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Unitary Matrix Integrals, Primitive Factorizations, and Jucys-Murphy Elements

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Abstract. A factorization of a permutation into transpositions is called "primitive" if its factors are weakly ordered. We discuss the problem of enumerating primitive factorizations of permutations, and its place in the hierarchy of previously studied factorization problems. Several formulas enumerating minimal primitive and possibly non-minimal primitive factorizations are presented, and interesting connections with Jucys-Murphy elements, symmetric group characters, and matrix models are described.

Résumé. Une factorisation en transpositions d'une permutation est dite "primitive" si ses facteurs sont ordonnés. Nous discutons du problème de l'énumération des factorisations primitives de permutations, et de sa place dans la hiérarchie des problèmes de factorisation précédemment étudiés. Nous présentons plusieurs formules énumérant certaines classes de factorisations primitives, et nous soulignons des connexions intéressantes avec les éléments Jucys-Murphy, les caractéres des groupes symétriques, et les modèles de matrices.

Keywords: Primitive factorizations, Jucys-Murphy elements, matrix integrals.

1 Introduction

1.1 Polynomial integrals on unitary groups

Let $U(N)$ denote the group of $N \times N$ complex unitary matrices $U = [u_{ij}]_{1 \leq i,j \leq N}$. By a *polynomial function* on $U(N)$ we mean a function of the form

$$p(U) = \sum_{m,n \geq 0} \sum_{I,J,I',J'} c(I,J,I',J') U_{IJ} \overline{U}_{I'J'}, \tag{1}$$

where

$$\begin{matrix} I = (i_1, \dots, i_m) & I' = (i'_1, \dots, i'_n) \\ J = (j_1, \dots, j_m) & J' = (j'_1, \dots, j'_n) \end{matrix} \tag{2}$$

are multi-indices,

$$U_{IJ} \overline{U}_{I'J'} = u_{i_1 j_1} \dots u_{i_m j_m} \overline{u_{i'_1 j'_1} \dots u_{i'_n j'_n}} \tag{3}$$

is the corresponding monomial in matrix entries, and only finitely many of the coefficients $c(I, J, I', J') \in \mathbb{C}$ are non-zero. A *polynomial integral* over $U(N)$ is the integral of a polynomial function on $U(N)$ against the normalized Haar measure.

The computation of polynomial integrals over $U(N)$ is of interest from many points of view, including mathematical physics (nuclear physics, lattice gauge theory, quantum transport and quantum information), random matrix theory (matrix models, asymptotic freeness of random matrices), number theory (stochastic models of the Riemann zeta function), and algebraic combinatorics (integral representations of structure constants in the ring of symmetric functions), see [10] for references to the large body of literature on matrix integrals of this type. Nevertheless, the evaluation of such integrals is a problem of substantial complexity that is not yet fully understood.

We wish to develop a general theory of polynomial integrals over $U(N)$. By linearity of the integral, we have

$$\int_{U(N)} p(U)dU = \sum_{m,n\geq 0} \sum_{I,J,I',J'} c(I,J,I',J') \int_{U(N)} U_{IJ}\overline{U}_{I'J'}dU, \tag{4}$$

so we consider the problem of evaluating *monomial integrals*

$$\int_{U(N)} U_{IJ}\overline{U}_{I'J'}dU. \tag{5}$$

Monomial integrals are already of great interest in mathematical physics, see e.g. [3]. An easy argument involving the invariance of Haar measure shows that (5) can be non-zero only for $m = n$ (i.e. the multi-indices I, J are of the same length as the multi-indices I', J'). Furthermore, when $m = n \leq N$ (i.e. the degree of the monomial to be integrated is at most the dimension of the matrices being integrated over), the integral (5) can be decomposed into a double sum over the symmetric group $S(n)$ of the form

$$\int_{U(N)} U_{IJ}\overline{U}_{I'J'}dU = \sum_{(\sigma,\tau)\in S(n)\times S(n)} [I = \sigma(I')][J = \tau(J')]\mathsf{W}_{\sigma\tau}. \tag{6}$$

This integration formula has two ingredients: a combinatorial "Wick-like" rule — sum over pairs of permutations (σ, τ) such that σ maps the multi-index I' to the multi-index I and τ maps the multi-index J' to the multi-index J — together with a certain "weight" $\mathsf{W}_{\sigma\tau}$ associated to each admissible pair of permutations. These weights have a remarkable combinatorial interpretation as generating functions enumerating certain factorizations in the symmetric group; the resulting connections with algebraic combinatorics are the focus of this extended abstract prepared by the authors for FPSAC 2010.

1.2 Primitive factorizations and Weingarten numbers

Let $S(\infty)$ denote the group of finitary permutations of the natural numbers $\{1, 2, 3, \dots\}$, with $S(n) \leq S(\infty)$ the subgroup of permutations of $[1, n] = \{1, \dots, n\}$. An ordered sequence of transpositions

$$(s_1\ t_1)(s_2\ t_2)\dots(s_k\ t_k), \quad s_i < t_i, \tag{7}$$

is said to be a *factorization* of $\pi \in S(\infty)$ if

$$\pi = (s_1\ t_1) \circ (s_2\ t_2) \circ \dots \circ (s_k\ t_k). \tag{8}$$

A factorization is called *primitive* (more precisely, *right primitive*) if the inequalities

$$t_1 \leq t_2 \leq \cdots \leq t_k \tag{9}$$

hold in (8). Consider the quantities

$$\begin{aligned} h_{k,\pi}(n) &= \#\{\text{factorizations of } \pi \text{ into } k \text{ transpositions from } S(n)\} \\ w_{k,\pi}(n) &= \#\{\text{primitive factorizations of } \pi \text{ into } k \text{ transpositions from } S(n)\}. \end{aligned} \tag{10}$$

The numbers $h_{k,\pi}(n)$ are known as (disconnected) ***Hurwitz numbers***, and are of much interest in enumerative geometry, see e.g. [12]. We will call the numbers $w_{k,\pi}(n)$ ***Weingarten numbers***, see [2, 10] for the origin of this name. The primitive factorizations counted by Weingarten numbers have previously been considered by combinatorialists, both in relation to the enumeration of chains in noncrossing partition lattices [1, 14] and for their own sake [5]. Our approach to polynomial integrals over unitary groups is based on the remarkable fact that the weights appearing in the integration formula (6) are generating functions for Weingarten numbers.

Theorem 1 ([10, 11]) *For any $n \leq N$ and $\pi \in S(n)$ we have*

$$N^n \mathsf{W}_{\sigma\tau} = \sum_{k \geq 0} w_{k,\pi}(n) \left(\frac{-1}{N} \right)^k,$$

where $\pi = \sigma \circ \tau^{-1}$.

2 Jucys-Murphy elements

2.1 Centrality

Let $C_\mu \subset S(\infty)$ denote the conjugacy class of permutations of reduced cycle type μ (μ is a Young diagram). For instance, $C_{(0)}$ is the class of the identity permutation, $C_{(1)}$ is the class of transpositions, and more generally $C_{(r)}$ is the class of $(r+1)$-cycles. Note that $|\mu|$ is the minimal length of a factorization of π into transpositions. The conjugacy classes of $S(n)$ are $C_\mu(n) := C_\mu \cap S(n)$. Let $\mathcal{Z}(n)$ denote the centre of the group algebra $\mathbb{C}[S(n)]$. Then $\{C_\mu(n)\}$ is the canonical basis of $\mathcal{Z}(n)$, where $C_\mu(n)$ is identified with the formal sum of its elements, so $\mathcal{Z}(n)$ is referred to as the *class algebra* of $S(n)$.

Multiplying k copies of the class of transpositions, we obtain

$$\underbrace{C_{(1)}(n) C_{(1)}(n) \ldots C_{(1)}(n)}_{k \text{ times}} = \sum_{\mu} h_{k,\mu}(n) C_\mu(n), \tag{11}$$

where clearly $h_{k,\mu}(n) = h_{k,\pi}(n)$ for any $\pi \in C_\mu(n)$. In other words, $h_{k,\pi}(n)$ depends on π only up to conjugacy class. That this also holds for Weingarten numbers is not so obvious. To see that Weingarten numbers are central, we consider the enumeration of *strictly primitive factorizations*, i.e. factorizations

$$\pi = (s_1\ t_1) \circ (s_2\ t_2) \circ \cdots \circ (s_k\ t_k) \tag{12}$$

such that $t_1 < t_2 < \cdots < t_k$. One may show by a direct combinatorial argument that any permutation $\pi \in C_\mu$ admits a *unique* strictly primitive factorization, and that this unique factorization has length $|\mu|$.

This combinatorial fact can be written algebraically as follows. Consider the *Jucys-Murphy* elements $J_1, J_2, \dots, J_t, \dots \in \mathbb{C}[S(\infty)]$ defined by

$$J_t = \sum_{s<t} (s\ t). \tag{13}$$

Let Ξ_n denote the alphabet $\{\{J_1, J_2, \dots, J_n, 0, 0, \dots\}\}$. Then

$$\begin{aligned} e_k(\Xi_n) &= \sum_{\pi \in S(n)} \#\{\text{length } k \text{ strictly primitive factorizations of } \pi\}\pi \\ h_k(\Xi_n) &= \sum_{\pi \in S(n)} \#\{\text{length } k \text{ primitive factorizations of } \pi\}\pi, \end{aligned} \tag{14}$$

where

$$\begin{aligned} e_k &= \sum_{t_1<t_2<\dots<t_k} x_{t_1} x_{t_2} \dots x_{t_k} \\ h_k &= \sum_{t_1 \le t_2 \le \dots \le t_k} x_{t_1} x_{t_2} \dots x_{t_k} \end{aligned} \tag{15}$$

are the elementary and complete symmetric functions in commuting variables $x_1, x_2, \dots$. The fact that each $\pi \in C_\mu$ admits a unique strictly primitive factorization, and that this factorization has length π, translates into the identity

$$e_k(\Xi_n) = \sum_{|\mu|=k} C_\mu(n) \in \mathcal{Z}(n), \tag{16}$$

which was first obtained by Jucys [8] (see also [4]). On the other hand, the algebra Λ of symmetric functions is precisely the polynomial algebra $\Lambda = \mathbb{C}[e_1, e_2, \dots]$ in the elementary symmetric functions, so we conclude that the substitution $f \mapsto f(\Xi_n)$ defines a specialization $\Lambda \to \mathcal{Z}(n)$ from the algebra of symmetric functions to the class algebra. In particular, $h_k(\Xi_n) \in \mathcal{Z}(n)$, and we can write

$$h_k(\Xi_n) = \sum_\mu w_{k,\mu}(n) C_\mu(n), \tag{17}$$

where $w_{k,\mu}(n) = w_{k,\pi}(n)$ for any $\pi \in C_\mu(n)$.

2.2 Character theory

Since any permutation is either even or odd, the Hurwitz and Weingarten numbers $h_{k,\mu}(n), w_{k,\mu}(n)$ can only be non-zero for k of the form $k = |\mu| + 2g$ for integer $g \ge 0$. We thus introduce the notation

$$\begin{aligned} \tilde{h}_{g,\mu}(n) &:= h_{|\mu|+2g,\mu}(n) \\ \tilde{w}_{g,\mu}(n) &:= w_{|\mu|+2g,\mu}(n). \end{aligned} \tag{18}$$

In particular, Theorem 1 reads

$$(-1)^{|\mu|} N^{n+|\mu|} \mathsf{W}_{\sigma\tau} = \sum_{g \ge 0} \frac{\tilde{w}_{g,\mu}(n)}{N^{2g}}, \tag{19}$$

where $n \leq N$ and $\sigma \circ \tau^{-1} \in C_\mu(n)$. Using the character theory of $S(n)$, Jackson [7] and Shapiro-Shapiro-Vainshtein [13] obtained the remarkable formula

$$\tilde{h}_{g,(n-1)}(n) = \frac{1}{n!}\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\left(\binom{n}{2} - jn\right)^{n-1+2g} \tag{20}$$

for the number of factorizations of a full cycle (i.e. an element of $C_{(n-1)}(n)$) into $n-1+2g$ transpositions. Here we will explain how properties of Jucys-Murphy elements in irreducible representations of $\mathbb{C}[S(n)]$ may be used to obtain an analogous formula for the Weingarten number $\tilde{w}_{g,(n-1)}(n)$.

Our point of departure is the remarkable expansion

$$f(\Xi_n) = \sum_{\lambda \vdash n} \frac{f(A_\lambda)}{H_\lambda}\chi^\lambda \tag{21}$$

obtained by Jucys [8], of the symmetric function $f \in \Lambda$ evaluated at Ξ_n in terms of the characters

$$\chi^\lambda := \sum_\mu \chi^\lambda(C_\mu(n))C_\mu(n) \tag{22}$$

of the irreducible (complex, finite-dimensional) representations of $\mathbb{C}[S(n)]$. Here A_λ denotes the alphabet of contents of the Young diagram λ, and H_λ is the product of its hook-lengths. This can be viewed as an analogue of the formula of Burnside which expresses the connection coefficients of $\mathcal{Z}(n)$ in terms of irreducible characters.

Consider the ordinary generating function

$$\Phi(z;n) = \sum_{k \geq 0} h_k(\Xi_n)z^k, \tag{23}$$

which is an element of the algebra $\mathcal{Z}(n)[[z]]$ of formal power series in one indeterminate z over the class algebra $\mathcal{Z}(n)$. Plugging in the character expansion

$$h_k(\Xi_n) = \sum_{\lambda \vdash n} \frac{h_k(A_\lambda)}{H_\lambda}\chi^\lambda \tag{24}$$

and changing order of summation, we obtain

$$\Phi(z;n) = \sum_{\lambda \vdash n} \frac{\chi^\lambda}{H_\lambda \prod_{\Box \in \lambda}(1 - c(\Box)z)}, \tag{25}$$

where $c(\Box)$ denotes the content of a cell $\Box \in \lambda$ and we have made us of the generating function

$$\sum_{k \geq 0} h_k(x_1, x_2, \dots)z^k = \prod_{i \geq 1} \frac{1}{1 - x_i z} \tag{26}$$

for the complete symmetric functions. Thus we obtain the formula

$$\Phi_\mu(z;n) = \sum_{\lambda \vdash n} \frac{\chi^\lambda(C_\mu(n))}{H_\lambda \prod_{\Box \in \lambda}(1 - c(\Box)z)} \tag{27}$$

for the ordinary generating function

$$\Phi_\mu(z;n) = \sum_{k\geq 0} w_{k,\mu}(n) z^k \tag{28}$$

of Weingarten numbers. Note that by Theorem 1, this corresponds to the character expansion

$$\mathsf{W}_{\sigma\tau} = \sum_{\lambda\vdash n} \frac{\chi^\lambda(C_\mu(n))}{H_\lambda \prod_{\Box\in\lambda}(N + c(\Box))} \tag{29}$$

(where $n \leq N$ and $\sigma \circ \tau^{-1} \in C_\mu(n)$), which is well known in the physics literature and was first rigorously obtained in [2] by a different argument.

Up until this point, the partition μ has been generic, but now we restrict to the special case $\mu = (n-1)$, the class of a full cycle in $S(n)$. A classical result from representation theory informs us that the trace of $C_{(n-1)}(n)$ in an irreducible representation can only be non-zero in "hook" representations:

$$\chi^\lambda(C_{(n-1)}(n)) = \begin{cases} (-1)^r, \text{ if } \lambda = (n-r, 1^r) \\ 0, \text{ otherwise} \end{cases}. \tag{30}$$

Now, the content alphabet of a hook diagram may be obtained immediately,

$$A_{(n-r,1^r)} = \{0, 1, \ldots, n-r-1\} \sqcup \{-1, \ldots, -r\}. \tag{31}$$

so that

$$\Phi_{(n-1)}(z;n) = \sum_{r=0}^{n-1} \frac{(-1)^r}{H_{(n-r,1^r)} \prod_{i=1}^{n-r-1}(1-iz) \prod_{j=1}^{r}(1+jz)}. \tag{32}$$

For example, if $n = 4$, this is a rational function of the form

$$\begin{aligned} \Phi_{(3)}(z;n) =& \frac{\text{const.}}{(1-z)(1-2z)(1-3z)} + \frac{\text{const.}}{(1-z)(1-2z)(1+z)} \\ &+ \frac{\text{const.}}{(1-z)(1+z)(1+2z)} + \frac{\text{const.}}{(1+z)(1+2z)(1+3z)}. \end{aligned} \tag{33}$$

Thus, as an irreducible rational function, $\Phi_{(n-1)}(z;n)$ has the form

$$\Phi_{(n-1)}(z;n) = \frac{\sum_{i=0}^{n-1} c_i z^i}{\prod_{i=1}^{n-1}(1 - i^2 z^2)} \tag{34}$$

where $c_0, \ldots, c_{n-1} \in \mathbb{C}$ are some constants to be determined momentarily.

Before finding the above coefficients, let us consider the generating function

$$\frac{1}{\prod_{i=1}^{n}(1 - i^2 u)} = \sum_{g\geq 0} h_g(1^2, \ldots, n^2) u^g. \tag{35}$$

The coefficients in this generating function are complete symmetric functions evaluated on the alphabet $\{1^2,\dots,n^2\}$ of square integers. Reason dictates that they ought to be close relatives of the Stirling numbers

$$S(n+g,n)=h_g(1,\dots,n). \tag{36}$$

The Stirling number $S(a,b)$ has the following combinatorial interpretation: it counts the number of partitions

$$\{1,\dots,a\}=V_1\sqcup\dots\sqcup V_b \tag{37}$$

of an a-element set into b disjoint non-empty subsets. Stirling numbers are given by the explicit formula

$$S(a,b)=\sum_{j=0}^{b}(-1)^{b-j}\frac{j^a}{j!(b-j)!}. \tag{38}$$

The numbers

$$T(n+g,n)=h_g(1^2,\dots,n^2) \tag{39}$$

are known as *central factorial numbers*. The central factorial numbers were studied classically by Carlitz and Riordan, see [15, Exercise 5.8] for references. They have the following combinatorial interpretation: $T(a,b)$ counts the number of partitions

$$\{1,1',\dots,a,a'\}=V_1\sqcup\dots\sqcup V_b \tag{40}$$

of a set of a marked and a unmarked points into b disjoint non-empty subsets such that[(i)], for each block V_j, if i is the least integer such that either i or i' appears in V_j, then $\{i,i'\}\subseteq V_j$. Central factorial numbers are given by the explicit formula

$$T(a,b)=2\sum_{j=0}^{b}(-1)^{b-j}\frac{j^{2a}}{(b-j)!(b+j)!}. \tag{41}$$

Now let us determine the unknown constants $c_0,\dots,c_{n-1}$. By the above discussion, the generating function $\Phi_{(n-1)}(z;n)$ has the form

$$\Phi_{(n-1)}(z;n)=(c_0+c_1z+\dots+c_{n-1}z^{n-1})\sum_{g\geq 0}T(n-1+g,n-1)z^{2g}. \tag{42}$$

On the other hand,

$$\begin{aligned}\Phi_{(n-1)}(z;n)&=\sum_{k\geq 0}w_{k,(n-1)}(n)z^k\\&=\sum_{g\geq 0}\tilde{w}_{g,(n-1)}(n)z^{n-1+2g}\\&=\tilde{w}_{0,(n-1)}(n)z^{n-1}+\tilde{w}_{1,(n-1)}(n)z^{n+1}+\dots.\end{aligned} \tag{43}$$

(i) Bálint Virág (personal communication) gave a colourful description of this condition, which is actually quite a useful mnemonic: "the most important guy gets to bring his wife."

Consequently, we must have $c_0 = \cdots = c_{n-2} = 0, c_{n-1} = \tilde{w}_{0,(n-1)}(n)$, the number of primitive factorizations of the cyclic permutation $\xi[1,n] = (1\ 2\ \ldots\ n)$ into the minimal number $n-1$ of transpositions. It is not difficult to show (see [5, 10]) bijectively that the number of minimal primitive factorizations of the cycle $\xi[1,n]$ is the Catalan number $\mathrm{Cat}_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$. In fact, a stronger result from [10] asserts that the number $\tilde{w}_{0,\mu}(n)$ of minimal primitive factorizations of an arbitrary permutation of reduced cycle type μ is a product of Catalan numbers,

$$\tilde{w}_{0,\mu}(n) = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i}, \tag{44}$$

so that the function

$$\pi \mapsto \#\{\text{minimal primitive factorizations of } \pi\} \tag{45}$$

is a central multiplicative function on $S(\infty)$ (note that, via Theorem 1, this result corresponds to the first-order estimate

$$(-1)^{|\mu|} N^{n+|\mu|} \mathsf{W}_{\sigma\tau} = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i} + O\left(\frac{1}{N^2}\right), \tag{46}$$

where $\sigma \circ \tau^{-1} \in C_\mu(n)$). Thus we have proved the following analogue of (20) for primitive factorizations.

Theorem 2 *For any $g \geq 0$, the number of primitive factorizations of a full cycle from $S(n)$ into $n-1+2g$ transpositions is*

$$\tilde{w}_{g,(n-1)}(n) = \mathrm{Cat}_{n-1} \cdot T(n-1+g, n-1),$$

where $T(a,b)$ denotes the Carlitz-Riordan central factorial number. Equivalently, we have the generating function

$$\Phi_{(n-1)}(z;n) = \frac{\mathrm{Cat}_{n-1}\, z^{n-1}}{(1-1^2z^2)\ldots(1-(n-1)^2z^2)}.$$

Via Theorem 1, Theorem 2 corresponds to the exact integration formula

$$\mathsf{W}_{\sigma\tau} = \frac{(-1)^{n-1}\,\mathrm{Cat}_{n-1}}{N(N^2-1^2)\ldots(N^2-(n-1)^2)}, \quad \sigma \circ \tau^{-1} \in C_\mu(n), \tag{47}$$

which was first stated by Collins in [2].

3 Conclusion

We have discussed the close relationship between the problem of computing polynomial integrals over unitary groups and the enumeration of primitive factorizations of permutations. In particular, the problem was completely solved for full cycles, and the central factorial numbers of Carlitz and Riordan made a surprising appearance and were given a new combinatorial interpretation. It seems that Hurwitz numbers and Weingarten numbers are remarkably similar in character. For example, writing (20) and Theorem 2 in terms of exponential generating functions yields

$$\begin{aligned} \tilde{h}_{g,(n-1)}(n) &= n^{n-2} n^{2g} \binom{n-1+2g}{n-1} \left[\frac{z^{2g}}{(2g)!}\right] \left(\frac{\sinh z/2}{z/2}\right)^{n-1} \\ \tilde{w}_{g,(n-1)}(n) &= \mathrm{Cat}_{n-1} \binom{2n-2+2g}{2n-2} \left[\frac{z^{2g}}{(2g)!}\right] \left(\frac{\sinh z/2}{z/2}\right)^{2n-2}. \end{aligned} \tag{48}$$

On one hand, the multiplicative form of Theorem 2 suggests the existence of an underlying bijective explanation, and on the other computer calculations performed by Valentin Féray (personal communication) suggest that such a bijection could be very complex. Further similarities between Hurwitz numbers and Weingarten numbers are the subject of work in progress [6]. Let us finish by pointing out that the first author has extended many of the results presented here to the setting of polynomial integrals over orthogonal groups [9].

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