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Words and Noncommutative Invariants of the Hyperoctahedral Group

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Abstract. Let $\mathcal{B}_n$ be the hyperoctahedral group acting on a complex vector space $\mathcal{V}$. We present a combinatorial method to decompose the tensor algebra $T(\mathcal{V})$ on $\mathcal{V}$ into simple modules via certain words in a particular Cayley graph of $\mathcal{B}_n$. We then give combinatorial interpretations for the graded dimension and the number of free generators of the subalgebra $T(\mathcal{V})^{\mathcal{B}_n}$ of invariants of $\mathcal{B}_n$, in terms of these words, and make explicit the case of the signed permutation module. To this end, we require a morphism from the Mantaci-Reutenauer algebra onto the algebra of characters due to Bonnafé and Hohlweg.

Résumé. Soit $\mathcal{B}_n$ le groupe hyperoctaédral agissant sur un espace vectoriel complexe $\mathcal{V}$. Nous présentons une méthode combinatoire donnant la décomposition de l'algèbre $T(\mathcal{V})$ des tenseurs sur $\mathcal{V}$ en modules simples via certains mots dans un graphe de Cayley donné. Nous donnons ensuite des interprétations combinatoires pour la dimension graduée et le nombre de générateurs libres de la sous-algèbre $T(\mathcal{V})^{\mathcal{B}_n}$ des invariants de $\mathcal{B}_n$, en termes de ces mots, et explicitons le cas du module de permutation signé. À cette fin, nous utilisons un morphisme entre l'algèbre de Mantaci-Reutenauer et l'algèbre des charactères introduit par Bonnafé et Hohlweg.

Keywords: Tensor algebras, invariants of finite groups, hyperoctahedral group, signed permutation module, Cayley graph, words.

1 Introduction

Let $\mathcal{V}$ be a vector space over the field $\mathbb{C}$ of complex numbers with basis $\{x_1, x_2, \ldots, x_n\}$. Then the tensor algebra

$$T(\mathcal{V}) = \mathbb{C} \oplus \mathcal{V} \oplus \mathcal{V}^{\otimes 2} \oplus \mathcal{V}^{\otimes 3} \oplus \cdots$$

can be identified with the ring $\mathbb{C}\langle \mathbf{x} \rangle$ of polynomials in noncommutative variables $\mathbf{x} = x_1, x_2, \ldots, x_n$, where we use the notation $\mathcal{V}^{\otimes d}$ to represent the d-fold tensor space. Any action of a finite group G on $\mathcal{V}$ can be extended to the tensor algebra and the graded character can be found in terms by what we might identify as an analogue of MacMahon's Master Theorem [10] for the tensor space,

$$\chi_{\mathcal{V}^{\otimes d}}(g) = \mathrm{tr}(M(g))^d = \left[q^d\right] \ \frac{1}{1 - \mathrm{tr}\, M(g) q} \ ,$$

where $\left[q^d\right]$ represents taking the coefficient of q^d in the expression to the right and $M(g)$ is a matrix which represents the action of the group element g on a basis of $\mathcal{V}$. In particular, we consider the *algebra* *of invariants of* G, denoted $T(\mathcal{V})^G$, as the subspace of elements of $T(\mathcal{V})$ which are fixed under the action of G. The analogue of Molien's Theorem [6] for the tensor algebra allows us to calculate the graded dimension of this space

$$\mathcal{P}(T(\mathcal{V})^G) = \sum_{d\geq 0} \dim(\mathcal{V}^{\otimes d})^G q^d = \frac{1}{|G|} \sum_{g\in G} \frac{1}{1 - \operatorname{tr} M(g)\, q} \,. \tag{1.1}$$

It is well-known that the algebra of invariants of G is freely generated [7, 8] by an infinite set of generators (except when G is scalar) [6].

These algebraic tools do not clearly show the underlying combinatorial structure of these algebras. Our main goal is to find a combinatorial method to decompose $T(\mathcal{V})$ into simple G-modules. The idea is to associate to a module $\mathcal{V}$ of G, a special subalgebra of the group algebra together with a surjective morphism of algebras into the algebra of characters of G. Then we get as a consequence a combinatorial way to decompose $T(\mathcal{V})$ by counting words generated by a particular Cayley graph of the group G. To compute the graded dimension of $T(\mathcal{V})^G$, it then suffices to look at the multiplicity of the trivial module in $T(\mathcal{V})$. This leads to combinatorial descriptions for the graded dimension and the number of free generators of the algebras of invariants of G, which unifie their interpretations.

At this point, we treated the cases of the cyclic, dihedral and symmetric groups [3]. For the symmetric group, the main bridge to link the words in a particular Cayley graph and the decomposition of the tensor algebra into simple modules is a morphism from the theory of the descent algebra [12, 15]. In order to handle cases beyond those already considered, we must find a relation between the group algebra and the algebra of characters.

We present in this paper the case of the hyperoctahedral group $\mathcal{B}_n$, where the main bridge comes from a surjective morphism from the Mantaci-Reutenauer algebra [11] onto the characters of $\mathcal{B}_n$ due to Bonnafé and Hohlweg [1]. More precisely, we present a combinatorial way to decompose the $\mathcal{B}_n$-module $T(\mathcal{V})$ into simple modules using words in a Cayley graph of $\mathcal{B}_n$ and study the subalgebra $T(\mathcal{V})^{\mathcal{B}_n}$ of invariants. This technique applies to modules that can be realized in the Mantaci-Reutenauer algebra, for example for modules indexed by bipartitions of hook shapes, and we make explicit the case of the signed permutation module $\mathcal{V}_{[n-1],[1]}$. We also give combinatorial descriptions for the graded dimensions and the number of free generators of the algebra $T(\mathcal{V}_{[n-1],[1]})^{\mathcal{B}_n}$ of invariants, using words in a particular Cayley graph of $\mathcal{B}_n$. Finally, we present an application to set partitions, since the dimension of $T(\mathcal{V}_{[n-1],[1]})^{\mathcal{B}_n}$ is also given by the set partitions of at most n even parts [2].

The paper is organized as follows. We recall in Section 2.1 the definition of a Cayley graph, and introduce its weighted version. Section 2.2 fixes some notation about bipartitions and bitableaux. Section 2.3 is dedicated to the hyperoctahedral group and recalls its representation theory. In Section 2.4, we describe the generalized Robinson-Schensted correspondence from [16, 5] needed in the statement of the Main Theorem. The bridge between the words in a particular Cayley graph of $\mathcal{B}_n$ and the decomposition of the tensor algebra is a morphism from the Mantaci-Reutenauer algebra into the character algebra of $\mathcal{B}_n$, which we present in Section 2.5. In Section 3, we prove the Main Theorem which gives a combinatorial way to decompose the tensor algebra on any $\mathcal{B}_n$-module into simple modules. As a consequence, we give in Section 4 a combinatorial way to compute the graded dimension of the space of invariants of $\mathcal{B}_n$, and give a description for its number of free generators as an algebra. We then investigate in Sections 3.1 and 4.1 the case of the signed permutation module and give an application to set partitions in Section 4.2.

2 Preliminairies

2.1 Cayley graph

For our purpose, let us recall the definition of a Cayley graph. Let G be a finite group and let $S \subseteq G$ be a set of group elements. The *Cayley graph* associated with (G, S) is defined as the oriented graph $\Gamma = \Gamma(G, S)$ having one vertex for each element of G and the edges associated with elements in S. Two vertices g_1 and g_2 are joined by a directed edge associated to $s \in S$ if $g_2 = g_1 s$. If the resulting Cayley graph of G is connected, then the set S generates G.

A path along the edges of Γ corresponds to a word in the alphabet S. We denote by S^* the *free monoid* on S, *i.e.* the set of all words in the alphabet S. Naturally, the *length* of a word is the number of its letters. We say that a word *reduces to an element* $g \in G$ in the Cayley graph Γ if it corresponds to a path along the edges from the vertex labelled by the identity to the one labelled by g. Such a word, when simplified with respect to the group relations, corresponds to the reduced word g. We denote by $\mathcal{W}_d(g)$ the set of words of length d which reduce to g. A word w is called a *prefix* of a word u if there exits a word v such that $u = wv$. The prefix is *proper* if v is not the empty word. We say that a word *does not cross the identity* if it has no proper prefix which reduces to the identity.

We also consider *weighted Cayley graphs*, where we associate a weight $\nu(s)$ to each letter $s \in S$. We define the *weight of a word* $w = s_1 s_2 \cdots s_r$ in S^* to be the product of the weights of its letter,

$$\nu(w) = \nu(s_1)\nu(s_2)\cdots\nu(s_r).$$

For sake of simplicity, we use undirected edges to represent bidirectional edges and nonlabelled edges to represent edges of weight one.

Example 2.1 *The Cayley graph of the hyperoctahedral group* $\mathcal{B}_2 = \{12, 21, 1\bar{2}, 2\bar{1}, \bar{1}2, \bar{2}1, \bar{1}\bar{2}, \bar{2}\bar{1}\}$ *of signed permutations of* $\{1, 2\}$ *with generators* $\bar{1}2$ *and* $\bar{2}1$ *of weigth one is represented in Figure 1.*

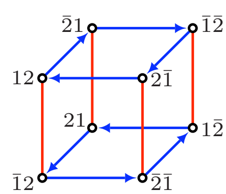

Fig. 1: $\Gamma(\mathcal{B}_2, \{\bar{1}2, \bar{2}1\})$.

2.2 Bipartitions and bitableaux

To fix the notation, we recall some definitions. A *partition* λ of a positive integer n is a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ of positive integers such that $n = |\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell$. We write $\lambda = [\lambda_1, \lambda_2, \ldots \lambda_\ell] \vdash n$. It is natural to represent a partition by its *Young diagram* which is the finite subset $\mathrm{diag}(\lambda) = \{(a, b) \,|\, 0 \leq a \leq \ell - 1 \text{ and } 0 \leq b \leq \lambda_{a+1} - 1\}$ of $\mathbb{N}^2$. Visually, each element of $\mathrm{diag}(\lambda)$ corresponds to the bottom left corner of a box of dimension 1×1 in $\mathbb{N}^2$.

A *bipartition of* n, denoted $\boldsymbol{\lambda} \vdash n$, is a couple $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ of partitions such that $|\boldsymbol{\lambda}| = |\lambda_1| + |\lambda_2| = n$.

Example 2.2 *The bipartitions of 2 are*

$$\ydiagram{2}, \emptyset \qquad \ydiagram{1,1}, \emptyset \qquad \ydiagram{1}, \ydiagram{1} \qquad \emptyset, \ydiagram{1,1} \qquad \emptyset, \ydiagram{2}\ .$$

A *tableau* t of shape $\lambda \vdash n$ with values in $T = \{1, 2, \ldots, n\}$ is a function $t : \mathrm{diag}(\lambda) \to T$. We denote by $\mathrm{sh}(t)$ the shape of t. We can visualize it by filling each box c of $\mathrm{diag}(\lambda)$ with the value $t(c)$. A *standard Young tableau* of shape $\lambda \vdash n$ is a tableau with filling $\{1, 2, \ldots, n\}$ and strictly increasing values along each row and each column.

A *bitableau* is a pair $\mathbf{T} = (t_1, t_2)$ of tableaux. The *shape* of a bitableau is the couple $\mathrm{sh}(\mathbf{T}) = (\mathrm{sh}(t_1), \mathrm{sh}(t_2))$. A *standard Young bitableau* is a bitableau $\mathbf{T} = (t_1, t_2)$ where t_1 and t_2 have strictly increasing values along each row and each column, $|\mathrm{sh}(\mathbf{T})| = n$ and the filling of t_1 and t_2 is the set $\{1, 2, \ldots, n\}$. We denote by $\mathrm{SYB}(\boldsymbol{\lambda})$ the set of standard Young bitableaux of shape $\boldsymbol{\lambda}$ and by SYB_n the set of standard Young bitableaux with n boxes.

Example 2.3 *The standard Young bitableaux of shape* $\boldsymbol{\lambda} \vdash 2$ *are*

$$\begin{ytableau}1&2\end{ytableau}, \emptyset \qquad \begin{ytableau}2\\1\end{ytableau}, \emptyset \qquad \begin{ytableau}1\end{ytableau}, \begin{ytableau}2\end{ytableau} \qquad \begin{ytableau}2\end{ytableau}, \begin{ytableau}1\end{ytableau} \qquad \emptyset, \begin{ytableau}2\\1\end{ytableau} \qquad \emptyset, \begin{ytableau}1&2\end{ytableau}\ .$$

2.3 The hyperoctahedral group $\mathcal{B}_n$

Denote by $[n]$ the set $\{1, 2, \ldots, n\}$ and by $\overline{m}$ the integer $-m$. The *hyperoctahedral group* is the group of signed permutations of $[n]$ of order $2^n n!$ which can be seen as the wreath product of the cyclic group of order two $\mathbb{Z}/2\mathbb{Z}$ with the symmetric group $\mathcal{S}_n$ of permutations of $[n]$. We will often represent an element π of $\mathcal{B}_n$ as a word

$$\pi = \pi(1)\pi(2)\cdots\pi(n),$$

where each $\pi(i)$ is an integer whose absolute value is in $[n]$. Note that if we forget the signs in π, we get a permutation of $[n]$. We denote by e the identity element in the hyperoctahedral group.

Example 2.4 $\bar{1}7\bar{6}\bar{5}243$ *is an element of* $\mathcal{B}_7$.

Since the conjugacy classes of $\mathcal{B}_n$ are characterized by bipartitions of n (see [9], Appendix B), it is natural to index the simple modules of $\mathcal{B}_n$ with bipartitions $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ such that $|\lambda_1| + |\lambda_2| = n$. We denote them by $\mathcal{V}_{\boldsymbol{\lambda}}$ with associated irreducible characters $\chi_{\boldsymbol{\lambda}}$. In particular, $\mathcal{V}_{[n],\emptyset}$ is the trivial module and $\mathcal{V}_{[n-1],[1]}$ the signed permutation module (see Example 4.4). Let us denote by $\mathbb{Z}\mathrm{Irr}(\mathcal{B}_n)$ the *algebra of characters* of $\mathcal{B}_n$.

2.4 Generalized Robinson-Schensted correspondence

The *Robinson-Schensted correspondence* [13, 14] is a bijection between the elements σ of the symmetric group $\mathcal{S}_n$ and pairs $(P(\sigma), Q(\sigma))$ of standard Young tableaux of the same shape. In this section, we present a generalization of this correspondence to the hyperoctahedral group defined as in [16, 5].

Consider the element π of $\mathcal{B}_n$ as a word. Define $\mathbf{P}(\pi)$ to be the standard Young bitableau $(P^+(\pi), P^-(\pi))$ where $P^+(\pi)$ and $P^-(\pi)$ are the insertion tableaux (from the Robinson-Schensted correspondence) of π with respectivley positive and negative letters of π. Similarly, $\mathbf{Q}(\pi) = (Q^+(\pi), Q^-(\pi))$ is the standard Young bitableau where $Q^+(\pi)$ and $Q^-(\pi)$ are the recording tableaux of π for the insertion of respectivley positive and negative letters of π. The map

$$\pi \longleftrightarrow (\mathbf{P}(\pi), \mathbf{Q}(\pi))$$

is a bijection from $\mathcal{B}_n$ onto the set of all pairs of standard Young bitableaux of the same shape. We say that $\mathbf{P}(\pi)$ and $\mathbf{Q}(\pi)$ are respectively the *insertion* and *recording bitableaux* of π.

Example 2.5 *Consider the element* $\bar{1}7\bar{6}\bar{5}243$ *of* $\mathcal{B}_7$*. Then we find*

$$\mathbf{P}(\pi) = \begin{array}{l}\boxed{7}\\ \boxed{4}\\ \boxed{2}\boxed{3}\end{array}, \begin{array}{l}\boxed{6}\\ \boxed{1}\boxed{5}\end{array} \qquad \textit{and} \qquad \mathbf{Q}(\pi) = \begin{array}{l}\boxed{7}\\ \boxed{5}\\ \boxed{2}\boxed{6}\end{array}, \begin{array}{l}\boxed{4}\\ \boxed{1}\boxed{3}\end{array}.$$

2.5 Mantaci-Reutenauer algebra and special morphism

Surprisingly, the key to prove our main result comes from a morphism from the Mantaci-Reutenauer algebra onto the characters of $\mathcal{B}_n$ due to Bonnafé and Hohlweg [4].

A *signed composition of* n, denoted $\mathbf{c} \models n$, is a sequence of nonzero integers $\mathbf{c} = (c_1, c_2, \ldots, c_k)$ such that $|\mathbf{c}| = |c_1| + |c_2| + \cdots + |c_k| = n$. Following Mantaci and Reutenauer [11], we associate to each element $\pi \in \mathcal{B}_n$ a *descent composition* $\mathbf{Des}(\pi)$ constructed by recording the length of the increasing runs (in absolute value) with constant sign, and then recording that sign.

Example 2.6 *The descent composition of* $\bar{1}7\bar{6}\bar{5}243 \in \mathcal{B}_7$ *is* $\mathbf{Des}(\bar{1}.7.\bar{6}.\bar{5}.24.3) = (\bar{1}, 1, \bar{1}, \bar{1}, 2, 1)$.

The *descent composition* $\mathbf{Des}(\mathbf{T})$ of a standard Young bitableau $\mathbf{T} = (t^+, t^-)$ with n boxes is defined in [1] in the following way. First, look for maximal subwords $j\ j+1\ j+2\cdots k$ of consecutive letters of the word $12\cdots n$ such that either the numbers $j, j+1, j+2, \ldots, k$ can be read in this order in t^+ when one goes from left to right and top to bottom, or they can be read in t^- in the same manner. The concatenation of these subwords is the word $12\cdots n$ and the descent composition $\mathbf{Des}(\mathbf{T})$ is the signed composition of n obtained by recording the lengths of these subwords, and the sign of their tableau.

Example 2.7 *Consider the bitableau* $\mathbf{T}= \begin{array}{l}\boxed{7}\\ \boxed{5}\\ \boxed{2}\boxed{6}\end{array}, \begin{array}{l}\boxed{4}\\ \boxed{1}\boxed{3}\end{array}$. *The partition of* 1234567 *in maximal subwords is* $1|2|3|4|56|7$ *hence we can deduce that* $\mathbf{Des}(\mathbf{T}) = (\bar{1}, 1, \bar{1}, \bar{1}, 2, 1)$.

Given a signed composition $\mathbf{c} \models n$, define the element of the group algebra of $\mathcal{B}_n$

$$D_{\mathbf{c}} = \sum_{\substack{\pi \in \mathcal{B}_n \\ \mathbf{Des}(\pi) = \mathbf{c}}} \pi.$$

These elements form a basis of the Mantaci-Reutenauer algebra $\mathcal{MR}_n$, which is a subalgebra of the group algebra of $\mathcal{B}_n$ containing the Solomon's descent algebra of $\mathcal{B}_n$ [11]. Given a standard Young bitableau $\mathbf{T}$ with n boxes, define the element of the group algebra of $\mathcal{B}_n$

$$Z_{\mathbf{T}} = \sum_{\substack{\pi \in \mathcal{B}_n \\ \mathbf{Q}(\pi) = \mathbf{T}}} \pi,$$

where $\mathbf{Q}(\pi)$ corresponds to the recording bitableau resulting from the generalized Robinson-Schensted correspondence. These elements are linearly independent and the space $\mathcal{Q}_n$ that they span is called the *coplactic space*, introduced by Bonnafé and Hohlweg [4]. Note that this space in not an algebra in general. By Lemma 5.7 of [1], the descent composition of an element $\pi \in \mathcal{B}_n$ coincides with the descent composition of its recording bitableau $\mathbf{Q}(\pi)$. Therefore we can rewrite $D_{\mathbf{c}}$ as

$$D_{\mathbf{c}} = \sum_{\substack{\pi \in \mathcal{B}_n \\ \mathbf{Des}(\pi)=\mathbf{c}}} \pi = \sum_{\substack{\mathbf{T} \in \mathrm{SYB}_n \\ \mathbf{Des}(\mathbf{T})=\mathbf{c}}} Z_{\mathbf{T}}, \tag{2.1}$$

hence $\mathcal{MR}_n \subseteq \mathcal{Q}_n$ (see [1] Corollary 5.8). There is a surjective algebra morphism from the Mantaci-Reutenauer algebra onto the character algebra $\Theta : \mathcal{MR}_n \to \mathbb{Z}\mathrm{Irr}(\mathcal{B}_n)$ due to Bonnafé and Hohlweg [4], and a linear map

$$\tilde{\Theta} : \mathcal{Q}_n \to \mathbb{Z}\mathrm{Irr}(\mathcal{B}_n) \tag{2.2}$$

defined by $\tilde{\Theta}(Z_{\mathbf{T}}) = \chi_{\mathrm{sh}(\mathbf{T})}$ such that $\tilde{\Theta}$ restricted to $\mathcal{MR}_n$ corresponds to Θ.

3 Decomposition of $T(\mathcal{V})$ into simple modules

In this section, we develop a combinatorial method to decompose the d-fold tensor of any $\mathcal{B}_n$-module into simple modules. To achieve this, we use the algebra morphism $\Theta : \mathcal{MR}_n \to \mathbb{Z}\mathrm{Irr}(\mathcal{B}_n)$ introduced in Section 2.5 from the Mantaci-Reutenauer algebra onto the algebra of characters of $\mathcal{B}_n$. The next proposition says that the multiplicity of a simple module in the d-fold tensor of any module $\mathcal{V}$ is given as some coefficients in f^d, where f is an element of $\mathcal{MR}_n$ whose image under Θ is the character of $\mathcal{V}$.

Proposition 3.1 *Let $\mathcal{V}$ be a $\mathcal{B}_n$-module such that $\Theta(f) = \chi_{\mathcal{V}}$, for some element f in $\mathcal{MR}_n$. For $\boldsymbol{\lambda} \vdash n$, the multiplicity of $\mathcal{V}_{\boldsymbol{\lambda}}$ in $\mathcal{V}^{\otimes d}$ is equal to*

$$\sum_{\mathbf{T} \in \mathrm{SYB}(\boldsymbol{\lambda})} [Z_{\mathbf{T}}] f^d,$$

where $[Z_{\mathbf{T}}] f^d$ means taking the coefficient of $Z_{\mathbf{T}}$ in f^d.

Proof: By Equation (2.1), we can write f^d as

$$f^d = \sum_{\boldsymbol{\lambda} \vdash n} \sum_{\mathbf{T} \in \mathrm{SYB}(\boldsymbol{\lambda})} c_{\mathbf{T}} Z_{\mathbf{T}}.$$

Applying the linear map (2.2), we get

$$\tilde{\Theta}(f^d) = \sum_{\boldsymbol{\lambda} \vdash n} \sum_{\mathbf{T} \in \mathrm{SYB}(\boldsymbol{\lambda})} c_{\mathbf{T}} \tilde{\Theta}(Z_{\mathbf{T}}) = \sum_{\boldsymbol{\lambda} \vdash n} \sum_{\mathbf{T} \in \mathrm{SYB}(\boldsymbol{\lambda})} c_{\mathbf{T}} \chi_{\boldsymbol{\lambda}}.$$

Since the restriction of $\tilde{\Theta}$ to $\mathcal{MR}_n$ is Θ, we get $\tilde{\Theta}(f^d) = \Theta(f^d) = \Theta(f)^d = {\chi_{\mathcal{V}}}^d$ and thus

$$[\chi_{\boldsymbol{\lambda}}]{\chi_{\mathcal{V}}}^d = \sum_{\mathbf{T} \in \mathrm{SYB}(\boldsymbol{\lambda})} c_{\mathbf{T}} = \sum_{\mathbf{T} \in \mathrm{SYB}(\boldsymbol{\lambda})} [Z_{\mathbf{T}}] f^d.$$

$\square$

The subsequent theorem provide us with an interesting interpretation for the multiplicity of $\mathcal{V}_{\boldsymbol{\lambda}}$ in the d-fold tensor of a $\mathcal{B}_n$-module. This multiplicity is the weighted sum of words in a particular Cayley graph of $\mathcal{B}_n$ which reduce to $\pi_{\mathbf{T}}$, an element of $\mathcal{B}_n$ having recording bitableau $\mathbf{T}$ of shape $\boldsymbol{\lambda}$ (after performing the generalized Robinson-Schensted correspondence). But first, the following key lemma will allow us to link some coefficients of an element of the group algebra to some weighted words in a Cayley graph of G.

Lemma 3.2 ([3]) *Let $\Gamma(G,\{s_1,s_2,\dots,s_r\})$ be a Cayley graph of G with weights $\nu(s_i)=\nu_i$. Then the coefficient of $\pi\in G$ in the element $(\nu_1 s_1+\nu_2 s_2+\cdots+\nu_r s_r)^d$ of the group algebra $\mathbb{C}G$ equals*

$$\sum_{w\in\mathcal{W}_d(\pi)}\nu(w).$$

Before stating the Main Theorem, we need to recall the following. The *support* of an element f of the group algebra of $\mathcal{B}_n$ is defined by $\mathrm{supp}(f)=\{\pi\in\mathcal{B}_n|[\pi]f\neq 0\}$, where $[\pi]f$ is the coefficient of π in f.

Theorem 3.3 *Let $\mathcal{V}$ be a $\mathcal{B}_n$-module such that $\Theta(f)=\chi_{\mathcal{V}}$, for some element f of $\mathcal{MR}_n$, and consider the Cayley graph $\Gamma(\mathcal{B}_n,\mathrm{supp}(f))$ with weights $\nu(\pi)=[\pi](f)$ for each $\pi\in\mathrm{supp}(f)$. For $\boldsymbol{\lambda}\vdash n$, the multiplicity of $\mathcal{V}_{\boldsymbol{\lambda}}$ in $\mathcal{V}^{\otimes d}$ is equal to*

$$\sum_{\mathbf{T}\in\mathrm{SYB}(\boldsymbol{\lambda})}\ \sum_{w\in\mathcal{W}_d(\pi_{\mathbf{T}})}\nu(w),$$

where $\pi_{\mathbf{T}}\in\mathcal{B}_n$ is such that $\mathbf{Q}(\pi_{\mathbf{T}})=\mathbf{T}$ and $\mathcal{W}_d(\pi_{\mathbf{T}})$ is the set of words of length d which reduce to $\pi_{\mathbf{T}}$.

Proof: From Proposition 3.1, the multiplicity of $\mathcal{V}_{\boldsymbol{\lambda}}$ in $\mathcal{V}^{\otimes d}$ is

$$\sum_{\mathbf{T}\in\mathrm{SYB}(\boldsymbol{\lambda})}[Z_{\mathbf{T}}]f^d.$$

Since by definition $\pi\in\mathrm{supp}(Z_{\mathbf{T}})$ if and only if π has recording bitableau $\mathbf{T}$, the coefficient of $Z_{\mathbf{T}}$ in f^d is also the coefficient of $\pi_{\mathbf{T}}$ in f^d with $\mathbf{Q}(\pi_{\mathbf{T}})=\mathbf{T}$ and the result follows from Lemma 3.2. □

3.1 Decomposition of $T(\mathcal{V}_{[n-1],[1]})$ into simple modules

When the hyperoctahedral group $\mathcal{B}_n$ acts as a reflection group on the ring of polynomials in n noncommutative variables, this action corresponds to the signed permutation module $\mathcal{V}_{[n-1],[1]}$. We use the following two corollaries of Proposition 3.1 and Theorem 3.3 respectively, for establishing a connection between the multiplicity of a simple module in ${\mathcal{V}_{[n-1],[1]}}^{\otimes d}$ and words of length d in a particular Cayley graph of $\mathcal{B}_n$. To this end, consider the basis element $D_{(\bar{1},n-1)}$ of the Mantaci-Reutenauer algebra $\mathcal{MR}_n$, which is the sum of all elements of $\mathcal{B}_n$ having descent composition $(\bar{1},n-1)$. Since

$$\Theta\big(D_{(\bar{1},n-1)}\big)=\tilde{\Theta}\Big(Z_{\boxed{2}\boxed{3}\boxed{4}\boxed{\cdots}\boxed{n}\,,\,\boxed{1}}\Big)=\chi_{[n-1],[1]},$$

we have the following formulas for the multiplicity.

Corollary 3.4 *For $\boldsymbol{\lambda}\vdash n$, the multiplicity of $\mathcal{V}_{\boldsymbol{\lambda}}$ in ${\mathcal{V}_{[n-1],[1]}}^{\otimes d}$ is equal to*

$$\sum_{\mathbf{T}\in\mathrm{SYB}(\boldsymbol{\lambda})}[Z_{\mathbf{T}}]{D_{(\bar{1},n-1)}}^d.$$

Corollary 3.5 *Consider the Cayley graph $\Gamma(\mathcal{B}_n, \mathrm{supp}(D_{(\bar{1},n-1)}))$. For $\boldsymbol{\lambda} \vdash n$, the multiplicity of $\mathcal{V}_{\boldsymbol{\lambda}}$ in ${\mathcal{V}_{[n-1],[1]}}^{\otimes d}$ is equal to*

$$\sum_{\mathbf{T} \in \mathrm{SYB}(\boldsymbol{\lambda})} |\mathcal{W}_d(\pi_{\mathbf{T}})|,$$

where $\pi_{\mathbf{T}} \in \mathcal{B}_n$ is such that $\mathbf{Q}(\pi_{\mathbf{T}}) = \mathbf{T}$.

Example 3.6 *Using Corollary 3.4, the $\mathcal{B}_3$-module ${\mathcal{V}_{[2],[1]}}^{\otimes 4}$ decomposes into simple modules as*

$${\mathcal{V}_{[2],[1]}}^{\otimes 4} \cong 4\,\mathcal{V}_{[3],\emptyset} \oplus 7\,\mathcal{V}_{[2,1],\emptyset} \oplus 3\,\mathcal{V}_{[1,1,1],\emptyset} \oplus 10\,\mathcal{V}_{[1],[2]} \oplus 10\,\mathcal{V}_{[1],[1,1]}.$$

Indeed, the element ${D_{(\bar{1},2)}}^4$ of the Mantaci-Reutenauer algebra equals

$$4\,Z_{\boxed{1}\boxed{2}\boxed{3},\emptyset} + 3\,Z_{\begin{smallmatrix}\boxed{3}\\ \boxed{1}\boxed{2}\end{smallmatrix},\emptyset} + 4\,Z_{\begin{smallmatrix}\boxed{2}\\ \boxed{1}\boxed{3}\end{smallmatrix},\emptyset} + 3\,Z_{\begin{smallmatrix}\boxed{3}\\ \boxed{2}\\ \boxed{1}\end{smallmatrix},\emptyset} + 5\,Z_{\boxed{3},\boxed{2}\boxed{1}} + 2\,Z_{\boxed{2},\boxed{1}\boxed{3}} + 3\,Z_{\boxed{1},\boxed{2}\boxed{3}} + 5\,Z_{\boxed{3},\begin{smallmatrix}\boxed{2}\\ \boxed{1}\end{smallmatrix}} + 3\,Z_{\boxed{1},\begin{smallmatrix}\boxed{3}\\ \boxed{2}\end{smallmatrix}} + 2\,Z_{\boxed{2},\begin{smallmatrix}\boxed{3}\\ \boxed{1}\end{smallmatrix}}$$

and is sent to

$$4\,\chi_{[3],\emptyset} + 7\,\chi_{[2,1],\emptyset} + 3\,\chi_{[1,1,1],\emptyset} + 10\,\chi_{[1],[2]} + 10\,\chi_{[1],[1,1]}$$

via the map $\tilde{\Theta}$. Table 1 shows how these multiplicities can also be computed using Corollary 3.5 by considering words of length four in the Cayley graph of $\mathcal{B}_3$ with generators $\{\bar{1}23, \bar{2}13, \bar{3}12\}$.

$V_{\boldsymbol{\lambda}}$	$\mathbf{T} \in \mathrm{SYB}(\boldsymbol{\lambda})$	$\pi_{\mathbf{T}} \in \mathcal{B}_3$ $\mathbf{Q}(\pi_{\mathbf{T}}) = \mathbf{T}$	$\mathcal{W}_4(\pi_{\mathbf{T}})$	mult. of $V_{\boldsymbol{\lambda}}$ in ${V_{[2],[1]}}^{\otimes 4}$
$V_{[3],\emptyset}$	$\boxed{1}\boxed{2}\boxed{3}\,,\emptyset$	123	*aaaa* *abab* *baba* *bbbb*	4
$V_{[2,1],\emptyset}$	$\begin{smallmatrix}\boxed{3}\\ \boxed{1}\boxed{2}\end{smallmatrix}\,,\emptyset$	132	*acab* *caba* *cbbb*	
	$\begin{smallmatrix}\boxed{2}\\ \boxed{1}\boxed{3}\end{smallmatrix}\,,\emptyset$	213	*aaba* *abbb* *baaa* *bbab*	7
$V_{[1,1,1],\emptyset}$	$\begin{smallmatrix}\boxed{3}\\ \boxed{2}\\ \boxed{1}\end{smallmatrix}\,,\emptyset$	321	*baca* *bcbb* *ccab*	3
$V_{[1],[2]}$	$\boxed{1}\,,\boxed{2}\boxed{3}$	$3\bar{1}\bar{2}$	*bbca* *bccb* *cccc*	
	$\boxed{2}\,,\boxed{1}\boxed{3}$	$\bar{1}3\bar{2}$	*bcac* *ccbc*	
	$\boxed{3}\,,\boxed{1}\boxed{2}$	$\bar{1}\bar{2}3$	*aabb* *abba* *baab* *bbaa* *cacc*	10
$V_{[1],[1,1]}$	$\boxed{1}\,,\begin{smallmatrix}\boxed{3}\\ \boxed{2}\end{smallmatrix}$	$3\bar{2}\bar{1}$	*bcab* *caca* *ccbb*	
	$\boxed{2}\,,\begin{smallmatrix}\boxed{3}\\ \boxed{1}\end{smallmatrix}$	$\bar{2}3\bar{1}$	*acac* *cbbc*	
	$\boxed{3}\,,\begin{smallmatrix}\boxed{2}\\ \boxed{1}\end{smallmatrix}$	$\bar{2}\bar{1}3$	*aaab* *abaa* *babb* *bbba* *cabc*	10

Tab. 1: Decomposition of ${\mathcal{V}_{[2],[1]}}^{\otimes 4}$ using words in $\Gamma(\mathcal{B}_3, \{a, b, c\})$ where $a = \bar{1}23$, $b = \bar{2}13$ and $c = \bar{3}12$.

4 Algebra $T(\mathcal{V})^{\mathcal{B}_n}$ of invariants of $\mathcal{B}_n$

As a consequence of Theorem 3.3, we have a combinatorial interpretation for the graded dimension of the algebra $T(\mathcal{V})^{\mathcal{B}_n}$ of invariants of $\mathcal{B}_n$ in terms of words in a particular Cayley graph of $\mathcal{B}_n$.

Corollary 4.1 *Let $\mathcal{V}$ be a $\mathcal{B}_n$-module such that $\theta(f) = \chi_{\mathcal{V}}$, for some $f \in \mathcal{MR}_n$, and consider the Cayley graph $\Gamma(\mathcal{B}_n, \mathrm{supp}(f))$ with weight $\nu(\pi) = [\pi](f)$ for each $\pi \in \mathrm{supp}(f)$. Then*

$$\dim \left(\mathcal{V}^{\otimes d}\right)^{\mathcal{B}_n} = \sum_{w \in \mathcal{W}_d(e)} \nu(w).$$

Proof: The dimension of the space of invariants of $\mathcal{B}_n$ in $\mathcal{V}^{\otimes d}$ is equal to the multiplicity of the trivial module in $\mathcal{V}^{\otimes d}$. Then the result follows from Theorem 3.3. □

Another interesting result is that the number of free generators of the algebra of invariants of $\mathcal{B}_n$ can be counted by some special words in a particular Cayley graph of $\mathcal{B}_n$. These are the weighted words corresponding to paths which begin and end at the identity vertex, but without crossing the identity vertex.

Proposition 4.2 *Let $\mathcal{V}$ be a $\mathcal{B}_n$-module such that $\theta(f) = \chi_{\mathcal{V}}$, for some $f \in \mathcal{MR}_n$. Then the number of free generators of $T(\mathcal{V})^{\mathcal{B}_n}$ as an algebra are counted by the words which reduce to the identity without crossing the identity in the Cayley graph $\Gamma(\mathcal{B}_n, \mathrm{supp}(f))$ with weight $\nu(\pi) = [\pi](f)$ for each $\pi \in \mathrm{supp}(f)$.*

4.1 Algebra $T(\mathcal{V}_{[n-1],[1]})^{\mathcal{B}_n}$ of invariants of $\mathcal{B}_n$

We have an interpretation for the graded dimension of the space $T(\mathcal{V}_{[n-1],[1]})^{\mathcal{B}_n}$ of invariants of $\mathcal{B}_n$ in terms of paths starting from and ending at the identity vertex in the Cayley graph of $\mathcal{B}_n$ generated by the elements of $\mathcal{B}_n$ having descent composition $(\bar{1}, n-1)$. As a consequence of Corollary 4.1, we can easily compute these dimensions since

$$\Theta\big(D_{(\bar{1},n-1)}\big) = \chi_{[n-1],[1]}.$$

Corollary 4.3 *The dimension of $({\mathcal{V}_{[n-1],[1]}}^{\otimes d})^{\mathcal{B}_n}$ is equal to the number of words of length d which reduce to the identity in the Cayley graph $\Gamma(\mathcal{B}_n, \mathrm{supp}(D_{(\bar{1},n-1)}))$.*

Example 4.4 *When the group $\mathcal{B}_3$ acts on the polynomial ring $\mathbb{C}\langle x_1, x_2, x_3\rangle$ by $\pi(x_i) = sgn(\pi(i))\, x_{|\pi(i)|}$, the space $\mathbb{C}\langle x_1, x_2, x_3\rangle_4^{\mathcal{B}_3} \cong ({\mathcal{V}_{[2],[1]}}^{\otimes 4})^{\mathcal{B}_3}$ of invariants of $\mathcal{B}_3$ has a monomial basis indexed by the set partitions of $[4]$ with at most 3 parts of even cardinality (see Section 4.2):*

$$\begin{aligned}
\mathbf{m}_{\{1234\}}(x_1,x_2,x_3) &= x_1x_1x_1x_1 + x_2x_2x_2x_2 + x_3x_3x_3x_3,\\
\mathbf{m}_{\{12,34\}}(x_1,x_2,x_3) &= x_1x_1x_2x_2 + x_1x_1x_3x_3 + x_2x_2x_1x_1 + x_2x_2x_3x_3 + x_3x_3x_1x_1 + x_3x_3x_2x_2,\\
\mathbf{m}_{\{13,24\}}(x_1,x_2,x_3) &= x_1x_2x_1x_2 + x_1x_3x_1x_3 + x_2x_1x_2x_1 + x_2x_3x_2x_3 + x_3x_1x_3x_1 + x_3x_2x_3x_2,\\
\mathbf{m}_{\{14,23\}}(x_1,x_2,x_3) &= x_1x_2x_2x_1 + x_1x_3x_3x_1 + x_2x_1x_1x_2 + x_2x_3x_3x_2 + x_3x_1x_1x_3 + x_3x_2x_2x_3.
\end{aligned}$$

As recorded in Table 2, its cardinality equals the one of the set

$$\{aaaa, abab, baba, bbbb\}$$

of words of length 4 in the letters $a = \bar{1}23$, $b = \bar{2}13$ and $c = \bar{3}12$ which reduce to the identity in the Cayley graph $\Gamma(\mathcal{B}_3, \{a, b, c\})$.

In general, for any module $\mathcal{V}$, the algebra $T(\mathcal{V})^{\mathcal{B}_n}$ of invariants of $\mathcal{B}_n$ is freely generated [6], therefore we have the following relation between its Poincaré series and the generating series $\mathcal{F}(T(\mathcal{V})^{\mathcal{B}_n})$ counting the number of its free generators:

$$\mathcal{P}(T(\mathcal{V})^{\mathcal{B}_n}) = \frac{1}{1 - \mathcal{F}(T(\mathcal{V})^{\mathcal{B}_n})}. \tag{4.1}$$

The next corollary of Proposition 4.2 presents a nice interpretation for the number of these free generators.

Corollary 4.5 *The number of free generators of $T(\mathcal{V}_{[n-1],[1]})^{\mathcal{B}_n}$ as an algebra are counted by the words which reduce to the identity without crossing the identity in the Cayley graph $\Gamma(\mathcal{B}_n, \mathrm{supp}(D_{(\bar{1},n-1)}))$.*

Example 4.6 *The free generators of $T(\mathcal{V}_{[2],[1]})^{\mathcal{B}_3}$ are counted by the number of words which reduce to the identity without crossing the identity in the Cayley graph $\Gamma(\mathcal{B}_3, \{a, b, c\})$ where $a = \bar{1}23$, $b = \bar{2}13$ and $c = \bar{3}12$. They are*

		abaaab	*abbabb*	*abbbba*	*baaaba*	
		baabbb	*babbab*	*bbaabb*	*bbabba*	
	abab	*bbbaab*	*abcabc*	*acacac*	*accbbc*	
aa	*baba*	*bbcacc*	*bcabca*	*bcacca*	*bccbcc*	$\cdots$
	bbbb	*baccab*	*cacaca*	*caccbb*	*cbbcac*	
		cbccbc	*ccbbca*	*ccbccb*	*cccccc*	

Using relation (4.1) *and the analogue of Molien's Theorem* (1.1)*, the generating series for the number of free generators is given by*

$$\begin{aligned}
\mathcal{F}(T(\mathcal{V}_{[2],[1]})^{\mathcal{B}_3}) &= 1 - \mathcal{P}(T(\mathcal{V}_{[2],[1]})^{\mathcal{B}_3})^{-1} \\
&= 1 - \left(\frac{1}{48} \left\{ \frac{1}{(1-3q)} + \frac{15}{(1-q)} + 16 + \frac{15}{(1+q)} + \frac{1}{(1+3q)} \right\} \right)^{-1} \\
&= \frac{q^2 - 6q^4}{1 - 9q^2 + 3q^4},
\end{aligned}$$

with series expansion $q^2 + 3q^4 + 24q^6 + 207q^8 + 1791q^{10} + 15498q^{12} + 134109q^{14} + 1160487q^{16} + \cdots$

4.2 Applications to set partitions

A *set partition of* $[n]$, denoted by $A \vdash [n]$, is a family of disjoint nonempty subsets $A_1, A_2, \ldots, A_k \subseteq [n]$ such that $A_1 \cup A_2 \cup \ldots \cup A_k = [n]$. The subsets A_i are called the *parts* of A. The algebra

$$T(\mathcal{V}_{[n-1],[1]})^{\mathcal{B}_n} \cong \mathbb{C}\langle \mathbf{x} \rangle^{\mathcal{B}_n}$$

corresponds to the space of polynomials in noncommutative variables $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ which are invariant under the action of $\mathcal{B}_n$ defined in Example 4.4. Using the fact that a monomial basis for the space $\mathbb{C}\langle \mathbf{x} \rangle^{\mathcal{B}_n}$ of invariants of $\mathcal{B}_n$ is indexed by the set partitions with at most n parts of even cardinality, a closed formula for the Poincaré series of $T(\mathcal{V}_{[n-1],[1]})^{\mathcal{B}_n}$ has been proved in [2] and is given by

$$\mathcal{P}(T(\mathcal{V}_{[n-1],[1]})^{\mathcal{B}_n}) = 1 + \sum_{k=1}^{n} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1) q^{2k}}{(1-q^2)\,(1-4\,q^2) \cdots (1-k^2\,q^2)}.$$

The words considered in the Cayley graph of $\mathcal{B}_n$, with generators the elements having descent composition $(\bar{1}, n-1)$, have a different nature to that of set partitions. But from Corollary 4.3, we can show for instance the following result.

Corollary 4.7 *The number of set partitions of* $[2d]$ *into at most* n *parts is the number of words of length* $2d$ *which reduce to the identity in the Cayley graph* $\Gamma(\mathcal{B}_n, \mathrm{supp}(D_{(\bar{1},n-1)}))$.

5 Appendix

The table in this section represents the words of length 2, 3 and 4 which reduce to a specific element in the Cayley graph $\Gamma(\mathcal{B}_3, \{\bar{1}23, \bar{2}13, \bar{3}12\})$.

123	132	231	213	312	321	$23\bar{1}$	$13\bar{2}$	$12\bar{3}$	$2\bar{1}3$	$1\bar{2}3$	$1\bar{3}2$
aa			ba	ca							
									aba bbb	abb bba	acb cba
$aaaa$ $abab$ $baba$ $bbbb$	$acab$ $caba$ $cbbb$	$bcab$ $caca$ $ccbb$	$aaba$ $abbb$ $baaa$ $bbab$	$aaca$ $acbb$ $caaa$ $cbab$	$baca$ $bcbb$ $ccab$						
$\bar{1}23$	$\bar{2}13$	$\bar{3}12$	$32\bar{1}$	$31\bar{2}$	$21\bar{3}$	$3\bar{1}2$	$3\bar{2}1$	$2\bar{3}1$	$\bar{1}32$	$\bar{2}31$	$\bar{3}21$
aaa bab	aab baa	aac caa				aca cbb	bca ccb	bcb cca	cab	cac	bac
$3\bar{1}\bar{2}$	$2\bar{1}\bar{3}$	$1\bar{2}\bar{3}$	$\bar{1}3\bar{2}$	$\bar{1}2\bar{3}$	$\bar{2}1\bar{3}$	$\bar{1}\bar{2}3$	$\bar{1}\bar{3}2$	$\bar{2}\bar{3}1$	$3\bar{2}\bar{1}$	$2\bar{3}\bar{1}$	$1\bar{3}\bar{2}$
						bb	cb	cc			
$bbca$ $bccb$ $cccc$	$bccc$ $cbca$ $cccb$	$accc$ $cbcb$ $ccca$	$bcac$ $ccbc$	$bcbc$ $ccac$	$acbc$ $cbac$	$aabb$ $abba$ $baab$ $bbaa$ $cacc$	$aacb$ $acbc$ $bacc$ $caab$ $cbaa$	$aacc$ $baca$ $bcba$ $caac$ $ccaa$	$bcab$ $caca$ $ccbb$	$abca$ $accb$ $cbcc$	$abcb$ $acca$ $bbcc$
$\bar{2}3\bar{1}$	$\bar{3}2\bar{1}$	$\bar{3}1\bar{2}$	$\bar{2}\bar{1}\bar{3}$	$\bar{3}\bar{1}\bar{2}$	$\bar{3}\bar{2}\bar{1}$	$\bar{3}2\bar{1}$	$\bar{2}3\bar{1}$	$\bar{1}3\bar{2}$	$\bar{3}1\bar{2}$	$\bar{2}1\bar{3}$	$\bar{1}2\bar{3}$
			ab	ac	bc						
						abc	acc	bcc	bbc	cbc	ccc
$acac$ $cbbc$	$abac$ $bbbc$	$abbc$ $bbac$	$aaab$ $abaa$ $babb$ $bbba$ $cabc$	$aaac$ $acaa$ $babc$ $cabb$ $cbba$	$aabc$ $baac$ $bcaa$ $cacb$ $ccba$						

Tab. 2: Words in $\Gamma(\mathcal{B}_3, \{a, b, c\})$, where $a = \bar{1}23$, $b = \bar{2}13$ and $c = \bar{3}12$.

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