FPSAC 2010, San Francisco, USA DMTCS proc. **AN**, 2010, 802–808

An algorithm which generates linear extensions for a generalized Young diagram with uniform probability

Kento NAKADA and Shuji OKAMURA

K. NAKADA : Wakkanai Hokusei Gakuen University, Faculty of Integrated Media. nakada@wakhok.ac.jp
S. OKAMURA : Osaka Prefectural College of Technology. okamura@ipc.osaka-pct.ac.jp

Abstract. The purpose of this paper is to present an algorithm which generates linear extensions for a generalized Young diagram, in the sense of D. Peterson and R. A. Proctor, with uniform probability. This gives a proof of a D. Peterson's hook formula for the number of reduced decompositions of a given minuscule elements.

Résumé. Le but de ce papier est présenter un algorithme qui produit des extensions linéaires pour un Young diagramme généralisé dans le sens de D. Peterson et R. A. Proctor, avec probabilité constante. Cela donne une preuve de la hook formule d'un D. Peterson pour le nombre de décompositions réduites d'un éléments minuscules donné.

Keywords: Generalized Young diagrams, Algorithm, linear extension, Kac-Moody Lie algebra

1 Introduction

In [3], C. Greene, A. Nijenhuis, and H. S. Wilf constructed an algorithm which generates standard tableaux for a given Young diagram with uniform probability. This provides a proof of the hook formula [2] for the number of the standard tableaux of a Young diagram, which is originally due to J. S. Frame, G. de B. Robinson, and R. M. Thrall.

As a generalization of the result of [3], the second author constructed an algorithm, in his master's thesis [9], which generates standard tableau of a given generalized Young diagram. Here, a "generalized Young diagram" is one in the sense of D. Peterson and R. A. Proctor. Similary, this result provides a proof of the hook formula for the number of the standard tableaux of a generalized Young diagram. The purpose of this paper is to present the following theorem:

Theorem 1.1 *Let λ be a finite pre-dominant integral weight over a simply-laced Kac-Moody Lie algebra. Let L be a linear extension of the diagram $\mathrm{D}(\lambda)$ of λ. Then the algorithm A for $\mathrm{D}(\lambda)$ generates L with the probability:*

$$\mathrm{Prob}_{\mathrm{D}(\lambda)}(L) = \frac{\prod_{\beta\in\mathrm{D}(\lambda)} \mathrm{ht}\,(\beta)}{(\#\mathrm{D}(\lambda))!}.$$

Here, λ is a certain integral weight (see section 5), $\mathrm{D}(\lambda)$ a certain set of positive real roots determined by λ (section 5), linear extension is a certain sequence of elements of $\mathrm{D}(\lambda)$ (section 2 and 5). $\mathrm{Prob}_{\mathrm{D}(\lambda)}(L)$ the probability we get L by the algorithm A for a diagram $\mathrm{D}(\lambda)$ (section 2), and $\mathrm{ht}\,(\beta)$ denotes the height of β.

2 An algorithm for a graph $(\Gamma; \rightarrow)$

Let $\Gamma = (\Gamma; \rightarrow)$ be a finite simple directed acyclic graph, where $\rightarrow$ denotes the adjacency relation of Γ.

Definition 2.1 *Put* $d := \#\Gamma$. *A bijection* $L : \{1, \cdots, d\} \longrightarrow \Gamma$ *is said to be a* linear extension of $(\Gamma; \rightarrow)$ *if:*

$$L(k) \rightarrow L(l) \textit{ implies } k > l, \quad k, l \in \{1, \cdots, d\}.$$

The set of linear extensions of $(\Gamma; \rightarrow)$ *is denoted by* $\mathcal{L}\,(\Gamma; \rightarrow)$.

For a given $v \in \Gamma$, we define a set $\mathrm{H}_{\Gamma}\,(v)^{+}$ by:

$$\mathrm{H}_{\Gamma}\,(v)^{+} := \left\{ v' \in \Gamma \,\middle|\, v \rightarrow v' \right\}.$$

For a given Γ, we call the following algorithm the *algorithm* A *for* Γ:

GNW1. Set $i := 0$ and set $\Gamma_0 := \Gamma$.

GNW2. (Now Γ_i has $d - i$ nodes.) Set $j := 1$ and pick a node $v_1 \in \Gamma_i$ with the probability $1/(d - i)$.

GNW3. If $\#\mathrm{H}_{\Gamma_i}\,(v_j)^{+} \neq 0$, pick a node $v_{j+1} \in \mathrm{H}_{\Gamma_i}\,(v_j)^{+}$ with the probability $1/\#\mathrm{H}_{\Gamma_i}\,(v_j)^{+}$. If not, go to GNW5.

GNW4. Set $j := j + 1$ and return to GNW3.

GNW5. (Now $\#\mathrm{H}_{\Gamma_i}\,(v_j)^{+} = 0$.) Set $L(i+1) := v_j$ and set $\Gamma_{i+1} := \Gamma_i \setminus v_j$ (the graph deleted v_j from Γ_i).

GNW6. Set $i := i + 1$. If $i < d$, return to GNW2; if $i = d$, terminate.

Then, by the definition of the algorithm A for Γ, the map $L : i \mapsto L(i)$ generated above is a linear extension of Γ. We denote by $\mathrm{Prob}_{\Gamma}(L)$ the probability we get $L \in \mathcal{L}\,(\Gamma)$ by the algorithm A.

3 Case of Young diagrams

When we draw a Young diagram, we use *nodes* instead of *cells* like FIGURE 3.1(left) below:

Definition 3.1 *We equip the set* $\mathbb{Y} := \mathbb{N} \times \mathbb{N}$ *with the partial order:*

$$(i, j) \leq (i', j') \iff i \geq i' \text{ and } j \geq j'.$$

A finite order filter Y *of* $\mathbb{Y}$ *is called a* Young diagram.

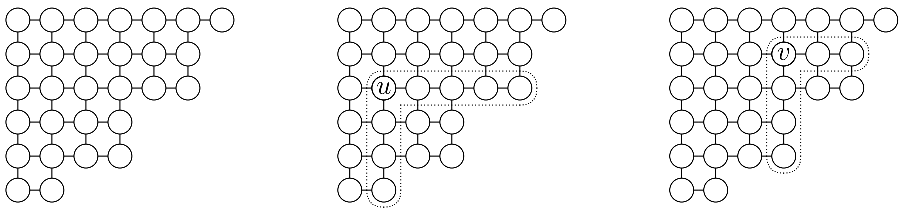

Fig. 3.1: a Young diagram and Hooks of u and v

Definition 3.2 *Let Y be a Young diagram. Let $v = (i,j) \in Y$. We define the subset $\mathrm{H}_Y(v)$ of Y by:*

$$\mathrm{Arm}(v) := \left\{ (i',j') \in Y \,\middle|\, i = i' \text{ and } j < j' \right\}.$$
$$\mathrm{Leg}(v) := \left\{ (i',j') \in Y \,\middle|\, i < i' \text{ and } j = j' \right\}.$$

$$\mathrm{H}_Y(v) := \{v\} \sqcup \mathrm{Arm}(v) \sqcup \mathrm{Leg}(v).$$
$$\mathrm{H}_Y(v)^+ := \mathrm{Arm}(v) \sqcup \mathrm{Leg}(v).$$

The set $\mathrm{H}_Y(v)$ is called the hook of $v \in Y$ *(see FIGURE 3.1(right)).*

For $v, v' \in Y$, we define a relation $v \to v'$ by $v' \in \mathrm{H}_Y(v)^+$. Then $(Y; \to)$ is a finite simple directed acyclic graph.

Then we have the following theorem:

Theorem 3.3 (C. Greene, A. Nijenhuis, and H. S. Wilf [3]) *Let $(Y; \to)$ be a graph defined above for a Young diagram Y. Let $L \in \mathcal{L}(Y; \to)$. Then the algorithm A for $(Y; \to)$ generates L with the probability*

$$\mathrm{Prob}_{(Y;\leq)}(L) = \frac{\prod_{v \in Y} \#\mathrm{H}_Y(v)}{\#Y!}. \tag{3.1}$$

Since the right hand side of (3.1) is independent from the choice of $L \in \mathcal{L}(Y; \to)$, we have:

Corollary 3.4 *Let $(Y; \to)$ be a graph for a Young diagram Y. Then we have:*

$$\#\mathcal{L}(Y; \to) = \frac{\#Y!}{\prod_{v \in Y} \#\mathrm{H}_Y(v)}.$$

This gives a proof of hook length formula [2] for the number of standard tableaux for a Young diagram.

4 Case of shifted Young diagrams

Definition 4.1 *We equip the $\mathbb{S} := \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} \,\middle|\, i \leq j \right\}$ with the partial order:*

$$(i,j) \leq (i',j') \Longleftrightarrow i \geq i' \text{ and } j \geq j'.$$

A finite order filter S of $\mathbb{S}$ is called a shifted Young diagram.

Definition 4.2 *Let S be a shifted Young diagram. Let $v = (i, j) \in S$. We define the subset $\mathrm{H}_S(v)$ of S by:*

$$\mathrm{Arm}_S(v) := \left\{ (i', j') \in S \,\middle|\, i = i' \text{ and } j < j' \right\}.$$
$$\mathrm{Leg}_S(v) := \left\{ (i', j') \in S \,\middle|\, i < i' \text{ and } j = j' \right\}.$$
$$\mathrm{Tail}_S(v) := \left\{ (i', j') \in S \,\middle|\, j + 1 = i' \text{ and } j < j' \right\}.$$

$$\mathrm{H}_S(v) := \{v\} \sqcup \mathrm{Arm}_S(v) \sqcup \mathrm{Leg}_S(v) \sqcup \mathrm{Tail}_S(v).$$
$$\mathrm{H}_S(v)^+ := \mathrm{Arm}_S(v) \sqcup \mathrm{Leg}_S(v) \sqcup \mathrm{Tail}_S(v).$$

The set $\mathrm{H}_S(v)$ is called the hook of $v \in S$ *(see FIGURE 4.1).*

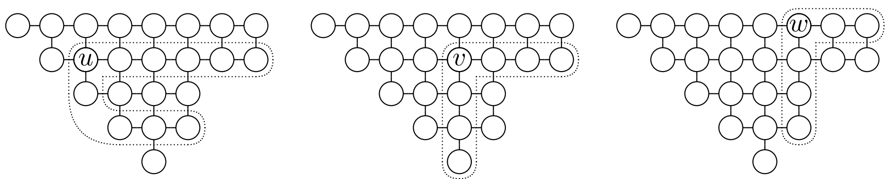

Fig. 4.1: Hooks of u, v , and w.

For $v, v' \in Y$, we define a relation $v \to v'$ by $v' \in \mathrm{H}_Y(v)^+$. Then $(Y; \to)$ is a finite simple directed acyclic graph.

Then we have the following theorem:

Theorem 4.3 (B. E. Sagan [11]) *Let $(S; \to)$ be a graph defined above for a shifted Young diagram S. Let $L \in \mathcal{L}(S; \to)$. Then the algorithm* A *for $(S; \to)$ generates L with the probability*

$$\mathrm{Prob}_{(Y;\leq)}(L) = \frac{\prod_{v \in S} \#\mathrm{H}_S(v)}{\#S!}. \tag{4.1}$$

Since the right hand side of (4.1) is independent from the choice of $L \in \mathcal{L}(S; \to)$, we have:

Corollary 4.4 *Let $(S; \to)$ be a graph for a shifted Young diagram S. Then we have:*

$$\#\mathcal{L}(S; \to) = \frac{\#S!}{\prod_{v \in S} \#\mathrm{H}_S(v)}.$$

This gives a proof of hook length formula [12] for the number of standard tableaux for a shifted Young diagram.

5 General case

In this section, we fix a simply-laced Kac-Moody Lie algebra $\mathfrak{g}$ with a simple root system $\Pi = \left\{ \alpha_i \,\middle|\, \in I \right\}$. For all undefined terminology in this section, we refer the reader to [4] [5].

Definition 5.1 *An integral weight λ is said to be* pre-dominant *if:*

$$\langle \lambda,\, \beta^\vee \rangle \geq -1 \quad \textit{for each } \beta^\vee \in \Phi_+^\vee,$$

where $\Phi_+^\vee$ denotes the set of positive real coroots. The set of pre-dominant integral weights is denoted by $P_{\geq -1}$. For $\lambda \in P_{\geq -1}$, we define the set $\mathrm{D}(\lambda)$ by:

$$\mathrm{D}(\lambda) := \left\{ \beta \in \Phi_+ \,\middle|\, \langle \lambda,\, \beta^\vee \rangle = -1 \right\}.$$

The set $\mathrm{D}(\lambda)$ is called the diagram of λ. *If $\#\mathrm{D}(\lambda) < \infty$, then λ is called* finite.

5.1 Hooks

Definition 5.2 *Let $\lambda \in P_{\geq -1}$ and $\beta \in \mathrm{D}(\lambda)$. We define the set $\mathrm{H}_\lambda(\beta)$ by:*

$$\mathrm{H}_\lambda(\beta) := \mathrm{D}(\lambda) \cap \Phi(s_\beta),$$
$$\mathrm{H}_\lambda(\beta)^+ := \mathrm{H}_\lambda(\beta) \setminus \{\beta\}.$$

where $\Phi(s_\beta)$ denotes the inversion set of the reflection corresponding to β:

$$\Phi(s_\beta) = \left\{ \gamma \in \Phi_+ \,\middle|\, s_\beta(\gamma) < 0 \right\}.$$

Proposition 5.3 (see [6],[8]) *Let $\lambda \in P_{\geq -1}$ be finite and $\beta^\vee \in \mathrm{D}(\lambda)$. Then we have:*

1. $\#\mathrm{H}_\lambda(\beta) = \mathrm{ht}(\beta)$.
2. *If $\gamma \in \mathrm{H}_\lambda(\beta)$, then $\gamma \leq \beta$.*

By proposition 5.3 (2), defining $\beta \to \gamma$ by $\gamma \in \mathrm{H}_\lambda(\beta)^+$, the graph $(\mathrm{D}(\lambda); \to)$ is acyclic.

5.2 Main Theorem and Corollaries

Theorem 5.4 (see [8][9]) *Let $\lambda \in P_{\geq -1}$ be finite. Let $L \in \mathcal{L}(\mathrm{D}(\lambda); \to)$. Then the algorithm* A *for $(\mathrm{D}(\lambda); \to)$ generates L with the probability*

$$\mathrm{Prob}_{(\mathrm{D}(\lambda);\to)}(L) = \frac{\prod_{\beta \in \mathrm{D}(\lambda)} \mathrm{ht}(\beta)}{\#\mathrm{D}(\lambda)!}. \tag{5.1}$$

Remark 5.5 *The original statement of theorem 5.4 was proved by the second author [9]. The proof in [9] was done by case-by case argument. On the other hand, the proof in [8] is given by an application of the very special case of the colored hook formula [6].*

Since the right hand side of (5.1) is independent from the choice of $L \in \mathcal{L}(\mathrm{D}(\lambda); \to)$, we have:

Corollary 5.6 *Let $\lambda \in P_{\geq -1}$ be finite. Then we have:*

$$\#\mathcal{L}(\mathrm{D}(\lambda); \to) = \frac{\#\mathrm{D}(\lambda)}{\prod_{\beta \in \mathrm{D}(\lambda)} \mathrm{ht}(\beta)}.$$

Corollary 5.6 gives a proof of Peterson's hook formula for the number of reduced decompositions of minuscule [1][6] element, in simply-laced case. Another proof of Peterson's hook formula is given in [6].

Remark 5.7 *The finite pre-dominant integral weights λ are identified with the minuscule elements w [6]. And, we have $\mathrm{D}(\lambda) = \Phi(w)$ [6]. Furthermore, the linear extensions of $\mathrm{D}(\lambda)$ are identified with the reduced decompositions of w [6] by the following one-to-one correspondence:*

$$\mathrm{Red}(w) \ni (s_{i_1}, s_{i_2}, \cdots, s_{i_d}) \longleftrightarrow L \in \mathcal{L}(\mathrm{D}(\lambda)), \qquad L(k) = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \mathrm{D}(\lambda) \ (k = 1, \cdots d),$$

where $\mathrm{Red}(w)$ denotes the set of reduced decompositions of w, $d = \ell(w)$ the length of w. Hence, corollary 5.6 is equivalent with the Peterson's hook formula:

$$\#\mathrm{Red}(w) = \frac{\ell(w)!}{\prod_{\beta \in \Phi(w)} \mathrm{ht}(\beta)}.$$

Remark 5.8 *A Young diagram is realized as a diagram for some pre-dominant integral weight over a root system of type A. Similarly, a shifted Young diagram is realized as a diagram for some pre-dominant integral weight over a root system of type D.*

There are 15 classes of generalized Young diagrams (over simply-laced Kac-Moody Lie algebras). We note that many of them are realized over root systems of indefinite types (see [10]).

Remark 5.9 *The first author has also succeeded in proving Theorem 1.1 in the case of a root system of type B by a certain similar algorithm [7].*

References

[1] J. B. Carrell, *Vector fields, flag varieties and Schubert calculus,* Proc. Hyderabad Conference on Algebraic Groups (ed. S.Ramanan), Manoj Prakashan, Madras, 1991.

[2] J. S. Frame, G. de B. Robinson, and R. M. Thrall, *The hook graphs of symmetric group,* Canad. J. Math. **6** (1954),316-325.

[3] C. Greene, A. Nijenhuis, and H. S. Wilf, *A probabilistic proof of a formula for the number of Young tableaux of a given shape*, Adv. in Math. **31** (1979), 104-109.

[4] V. G. Kac, "Infinite Dimentional Lie Algebras," Cambridge Univ. Press, Cambridge, UK, 1990.

[5] R. V. Moody and A. Pianzola, "Lie Algebras With Triangular Decompositions," Canadian Mathematical Society Series of Monograph and Advanced Text, 1995.

[6] K. Nakada, *Colored hook formula for a generalized Young diagram,* Osaka J. of Math. **Vol.45 No.4** (2008), 1085-1120.

[7] K. Nakada, *Another proof of hook formula for a shifted Young diagram*, in preparation.

[8] K. Nakada and S. Okamura, *Uniform generation of standard tableaux of a generalized Young diagram*, preprint.

[9] S. Okamura, *An algorithm which generates a random standard tableau on a generalized Young diagram* (in Japanese), master's thesis, Osaka university, 2003.

[10] R. A. Proctor, *Dynkin diagram classification of λ-minuscule Bruhat lattices and of d-complete posets,* J.Algebraic Combin. **9** (1999), 61-94.

[11] B. E. Sagan, *On selecting a random shifted Young tableaux*, J. Algorithm **1** (1980), 213-234.

[12] R. M. Thrall, *A combinatorial problem,* Mich.Math.J. **1** (1952), 81-88.