4 uses the fact that supercharacters of $U_n(q)$ decompose into tensor products of simpler characters to supply algorithms for computing restrictions and superinductions of supercharacters. These results generalize restriction results in [21], and make use of a new generalization of the inflation functor to supercharacters of pattern groups.

This paper builds on [18] and [20] by giving restriction and superinduction formulas for larger families of groups. These formulas are computable, and are being implemented in Python as part of an honors thesis at the University of Colorado. Other recent work in this area worth mentioning includes extensions by André and his collaborators to supercharacter theories of other types [5] and over other rings [6], explorations of all supercharacter theories for a given group by Hendrickson in his thesis [16], and an intriguing unexplored connection to L-packets in the work of Drinfeld and Boyarchenko [11]. This abstract omits the proofs, since they relatively straight-forward once the results are known.

2 Preliminaries

This section reviews the combinatorics needed for the main results, gives a brief introduction to the supercharacter theory of pattern groups, and recalls the ring of symmetric functions in non-commuting variables.

2.1 $\mathbb{F}_q$-labeled set-partitions

For $S \subseteq \{1, 2, \ldots, n\}$, let
$$\mathcal{S}_S = \{\text{set-partitions of } S\},$$
and
$$\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n, \qquad \text{where} \qquad \mathcal{S}_n = \mathcal{S}_{\{1,2,\ldots,n\}}.$$

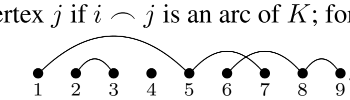
An *arc* $i \frown j$ of $K \in \mathcal{S}_S$ is a pair $(i, j) \in S \times S$ such that

(1) $i < j$,

(2) i and j are in the same part of K,

(3) if k is in the same part as i and $i < k \leq j$, then $k = j$.

Thus, if we order each part in increasing order, then the arcs are pairs of adjacent elements in each part. For example,
$$\{1, 5, 7\} \cup \{2, 3\} \cup \{4\} \cup \{6, 8, 9\} \in \mathcal{S}_9$$
has arcs $1 \frown 5$, $5 \frown 7$, $2 \frown 3$, $6 \frown 8$, and $8 \frown 9$. We can also represent the set partition K as a diagram consisting of $|K|$ vertices with an edge connecting vertex i to vertex j if $i \frown j$ is an arc of K; for example,

$$\{1, 5, 7\} \cup \{2, 3\} \cup \{4\} \cup \{6, 8, 9\} \qquad \longleftrightarrow$$


The *arc set* $A(K)$ of $K \in \mathcal{S}_S$ is
$$A(K) = \{\text{arcs of } K\}.$$

A *crossing* of $K \in \mathcal{S}_S$ is a pair of arcs $(i \frown k, j \frown l) \in A(K) \times A(K)$ such that $i < j < k < l$. The *crossing set* $C(K)$ of K is
$$C(K) = \{\text{crossings of } K\}.$$

For example, if $K = \{1, 5, 7\} \cup \{2, 3\} \cup \{4\} \cup \{6, 8, 9\}$, then K has one crossing $(5 \frown 7, 6 \frown 8)$, as is easily observed in the above diagrammatic representation of K.

An $\mathbb{F}_q$-*labeled set-partition of* S is a pair (λ, τ_λ), where λ is a set-partition of S and $\tau_\lambda : A(\lambda) \to \mathbb{F}_q^\times$ is a labeling of the arcs by nonzero elements of $\mathbb{F}_q$. By convention, if $\tau_\lambda(i \frown j) = a$, then we write the arc as $i \overset{a}{\frown} j$. Thus, we can typically suppress the labeling function in the notation. Let
$$\mathcal{S}_S(q) = \{\mathbb{F}_q\text{-labeled set-partitions of } S\},$$

and

$$\mathcal{S}(q) = \bigcup_{n\geq 0} \mathcal{S}_n(q), \qquad \text{where} \quad \mathcal{S}_n(q) = \mathcal{S}_{\{1,2,\ldots,n\}}(q).$$

Note that if $s_n(q) = |\mathcal{S}_n(q)|$, then the generating function

$$\sum_{n\geq 0} s_n(q) \frac{x^n}{n!} = e^{\frac{e^{(q-1)x}-1}{q-1}}$$

is a q-analogue of the usual exponential generating function of the Bell numbers (where $q = 2$ gives the usual generating function).

Suppose $S \subseteq T \subseteq \{1, 2, \ldots, n\}$. Then there is a function

$$\begin{array}{cccc} \langle\cdot\rangle_T : & \left\{ \begin{array}{c} \mathbb{F}_q\text{-labeled} \\ \text{set-partitions of } S \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \mathbb{F}_q\text{-labeled} \\ \text{set-partitions of } T \end{array} \right\} \\ & \lambda & \mapsto & \langle\lambda\rangle_T \end{array}$$

where $\langle\lambda\rangle_T$ is the unique $\mathbb{F}_q$-labeled set-partition of T with arc set $A(\lambda)$ and labeling function τ_λ. We will use the convention that $\langle\lambda\rangle_n = \langle\lambda\rangle_{\{1,2,\ldots,n\}}$.

2.2 Pattern groups

For $n \in \mathbb{Z}_{\geq 1}$, let $U_n(q)$ be the group of $n \times n$ unipotent upper-triangular matrices with entries in $\mathbb{F}_q$. Given a poset $\mathcal{P}$ of $\{1, 2, \ldots, n\}$, the *pattern group* $U_\mathcal{P}(q)$ is

$$U_\mathcal{P}(q) = \{u \in U_n(q) \mid u_{ij} \neq 0 \text{ implies } i \preceq j \text{ in } \mathcal{P}\}.$$

Remark. If $T_n(q)$ is the group of $n \times n$ diagonal matrices with entries in $\mathbb{F}_q^\times$, then the set of pattern subgroups of $U_n(q)$ can be characterized as the set of subgroups fixed by the conjugation action of $T_n(q)$ on $U_n(q)$.

Consider the injective map

$$\begin{array}{ccc} \mathcal{S}_n & \longrightarrow & \left\{ \begin{array}{c} \text{Posets of} \\ \{1, 2, \ldots, n\} \end{array} \right\} \\ K & \mapsto & \mathcal{P}_K \end{array}$$

where $i \prec j$ in $\mathcal{P}_K$ if and only if $i < j$ and both i and j are in the same part of K.

A pattern subgroup $U_\mathcal{P}(q)$ is a *parabolic subgroup* of $U_n(q)$ if there exists $K \in \mathcal{S}_n$ such that $\mathcal{P} = \mathcal{P}_K$. Note that if $K = K_1 \cup K_2 \cup \cdots \cup K_\ell$ is the decomposition of K into parts, then

$$U_{\mathcal{P}_K}(q) \cong U_{|K_1|}(q) \times U_{|K_2|}(q) \times \cdots \times U_{|K_\ell|}(q).$$

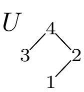
Thus, the parabolic subgroups of $U_\mathcal{P}(q)$ are reminiscent of the Young subgroups of the symmetric groups S_n or parabolic subgroups of a reductive groups of Lie type (such as the general linear group $\mathrm{GL}_n(q)$). In fact, we will follow this analogy into the supercharacter theory of $U_n(q)$. To simplify notation, we will typically write

$$U_K(q) = U_{\mathcal{P}_K}(q), \qquad \text{for } K \in \mathcal{S}_n.$$

Remark. These subgroups are not generally block diagonal. For example,

$$U_{\mathcal{P}_{\{1,3,5\}\cup\{2,4\}}} = \left\{ \left(\begin{array}{ccccc} 1 & 0 & * & 0 & * \\ 0 & 1 & 0 & * & 0 \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \middle| * \in \mathbb{F}_q \right\} \cong U_3(q) \times U_2(q).$$

However, parabolic subgroups do not include all possible copies of pattern subgroups isomorphic to a direct product of $U_k(q)$'s. For example,

U 

$$= \left\{ \left(\begin{array}{cccc} 1 & 0 & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| * \in \mathbb{F}_q \right\} \cong U_3(q) \times U_2(q)$$

is not a parabolic subgroup of $U_4(q)$.

2.3 A supercharacter theory for pattern groups

Given a group G, a supercharacter theory is an approximation to the usual character theory. To be more precise, a supercharacter theory consists of a set of superclasses $\mathcal{K}$ and a set of supercharacters $\mathcal{X}$, such that

(a) the set $\mathcal{K}$ is a partition of G such that each part is a union of conjugacy classes,

(b) the set $\mathcal{X}$ is a set of characters such that each irreducible character appears as the constituent of exactly one supercharacter,

(c) the supercharacters are constant on superclasses,

(d) $|\mathcal{K}| = |\mathcal{X}|$,

(e) the identity element 1 of G is in its own superclass, and the trivial character $\mathbb{1}$ of G is a supercharacter.

This general notion of a supercharacter theory was introduced by Diaconis and Isaacs [12] to generalize work of André and Yan on the character theory of $U_n(q)$.

Remark. The definition includes a reasonable amount of redundancy, as explored in [12, 16].

Diaconis and Isaacs extended the construction of André of a supercharacter theory for $U_n(q)$ to a larger family of groups called algebra groups. We will review the construction for pattern groups (a subset of the set of algebra groups). Let $\mathcal{P}$ be a poset of $\{1, 2, \ldots, n\}$ and let

$$\mathfrak{n}_{\mathcal{P}}(q) = U_{\mathcal{P}}(q) - 1,$$

which is an $\mathbb{F}_q$-algebra.

Fix a nontrivial homomorphism $\vartheta : \mathbb{F}_q^+ \to \mathbb{C}^\times$. The pattern group $U_{\mathcal{P}}(q)$ acts on the left and right on both $\mathfrak{n}_{\mathcal{P}}(q)$ and the dual space $\mathfrak{n}_{\mathcal{P}}(q)^*$, and the two-sided orbits lead to the sets $\mathcal{K}$ and $\mathcal{X}$ by the following rules. The superclasses are given

$$\begin{array}{ccc} U_{\mathcal{P}}(q)\backslash \mathfrak{n}_{\mathcal{P}}(q)/U_{\mathcal{P}}(q) & \longleftrightarrow & \mathcal{K} \\ U_{\mathcal{P}}(q) X U_{\mathcal{P}}(q) & \mapsto & 1 + U_{\mathcal{P}}(q) X U_{\mathcal{P}}(q), \end{array}$$

and the supercharacters are given by

$$\begin{array}{ccc} U_{\mathcal{P}}(q)\backslash \mathfrak{n}_{\mathcal{P}}(q)^*/U_{\mathcal{P}}(q) & \longleftrightarrow & \mathcal{X} \\ U_{\mathcal{P}}(q)\lambda U_{\mathcal{P}}(q) & \mapsto & \chi^\lambda = \dfrac{|\lambda U_{\mathcal{P}}(q)|}{|U_{\mathcal{P}}(q)\lambda U_{\mathcal{P}}(q)|} \displaystyle\sum_{\mu \in U_{\mathcal{P}}(q)\lambda U_{\mathcal{P}}(q)} \vartheta \circ \mu. \end{array}$$

The corresponding $U_{\mathcal{P}}$-modules are given by

$$V^\lambda = \mathbb{C}\text{-span}\{v_\mu \mid \mu \in U_{\mathcal{P}}\lambda\}, \tag{2.1}$$

with action

$$g v_\mu = \vartheta\big((g\mu)(1-g)\big) v_{g\mu}, \qquad \text{for } g \in U_{\mathcal{P}} \text{ and } \mu \in U_{\mathcal{P}}\lambda.$$

Example. The group $U_n(q)$ was the original motivation for studying supercharacter theories. The following results are due to André, Yan, and Arias-Castro–Diaconis–Stanley. The number of superclasses is

$$|\mathcal{K}| = |\mathcal{X}| = |\mathcal{S}_n(q)|,$$

where, for example,

$$\begin{array}{ccc} \mathcal{S}_n(q) & \longrightarrow & \mathcal{K} \\ \mu & \mapsto & u_\mu, \end{array} \qquad \text{and} \quad (u_\mu)_{ij} = \begin{cases} 1, & \text{if } i = j, \\ \tau_\mu(i \frown j), & \text{if } i \frown j \in A(\mu), \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding supercharacter formula for $\lambda, \mu \in \mathcal{S}_n(q)$ is

$$\chi^\lambda(u_\mu) = \begin{cases} \displaystyle\prod_{i \frown l \in A(\lambda)} \frac{q^{l-i-1}\vartheta\big(\tau_\lambda(i \frown l)\tau_\mu(i \frown l)\big)}{q^{|\{j \frown k \in A(\mu) \mid i<j<k<l\}|}}, & \begin{array}{l}\text{if } i<j<k,\ i \frown k \in A(\lambda) \\ \text{implies } i \frown j,\ j \frown k \notin A(\mu),\end{array} \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

where $\tau_\mu(i \frown j) = 0$ if $i \frown l \notin A(\mu)$ (see [14] for the corresponding formula for arbitrary pattern groups). Note that the degree of each character is

$$\chi^\lambda(1) = \prod_{i \frown l \in A(\lambda)} q^{l-i-1}. \tag{2.3}$$

It follows directly from the formula that the supercharacters factor nicely

$$\chi^\lambda = \prod_{i \overset{a}{\frown} l \in A(\lambda)} \chi^{\langle i \overset{a}{\frown} l \rangle_n}.$$

It also follows from (2.2) and (2.3) that χ^λ is linear if and only if

$$i \frown k \in A(\lambda) \qquad \text{implies} \qquad k = i+1.$$

The set $C(\lambda)$ measures how close the supercharacter χ^λ is to being irreducible. In fact,

$$\langle \chi^\lambda, \chi^\mu \rangle = q^{|C(\lambda)|} \delta_{\lambda\mu}, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on characters.

Remark. If instead of considering $U_n(q)$-orbits on $\mathfrak{n}_n(q)$ and $\mathfrak{n}_n(q)^*$, we consider orbits of the full Borel subgroup $B_n(q) = T_n(q)U_n(q)$ on these spaces, then the corresponding supercharacter theory no longer depends on the finite field q. In this case, the combinatorics reduces to considering set-partitions rather than $\mathbb{F}_q$-labeled set-partitions.

Supercharacters satisfy a variety of nice properties, as described in [12]. The above construction satisfies

(a) The product of two supercharacters is a $\mathbb{Z}_{\geq 0}$-linear combination of supercharacters.

(b) The restriction of a supercharacter from one pattern group to a pattern subgroup is a $\mathbb{Z}_{\geq 0}$-linear combination of supercharacters.

However, it is not true that the induction functor sends a supercharacter to a $\mathbb{Z}_{\geq 0}$-linear combination of supercharacters. In fact, an induced supercharacter is generally no longer even a superclass function.

Diaconis and Isaacs therefore define a map superinduction on supercharacters that is adjoint to restriction with respect to the usual inner product on class functions; it turns out that this function averages over superclasses in the same way induction averages over conjugacy classes. In particular, if $H \subseteq G$ are pattern groups (or more generally algebra groups), then *superinduction* is the function

$$\begin{array}{cccc} \mathrm{SInd}: & \left\{ \begin{array}{c} \text{Superclass functions} \\ \text{of } H \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Superclass functions} \\ \text{of } G \end{array} \right\} \\ & \chi & \mapsto & \mathrm{SInd}_H^G(\chi), \end{array}$$

where

$$\mathrm{SInd}_H^G(\chi)(g) = \frac{1}{|G||H|} \sum_{\substack{x,y \in G \\ x(g-1)y+1 \in H}} \chi(x(g-1)y+1), \qquad \text{for } g \in G.$$

Unfortunately, while SInd sends superclass functions to superclass functions, it sends supercharacters to $\mathbb{Z}_{\geq 0}[1/q]$-linear combinations of supercharacters (where q comes from the underlying finite field). In fact, the image is not even generally a character. See also [18] for a further exploration of the relationship between superinduction and induction.

2.4 The ring of symmetric functions in non-commutative variables

Fix a set $X = \{X_1, X_2, \ldots\}$ of countably many non-commuting variables. For $K = K_1 \cup K_2 \cup \cdots \cup K_\ell \in \mathcal{S}_n$, define the *monomial symmetric function*

$$m_K(X) = \sum_{\substack{k=(k_1,k_2,\ldots,k_\ell) \in \mathbb{Z}_{\geq 1}^\ell \\ k_i \neq k_j, 1 \leq i < j \leq \ell}} X_{\pi_1(k)} X_{\pi_2(k)} \cdots X_{\pi_\ell(k)}, \qquad \text{where } \pi_j(k) = k_i \text{ if } j \in K_i.$$

The *space of symmetric functions in non-commuting variables of homogeneous degree* n is

$$\mathrm{NCSym}_n(X) = \mathbb{C}\text{-span}\{m_K(X) \mid K \in \mathcal{S}_n\},$$

and the *ring of symmetric functions in non-commuting variables* is

$$\mathrm{NCSym} = \bigoplus_{n\geq 0} \mathrm{NCSym}_n(X),$$

where a possible multiplication is given by usual polynomial products. However, note that if $K = \{a_1 < a_2 < \cdots < a_m\} \cup \{b_1 < b_2 < \cdots < b_n\} \in \mathcal{S}_{m+n}$ with $\sigma = (a_1, a_2, \cdots, a_{k_m}, b_1, b_2, \ldots, b_n)$ the corresponding permutation of $m+1$ elements, then we could "shuffle" two words according to K,

$$(X_{i_1} X_{i_2} \cdots X_{i_m}) *_K (X_{i_{m+1}} \cdots X_{i_{m+n}}) = X_{i_{\sigma^{-1}(1)}} X_{i_{\sigma^{-1}(2)}} \cdots X_{i_{\sigma^{-1}(m+n)}}.$$

These operations give a variety of alternate shuffle products for NCSym.

The ring NCSym naturally generalizes the usual ring of symmetric functions [17], but is different from other generalizations such as the ring of noncommutative symmetric functions studied in, for example, [15]. The ring NCSym was introduced by Wolf [22], and further explored by Rosas and Sagan [19]. There has been recent interest in the Hopf structure of NCSym and its Hopf dual – for example, [9, 10]. In particular, [9] show that it has a representation theoretic connection with partition lattice algebras. This paper suggests that the supercharacter theory of $U_n(q)$ also has a representation theoretic connection to NCSym in a way that is more analogous to how the ring of symmetric functions dictates the representation theory of S_n. However, the precise nature of this connection remains open. In particular, it is not clear whether the Hopf structure of NCSym translates naturally into a representation theoretic Hopf structure for the supercharacters of $U_n(q)$.

3 The ring of unipotent superclass functions

This section explores the relationship between NCSym and the space of supercharacters

$$\mathcal{C}(q) = \bigoplus_{n\geq 0} \mathcal{C}_n(q), \qquad \text{where} \quad \mathcal{C}_n(q) = \mathbb{C}\text{-span}\{\chi^\lambda \mid \lambda \in \mathcal{S}_n(q)\}.$$

3.1 Parabolic subgroups and set-partition combinatorics

There are different copies of $U_m(q) \times U_n(q)$ as subgroups of $U_{m+n}(q)$ which are not related via an inner automorphism of $U_{m+n}(q)$. In fact, for every $K = K_1 \cup K_2 \in \mathcal{S}_{m+n}$ with $|K_1| = m$ and $|K_2| = n$, $U_{m+n}(q)$ has a parabolic subgroup $U_m(q) \times_K U_n(q) = U_K(q) \cong U_m(q) \times U_n(q)$.

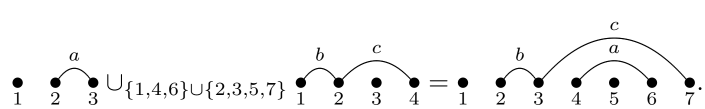
Thus, the space $\mathcal{C}$ has a variety of different products. For $\lambda \in \mathcal{S}_m(q)$, $\mu \in \mathcal{S}_n(q)$, and $K = K_1 \cup K_2 \in \mathcal{S}_{m+n}$ with $|K_1| = m$ and $|K_2| = n$, define

$$\chi^\lambda *_K \chi^\mu = \mathrm{SInd}_{U_m(q)\times_K U_n(q)}^{U_{m+n}(q)}(\chi^\lambda \times \chi^\mu).$$

There is a related map

$$\begin{array}{cccc} \cup_K : & \mathcal{S}_m(q) \times \mathcal{S}_n(q) & \longrightarrow & \mathcal{S}_{m+n}(q) \\ & (\lambda, \mu) & \mapsto & \lambda \cup_K \mu, \end{array}$$

where $\lambda \cup_K \mu = \lambda' \cup \mu'$ with $\lambda' \in \mathcal{S}_{K_1}(q)$ and $\mu' \in \mathcal{S}_{K_2}(q)$ the same $\mathbb{F}_q$-labeled set-partitions as λ and μ respectively, but with $\{1, 2, \ldots, m\}$ relabeled as K_1 and $\{1, 2, \ldots, n\}$ relabeled as K_2. For example,



It will follow from Corollary 4.10 that $\chi^{\lambda\cup_K\mu}$ is always a nonzero constituent of $\chi^\lambda *_K \chi^\mu$.

3.2 A characteristic map for supercharacters

For $\mu \in \mathcal{S}_n(q)$, let $\kappa_\mu : U_n \to U_n$ be the superclass characteristic function given by

$$\kappa_\mu(u) = \begin{cases} 1, & \text{if } u \text{ is in the same superclass as } u_\mu, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$z_\mu = \frac{|U_n|}{|U_n(u_\mu - 1)U_n|}$$

Proposition 3.1 *For $\mu \in \mathcal{S}_m(q)$ and $\nu \in \mathcal{S}_n(q)$,*

$$\mathrm{SInd}_{U_m \times_K U_n}^{U_{m+n}}\big((z_\mu \kappa_\mu) \otimes (z_\nu \kappa_\nu)\big) = z_{\mu \cup_K \nu} \kappa_{\mu \cup_K \nu}.$$

Let NCSym be the ring of symmetric functions in non-commuting variables. Let

$$\{p_\lambda \mid \lambda \in \mathcal{S}\}$$

be any basis that satisfies

$$p_\lambda *_K p_\mu = p_{\lambda \cup_K \mu}$$

for all $K = K_1 \cup K_2 \in \mathcal{S}$ with $|K_1| = |\lambda|$ and $|K_2| = |\mu|$. Note that NCSym in fact has several of such bases, such as $\{p_\lambda\}$ in [19] and $\{x_\lambda\}$ in [9].

Corollary 3.2 *The function*

$$\begin{array}{rccc} \mathrm{ch}: & \mathcal{C}(2) & \longrightarrow & \mathrm{NCSym} \\ & \kappa_\mu & \mapsto & \frac{1}{z_\mu} p_\mu \end{array}$$

is an isometric algebra isomorphism.

Questions. This result immediately raises the following questions.

(1) Does the Hopf algebra structure of NCSym transfer in a representation theoretic way to $\mathcal{C}$?

(2) What is the correct choice of basis p_μ? In particular, the $\{p_\lambda\}$ of [19] do not seem to give a nice Hopf structure to $\mathcal{C}$.

(3) Is there a corresponding NCSym-space for $q > 2$?

Questions (1) and (2) presumably need simultaneous answers, and question (3) suggests there might be an analogue of the ring symmetric functions corresponding to wreath products.

4 Representation theoretic structure constants

This section explores the computation of structure constants in $\mathcal{C}$. We begin with a family of natural embedding maps of $\mathcal{C}_m(q) \subseteq \mathcal{C}_n(q)$ for $m \leq n$ using a generalization of the inflation functor, and then give algorithms for computing restrictions from $\mathcal{C}_{m+n}(q)$ to $\mathcal{C}_m(q) \otimes \mathcal{C}_n(q)$. To finish the computations we require a method for decomposing tensor products $\mathcal{C}_n(q) \otimes \mathcal{C}_n(q) \to \mathcal{C}_n(q)$. We conclude with a discussion of the corresponding superinduction coefficients. In this section we will assume a fixed q, and suppress the q from the notation; that is $U_n = U_n(q)$, etc.

4.1 Superinflation of characters

Let $T \subseteq G$ be pattern groups with corresponding algebras $\mathfrak{t}$ and $\mathfrak{g}$, respectively. There exists a surjective projection

$$\begin{array}{rccc} \pi: & \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp & \longrightarrow & \mathfrak{t} \\ & X + Y & \mapsto & X, \end{array}$$

with a corresponding inflation map

$$\begin{array}{rccc} \mathrm{Inf}_{\mathfrak{t}}^{\mathfrak{g}}: & \mathfrak{t}^* & \longrightarrow & \mathfrak{g}^* \\ & \mu & \mapsto & \mu \circ \pi. \end{array}$$

The *superinflation* map on supermodules is given by

$$\begin{array}{cccc} \mathrm{Sinf}_T^G: & \left\{ \begin{array}{c} \text{Supermodules} \\ \text{of } T \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Supermodules} \\ \text{of } G \end{array} \right\} \\ & V^{\mu} & \mapsto & V^{\mathrm{Inf}_{\mathfrak{t}}^{\mathfrak{g}}(\mu)}, \end{array}$$

where supermodules are as in (2.1).

Note that superinflation takes supermodules to supermodules, just as the usual inflation map on characters takes irreducible characters to irreducible characters. Recall, the usual inflation map is constructed from a surjection $\pi : G \to T$ is given by

$$\begin{array}{cccc} \mathrm{Inf}_T^G: & \{T\text{-modules}\} & \longrightarrow & \{G\text{-modules}\} \\ & V & \mapsto & \mathrm{Inf}_T^G(V), \end{array}$$

where $gv = \pi(g)v$ for $g \in G$, $v \in \mathrm{Inf}_T^G(V)$. The following proposition says that superinflation is inflation whenever possible.

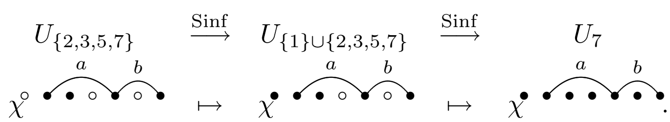
Proposition 4.1 *Suppose G is a pattern group with pattern subgroups T and H such that $G = T \ltimes H$. Then for any supermodule V^{λ} of T,*

$$\mathrm{Sinf}_T^G(V^{\lambda}) \cong \mathrm{Inf}_T^G(V^{\lambda}).$$

We will be primarily be interested in the superinflation function between parabolic subgroups of $U_n(q)$. In this case, if $U_K(q) \subseteq U_L(q)$, then

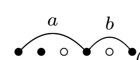
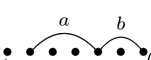
$$\mathrm{Sinf}_{U_K(q)}^{U_L(q)}(\chi^{\lambda}) = \chi^{\langle \lambda \rangle_L}.$$

For example,



Thus, superinflation allows us to embed $C_m(q) \subseteq C_n(q)$ for all $m < n$, although this embedding still depends on the embedding of $U_m(q)$ inside $U_n(q)$.

Remark. While the superinflation function does match up with the usual inflation when possible, it does not generally behave as nicely as the usual inflation function. In particular, it is no longer generally true that $\mathrm{Res}_T^G \circ \mathrm{Sinf}_T^G(\chi) = \chi$ for χ a class function of T. For example,

χ  $(1) = q^1 \neq q^3 = \chi$  $(1).$

4.2 Restrictions

In this section we give algorithms for computing restrictions between parabolic subgroups of $U_n(q)$. Since supercharacters decompose into tensor products of arcs, for $\lambda \in \mathcal{S}_n(q)$,

$$\chi^{\lambda} = \prod_{i \overset{a}{\frown} l \in A(\lambda)} \chi^{\langle i \overset{a}{\frown} l \rangle_n},$$

our strategy is to compute restrictions to for each $\chi^{\langle i \overset{a}{\frown} l \rangle_n}$. We then use a tensor product result in Section 4.3 to glue back together the resulting restrictions.

We begin with two observations, and then Theorem 4.4 gives a general algorithm. Recall that for $K = K_1 \cup K_2 \cup \cdots \cup K_{\ell} \in \mathcal{S}_n$, U_K is a subgroup of $U_n(q)$ isomorphic to

$$U_{|K_1|} \times U_{|K_2|} \times \cdots \times U_{|K_{\ell}|}.$$

Proposition 4.2 *Let $U_K \subseteq U_L$ be parabolic subgroups of U_n with $L = L_1 \cup L_2 \cup \cdots \cup L_{\ell} \in \mathcal{S}_n$. Then*

$$\mathrm{Res}_{U_K}^{U_L}(\chi^{\lambda_1} \times \cdots \times \chi^{\lambda_{\ell}}) = \mathrm{Res}_{U_{K_1}}^{U_{L_1}}(\chi^{\lambda_1}) \times \mathrm{Res}_{U_{K_2}}^{U_{L_2}}(\chi^{\lambda_2}) \times \cdots \times \mathrm{Res}_{U_{K_{\ell}}}^{U_{L_{\ell}}}(\chi^{\lambda_{\ell}})$$

where U_{K_j} is the parabolic subgroup of U_{L_j} corresponding to the vertices L_j.

The next proposition gives information about each factor in Proposition 4.2.

Proposition 4.3 *For $i < l$, $a \in \mathbb{F}_q^\times$ and $K = K_1 \cup K_2 \cup \ldots \cup K_\ell \in \mathcal{S}_n$,*

$$\mathrm{Res}^{U_n}_{U_K}(\chi^{\langle i \overset{a}{\frown} l\rangle_n}) = \frac{\mathrm{Res}^{U_n}_{U_{K_1}}(\chi^{\langle i \overset{a}{\frown} l\rangle_n})}{q^{|\{i<k<l \mid k \notin K_1\}|}} \times \frac{\mathrm{Res}^{U_n}_{U_{K_2}}(\chi^{\langle i \overset{a}{\frown} l\rangle_n})}{q^{|\{i<k<l \mid k \notin K_2\}|}} \times \cdots \times \frac{\mathrm{Res}^{U_n}_{U_{K_\ell}}(\chi^{\langle i \overset{a}{\frown} l\rangle_n})}{q^{|\{i<k<l \mid k \notin K_\ell\}|}}$$

For $S \subseteq \{1, 2, \ldots, n\}$, let

$$U_S = \{u \in U_n \mid u_{ij} \neq 0 \text{ implies } i, j \in S\}.$$

Note that while U_S is not itself a parabolic subgroup of U_n, it is isomorphic to the parabolic subgroup $U_{\langle S\rangle_n}$.

Theorem 4.4 *Let $S \subseteq \{1, 2, \ldots, n\}$. Then for $1 \leq i < l \leq n$ and $a \in \mathbb{F}_q^\times$,*

$$\mathrm{Res}^{U_n}_{U_S}(\chi^{\langle i \overset{a}{\frown} l\rangle_n}) = \begin{cases} q^{|\{i<k<l \mid k \notin S\}|}\chi^{\langle i \overset{a}{\frown} l\rangle_S}, & \text{if } i, l \in S, \\ q^{|\{i<k<l \mid k \notin S\}|}\Bigg(\mathbb{1} + \sum\limits_{\substack{i<j<l,\, j \in S \\ b \in \mathbb{F}_q^\times}} \chi^{\langle j \overset{b}{\frown} l\rangle_S}\Bigg), & \text{if } i \notin S, l \in S, \\ q^{|\{i<k<l \mid k \notin S\}|}\Bigg(\mathbb{1} + \sum\limits_{\substack{i<k<l,\, k \in S \\ b \in \mathbb{F}_q^\times}} \chi^{\langle i \overset{b}{\frown} k\rangle_S}\Bigg), & \text{if } i \in S, l \notin S, \\ q^{|\{i<k'<l \mid k' \notin S\}|}\Bigg((|S \cap [i,l]|(q-1)+1)\mathbb{1} + (q-1)\sum\limits_{\substack{i<j'<k'<l \\ j',k' \in S,\, c \in \mathbb{F}_q^\times}} \chi^{\langle j' \overset{c}{\frown} k'\rangle_S}\Bigg), & \text{if } i, l \notin S. \end{cases}$$

Example. Let $n = 7$, $j = 2$, $k = 5$, so that

$$U_{[2,5]} = U_{\{2,3,4,5\}} = \left\{ \begin{pmatrix} 1&0&0&0&0&0&0 \\ 0&1&*&*&*&0&0 \\ 0&0&1&*&*&0&0 \\ 0&0&0&1&*&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{pmatrix} \right\} \subseteq \left\{ \begin{pmatrix} 1&*&*&*&*&*&* \\ 0&1&*&*&*&*&* \\ 0&0&1&*&*&*&* \\ 0&0&0&1&*&*&* \\ 0&0&0&0&1&*&* \\ 0&0&0&0&0&1&* \\ 0&0&0&0&0&0&1 \end{pmatrix} \right\} = U_7$$

Then

$$\mathrm{Res}^{U_7}_{U_{[2,5]}}(\chi^{\overset{a}{\overgroup{\bullet\bullet\bullet\bullet\bullet}}\bullet\bullet}) = \chi^{\circ\bullet\bullet\bullet\bullet\circ\circ} + \sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\overset{b}{\overgroup{\bullet\bullet\bullet\bullet}}\circ\circ} + \sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\bullet\overset{b}{\overgroup{\bullet\bullet\bullet}}\circ\circ} + \sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\bullet\bullet\overset{b}{\overgroup{\bullet\bullet}}\circ\circ}$$

$$\begin{aligned} \mathrm{Res}^{U_7}_{U_{[2,5]}}(\chi^{\overset{a}{\overgroup{\bullet\bullet\bullet\bullet\bullet\bullet\bullet}}}) = q\Bigg(&(4q-3)\chi^{\circ\bullet\bullet\bullet\bullet\circ\circ} + (q-1)\sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\overset{b}{\overgroup{\bullet\bullet}}\bullet\bullet\circ\circ} + (q-1)\sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\bullet\overset{b}{\overgroup{\bullet\bullet\bullet}}\circ\circ} \\ &+ (q-1)\sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\bullet\bullet\overset{b}{\overgroup{\bullet\bullet}}\circ\circ} + (q-1)\sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\overset{b}{\overgroup{\bullet\bullet\bullet\bullet}}\circ\circ} + (q-1)\sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\overset{b}{\overgroup{\bullet\bullet\bullet}}\bullet\circ\circ} \\ &+ (q-1)\sum_{b \in \mathbb{F}_q^\times} \chi^{\circ\bullet\overset{b}{\overgroup{\bullet\bullet}}\bullet\circ\circ}\Bigg). \end{aligned}$$

4.3 Tensor products

We have seen in the previous section that when we decompose supercharacters into tensor products of irreducible characters, the restriction rules are manageable to compute. This section explains how to glue back together the resulting products of characters. The main result – Corollary 4.6 – has been computed by André for large primes in [1, Lemmas 6–8] and for arbitrary primes by Yan in [23, Propositions 7.2-7.5], but we reprove it here quickly using the machinery developed in this paper.

We begin with a lemma that further establishes the relationship between tensor products and restrictions. For $H \subseteq G$ and χ a superclass function of G, let

$$\mathrm{SinfRes}^G_H(\chi) = \mathrm{Sinf}^G_H \mathrm{Res}^G_H(\chi).$$

Lemma 4.5 *For $i<j<k<l$,*

$$\begin{aligned}
\chi^{\langle i \overset{a}{\frown} k\rangle_n} \otimes \chi^{\langle i \overset{b}{\frown} l\rangle_n} &= \mathrm{SinfRes}^{U_n}_{U_{[i+1,l]}}(\chi^{\langle i \overset{a}{\frown} k\rangle_n}) \otimes \chi^{\langle i \overset{b}{\frown} l\rangle_n}, && a,b \in \mathbb{F}_q^\times,\\
\chi^{\langle i \overset{a}{\frown} l\rangle_n} \otimes \chi^{\langle j \overset{b}{\frown} l\rangle_n} &= \chi^{\langle i \overset{a}{\frown} l\rangle_n} \otimes \mathrm{SinfRes}^{U_n}_{U_{[i,l-1]}}(\chi^{\langle j \overset{b}{\frown} l\rangle_n}), && a,b \in \mathbb{F}_q^\times,\\
\chi^{\langle i \overset{a}{\frown} l\rangle_n} \otimes \chi^{\langle i \overset{-a}{\frown} l\rangle_n} &= \mathrm{SinfRes}^{U_n}_{U_{[i+1,l]}}(\chi^{\langle i \overset{a}{\frown} l\rangle_n}) \otimes \mathrm{SinfRes}^{U_n}_{U_{[i,l-1]}}(\chi^{\langle i \overset{-a}{\frown} l\rangle_n}), && a \in \mathbb{F}_q^\times,\\
\chi^{\langle i \overset{a}{\frown} l\rangle_n} \otimes \chi^{\langle i \overset{b}{\frown} l\rangle_n} &= \chi^{\langle i \overset{a+b}{\frown} l\rangle_n} \otimes \mathrm{SinfRes}^{U_n}_{U_{[i+1,l-1]}}(\chi^{\langle i \overset{a+b}{\frown} l\rangle_n}), && a,b \in \mathbb{F}_q^\times, b \neq -a.
\end{aligned}$$

Combining Lemma 4.6 with Theorem 4.4 we obtain the following corollary.

Corollary 4.6 *For $i<k$, $j<l$, $a,b\in\mathbb{F}_q^\times$, and $\{i,k\}\neq\{j,l\}$,*

$$\chi^{\langle i \overset{a}{\frown} k\rangle_n} \otimes \chi^{\langle j \overset{b}{\frown} l\rangle_n} = \begin{cases} \chi^{\langle \{i \overset{a}{\frown} k\} \cup \{j \overset{b}{\frown} l\}\rangle_n}, & \text{if } \{i,k\}\cap\{j,l\}=\emptyset,\\ \chi^{\langle i \overset{a}{\frown} j \overset{b}{\frown} l\rangle_n}, & \text{if } i<j=k<l,\\ \chi^{\langle i \overset{b}{\frown} l\rangle_n} + \displaystyle\sum_{\substack{i<j'<k\\ c\in\mathbb{F}_q^\times}} \chi^{\langle \{j' \overset{c}{\frown} k\} \cup \{i \overset{b}{\frown} l\}\rangle_n}, & \text{if } i=j<k<l,\\ \chi^{\langle i \overset{a}{\frown} l\rangle_n} + \displaystyle\sum_{\substack{j<k'<l\\ c\in\mathbb{F}_q^\times}} \chi^{\langle \{i \overset{a}{\frown} l\} \cup \{j \overset{c}{\frown} k'\}\rangle_n}, & \text{if } i<j<k=l,\end{cases}$$

For $i<l$, $a,b\in\mathbb{F}_q^\times$,

$$\chi^{\langle i \overset{a}{\frown} l\rangle_n} \otimes \chi^{\langle i \overset{b}{\frown} l\rangle_n} = \begin{cases} \mathbb{1} + \displaystyle\sum_{\substack{i<j'<l\\ c\in\mathbb{F}_q^\times}} \chi^{\langle i \overset{c}{\frown} j'\rangle_n} + \sum_{\substack{i<k'<l\\ c\in\mathbb{F}_q^\times}} \chi^{\langle k' \overset{c}{\frown} l\rangle_n} + \sum_{\substack{i<j',k'<l\\ c,d\in\mathbb{F}_q^\times}} \chi^{\langle \{i \overset{c}{\frown} j'\} \cup \{k' \overset{d}{\frown} l\}\rangle_n}, & \text{if } b=-a,\\ ((l-i-1)(q-1)+1)\chi^{\langle i \overset{a+b}{\frown} l\rangle_n} + (q-1)\displaystyle\sum_{\substack{i<j'<k'<l\\ c\in\mathbb{F}_q^\times}} \chi^{\langle \{j' \overset{c}{\frown} k'\} \cup \{i \overset{a+b}{\frown} l\}\rangle_n}, & \text{otherwise.}\end{cases}$$

Remark. The coefficients of the tensor products are not understood in general, although it is clear from Corollary 4.6 that they are polynomial in q.

4.4 Superinduction

Let $S\subseteq\{1,2,\ldots,n\}$. If $\mu\in\mathcal{S}_S(q)$ and $\lambda\in\mathcal{S}_n(q)$, then by Frobenius reciprocity,

$$\langle \chi^\lambda, \mathrm{SInd}^{U_n}_{U_S}(\chi^\mu)\rangle_{U_n} = \langle \mathrm{Res}^{U_n}_{U_S}(\chi^\lambda), \chi^\mu\rangle_{U_S}.$$

Thus, if

$$\mathrm{SInd}^{U_n}_{U_S}(\chi^\mu) = \sum_\nu a^\nu_\mu \chi^\nu \quad \text{and} \quad \mathrm{Res}^{U_n}_{U_S}(\chi^\lambda) = \sum_\gamma b^\lambda_\gamma \chi^\gamma,$$

then by (2.4)

$$q^{|C(\lambda)|} a^\lambda_\mu = q^{|C(\mu)|} b^\lambda_\mu,$$

where $C(\nu)$ is the set of crossings of ν. Therefore,

$$\mathrm{SInd}^{U_n}_{U_S}(\chi^\mu) = \sum_\nu a^\nu_\mu \chi^\nu = \sum_\nu q^{|C(\mu)|-|C(\nu)|} b^\nu_\mu \chi^\nu.$$

In general, if $U_K\subseteq U_n$ with $K\in\mathcal{S}_n$, then

$$\mathrm{SInd}^{U_n}_{U_K}(\chi^\mu) = \sum_\nu q^{|C_K(\mu)|-|C(\nu)|} b^\nu_\mu \chi^\nu, \tag{4.1}$$

where $C_K(\nu)$ is the set of crossings that occur within the same parts of K.

With this discussion, we obtain the following corollary of Sections 4.2 and 4.3. When combined with Corollary 4.10, below, these results give a reasonably direct way to compute superinduction for some cases.

Corollary 4.7 *Let $K = \{1, 2, \ldots, k\} \cup \{k+1, k+2, \ldots, n\} \in \mathcal{S}_n$ be a set-partition with two parts. Then*

$$\mathrm{SInd}_{U_K}^{U_n}(\mathbb{1}) = \sum_{\substack{\lambda \in \mathcal{S}_n(q) \\ \text{if } i \frown j \in \lambda, \\ \text{then } i \not\sim j \in K}} q^{-|C(\lambda)|}\chi^{\lambda},$$

where $i \sim j$ if and only if i and j are in the same part in K.

Corollary 4.7 has some immediate combinatorial consequences. Let

$$SG_{n\times m} = \{a \in M_{n\times n}(\{0,1\}) \mid a \text{ has at most one 1 and every row and column}\}$$

be the set of $m \times n$ 0-1 matrices with at most one 1 in every row an column. Define statistics for $w \in SG_{m\times n}$

$$\begin{aligned} \mathrm{ones}(w) &= |\{(i,j) \in [1,n] \times [1,m] \mid w_{ij} = 1\}| \\ \mathrm{sow}(w) &= |\{(j,k) \in [1,n] \times [1,m] \mid w_{jk} = 0, w_{ik} = 1 \text{ for some } i < j \text{ or } w_{jl} = 1 \text{ for some } k < l\}|. \end{aligned}$$

For example, if

$$w = \begin{pmatrix} \underline{0} & 1 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{0} & 1 \\ 0 & \underline{0} & 0 & \underline{0} \end{pmatrix}, \qquad \text{then} \qquad \begin{matrix} \mathrm{ones}(w) = 2 \\ \mathrm{sow}(w) = 6. \end{matrix}$$

Corollary 4.8 *Let m and n be positive integers. Then*

(a) $q^{mn} = \displaystyle\sum_{w \in SG_{m\times n}} (q-1)^{\mathrm{ones}(w)} q^{\mathrm{sow}(w)}$

(b) $0 = \displaystyle\sum_{w \in SG_{m\times n}} (-1)^{w_{1n}} (q-1)^{\mathrm{ones}(w)} q^{\mathrm{sow}(w)}.$

We conclude with some observations relating superinduction to these superpermutation "characters." The first corollary follows from Frobenius reciprocity.

Corollary 4.9 *Let $H \subseteq G$ be pattern groups, and let $\mu \in (H-1)$. If $\chi^{\mu}(1)\mathrm{Sinf}_H^G(\chi^{\mu})(h) = \mathrm{Sinf}_H^G(\chi^{\mu})(1)\chi^{\mu}(h)$, for all $h \in H$, then*

$$\mathrm{SInd}_H^G(\chi^{\mu}) = \frac{\chi^{\mu}(1)}{\mathrm{Sinf}_H^G(\chi^{\mu})(1)}\mathrm{Sinf}_H^G(\chi^{\mu}) \otimes \mathrm{SInd}_H^G(\mathbb{1}).$$

The assumption in Corollary 4.9 is not so unusual. In fact,

Corollary 4.10 *Let $U_K \subseteq U_L$ be parabolic subgroups of U_n, where $K = K_1 \cup K_2 \cup \cdots \cup K_\ell, L \in \mathcal{S}_n$. Then for $\mu \in \mathcal{S}_{K_1}(q) \times \mathcal{S}_{K_2}(q) \times \cdots \times \mathcal{S}_{K_\ell}(q)$,*

$$\mathrm{SInd}_{U_K}^{U_L}(\chi^{\mu}) = \frac{\chi^{\mu}(1)}{\mathrm{Sinf}_{U_K}^{U_L}(\chi^{\mu})(1)}\mathrm{Sinf}_{U_K}^{U_L}(\chi^{\mu}) \otimes \mathrm{SInd}_{U_K}^{U_L}(\mathbb{1}).$$

Remark. While the assumption in Corollary 4.9 is sufficient, it is not necessary. For example, if

$$H = \left\{\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} \subseteq U_3 = \left\{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}\right\}.$$

then for these groups,

$$\chi^{\left(\begin{smallmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \qquad \text{and} \qquad \mathrm{Sinf}_H^{U_3}\left(\chi^{\left(\begin{smallmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)}\right)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

However, it remains true that

$$\mathrm{SInd}_H^G\left(\chi^{\left(\begin{smallmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)}\right) = q^{-1}\mathrm{Sinf}_H^G\left(\chi^{\left(\begin{smallmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}\right)}\right) \otimes \mathrm{SInd}_H^G(\mathbb{1}).$$

In fact, the conclusion of Corollary 4.9 may be true for all pattern groups; I know of no counter-example.

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On k-simplexes in $(2k-1)$-dimensional vector spaces over finite fields

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Abstract. We show that if the cardinality of a subset of the $(2k-1)$-dimensional vector space over a finite field with q elements is $\gg q^{2k-1-\frac{1}{2k}}$, then it contains a positive proportional of all k-simplexes up to congruence.

Résumé. Nous montrons que si le cardinal d'un sous-ensemble de la $(2k-1)$-dimensional espace vectoriel sur un corps fini à q éléments est de $\gg q^{2k-1-\frac{1}{2k}}$, alors qu'il contient une proportion de l'ensemble des k-simplexes de congruence.

Keywords: distance problem, finite Euclidean graphs, finite non-Euclidean graphs, spectral graphs

1 Introduction

A classical result due to Furstenberg, Katznelson and Weiss (6) states that if $E \subset \mathbb{R}^2$ has positive upper Lebesgue density, then for any $\delta > 0$, the δ-neighborhood of E contains a congruent copy of a sufficiently large dilate of every three-point configuration. For higher dimensional simplexes, Bourgain (4) showed that if $E \subset \mathbb{R}^d$ has positive upper density, and Δ is a k-simplex with $k < d$, then E contains a rotated and translated image of every large dilate of Δ. The cases $k = d$ and $k = d+1$ still remain open. Magyar (10; 11) studied related problems in the integer lattice $\mathbb{Z}^d$. He showed (11) that if $d > 2k+4$, and $E \subset \mathbb{Z}^d$ has positive upper densitiy, then all large (depending on density of E) dilates of a k-simplex in $\mathbb{Z}^d$ can be embedded in E.

Hart and Iosevich (7) made the first investigation in an analog of this question in finite field geometries. They showed that if $E \subset \mathbb{F}_q^d$, $d \geq \binom{k+1}{2}$ of cardinality $|E| \geq Cq^{\frac{kd}{k+1}} q^{\frac{k}{2}}$ for a sufficiently large constant $C > 0$, then E contains an isometric copy of every k-simplex. Using graph theoretic method, the author (14) showed that the same result holds for $d \geq 2k$ and $|E| \gg q^{\frac{d-1}{2}+k}$ (cf. Theorem 1.4 in (14)).

Note that serious difficulties arise when the size of the simplex is sufficiently large with respect to the ambient dimension. Even in the case of triangles, the result in (14) is only non-trivial for $d \geq 4$. In (5), Covert, Hart, Iosevich, and Uriarte-Tuero addressed the case of triangles in two-dimensional vector spaces over finite fields. They showed that if E has density $\geq \rho$, for some $\frac{C}{\sqrt{q}} \leq \rho \leq 1$ with a sufficiently large constant $C > 0$, then the set of triangles determined by E, up to congruence, has density $\geq c\rho$. In (15), the author studied the remaining case; triangles in three-dimensional vector spaces over finite fields. Using a combination of graph theory method and Fourier analysis, the author showed that if $E \subset \mathbb{F}_q^d$, $d \geq 3$, such that $|E| \gg q^{\frac{d+2}{2}}$, then E determines almost all triangles up to congruence. The arguments in (15), however, do not work for $k \geq 5$.

In this paper, we will study the case of k-simplexes in $(2k-1)$-dimensional vector spaces with $k \geq 3$. Given $E_1, \ldots, E_k \subset \mathbb{F}_q^d$, where $\mathbb{F}_q$ is a finite field of q elements, define

$$T_k(E_1, \ldots, E_k) = \{(x_1, \ldots x_k) \in E_1 \times \ldots \times E_k\} / \sim \tag{1.1}$$

with the equivalence relation $\sim$ such that $(x_1, \ldots, x_k) \sim (x'_1, \ldots, x'_k)$ if there exist $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$, the set of d-by-d orthogonal matrices over $\mathbb{F}_q$ with determinant 1, such that

$$(x'_1, \ldots, x'_k) = (O(x_1) + \tau, \ldots, O(x_k) + \tau). \tag{1.2}$$

The main result of this paper is the following.

Theorem 1.1 *Let $E \subset \mathbb{F}_q^{2k-1}$ with $k \geq 3$, and suppose that*

$$|E| \gg q^{2k-1-\frac{1}{2k}}.$$

There exists $c > 0$ such that

$$|T_{k+1}(E)| \geq cq^{\binom{k+1}{2}}.$$

In other words, we always get a positive proportion of all k-simplexes if $E \gg q^{k-1-\frac{1}{2k}}$ and $k \geq 3$. The rest of this short paper is organized as follows. In Section 2, we establish some results about the occurrences of colored subgraphs in a pseudo-random coloring of a graph. In Section 3, we construct our main tools to study simplexes in vector spaces over finite fields, the finite Euclidean and non-Euclidean graphs. We then prove our main result, Theorem 1.1 in Section 4.

2 Subgraphs in expanders

We call a graph $G = (V, E)$ (n, d, λ)-graph if G is a d-regular graph on n vertices with the absolute values of each of its eigenvalues but the largest one are at most λ. Suppose that a graph G of order n is colored by t colors. Let G_i be the induced subgraph of G on the i^{th} color. We call a t-colored graph G (n, d, λ)-r.c. (regularly colored) graph if G_i is an (n, d, λ)-regular graph for each color $i \in \{1, \ldots, t\}$. In this section, we will study the occurrences of colored subgraphs in an (n, d, λ)-r.c. graph.

2.1 Colored subgraphs

It is well-known that if $\lambda \ll d$ then an (n, d, λ)-graph behaves similarly as a random graph $G_{n,d/n}$. Precisely, we have the following result.

Theorem 2.1 *((1, Theorem 9.2.4)) Let G be an (n, d, λ)-graph. For a vertex $v \in V$ and a subset B of V, denote by $N(v)$ the set of all neighbors of v in G, and let $N_B(v) = N(v) \cap B$ denote the set of all neighbors of v in B. For every subset B of V, we have*

$$\sum_{v \in V} (|N_B(v)| - \frac{d}{n}|B|)^2 \leqslant \frac{\lambda^2}{n}|B|(n - |B|). \tag{2.1}$$

The following result is an easy corollary of Theorem 2.1.

Theorem 2.2 *((1, Corollary 9.2.5)) Let G be an (n, d, λ)-graph. For every set of vertices B and C of G, we have*

$$|e(B, C) - \frac{d}{n}|B||C|| \leqslant \lambda\sqrt{|B||C|}, \tag{2.2}$$

where $e(B, C)$ is the number of edges in the induced bipartite subgraph of G on (B, C) (i.e. the number of ordered pair (u, v) where $u \in B$, $v \in C$ and uv is an edge of G).

Let H be a fixed graph of order s with r edges. Let $\mathrm{Aut}(H)$ be an automorphism group of H. It is well-known that for every constant $p \in (0, 1)$, the random graph $G(n, p)$ contains

$$(1 + o(1))p^r(1 - p)^{\binom{s}{2} - r}\frac{n^s}{|\mathrm{Aut}(H)|} \tag{2.3}$$

induced copies of H. Alon extended this result to (n, d, λ)-graph. He proved that every large subset of the set of vertices of a (n, d, λ)-graph contains the "correct" number of copies of any fixed small subgraph.

Theorem 2.3 *((9, Theorem 4.10)) Let H be a fixed graph with r edges, s vertices and maximum degree Δ, and let $G = (V, E)$ be an (n, d, λ)-graph, where, say, $d \leqslant 0.9n$. Let $m < n$ satisfies $m \gg \lambda \left(\frac{n}{d}\right)^{\Delta}$. Then, for every subset $U \subset V$ of cardinality m, the number of (not necessrily induced) copies of H in U is*

$$(1 + o(1))\frac{m^s}{|\mathrm{Aut}(H)|}\left(\frac{d}{n}\right)^r. \tag{2.4}$$

In (14), we observed that Theorem 2.3 can be extended to (n, d, λ)-r.c. graph. Precisely, we showed that every large subset of the set of vertices of an (n, d, λ)-r.c. graph contains the "correct" number of copies of any fixed small colored graph. We present here a multiset version of this statement.

Theorem 2.4 *Let H be a fixed t-colored graph with r edges, s vertices, maximum degree Δ (with the vertex set is ordered), and let G be a t-colored graph of order n. Suppose that G is an (n, d, λ)-r.c graph, where, say, $d \ll n$. Let $E_1, \ldots, E_s \subset V$ satisfy $|E_i| \gg \lambda \left(\frac{n}{d}\right)^{\Delta}$. Then the number of (not necessrily induced) copies of H in $E_1 \times \ldots \times E_s$ (one vertex in each set) is*

$$(1 + o(1)) \prod_{i=1}^{s} |E_i| \left(\frac{d}{n}\right)^{r}. \tag{2.5}$$

The proof of this theorem is similar to the proofs of (9, Theorem 4.10) and (14, Theorem 2.3). Note that going from one color formulation ((9, Theorem 4.10)) and one set formulation ((14, Theorem 2.3)) to a multicolor-multiset formulation (Theorem 2.4) is just a matter of inserting different letters in a couple of places.

2.2 Colored stars

Given any k colors $r_1, \ldots, r_k$, a k-star of type $(r_1, \ldots, r_k)$ has $k+1$ vertices, one center vertex x_0 and k leaves $x_1, \ldots, x_k$, with the edge (x_0, x_i) is colored by the color r_i. The following result gives us an estimate for the number of colored k-stars in an (n, d, λ)-r.c. graph G (see (15) for an earlier version).

Theorem 2.5 *Let G be an (n, d, λ)-r.c. graph. Given any k colors $r_1, \ldots, r_k$ in the color set. Suppose that $E_0, E_1, \ldots, E_k \subset V(G)$ with*

$$|E_0|^2 \prod_{i \in I} |E_i| \gg \left(\frac{n}{d}\lambda\right)^{2|I|} \tag{2.6}$$

for all $I \subset \{1, \ldots, k\}$, $|I| \geq 2$, and

$$|E_0||E_i| \gg \left(\frac{n}{d}\lambda\right)^{2} \tag{2.7}$$

for all $i \in \{1, \ldots, k\}$. Let $e_{\{r_1, \ldots, r_k\}}(E_0; \{E_1, \ldots, E_k\})$ denote the number of k-stars of type $(r_1, \ldots, r_k)$ in $E_0 \times E_1 \times \ldots \times E_k$ (with the center in E_0). We have

$$e_{\{r_1, \ldots, r_k\}}(E_0; \{E_1, \ldots, E_k\}) = (1 + o(1)) \left(\frac{d}{n}\right)^{k} \prod_{i=0}^{k} |E_i|, \tag{2.8}$$

where k is fixed and $n, d, \lambda \gg 1$.

Proof: The proof proceeds by induction. We first consider the bast case, $k = 1$. Since $|E| \gg \frac{n}{d}\lambda$ and the number of 1-stars of type a in $E_0 \times E_1$ is just the number of a-colored edges in $E_0 \times E_1$, the statement follows immediately from Theorem 2.2 and (2.7).

Suppose that the statement holds for all colored l-stars with $l < k$. For a vertex $v \in V$ and a color r, let $N_E^r(v)$ denote the set of all r-colored neighbors of v in E. From Theorem 2.1, we have

$$\sum_{v \in E_0} (|N_{E_i}^{r_i}(v)| - \frac{d}{n}|E_i|)^2 \leqslant \sum_{v \in V} (|N_{E_i}^{r_i}(v)| - \frac{d}{n}|E_i|)^2 \leqslant \frac{\lambda^2}{n}|E_i|(n - |E_i|) \leqslant \lambda^2 |E_i|. \tag{2.9}$$

For $k \geqslant 2$, by the Cauchy-Schwartz inequality, we have

$$\prod_{i=1}^{k}\left(\sum_{j=1}^{n} a_{i.j}^2\right) \geqslant \left(\sum_{j=1}^{n}\prod_{i=1}^{k-1} a_{i,j}^2\right)\left(\sum_{j=1}^{n} a_{k.j}^2\right) \geqslant \left(\sum_{j=1}^{n}\prod_{i=1}^{k} a_{i,j}\right)^2. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\left(\sum_{v\in E_0}\prod_{i=1}^{k}(N_{E_i}^{r_i}(v) - \frac{d}{n}|E_i|)\right)^2 \leqslant \prod_{i=1}^{k}\sum_{v\in E_0}(|N_{E_i}^{r_i}(v)| - \frac{d}{n}|E_i|)^2 \leqslant \lambda^{2k}\prod_{i=1}^{k}|E_i|.$$

It can be written as

$$\left|\sum_{I\subset\{1,\dots,k\}}(-1)^{k-|I|}\left(\frac{d}{n}\right)^{k-|I|}\prod_{j\notin I}|E_j|\sum_{v\in E_0}\prod_{i\in I}N_{E_i}^{r_i}(v)\right| \leqslant \lambda^k\sqrt{\prod_{i=1}^{k}|E_i|}. \tag{2.11}$$

For any $I \subset \{1,\dots,k\}$ with $0 < |I| < k$, by the induction hypothesis, we have

$$\sum_{v\in E_0}\prod_{i\in I}N_{E_i}^{r_i}(v) = e_I(E_0;\{E_i\}_{i\in I}) = (1+o(1))\left(\frac{d}{n}\right)^{|I|}|E_0|\prod_{i\in I}|E_i|. \tag{2.12}$$

Putting (2.11) and (2.12) together, we have

$$\left|\sum_{v\in E_0}\prod_{i=1}^{k}N_{E_i}^{r_i}(v) - (1+o(1))\left(\frac{d}{n}\right)^k\prod_{i=0}^{k}|E_i|\right| \leqslant \lambda^k\sqrt{\prod_{i=1}^{k}|E_i|}.$$

Since $|E_0|^2\prod_{i=1}^{k}|E_i| \gg (\frac{n}{d}\lambda)^{2k}$, the left hand side is dominated by $(1+o(1))\left(\frac{d}{n}\right)^k\prod_{i=0}^{k}|E_i|$. This implies that

$$e_{\{r_1,\dots,r_k\}}(E_0;\{E1,\dots,E_k\}) = \sum_{v\in E_0}\prod_{i=1}^{k}N_{E_i}^{r_i}(v) = (1+o(1))\left(\frac{d}{n}\right)^k\prod_{i=0}^{k}|E_i|,$$

completing the proof of the theorem. □

3 Finite Euclidean and non-Euclidean graphs

In this section, we construct our main tools to study simplexes in vector spaces over finite fields, the graphs associated to finite Euclidean and non-Euclidean spaces. The construction of finite Euclidean graphs follows one of Medrano et al. in (12) and the construction of finite non-Euclidean graphs follows one of Bannai, Shimabukuro, and Tanaka in (3).

3.1 Finite Euclidean graphs

Let $\mathbb{F}_q$ be a finite field with q elements where $q \gg 1$ is an odd prime power. For any $x = (x_1, \ldots, x_d) \in \mathbb{F}_q^d$, let

$$\|x\| = x_1^2 + \ldots + x_d^2.$$

For a fixed $a \in \mathbb{F}_q$, the finite Euclidean graph $G_q(a)$ in $\mathbb{F}_q^d$ is defined as the graph with the vertex set $\mathbb{F}_q^d$, and the edge set

$$\{(x, y) \in \mathbb{F}_q^d \times \mathbb{F}_q^d \mid x \neq y, ||x - y|| = a\}.$$

Medrano et al. (12) studied the spectrum of these graphs and showed that these graphs are asymptotically Ramanujan graphs. Precisely, they proved the following result.

Theorem 3.1 *((12)) The finite Euclidean graph $G_q(a)$ is a regular graph with $n(q, a) = q^d$ vertices of valency*

$$k(q,a) = \begin{cases} q^{d-1} + \chi((-1)^{(d-1)/2}a)q^{(d-1)/2} & a \neq 0, \ d \textit{ odd,} \\ q^{d-1} - \chi((-1)^{d/2}))q^{(d-2)/2} & a \neq 0, \ d \textit{ even,} \\ q^{d-1} & a = 0, \ d \textit{ odd,} \\ q^{d-1} - \chi((-1)^{d/2}))(q-1)q^{(d-2)/2} & a = 0, \ d \textit{ even.} \end{cases}$$

where χ is the quadratic character

$$\chi(a) = \begin{cases} 1 & a \neq 0, \ a \textit{ is square in } \mathbb{F}_q, \\ -1 & a \neq 0, \ a \textit{ is nonsquare in } \mathbb{F}_q, \\ 0 & a = 0. \end{cases}$$

Let λ be any eigenvalues of the graph $G_q(a)$ with $\lambda \neq$ valency of the graph then

$$|\lambda| \leq 2q^{\frac{d-1}{2}}. \tag{3.1}$$

3.2 Finite non-Euclidean graphs

Let $V = \mathbb{F}_q^{2k-1}$ be the $(2k-1)$-dimensional vector space over the finite field $\mathbb{F}_q$ (q is an odd prime power). For each element x of V, we denote the 1-dimensional subspace containing x by $[x]$. Let Ω be the set of all square type non-isotropic 1-dimensional subspaces of V with respect to the quadratic form $Q(x) = x_1^2 + \ldots + x_{2k-1}^2$. The simple orthogonal group $O_{2k-1}(\mathbb{F}_q)$ acts transtively on Ω, and yields a symmetric association scheme $\Psi(O_{2k-1}(\mathbb{F}_q), \Omega)$ of class $(q+1)/2$. The relations of $\Psi(O_{2k-1}(\mathbb{F}_q), \Omega)$ are given by

$$\begin{aligned} R_1 &= \{([U],[V]) \in \Omega \times \Omega \mid (U+V) \cdot (U+V) = 0\}, \\ R_i &= \{([U],[V]) \in \Omega \times \Omega \mid (U+V) \cdot (U+V) = 2 + 2\nu^{-(i-1)}\} \ (2 \leqslant i \leqslant (q-1)/2) \\ R_{(q+1)/2} &= \{([U],[V]) \in \Omega \times \Omega \cdot (U+V) \cdot (U+V) = 2\}, \end{aligned}$$

where ν is a generator of the field $\mathbb{F}_q$ and we assume $U \cdot U = 1$ for all $[U] \in \Omega$ (see (2) for more details).

The graphs (Ω, R_i) are asymptotic Ramanujan for large q. The following theorem summaries the results from Section 2 in (3) in a rough form.

Theorem 3.2 *(3) The graphs (Ω, R_i) $(1 \leq i \leq (q+1)/2)$ are regular of order $q^{2k-2}(1+o_q(1))/2$ and valency Kq^{2k-3}. Let λ be any eigenvalue of the graph (Ω, R_i) with $\lambda \neq$ valency of the graph then*

$$|\lambda| \leq kq^{(2k-3)/2},$$

for some $k, K > 0$ (In fact, we can show that $k = 2 + o_q(1)$ and $K = 1 + o_q(1)$ or $1/2 + o_q(1)$).

4 Proof of Theorem 1.1

We now give a proof of Theorem 1.1. For any $\{a_{ij}\}_{1 \leq i < j \leq k+1} \in \mathbb{F}_q^{\binom{k+1}{2}}$, define

$$T_{\{a_{ij}\}_{1 \leq i < j \leq k+1}}(E) = \{(x_i)_{i=1}^{k+1} \in E^{k+1} : \|x_i - x_j\| = a_{ij}\}.$$

Hart and Iosevich (7) observed that in vector spaces over finite fields, a (non-degenerate) simplex is defined uniquely (up to translation and rotation) by the norms of its edges.

Lemma 4.1 *((7)) Let P be a (non-degenerate) simplex with vertices $V_0, V_1, \ldots, V_k$ where $V_j \in \mathbb{F}_q^d$. Let P' be another (non-degenerate) simplex with vertices $V_0', \ldots, V_k'$. Suppose that*

$$\|V_i - V_j\| = \|V_i' - V_j'\| \tag{4.1}$$

for all i, j. There exists $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$ such that $\tau + O(P) = P'$.

Therefore, it suffices to show that if $E \subset \mathbb{F}_q^{2k-1}$ $(k \geq 3)$ of cardinality $|E| \gg q^{2k-1-\frac{1}{2k}}$, then

$$\left|\left\{\{a_{ij}\}_{1 \leq i < j \leq k+1} \in \mathbb{F}_q^{\binom{k+1}{2}} : |T_{\{a_{ij}\}_{1 \leq i < j \leq k+1}}(E)| > 0\right\}\right| \geqslant cq^{\binom{k+1}{2}}. \tag{4.2}$$

Consider the set of colors $L = \{c_0, \ldots, c_{q-1}\}$ corresponding to elements of $\mathbb{F}_q$. We color the complete graph G_q with the vertex set $\mathbb{F}_q^{2k-1}$, by q colors such that $(x, y) \in \mathbb{F}_q^{2k-1} \times \mathbb{F}_q^{2k-1}$ is colored by c_i whenever $||x - y|| = i$.

Suppose that $|E| \gg q^{2k-1-\frac{1}{2k}}$, we have

$$|E| \gg \left(\frac{q^{2k-1} \cdot 2q^{k-1}}{q^{2k-2}(1+o(1))}\right)^{\frac{2i}{i+2}},$$

for all $2 \leq i \leq k$. From Theorem 3.1, G_q is a $(q^{2k-1}, q^{2k-2}(1+o(1)), 2q^{k-1})$-r.c. graph when $k \geq 3$. Therefore, applying Theorem 2.5 for the number of k-stars of type $(a_{12}, \ldots, a_{1(k+1)})$ in E^{k+1}, we have

$$e_{a_{12},\ldots,a_{1(k+1)}}(E; \{E, \ldots, E\}) = \left(\frac{q^{2k-2}(1+o(1))}{q^{2k-1}}\right)^k |E|^{k+1}(1+o(1)) = \frac{|E|^{k+1}(1+o(1))}{q^k},$$

for any $a_{12}, \ldots, a_{1(k+1)} \in \mathbb{F}_q$.

Let $\mathbb{F}_\square^*$ denote the set of non-zero squares in $\mathbb{F}_q$. For any $a_{12}, \ldots, a_{1(k+1)} \in \mathbb{F}_\square^*$, then

$$\begin{aligned} |\{(x_1, \ldots, x_{k+1}) \in E^{k+1} : \|x_1 - x_i\| = a_{1i}\}| &= e_{a_{12},\ldots,a_{1(k+1)}}(E; \{E, \ldots, E\}) \\ &= \frac{|E|^{k+1}(1+o(1))}{q^k}. \end{aligned}$$

By the pigeon-hole principle, there exists $x_1 \in E$ such that

$$|\{(x_2, \ldots, x_{k+1}) \in E^k : \|x_1 - x_i\| = a_{1i}\}| = \frac{|E|^k(1+o(1))}{q^k}.$$

Let $S_t = \{v \in \mathbb{F}_q^{2k-1} : ||v|| = t\}$ denote the sphere of radius t in $\mathbb{F}_q^{2k-1}$, then

$$|S_t| = q^{2k-2}(1+o(1))$$

for any $t \in \mathbb{F}_q$. Let

$$E_i = \{v \in E : \|x_1 - v\| = a_{1i}\} \subset S_{a_i},\ 2 \le i \le k+1,$$

then

$$|E_2| \ldots |E_{k+1}| = \frac{|E|^k(1+o(1))}{q^k},$$

and

$$|E_2|, \ldots, |E_{k+1}| \le O(q^{2k-2}).$$

This implies that

$$|E_i| \ge \Omega\left(\frac{|E|^k(1+o(1))}{q^{k+(2k-2)(k-1)}}\right) \gg q^{2k-\frac{5}{2}}.$$

There are $(q-1)^k/2^k$ possibilities of $a_{12}, \ldots, a_{1(k+1)} \in \mathbb{F}_\square^*$. From Lemma 4.1, it suffices to show that $T_k(E_2, \ldots, E_{k+1}) \ge cq^{\binom{k}{2}}$ for some $c > 0$. Let

$$E_i' = \{[x] : x \in E_i\} \subset \Omega$$

where Ω is the set of all square type non-isotropic 1-dimensional subspaces of $\mathbb{F}_q^{2k-1}$ with respect to the quadratic form $Q(x) = x_1^2 + \ldots + x_{2k-1}^2$. Since each line through origin in $\mathbb{F}_q^{2k-1}$ intersects the unit sphere S_1 at two points, $|E_i'| \ge |E_i|/2 \gg q^{2k-\frac{5}{2}}$. Suppose that $([U],[V]) \in E_i' \times E_j'$ is an edge of (Ω, R_l), $2 \le l \le (q-1)/2$. Then

$$(U+V)\cdot(U+V) = 2 + \alpha_l,$$

where $\alpha_l = 2\nu^{-(l-1)}$. Since $U \cdot U = V \cdot V = 1$, we have $(U-V)\cdot(U-V) = 2 - \alpha_l$. The distance between U and V (in $E_i' \times E_j'$) is either $(U+V)\cdot(U+V)$ or $(U-V)\cdot(U-V)$. Hence,

$$||U - V|| \in \{2+\alpha_l, 2-\alpha_l\}. \tag{4.3}$$

Consider the set of colors $L = \{r_1, \ldots, r_{(q+1)/2}\}$ corresponding to classes of the association scheme $\Psi(O_{2k-1}(\mathbb{F}_q), \Omega)$. We color the complete graph P_q with the vertex set Ω, by $(q+1)/2$ colors such that $([U],[V]) \in \Omega \times \Omega$ is colored by r_i whenever $([U],[V]) \in R_i$.

From Theorem 3.2, P_q is a $((1+o(1))q^{2k-2}/2, Kq^{2k-3}, kq^{(2k-3)/2})$-r.c. graph when $k \ge 3$. Since $|E_i'| \gg q^{2k-\frac{5}{2}}$, we have

$$|E_i'| \gg kq^{(2k-3)/2}\left(\frac{(1+o(1))q^{2k-2}/2}{Kq^{2k-3}}\right)^{k-1}.$$

Therefore, applying Theorem 2.4 for colored k-complete subgraphs of P_q then P_q contains all possible colored k-complete subgraphs. From (4.3), $([U],[V])$ is colored by r_l $(2 \le l \le (q-1)/2)$ then $||U-V|| \in \{2+\alpha_l, 2-\alpha_l\}$. Hence, $T_k(E_2, \ldots, E_{k+1}) \ge cq^{\binom{k}{2}}$ for some $c > 0$. The theorem follows.

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A promotion operator on rigged configurations

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In a recent paper, Schilling proposed an operator $\overline{\mathrm{pr}}$ on unrestricted rigged configurations RC, and conjectured it to be the promotion operator of the type A crystal formed by RC. In this paper we announce a proof for this conjecture.

Keywords: promotion operator, rigged configurations, affine crystals

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1 Introduction

Rigged configurations appear in the Bethe Ansatz study of exactly solvable lattice models as combinatorial objects to index the solutions of the Bethe equations [5, 6]. Based on work by Kerov, Kirillov and Reshetikhin [5, 6], it was shown in [7] that there is a statistic preserving bijection Φ between Littlewood-Richardson tableaux and rigged configurations. The description of the bijection Φ is based on a quite technical recursive algorithm.

Littlewood-Richardson tableaux can be viewed as highest weight crystal elements in a tensor product of Kirillov–Reshetikhin (KR) crystals of type $A_n^{(1)}$. KR crystals are affine finite-dimensional crystals corresponding to affine Kac–Moody algebras, in the setting of [7] of type $A_n^{(1)}$. The highest weight condition is with respect to the finite subalgebra A_n. The bijection Φ can be generalized by dropping the highest weight requirement on the elements in the KR crystals [1], yielding the set of crystal paths $\mathcal{P}$. On the corresponding set of unrestricted rigged configurations RC, the A_n crystal structure is known explicitly [12]. One of the remaining open questions is to define the full affine crystal structure on the level of rigged configurations. Given the affine crystal structure on both sides, the bijection Φ has a much more conceptual interpretation as an affine crystal isomorphism.

In type $A_n^{(1)}$, the affine crystal structure can be defined using the promotion operator pr, which corresponds to the Dynkin diagram automorphism mapping node i to $i+1$ modulo $n+1$. On crystals, the promotion operator is defined using jeu-de-taquin [13, 15]. In [12], Schilling proposed an algorithm $\overline{\mathrm{pr}}$ on RC and conjectured [12, Conjecture 4.12] that $\overline{\mathrm{pr}}$ corresponds to the promotion operator pr under the bijection Φ. Several necessary conditions of promotion operators were established and it was shown that in special cases $\overline{\mathrm{pr}}$ is the correct promotion operator.

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In this paper, we show in general that $\Phi \circ \mathrm{pr} \circ \Phi^{-1} = \overline{\mathrm{pr}}$ (i.e., Φ is the intertwiner between pr and $\overline{\mathrm{pr}}$):

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}"'] & \mathrm{RC} \arrow[d, "\overline{\mathrm{pr}}"] \\ \mathcal{P} \arrow[r, "\Phi"'] & \mathrm{RC}. \end{tikzcd}$$

Thus $\overline{\mathrm{pr}}$ is indeed the promotion on RC and Φ is an affine crystal isomorphism.

Another reformulation of the bijection from tensor product of crystals to the rigged configurations in terms of the energy function of affine crystals and the inverse scattering formalism for the periodic box ball systems was given in [8, 9, 10, 11].

This paper is orangized as follows. In Section 2, we review the definitions of crystal paths and rigged configurations and state the main result in Theorem 2.27. In Section 3, we outline the proof by a running example. Due to the limitation of space, details of the proof are not included. A long version of this paper that contains all technical details is in progress [17] and will appear elsewhere.

2 Preliminaries and the main result

Throughout this paper the positive integer n stands for the rank of the Lie algebra A_n. Let $\overline{I} = [n]$ be the index set of the Dynkin diagram of type A_n. Let $\mathcal{H} = \overline{I} \times \mathbb{Z}_{>0}$ and define B_n to be a finite sequence of pairs of positive integers

$$B_n = ((r_1, s_1), \ldots, (r_K, s_K))$$

with $(r_i, s_i) \in \mathcal{H}$ and $1 \leq i \leq K$. We use L_n as a finite (multi-)set of pairs of positive integers

$$L_n = \{(r_1, s_1), \ldots, (r_K, s_K)\}$$

with $(r_i, s_i) \in \mathcal{H}$ and $1 \leq i \leq K$. We omit the subscript n when its value is irrelevant or clear from the context. We also write $L(B)$ for the underlying (multi-)set of B. When L is used this way, it is called the **multiplicity array** of B.

B represents a sequence of rectangles where the i-th rectangle is of height r_i and width s_i. We sometimes use the phrase "leftmost rectangle" (resp."rightmost rectangle") to mean the first (resp. last) pair in the list. We use $|B| = K$ for the number of pairs in B, and $B[i] = (r_i, s_i)$ as the i-th pair in B. Similarly, we can think of L as a (multi-)set of rectangles. It is sometimes useful (especially in the setting of rigged configurations) to consider the multiplicity of a given $(a, i) \in \mathcal{H}$ in L by setting $L_i^{(a)} = \#\{(r, s) \in L \mid r = a, s = i\}$.

Given a sequence of rectangles B, we will use the following operations for successively removing boxes from it. In the following subsections, we define the set of paths $\mathcal{P}(B)$ and rigged configurations $\mathrm{RC}(L(B))$, and discuss the analogous operations defined on $\mathcal{P}(B)$ and $\mathrm{RC}(L(B))$. They are used to define the bijection Φ between $\mathcal{P}(B)$ and $\mathrm{RC}(L(B))$ recursively. The proof exploits this recursion.

Definition 2.1 *[1, Section 4.1,4.2].*

1. *If $B = ((1,1), B')$, let $\mathrm{lh}(B) = B'$. This operation is called **left-hat**.*
2. *If $B = ((r,s), B')$ with $s \geq 2$, let $\mathrm{ls}(B) = ((r,1),(r,s-1),B')$. This operation is called **left-split**.*

3. *If $B = ((r,1), B')$ with $r \geq 2$, let* $\mathrm{lb}(B) = ((1,1),(r-1,1),B')$. *This operation is called* ***box-split****.*

2.1 Inhomogeneous lattice paths

Definition 2.2 *Given $(r,s) \in \mathcal{H}$, define $\mathcal{P}_n(r,c)$ to be the set of semi-standard Young tableaux of (rectangular) shape (s^r) over the alphabet $\{1,2,\ldots,n+1\}$.*

Recall that for each semi-standard Young tableau t, we can associate a ambient weight $\mathrm{wt}(t) = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ where λ_i is the number of times that i appears in t. Moreover, $\mathcal{P}_n(r,s)$ is endowed with a type A_n-crystal structure, with the Kashiwara operator e_a, f_a for $1 \leq a \leq n$ defined by the signature rule. For a detailed discussion see for example [4, Chapters 7 and 8].

Definition 2.3 *Given a sequence B_n as defined above, $\mathcal{P}_n(B_n) = \mathcal{P}_n(r_1,s_1) \otimes \cdots \otimes \mathcal{P}_n(r_K,s_K)$.*

As a set $\mathcal{P}_n(B_n)$ is a sequence of rectangular semi-standard Young tableaux. It is also endowed with a crystal structure through the tensor product rule. The Kashiwara operators e_a, f_a for $1 \leq a \leq n$ naturally extend from semi-standard tableaux to a list of tableaux using signature rule. For $b_1 \otimes b_2 \otimes \cdots \otimes b_K \in \mathcal{P}_n(B_n)$, $\mathrm{wt}(b_1 \otimes b_2 \otimes \cdots \otimes b_K) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2) + \cdots + \mathrm{wt}(b_K)$. We note here that the convention of tableaux tensor product in this paper follows that of [1, Section 2], which is opposite to the Kashiwara's original convention [2].

Definition 2.4 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ be an list of nonnegative integers, define*

$$\mathcal{P}_n(B_n, \lambda) = \{p \in \mathcal{P}_n(B_n) \mid \mathrm{wt}(p) = \lambda\}$$

Example 2.5 *Let $B_3 = ((2,2),(1,2),(3,1))$. Let*

$$p = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array}$$

Then p is an element of $\mathcal{P}(B_3)$ and $\mathrm{wt}(p) = (3,4,1,1)$.

As in the above example, we often omit the subscript n, writing $\mathcal{P}$ instead of $\mathcal{P}_n$ when n is irrelevant or clear from the discussion.

We often refer to a rectangular tableau just as a "rectangle" when there is no ambiguity, for example, the leftmost rectangle in p of above example is the tableau:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}\,.$$

The following maps on $\mathcal{P}(B)$ are the counterparts to the maps lh, lb and ls defined on B.

Definition 2.6 *[1, Sections 4.1,4.2]..*

1. *Let $b = c \otimes b' \in \mathcal{P}((1,1),B')$. Then $\mathrm{lh}(b) = b' \in \mathcal{P}(B')$.*

2. *Let $b = c \otimes b' \in \mathcal{P}((r,s),B')$, where $c = c_1 c_2 \cdots c_s$ and c_i denotes the i-th column of c. Then* $\mathrm{ls}(b) = c_1 \otimes c_2 \cdots c_s \otimes b'$.

3. *Let* $b = \begin{array}{|c|}\hline b_1 \\ \hline b_2 \\ \hline \vdots \\ \hline b_r \\ \hline\end{array} \otimes b' \in \mathcal{P}((r,1), B')$, *where* $b_1 < \cdots < b_r$. *Then* $\mathrm{lb}(b) = \boxed{b_r} \otimes \begin{array}{|c|}\hline b_1 \\ \hline \vdots \\ \hline b_{r-1} \\ \hline\end{array} \otimes b'$.

2.2 Rigged configurations

This section follows [12, Section 3.1]. In this paper we only consider rigged configurations of type $A_n^{(1)}$.

The (highest-weight) rigged configurations are indexed by the multiplicity array L and a dominant weight Λ. The sequence of partitions $\nu = \{\nu^{(a)} \mid a \in \overline{I}\}$ is a (L, Λ)**-configuration** if

$$\sum_{(a,i)\in\mathcal{H}} i m_i^{(a)} \alpha_a = \sum_{(a,i)\in\mathcal{H}} i L_i^{(a)} \Lambda_a - \Lambda, \tag{2.1}$$

where $m_i^{(a)}$ is the number of parts of length i in partition $\nu^{(a)}$, α_a is the a-th simple root and Λ_a is the a-th fundamental weight. Denote the set of all (L, Λ)-configurations by $\mathrm{C}(L, \Lambda)$. The **vacancy number** of a configuration is defined as

$$p_i^{(a)} = \sum_{j\geq 1} \min(i,j) L_j^{(a)} - \sum_{(b,j)\in\mathcal{H}} (\alpha_a|\alpha_b) \min(i,j) m_j^{(b)}.$$

Here $(\cdot|\cdot)$ is the normalized invariant form on the weight lattice P such that $A_{ab} = (\alpha_a|\alpha_b)$ is the Cartan matrix (of type A_n in our case). The (L, Λ)-configuration ν is **admissible** if $p_i^{(a)} \geq 0$ for all $(a,i) \in \mathcal{H}$, and the set of admissible (L, Λ)-configurations is denoted by $\overline{\mathrm{C}}(L, \Lambda)$.

A partition p can be viewed as a linear ordering $(p, \gg)$ of a finite multiset of positive integers, where elements of different values are ordered by their value, and elements the same value are given an arbitrary ordering. Implicitly, when we draw a Young diagram of p, we are giving such an ordering. A labeling of a partition p is then a map $J^{(p)} : (p, \gg) \to \mathbb{Z}_{\geq 0}$ satisfying that if $i, j \in p$ are of the same value and $i \gg j$ then $J^{(p)}(i) \geq J^{(p)}(j)$ as integers. The pairs $(x, J^{(p)}(x))$ are referred to as strings; x is referred to as the length or size of the string and $J^{(p)}(x)$ as the **label** or **quantum number**.

A rigging J of an (admissible) (L, Λ)-configuration ν is a sequence of maps $J = (J^{(a)})$, each $J^{(a)}$ is a labeling of the partitions $\nu^{(a)}$ with the extra requirement that for any part $i \in \nu^{(a)}$

$$0 \leq J^{(a)}(i) \leq p_i^{(a)}.$$

The difference $cJ^{(a)}(i) = p_i^{(a)} - J^{(a)}(i)$ is referred to as the **colabel** or **coquantum number** of the part i. A part is said to be **singular** if its colabel is 0. Since cJ and J uniquely determine each other, we sometimes refer to a string by $(x, cJ^{(p)}(x))$ when it is more convenient.

Definition 2.7 *The pair (ν, J) described above is called a (restricted-)**rigged configuration**. The set of all rigged (L, Λ)-configurations is denoted by $\overline{\mathrm{RC}}_n(L, \Lambda)$. In addition, define $\overline{\mathrm{RC}}(L) = \bigcup_{\Lambda\in P^+} \overline{\mathrm{RC}}(L, \Lambda)$.*

The equation (2.1) provides an obvious way of defining weight function on $\overline{\mathrm{RC}}(L)$. Namely, for $rc \in \overline{\mathrm{RC}}(L)$

$$\mathrm{wt}(rc) = \sum_{(a,i)\in\mathcal{H}} i L_i^{(a)} \Lambda_a - \sum_{(a,i)\in\mathcal{H}} i m_i^{(a)} \alpha_a \tag{2.2}$$

When working with rigged configuration, it is often convenient to take the fundamental weights as basis for the weight space. On the other hand, we presented weights in the ambient weight space when working with lattice paths. Conceptually, this distinction is not necessary, as weights can be considered as abstract vectors in the weight space. Identifying the fundamental weight Λ_i with $(1^i, 0^{n+1-i})$ we can switch from one representation to the other.

Remark 2.8 *From the above definition, it is clear that* $\mathrm{RC}(L)$ *is not sensitive to the ordering of the rectangles in* L.

Definition 2.9 *Let* L *be a multiplicity array. Define the set of* ***unrestricted rigged configurations*** $\mathrm{RC}(L)$ *as the closure of* $\overline{\mathrm{RC}}(L)$ *under the operators* f_a, e_a *for* $a \in \overline{I}$*, with* f_a, e_a *given by:*

1. *Define* $e_a(\nu, J)$ *by removing a box from a string of length* k *in* $(\nu, J)^{(a)}$ *leaving all colabels fixed and increasing the new label by one. Here* k *is the length of the string with the smallest negative rigging of smallest length. If no such string exists,* $e_a(\nu, J)$ *is undefined.*

2. *Define* $f_a(\nu, J)$ *by adding a box to a string of length* k *in* $(\nu, J)^{(a)}$ *leaving all colabels fixed and decreasing the new label by one. Here* k *is the length of the string with the smallest non positive rigging of largest length. If no such string exists, add a new string of length one and label -1. If the result is not a valid unrestricted rigged configuration* $f_a(\nu, J)$ *is undefined.*

The weight function (2.2) defined on $\overline{\mathrm{RC}}(L)$ extends to $\mathrm{RC}(L)$ with no change.

As their names suggested, f_a and e_a are indeed the Kashiwara operators with respect to the weight function above, and define a crystal structure on $\mathrm{RC}(L)$. This was proved in [12].

From the definition of f_a, it is clear that the labels of parts in an unrestricted rigged configuration may be negative. It is natural to ask what shapes and labels can appear in an unrestricted rigged configuration. There is an explicit characterization of $\mathrm{RC}(L)$ which answers this question [1, Section 3]. The statement is rather long and is not directly used by our proof, so we will just give rough outline and leave the interested reader to the original paper for detail: In the definition of $\overline{\mathrm{RC}}(L)$, we required that the vacancy number associated to each part non-negative. We drop this requirement for $\mathrm{RC}(L)$. Yet a vacancy numbers in $\mathrm{RC}(L)$ still serves as the upper bound of the label, much like the role a vacancy number plays for a restricted rigged configuration. For restricted rigged configurations, the lower bound for the label of a part is uniformly 0. For unrestricted rigged configurations, this is not the case. The characterization gives a way on how to find lower bound for each part.

Example 2.10 *Here is an example on how we normally visualize a restricted/unrestricted rigged configuration. Let* $B_3 = ((2,2),(1,2),(3,1))$*, and* $L_3 = L(B_3)$*. Then*

$$\mathrm{rc} = \quad \boxed{}\boxed{}\ -1 \qquad \boxed{}\ 1 \qquad \boxed{}\ -1$$

is an element of $\mathrm{RC}(L_3, -\Lambda_1 + 3\Lambda_2)$.

In this example, the sequence of partitions ν *is ((2),(1),(1)). The number that follows each part is the label assigned to this part by* J*. The vacancy numbers associated to these parts are* $p_2^{(1)} = -1$, $p_1^{(2)} = 1$, *and* $p_1^{(3)} = 0$*. Note that the labels are all less than or equal to the vacancy numbers, in the case that they are equal, e.g. parts in* $\nu^{(1)}$ *and* $\nu^{(2)}$*, those parts are called singular as restricted rigged configuration.*

In this example $rc \in \mathrm{RC} \setminus \overline{\mathrm{RC}}$*, the idea is the same for* $rc \in \overline{\mathrm{RC}}$.

The following maps on $\mathrm{RC}(L(B))$ are the counterparts to the lh, lb and ls map defined on B.

Definition 2.11 *[1, Section 4.1,4.2] .*

1. *Let $rc = (\nu, J) \in \mathrm{RC}(L(B))$. Then $\overline{\mathrm{lh}}(rc) \in \mathrm{RC}(L(\mathrm{lh}(B)))$ is defined by: first set $\ell^{(0)} = 1$ and then repeat the following process for $a = 1, 2, \ldots, n-1$ or until stopped. Find the smallest index $i \geq \ell^{(a-1)}$ such that $J^{(a)}(i)$ is singular. If no such i exists, set $\mathrm{rk}(\nu, J) = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue with $a+1$. Set all undefined $\ell^{(a)}$ to ∞.*

   The new rigged configuration $(\tilde{\nu}, \tilde{J}) = \overline{\mathrm{lh}}(\nu, J)$ is obtained by removing a box from the selected strings and making the new strings singular again.

2. *Let $rc = (\nu, J) \in \mathrm{RC}(L(B))$. Then $\overline{\mathrm{ls}}(rc) \in \mathrm{RC}(L(\mathrm{ls}(B)))$ is the same as (ν, J).*

3. *Let $rc = (\nu, J) \in \mathrm{RC}(L(B))$. Then $\overline{\mathrm{lb}}(rc) \in \mathrm{RC}(L(\mathrm{ls}(B)))$ is defined by adding singular strings of length 1 to $(\nu, J)^{(a)}$ for $1 \leq a < r$. Note that the vacancy numbers remain unchanged under $\overline{\mathrm{lb}}$.*

2.3 The bijection between $\mathcal{P}(B)$ and $RC(L(B))$

The map $\Phi : \mathcal{P}(B, \lambda) \to \mathrm{RC}(L(B), \lambda)$ is defined recursively by various commutative diagrams. Note that it is possible to go from $B = ((r_1, s_1), (r_2, s_2), \ldots, (r_K, s_K))$ to the empty crystal via successive application of lh, ls and lb. For further details see [1, Section 4].

Definition 2.12 *Define the map $\Phi : \mathcal{P}(B, \lambda) \to \mathrm{RC}(L(B), \lambda)$ such that the empty path maps to the empty rigged configuration and such that the following conditions hold:*

1. *Suppose $B = ((1,1), B')$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}(B,\lambda) & \xrightarrow{\Phi} & \mathrm{RC}(L(B),\lambda) \\ {\scriptstyle \mathrm{lh}}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\ \bigcup\limits_{\mu \in \lambda^-} \mathcal{P}(\mathrm{lh}(B),\mu) & \xrightarrow[\Phi]{} & \bigcup\limits_{\mu \in \lambda^-} \mathrm{RC}(L(\mathrm{lh}(B)),\mu) \end{array}$$

   where λ^- is the set of all partitions obtained from partitionλ by removing a box.

2. *Suppose $B = ((r,s), B')$ with $s \geq 2$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}(B,\lambda) & \xrightarrow{\Phi} & \mathrm{RC}(L(B),\lambda) \\ {\scriptstyle \mathrm{ls}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\mathrm{ls}}} \\ \mathcal{P}(\mathrm{ls}(B),\lambda) & \xrightarrow[\Phi]{} & \mathrm{RC}(L(\mathrm{ls}(B)),\lambda) \end{array}$$

3. *Suppose $B = ((r,1), B')$ with $r \geq 2$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}(B,\lambda) & \xrightarrow{\Phi} & \mathrm{RC}(L(B),\lambda) \\ {\scriptstyle \mathrm{lb}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\mathrm{lb}}} \\ \mathcal{P}(\mathrm{lb}(B),\lambda) & \xrightarrow[\Phi]{} & \mathrm{RC}(L(\mathrm{lb}(B)),\lambda) \end{array}$$

2.4 Promotion operators

The **promotion** operator pr on $\mathcal{P}_n(B)$ is defined in [15, page 164]. For the purpose of our proof, we will phrase it as a composition of one **lifting** operator and then several **sliding** operators defined on $\mathcal{P}_n(B)$.

Definition 2.13 *The lifting operator l on $\mathcal{P}_n(B)$ lifts $p \in \mathcal{P}_n(B)$ to $l(p) \in \mathcal{P}_{n+1}(B)$ by adding 1 to each box in each rectangle of p.*

Definition 2.14 *Given $p \in \mathcal{P}_{n+1}(B)$, the sliding operator ρ is defined as the following algorithm: find in p the rightmost rectangle that contains $n+2$, remove one appearance of $n+2$, apply jeu-de-taquin on this rectangle to move the empty box to the opposite corner, fill in 1 in this empty box. If no rectangle contains $n+2$, then ρ is the identity map.*

Definition 2.15

$$\mathrm{pr}(p) = \rho^m \circ l(p)$$

where m is the total number of $n+2$ in p.

This promotion operator is used to construct the affine crystal structure on $\mathcal{P}(B)$. See [15, page 164] for a detailed discussion.

The proposed promotion operator $\overline{\mathrm{pr}}$ on $\mathrm{RC}_n(L)$ is defined in [12, Definition 4.8]. To draw the parallel with pr we will phrase it as a composition of one lifting operator and then several sliding operators defined on $\mathrm{RC}(L)$.

Definition 2.16 *The lifting operator $\bar{l}$ on $\mathrm{RC}_n(L)$ lifts $\mathrm{rc} \in \mathrm{RC}_n(L)$ to $\bar{l}(\mathrm{rc}) \in \mathrm{RC}_{n+1}(L)$ by setting $\bar{l}(\mathrm{rc}) = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_{n+1}^{\lambda_{n+1}}(\mathrm{rc})$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ is the ambient weight of rc $\sum_{i \in [n+1]} \lambda_i = \sum_{(r,s)\in L} r \cdot s$. Notice here we use the fact that $\mathrm{RC}_n(L)$ is naturally embedded in $\mathrm{RC}_{n+1}(L)$ by simply treating the $(n+1)_{st}$ partition $v^{(n+1)}$ to be $\emptyset$.*

Definition 2.17 *[12, Definition 4.8] Given $\mathrm{rc} \in \mathrm{RC}_{n+1}(L)$, the sliding operator $\overline{\rho}$ is defined as the following algorithm: Find the smallest singular string in $\mathrm{rc}^{(n+1)}$. Let the length be $\ell^{(n+1)}$. Repeatedly find the smallest singular string in $(\nu', J')^{(k)}$ of length $\ell^{(k)} \geq \ell^{(k+1)}$ for all $1 \leq k < n+1$. Shorten the selected strings by one and make them singular again.*

Definition 2.18

$$\overline{\mathrm{pr}}(\mathrm{rc}) = \overline{\rho}^m \circ \bar{l}(\mathrm{rc})$$

where m is the number of boxes in $\mathrm{rc}^{(n+1)}$.

Remark 2.19 *It is a easy matter to show that $\bar{l} = \Phi(l)$ (that is $\bar{l} = \Phi \circ l \circ \Phi^{-1})$). Indeed, we could have defined $l(p) = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_{n+1}^{\lambda_{n+1}}(p)$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ is the weight of p.*

There is a question in Definition 2.14 on whether a sequence of m ρ operators can be always applied. The same question can be asked for Definition 2.17. The following are examples on how things could go wrong:

Example 2.20 *Let*

$$p = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & 4 \\ \hline \end{array} \in \mathcal{P}_3((2,2)) \tag{2.3}$$

If we try to construct $\rho(p)$, we realize that after removing a copy of 4 and move the empty box to the upper left corner we get $\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & 4 \\ \hline\end{array}$, *and filling the empty box with 1 will violate the column-strictness of semi-standard Young tableaux.*

On the RC *side, let*

$$\mathrm{rc} = \quad \begin{array}{|c|c|}\hline & \\ \hline\end{array}\ 0 \quad \begin{array}{|c|c|}\hline & \\ \hline\end{array}\ 0 \quad \emptyset \in \mathrm{RC}_3((2,2))$$

We see that $\overline{\rho}(\mathrm{rc})$ is not well-defined.

Therefore, ρ and $\overline{\rho}$ are partial functions on $\mathcal{P}_{n+1}$ and RC_{n+1}. This, however, will not cause problems in our discussion because of the following two remarks.

Remark 2.21 *ρ is well-defined on $\rho^k(\mathrm{Img}(l))$ for any k. This follows from the well known fact that if T is a semi-standard rectangular tableau, and if we remove all cells that contain the largest number (which is a horizontal strip in the last row) and apply "jeu de taquin" to move these empty cells to the upper left corner, then these empty cells form a horizontal strip.*

Thus we could have just restricted the domain of ρ to:

Definition 2.22

$$\mathrm{Dom}(\rho) = \bigcup_{k=0,1,2\dots} \rho^k(\mathrm{Img}(l)) \tag{2.4}$$

Remark 2.23 *It is not known at this stage that $\overline{\rho}$ is fully defined on $\Phi(\mathrm{Dom}(\rho))$. In fact, it is a consequence of our proof.*

2.5 Combinatorial R-matrix and right-split

Let $B = ((r_1, s_1), \dots, (r_K, s_K))$ be a sequence of rectangles, and let $\sigma \in S_K$ be a permutation of K letters. The R-matrix is the affine crystal isomorphism $\mathrm{R} : \mathcal{P}(B) \to \mathcal{P}(\sigma(B))$, which sends $u_1 \otimes \cdots \otimes u_K$ to $u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(K)}$, where $u_i \in \mathcal{P}(r_i, s_i)$ is the unique tableau of content $(s_i^{r_i})$ and $\sigma(B) = ((r_{\sigma(1)}, s_{\sigma(1)}), \dots, (r_{\sigma(K)}, s_{\sigma(K)}))$. It was shown in [7, Lemma 8.5] that $\Phi \circ R = \overline{\mathrm{R}} \circ \Phi$, where $\overline{\mathrm{R}}$ is the identity map. Together with the fact that R preserves the crystal structure [12] this implies the following result.

Remark 2.24 *The following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{P}(B) & \xrightarrow{\Phi} & \mathrm{RC}(L(B)) \\
{\scriptstyle \mathrm{R}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\mathrm{R}}=\mathrm{id}} \\
\mathcal{P}(\sigma(B)) & \xrightarrow[\Phi]{} & \mathrm{RC}(L(\sigma(B))).
\end{array}$$

Remark 2.25 *Since any $\sigma \in S_K$ is generated by simple transpositions, it suffices to work with cases where $|B|$=2, and σ being the only simple transposition is S_2.*

Definition 2.26 rs, $\overline{\mathrm{rs}}$ *are called **right-split**.* rs *operates on sequences of rectangles as follow: Let $B = ((r_1, s_1), \dots, (r_K, s_K))$, and suppose $s_K > 1$ (i.e, the rightmost rectangle is not a single column). Then* $\mathrm{rs}(B) = ((r_1, s_1), \dots, (r_K, s_K - 1), (r_K, 1))$*, that is,* rs *splits one column off the rightmost rectangle.*

$\overline{\mathrm{rs}}$ *operates on* $RC(L(B))$ *as follow: If* $\mathrm{rc} \in \mathrm{RC}(L(B))$*, then* $\overline{\mathrm{rs}}(\mathrm{rc}) \in \mathrm{RC}(L(\mathrm{rs}(B)))$ *is obtained by increasing the riggings by 1 for all parts in* $\nu^{(r_K)}$ *of size less than* s_K*. Observe that this will leave the co-riggings of all parts unchanged.*

rs*, which operates on* $\mathcal{P}(B)$*, is defined to be* $\Phi \circ \overline{\mathrm{rs}} \circ \Phi^{-1}$.

2.6 The main result

We now state the main result of this paper.

Theorem 2.27 *Let* $B = ((r_1, s_1), \ldots, (r_K, s_K))$ *be a sequence of rectangles, and* $\mathcal{P}(B)$*,* $RC(L(B))$*,* Φ*,* pr*, and* $\overline{\mathrm{pr}}$ *as given as above. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}(B) & \xrightarrow{\Phi} & \mathrm{RC}(L(B)) \\ {\scriptstyle \mathrm{pr}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\mathrm{pr}}} \\ \mathcal{P}(B) & \xrightarrow[\Phi]{} & \mathrm{RC}(L(B)). \end{array} \tag{2.5}$$

3 The outline of the proof

In this part, we outline the proof by a running example. The details of the proofs can be found in [17].

Recall we want to prove (2.5). By Remark 2.19 it suffices for us to show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Dom}(\rho) & \xrightarrow{\Phi} & \Phi(\mathrm{Dom}(\rho)) \\ {\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\rho}} \\ \mathrm{Dom}(\rho) & \xrightarrow[\Phi]{} & \Phi(\mathrm{Dom}(\rho)). \end{array}$$

In particular, we need to show that $\overline{\rho}$ is defined on $\Phi(\mathrm{Dom}(\rho))$.

As an abbreviation, for any $p \in \mathrm{Dom}(\rho)$, we use $\mathcal{D}(p)$ to mean the following statement: "$\overline{\rho}(\Phi(p))$ is well-defined and the diagram

$$\begin{array}{ccc} p & \xrightarrow{\Phi} & \bullet \\ {\scriptstyle \rho}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\rho}} \\ \bullet & \xrightarrow[\Phi]{} & \bullet \end{array}$$

commutes".

For $p, q \in \mathrm{Dom}(\rho)$ we write $\mathcal{D}(p) \rightsquigarrow \mathcal{D}(q)$ to mean that $\mathcal{D}(p)$ reduces to $\mathcal{D}(q)$, that is, $\mathcal{D}(q)$ is a sufficient condition for $\mathcal{D}(p)$.

We will use the following $p \in \mathcal{P}_3((2,2),(3,2),(2,2))$ as the starting point of the running example:

$$p = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 4 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}.$$

After lifting to $\mathcal{P}_4$ we have:

$$l(p) \;=\; \begin{array}{|c|c|}\hline 3&3\\\hline 5&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array} \in \mathrm{Dom}(\rho).$$

Our goal is to show $\mathcal{D}(l(p))$ by a sequence of reductions. Note that the rightmost 5 (which is $n+2$ where n=3) appears in the second rectangle. Thus ρ acts on the second rectangle. The motivation behind our reductions is to try to get rid of boxes from the left and make ρ act on the leftmost rectangle:

Step 1

$$\mathcal{D}(\begin{array}{|c|c|}\hline 3&3\\\hline 5&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}) \overset{\mathrm{ls}}{\rightsquigarrow} \mathcal{D}(\begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array})$$

This is called a ls-reduction.

Step 2

$$\mathcal{D}(\begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}) \overset{\mathrm{lb}}{\rightsquigarrow} \mathcal{D}(\boxed{5} \otimes \boxed{3} \otimes \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array})$$

This is called a lb-reduction.

Step 3

$$\mathcal{D}(\boxed{5} \otimes \boxed{3} \otimes \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}) \overset{\mathrm{lh}}{\rightsquigarrow} \mathcal{D}(\boxed{3} \otimes \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array})$$

This is called a lh-reduction.

Step 4 Another application of lh-reduction.

$$\mathcal{D}(\boxed{3} \otimes \begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}) \overset{\mathrm{lh}}{\rightsquigarrow} \mathcal{D}(\begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array})$$

We repeat above reductions until the rightmost tableau containing 5 becomes the first tableau in the list. After that we want to further simplify the list, if possible, to get rid of boxes from right by pushing them column-by-column to the left using the R-matrix map R, until we reach the place where can prove $\mathcal{D}(\bullet)$ directly:

Step 8

$$\mathcal{D}(\begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}) \overset{\mathrm{rs}}{\rightsquigarrow} \mathcal{D}(\begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array})$$

This is called a rs-reduction.

Step 9

$$\mathcal{D}\left(\begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array}\otimes\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\overset{\mathrm{R}}{\rightsquigarrow}\mathcal{D}\left(\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)$$

This is called a R-reduction.

Now since the rectangle that ρ acts on is no longer the leftmost one we can go back to Step 1. Repeat above steps until ρ acts on the leftmost rectangle again, then we need one more R-reduction:

Step 13

$$\mathcal{D}\left(\begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline 4&5\\\hline\end{array}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\overset{\mathrm{R}}{\rightsquigarrow}\mathcal{D}\left(\begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline 4&4\\\hline\end{array}\right)$$

Repeating, we will eventually reach one of the following two cases (not mutually exclusive):

- Base case 1: p is a single rectangle that contains $n+2$ (which is 5 in the above example); or
- Base case 2: $p = S \otimes q$ where S is a single column that contains $n+2$ (which is 5 in the above example), and $n+2$ does not appear in q.

In the above example, we reached the second case.

Step 14 Now we have to prove this base case directly:

$$\mathcal{D}\left(\begin{array}{|c|}\hline 3\\\hline 5\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline 4&4\\\hline\end{array}\right)$$

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Permutations with Kazhdan-Lusztig polynomial $P_{id,w}(q) = 1 + q^h$

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Abstract. Using resolutions of singularities introduced by Cortez and a method for calculating Kazhdan-Lusztig polynomials due to Polo, we prove the conjecture of Billey and Braden characterizing permutations w with Kazhdan-Lusztig polynomial $P_{id,w}(q) = 1 + q^h$ for some h.

Résumé. On démontre la conjecture de Billey et Braden sur les permutations w pour lesquelles le polynôme de Kazhdan-Lusztig $P_{id,w}(q) = 1 + q^h$ pour un entier h. On emploie une résolution des singularités présentées par Cortez et une méthode de Polo pour calculer ces polynômes.

Keywords: Kazhdan-Lusztig polynomials, Schubert varieties

1 Introduction

The results mentioned in this extended abstract have been published in [33] along with most of the introductory material. We explain here the alternative approach mentioned in [33, Remark 4.7]. This approach recasts some of the geometry into combinatorial language, but the details in the proofs of the lemmas will be essentially the same.

Kazhdan-Lusztig polynomials are polynomials $P_{u,w}(q)$ in one variable associated to each pair of elements u and w in the symmetric group S_n (or more generally in any Coxeter group). They have an elementary definition in terms of the Hecke algebra [24, 21, 9] and numerous applications in representation theory, most notably in [24, 1, 13], and the geometry of homogeneous spaces [25, 17]. While their definition makes it fairly easy to compute any particular Kazhdan-Lusztig polynomial, on the whole they are poorly understood. General closed formulas are known [5, 10], but they are fairly complicated; furthermore, although Kazhdan-Lusztig polynomials are known to be positive (for S_n and other Weyl groups), these formulas have negative signs. For S_n, positive formulas are known only for 3412 avoiding permutations [26, 27], 321-hexagon avoiding permutations [7], and some isolated cases related to the generic singularities of Schubert varieties [8, 29, 16, 32].

One important interpretation of Kazhdan-Lusztig polynomials is as local intersection homology Poincaré polynomials for Schubert varieties. This interpretation, originally established by Kazhdan and

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Lusztig [25], shows, in an entirely non-constructive manner, that Kazhdan-Lusztig polynomials have non-negative integer coefficients and constant term 1. Furthermore, as shown by Deodhar [17], $P_{id,w}(q) = 1$ (for S_n) if and only if the Schubert variety X_w is smooth, and, more generally, $P_{u,w}(q) = 1$ if and only if X_w is smooth over the Schubert cell X_u°.

The purpose of this paper is to prove the following theorem.

Theorem 1.1 *Suppose the singular locus of X_w has exactly one irreducible component, and w avoids the patterns* 653421*,* 632541*,* 463152*,* 526413*,* 546213*, and* 465132*. Then $P_{id,w}(1) = 2$.*

More precisely, when the hypotheses are satisfied, $P_{id,w}(q) = 1 + q^h$ where h is the minimum height of a 3412 *embedding, with $h = 1$ if no such embedding exists.*

Here, a 3412 embedding is a sequence of indices $i_1 < i_2 < i_3 < i_4$ such that $w(i_3) < w_(i_4) < w(i_1) < w(i_2)$, and its height is $w(i_1) - w(i_4)$. Given the first part of the theorem, the second part can be immediately deduced from the unimodality of Kazhdan-Lusztig polynomials [22, 12] and the calculation of the Kazhdan-Lusztig polynomial at the unique generic singularity [8, 29, 16]. Indeed, unimodality and this calculation imply the following corollary.

Corollary 1.2 *Suppose w satisfies the hypotheses of Theorem 1.1. Let X_v be the singular locus of X_w. Then $P_{u,w}(q) = 1 + q^h$ (with h as in Theorem 1.1) if $u \leq v$ in Bruhat order, and $P_{u,w}(q) = 1$ otherwise.*

The permutation v and the singular locus in general has a combinatorial description given in Theorem 2.1, which was originally proved independently in [8, 16, 23, 28]. This description is used in our proof. Furthermore, Billey and Weed recently found a combinatorial version [33, Theorem A.1] of Theorem 1.1, replacing the geometric condition that X_w has one irreducible component with sixty additional patterns.

Theorem 1.1 was conjectured by Billey and Braden [6]. They claim to have a proof for the converse in their paper. An outline of their proof is as follows. If $P_{id,w}(1) = 1$ then X_w is nonsingular [17]. The methods for calculating Kazhdan-Lusztig polynomials due to Braden and MacPherson [12] show that $P_{id,w}(1) \leq 2$ implies that the singular locus of X_w has at most one component. That $P_{id,w}(1) \leq 2$ implies the pattern avoidance conditions follows from [6, Thm. 1] and the computation of Kazhdan-Lusztig polynomials for the six pattern permutations.

Example 1.3 *To illustrate the theorem, $P_{id,643521}(q) = 1 + q$ (as* 643521 *has no* 3412 *embedding), $P_{id,254613}(q) = 1 + q$ (as $h = 1$), $P_{id,2657413}(q) = 1 + q^2$, and $P_{id,564312}(q) = 1 + q^3$. On the other hand, $P_{id,34512}(q) = 1 + 2q$ (as the singular locus of X_{34512} has three irreducible components), and $P_{id,2574163}(q) = 1 + q + q^2$ (as* 2574163 *does not avoid* 463152*).*

The proof of Theorem 1.1 outlined in this abstract requires two cases. When w has no 3412 embedding, we analyze the algorithm of Lascoux [26] for calculating Kazhdan-Lusztig polynomials for such w. For w containing a 3412 embedding, we use a resolution of singularities for Schubert varieties introduced by Cortez [16]. In general, the maps introduced by Cortez [16] do not necessarily come from a smooth variety, but they are actual resolutions for w satisfying the conditions of Theorem 1.1. A Bialynicki-Birula decomposition [3, 4, 14] of the resolution gives us a combinatorial formula purely in terms of permutations for the Poincaré polynomials for the fibers of the resolution. Polo [30] gave a combinatorial interpretation of the Decomposition Theorem [2] which allows us to then calculate Kazhdan-Lusztig polynomials from these Poincaré polynomials. This calculation is in the spirit of Deodhar's approach [18] to calculating

Kazhdan-Luzstig polynomials $P_{u,w}(q)$ from a reduced expression for w, but our calculation is simpler in this particular case.

Corollary 1.2 suggests the problem of describing all pairs u and w for which $P_{u,w}(1) = 2$. It seems possible to extend the methods of this paper to characterize such pairs; presumably X_u would need to lie in no more than one component of the singular locus of X_w, and $[u, w]$ would need to avoid certain intervals (see Section 2.3). Our methods in theory extend to more permutations, but any further extension to characterize w for which $P_{id,w}(1) = 3$ is likely to be extremely combinatorially intricate. An extension to other Weyl groups would also be interesting, not only for its intrinsic value, but because methods for proving such a result may suggest methods for proving any (currently nonexistent) conjecture combinatorially describing the singular loci of Schubert varieties for these other Weyl groups.

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2 Preliminaries

2.1 The symmetric group and Bruhat order

We begin by setting notation and basic definitions. We let S_n denote the symmetric group on n letters. We let $s_i \in S_n$ denote the adjacent transposition which switches i and $i + 1$; the elements s_i for $i = 1, \dots, n - 1$ generate S_n. Given an element $w \in S_n$, its **length**, denoted $\ell(w)$, is the minimal number of generators such that w can be written as $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. An **inversion** in w is a pair of indices $i < j$ such that $w(i) > w(j)$. The length of a permutation w is equal to the number of inversions it has.

Unless otherwise stated, permutations are written in one-line notation, so that $w = 3142$ is the permutation such that $w(1) = 3$, $w(2) = 1$, $w(3) = 4$, and $w(4) = 2$.

Given a permutation $w \in S_n$, the **graph** of w is the set of points $(i, w(i))$ for $i \in \{1, \dots, n\}$. We will draw graphs according to the Cartesian convention, so that $(0, 0)$ is at the bottom left and $(n, 0)$ the bottom right.

The **rank function** r_w is defined by

$$r_w(p, q) = \#\{i \mid 1 \leq i \leq p, 1 \leq w(i) \leq q\}$$

for any $p, q \in \{1, \dots, n\}$. We can visualize $r_w(p, q)$ as the number of points of the graph of w in the rectangle defined by $(1, 1)$ and (p, q). There is a partial order on S_n, known as **Bruhat order**, which can be defined as the reverse of the natural partial order on the rank function; explicitly, $u \leq w$ if $r_u(p, q) \geq r_w(p, q)$ for all $p, q \in \{1, \dots, n\}$. The Bruhat order and the length function are closely related. If $u < w$, then $\ell(u) < \ell(w)$; moreover, if $u < w$ and $j = \ell(w) - \ell(u)$, then there exist (not necessarily adjacent) transpositions $t_1, \dots, t_j$ such that $u = t_j \cdots t_1 w$ and $\ell(t_{i+1} \cdots t_1 w) = \ell(t_i \cdots t_1 w) - 1$ for all i, $1 \leq i < j$. For a thorough exposition covering various definitions and properties of Bruhat order see [9, Chap. 2].

2.2 Schubert varieties

Now we briefly define Schubert varieties. A **(complete) flag** $F_\bullet$ in $\mathbb{C}^n$ is a sequence of subspaces $\{0\} \subseteq F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n$, with $\dim F_i = i$. As a set, the **flag variety** $\mathcal{F}_n$ has one point for every flag in $\mathbb{C}^n$. The flag variety $\mathcal{F}_n$ has an algebraic and geometric structure as $GL(n)/B$, where B is the group of invertible upper triangular matrices, as follows. Given a matrix $g \in GL(n)$, we can associate to it the flag $F_\bullet$ with F_i being the span of the first i columns of g. Two matrices g and g' represent the same flag if and only if $g' = gb$ for some $b \in B$, so complete flags are in one-to-one correspondence with left B-cosets of $GL(n)$.

Fix an ordered basis $e_1, \dots, e_n$ for $\mathbb{C}^n$, and let $E_\bullet$ be the flag where E_i is the span of the first i basis vectors. Given a permutation $w \in S_n$, the **Schubert cell** associated to w, denoted X_w°, is the subset of $\mathcal{F}_n$ corresponding to the set of flags

$$\{F_\bullet \mid \dim(F_p \cap E_q) = r_w(p,q) \; \forall p, q\}. \tag{1}$$

The conditions in 1 are called **rank conditions** The **Schubert variety** X_w is the closure of the Schubert cell X_w°; its points correspond to the flags

$$\{F_\bullet \mid \dim(F_p \cap E_q) \geq r_w(p,q) \; \forall p, q\}.$$

Bruhat order has an alternative definition in terms of Schubert varieties; the Schubert variety X_w is a union of Schubert cells, and $u \leq w$ if and only if $X_u^\circ \subset X_w$. In each Schubert cell X_w° there is a **Schubert point** e_w, which is the point associated to the permutation matrix w; in terms of flags, the flag $E_\bullet^{(w)}$ corresponding to e_w is defined by $E_i^{(w)} = \mathbb{C}\{e_{w(1)}, \dots, e_{w(i)}\}$. The Schubert cell X_w° is the orbit of e_w under the left action of the group B.

Many of the rank conditions in (1) are actually redundant. Fulton [20] showed that for any w there is a minimal set, called the **coessential set**(i), of rank conditions which suffice to define X_w. To be precise, the coessential set is given by

$$\mathrm{Coess}(w) = \{(p,q) \mid w(p) \leq q < w(p+1), w^{-1}(q) \leq p < w^{-1}(q+1)\},$$

and a flag $F_\bullet$ corresponds to a point in X_w if and only if $\dim(F_p \cap E_q) \geq r_w(p,q)$ for all $(p,q) \in \mathrm{Coess}(w)$.

While we have distinguished between points in flag and Schubert varieties and the flags they correspond to here, we will freely ignore this distinction in the rest of the paper.

2.3 Pattern avoidance and interval pattern avoidance

Let $v \in S_m$ and $w \in S_n$, with $m \leq n$. A **(pattern) embedding** of v into w is a set of indices $i_1 < \cdots < i_m$ such that the entries of w in those indices are in the same relative order as the entries of v. Stated precisely, this means that, for all $j, k \in \{1, \dots, m\}$, $v(j) < v(k)$ if and only if $w(i_j) < w(i_k)$. A permutation w is said to **avoid** v if there are no embeddings of v into w.

Now let $[x, v] \subseteq S_m$ and $[u, w] \subseteq S_n$ be two intervals in Bruhat order. An **(interval) (pattern) embedding** of $[x, v]$ into $[u, w]$ is a simultaneous pattern embedding of x into u and v into w using the same set of indices $i_1 < \cdots < i_m$, with the additional property that $[x, v]$ and $[u, w]$ are isomorphic as posets. For the last condition, it suffices to check that $\ell(v) - \ell(x) = \ell(w) - \ell(u)$ [34, Lemma 2.1].

Note that given the embedding indices $i_1 < \cdots < i_m$, any three of the four permutations x, v, u, and w determine the fourth. Therefore, for convenience, we sometimes drop u from the terminology and discuss embeddings of $[x, v]$ in w, with u implied. We also say that w **(interval) (pattern) avoids** $[x, v]$ if there are no interval pattern embeddings of $[x, v]$ into $[u, w]$ for any $u \leq w$.

(i) Fulton [20] indexes Schubert varieties in a manner reversed from our indexing as it is more convenient in his context. As a result, his Schubert varieties are defined by inequalities in the opposite direction, and he defines the **essential set** with inequalities reversed from ours. Our conventions also differ from those of Cortez [15] in replacing her $p - 1$ with p.

2.4 Singular locus of Schubert varieties

Now we describe combinatorially the singular loci of Schubert varieties. The results of this section are due independently to Billey and Warrington [8], Cortez [15, 16], Kassel, Lascoux, and Reutenauer [23], and Manivel [28].

Stated in terms of interval pattern embeddings as in [34, Thm. 6.1], the theorem is as follows. Permutations are given in 1-line notation. We use the convention that the segment "$j, \cdots, i$" means $j, j-1, j-2, \ldots, i+1, i$. In particular, if $j < i$ then the segment is empty.

Theorem 2.1 *The Schubert variety X_w is singular at $e_{u'}$ if and only if there exists u with $u' \leq u < w$ such that one of the following (infinitely many) intervals embed in $[u, w]$:*

- *I:* $\big[(y+1), z, \cdots, 1, (y+z+2), \cdots, (y+2); \quad (y+z+2), (y+1), y, \cdots, 2, (y+z+1), \cdots, (y+2), 1\big]$ *for some integers $y, z > 0$.*
- *IIA:* $\big[(y+1), \cdots, 1, (y+3), (y+2), (y+z+4), \cdots, (y+4); \quad (y+3), (y+1), \cdots, 2, (y+z+4), 1, (y+z+3), \cdots, (y+4), (y+2)\big]$ *for some integers $y, z \geq 0$.*
- *IIB:* $\big[1, (y+3), \cdots, 2, (y+4); \quad (y+3), (y+4), (y+2), \cdots, 3, 1, 2\big]$ *for some integer $y > 1$.*

Equivalently, the irreducible components of the singular locus of X_w are the subvarieties X_u for which one of these intervals embeds in $[u, w]$.

2.5 Bialynicki-Birula decompositions

Given a $\mathbb{C}^*$ action on a smooth complex projective variety Y with finitely many fixed points, Bialynicki-Birula [3, 4] defined a decomposition of Y into cells, which he showed are each isomorphic to $\mathbb{C}^n$ for some n. More precisely, given a $\mathbb{C}^*$-fixed point p, we can associate the cell

$$Y_p^\circ := \{y \in Y \mid \lim_{t \to 0} t \cdot y = p\}.$$

In the case where Y is the flag variety, there is a $\mathbb{C}^*$ action whose fixed points are the Schubert points and whose resulting cells are the Schubert cells. Therefore, even though Schubert varieties are not smooth, they have a Bialynicki-Birula decomposition.

Given a $\mathbb{C}^*$-equivariant resolution of singularities $\pi : Z \to X_w$, we also have a Bialynicki-Birula decomposition of Z. Furthermore, if we let P_u denote the set of $\mathbb{C}^*$-fixed points of Z in $\pi^{-1}(e_u)$, we have a cell decomposition

$$\pi^{-1}(X_u^\circ) = \bigsqcup_{p \in P_u} Y_p^\circ,$$

and a decomposition of the fiber $\pi^{-1}(e_u)$ into cells $\pi^{-1}(e_u) \cap Y_p^\circ$ which are respectively of dimensions $\dim(Y_p^\circ) - \dim(X_u^\circ)$.

Therefore, the homology Poincaré polynomial for $\pi^{-1}(e_u)$ is

$$H_{u,\pi}(q) = \sum_{p \in P_u} q^{\dim(Y_p^\circ) - \ell(u)}.$$

(Technically, the degrees should be doubled, but as we have halved the degrees since all cells will be ($\mathbb{R}$)-even-dimensional and this will match the usual degrees for Kazhdan-Lusztig polynomials.)

2.6 The Decomposition Theorem

From the homology Poincaré polynomials $H_{u,\pi}$ for a resolution $\pi : Z \to X_w$ we can, following Polo [30], use the Decomposition Theorem [2] to calculate Kazhdan-Lusztig polynomials. More specifically, given such a resolution,

$$H_{u,\pi}(q) = P_{u,w}(q) + \sum_{u \le v < w} q^{\ell(w)-\ell(v)} E_v(q) P_{u,v}(q).$$

In this statement, $E_v(q)$ are Laurent polynomials in $q^{\frac{1}{2}}$ to be determined later; the Laurent polynomials $E_v(q)$ depend only on v and π and not on u, have with positive integer coefficients, and satisfy the identity $E_v(q) = E_v(q^{-1})$.

One case of the Decomposition Theorem is well-known in the theory of Kazhdan-Lusztig polynomials. When Z is the full Bott-Samelson resolution of X_w constructed from a reduced word decomposition $w = s_{i_1} \cdots s_{i_\ell}$, the fixed points of Z are indexed by the $2^{\ell(w)}$ subwords of this reduced word. One method of indexing leads to $\dim(Y_p^\circ) - \dim(X_u^\circ)$ being Deodhar's defect statistic [18], so that $H_{u,\pi}$ is precisely the sum, taken over subwords of our defining reduced word, of q raised to the number of defects in the subword. Rearranged, the formula above is precisely Deodhar's formula, and $E_v(q)$ represents the inadmissible masks.

Unfortunately the full Bott-Samelson resolution and Deodhar's approach is too difficult to analyze in this case. Instead we use a resolution of singularities due to Cortez [16] and calculate $H_{u,\pi}$ for this resolution π and certain crucial permutations u. This will give us enough information to calculate E_v for those resolutions and determine $P_{id,w}(q)$ when w satisfies the conditions of Theorem 1.1.

3 The covexillary case

A permutation w is **covexillary** if it avoids 3412. Generalizing a formula of Lascoux and Schützenberger in the case where w has only one ascent, Lascoux [26] gave a formula for the Kazhdan-Lusztig polynomials $P_{u,w}(q)$ which applies whenever w is covexillary. This formula proceeds by constructing a rooted tree T_w from w with nonnegative integer labels for the leaves of this tree based on how far u and w are from each other. Given an edge labelling L of a tree by nonnegative integers, let $s(L)$ be the sum of the edge labels. Then Lascoux shows that

$$P_{u,w} = \sum_L q^{s(L)},$$

where the sum is over all nondecreasing edge labellings of T_w which are bounded by the labels for the leaves.

A Schubert variety X_w for a covexillary permutation w has one component in its singular locus precisely when the labelling of the rooted tree T_w for id has only one leaf λ which is not labeled 0. Furthermore, the following lemmas hold.

Lemma 3.1 *Suppose w avoids 632541. Then no single branch of T_w is two edges long by itself. (In other words, every leaf is adjacent to a internal node with at least two children.)*

Lemma 3.2 *Suppose w avoids 653421. Then no leaf of T_w has a label greater than 1.*

In consequence, when the singular locus of X_w has one component and w avoids 3412, 632541, and 653421, one must label all the edges of T_w by 0, except for the edge above λ which can be labelled 0 or 1. Therefore, $P_{id,w}(q) = 1 + q$.

4 The 3412 containing case

In this section we treat the case where w contains a 3412 pattern. We use a resolution of singularities defined by Cortez and the machinery mentioned above of a Bialynicki-Birula decomposition followed by an application of the Decomposition Theorem.

4.1 Cortez's resolution

We begin with some definitions necessary for defining a variety Z and a $\mathbb{C}^*$-equivariant map $\pi : Z \to X_w$ which we will show is a resolution of singularities. Our notation and terminology generally follows that of Cortez [16]. Given an embedding $i_1 < i_2 < i_3 < i_4$ of 3412 into w, we call $w(i_1) - w(i_4)$ its **height** (*hauteur*), and $w(i_2) - w(i_3)$ its **amplitude**. Among all embeddings of 3412 in w, we take the ones with minimum height, and among embeddings of minimum height, we choose one with minimum amplitude. As we will be continually referring this particular embedding, we denote the indices of this embedding by $a < b < c < d$ and entries of w at these indices by $\alpha = w(a)$, $\beta = w(b)$, $\gamma = w(c)$, and $\delta = w(d)$. We let $h = \alpha - \delta$ be the height of this embedding.

Let α' be the largest number such that $w^{-1}(\alpha') < w^{-1}(\alpha' - 1) < \cdots < w^{-1}(\alpha + 1) < w^{-1}(\alpha)$ and δ' the smallest number such that $w^{-1}(\delta) < w^{-1}(\delta - 1) < \cdots < w^{-1}(\delta')$. Also let $a' = w^{-1}(\alpha')$ and $d' = w^{-1}(\delta')$. Now let $\kappa = \delta' + \alpha' - \alpha$, let I denote the set of simple transpositions $\{s_{\delta'}, \cdots, s_{\alpha'-1}\}$, and let J be $I \setminus \{s_\kappa\}$. Furthermore, let $v = w_0^J w_0^I w$, where w_0^J and w_0^I denote the longest permutations in the parabolic subgroups of S_n generated by J and I respectively.

Example 4.1 *Suppose $w = 817396254 \in S_9$. Then $a = 3$, $b = 5$, $c = 7$, and $d = 8$, while $\alpha = 7$, $\beta = 9$, $\gamma = 2$, and $\delta = 5$. We also have $h = 2$, $\alpha' = 8$ and $\delta' = 4$. Hence $\kappa = 5$ and $v = 514398276$.*

Now consider the variety $Z = P_I \times^{P_J} X_v$. By definition, Z is a quotient of $P_I \times X_v$ under the free action of P_J where $q \cdot (p, x) = (pq^{-1}, q \cdot x)$ for any $q \in P_J$, $p \in P_I$, and $x \in X_v$. We have a map $\pi : Z \to X_w$ defined by $\pi(p, x) = p \cdot x$; note this is well-defined. The map π is birational and surjective [16, Proposition 4.4]. However, Z is not smooth in general, as X_v need not be smooth. Nevertheless, we show the following for our case.

Lemma 4.2 *Suppose the singular locus of X_w has only one component and w avoids 463152. Then Z is smooth.*

Cortez [16] introduced the variety Z along with several other varieties (constructed by defining $\kappa = \delta' + \alpha' - \alpha + i - 1$ for $i = 1, \ldots, h$) to help in describing the singular locus of Schubert varieties[(ii)]. A virtually identical proof would follow from analyzing the resolution given by $i = h$ instead of $i = 1$ as we are doing, but the other choices of i will give maps which are harder to analyze as they have more complicated fibers.

4.2 Calculations for $H_{\pi,u}$

We now need to identify the fixed points of Z under the $\mathbb{C}^*$ action, calculate the dimensions of the cells associated with them, and classify them according to the fixed point e_u they map to under π. The fixed points of Z are precisely $\{(\sigma, e_\tau)\}$, where σ is in W_I, the parabolic subgroup of S_n generated by s_k for

(ii) Cortez's choice of 3412 embedding in [16] is slightly different from ours. For technical reasons she chooses one of minimum amplitude among those satisfying a condition she calls "well-filled" (*bien remplie*). As she notes, 3412 embeddings of minimum height are automatically "well-filled".

$k \in I$ (considered as a subgroup of GL_n in the usual way), and $\tau \leq v$ in Bruhat order on S_n. Several such pairs (σ, τ) will be in the same P_J orbit, so they will represent the same point in Z. We can eliminate this duplication by choosing one σ from each left W_J coset. For convenience, we will choose the one which is minimal in Bruhat order; each coset has a unique minimal element since W_J is parabolic. Furthermore, $\pi(\sigma, e_\tau) = e_u$ if and only if $\sigma\tau = u$.

When u is minimal in its right W_I coset, then the dimension of the cell associated to $(\sigma, e_\tau) \in \pi^{-1}(e_u)$ is $\ell(u) + \ell(\sigma)$. When u is not minimal in its right W_I coset, then the dimension of the cell is harder to calculate, but since π is P_I-equivariant, the fiber of $e_{u'}$ is the same as the fiber of e_u whenever u' and u are in the same right W_I coset. Therefore, given $u \leq w$, let u' denote the minimal element of its right W_I coset. Then

$$H_{\pi,u} = \sum_{(\sigma,\tau)} q^{\ell(\sigma)},$$

where $\sigma \in W_I$ is minimal in its left W_J coset, $\tau \leq v$, and $\sigma\tau = u'$.

It would be interesting to give a more direct formula for $H_{\pi,u}$ in general; hopefully this formula would mimic that of Deodhar for the full Bott-Samelson resolution by placing some defect-like statistic in the exponent of q.

Now we have the following combinatorial lemmas.

Lemma 4.3 *Suppose that the singular locus of X_w has only one component and w avoids* 546213. *If $\sigma \in P_I$, $\tau \leq v$, and $\sigma\tau = id$, then $\{1, \ldots, \kappa - 1\} \subseteq \sigma(\{1, \ldots, \kappa\})$.*

Lemma 4.4 *Suppose that the singular locus of X_w has only one component and w avoids* 465132. *If $\sigma \in P_I$, $\tau \leq v$, and $\sigma\tau = id$, then $\sigma(\{1, \ldots, \kappa\}) \subseteq \{1, \ldots, \kappa + h\}$.*

In the case where $h = 1$, this shows that $H_{id,\pi}(q) = 1 + q$, since the only admissible σ are the identity and the adjacent transposition s_κ. This shows that $P_{id,w}(q) = 1 + q$. Otherwise, $H_{id,\pi}(q) = 1 + q + \cdots + q^h$. In this case, let $\xi \in S_n$ be the cycle $(\gamma, \delta + 1, \delta + 2, \ldots, \alpha = \delta + h)$, and let $\rho = \xi w$. We then have the following lemma.

Lemma 4.5 *Assume that the singular locus of X_w has only one component, that $h > 1$, and that w avoids* 526413. *Then $H_{\pi,u}(1) > 1$ only if $u \leq \rho$, $\ell(w) - \ell(\rho) = h$, and $H_{\pi,\rho} = 1 + q + \cdots + q^{h-1}$.*

From these lemmas it follows by a calculation similar to one by Polo [30, Proposition 2.4(b)] that, in the case $h > 1$,

$$E_u(q) = 0 \text{ for } u \neq \rho,$$

$$E_\rho(q) = q^{1-\frac{h}{2}} + \cdots + q^{\frac{h}{2}-1},$$

and therefore

$$P_{id,w} = 1 + q^h.$$

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Combinatorial Formula for the Hilbert Series of bigraded S_n-modules

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Abstract. We introduce a combinatorial way of calculating the Hilbert series of bigraded S_n-modules as a weighted sum over standard Young tableaux in the hook shape case. This method is based on Macdonald formula for Hall-Littlewood polynomial and extends the result of A. Garsia and C. Procesi for the Hilbert series when $q = 0$. Moreover, we give the way of associating the fillings giving the monomial terms of Macdonald polynomials to the standard Young tableaux.

Résumé. Nous introduisons une méthode combinatoire pour calculer la série de Hilbert de modules bigradués de S_n comme une somme pondérée sur les tableaux de Young standards à la forme crochet. Cette méthode se fonde sur la formule Macdonald pour les polynômes Hall-Littlewood et généralise un résultat de A. Garsia et C. Procesi pour la série de Hilbert dans le cas $q = 0$. De plus, nous proposons une méthode pour associer aux tableaux de Young standards les remplissages des monomes des polynômes de Macdonald.

Keywords: combinatorial formula, Hilbert series, Garsia-Haiman modules

1 Introduction

In 1988 (Mac), Macdonald introduced a family of symmetric functions with two variables that are known as the Macdonald polynomials which becomes a basis for the space of symmetric functions. Upon introducing these polynomials, Macdonald conjectured that the coefficients of the Schur expansion of Macdonald polynomials are polynomials in the parameters q and t with nonnegative integer coefficients. To prove the positivity conjecture of Macdonald polynomials, Garsia and Haiman introduced certain bigraded S_n modules M_μ (GH93) and Haiman proved that *the bigraded Frobenius characteristic* $\mathcal{F}(M_\mu)$, which by definition is simply the image of the bigraded character of M_μ under the Frobenius map, is the transformed Macdonald polynomials, i.e.,

$$\mathcal{F}_{M_\mu}(x;q,t) = \tilde{H}_\mu(x;q,t),$$

where $\tilde{H}_\mu(x;q,t)$ is the *modified Macdonald polynomials* (HHL05). For the Garsia-Haiman module M_μ, if we define $\mathcal{H}_{h,k}(M_\mu)$ to be the subspace of M_μ spanned by its bihomogeneous elements of degree h in x and degree k in y, we can write a bivariate Hilbert series such as

$$\mathcal{H}_{M_\mu}(q,t) = \sum_{h=0}^{n(\mu)} \sum_{k=0}^{n(\mu')} t^h q^k \mathrm{dim}(\mathcal{H}_{h,k}(M_\mu)).$$

Noting that the degree of the S_n character χ_λ is given by $< p_1^n, s_\lambda >$, where $< \ , \ >$ is the usual inner product on symmetric functions and p_k is the k^{th} power sum, we may write

$$\mathcal{H}_{M_\mu}(q,t) = < p_1^n, \mathcal{F}_{M_\mu} > .$$

Therefore, the coefficient of $x_1 x_2 \cdots x_n$ of $\tilde{H}_\mu(x;q,t)$ gives the Hilbert series of Garsia-Haiman module M_μ.

On the other hand, Haglund, Haiman and Loehr found the combinatorial formula for the monomial expansion of $\tilde{H}_\mu[X;q,t]$ given by (HHL05)

$$\tilde{H}_\mu(x;q,t) = \sum_{\sigma:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\mu,\sigma)} t^{\mathrm{maj}(\mu,\sigma)} x^\sigma \tag{1}$$

where the definitions of inv(μ,σ) and maj(μ,σ) are given in Section 2. This combinatorial formula gives a way of calculating the Hilbert series $\mathcal{H}_{M_\mu}(q,t)$ as a sum of $n!$ monomials. In this paper, we introduce a combinatorial formula for this Hilbert series which can be calculated over only the standard Young tableaux of shape μ when μ has a hook shape. This combinatorial formula is motivated by the formula for the 2 column shape case which is conjectured by Haglund and proved by Garsia and Haglund.

2 The Formula

We begin by recalling definitions of q-analogs :

$$[n]_q = 1 + q + \cdots + q^{n-1},$$
$$[n]_q! = [1]_q \cdots [n]_q.$$

Given a sequence $(\mu_1, \mu_2, \dots)$ of nonincreasing, nonnegative integers with $\sum_i \mu_i = n$, we say μ is a *partition* of n, denoted by either $|\mu| = n$ or $\mu \vdash n$. And let

$$\mathrm{dg}(\mu) = \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : j \leq \mu_i\}$$

be its Young (or Ferrers) diagram, whose elements are called *cells*. For simplicity, we henceforth write μ instead of dg(μ) when it will not cause confusion. A *filling* is a function $\sigma : \mu \to \mathbb{Z}_+$ assigning integer entries to the cells of μ. A *semi-standard Young tableau* is a filling which is weakly increasing along each row of μ and strictly increasing along each column. A semi-standard Young tableau is *standard* if it is a bijection from μ to $[n] = \{1, 2, \dots, n\}$. We define

SYT(μ)={ standard Young tableaux $T : \mu \to [n]$, T bijection }.

A *descent* of σ is a pair of entries $\sigma(u) > \sigma(v)$, where the cell u is immediately above v. Define

$$\mathrm{Des}(\sigma, \mu) = \{u \in \mu : \sigma(u) > \sigma(v) \quad \mathrm{a} \quad \mathrm{descent}\}.$$

The *arm* of a cell $u \in \mu$ is the number of cells strictly to the right of u in the same row, and its *leg* is the number of cells strictly above u in the same column. Then define

$$\mathrm{maj}(\sigma, \mu) = \sum_{u \in \mathrm{Des}(\sigma,\mu)} (\mathrm{leg}(u) + 1).$$

Three cells $u, v, w \in \mu$ are said to form a *triple* if they are situated as shown below,

$$\begin{array}{|c|}\hline u \\ \hline v \\ \hline \end{array} \qquad \begin{array}{|c|}\hline w \\ \hline \end{array}$$

namely, v is directly below u, and w is in the same row as u, to its right. Let σ be a filling and let x, y, z be the entries of σ in the cells of a triple (u, v, w) :

$$\begin{array}{|c|}\hline y \\ \hline x \\ \hline \end{array} \qquad \begin{array}{|c|}\hline z \\ \hline \end{array}$$

If a path starting from the smallest entry to the largest entry rotates in counter clockwise way, then the triple is called an *inversion* triple. Otherwise, it is called a *coinversion* triple. Define

inv(σ, μ)= number of inversion triples in μ with σ filling,
coinv(σ, μ) = number of coinversion triples in μ with σ filling.

For a hook $\mu' = (n - s, 1^s)$, we define

$$\tilde{G}_{\mu'}(q,t) = \sum_{T \in \mathrm{SYT}(\mu')} \prod_{i=1}^{n} [a_i(T)]_t \cdot [s]_q! \left(1 + \sum_{j=1}^{s} q^j t^{\alpha_j(T)}\right)$$

where $a_i(T)$ and $\alpha_i(T)$ are calculated in the following way : to construct $T \in \mathrm{SYT}(\mu')$, staring with the cell containing 1, add cells containing $2, 3, \dots, i$, one at a time. After adding the cell containing i, $a_i(T)$ counts the number of columns which have the same height with the column containing the square just added with i. And $\alpha_j(T)$ is the number of cells in the first row with column height 1 (i.e., strictly to the right of the first column) having bigger element than the element in $(s-j+1,1)$ cell. Then, for $\tilde{F}_{(n-s,1^s)'}(q,t) = \sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma,\mu')} t^{\mathrm{coinv}(\sigma,\mu')}$, where $\tilde{F}_{\mu'}(q,t) = t^{n(\mu)} F_\mu\left(\frac{1}{t},q\right)$[(i)], $n(\mu) = \sum_{i\geq q}(i-1)\mu_i$, we have the following theorem :

Theorem 2.1

$$\tilde{F}_{(n-s,1^s)'}(q,t) = \tilde{G}_{(n-s,1^s)'}(q,t).$$

Example 2.2 ($\mu = (2,1)$ case) *We calculate $\tilde{F}_{(2,1)}(q,t) = \sum_{\sigma\in S_3} q^{maj(\sigma,\mu')} t^{coinv(\sigma,\mu')}$ first.*

$$\begin{array}{|c|c}\hline 1 \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 1 \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 2 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 2 \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\quad \begin{array}{|c|c}\hline 3 \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$$

From the above tableaux, reading from the left, we get

$$\tilde{F}_{(2,1)}(q,t) = t+1+qt+1+qt+q = 2+q+t+2qt. \tag{2}$$

Now we consider $\tilde{G}_{(2,1)}(q,t)$ over the two standard tableaux.

$$T_1 = \begin{array}{|c|c}\hline 2 \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\ , \qquad T_2 = \begin{array}{|c|c}\hline 3 \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

For the SYT T_1, if we add 1, there is only one column with height 1, so we multiply 1. And if we add 2, since it is going on the top of the square with 1, it makes a column with height 2 and so there is only one column with height 2 which gives us factor 1 again. Adding the square with 3, since there is one column with height 1 and the column containing the square with 3 is height 1, it again gives the factor 1. Hence for this tableau, the first factor is 1. For $\alpha_j(T_1)$, we compare the element in the first row to the right of the square with 2, and that is 3 which is bigger than 2, so it gives $\alpha_1(T_1) = 0$ which contributes the factor $(1+qt)$. Hence, T_1 gives the $1\cdot(1+qt)$. Now we consider T_2. If we add the second square with 2, then it makes two columns with height 1, so we get $a_2(T_2) = [2]_t = (1+t)$. Adding the last square gives the factor 1, so the first factor is $(1+t)$. If we consider the second factor, since 3 is the biggest element in this case, the power of t becomes 0 and that makes $\alpha_1(T_2) = 0$. Hence from T_2, we get $(1+t)(1+q)$. If we add two polynomials from two standard young tableaux of shape $\mu = (2,1)$, we get

$$\tilde{G}_{(2,1)}(q,t) = 1\cdot(1+qt) + (1+t)(1+q) = 1+qt+1+t+q+qt = 2+q+t+2qt$$

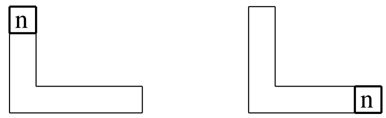
which is equal to (2), i.e., $\tilde{F}_{(2,1)}(q,t) = \tilde{G}_{(2,1)}(q,t)$.

Proof: We first note the Garsia-Haiman recursion for the Hilbert series of the hooks (GH96) : for $\mu = (s+1,1^{n-s-1})$,

$$F_\mu(q,t) = [n-s-1]_t F_{(s+1,1^{n-s-2})} + \binom{n-1}{s} t^{n-s-1}[n-s-1]_t![s]_q! + q[s]_q F_{(s,1^{n-s-1})}. \tag{3}$$

(i) Note that $F_\mu(q,t)$ denotes the Hilbert series of Garsia-Haiman module M_μ.

We derive the recursion formula for $\tilde{G}_{\mu'}(q,t)$ over standard tableaux by fixing the position of the cell with the largest number n :



Let's first start from a SYT of shape $(n-s, 1^{s-1})$ and say

$$\tilde{G}_{(n-s,1^{s-1})}(q,t) = \sum_{T \in \mathrm{SYT}((n-s,1^{s-1}))} \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left(1 + \sum_{j=1}^{s-1} q^j t^{\alpha_j(T)} \right)$$

and put the cell with n on the top of the first column. Then, since there is no other column with height $s+1$, adding the cell with n on the top of the first column gives $a_n = [1]_t$ which doesn't change the first part of the above formula. Now as for the q part, we will have an additional factor of $[s]_q$, and all the q powers in the last parenthesis will be increased by 1 and it will have additional q from the top cell of the first column. The exponent of t with that q is 0 since n is the largest possible number. Hence, for the first case tableaux, the formula we get becomes

$$\sum_{T \in \mathrm{SYT}((n-s,1^{s-1}))} \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left[1 + q \left(1 + \sum_{j=1}^{s-1} q^j t^{\alpha_j(T)} \right) \right]$$

$$= \left(\sum_{T \in \mathrm{SYT}(n-s,1^{s-1})} \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \right) + q[s]_q \tilde{G}_{(n-s,1^{s-1})}(q,t)$$

and in terms of $\tilde{G}_{(n-s,1^{s-1})}(q,t)$, this is equal to

$$[s]_q! \tilde{G}_{(n-s,1^{s-1})}(0,t) + q[s]_q \tilde{G}_{(n-s,1^{s-1})}(q,t).$$

In the second case, we start from a SYT of shape $(n-s-1, 1^s)$ and add the cell with n in the end of the first row. This increases the number of columns with height 1 from $n-s-2$ to $n-s-1$, so contributes the t factor $a_n = [n-s-1]_t$. Since it doesn't affect the first column, we don't get any extra q factor, but having the largest number n in the first row increases all the α_j's by 1. In other words, if we let the formula for the SYT of shape $(n-s-1, 1^s)$ as

$$\tilde{G}_{(n-s-1,1^s)}(q,t) = \sum_{T \in \mathrm{SYT}((n-s-1,1^s))} \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left(1 + \sum_{j=1}^{s} q^j t^{\alpha_j(T)} \right)$$

then by adding the cell with n in the end of the first row, it changes to

$$\sum_{T \in \mathrm{SYT}(n-s-1,1^s)} [n-s-1]_t \cdot \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left(1 + \sum_{j=1}^{s-1} q^j t^{\alpha_j(T)+1} \right)$$

$$= \sum_{T \in \mathrm{SYT}(n-s-1,1^s)} [n-s-1]_t \cdot \prod_{i=1}^{n-1} [a_i(T)]_t \cdot [s]_q! \left[t \left(1 + \sum_{j=1}^{s} q^j t^{\alpha_j(T)} \right) + (1-t) \right].$$

Thus, in terms of $\tilde{G}_{(n-s-1,1^s)}$, this can be expressed as

$$t[n-s-1]_t \tilde{G}_{(n-s-1,1^s)}(q,t) + (1-t)[n-s-1]_t [s]_q! \tilde{G}_{(n-s-1,1^s)}(0,t).$$

In conclusion, the recursive formula is the following :

$$\begin{aligned} \tilde{G}_{(n-s,1^s)}(q,t) &= q[s]_q \tilde{G}_{(n-s,1^{s-1})}(q,t) + t[n-s-1]_t \tilde{G}_{(n-s-1,1^s)}(q,t) \\ &+ [s]_q!(\tilde{G}_{(n-s,1^{s-1})}(0,t) + (1-t^{n-s-1})\tilde{G}_{(n-s-1,1^s)}(0,t)). \end{aligned}$$

To compare it to the Hilbert series $F_\mu(q,t)$, we do the transformations $\tilde{G}_{\mu'}(q,t) = G_\mu\left(\frac{1}{t}, q\right) t^{n(\mu)}$, and we get the recursion formula for $G_\mu(q,t)$

$$\begin{aligned} G_{(s+1,1^{n-s-1})}(q,t) &= q[s]_q G_{(s,1^{n-s-1})}(q,t) + [n-s-1]_t G_{(s+1,1^{n-s-2})}(q,t) \\ &+ [s]_q!(G_{(s,1^{n-s-1})}(0,t) + (t^{n-s-1}-1) G_{(s+1,1^{n-s-2})}(0,t)). \end{aligned}$$

By calculation, we get

$$G_{(s,1^{n-s-1})}(0,t) + G_{(s+1,1^{n-s-2})}(0,t) = [n-s-1]_t! [s]_q! \binom{n-1}{s} t^{n-s-1}.$$

Thus the recursion formula for $G_\mu(q,t)$ simplifies to

$$\begin{aligned} G_{(s+1,1^{n-s-1})}(q,t) &= q[s]_q G_{(s,1^{n-s-1})}(q,t) + [n-s-1]_t G_{(s+1,1^{n-s-2})}(q,t) \\ &+ [n-s-1]_t! [s]_q! \binom{n-1}{s} t^{n-s-1}. \end{aligned} \tag{4}$$

We compare two recursions (3) and (4) and we can confirm that $F_\mu(q,t)$ and $G_\mu(q,t)$ both satisfy the same recursion. Based on the fact that $F_{(2,1)}(q,t) = G_{(2,1)}(q,t)$ (note that we confirmed it in the previous example), we conclude that $F_\mu(q,t) = G_\mu(q,t)$ which implies $\tilde{F}_{\mu'}(q,t) = \tilde{G}_{\mu'}(q,t)$. □

Remark 2.3 *We can construct a combinatorial way of calculating $F_\mu(q,t)$ directly over the standard Young tableaux :*

$$F_\mu(q,t) = \sum_{T \in SYT(\mu)} \prod_{i=1}^{n} [a_i(T)]_t [\mu_1 - 1]_q! \left(\sum_{j=1}^{\mu_1 - 1} q^{j-1} t^{b_j(T)} + q^{\mu_1 - 1} \right)$$

where $a_i(T)$ counts the number of rows having the same width with the row containing i as adding the cell i, from 1 to n, and $b_j(T)$ counts the number of cells in the first column in rows strictly higher than row 1 containing bigger numbers than the element in the cell $(1, j+1)$.

3 Association with the Fillings

For the association with fillings, we are going to introduce a *grouping table*. For the general hook of shape $\mu = (s, 1^{n-s})$, the way that we make the *grouping table* is the following : first we choose s many numbers including 1 and n, in all possible ways. Note that for this we have $\binom{n-2}{s-2}$ many choices. The unchosen $n-s$ many numbers will be placed in the first column above the $(1,1)$ cell, in all possible ways, and the chosen s many numbers will come in the first row, in all possible ways. Then, this set of fillings will correspond to one standard Young tableau. We read out the polynomial corresponding the standard tableau in the following way : keeping in mind that we are calculating $q^{\text{inv}}t^{\text{maj}}$, since the permutations in the first column without including the $(1,1)$ cell give $[n-s]_t!$ factor and the permutations in the first row without the $(1,1)$ cell give $[s-1]_q!$ factor, we just consider s many different cases as we change the element coming in the $(1,1)$ cell by the chosen ones and each will give $q^a t^b [n-s]_t! [s-1]_q!$ where a is the number of elements in the first row to the right of $(1,1)$ cell which are smaller than the element in the $(1,1)$ cell, and b is the number of elements in the first column above $(1,1)$ cell which are bigger than the one in the $(1,1)$ cell. We repeat choosing s many numbers within 1 to $n-k$ including 1 and $n-k$, where k changes from 1 to $n-s+1$. Then the rest of the procedure will be the same. We consider an example for $\mu = (3,1,1)$. The corresponding grouping table is the following :

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\hline
\times & \times & \circ & \circ & \times \\
\hline
\times & \circ & \times & \circ & \times \\
\hline
\times & \circ & \circ & \times & \times \\
\hline
\times & \times & \circ & \times & \circ \\
\circ & \times & \times & \circ & \times \\
\hline
\times & \circ & \times & \times & \circ \\
\circ & \times & \circ & \times & \times \\
\hline
\times & \times & \times & \circ & \circ \\
\circ & \times & \times & \times & \circ \\
\circ & \circ & \times & \times & \times \\
\hline
\end{array}
$$

The rows between dividing lines will be grouped in the same set and the entries marked by $\times$ will come on the row and permute in all possible ways, and the entries marked by $\circ$ will come on the column not including the $(1,1)$ cell, and permute all possible ways. All the fillings obtained by these permutations of row and column will correspond to one standard tableau. For instance, from the first grouping $\times \quad \times \quad \circ \quad \circ \quad \times$, we get 12 different fillings.

$$
\begin{array}{l}4\\3\\1\,2\,5\end{array}\quad
\begin{array}{l}4\\3\\1\,5\,2\end{array}\quad
\begin{array}{l}3\\4\\1\,2\,5\end{array}\quad
\begin{array}{l}3\\4\\1\,5\,2\end{array}\quad , \quad
\begin{array}{l}4\\3\\2\,1\,5\end{array}\quad
\begin{array}{l}4\\3\\2\,5\,1\end{array}\quad
\begin{array}{l}3\\4\\2\,1\,5\end{array}\quad
\begin{array}{l}3\\4\\2\,5\,1\end{array}
$$

$$
\begin{array}{l}4\\3\\5\,1\,2\end{array}\quad
\begin{array}{l}4\\3\\5\,1\,3\end{array}\quad
\begin{array}{l}3\\4\\5\,1\,2\end{array}\quad
\begin{array}{l}3\\4\\5\,2\,1\end{array}
$$

If we calculate $t^{\text{maj}}q^{\text{inv}}$ from the left of the top line, we get

$$t^3 + qt^3 + t^2 + qt^2 + qt^3 + q^2t^3 + qt^2 + q^2t^2 + q^2t + q^3t + q^2 + q^3 = (1+t)(1+q)(t^2 + qt^2 + q^2)$$

which will correspond to the standard tableau

$$\begin{ytableau} 5 \\ 4 \\ 1 & 2 & 3 \end{ytableau}$$

Noting that the permutations of either column or row not including the $(1,1)$ cell will give $(1+t)$ from the permutations of the column, and $(1+q)$ from the permutations of the row, the monomial coefficient factors $q^a t^b$ multiplied by $(1+t)(1+q)$ will be calculated by the following way : say k is placed in the $(1,1)$ cell, then a is the number of chosen elements (i.e., $\times$ marked in the table) strictly smaller than k (or number of $\times$'s strictly to the left of k), and b is the number of unchosen elements (i.e., $\circ$ marked in the table) strictly bigger than k (or number of $\circ$'s strictly to the right of k). Following this method, we read out the following polynomials from the grouping table, from the second line

$$\begin{aligned}
&(1+t)(1+q)(t^2+qt+q^2),\\
&(1+t)(1+q)(t^2+q+q^2),\\
&(1+t)^2(1+q)(t+qt+q^2),\\
&(1+t)^2(1+q)(t+q+q^2),\\
&(1+t)(1+t+t^2)(1+q)(1+q+q^2).
\end{aligned}$$

And the corresponding standard tableaux are from the top

$$\begin{ytableau} 5 \\ 3 \\ 1 & 2 & 4 \end{ytableau} \qquad \begin{ytableau} 4 \\ 3 \\ 1 & 2 & 5 \end{ytableau} \qquad \begin{ytableau} 5 \\ 2 \\ 1 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 4 \\ 2 \\ 1 & 3 & 5 \end{ytableau} \qquad \begin{ytableau} 3 \\ 2 \\ 1 & 4 & 5 \end{ytableau}$$

We use the modified Garsia-Procesi tree (GP92) to find the corresponding standard Young tableau given the polynomial from the group of fillings. For the running example for $\mu = (3,1,1)$, the modified Garsia-Procesi tree is given in the Figure 1. The way of finding the standard tableau from the modified Garsia-Procesi tree is the following : given the polynomial from the fillings, either of the form $(1+t)(1+q)(t^a+qt^b+q^2)$ or $(1+t)^2(1+q)(t^a+qt^b+q^2)$, compare (a,b) with the numbers on the right-top of bottom leaves in the tree. Finding the same numbers in the tree, trace back the tree from the bottom to top filling the cells from 1 to n (here $n=5$) as the tree adds the cells. Then on the top of the tree, we get the corresponding standard tableau giving exactly the same polynomial as we calculated before.

Proposition 3.1 *The grouping table gives the complete Hilbert series.*

Proof: By the way of constructing the grouping table that we don't count the same filling multiple times, we only need to check that the number of fillings that are counted in the grouping table is $n!$. From the permutations on the first column and the first row not including $(1,1)$ we count $(s-1)!(n-s)!$. In the grouping table, the set with k lines will be $\binom{n-(k-1)}{s-2}$ many and each line represents s different fillings. Adding up them all, the number of fillings that the grouping table counts is

$$s!(n-s)!\left(\binom{n-2}{s-1}+2\binom{n-3}{s-2}+\cdots+(n-s)\binom{s-1}{s-2}+(n-s+1)\binom{s-2}{s-2}\right).$$

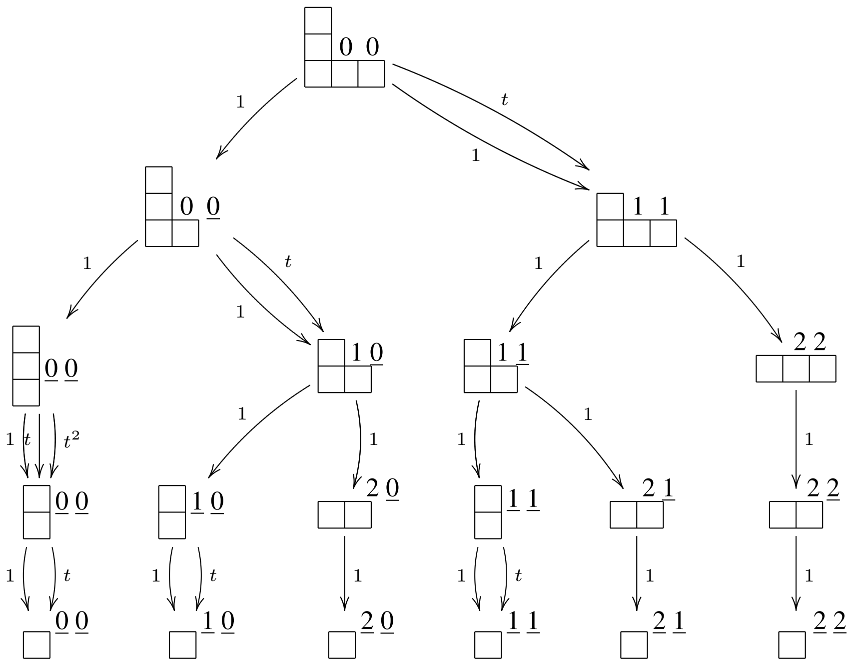


Fig. 1: The modified Garsia-Procesi tree for $\mu = (3, 1, 1)$.

Therefore, we want to show the following identity

$$\binom{n}{s} = \binom{n-2}{s-1} + 2\binom{n-3}{s-2} + \cdots + (n-s)\binom{s-1}{s-2} + (n-s+1)\binom{s-2}{s-2}. \tag{5}$$

Note the identity

$$\sum_{j=k}^{n-1} \binom{j}{k} = \binom{n}{k}. \tag{6}$$

Then, by applying (6) twice, the right hand side of (5) becomes

$$\begin{aligned} &\binom{n-2}{s-1} + 2\binom{n-3}{s-2} + \cdots + (n-s)\binom{s-1}{s-2} + (n-s+1)\binom{s-2}{s-2} \\ = \quad &\left(\binom{n-2}{s-1} + \binom{n-3}{s-2} + \cdots + \binom{s-1}{s-2} + \binom{s-2}{s-2}\right) \\ &+ \left(\binom{n-3}{s-2} + \cdots + \binom{s-1}{s-2} + \binom{s-2}{s-2}\right) + \cdots + \binom{s-2}{s-2} \end{aligned}$$

$$= \binom{n-1}{s-1} + \binom{n-2}{s-1} + \cdots + \binom{s}{s-1} + \binom{s-1}{s-1}$$
$$= \binom{n}{s},$$

which is the left hand side of (5). This shows that we considered all $n!$ possible fillings, hence the grouping table gives the complete Hilbert series. □

Proposition 3.2 *The grouping table gives the association with the fillings corresponding to the standard tableaux.*

Proof: Remind that for the hook of shape $\mu = (s, 1^{n-s})$, the Hilbert series will be expressed as the following.

$$F_\mu(q,t) = [n-s]_t![s-1]_q! \sum_{j_1=2}^{n-s+1} \cdots \sum_{j_{s-1}=j_{s-2}+1}^{n} [j_1-1]_t(t^{b_1} + qt^{b_2} + \cdots + q^{s-2}t^{b_{s-1}} + q^{s-1}) \quad (7)$$

where b_i is the number of elements in the first column above the $(1,1)$ cell which are bigger than j_i. We start from the case where we have 1-lined set in the grouping table. Note that by knowing the tail part of the standard tableaux in the Garsia-Procesi tree, we know that in this case, the standard tableaux look like Figure 2. Then there are $\binom{n-2}{s-2}$ possibilities for the choice of the rest of the $(s-2)$ elements in the

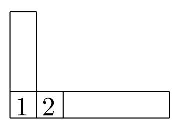


Fig. 2: SYT corresponding to 1-lined set in grouping table

first row. Let's say $1, a_1, \ldots, a_{s-2}, n$, $a_i < a_j$ for $i < j$, are chosen in the grouping table. If 1 comes in the $(1,1)$ cell, then all the elements in the first row to the right and all the elements in the first column above 1 are bigger than 1, so the monomial factor will be t^{n-s}. And if a_1 comes in the $(1,1)$ cell, then we gain one power of q since 1 will be to the right of 2 in the first row, and the power of t will depend on a_i. Similarly, as a_i comes in the $(1,1)$ cell, as i gets larger by 1, the power of q will be increased by 1, and finally when n comes in the $(1,1)$ cell, since there are no bigger elements than n, it doesn't have any t powers and the power of q will be $s-1$, since all the rest of the chosen numbers are smaller than n. So this case gives the following form of polynomial

$$[n-s]_t![s-1]_q!(t^{n-s} + qt^{b_1} + \cdots + q^{s-2}t^{b_{s-2}} + q^{s-1})$$

where b_i is the number of elements in the unchosen ones which are bigger than a_i. Note that the fact that we don't have any repeated lines in the grouping table guarantees that we don't get the same polynomials multiple times, since the power of t, b_i, is the number of unchosen ones to the right of a_i in the grouping table. Secondly, consider the two-lined sets in the grouping table. Again, by the tail looking of the

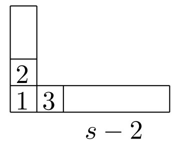


Fig. 3: SYT corresponding to 2-lined set in grouping table

standard tableaux in the tree, we know that this case takes care of the standard tableaux of the kind in Figure 3. Then there are $\binom{n-3}{s-2}$ many choices for the $s-2$ elements in the rest of the first row. Let's say we have chosen $1, a_1, \ldots, a_{s-2}, n-1$ in the first line. Then, by the construction, $2, a_1+1, \ldots, a_{s-2}+1, n$ will be chosen as well in the second line. Notice that in the grouping table, the lost of one $\circ$ under n in the

1	2	a_1	$\ldots$	a_{s-2}	$n-1$	n
$\times$	[			]	$\times$	$\circ$
$\circ$	$\times$	[			]	$\times$

Tab. 1: 2-lined set of the grouping table.

second line means that all the monomial factors from the first line have 1 more power of t than the ones from the second line, hence we get $(1+t)$ factor after we sum them up all. So the polynomial that we get from this case is the following

$$[n-s]_t![s-1]_q!(1+t)(t^{n-s-1} + qt^{b_1} + \cdots + q^{s-2}t^{b_{s-s}} + q^{s-1})$$

where b_i is the number of circles in the grouping table to the right of $a_i + 1$. Now, we consider a general case when we have k-lined set in the grouping table. This case takes care of the form of standard tableaux in Figure 4. Again, there are $\binom{n-(k+1)}{s-2}$ different possibilities for the different choice of numbers

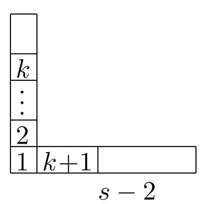


Fig. 4: SYT corresponding to k-lined set in grouping table

coming in the rest of the first column. Let's say we choose $1, a_1, \ldots, a_{s-2}, n-k+1$ in the first line, then $2, a_1+1, \ldots, a_{s-2}+1, n-k+2$ will be chosen in the second line, and finally in the k^{th} line, $k, a'_1, \ldots, a'_{s-2}, n$ will be chosen where $a'_i = a_i + k - 1$. Keeping in mind that the right most consecutive circles in the same line will give the common t powers and having the same number of circles and the same pattern means that the k lines have the common factor which comes from the k^{th} line, this case gives the polynomials as follows

$$[n-s]_t![s-1]_q![k]_t(t^{n-s-(k-1)} + qt^{b_1} + \cdots + q^{s-2}t^{b_{s-2}} + q^{s-1})$$

where b_i is the number of circles to the right of a'_i. In the last set of the grouping table, we choose $1, 2, \dots, s$ in the first line and $n-s+1, \dots, n$ in the last (which is $(n-s+1)^{\text{th}}$) line. This whole set of fillings will correspond to the standard tableaux of the kind in Figure 5. The fact that there is no $\circ$ between

$$\begin{array}{|c|cccc}
\hline
n-s+1 & & & & \\
\hline
\vdots & & & & \\
\hline
2 & & & & \\
\hline
1 & \multicolumn{1}{|c|}{n-s+2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{n} \\
\hline
\end{array}$$

Fig. 5: SYT corresponding to the last line in grouping table

$times$ marks means there is no t powers combined with q and the consecutive circles to the right of the chosen elements will give the common t factors. Adding up them all will give

$$[n-s]_t![s-1]_q![n-s+1]_t(1+q+q^2+\cdots+q^{s-1}).$$

By comparing the polynomials coming from the grouping table and the polynomials added in the Hilbert series, we can confirm that the sets in the grouping table give the polynomials corresponding standard Young tableaux. Since we know that the grouping table gives the complete Hilbert series by Proposition 3.1, we conclude that the grouping table gives the association with fillings to the standard tableaux. $\square$

Remark 3.3 *We note that the grouping table doesn't give the information about what the right corresponding standard tableau is. But since we can know which polynomial the standard tableau gives, once the modified Garsia-Procesi tree is given, by using the powers of t combined with q's, we can trace back the tree to construct the corresponding the standard tableau, as we did in the example for $\mu = (3,1,1)$.*

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