On the diagonal ideal of $(\mathbb{C}^2)^n$ and q,t-Catalan numbers

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Abstract. Let I_n be the (big) diagonal ideal of $(\mathbb{C}^2)^n$. Haiman proved that the q,t-Catalan number is the Hilbert series of the graded vector space $M_n = \bigoplus_{d_1,d_2}(M_n)_{d_1,d_2}$ spanned by a minimal set of generators for I_n. We give simple upper bounds on dim $(M_n)_{d_1,d_2}$ in terms of partition numbers, and find all bi-degrees (d_1, d_2) such that dim $(M_n)_{d_1,d_2}$ achieve the upper bounds. For such bi-degrees, we also find explicit bases for $(M_n)_{d_1,d_2}$.

Résumé. Soit I_n l'idéal de la (grande) diagonale de $(\mathbb{C}^2)^n$. Haiman a démontré que le q,t-nombre de Catalan est la série de Hilbert de l'espace vectoriel gradué $M_n = \bigoplus_{d_1,d_2}(M_n)_{d_1,d_2}$ engendré par un ensemble minimal de générateurs de I_n. Nous obtenons des bornes supérieures simples pour dim $(M_n)_{d_1,d_2}$ en termes de nombres de partitions, ainsi que tous les bi-degrés (d_1, d_2) pour lesquels ces bornes supérieures sont atteintes. Pour ces bi–degrés, nous trouvons aussi des bases explicites de $(M_n)_{d_1,d_2}$.

Keywords: q,t-Catalan number, diagonal ideal

1 introduction

1.1 Background

The goal of this paper is to study the q,t-Catalan numbers and the (thick) diagonal ideal in $(\mathbb{C}^2)^n$, and discuss some technique that we have developed recently.

Let n be a positive integer. Consider the set of n-tuples $\{(x_i, y_i)\}_{1\leq i\leq n}$ in the plane $\mathbb{C}^2$. They form an affine space $(\mathbb{C}^2)^n$ with coordinate ring $\mathbb{C}[\mathbf{x},\mathbf{y}] = \mathbb{C}[x_1, y_1, ..., x_n, y_n]$. There is a natural symmetric group S_n acting on $\mathbb{C}[\mathbf{x},\mathbf{y}]$ by permuting the coordinates in $\mathbf{x},\mathbf{y}$ simultaneously. With this group action, a polynomial $f \in \mathbb{C}[\mathbf{x},\mathbf{y}]$ is called *alternating* if

$$\sigma(f) = \mathrm{sgn}(\sigma)f \quad \text{for all } \sigma \in S_n.$$

Define $\mathbb{C}[\mathbf{x},\mathbf{y}]^\epsilon$ to be the vector space of alternating polynomials in $\mathbb{C}[\mathbf{x},\mathbf{y}]$.

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There is a more combinatorial way to describe $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\epsilon}$. Denote by $\mathbb{N}$ the set of nonnegative integers. Let $\mathfrak{D}_n$ be the set of n-tuples $D = \{(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n)\} \subset \mathbb{N} \times \mathbb{N}$. For $D \in \mathfrak{D}_n$, define

$$\Delta(D) := \det \begin{bmatrix} x_1^{\alpha_1} y_1^{\beta_1} & x_1^{\alpha_2} y_1^{\beta_2} & ... & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & x_n^{\alpha_2} y_n^{\beta_2} & ... & x_n^{\alpha_n} y_n^{\beta_n} \end{bmatrix}$$

Then $\{\Delta(D)\}_{D \in \mathfrak{D}_n}$ forms a basis for the $\mathbb{C}$-vector space $\mathbb{C}[\mathbf{x}, \mathbf{y}]^{\epsilon}$.

It is easy to see that any alternating polynomial vanishes on the thick diagonal of $(\mathbb{C}^2)^n$. (By thick diagonal we mean the set of n-tuples of points in $\mathbb{C}^2$ where at least two points coincide.) A theorem of Haiman asserts that the converse is also true: any polynomial that vanishes on the diagonal of $(\mathbb{C}^2)^n$ can be generated by alternating polynomials, i.e.

$$\bigcap_{1 \leq i < j \leq n} (x_i - x_j, y_i - y_j) = \text{ideal generated by } \Delta(D)\text{'s}.$$

We call the above ideal the **diagonal ideal** and denote it by I_n. the number of minimal generators of I_n, which is the same as the dimension of the vector space $M_n = I_n/(\mathbf{x}, \mathbf{y})I_n$, is equal to the n-th Catalan number. The space M_n is doubly graded as $\oplus_{d_1, d_2} (M_n)_{d_1, d_2}$. The q, t-Catalan number, originally introduced by A.M.Garsia and M.Haiman in [4], can be defined as

$$C_n(q, t) = \sum_{d_1, d_2} t^{d_1} q^{d_2} \dim(M_n)_{d_1, d_2}.$$

The q, t-Catalan number $C_n(q, t)$ also has a combinatorial interpretation using Dyck paths. To be more precise, take the $n \times n$ square whose southwest corner is $(0, 0)$ and northeast corner is (n, n). Let $\mathcal{D}_n$ be the collection of Dyck paths, i.e. lattice paths from $(0, 0)$ to (n, n) that proceed by NORTH or EAST steps and never go below the diagonal. For any Dyck path Π, let $a_i(\Pi)$ be the number of squares in the i-th row that lie in the region bounded by Π and the diagonal. A.M.Garsia and J.Haglund ([2], [3]) among others showed that

$$C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{\text{dinv}(\Pi)},$$

where

$$\text{dinv}(\Pi) := |\{(i, j) \,|\, i < j \text{ and } a_i(\Pi) = a_j(\Pi)\}| + |\{(i, j) \,|\, i < j \text{ and } a_i(\Pi) + 1 = a_j(\Pi)\}|.$$

Haiman posed a question asking for a rule that associate to each Dyck path Π an element $D(\Pi) \in \mathfrak{D}_n$ such that $\deg_{\mathbf{x}} \Delta(D(\Pi)) = \text{area}(\Pi)$, $\deg_{\mathbf{y}} \Delta(D(\Pi)) = \text{dinv}(\Pi)$, and that the set $\{\Delta(D(\Pi))\}$ generates I_n. The last condition is equivalent to requiring the images $\{\overline{\Delta(D(\Pi))}\}$ form a basis of M_n). It is natural to ask the following more general question:

Question 1.1 *Given a bi-degree (d_1, d_2), is there a combinatorially significant construction of the basis for each $(M_n)_{d_1, d_2}$?*

1.2 Main results

This paper initiates the approach to the study of M_n by comparing it with $M_{n'}$ for a large integer n'. On the one hand, there is a natural map $M_n \to M_{n'}$ for any $n' > n$. On the other hand, for n' sufficiently large, the basis of $(M_{n'})_{d_1',d_2}$ becomes "stable" if we fix d_2 and fix

$$k = \binom{n}{2} - d_1 - d_2 = \binom{n'}{2} - d_1' - d_2.$$

Therefore we can take the "limit" of such basis for $n' \to \infty$. This basis is indexed by the partitions of k. As a consequence, $(M_{n'})_{d_1',d_2}$ can be imbedded as a subspace of the polynomial ring with infinite many variables $\mathbb{C}[\rho_1, \rho_2, \dots]$. The induced map

$$\bar{\varphi} : (M_n)_{d_1,d_2} \to \mathbb{C}[\rho_1, \rho_2, \dots],$$

which will be defined explicitly in subsection 1.2.3, provides a powerful tool to study M_n.

1.2.1 Asymptotic behavior when $k \ll n$

We shall show that if $k \ll n$, then $(M_n)_{d_1,d_2}$ has a basis $\{\overline{\Delta(D)}\}$ where D are so-called **minimal staircase forms** that will be defined later.

The essential step is to observe the following three linear relations that turn the questions into combinatorial games First we introduce some notations.

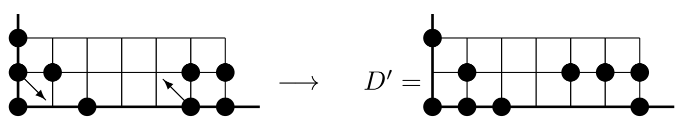
• For $D = \{P_1, \dots, P_n\} \in \mathfrak{D}_n$ where $P_i = (\alpha_i, \beta_i)$, define $|P_i| = \alpha_i + \beta_i$.

Relation 1. Given positive integers $1 \le i \ne j \le n$ such that $|P_i| = i-1$, $|P_{i+1}| = i$, $|P_j| = j-1$, $|P_{j+1}| = j$, $\beta_i > 0$, $\alpha_j > 0$ (we assume $|P_{n+1}| = n$). Let D' be obtained from D by moving P_i to southeast and P_j to northwest, i.e.

$$D' = \{P_1, \dots, P_{i-1}, P_i + (1,-1), P_{i+1}, \dots, P_{j-1}, P_j + (-1,1), P_{j+1}, \dots, P_n\}.$$

Then $\overline{\Delta(D)} = \overline{\Delta(D')}$.

Example: $n = 9, i = 2, j = 6$.

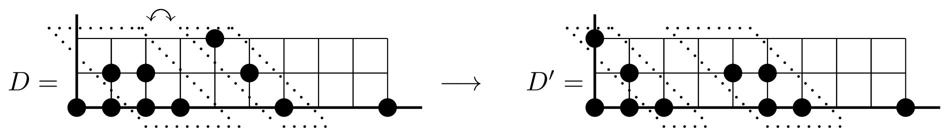
$D =$  $\longrightarrow$ $D' =$

Relation 2. Given positive integers h, ℓ and m such that $2 \le h < h+\ell+m \le n+1$, $|P_h| = h-1$, $|P_{h+\ell}| = h+\ell-1$, $|P_{h+\ell+m}| = h+\ell+m-1$ (by convention, the last equality holds if $h+\ell+m = n+1$) and $\alpha_{h+\ell}, \dots, \alpha_{h+\ell+m-1} \ge \ell$. Let D' be obtained from D by moving the m points $P_{h+\ell}, \dots, P_{h+\ell+m-1}$ to the left by ℓ units and moving the ℓ points $P_h, \dots, P_{h+\ell-1}$ to the right by m units, i.e.

$$\begin{aligned} D' = \{&P_1, P_2, \dots, P_{h-1}, P_{h+\ell} - (\ell,0), P_{h+\ell+1} - (\ell,0), \dots, P_{h+\ell+m-1} - (\ell,0),\\ &P_h + (m,0), P_{h+1} + (m,0), \dots, P_{h+\ell-1} + (m,0), P_{h+\ell+m}, \dots, P_n\}. \end{aligned}$$

Then $\overline{\Delta(D)} = \overline{\Delta(D')}$.

Example: $n = 10, h = 3, \ell = 4, m = 3$.

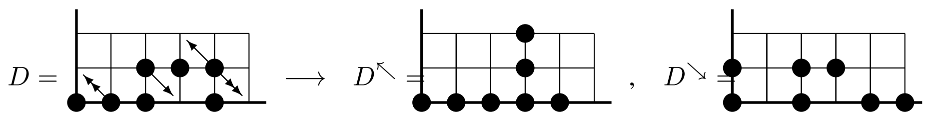


Relation 3. Given positive integers j and s. Suppose P_{s_0} is the last point in D satisfying $|P_i| = i - 1$. Define $j = (s_0 - 1 - |P_{s_0}|) + (s_0 - |P_{s_0+1}|) + \cdots + (n - 1 - |P_n|)$. Suppose $|P_i| = i - 1$ for $1 \le i \le j + 2$, $P_2 = (1, 0)$, $s_0 \le s \le n$, and $\alpha_s, \beta_s \ge 1$. Let

$$D^{\nwarrow} = \{P_1, \ldots, P_{j+1}, P_{j+2} + (1, -1), P_{j+3}, \ldots, P_{s-1}, P_s + (-1, 1), P_{s+1}, \ldots, P_n\},$$
$$D^{\searrow} = \{P_1, (0, 1), P_3, \ldots, P_{s-1}, P_s + (1, -1), P_{s+2}, \ldots, P_n\}.$$

Then $2\overline{\Delta(D)} = \overline{\Delta(D^{\nwarrow})} + \overline{\Delta(D^{\searrow})}$.

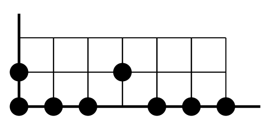
Example: $n = 9, i = 2, j = 6$.



We call $D = \{P_1, \ldots, P_n\}$ a *minimal staircase form* if $|P_i| = i - 1$ or $i - 2$ for every $1 \le i \le n$. For a minimal staircase form D, let $\{i_1 < i_2 < \cdots < i_\ell\}$ be the set of i's such that $|P_i| = i - 1$, we define the *partition type* of D to be the partition of $(\binom{n}{2} - \sum |P_i|)$ consisting of all the positive integers in the sequence

$$(i_1 - 1, i_2 - i_1 - 1, i_3 - i_2 - 1, \ldots, i_\ell - i_{\ell-1} - 1, n - i_\ell).$$

Example: Let $n = 8$ and $D = \{P_1, \ldots, P_8\}$ satisfying $(|P_1|, \ldots, |P_8|) = (0, 1, 1, 2, 4, 4, 5, 6)$. Then D is a minimal staircase form. The set $\{i \,\big|\, |P_i| = i - 1\}$ equals $\{1, 2, 5\}$. The positive integers in the sequence $(1 - 1, 2 - 1 - 1, 5 - 2 - 1, 8 - 5)$ are $(2, 3)$, so the partition type of D is $(2, 3)$.



Let $p(k)$ denote the number of partitions of an integer k and Π_k denote the set of partitions of k.

Theorem 1.2 *Let k be any positive integer. There are positive constants $c_1 = 8k + 5, c_2 = 2k + 1$ such that the following holds:*

For integers n, d_1, d_2 satisfying $n \ge c_1$, $d_1 \ge c_2 n$, $d_2 \ge c_2 n$ and $d_1 + d_2 = \binom{n}{2} - k$, the vector space $(M_n)_{d_1,d_2}$ has dimension $p(k)$, and the $p(k)$ elements

$$\big\{\text{a minimal staircase form of bi-degree } (d_1, d_2) \text{ and of partition type } \mu\big\}_{\mu \in \Pi_k}$$

form a basis of $(M_n)_{d_1,d_2}$.

Note that N.Bergeron and Z.Chen have found explicit bases for $(M_n)_{d_1,d_2}$ for certain bi-degrees using a different method [1].

1.2.2 For arbitrary k and n

Denote by $p(k)$ the partition number of k and by convention $p(0) = 1$ and $p(k) = 0$ for $k < 0$. Denote by $p(b, k)$ the partition number of k into no more than b parts, and by convention $p(0, k) = 0$ for $k > 0$, $p(b, 0) = 1$ for $b \geq 0$. One of our main results is as follows.

Theorem 1.3 *Let d_1, d_2 be non-negative integers d_1, d_2 with $d_1 + d_2 \leq \binom{n}{2}$. Define $k = \binom{n}{2} - d_1 - d_2$ and $\delta = \min(d_1, d_2)$. Then the coefficient of $q^{d_1} t^{d_2}$ in $C_n(q, t)$ is less than or equal to $p(\delta, k)$, and the equality holds if and only if one the following conditions holds:*

- $k \leq n - 3$, *or*
- $k = n - 2$ *and* $\delta = 1$, *or*
- $\delta = 0$.

This theorem is a consequence of Theorem C. It contains [8, Theorem 6] and a result of N.Bergeron and Z.Chen [1, Corollary 8.3.1] as special cases. In fact it proves [8, Conjecture 8]. We feel that the coefficient of $q^{d_1} t^{d_2}$ for general k can also be expressed in terms of partition numbers, only that the expression might be complicated. For example, we give the following conjecture which is verified for $6 \leq n \leq 10$.

Conjecture. Let n, d_1, d_2, δ, k be as in Theorem 1.3. If $n - 2 \leq k \leq 2n - 8$ and $\delta \geq k$, then the coefficient of $q^{d_1} t^{d_2}$ in $C_n(q, t)$ equals

$$p(k) - 2[p(0) + p(1) + \cdots + p(k - n + 1)] - p(k - n + 2).$$

As a corollary of Theorem 1.3 , we can compute some higher degree terms of the specialization at $t = q$.

Corollary 1.4

$$C_n(q, q) = \sum_{k=0}^{n-3} \left(p(k) \left(\binom{n}{2} - 3k + 1 \right) + 2 \sum_{i=1}^{k-1} p(i, k) \right) q^{\binom{n}{2} - k} + (\textit{lower degree terms}).$$

The following theorem immediately implies Theorem 1.3.

Theorem 1.5 *Let d_1, d_2 be non-negative integers d_1, d_2 with $d_1 + d_2 \leq \binom{n}{2}$. Define $k = \binom{n}{2} - d_1 - d_2$ and $\delta = \min(d_1, d_2)$. Then $\dim(M_n)_{d_1, d_2} \leq p(\delta, k)$, and the equality holds if and only if one the following conditions holds:*

- $k \leq n - 3$, *or*
- $k = n - 2$ *and* $\delta = 1$, *or*
- $\delta = 0$.

In case the equality holds, there is an explicit construction of a basis of $(M_n)_{d_1, d_2}$.

The idea of the construction of the basis in the above theorem consists of two parts:
(1) Prove that

$$\dim(M_n)_{d_1,d_2} \le p(\delta, k)$$

using a new characterization of q,t-Catalan numbers. The characterization is as follows, and is discovered independently by A. Woo [10].

Let $\mathfrak{D}_n^{catalan}$ be the set consisting of $D \subset \mathbb{N} \times \mathbb{N}$, where D contains n points satisfying the following conditions.

(a) If $(p, 0) \in D$ then $(i, 0) \in D, \forall i \in [0, p]$.

(b) For any $p \in \mathbb{N}$,

$$\#\{j \,|\, (p+1, j) \in D\} + \#\{j \,|\, (p, j) \in D\} \ge \max\{j \,|\, (p, j) \in D\} + 1.$$

(If $\{j \,|\, (p, j) \in D\} = \emptyset$, then we require that no point $(i, j) \in D$ satisfies $i \ge p$.) Denote by $\deg_x D$ (resp. $\deg_y D$) the sum of the first (resp. second) components of the n points in D.

Proposition 1.6 *The coefficient of $q^{d_1}t^{d_2}$ in the q,t-Catalan number $C_n(q,t)$ is equal to*

$$\#\{D \in \mathfrak{D}_n^{catalan} \mid \deg_x D = d_1, \deg_y D = d_2\}.$$

(2) Construct a set of $p(\delta, k)$ linearly independent elements in $(M_n)_{d_1,d_2}$. It seems difficult (as least to the authors) to test directly whether a given set of elements in $(M_n)_{d_1,d_2}$ are linearly independent. We define a map φ sending an alternating polynomial $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ to a polynomial ring

$$\mathbb{C}[\rho] := \mathbb{C}[\rho_1, \rho_2, \rho_3, \dots].$$

The map has two desirable properties: (i) for many f, $\varphi(f)$ can be easily computed, and (ii) for each bi-degree (d_1, d_2), φ induces a morphism $\bar{\varphi} : (M_n)_{d_1,d_2} \to \mathbb{C}[\rho]$ of $\mathbb{C}$-modules. Then we use the fact the linear dependency is easier to check in $\mathbb{C}[\rho]$ than in $(M_n)_{d_1,d_2}$. The map φ is defined as below.

1.2.3 Maps φ and $\bar{\varphi}$.

(a) Define the map $\varphi : \mathfrak{D}_n \to \mathbb{Z}[\rho]$ as follows. Let $D = \{(a_1, b_1), ..., (a_n, b_n)\} \in \mathfrak{D}_n$, $k = \binom{n}{2} - \sum_{i=1}^n (a_i + b_i)$, and define

$$\varphi(D) := (-1)^k \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n \left(\sum \rho_{w_1}\rho_{w_2} \cdots \rho_{w_{b_i}} \right),$$

where $(w_1, \dots, w_{b_i})$ in the sum $\sum \rho_{w_1}\rho_{w_2} \cdots \rho_{w_{b_i}}$ runs through the set

$$\{(w_1, \dots, w_{b_i}) \in \mathbb{N}^{b_i} \,|\, w_1 + ... + w_{b_i} = \sigma(i) - 1 - a_i - b_i\},$$

with the convention that

$$\sum \rho_{w_1} ... \rho_{w_{b_i}} = \begin{cases} 0 & \text{if } \sigma(i) - 1 - a_i - b_i < 0; \\ 0 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i > 0; \\ 1 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i = 0. \end{cases}$$

(b) Here is an equivalent definition of $\varphi(D)$. Define the weight of ρ_i to be i for $i \in \mathbb{N}^+$ and define the weight of $\rho_0 = 1$ to be 0. Naturally the weight of any monomial $c\rho_{i_1}...\rho_{i_n}$ ($c \in \mathbb{Z}$) is defined to be $i_1 + ... + i_n$. For $\mathrm{w} \in \mathbb{N}$ and a power series $f \in \mathbb{Z}[[\rho_1, \rho_2, \dots]]$, denote by $\{f\}_{\mathrm{w}}$ the sum of terms of weight-w in f, which is a polynomial. Define

$$h(b, \mathrm{w}) := \left\{(1 + \rho_1 + \rho_2 + \cdots)^b\right\}_{\mathrm{w}}, \quad b \in \mathbb{N}, \mathrm{w} \in \mathbb{Z}.$$

Naturally $h(b, \mathrm{w}) = 0$ if $\mathrm{w} < 0$. Also assume $(1 + \rho_1 + \rho_2 + \cdots)^0 = 1$. Then

$$\varphi(D) = (-1)^k \begin{vmatrix} h(b_1, -|P_1|) & h(b_1, 1 - |P_1|) & h(b_1, 2 - |P_1|) & \cdots & h(b_1, n-1-|P_1|) \\ h(b_2, -|P_2|) & h(b_2, 1 - |P_2|) & h(b_2, 2 - |P_2|) & \cdots & h(b_2, n-1-|P_2|) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(b_n, -|P_n|) & h(b_n, 1 - |P_n|) & h(b_n, 2 - |P_n|) & \cdots & h(b_n, n-1-|P_n|) \end{vmatrix}.$$

(c) Let $D_1, \dots, D_\ell \in D'$ be of the same bi-degree and $\sum_{i=1}^{\ell} c_i D_i$ be the formal sum for any $c_i \in \mathbb{C}$ $(1 \le i \le \ell)$. Define

$$\varphi(\sum_{i=1}^{\ell} c_i D_i) := \sum_{i=1}^{\ell} c_i \, \varphi(D_i).$$

For any bi-homogeneous alternating polynomials $f = \sum_{i=1}^{\ell} c_i \, \Delta(D_i) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$, we define

$$\varphi(f) := \varphi(\sum_{i=1}^{\ell} c_i D_i) = \sum_{i=1}^{\ell} c_i \, \varphi(D_i)$$

by abuse of notation. □

Proposition 1.7 *Fix any pair of nonnegative integers* (d_1, d_2)*, the map* φ *induces a well-defined linear map*

$$\bar{\varphi} : (M_n)_{d_1, d_2} \longrightarrow \mathbb{C}[\rho].$$

Moreover, this map $\bar{\varphi}$ is conjecturally injective. And our future work is to generalizing it to the case $I_n^m / (\mathbf{x}, \mathbf{y}) I_n^m$ for any positive integer m.

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